SATHYABAMA

INSTITUTE OF SCIENCE AND TECHNOLOGY

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SCHOOL OF BUILDING AND ENVIRONMENT

DEPARTMENT OF CIVIL ENGINEERING

UNIT – I - BASICS & STATICS OF PARTICLES– SCIA1101

I. Basics & Statics Of Particles

Introduction - Units and Dimensions - Laws of Mechanics - Vectors - Vectorial representation of forces and moments -Vector operations, Coplanar forces resolution and composition of forces - Equilibrium of particle - Forces in space - Equilibrium of a particle in space - Equivalent systems of forces - Principle of transmissibility - Single equivalent force.

INTRODUCTION

Engineering Mechanics is all about mechanical interaction between bodies which means we will learn how different bodies apply forces on one another and how they then balance to keep each other in equilibrium. The branch of physical science that deals with the state of rest or the state of motion is termed as Mechanics. The state of rest and state of motion of the bodies under the action of different forces has engaged the attention of theorists, mathematicians and scientists. Starting from the analysis of rigid bodies under gravitational force and simple applied forces the mechanics has grown to the analysis of robotics, aircrafts, spacecrafts under dynamic forces, atmospheric forces, temperature forces etc.

Engineering mechanics is the application of mechanics to solve problems involving common engineering elements. The engineering mechanics is mainly classified into two branches. They are

1. Statics
2. Dynamics

Statics - Statics deals with the forces on a body at rest.

Dynamics - Dynamics deals with the forces acting on a body when the body is in motion.

Dynamics further subdivided in to two sub branches.

They are:

- **Kinematics:** Deals the motion of a body without considering the forces causing the motion.
- **Kinetics:** Deals with the relation between the forces acting on the body and the resulting motion

IMPORTANCE OF MECHANICS TO ENGINEERING

- For designing and manufacturing of various mechanical tools and equipments
- For calculation and estimation of forces of bodies while they are in use.
- For designing and construing to dams, roads, sheds, structure, building etc.
- For designing a fabrication of rockets.

UNITS AND DIMENSIONS

Length (L), Mass (M) and Time (S) are the fundamental units in mechanics. The units of all other quantities may be expressed in terms of these basic units.

The three commonly used systems in engineering are

Metre—Kilogramme—Second (MKS) system,

Centimetre—Gramme—Second (CGS) system, and

Foot—Pound—Second (FPS) system.

The units of length, mass and time used in the system are used to name the systems. Using these basic units, the units for other quantities can be found.

Fundamental quantities

- The quantities that are independent of other quantities are called fundamental quantities.
- The units that are used to measure these fundamental quantities are called fundamental units.
- There are four systems of units namely C.G.S, M.K.S, F.P.S, and SI.

Derived quantities

- The quantities that are derived using the fundamental quantities are called derived quantities.
- The units that are used to measure these derived quantities are called derived units.

International System of Units

SI base units - The SI is founded on seven SI base units for seven base quantities assumed to be mutually independent, as given in Table below:

Table 1 Base Quantity and its units

BASE QUANTITY	NAME	SYMBOL
	SI BASED UNIT	
Length	Meter	m
Mass	Kilogram	kg
Time	Second	s
Electric Current	Ampere	A
Thermodynamic Temperature	Kelvin	K
Amount of Substance	Mole	mol
Luminous Intensity	Candela	cd

Table 2 Examples of SI derived units

Area	Square meter	**m^2**
Volume	Cubic meter	**m^3**
Speed, Velocity	Meter per second	**m/s**
Acceleration	Meter per second squared	**m/s^2**
Wave Number	Reciprocal meter	**m^{-1}**
Mass Density	Kilogram per cubic meter	**kg/m^3**
Specific Volume	Cubic meter per kilogram	**m^3/kg**
Current Density	Ampere per square meter	**A/m^2**
Magnetic Field Strength	Ampere per meter	**A/m**
Amount-of-substance Concentration	Mole per cubic meter	**mol/m^3**
Luminance	Candela per square meter	**cd/m^2**
Mass Fraction	Kilogram per kilogram, which may be represented by the number 1	**kg/kg = 1**

Table 3 Derived SI units with special names

Physical quantity	**SI unit**	**Symbol**
Frequency	hertz	Hz
Energy	joule	J
Force	newton	N
Power	watt	W
Pressure	pascal	Pa
Electric charge or quantity of electricity	coulomb	C
Electric potential difference and emf	volt	V
Electric resistance	ohm	Omega / Ω
Electric conductance	siemen	S
Electric capacitance	farad	F
Magnetic flux	weber	Wb
Inductance	henry	H
Magnetic flux density	tesla	T
Illumination	lux	Lx
Luminous flux	lumen	Lm

Table 4 Dimensional Formulas for Physical Quantities

Physical quantity	Unit	Dimensional formula
Acceleration or acceleration due to gravity	ms^{-2}	LT^{-2}
Angle (arc/radius)	rad	$M^oL^oT^o$
Angular displacement	rad	$M^ol^oT^o$
Angular frequency (angular displacement/time)	$rads^{-1}$	T^{-1}
Angular impulse (torque x time)	Nms	ML^2T^{-1}
Angular momentum (Iω)	kgm^2s^{-1}	ML^2T^{-1}
Angular velocity (angle/time)	$rads^{-1}$	T^{-1}

Area (length x breadth)	m^2	L^2

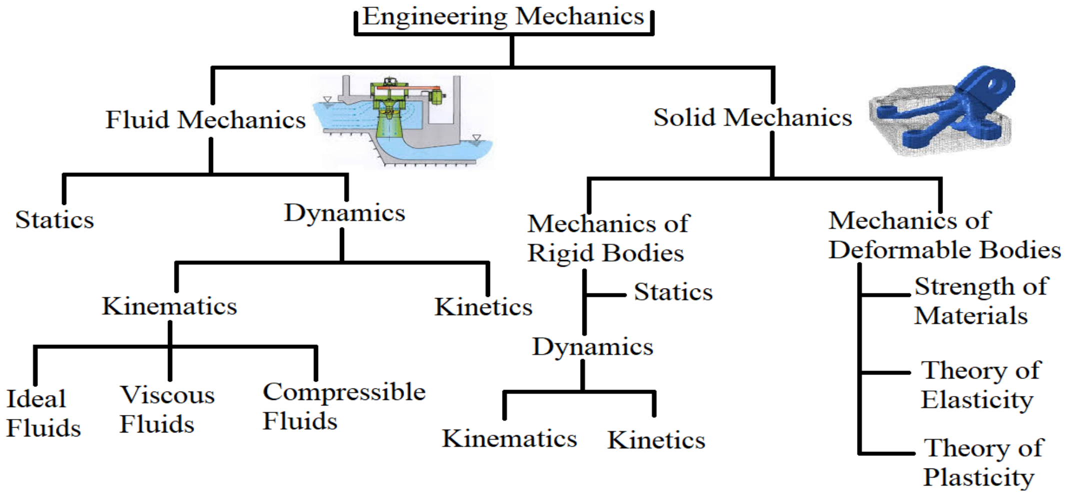


Fig. 1 Classification of Engineering Mechanics

LAWS OF MECHANICS

- Newton's First Law of Motion
- Newton's Second Law of Motion
- Newton's Third Law of Motion
- Newton's Law of Gravitation
- Parallelogram law of forces
- Principles of Transmissibility

Newton's first law of Motion

It states that each and every body continues in its state of rest or of uniform motion in a straight line unless it is compelled by external agency acting on it. This leads to the definition of force as the external agency which changes or tends to change the state of rest or uniform linear motion of the body.

Everybody continues in a state of rest or uniform motion in a straight line unless it is compelled to change that state by some external force acting on it.

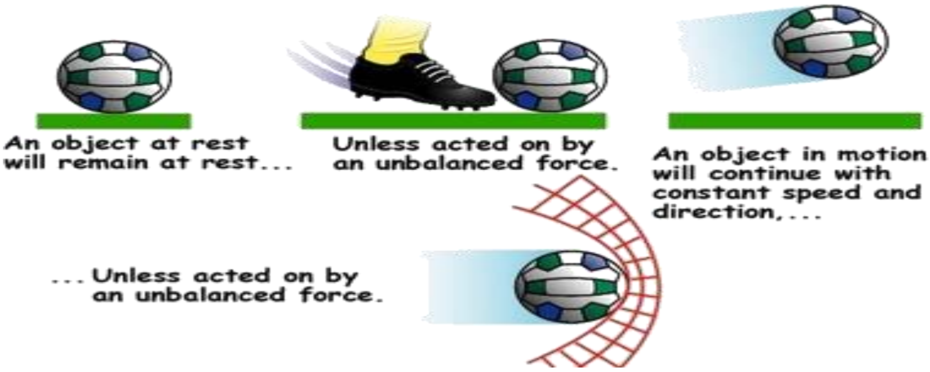


Fig. 2 Example for Newton's first law of Motion

Newton's Second Law

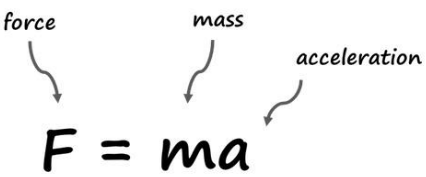
It states that the rate of change of momentum of a body is directly proportional to the impressed force and it takes place in the direction of the force acting on it. Thus according to this law,

Force ∝ Rate of change of momentum. But momentum = Mass × Velocity

As mass do not change, Force ∝ Mass × Rate of change of velocity

$$\textit{Force} \propto \textit{Mass} \times \textit{Acceleration}$$

$$F \propto m \times a$$



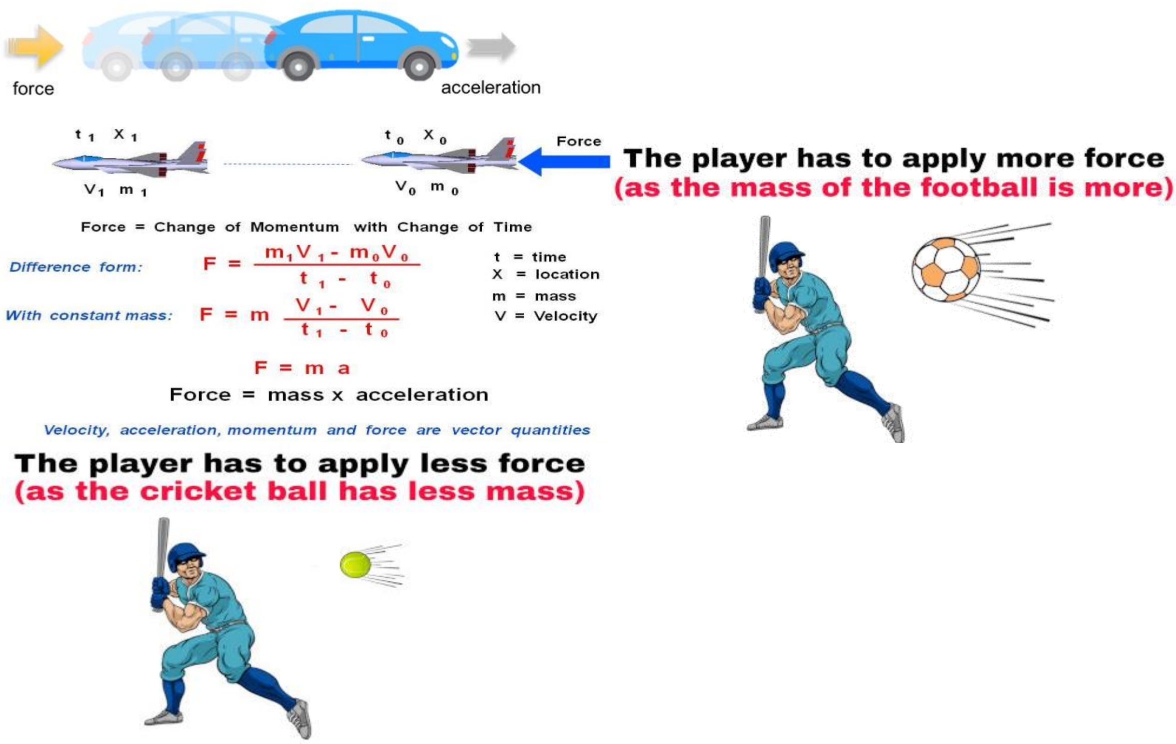


Fig. 3 Example for Newton's second law

Newton's Third Law

It states that for every action there is an equal and opposite reaction. Consider the two bodies in contact with each other. Let one body applies a force *F on another. According to this* law, the second body develops a reactive force *R which is equal in magnitude to force F and* acts in the line same as *F but in the opposite direction.*

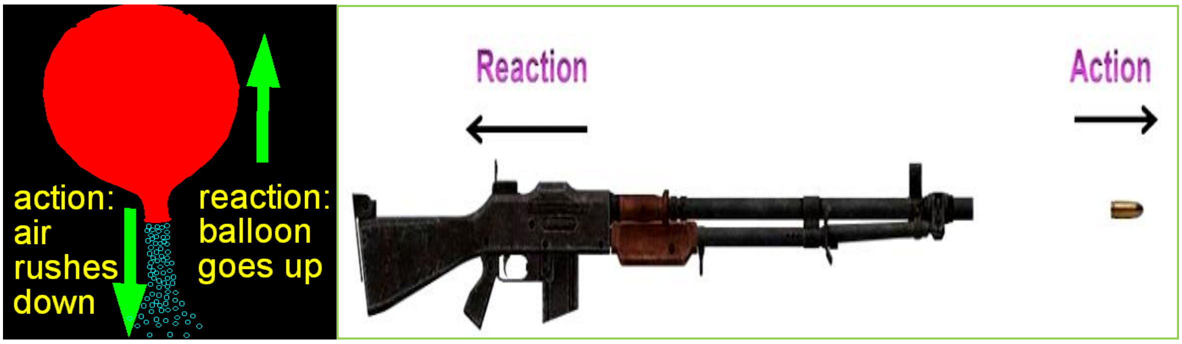


Fig. 4 Example for Newton's third law

Newton's Law of Gravitation

The force of attraction between any two bodies is directly proportional to their masses and inversely proportional to the square of the distance between them. According to this law, the force of attraction between the bodies of mass m_1 and mass m_2 at a distance d as shown in figure below is

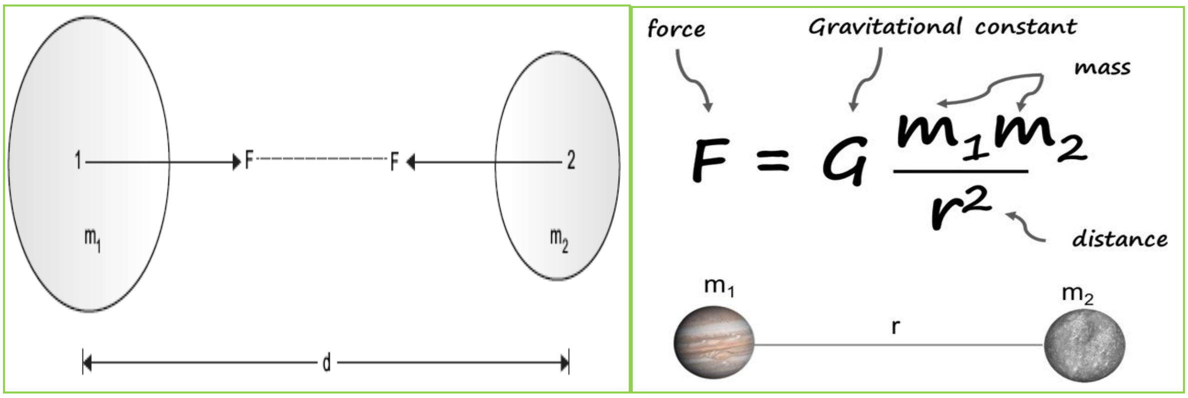


Fig. 5 Example for Newton's Law of Gravitation

$$\mathbf{F = G\, m_1\, m_2/d^2}$$

Where,

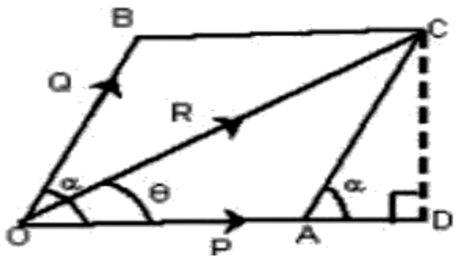
G is the constant of proportionality and is known as constant of gravitation.

(Or)

It states that two bodies will be attracted towards each other along their connecting line with a force which is directly proportional to the product of their masses and inversely proportional to the square of the distance between the centres.

Parallelogram law of forces

If two forces acting at a point be represented in magnitude and direction by the two adjacent sides of a parallelogram, then their resultant is represented in magnitude and direction by the diagonal of the parallelogram ram passing through that point.



The magnitude of Resultant force R

$$R = \sqrt{P^2 + Q^2 + 2PQcos\alpha}$$

The direction of Resultant force R with the force P

$$\theta = \tan^{-1}\left(\frac{Q\sin\alpha}{P + Q\cos\alpha}\right)$$

Fig. 6 Parallelogram law of forces

Principles of Transmissibility

The general principle states that the effect of force acting on a rigid body does not change if the force is moved along its line of action to another point on the body.

Example: Let F be the force acting on a rigid body at point A as shown in figure given below According to the law of transmissibility of force, this force has the same effect on the state of body as the force F applied at point B.

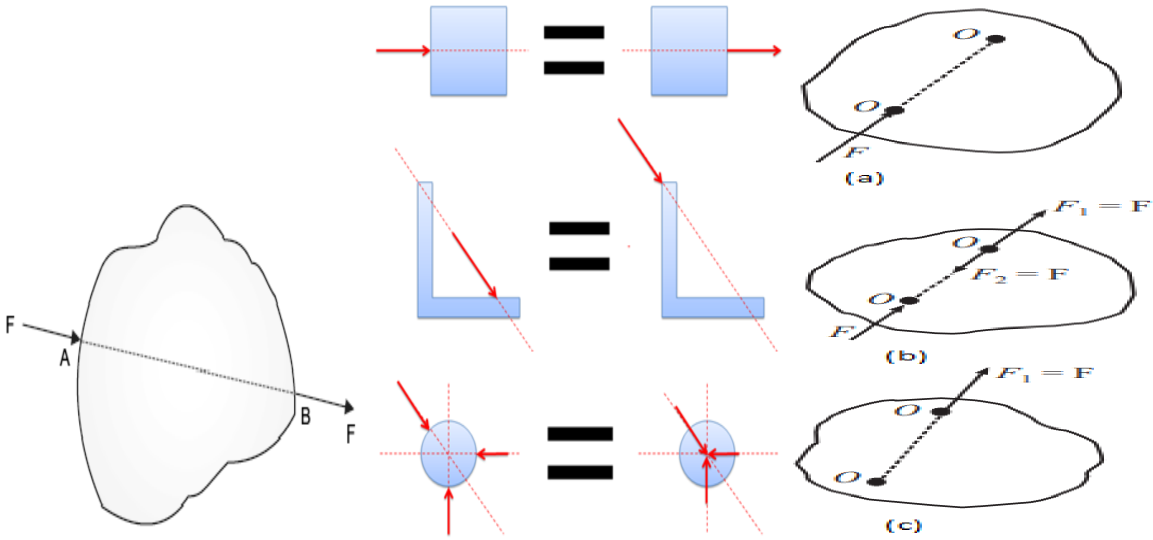


Fig. 7 Principles of Transmissibility

Triangle Law of forces

If two forces acting at a point are represented by the two sides of a triangle taken in order then their resultant force is represented by the third side taken in opposite order.

therefore, $R = \sqrt{P^2 + 2PQcos\Theta + Q^2}$

Above equation is the magnitude of the resultant vector.

therefore, $\phi = tan^{-1}(\frac{Qsin\Theta}{P+Qcos\Theta})$

Above equation is the direction of the resultant vector.

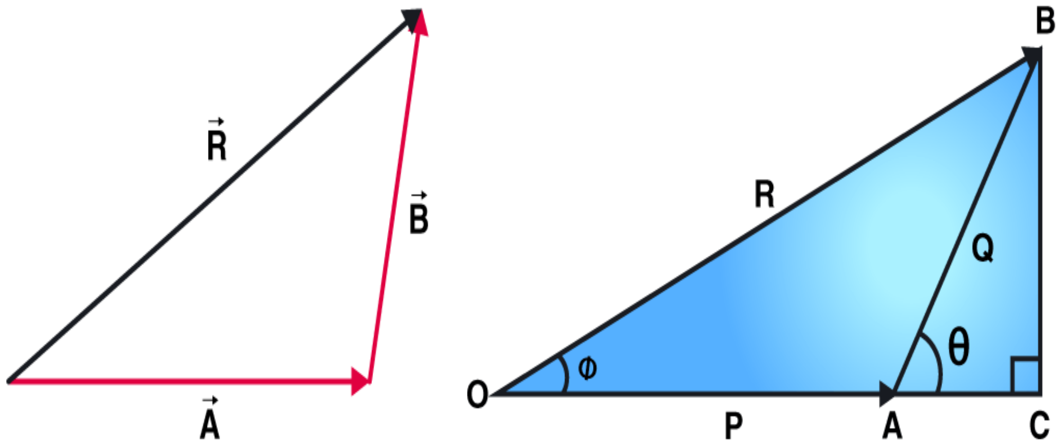


Fig. 8 Triangle law of forces

Polygon law of Forces

If a number of forces acting simultaneously on a particle be represented in magnitude and direction by the sides of a polygon taken in order then the resultant of all three forces may be represented in magnitude and direction by the closing side of the polygon taken in opposite order.

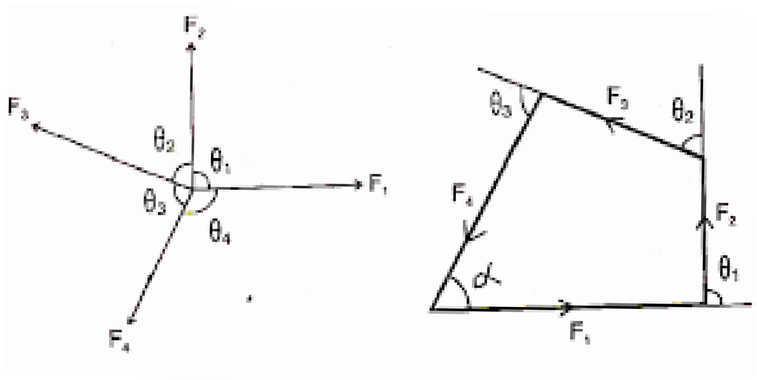


Fig. 9 Polygon law of forces

VECTORS

The vector quantities (or sometimes known as vectors) are those quantities which have both magnitude and direction such as force, displacement, velocity, acceleration, momentum etc. Following are the important features of vector quantities:

Since a vector is defined by the direction and magnitude, two vectors are equal if they have the same magnitude and direction. Thus in figure 2 vector $\vec{A}$ is equal to vector $\vec{B}$ and but not equal to vector $\vec{C}$ although all of them have the same magnitude.

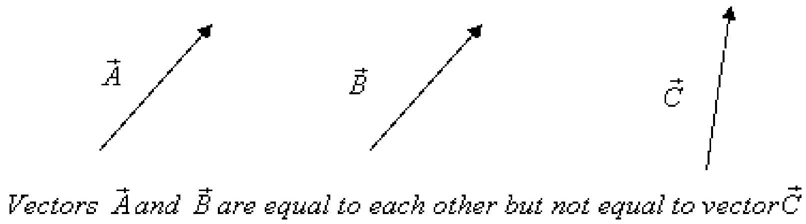


Fig. 10 Vector forces

In physical situations even two equal vectors may produce different effects depending on where they are located. For example take the force $\vec{F}$ applied on a disc. If applied on the rim it rotates the wheel at a speed different from when it is applied to a point nearer to the centre. Thus although it is the same force, applied at different points it produces different effects. On the other hand, imagine a thin rope wrapped on a wheel and being pulled out horizontally from the top. On the rope no matter where the force is applied, the effect is the same. Similarly we may push the wheel by applying the same force at the end of a stick with same result.

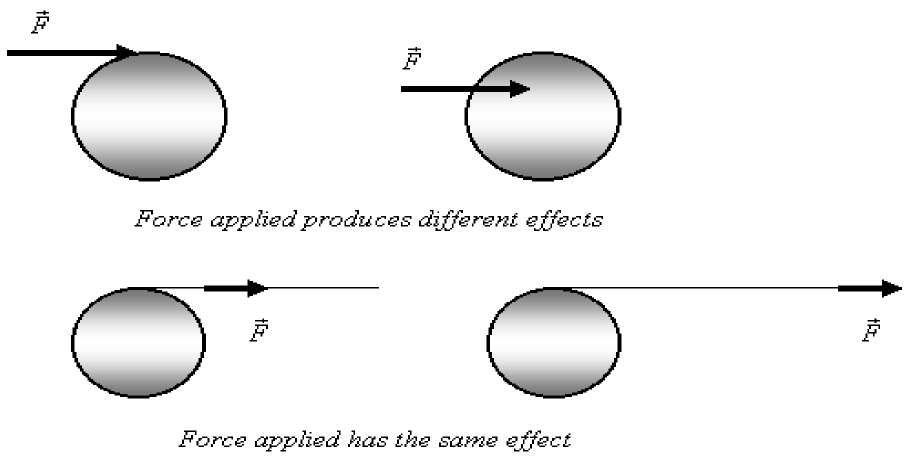


Fig. 11 Example for Vector force

VECTORIAL REPRESENTATION OF FORCES AND MOMENTS

1. ***Representation of a vector.*** A vector is represented by a directed line as shown in Fig.12. It may be noted that the length *OA* represents the magnitude of the vector $\vec{OA}$ The direction of the vector is $\vec{OA}$ is from *O* (*i.e*., starting point) to *A* (*i.e*., end point). It is also known as vector *P*.

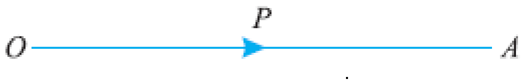


Fig. 12 Representation of a vector $\vec{OA}$

2. ***Unit vector.*** A vector, whose magnitude is unity, is known as unit vector.
3. ***Equal vectors.*** The vectors, which are parallel to each other and have same direction (*i.e*., same sense) and equal magnitude, are known as equal vectors.
4. ***Like vectors.*** The vectors, which are parallel to each other and have same sense but unequal magnitude, are known as like vectors.
5. ***Addition of vectors.*** Consider two vectors *PQ* and *RS*, which are required to be added as shown in Fig. 13.(a)

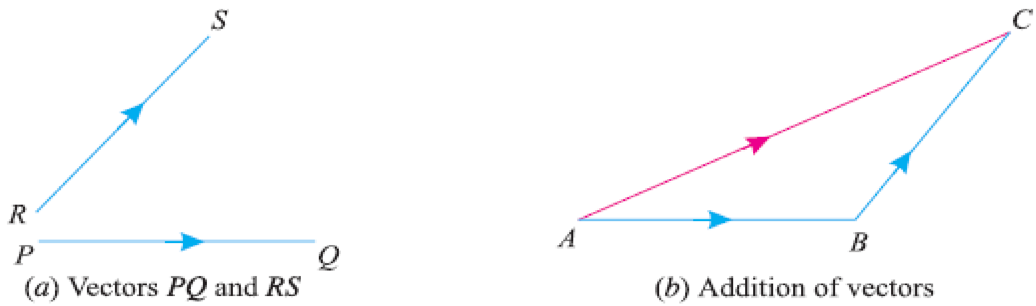


Fig. 13 Addition of vectors

Take a point *A*, and draw line *AB* parallel and equal in magnitude to the vector *PQ* to some convenient scale. Through *B*, draw *BC* parallel and equal to vector *RS* to the same scale. Join *AC* which will give the required sum of vectors *PQ* and *RS* as shown in Fig. 13. (*b*).

This method of adding the two vectors is called the Triangle Law of Addition of Vectors.

Similarly, if more than two vectors are to be added, the same may be done first by adding the two vectors, and then by adding the third vector to the resultant of the first two and so on. This method of adding more than two vectors is called Polygon Law of Addition of Vectors.

6. ***Subtraction of vectors***. Consider two vectors PQ and RS in which the vector RS is required to be subtracted as shown in Fig. 14 (a). Take a point *A*, and draw line *AB* parallel and equal in magnitude to the vector *PQ* to some convenient scale. Through *B*, draw *BC*

parallel and equal to the vector *RS*, but in *opposite direction*, to that of the vector *RS* to the same scale. Join *AC*, which will give the resultant when the vector *PQ* is subtracted from vector *RS* as shown in Fig. 14 (*b*).

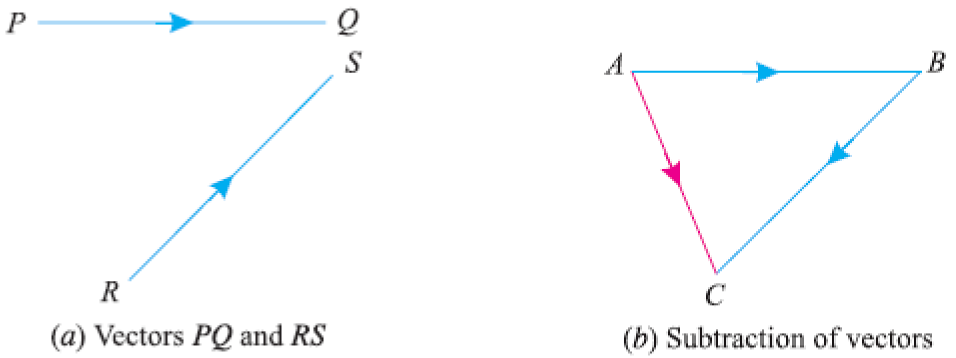
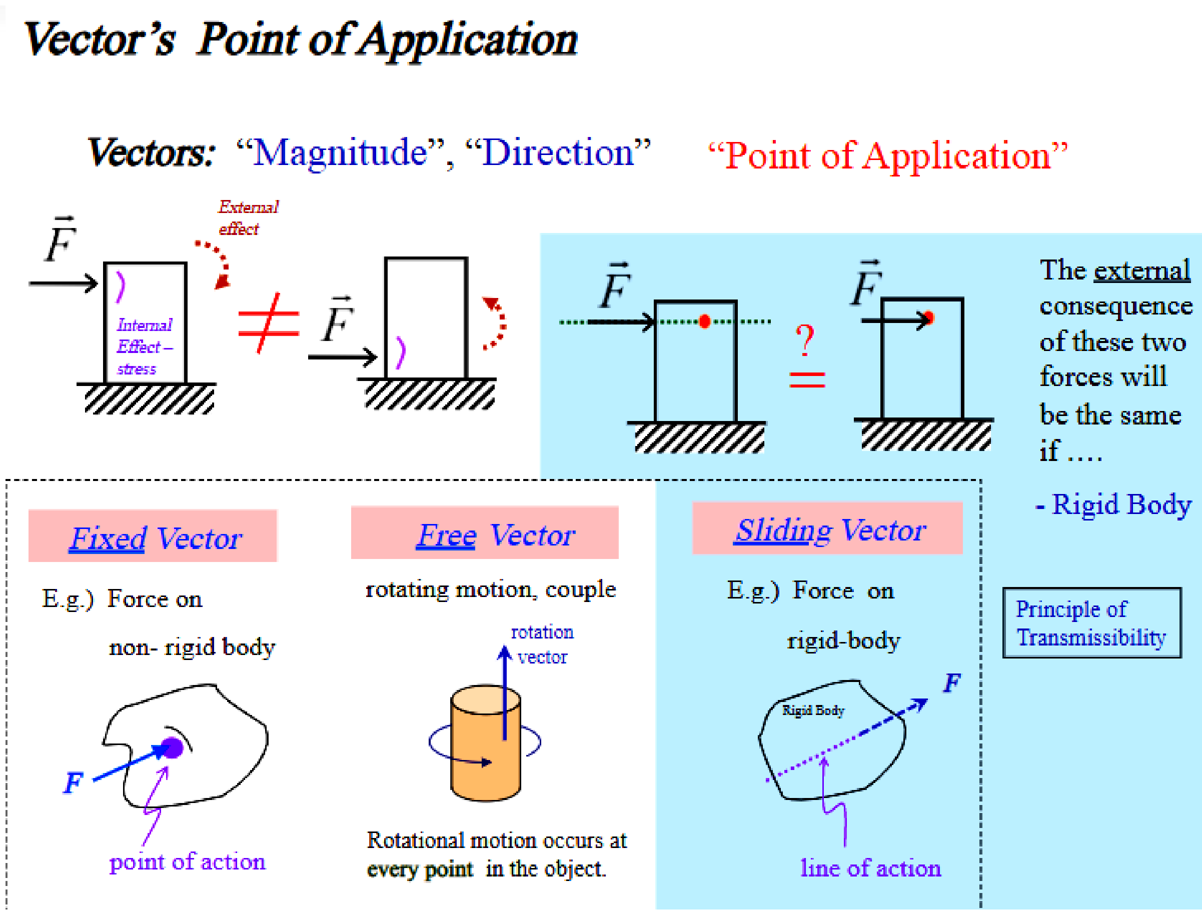


Fig. 14 Subtraction of vectors

VECTOR OPERATIONS, COPLANAR FORCES RESOLUTION AND COMPOSITION OF FORCES

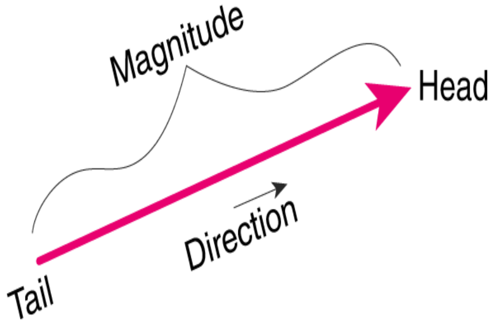


FORCES

A force is a measure of the action of one body or media on another (push or pull)

Force has:

- ❖ Magnitude
- ❖ Direction
- ❖ Point Of Application



Types of Forces:

- ❖ *External Forces* – It represents the action of other bodies on the rigid body
- ❖ *Internal Forces* – The forces which hold together the particles forming the rigid body

Effects of a force

A force may produce the following effects in a body, on which it acts :

- ➢ It may change the motion of a body. *i.e*. if a body is at rest, the force may set it in motion.
- ➢ And if the body is already in motion, the force may accelerate it.
- ➢ It may retard the motion of a body.
- ➢ It may retard the forces, already acting on a body, thus bringing it to rest or in equilibrium.
- ➢ It may give rise to the internal stresses in the body, on which it acts.

Characteristics of a force

In order to determine the effects of a force, acting on a body, we must know the following characteristics of a force:

- ➢ Magnitude of the force (*i.e*., 100 N, 50 N, 20 kN, 5 kN, etc.)
- ➢ The direction of the line, along which the force acts (*i.e.*, along *OX*, *OY*, at 30° North of East etc.). It is also known as line of action of the force.

- Nature of the force (*i.e.*, whether the force is push or pull). This is denoted by placing an arrow head on the line of action of the force.
- The point at which (or through which) the force acts on the body.

SYSTEM OF FORCES

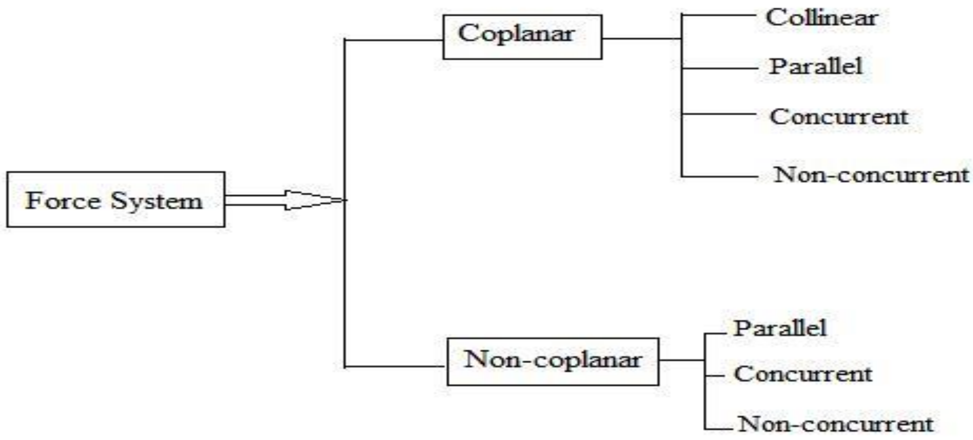
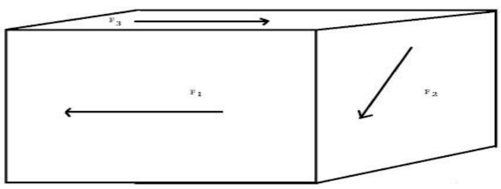


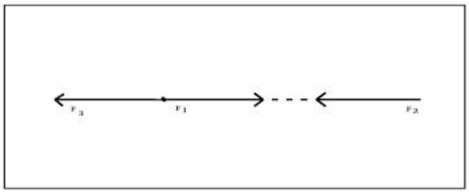
Fig. 15 System of forces

Coplanar force system - When the lines of action of all forces of a system lie on the same plane then the system is coplanar force system.

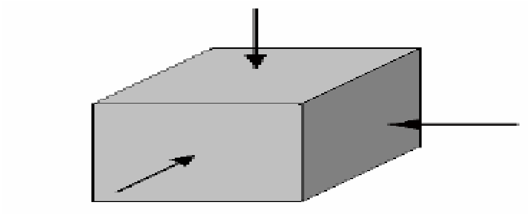
Non coplanar force system - The system in which the forces do not lie on the same plane is called non coplanar force system.



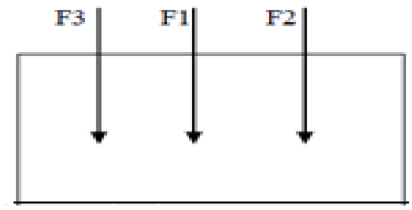
Collinear forces - The system in which the forces whose line of action lie on the same line and in same plane is called collinear force system.



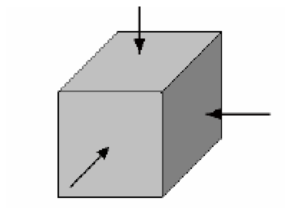
Concurrent force system - The system in which the forces meet at one point and lie in the same plane is called concurrent force system.



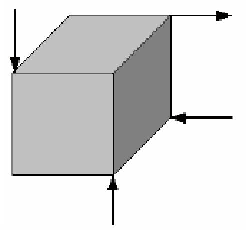
Parallel force system - In parallel force system the line of action of forces one parallel to each other.



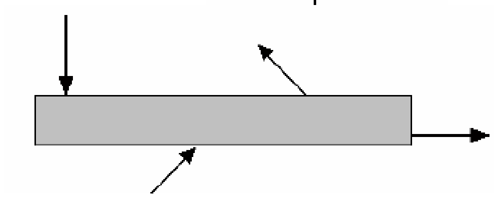
Concurrent Forces means that the forces all act at a single point like that illustrated where all the force acting on the cube pass through the centre.



Non - concurrent means the forces do not act at a single point. In this case the cube is likely to revolve as a result of the moments created.



Non - concurrent coplanar means that the force act in different lines but only on one plane, in other words two dimensions only. There will be a resultant force and a resultant moment of force but only in the plane of the paper.



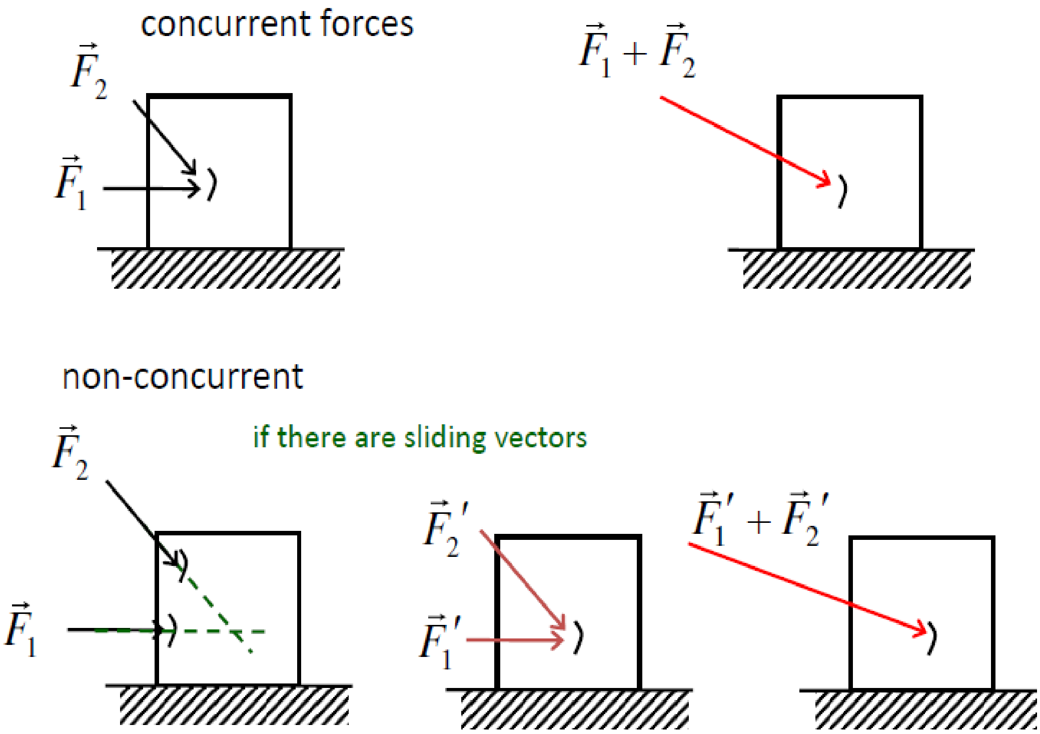


Fig. 16 Concurrent and Non-concurrent forces

Resolution of a force

Splitting up of a force into components along the fixed reference axis is called resolution of forces. The effect by single force and component forces remains the same.

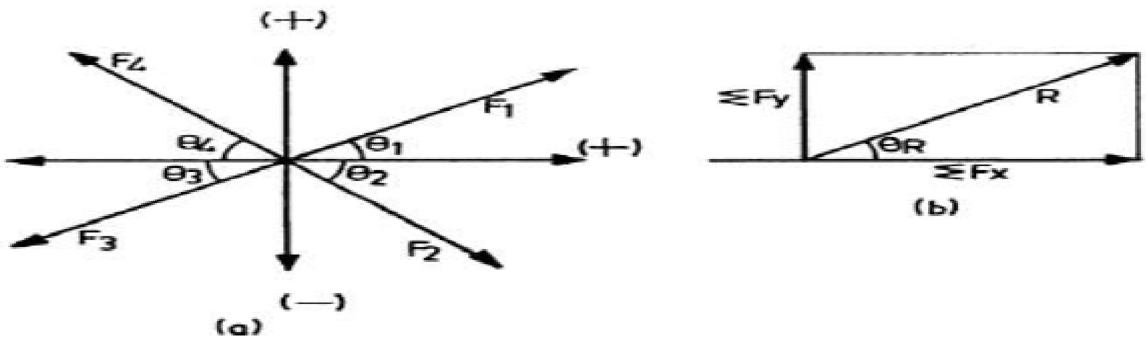


Fig. 17 Resolution of a force

Algebraic sum of horizontal components

$\Sigma F_x = F_1\cos\theta_1 + F_2\cos\theta_2 - F_3\cos\theta_3 - F_4\cos\theta_4$

Algebraic sum of vertical components

$\Sigma F_y = F_1 \sin\theta_1 - F_2 \sin\theta_2 - F_3 \sin\theta_3 + F_4 \sin\theta_4$

Resultant $R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2}$

Angle α mode by the resultant with x axis is given by

$\tan\alpha = \Sigma F_y / \Sigma F_x$

A vertical force has no horizontal component

A horizontal force has no vertical component

Q1. Forces *R*, *S*, *T*, *U* are collinear. Forces *R* and *T* act from left to right. Forces *S* and *U* act from right to left. Magnitudes of the forces *R*, *S*, *T*, *U* are 40 N, 45 N, 50 N and 55 N respectively. Find the resultant of *R*, *S*, *T*, *U*.

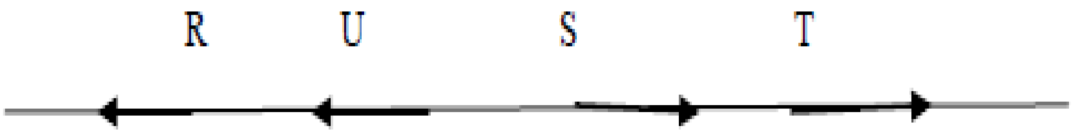
Given data:

R=40 N

S=45 N

T=50 N

U=55N



Resultant= -R-U+S+T=-40-55+45+50=0

Q2. Two forces of 100 N and 150 N are acting simultaneously at a point. What is the resultant of these two forces, if the angle between them is 45°?

Solution

Given:

First force (*F*1) = 100 N; Second force (*F*2) = 150 N and angle between *F*1 and *F*2 (θ) = 45°.

We know that the resultant force,

$$R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2\cos\theta}$$

$= (100)^2 + (150)^2 + 2 \times 100 \times 150 \cos 45°$ N

$= 10\,000 + 22\,500 + (30\,000 \times 0.707)$ N

$= 232$ N

Q3. Find the magnitude of the two forces, such that if they act at right angles, their resultant is √10 N. But if they Act at 60°, their resultant is √13 N.

Given : Two forces = F_1 and F_2.

First of all, consider the two forces acting at right angles. We know that when the angle between the two given forces is 90°, then the resultant force (R)

$$\sqrt{10} = \sqrt{F_1^2 + F_2^2}$$

or $$10 = F_1^2 + F_2^2 \quad \text{...(Squaring both sides)}$$

Similarly, when the angle between the two forces is 60°, then the resultant force (R)

$$\sqrt{13} = \sqrt{F_1^2 + F_2^2 + 2F_1F_2 \cos 60°}$$

$$\therefore \quad 13 = F_1^2 + F_2^2 + 2F_1F_2 \times 0.5 \quad \text{...(Squaring both sides)}$$

or $$F_1F_2 = 13 - 10 = 3 \quad \text{...(Substituting } F_1^2 + F_2^2 = 10)$$

We know that $(F_1 + F_{2]})^2 = F_1^2 + F_2^2 + 2F_1F_2 = 10 + 6 = 16$

$$\therefore \quad F_1 + F_2 = \sqrt{16} = 4 \quad \text{...}(i)$$

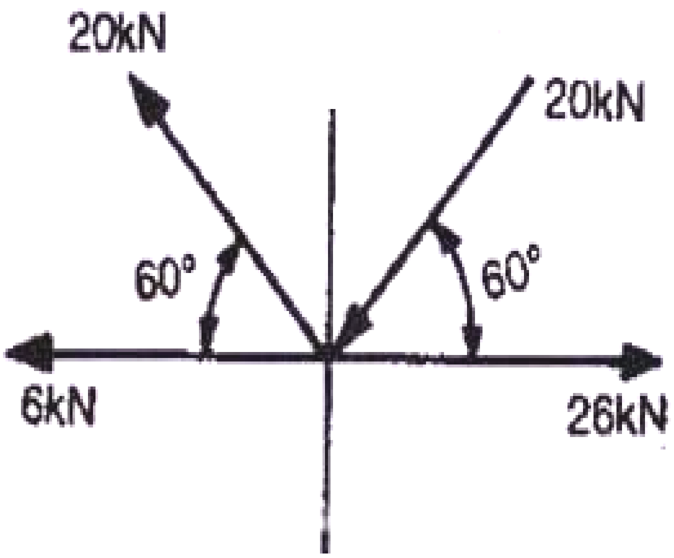
Similarly $(F_1 - F_2)^2 = F_1^2 + F_2^2 - 2F_1F_2 = 10 - 6 = 4$

$$\therefore \quad F_1 - F_2 = \sqrt{4} = 2 \quad \text{...}(ii)$$

Solving equations (i) and (ii),

$$F_1 = 3 \text{ N} \quad \text{and} \quad F_2 = 1 \text{ N} \quad \textbf{Ans.}$$

Q2. Find the resultant of the force system shown in Fig



Given data:

F1=20 KN ; θ_1=60°

F2=26 KN ; θ_2=0°

F3=6KN ; θ_3=00°

F4=20KN ; θ_4=60°

Solution:

Resolve the given forces horizontally and calculate the algebraic total of all the horizontal parts or

Σ H=-20cos60°+26cos0°-6cos0°-20cos60°=0

Resolve the given forces vertically and calculate the algebraic total of all the vertical parts or Σ V.

Σ V=-20sin60°±26sin0°±6sin0°+20 sin60°=0

R = √((∑H)^2+(V)^2)=0

Q 3. A triangle ABC has its side AB = 40 mm along positive x-axis and side BC = 30 mm along positive y-axis. Three forces of 40 N, 50 N and 30 N act along the sides AB, BC and CA respectively. Determine magnitude of the resultant of such a system of forces.

From the geometry of the figure, we find that the triangle *ABC* is a right angled triangle, in which the *side $AC = 50$ mm. Therefore

$$\sin\theta = \frac{30}{50} = 0.6$$

and $$\cos\theta = \frac{40}{50} = 0.8$$

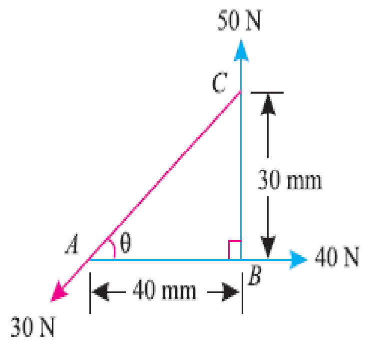


Fig. 2.3.

Resolving all the forces horizontally (*i.e.*, along *AB*),

$$\Sigma H = 40 - 30\cos\theta$$
$$= 40 - (30 \times 0.8) = 16 \text{ N}$$

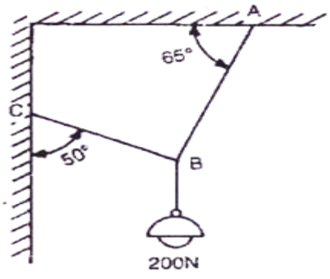
and now resolving all the forces vertically (*i.e.*, along *BC*)

$$\Sigma V = 50 - 30\sin\theta$$
$$= 50 - (30 \times 0.6) = 32 \text{ N}$$

We know that magnitude of the resultant force,

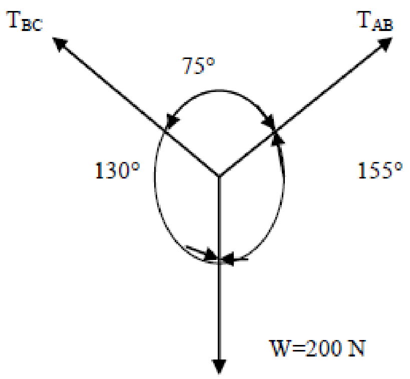
$$R = \sqrt{(\Sigma H)^2 + (\Sigma V)^2} = \sqrt{(16)^2 + (32)^2} = 35.8 \text{ N} \quad \textbf{Ans.}$$

Q 4. An electric light fixture weighting 200 N is supported as shown in Fig. Determine the tensile forces in the wires and BA and BC.



Solution:

Free body diagram(FBD):



By using lami theorem

T_{AB}/sin130°= T_{BC}/sin155°=200/sin75°

T_{AB}=200/sin75°* sin130°=**158.61N**

T_{BC}=200/sin75°* sin155°=**87.50N**

FORCES IN SPACE

Magnitude of a force F in space

$F=\sqrt{F_x^2+F_y^2+F_z^2}$

Components of a force in space

$F_x \quad = F \cos \theta_x$

$F_y \quad = F \cos \theta_y$

$F_z \quad = F \cos \theta_z$

Direction cosines

$\cos \theta_x = F_x / F$

$\cos \theta_y = F_y / F$

$\cos \theta_z = F_z / F$

Proportion of components

$$\frac{F_x}{x} = \frac{F_y}{y} = \frac{F_z}{z} = \frac{F}{d}$$

Moment of a force about an axis

$$M_x = zF_y \pm yF_z$$
$$M_y = zF_x \pm xF_z$$
$$M_z = yF_x \pm xF_y$$

EQUILIBRIUM OF A PARTICLE IN SPACE

In three dimension of space if the forces acting on the particle are resolved into their respective i, j, k components the equilibrium equation is written as,

ΣFxi+ΣFj+ΣFxk=0

The equation for equilibrium of a particle in space is,

ΣFx=0 ; ΣFy=0 ; ΣFz=0;

Resultant of Concurrent Force Systems in Space

Components of the resultant

$R_x=\Sigma F_x$, $R_y=\Sigma F_y$ and $R_z=\Sigma F_z$

Magnitude of the resultant

$R=\sqrt{R_x^2+R_y^2+R_z^2}$

Equilibrium of Concurrent Space Forces

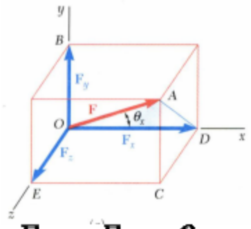
The resultant of all forces is zero

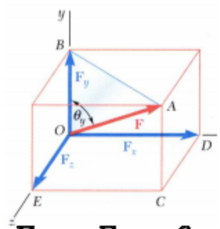
$\Sigma F_x=0$, $\Sigma F_y=0$ and $\Sigma F_z=0$

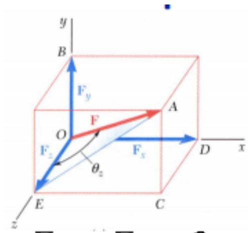
The sum of moment is zero

$\Sigma M_x=0$, ΣMy=0 and $\Sigma M_z=0$

Rectangular Components in Space







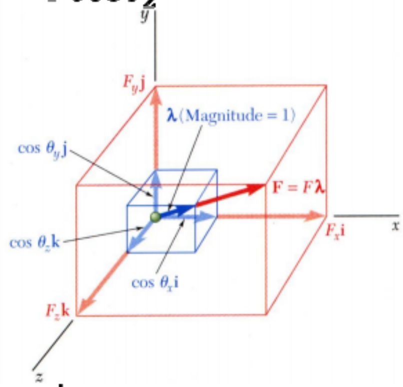
$F_x = F\cos\theta_x$ $\quad$ $F_y = F\cos\theta_y$ $\quad$ $F_z = F\cos\theta_z$

$$\boldsymbol{F} = F_x\boldsymbol{i} + F_y\boldsymbol{j} + F_z\boldsymbol{k}$$

$$\boldsymbol{F} = F\cos\theta_x\boldsymbol{i} + F\cos\theta_y\boldsymbol{j} + F\cos\theta_z\boldsymbol{k}$$

$$\boldsymbol{F} = F(\cos\theta_x\boldsymbol{i} + \cos\theta_y\boldsymbol{j} + \cos\theta_z\boldsymbol{k})$$

$$\boldsymbol{F} = F\boldsymbol{\lambda}$$

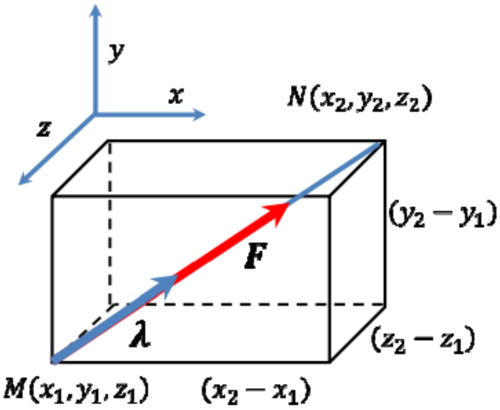


Where $\boldsymbol{\lambda} = \cos\theta_x\boldsymbol{i} + \cos\theta_y\boldsymbol{j} + \cos\theta_z\boldsymbol{k}$

$\boldsymbol{\lambda}$ is a unit vector along the line of action of $\boldsymbol{F}$ and $\cos\theta_x$, $\cos\theta_y$ and $\cos\theta_z$ are the direction cosine for $\boldsymbol{F}$

Direction of the force is defined by the location of two points

$M(x_1, y_1, z_1)$ and $N(x_2, y_2, z_2)$



$\boldsymbol{d}$ is the vector joining M and N

$$\boldsymbol{d} = d_x\boldsymbol{i} + d_y\boldsymbol{j} + d_z\boldsymbol{k}$$

$d_x = (x_2 - x_1)$ $\quad$ $d_y = (y_2 - y_1)$

$d_z = (z_2 - z_1)$

$$\boldsymbol{F} = F\boldsymbol{\lambda}$$

$$= F\left(\frac{d_x\boldsymbol{i} + d_y\boldsymbol{j} + d_z\boldsymbol{k}}{d}\right)$$

$F_x = F\dfrac{d_x}{d}$ $\qquad$ $F_y = F\dfrac{d_y}{d}$ $\qquad$ $F_z = F\dfrac{d_z}{d}$

Q1 Determine the magnitude of the resultant, its pointing and its direction cosines for the following system of non-coplanar, concurrent forces. 300 N (+3, -4, +6); 400 N (-2, +4, -5); 200 N (-4, +5, -3)

SOLUTION

Distance

$$d = \sqrt{x^2 + y^2 + z^2}$$

Components of given forces

$$\text{From } \frac{F_x}{x} = \frac{F_y}{y} = \frac{F_z}{z} = \frac{F}{d}$$

$$F_x = \frac{x\,F_x}{d} \qquad F_y = \frac{y\,F_y}{d} \qquad F_z = \frac{z\,F_z}{d}$$

Components of resultant

$$R_x = \Sigma F_x \qquad R_y = \Sigma F_y \qquad R_z = \Sigma F_z$$

Resultant

$$R = \sqrt{R_x{}^2 + R_y{}^2 + R_z{}^2}$$

Force F	Components of Distance x	y	z	Distance d	Components of Force F_x	F_y	F_z
300	3	-4	6	7.810	115.233	-153.644	230.466
400	-2	4	-5	6.708	-119.257	238.514	-298.142
200	-4	5	-3	7.071	-113.137	141.421	-84.853
SUM (R_x, R_y, and R_z)					-117.161	226.291	-152.529
RESULTANT							**R = 296.984**

Direction cosines of the resultant

$$\cos\theta_x = \frac{R_x}{R} = -0.394$$

$$\cos\theta_y = \frac{R_y}{R} = 0.762$$

$$\cos\theta_z = \frac{R_z}{R} = -0.514$$

EQUIVALENT FORCE SYSTEMS

Two forces are said to be equivalent if they have the same magnitude and direction (i.e. they are equal) and produce the same moment about any point O (i.e. same line of action).

The basic idea - Two force systems are equivalent if they result in the same resultant force and the same resultant moment.

$$\begin{cases} \sum \underline{\mathbf{F}} \text{ for system } 1 = \sum \underline{\mathbf{F}} \text{ for system } 2 \\ \sum \underline{\mathbf{M}}_O \text{ for system } 1 = \sum \underline{\mathbf{M}}_O \text{ for system } 2 \end{cases} \Leftrightarrow \text{The two force systems are equivalent}$$

Moving a force along its line of action - Moving a force along its line of action results in a new force system which is equivalent to the original force system

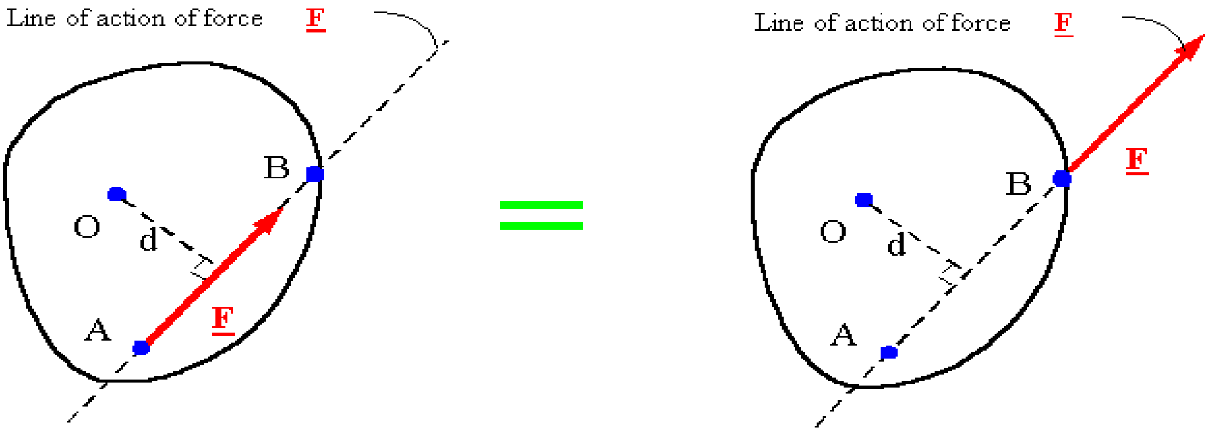


Fig. 17 Moving a force along its line of action

Moving a force off its line of action - If a force is moved off its line of action, a couple must be added to the force system so that the new system generates the same moment as the old system.

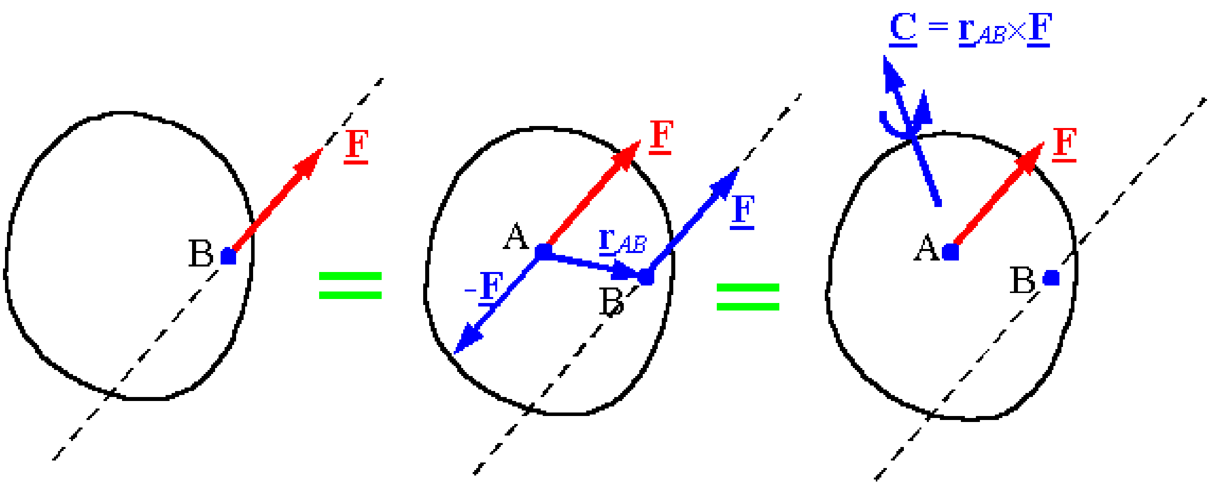
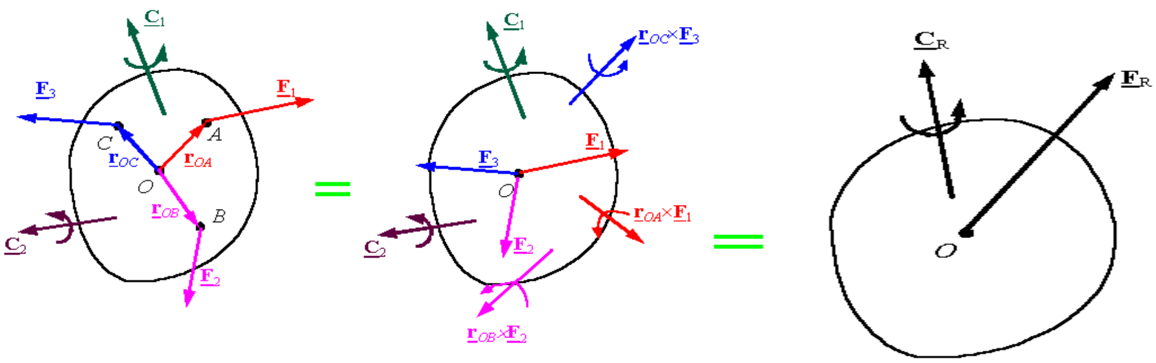


Fig. 18 Moving a force OFF its line of action

The resultant of a force and couple system - For any point *O*, every force and couple system can be made equivalent to a single force passing through *O* and a single couple. The single force passing through *O* is equal to the resultant force of the original system, and the couple is equal to the resultant moment of the original system around point *O*.



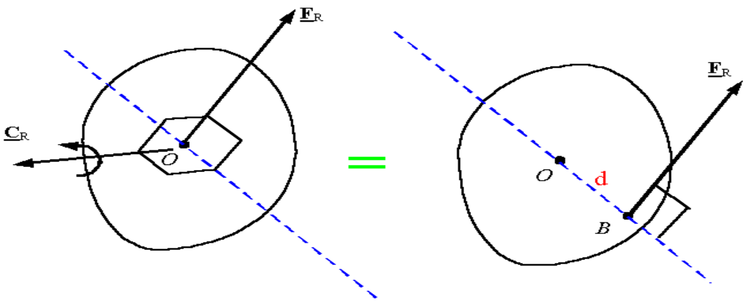
$$\begin{cases} \underline{\mathbf{F}}_R = \sum \underline{\mathbf{F}} \\ \underline{\mathbf{C}}_R = \sum \underline{\mathbf{M}}_O = \sum \underline{\mathbf{C}} + \sum \underline{\mathbf{r}} \times \underline{\mathbf{F}} \end{cases}$$

Fig. 18 Resultant of a force and couple

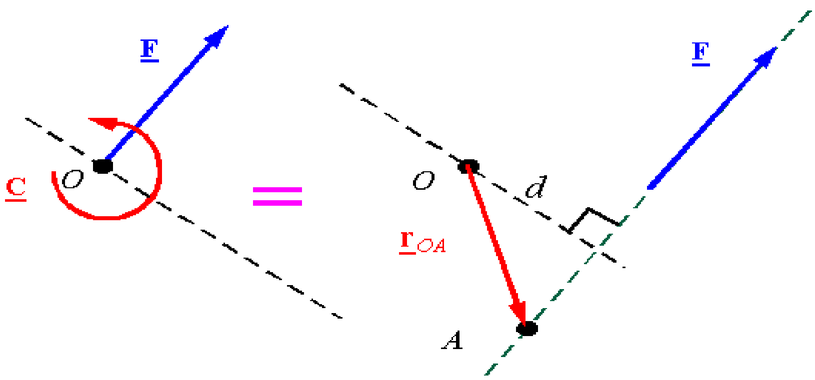
When can one reduce a force and couple system to a single force?

For a force and couple system if the resultant force and the resultant couple are perpendicular, then one can find an equivalent system with a single force and no couple. To obtain this system, move the resultant force a distance *d* along the line perpendicular to the plane of the resultant force and resultant couple until the resultant force creates a moment equivalent to the resultant couple.

$$F_R d = C_R \quad \Rightarrow \quad d = \frac{C_R}{F_R}$$



Note - All 2-D force systems can be reduced to a single force. To find the line of action of the force, the moment of the original system must be forced to be the same as the system with the single force.



$$\underline{\mathbf{C}} = \underline{\mathbf{r}}_{OA} \times \underline{\mathbf{F}}$$

$$d = \frac{C}{F}$$

Worked Out examples

1. The following forces act at a point :

i. 20 N inclined at 30° towards North of East,

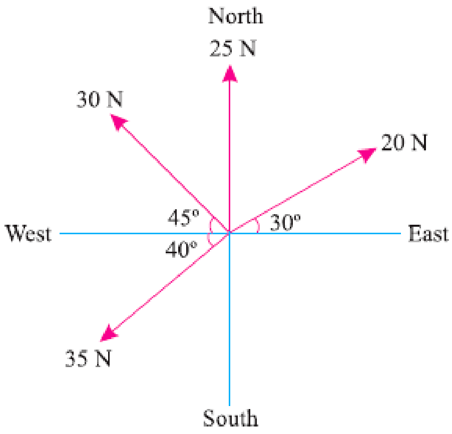
ii. 25 N towards North,

iii. 30 N towards North West, and

iv. 35 N inclined at 40° towards South of West.

Find the magnitude and direction of the resultant force.

Solution



Magnitude of the resultant force

Resolving all the forces horizontally *i.e.*, along East-West line,

ΣH = 20 cos 30° + 25 cos 90° + 30 cos 135° + 35 cos 220° N

= (20 × 0.866) + (25 × 0) + 30 (– 0.707) + 35 (– 0.766) N

= – 30.7 N ...(*i*)

And now resolving all the forces vertically *i.e.*, along North-South line,

ΣV = 20 sin 30° + 25 sin 90° + 30 sin 135° + 35 sin 220° N

= (20 × 0.5) + (25 × 1.0) + (30 × 0.707) + 35 (– 0.6428) N

= 33.7 N ...(ii)

We know that magnitude of the resultant force,

$$R = \sqrt{(\Sigma H)^2 + (\Sigma V)^2} = \sqrt{(-30.7)^2 + (33.7)^2} = 45.6 \text{ N}$$

Direction of the resultant force

Let θ = Angle, which the resultant force makes with the East.

We know that,

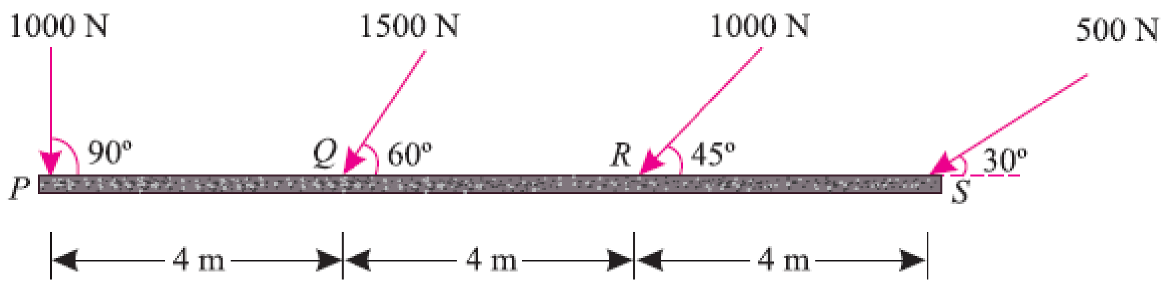
$$\tan\theta = \frac{\Sigma V}{\Sigma H} = \frac{33.7}{-30.7} = -1.098 \quad \text{or} \quad \theta = 47.7^\circ$$

Since ΣH is negative and ΣV is positive, therefore resultant lies between 90° and 180°.

Thus actual angle of the resultant = 180° – 47.7° = 132.3°

2. A horizontal line PQRS is 12 m long, where PQ = QR = RS = 4 m. Forces of 1000 N, 1500 N, 1000 N and 500 N act at P, Q, R and S respectively with downward direction. The lines of action of these forces make angles of 90°, 60°, 45° and 30° respectively with PS. Find the magnitude, direction and position of the resultant force.

Solution



Magnitude of the resultant force

Resolving all the forces horizontally,

ΣH = 1000 cos 90° + 1500 cos 60° + 1000 cos 45° + 500 cos 30° N

= (1000 × 0) + (1500 × 0.5) + (1000 × 0.707) + (500 × 0.866) N

= 1890 N ...(*i*)

and now resolving all the forces vertically,

ΣV = 1000 sin 90° + 1500 sin 60° + 1000 sin 45° + 500 sin 30° N

= (1000 × 1.0) + (1500 × 0.866) + (1000 × 0.707) + (500 × 0.5) N

= 3256 N ...(*ii*)

We know that magnitude of the resultant force,

$R = \sqrt{(\Sigma H)^2 + (\Sigma V)^2} = \sqrt{(1890)^2 + (3256)^2} = 3765$ N

Direction of the resultant force

Let θ = Angle, which the resultant force makes with *PS*.

$$\therefore \quad \tan\theta = \frac{\Sigma V}{\Sigma H} = \frac{3256}{1890} = 1.722$$

Note:
Since both the values of ΣH and ΣV are +ve, therefore resultant lies between 0° and 90°.

Position of the resultant force

Let *x* = Distance between *P* and the line of action of the resultant force.

Now taking moments* of the vertical components of the forces and the resultant force about P, and equating the same,

$$3256\,x = (1000 \times 0) + (1500 \times 0.866)\,4 + (1000 \times 0.707)8 + (500 \times 0.5)12$$

$$= 13\,852$$

$$x = \frac{13\,852}{3256} = 4.25 \text{ m}$$

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SCHOOL OF BUILDING AND ENVIRONMENT

DEPARTMENT OF CIVIL ENGINEERING

UNIT –II - EQUILIBRIUM OF RIGID BODIES – SCIA1101

II. EQUILIBRIUM OF RIGID BODIES

Free body diagram - Types of supports and their reactions - Requirements of stable equilibrium - Moments and Couples - Varignon's theorem - Equilibrium of Rigid bodies in two dimensions - Equilibrium of Rigid bodies in Three Dimensions.

FREE BODY DIAGRAM

Free body diagram is a diagram which shows all the forces acting at a rigid body involving **1) self weight, 2) Normal reactions, 3) frictional force, 4) Applied force, 5) External moment applied**.

In a rigid body mechanics, the concept of free body diagram is very useful to solve the problems.

Free body diagram for rigid bodies:

In order to draw the FBD for each member of a rigid body follow the instructions below:

- Isolate the object from its surroundings,
- Draw the outline of the object; consider all dimensions and angles,
- Include all forces and couple moments that the surroundings exert on the body. Forces include *loadings*, *support reactions* and *weights*. (See the support reaction section for detailed explanation)
- Known forces and moments should be labeled with their proper *magnitudes* and *directions*.
- Magnitudes and direction angles of unknown forces and moments should be represented with *letters*.

FBD is a sketch of the outlined shape of the body, which represents it being isolated from its surroundings.

- It is necessary to show all the forces and couple moments that the surroundings exert on the body so that these effects can be accounted for when equations of equilibrium are applied.
- Free Body Diagram As a general rule, if a support prevents translation of a body in a given direction, then a force is developed on the body in the opposite direction. Similarly, if rotation is prevented, a couple moment is exerted on the body.

- The problem becomes much simple if each body is considered in isolation i.e, separate from the surrounding body or bodies. Such a body which has been so separated or isolated from the surrounding bodies is called as Free Body.
- The sketch showing all the forces and moments acting on the body is called as the free body diagram.

It is a diagram of the body in which the bodies under consideration are freed from all contact surfaces and all the forces acting on it are clearly indicated.

Procedure for Drawing a FBD:

1. Draw outlined shape - Isolate rigid body from its surroundings
2. Show all the forces - Show all the external forces and couple moments. These typically include
   - Applied Loads
   - Support reactions
   - The weight of the body
3. Identify each force
   - Known forces should be labeled with proper magnitude and direction
   - Letters are used to represent magnitude and directions of unknown forces.

Examples

Consider the diagram shown in fig. We will draw the free body diagram at A, B and C and for the whole structure

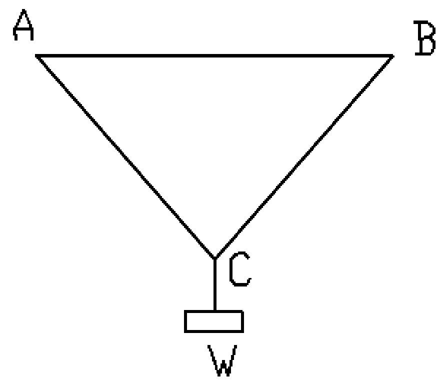
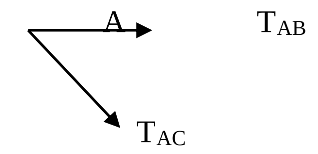


Figure shows the weight attached at C, Connected by a string ABCD.

Free body Diagram at A

The forces acting on A are

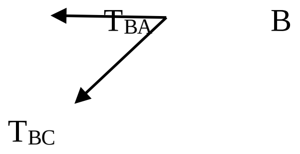
- Tension on string AB, Let it be T_{AB}
- Tension on string BC, Let it be T_{AC}



Free body Diagram at B

The forces acting on B are

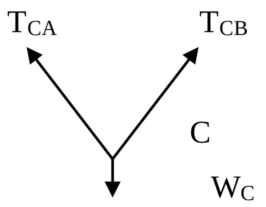
- Tension on string BA, Let it be T_{BA}
- Tension on string BC, Let it be T_{BC}



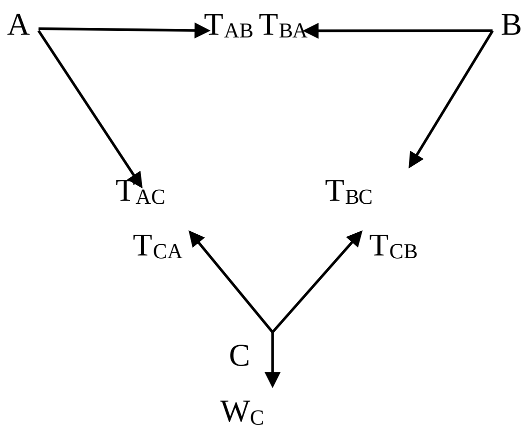
Free body Diagram at C

The forces acting on C are

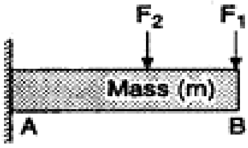
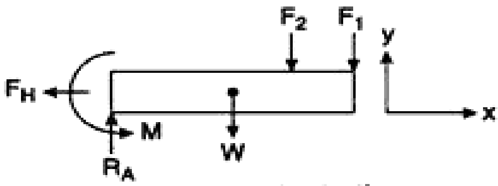
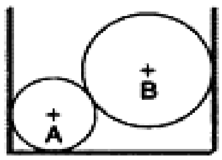
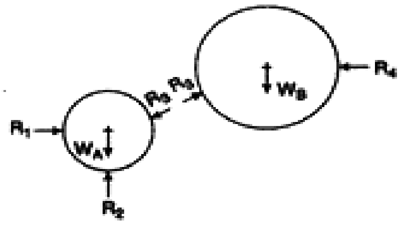
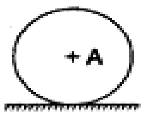
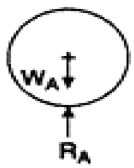
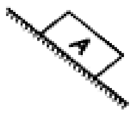
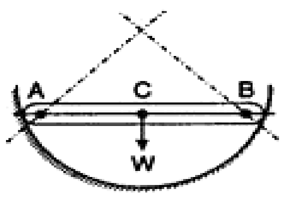
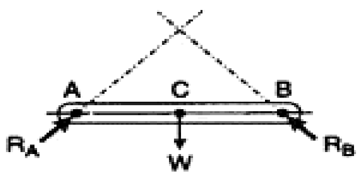
- Tension on string CA, Let it be T_{CA}
- Tension on string BC, Let it be T_{BA}
- Weight at C, Let it be W_c



Free body Diagram for ABC



EXAMPLES

Problem	Free Body Diagram
 a(I) Cantilever beam	 a(II) Free body diagram
 b(I) Two spheres in equilibrium	 b(II) Free body diagram
 (c) Ball resting on a surface	 c (II) Free body diagram
 d(I) Frictionless surface	 d(II) Free body diagram
 e(I) A bar placed in a hemispherical cup	 e(II) Free body diagram

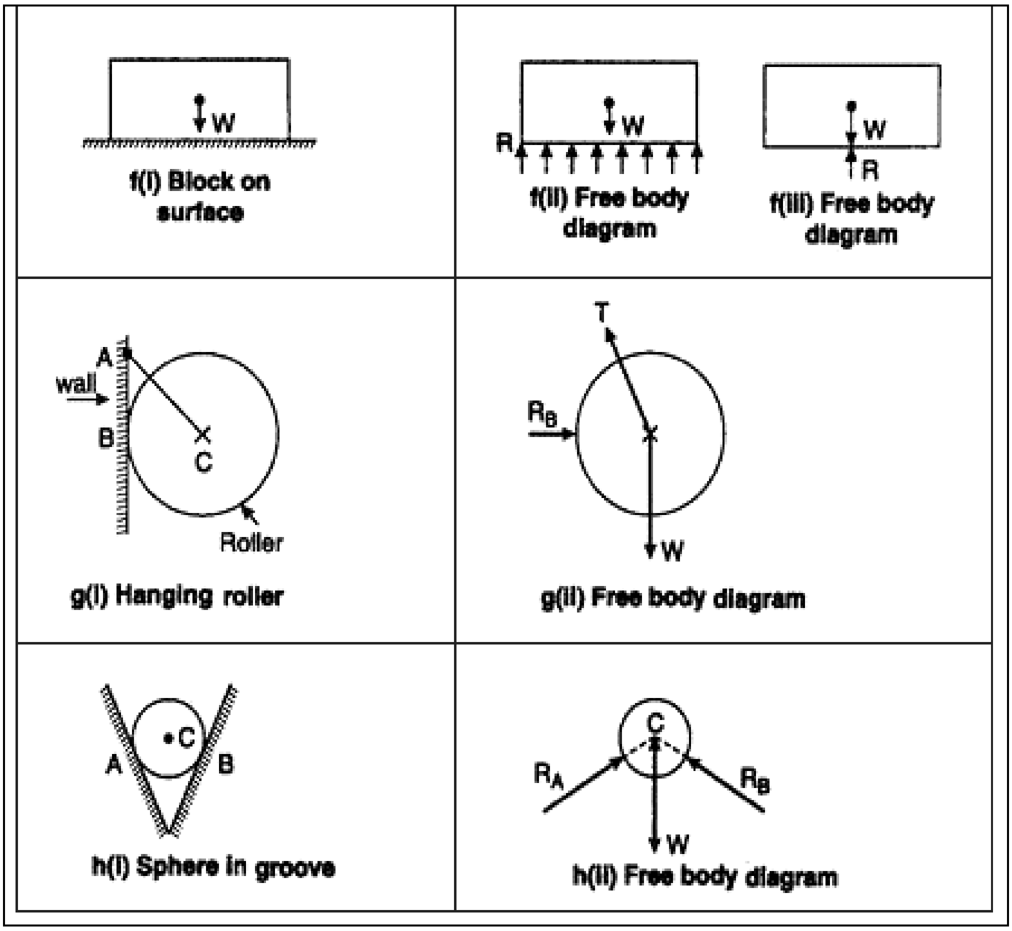
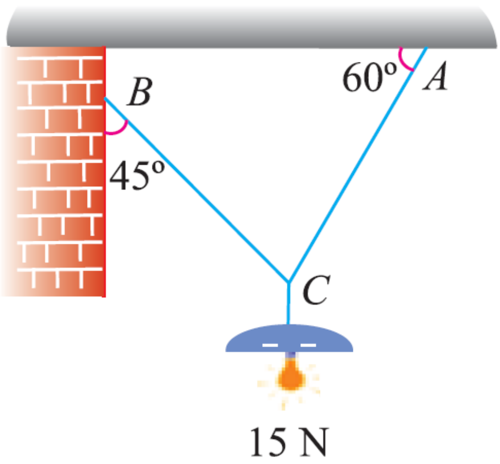


Fig. 1 Free body diagram examples

Worked out examples

An electric light fixture weighting 15 N hangs from a point C, by two strings AC and BC. The string AC is inclined at 60° to the horizontal and BC at 45° to the horizontal as shown in Figure. Using Lami's theorem, or otherwise, determine the forces in the strings AC and BC.



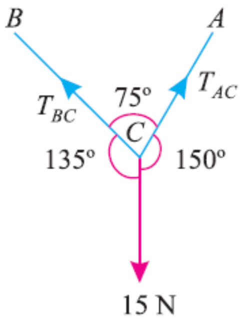
Given:

Weight at C = 15 N

Let TAC = Force in the string AC, and

TBC = Force in the string BC.

The system of forces is shown in Figure.



From the geometry of the figure, we find that angle between TAC and 15 N is 150° and angle between TBC and 15 N is 135°.

$$\therefore \quad \angle ACB = 180° - (45° + 60°) = 75°$$

Applying Lami's equation at C,

$$\frac{15}{\sin 75°} = \frac{T_{AC}}{\sin 135°} = \frac{T_{BC}}{\sin 150°}$$

or

$$\frac{15}{\sin 75°} = \frac{T_{AC}}{\sin 45°} = \frac{T_{BC}}{\sin 30°}$$

$$\therefore \quad T_{AC} = \frac{15 \sin 45°}{\sin 75°} = \frac{15 \times 0.707}{0.9659} = 10.98\,\text{N}$$

and

$$T_{BC} = \frac{15 \sin 30°}{\sin 75°} = \frac{15 \times 0.5}{0.9659} = 7.76\,\text{N}$$

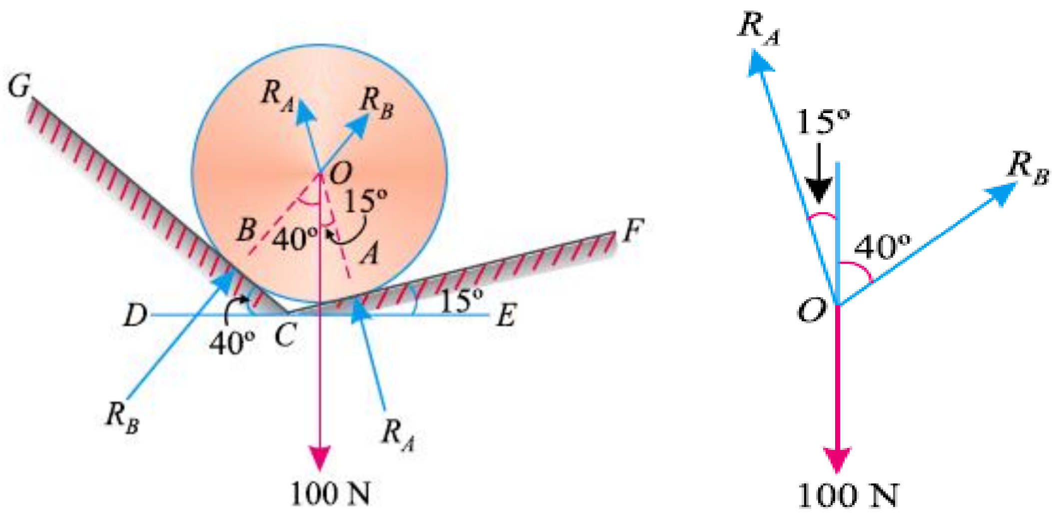
Worked out examples

A smooth circular cylinder of radius 1.5 meter is lying in a triangular groove, one side of which makes 15° angle and the other 40° angle with the horizontal. Find the reactions at the surfaces of contact, if there is no friction and the cylinder weights 100 N.

Solution

Given:

Weight of cylinder = 100 N



Let $\qquad R_A$ = Reaction at A, and

$\qquad R_B$ = Reaction at B.

The smooth cylinder lying in the groove is shown in Fig. 5.12 (*a*). In order to keep the system in equilibrium, three forces *i.e.* R_A, R_B and weight of cylinder (100 N) must pass through the centre of the cylinder. Moreover, as there is no *friction, the reactions R_A and R_B must be normal to the surfaces as shown in Fig. 5.12 (*a*). The system of forces is shown in Fig. 5.12 (*b*).

Applying Lami's equation, at O,

$$\frac{R_A}{\sin(180° - 40°)} = \frac{R_B}{\sin(180° - 15°)} = \frac{100}{\sin(15° + 40°)}$$

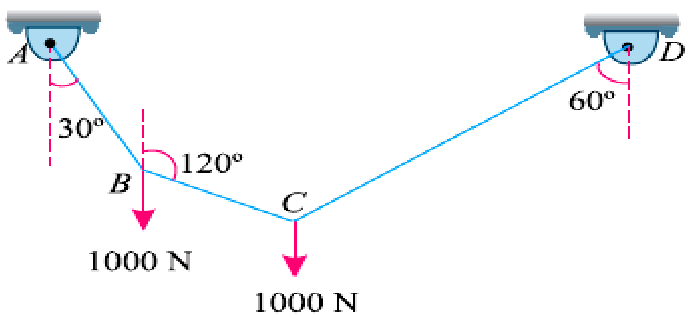
or
$$\frac{R_A}{\sin 40°} = \frac{R_B}{\sin 15°} = \frac{100}{\sin 55°}$$

∴
$$R_A = \frac{100 \times \sin 40°}{\sin 55°} = \frac{100 \times 0.6428}{0.8192} = 78.5 \text{ N} \quad \textbf{Ans.}$$

and
$$R_B = \frac{100 \times \sin 15°}{\sin 55°} = \frac{100 \times 0.2588}{0.8192} = 31.6 \text{ N} \quad \textbf{Ans.}$$

Worked out examples

A string ABCD, attached to fixed points A and D has two equal weights of 1000 N attached to it at B and C. The weights rest with the portions AB and CD inclined at angles as shown in Figure. Find the tensions in the portions AB, BC and CD of the string, if the inclination of the portion BC with the vertical is 120°.

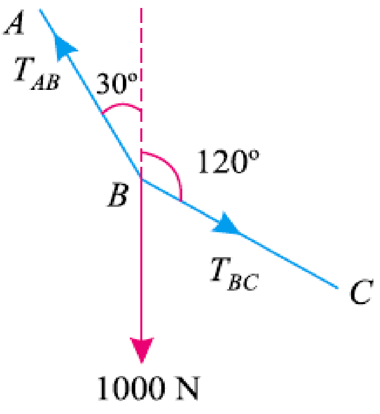


Solution

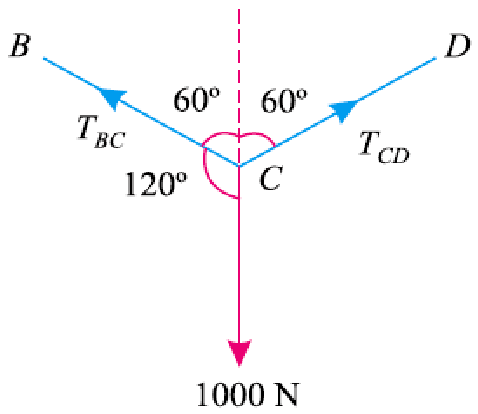
Given:

Load at B = Load at C = 1000 N

For the sake of convenience, let us split up the string *ABCD* into two parts. The system of forces at joints B and is shown in Figure (*a*) and (*b*).



(*a*) Joint *B*



(*b*) Joint *C*

Let T_{AB} = Tension in the portion *AB* of the string,

T_{BC} = Tension in the portion *BC* of the string, and

T_{CD} = Tension in the portion *CD* of the string.

Applying Lami's equation at joint *B*,

$$\frac{T_{AB}}{\sin 60^\circ} = \frac{T_{BC}}{\sin 150^\circ} = \frac{1000}{\sin 150^\circ}$$

$$\frac{T_{AB}}{\sin 60^\circ} = \frac{T_{BC}}{\sin 30^\circ} = \frac{1000}{\sin 30^\circ} \qquad ...[\because \sin(180^\circ - \theta) = \sin\theta]$$

$$\therefore \quad T_{AB} = \frac{1000 \sin 60^\circ}{\sin 30^\circ} = \frac{1000 \times 0.866}{0.5} = 1732 \text{ N} \quad \textbf{Ans.}$$

$$T_{BC} = \frac{1000 \sin 30^\circ}{\sin 30^\circ} = 1000 \text{ N} \quad \textbf{Ans.}$$

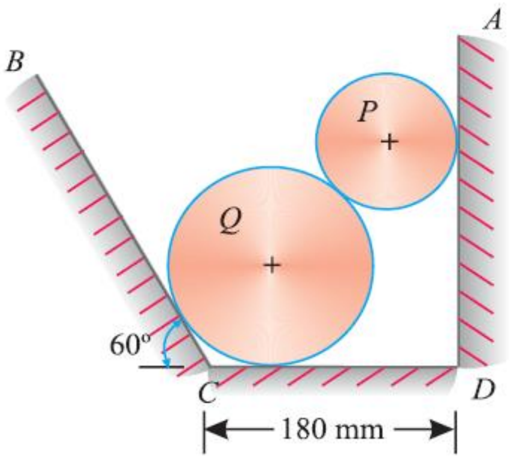
Again applying Lami's equation at joint *C*,

$$\frac{T_{BC}}{\sin 120^\circ} = \frac{T_{CD}}{\sin 120^\circ} = \frac{1000}{\sin 120^\circ}$$

$$\therefore \quad T_{CD} = \frac{1000 \sin 120^\circ}{\sin 120^\circ} = 1000 \text{ N} \quad \textbf{Ans.}$$

Worked out examples

Two cylinders P and Q rest in a channel as shown in Figure. The cylinder P has diameter of 100 mm and weighs 200 N, whereas the cylinder Q has diameter of 180 mm and weighs 500 N. If the bottom width of the box is 180 mm, with one side vertical and the other inclined at 60°, determine the reactions at all the four points of contact.



Solution

Given:

Diameter of cylinder P = 100 mm

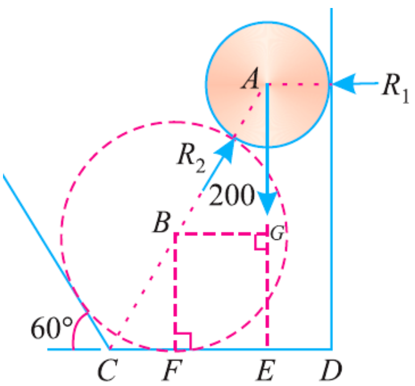
Weight of cylinder P = 200 N

Diameter of cylinder Q = 180 mm

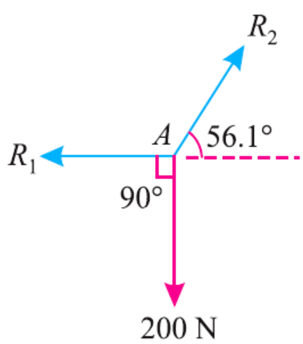
Weight of cylinder Q = 500 N and

Width of channel = 180 mm.

First of all, consider the equilibrium of the cylinder P. It is in equilibrium under the action of the following three forces which must pass through A i.e., the centre of the cylinder P as shown in Figure (a) below. The system of forces at A is shown in Figure (b) below.



(*a*) Free body diagram



(*b*) Force diagram

1. Weight of the cylinder (200 N) acting downwards.

2. Reaction (R_1) of the cylinder P at the vertical side.

3. Reaction (R_2) of the cylinder P at the point of contact with the cylinder Q.

From the geometry of the figure, we find that

$$ED = \text{Radius of cylinder } P = \frac{100}{2} = 50 \text{ mm}$$

Similarly $$BF = \text{Radius of cylinder } Q = \frac{180}{2} = 90 \text{ mm}$$

and $$\angle BCF = 60°$$

$$\therefore \quad CF = BF \cot 60° = 90 \times 0.577 = 52 \text{ mm}$$

$$\therefore \quad FE = BG = 180 - (52 + 50) = 78 \text{ mm}$$

and $$AB = 50 + 90 = 140 \text{ mm}$$

$$\therefore \quad \cos \angle ABG = \frac{BG}{AB} = \frac{78}{140} = 0.5571$$

or $$\angle ABG = 56.1°$$

Applying Lami's equation at A,

$$\frac{R_1}{\sin (90° + 56.1°)} = \frac{R_2}{\sin 90°} = \frac{200}{\sin (180° - 56.1°)}$$

$$\frac{R_1}{\cos 56.1°} = \frac{R_2}{1} = \frac{200}{\sin 56.1°}$$

$$R_1 = \frac{200 \cos 56.1°}{\sin 56.1°} = \frac{200 \times 0.5571}{0.830} = 134.2 \text{ N} \quad \textbf{Ans.}$$

$$R_2 = \frac{200}{\sin 56.1°} = \frac{200}{0.8300} = 240.8 \text{ N} \quad \textbf{Ans.}$$

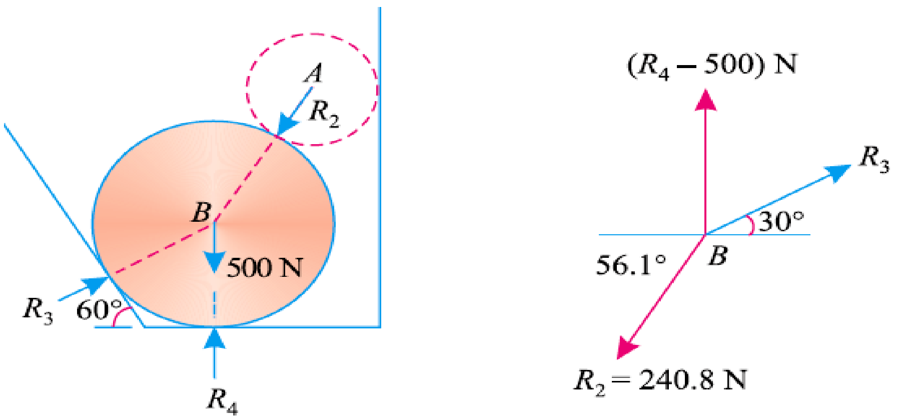
Now consider the equilibrium of the cylinder Q. It is in equilibrium under the action of the following four forces, which must pass through the centre of the cylinder as shown in Figure (a). The system of forces is shown in Figure (b).

1. Weight of the cylinder Q (500 N) acting downwards.

2. Reaction R_2 equal to 240.8 N of the cylinder P on cylinder Q.

3. Reaction R_3 of the cylinder Q on the inclined surface.

4. Reaction R_4 of the cylinder Q on the base of the channel.



(*a*) Free body diagram (*b*) Force diagram

A little consideration will show that the weight of the cylinder Q is acting downwards and the reaction R_4 is acting upwards. Moreover, their lines of action also coincide with each other.

∴ Net downward force = $(R_4 - 500)$ N

Applying Lami's equation at B,

$$\frac{R_3}{\sin(90° + 56.1°)} = \frac{240.8}{\sin 60°} = \frac{R_4 - 500}{\sin(180° + 30° - 56.1°)}$$

$$\frac{R_3}{\cos 56.1°} = \frac{240.8}{\sin 60°} = \frac{R_4 - 500}{\sin 26.1°}$$

$$\therefore \quad R_3 = \frac{240.8 \times \cos 56.1°}{\sin 60°} = \frac{240.8 \times 0.5577}{0.866} = 155 \text{ N} \quad \textbf{Ans.}$$

$$R_4 - 500 = \frac{240.8 \times \sin 26.1°}{\sin 60°} = \frac{240.8 \times 399}{0.866} = 122.3 \text{ N}$$

$$\therefore \quad R_4 = 122.3 + 500 = 622.3 \text{ N} \quad \textbf{Ans.}$$

TYPES OF SUPPORTS AND THEIR REACTIONS

In architectural structures, supports refer to the part of the structure which may help other parts to resist loads.

- ❖ Roller Supports
- ❖ Hinged Supports
- ❖ Fixed Supports

Roller Supports:

- ➢ Roller supports are free to rotate and translate along the surface upon which the roller rests. The surface can be horizontal, vertical, or sloped at any angle.

- The resulting reaction force is always a single force that is perpendicular to, and away from, the surface.
- Roller supports are commonly located at one end of long bridges.
- This allows the bridge structure to expand and contract with temperature changes.
- The expansion forces could fracture the supports at the banks if the bridge structure was "locked" in place.
- Roller supports can also take the form of rubber bearings, rockers, or a set of gears which are designed to allow a limited amount of lateral movement.
- A roller support cannot provide resistance to lateral forces. Imagine a structure on roller skates.
- It would remain in place as long as the structure must only support itself and perhaps a perfectly vertical load.
- As soon as a lateral load of any kind pushes on the structure it will roll away in response to the force.
- The lateral load could be a shove, a gust of wind or an earthquake.
- Since most structures are subjected to lateral loads it follows that a building must have other types of support in addition to roller supports.

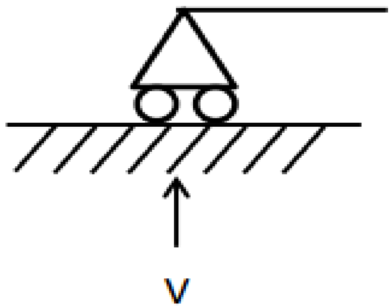
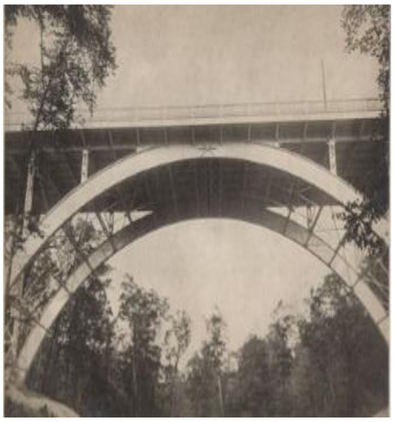


Fig. 2 Roller support

Hinged Supports:

- A hinged support can resist both vertical and horizontal forces but not a moment.
- They will allow the structural member to rotate, but not to translate in any direction.

- Many connections are assumed to be pinned connections even though they might resist a small amount of moment in reality.
- It is also true that a pinned connection could allow rotation in only one direction; providing resistance to rotation in any other direction.
- It is also used in doors to produce only rotation in a door.
- Hinge support reduces sensitivity to earthquake.



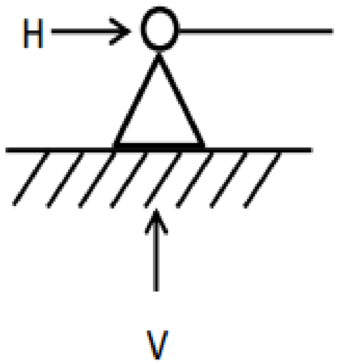


Fig. 3 Hinged supports

Fixed Support:

- Fixed support can resist vertical and horizontal forces as well as moment since they restrain both rotation and translation.
- They are also known as rigid support. For the stability of a structure there should be one fixed support.
- All three equations of equilibrium can be satisfied.
- A flagpole set into a concrete base is a good example of this kind of support. The representation of fixed supports always includes two forces (horizontal and vertical) and a moment.

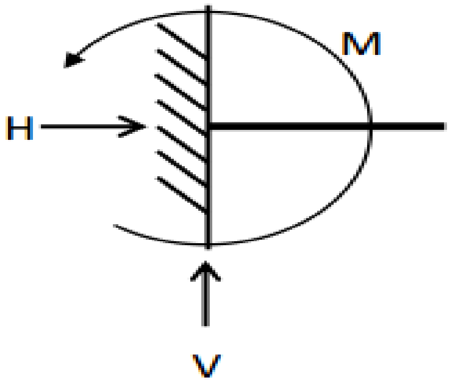


Fig. 3 Fixed supports

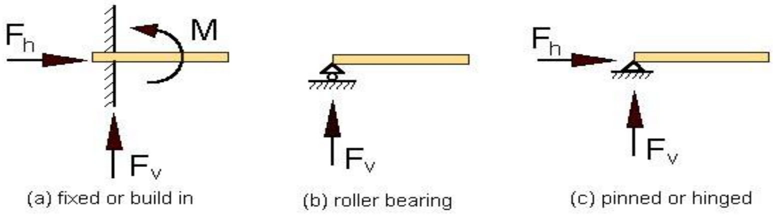


Fig. 4 all three supports

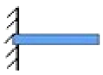
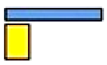
S.no	Types of Support	Representation by	Reaction Force	Resisting Load
1.	**Roller Support**		Vertical	Vertical loads
2.	**Pinned Support**		Horizontal and vertical	Vertical and horizontal loads
3.	**Fixed Support**		Horizontal, vertical and moments	All types of loads Horizontal, vertical and Moments
4.	**Simple Support**		Vertical	Vertical loads

Fig. 5 Supports and its respective reactions

TYPES OF LOADING

Though there are many types of loading, yet the following are important from the subject point of view:

- Concentrated or point load,
- Uniformly distributed load,
- Uniformly varying load.

CONCENTRATED OR POINT LOAD

A load, acting at a point on a beam is known as a concentrated or a point load as shown in Figure.

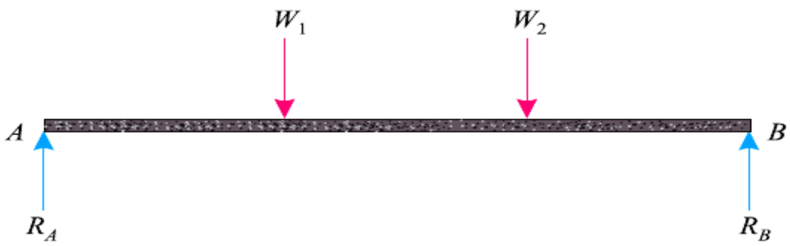


Fig. 6 concentrated or point load

UNIFORMLY DISTRIBUTED LOAD

A load, which is spread over a beam, in such a manner that each unit length is loaded to the same extent, is known as *uniformly distributed load (briefly written as U.D.L.) as shown in Figure*

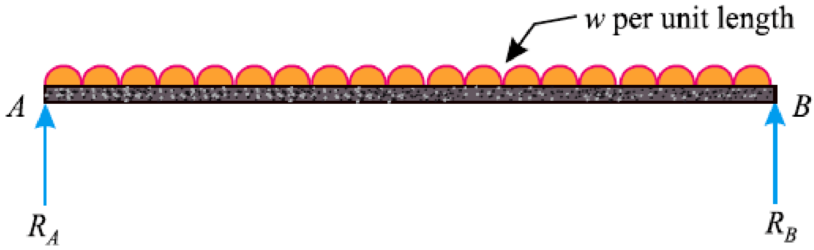


Fig. 7 uniformly distributed load

UNIFORMLY VARYING LOAD

- A load, which is spread over a beam, in such a manner that its extent varies uniformly on each unit length (say from *w_1 per unit length at one support to w_2 per unit length at the other support) is* known as *uniformly varying load as shown in Figure.*
- Sometimes, the load varies from zero at one support to w at the other. Such a load is also called triangular load.

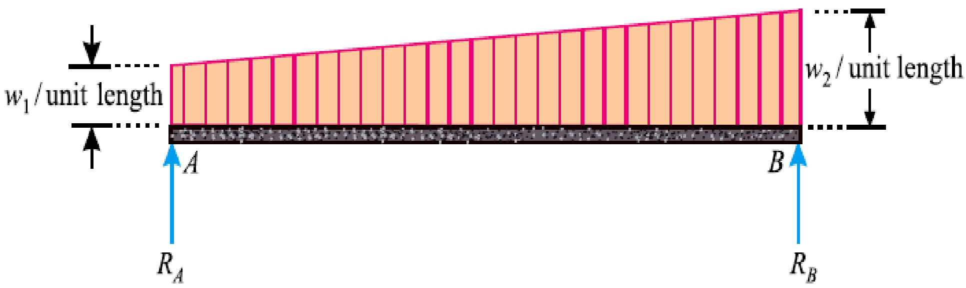


Fig. 8 uniformly varying load

MOMENTS AND COUPLES

A pair of two equal and unlike parallel forces (*i.e.* forces equal in magnitude, with lines of action parallel to each other and acting in opposite directions) is known as a couple. As a matter of fact, a couple is unable to produce any translatory motion (*i.e.*, motion in a straight line). But it produces a motion of rotation in the body, on which it acts. The simplest example of a couple is the forces applied to the key of a lock, while locking or unlocking it.

Arm of a couple

The perpendicular distance (a), between the lines of action of the two equal and opposite parallel forces, is known as arm of the couple as shown in Figure.

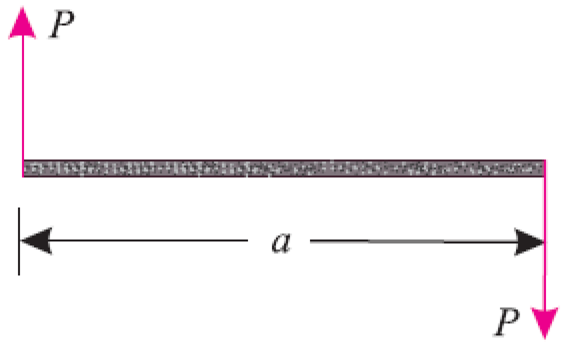


Fig. 9 arm of the couple

MOMENT OF A COUPLE

The moment of a couple is the product of the force (*i.e.*, one of the forces of the two equal and opposite parallel forces) and the arm of the couple.

Mathematically:

Moment of a couple = *P* × *a*

Where,

P = Magnitude of the force, and

a = Arm of the couple.

Classification of Couples

The couples may be, broadly, classified into the following two categories, depending upon their direction, in which the couple tends to rotate the body, on which it acts:

1) Clockwise couple, and
2) Anticlockwise couple

A couple, whose tendency is to rotate the body, on which it acts, in a clockwise direction, is known as a **clockwise couple** as shown in Figure (a). Such a couple is also called positive couple.

A couple, whose tendency is to rotate the body, on which it acts, in an anticlockwise direction, is known as an **anticlockwise couple** as shown in Figure (b). Such a couple is also called a negative couple.

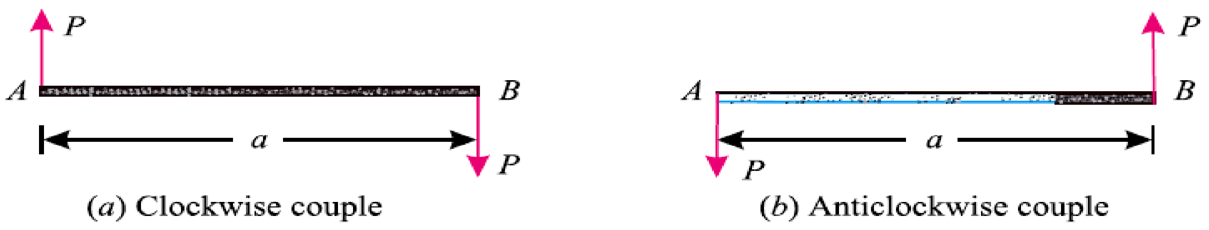


Fig. 10 Classification of Couples

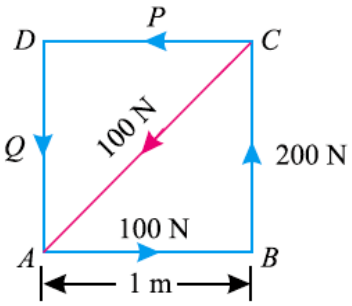
Characteristics of a couple

A couple (whether clockwise or anticlockwise) has the following characteristics:

- The algebraic sum of the forces, constituting the couple, is zero.
- The algebraic sum of the moments of the forces, constituting the couple, about any point is the same, and equal to the moment of the couple itself.
- A couple cannot be balanced by a single force. But it can be balanced only by a couple of opposite sense.
- Any no. of coplanar couples can be reduced to a single couple, whose magnitude will be equal to the algebraic sum of the moments of all the couples.

Worked out example

A square ABCD has forces acting along its sides as shown in Figure. Find the values of P and Q, if the system reduces to a couple. Also find magnitude of the couple, if the side of the square is 1 m.



Solution:

Values of P and Q

We know that if the system reduces to a couple, the resultant force in horizontal and vertical directions must be zero. Resolving the forces horizontally,

$$100 - 100 \cos 45° - P = 0$$

$$\therefore\ P = 100 - 100 \cos 45° \text{ N} = 100 - (100 \times 0.707) = \mathbf{29.3\ N}$$

Now resolving the forces vertically,

$$200 - 100 \sin 45° - Q = 0$$

$$\therefore\ Q = 200 - (100 \times 0.707) = \mathbf{129.3\ N}$$

Magnitude of the couple

We know that moment of the couple is equal to the algebraic sum of the moments about any point. Therefore moment of the couple (taking moments about *A*)

$$= (-200 \times 1) + (-P \times 1) = -200 - (29.3 \times 1) \text{ N.m} = \mathbf{-229.3\ N.m}$$

Since the value of moment is negative, therefore the couple is anticlockwise.

VARIGNON's THEOREM

Moment of a force about any point is equal to the sum of the moments of the components of that force about the same point. To prove this theorem, consider the force **R** acting in the plane of the body shown in Figure.1. The forces **P** and **Q** represent any two nonrectangular components of **R.** The moment of **R** about point O is

$\mathbf{M_0} = \mathbf{r} \times \mathbf{R}$

Because $\mathbf{R} = \mathbf{P} + \mathbf{Q}$, we may write

$\mathbf{r} \times \mathbf{R} = \mathbf{r} \times (\mathbf{P} + \mathbf{Q})$

Using the distributive law for cross products, we have

$\mathbf{M_0} = \mathbf{r} \times \mathbf{R} = \mathbf{r} \times \mathbf{P} + \mathbf{r} \times \mathbf{Q}$

This says that the moment of R about O equals the sum of the moments about O of its components P and Q.

This proves the theorem. Varignon's theorem need not be restricted to the case of two components, but it applies equally well to three or more where we take the clockwise moment sense to be positive.

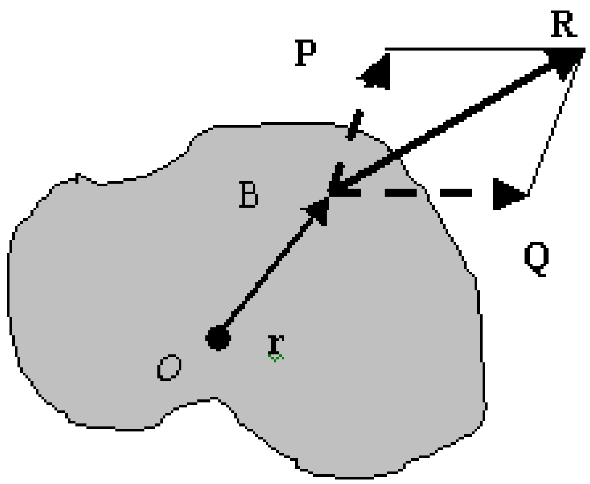


Fig. 11 Illustrating Varignon's theorem

Theorem of Varignon's

The moment of the resultant of two concurrent forces with respect to a centre in their plane is equal to the algebraic sum of the moments of the components with respect to some centre.

Introduction

In our day-to-day work, we see that whenever we apply a force on a body, it exerts a reaction, *e.g.,* when a ceiling fan is hung from a girder, it is subjected to the following two forces:

1. Weight of the fan, acting downwards, and
2. Reaction on the girder, acting upwards.

A little consideration will show, that as the fan is in equilibrium therefore, the above two forces must be equal and opposite. Similarly, if we consider the equilibrium of a girder supported on the walls, we see that the total weight of the fan and girder is acting through the supports of the girder on the walls. It is thus obvious, that walls must exert equal and upward reactions at the supports to maintain the equilibrium. The upward reactions, offered by the walls, are known as support reactions. As a matter of fact, the support reaction depends upon the type of loading and the support.

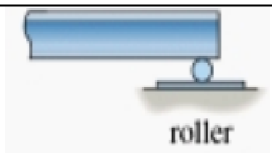
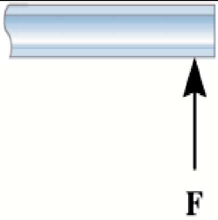
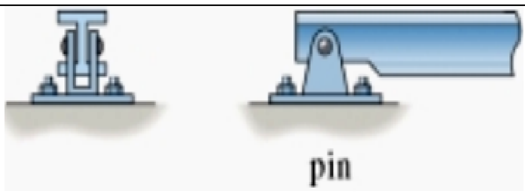
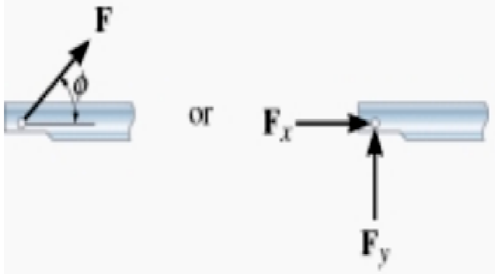
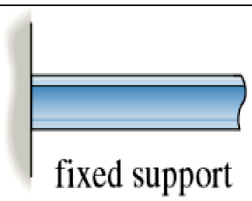
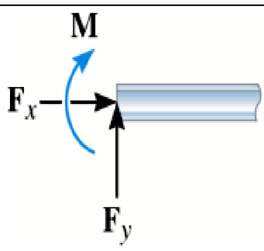
Constraints	Type and direction of forces produced
 The connection point on the bar can not move downward.	
 The joint can not move in vertical and horizontal directions.	
 The support prevents translation in vertical and horizontal directions and also rotation, Hence a couple moment is developed on the body in that direction as well.	

Fig. 12 Supports and Reactions

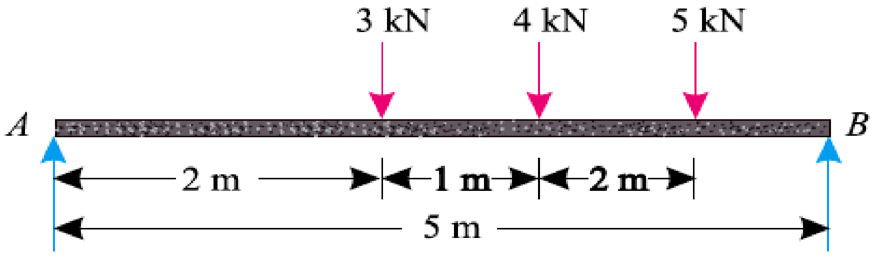
TYPES OF END SUPPORTS OF BEAMS

Though there are many types of supports, for beams and frames, yet the following three types of supports are important from the subject point of view:

1. Simply supported beams,
2. Roller supported beams, and
3. Hinged beams

Worked out examples

A simply supported beam AB of span 5 m is loaded as shown in Figure. Find the reactions at A and B.



Solution:

Given: Span (l) = 5 m

Let R_A = Reaction at A, and

R_B = Reaction at B.

The example may be solved either analytically or graphically. But we shall solve analytically only. We know that anticlockwise moment due to R_B about A

$= R_B \times l = R_B \times 5$

$= 5\ R_B$ kN-m ...(i)

And sum of the clockwise moments about A,

$= (3 \times 2) + (4 \times 3) + (5 \times 4)$

$= 38$ kN-m ...(ii)

Now equating anticlockwise and clockwise moments given in (i) and (ii),

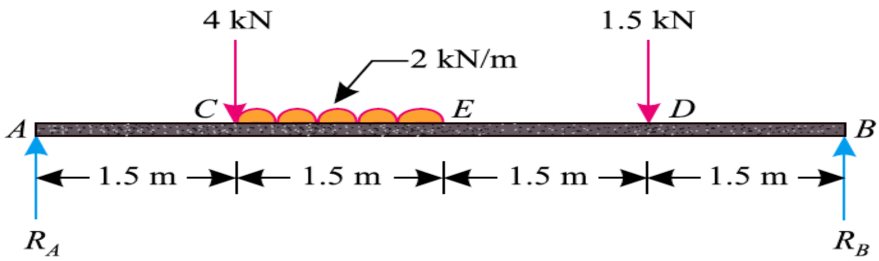
$5\ R_B = 38$

$$R_B = \frac{38}{5} = 7.6 \text{ kN}$$

$$R_A = (3 + 4 + 5) - 7.6 = 4.4 \text{ kN}$$

Worked out examples

A simply supported beam, AB of span 6 m is loaded as shown in Figure. Determine the reactions RA and RB of the beam.



Solution:

Given:

Span (l) = 6m

Let R_A = Reaction at A, and

R_B = Reaction at B.

The example may be solved either analytically or graphically. But we shall solve it analytically only.

We know that anticlockwise moment due to the reaction R_B about A.

$= R_B \times l = R_B \times 6 = 6\, R_B$ KN.m ...(i)

And sum of the clockwise moments about A

$= (4 \times 1.5) + (2 \times 1.5)\, 2.25 + (1.5 \times 4.5)$

$= 19.5$ KN.m ...(ii)

Equating anticlockwise and clockwise moments given in (i) and (ii),

$6\, R_B = 19.5$

$R_B = 19.5 / 6$

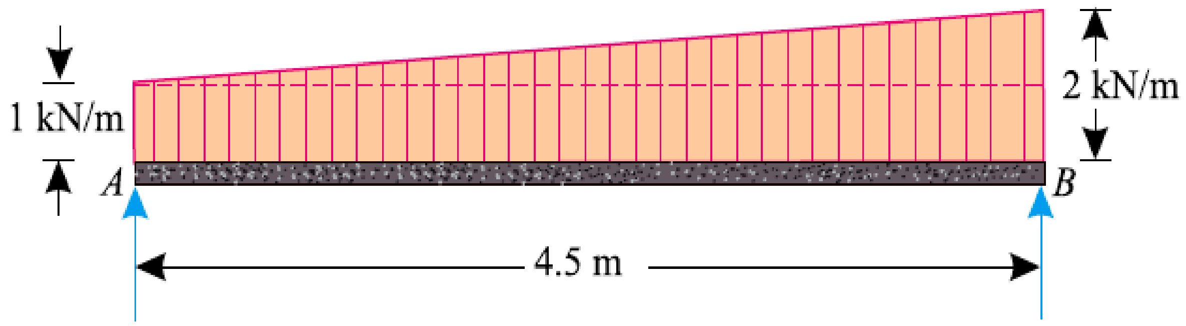
$\boldsymbol{R_B = 3.25}$ **KN**

And $R_A = 4 + (2 \times 1.5) + 1.5 - 3.25$

$\boldsymbol{R_A = 5.25}$ **KN**

Worked out examples

A simply supported beam AB of span 4.5 m is loaded as shown in Figure. Find the support reactions at A and B.



Solution:

Given: Span (l) = 4.5 m

Let R_A = Reaction at A, and

R_B = Reaction at B.

The uniformly distributed load of 2 kN/m for a length of 1.5 m (*i.e.*, between C and E) is assumed as an equivalent point load of $2 \times 1.5 = 3$ kN and acting at the centre of gravity of the load *i.e.*, at a distance of $1.5 + 0.75 = 2.25$ m from A.

The uniformly distributed load of 1 kN/m over the entire span is assumed as an equivalent point load of $1 \times 4.5 = 4.5$ kN and acting at the centre of gravity of the load *i.e.* at a distance of 2.25 m from A. Similarly, the triangular load in assumed as an equivalent point load of $4.5 \times \frac{0+1}{2} = 2.25$ kN and acting at the centre of gravity of the load *i.e.*, distance of $4.5 \times \frac{2}{3} = 3$ m from A.

We know that anticlockwise moment due to R_B about A

$= R_B \times l = R_B \times 4.5 = 4.5\ R_B$ kN-m ...(*i*)

And sum of clockwise moments due to uniformly varying load about A

$= (1 \times 4.5 \times 2.25) + (2.25 \times 3)$

$= 16.875$ kN-m ...(*ii*)

Now equating anticlockwise and clockwise moments given in (*i*) and (*ii*),

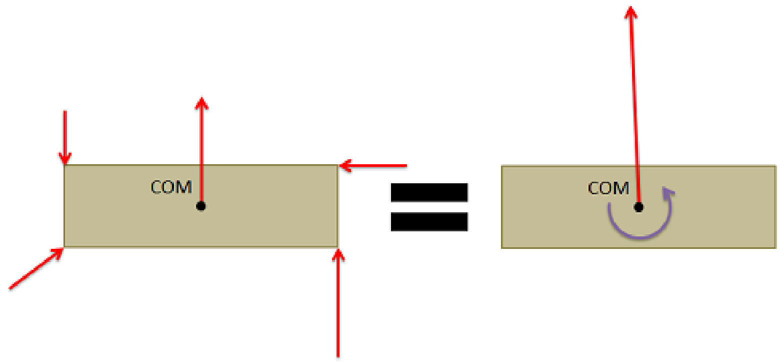
$$4.5\ R_B = 16.875$$

$$R_B = \frac{16.875}{4.5} = 3.75 \text{ kN}$$

$$R_A = [1 \times 4.5] + \left[4.5 \times \frac{0+1}{2}\right] - 3.75 = 3.0 \text{ kN}$$

Equivalent Force Couple System

Every set of forces and moments has an **equivalent force couple system**. This is a single force and pure moment (couple) acting at a single point that is **statically equivalent** to the original set of forces and moments.



Any set of forces on a body can be replaced by a single force and a single couple acting that is statically equivalent to the original set of forces and moments. This set of an equivalent force and a couple is known as the equivalent force couple system.

To find the equivalent force couple system, you simply need to follow the steps below.

1. First, choose a point to take the equivalent force couple system about. Any point will work, but the point you choose will affect the final values you find for the equivalent force couple system. Traditionally this point will either be the center of mass of the body or some connection point for the body.
2. Next resolve all the forces not acting though that point to a force and a couple acting at the point you chose.
3. To find the "force" part of the equivalent force couple system add together all the force vectors. This will give you the magnitude and the direction of the force in the equivalent force couple system.
4. To find the "couple" part of the equivalent force couple system, add together any moment vectors (this could be moments originally acting on the body or moments from the resolution of the forces into forces and couples). This will give you the magnitude and direction of the pure moment (couple) in the equivalent force couple system.

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SCHOOL OF BUILDING AND ENVIRONMENT

DEPARTMENT OF CIVIL ENGINEERING

UNIT – III - PROPERTIES OF SURFACES AND SOLIDS – SCIA1101

III. Properties of Surfaces and Solids

Determination of Areas - First moment of Area and the centroid - simple problems involving composite figures, Second moment of plane area - Parallel axis theorems and perpendicular axis theorems - Polar moment of Inertia Principal moments of Inertia of plane areas - Principle axes of inertia - relation to area moments of Inertia, Second moment of plane area of sections like C,I,T,Z etc. - Basic Concept of Mass moment of Inertia.

INTRODUCTION

An important part of the job of a skilled construction tradesperson involves making measurements based on instructions such as blueprints and then building based on those measurements. Before you begin construction, one of the challenges may be to take those measurements and to make calculations such as perimeter, area and volume. For example, to make a window frame, a glazier must calculate the perimeter around the glass in order to know how much trim will be needed. A reinforcing rod worker would need to calculate the total area of concrete coverage in order to determine the number of reinforcing rods to use.

This skill sheet reviews the steps in finding the perimeter, area and volume of simple two and three dimensional geometric figures, including:

1. Two dimensional figures
2. Finding the perimeter
3. Finding the area
4. Three dimensional figures
5. Finding the surface area
6. Finding the volume

TWO DIMENSIONAL GEOMETRIC FIGURES

A simple, closed, two dimensional (flat) figures with three or more straight sides is called a polygon. Triangles, squares, rectangles, and parallelograms (figures with 2 pair of opposite sides parallel) are all examples of polygons. A circle is also a flat, closed figure but it is a curve, consisting of points that are all the same distance from the centre.

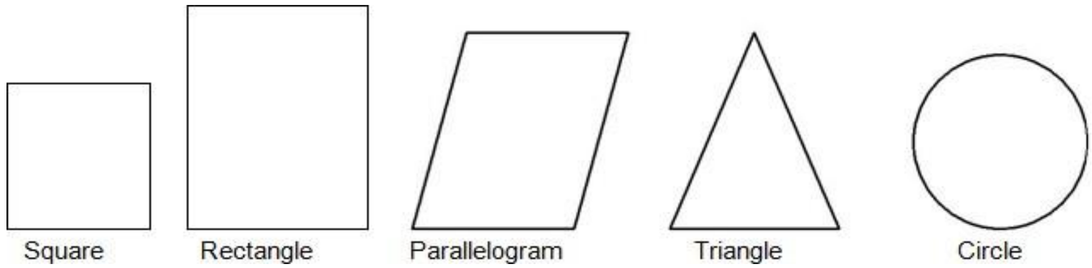


FIGURE 1: Some Simple Geometric Shapes

These figures can be measured in different ways.

a. Whenever we use measurements to make calculations with geometric figures, all measurements must be in the same linear units.

b. The units might be meters or centimeters, but they can't be a mix of meters and centimeters.

FINDING THE PERIMETER

The **perimeter** (P) of any polygon is the distance around its boundary. Perimeter is found by adding together the lengths of the sides.

Perimeter of a Rectangle

A **rectangle** is a polygon with four 90° (right angles) and with each pair of parallel sides the same length (see Figure 1).This means that we can find the perimeter of a rectangle by adding the lengths of the two long side to the lengths of the two shorter side.

The perimeter of a rectangle equals twice the length (l) added to twice the width (w). The formula is written in two forms:

$P = 2l + 2w$ or $P = 2(l + w)$

where

P is the perimeter, l is the length and w is the width of the rectangle.

Note: When finding perimeter, all units must be the same. If the length is measured in feet and the width in yards, one unit must be changed to that of the other.

Example: Find the perimeter of a house that is 30 m long and 16 m wide.

$P = 2l + 2w$

$= 2(30 \text{ m}) + 2(16 \text{ m})$

$= 60 \text{ m} + 32 \text{ m}$

$= 92 \text{ m}$

The perimeter is 92 m.

Example: Find the amount of fencing required to close in a space that is 400 yd wide and 1500 ft long.

Known:

$l = 1500$ ft

$w = 400$ yd $= 1200$ ft   400 yd x3 = 1200 ft

Find perimeter (P)

$P = 2(l + w)$

$= 2\,(1500 \text{ ft} + 1200 \text{ ft})$

$= 2(2700 \text{ ft})$

$= 5400 \text{ ft}$

The space will require 5400 ft of fencing.

Perimeter of a Square

A square is a rectangle with all four sides the same length.

To find the perimeter of a square, multiply the length by 4.

Perimeter of a square = $4l$

Example: How much baseboard trim is required for a bedroom that is 12 ft square? (If a room is 12 ft square, it measures 12 ft by 12 ft.)

$P = 4l$

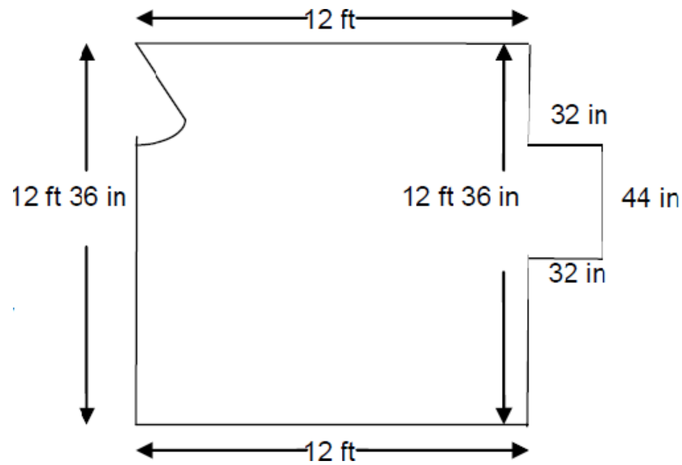
$= 4(12)$

$= 48 \text{ ft}$

48 ft of trim is required.

Often a space is more complicated than a simple square.

Example: If the bedroom has two door openings, each measuring 36 in. and a closet with two sides measuring 32 in. and a back length of 44 in, how much trim will be required now?



Solution

A diagram can help you with these calculations. When you have to find the perimeter or area and a diagram is not shown, it is helpful to draw one. First convert the 36 in door openings to feet.

36 in 12 = 3 ft

32 in + 32 in + 44 in = 108 in	Add the widths of the closet
108 in ÷ 12 = 9 ft	convert the inches to feet
= 48 ft - 2(3 ft) + 9 ft	Subtract the door openings and add the closet
= 48 ft - 6 ft + 9 ft	
= 51 ft	

51 ft of trim is needed.

To find the perimeter of an irregular shape, you basically add all the lengths together. Just make sure all the measurements are in the same units.

Finding the Length of an Unknown Side When the Perimeter Is Known

If you know the perimeter of a rectangle and the length of one side, you can find the other side.

1. Manipulate (or rearrange) the variables in the formula for perimeter so the letter for length or width is by itself on the left side.
2. Solve to find the unknown side.

Note: whatever you do to one side of the formula, you need to do to the numbers and letters on the other side.

Example: The perimeter of a window is 144 inches. The height of the window is 42 inches. What is the width?

P = 2l + 2 w	Fill in the quantities you are given.
144 = 2(42) + 2w	
144 = 84 + 2w	
84 + 2w = 144	Reverse the equation.
84 + 2w	Subtract 84 from both sides.
2w = 60	Divide both sides by 2.
w = 30	Write in the units, inches.

The width is 30 inches.

FINDING THE AREA

The **area** of a polygon is the measure of the surface inside the boundary. The units of area are squared units.

Area of a Rectangle

The area of a rectangle is the amount of surface enclosed within its boundaries of **length** and width**.**

Example: The area of a room is the amount of floor space it has.

Area is calculated by multiplying the length of the rectangle times its width.

The formula for area is:

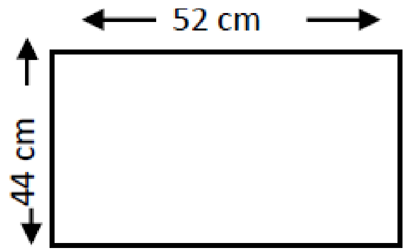
A = lw

Note: When finding the area of a rectangle, the units used to measure the length and the width must be the same. If the length is in meters, the width must also be in meters. If the units are different, one must be converted to the other before you can multiply.

Example: Find the area of a rectangle that is 52 cm long and 44 cm wide.

(The units are the same so we don’t have to convert.)

Draw the rectangle



Known: Find:

l = 52 cm Area

w= cm Use A = lw

A = lw

= 52 cm x 44 cm

= 2288 cm^2

Note: When two of the same units are multiplied together, such as the centimeters in our example, they become square units. Instead of writing square centimeters, you can use the short form of cm2 or sq cm. (Sq is the short form for square.) Four square feet is written 4 sq ft or 4 ft^2.

Example: Find the area of a space with length 5 m and width 142 cm.
We must convert one of the units so both are the same.

Known:
l = 5 m
w = 142 cm
w = 1.42 m

A = lw
= 5 m x 1.42 m
A= 7.1 m^2

Example: Find the floor space of a box that measures 60 inches long by 40 inch wide by 20 inches high.
(The information on height is not needed to answer this question.)

Known: l = 40 in
w = 20 in

Find: A

Use A= lw

A = lw

=60 x 40

=2400 sq in

Example: In order to calculate the quantity of terrazzo tile for a family room you need to calculate the area of the floor. If the family room measures 5 m by 3.5 m, what is the floor space to be covered?

Known: l = 5m

W = 3.5 m

Find: A

A = lw

=5 x 3.5

=17.5 m^2

The floor space to be covered is 17.5 m^2

Area of a Square

The four sides of a square are all the same length. To find the area of a square, square the length. (To square a number, multiply it by itself. Three squared is 3 x 3 = 9.)

Example: Find the area of a square with sides 15 ft long.

Known: l = 15

ft w =

15 ft

Find A

A = lw or l^2

A = 15 ft x 15 ft

A = 225 sq ft

Area of a Parallelogram

The area of a parallelogram is equal to the altitude or height times the base. The formula is:

A = ab or bh

Example: Find the area of a parallelogram with a height of 12 cm and a base of 15 cm.

THREE DIMENSIONAL FIGURES

A closed, solid geometric figure has three dimensions. It has length, width and height or depth. Some solid figures are the cube, the rectangular solid, the cylinder, the cone and the sphere.

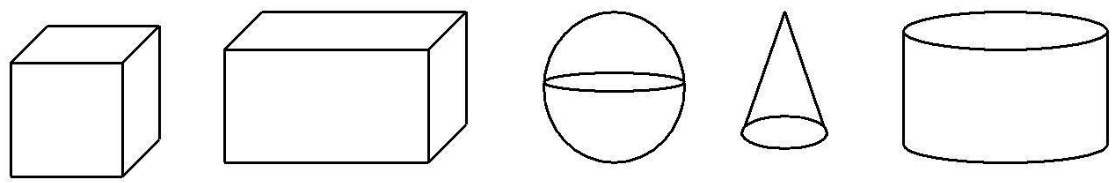


FIGURE 2: Solid Geometric Figures

SURFACE AREA OF THREE DIMENSIONAL FIGURES

The surface area of a three dimensional figure is the combined areas of all the outside surfaces or faces of the figure. When finding the surface area, all measurements must be in the same linear units. The answer will be in square units.

Finding the surface area of a rectangular solid

To find the total area of the outside surface of a rectangular solid, we have to find the areas of each face of the figure.

1. First find the area of the front surface by multiplying the length times the height.
   - The back surface is the same area, so multiply that answer by 2.
2. Next find the area of one side by multiplying the width times the height.
   - Since the opposite side is the same, multiply the answer by 2.
3. Now find the base by multiplying the length times the width.
   - The top is the same as the base, so multiply that answer by 2 also.

The formula is:

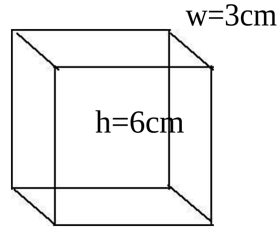
$A = 2lh + 2wh + 2lw$

or $A = 2(lh + wh + lw)$

or $A = 2(lh + wh + lw)$

Example: Find the total area of the outside surface of a rectangular solid 5 cm long, 3 cm wide and 6 cm high.

Draw and label the solid



Known:

l = 5 cm

w = 3 cm

h = 6 cm

Find:

Outside surface area of the solid $A = 2(lh + wh +$ lw)

$A = 2(lh + wh + lw)$

$= 2(5cm \times 6cm + 3cm \times 6cm$

$+ 5cm \times 3cm)$

$= 2(30\ cm^2 + 18\ cm^2 + 15\ cm^2)$

$= 2(63\ cm^2)$

$= 126\ cm^2$

Finding the surface area of a cube

A cube is made of six identical squares. Each edge is the same length, each side has the same area.

To find the area of a cube:

1. Find the area of one side (l^2) and multiply it by 6.

The formula is:

$A = 6(l^2)$

Example: Find the total surface area of a cube whose edges measure 10 in.

Known:

Edges of cube = 10 in

Find: A = 6(l

Surface area of cube 2)

$A = 6\ (l^2)$

$= 6(10^2)$

$= 6(100)$

$= 600$ sq in.

Finding the surface area of a cylinder

The surface area of a cylinder consists of the outside curved surface, which is actually a rectangle if it is straightened, and the circular areas at the top and bottom.

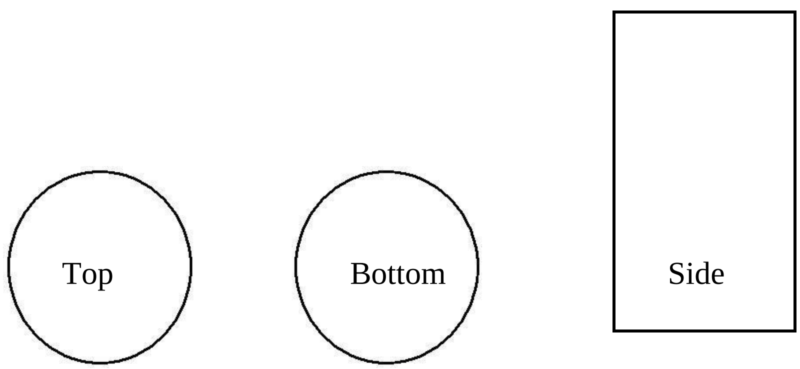


FIGURE 3: Finding the surface area of a cylinder

To find the surface area of a cylinder:

1. Find the area of each of the top and bottom circles.
2. Find the area of the rectangular side:
3. Add the areas together.

1. **To find the area of the top and bottom:** Use the formula $A = \pi r^2$. A cylinder has two circles (the top and the bottom), so we need to find the two areas, $2\pi r^2$.

Note: $\pi = 3.14$

2. **To find the area of the side of the cylinder (a rectangle):** Multiply the length times the width.

The formula is: $A = lw$.

This rectangle has a width equal to the height of the cylinder so substitute height (h) for the width.

The formula is now: $A = 2\ lh$.

The length of the rectangle is the same as the perimeters of the circles at the top and bottom. We find the perimeter of a circle using the formula $P = 2\pi r$. Substitute this formula for the length of the rectangle.

The formula becomes $A = 2\pi rh$.

3. **To find the area of the cylinder** add the areas of the top and bottom ($2\pi r^2$) to the area of the rectangle ($2\pi rh$).

$A = 2\pi r^2 + 2\pi rh$.

The formula is rearranged to become:

$A = 2\ \pi r(r + h)$

Note: $\pi = 3.14$

Example: Find the surface area of a cylinder when its radius is 8 ft and its height is 20 ft.

Known:

r of cylinder = 8 ft

h of cylinder = 20

ft

Find the surface area of the cylinder

$A = 2\ \pi r(r + h)$

$= (2 \times 3.14 \times 8)(8 + 20)$

$= (50.24)\ (28)$

$= 1406.72$ sq ft

=

<u>Finding the surface area of a sphere</u>

A sphere is a ball. The surface area of a sphere is equal to 4 times π times the radius squared. The formula is:

$A = 4\ \pi\ r^2$

Example: Find the surface area of a sphere with a radius of 5 cm.

Known

r = 5

cm

Find the surface area of the sphere

$A = 4 \pi r^2$

$= 4 \times 3.14 \times 5^2$

$= 314\ cm^2$

To find the cost of covering the outside surface of an object:

1. Find the surface area, and
2. Multiply it by the cost per unit area.

VOLUME OF THREE DIMENSIONAL GEOMETRIC FIGURES

The **volume** or capacity of a solid figure is the amount of space contained within its boundaries. To calculate volume, multiply length times width times depth. Since each linear measurement has a unit, the units in the answer become cubic units. For example, meters x meters x meters equal cubic meters. The short form for cubic units such as cubic inches is in^3 or cu in.

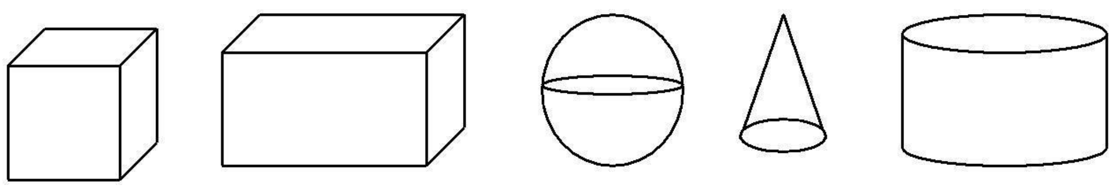


FIGURE Solid Geometric Figures

Volume of a rectangular solid

The volume of a rectangular solid equals the length times the width times the height.

$V = lwh$

Example: Find the volume of a rectangular solid 9 m long, 4 m wide and 3 m high.

$V = lwh$

$= 9 \times 4 \times 3$

$= 108\ m^3$

Volume of a cube

The volume of a cube equals the length of one edge cubed. The formula is:

$V = l^3$

Example: Find the volume of a cube whose length measures 2 m.

$V = l^3$

$= 2^3$

$= 8\ m^3$

Volume of a cylinder

The volume of a cylinder equals π times the square of the radius of the base times the height. The formula is:

$V = \pi r^2 h$

Example: Find the volume of a cylinder with a radius of 12 ft and a height of 72 in.

72 in ÷ 12 = 6 ft   Change the units of height to feet by dividing by 12.

Now use the formula.

$V = \pi r^2 h$

$= 3.14 \times 12^2 \times 6$

$= 2713$ cu ft

Volume of a sphere

The volume of a sphere equals 4/3 times π times the cube of the radius. The formula is:

$V = 4\pi r^3/3$

Example: Find the volume of a sphere with a radius of 10 inches.

$V = 4\pi r^3/3 = 4(3.14)\ 10^3/3 = 4186.67$ cu in

Second Moment

If any quantity is multiplied by the distance from the axis s-s twice, we have a second moment. Mass multiplied by a distance twice is called the moment of inertia but is really the second moment of mass. The symbol for both is confusingly a letter I.

$$I = A\ k^2$$

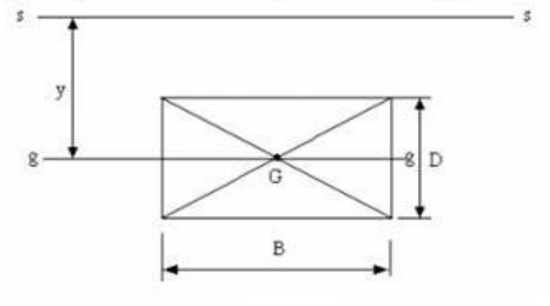
Parallel axis theorems

The moment of inertia of any object about an axis through its center of mass is the minimum moment of inertia for an axis in that direction in space. The moment of inertia about any axis parallel to that axis through the center of mass

If we wish to know the 2nd moment of area of a shape about an axis parallel to the one through the centroid (g-g), then the parallel axis theorem is useful.

The parallel axis theorem states $I_{ss} = I_{gg} + A(\bar{y})^2$

Consider a rectangle B by D and an axis s-s parallel to axis g-g.



$I_{gg} = \dfrac{BD^3}{12}$ $\quad A = BD \quad$ $I_{ss} = \dfrac{BD^3}{12} + BD\bar{y}^2$

Consider when s-s is the top edge.

$I_{ss} = \dfrac{BD^3}{12} + BD\bar{y}^2$ but $\bar{y} = \dfrac{D}{2}$ so $I_{ss} = \dfrac{BD^3}{12} + BD\left(\dfrac{D}{2}\right)^2 = \dfrac{BD^3}{12}$

This is the result obtained previously and confirms the method.

Perpendicular axis theorems

For a planar object, the moment of inertia about an axis perpendicular to the plane is the sum of the moments of inertia of two perpendicular axes through the same point in the plane of the object. The utility of this theorem goes beyond that of calculating moments of strictly planar objects. It is a valuable tool in the building up of the moments of inertia of three dimensional objects such as cylinders by breaking them up into planar disks and summing the moments of inertia of the composite disks.

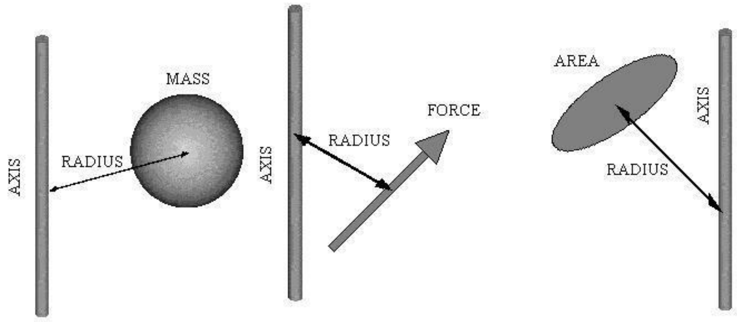
$$I_Z = I_X + I_Y$$

Composite bodies: If a body is composed of several bodies, to calculate the moment of inertia about a given axis one can simply calculate the moment of inertia of each part around the given axis and then add them to get the mass moment of inertia of the total body.

CENTROIDS AND FIRST MOMENTS OF AREA

A moment about a given axis is something multiplied by the distance from that axis measured at 90^o to the axis.

The moment of force is hence force times distance from an axis.
The moment of mass is mass times distance from an axis.
The moment of area is area times the distance from an axis.



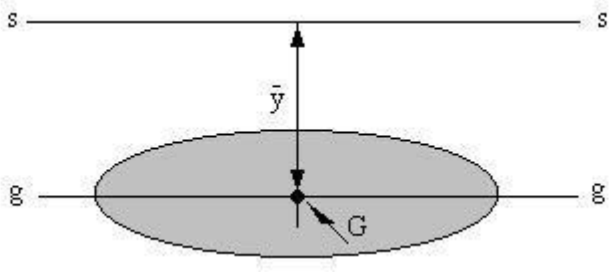
In the case of mass and area, the problem is deciding the distance since the mass and area are not concentrated at one point.

The point at which we may assume the mass concentrated is called the centre of gravity.

The point at which we assume the area concentrated is called the centroid.

Think of area as a flat thin sheet and the centroid is then at the same place as the centre of gravity. You may think of this point as one where you could balance the thin sheet on a sharp point and it would not tip off in any direction.

This section is mainly concerned with moments of area so we will start by considering a flat area at some distance from an axis as shown



The centroid is denoted G and its distance from the axis s-s is y. The axis drawn through G parallel to s-s is the axis g-g. The first moment of area about the axis s-s is the product of area A and distance.

1st moment of area = A

~~y~~ From this we may define the distance y.

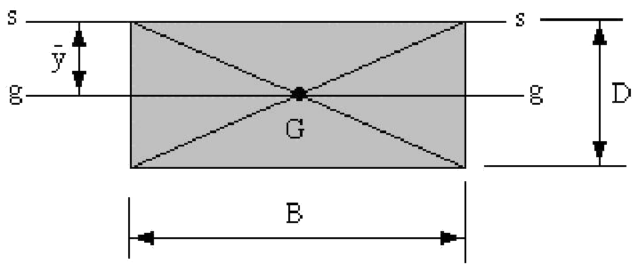
~~y~~ = 1st moment of area/Area.

For simple symmetrical shapes, the position of the centroid is obvious.

EXAMPLE 1

Find the formula for the first moment of area for rectangle about its longer edge given the dimensions are B and D.

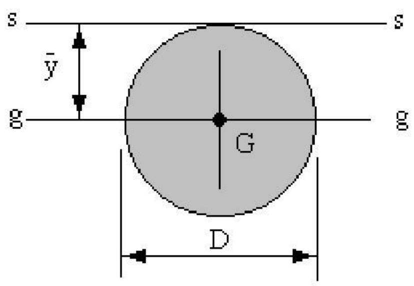
SOLUTION



y = D/2
A = BD

EXAMPLE 2

Find the formula for the 1st moment of area of a circular area about an axis touching its edge in terms of its diameter d.



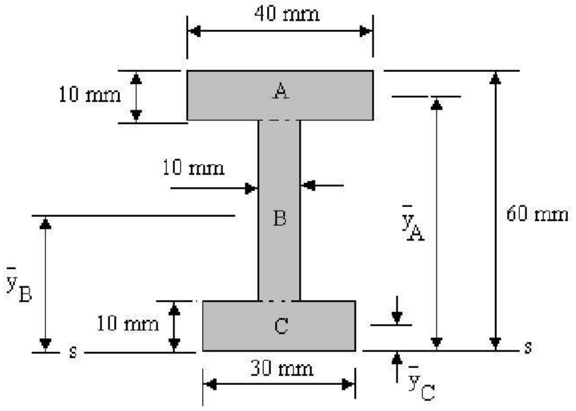
$\bar{y}$ = D/2 A = $\pi D^2/4$ 1st moment = A $\bar{y}$ = (πD2/4)/(D/2) = $\pi D^3/8$

COMPLEX AREAS

In order to find the moment of area of more complex shapes we divide them up into sections, solve each section separately and then add them together.

EXAMPLE 3

Calculate the 1st moment of area for the shape shown about the axis s-s and find



SOLUTION

The shape is not symmetrical so the centroid is not half way between the top and bottom edges. First determine the distance from the axis s-s to the centre of each

Part	Area	$\bar{y}$	Ay
A	400	55	22000
B	400	30	12000
C	300	5	1500
Total	1100		35500

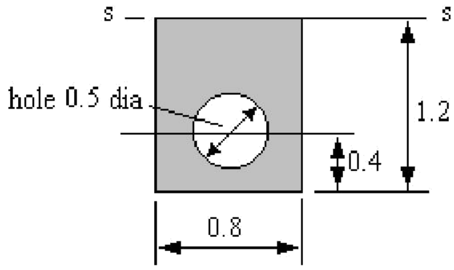
The total first moment of area is 35 500 mm^3.

This must also be given by A $\bar{y}$ for the whole section hence

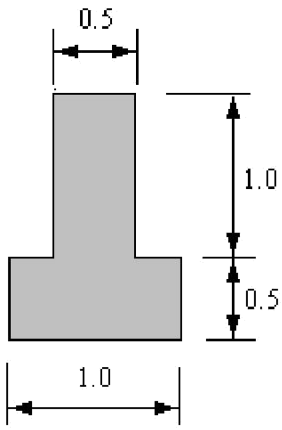
$\bar{y}$ = 35 500/1100 = 32.27 mm.

ASSIGNMENT PROBLEMS

1. Find the distance of the centroid from the axis s – s. All dimensions are in



2. Find the distance of the centroid from the bottom edge. All dimensions are in metres. (0.625 m)



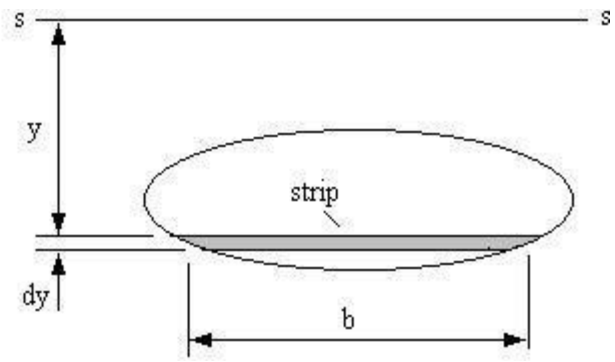
SECOND MOMENTS OF AREAS

If any quantity is multiplied by the distance from the axis s-s twice, we have a second moment. Mass multiplied by a distance twice is called the moment of inertia but is really the second moment of mass. We are concerned here with area only and the area multiplied by a distance twice is the second moment of area. The symbol for both is confusingly a letter I.

The above statement is over simplified. Unfortunately, both the mass and area are spread around and neither exists at a point. We cannot use the position of the centroid to calculate the 2nd moment of area. Squaring the distance has a greater effect on parts further from the axis than those nearer to it. The distance that gives the correct answer is called the **RADIUS OF GYRATION** and is denoted with a letter k. This is not the same as $\bar{y}$.

The simplest definition of the 2nd. moment of area is $I = A k^2$

Whilst standard formulae exist for calculating the radius of gyration of various simple shapes, we should examine the derivations from first principles. We do this by considering the area to be made up of lots of elementary strips of width b and height dy. The distance from the axis s-s to the strip is y.



The area of the strip = dA = b dy

1st moment of area of strip = y dA = by dy

2nd moment of area of strip = y2 dA = b y2 dy

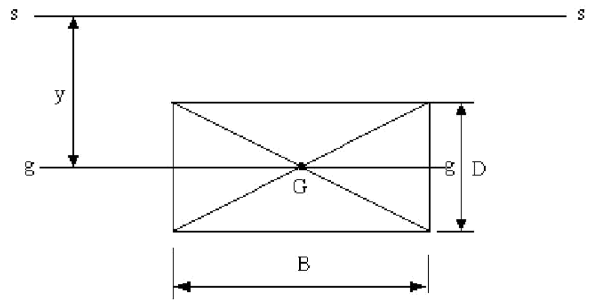
For the whole area, the 2nd moment of area is the sum of all the strips that make up the total area. This is found by integration.

$$\mathbf{I = \int b\, y2\, dy}$$

The limits of integration are from the bottom to the top of the area. This definition is important because in future work, whenever this expression is found, we may identify it as I and use standard formulae when it is required to evaluate it. We should now look at these.

EXAMPLE 4

Derive the standard formula for the second moment of area and radius of gyration for a rectangle of width B and depth D about an axis on its long edge.



b = constant = B

$$I = \int_0^{\overline{2}} by^2\,dy = B\int_0^{\overline{2}} y^2\,dy$$

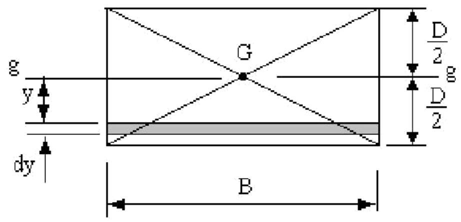
$$I = B\left[\frac{y^3}{3}\right]^D = \frac{BD^3}{3}$$

$$I = Ak^2 \qquad k = \sqrt{\frac{I}{A}} = \sqrt{\frac{BD^3}{3BD}} = 0.577D$$

Note $\bar{y}$ is 0.5 D and is not the same as k.

EXAMPLE 5

Derive the standard formula for the second moment of area and radius of gyration for a rectangle of width B and depth D about an axis through its centroid and parallel to the long edge.



b = constant = B

$$I = \int_{-\frac{D}{2}}^{\frac{D}{2}} by^2\, dy = B\int_{-\frac{D}{2}}^{\frac{D}{2}} y^2\, dy$$

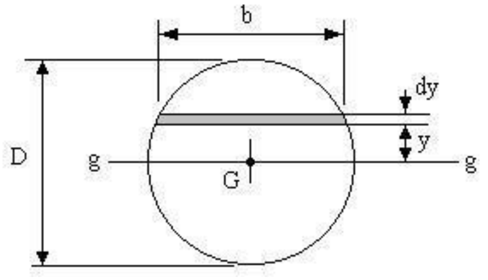
$$I = B\left[\frac{y^3}{3}\right]_{-\frac{D}{2}}^{\frac{D}{2}} = \frac{BD^3}{12}$$

$$k = \sqrt{\frac{I}{A}} = \sqrt{\frac{BD^3}{12BD}} = 0.289D$$

Note $\bar{y}$ is zero and not the same as

CIRCLES

The integration involved for a circle is complicated because the width of the strip b varies with distance y.



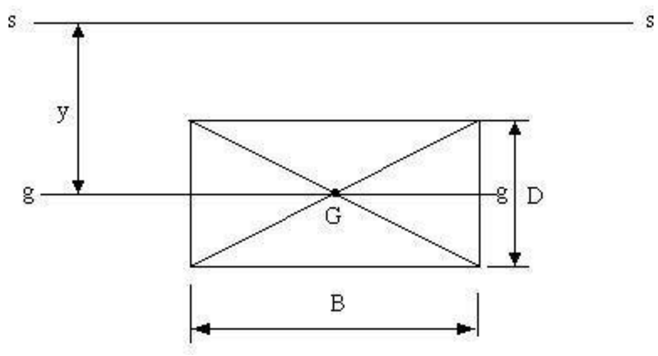
The solution yields the following result. **$I = \pi D^4/64 \quad k = D/4$**

PARALLEL AXIS THEOREM

If we wish to know the 2nd moment of area of a shape about an axis parallel to the one through the centroid (g-g), then the parallel axis theorem is useful.

The parallel axis theorem states **$I_{ss} = I_{gg} + A(\bar{y})^2$**

Consider a rectangle B by D and an axis s-s parallel to axis g-g.



$$I_{gg} = \frac{BD^3}{12} \qquad A = BD \qquad I_{ss} = \frac{BD^3}{12} + BD\bar{y}^2$$

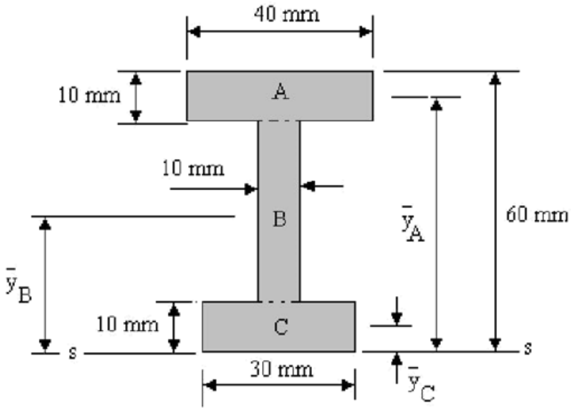
Consider when s-s is the top edge.

$$I_{ss} = \frac{BD^3}{12} + BD\bar{y}^2 \quad \text{but } \bar{y} = \frac{D}{2} \quad \text{so } I_{ss} = \frac{BD^3}{12} + BD\left(\frac{D}{2}\right)^2 = \frac{BD^3}{12}$$

This is the result obtained previously and confirms the method.

EXAMPLE 6

Calculate the 2nd moment of area for the same shape as in EXAMPLE 1.3. about the axis s-s



SOLUTION

The table shows the previous solution with extra columns added to calculate the second moment of area using the parallel axis theorem. In the new column calculate the second moment of area for each part (A, B and C) about each's own centroid using $BD^3/12$. In the next column calculate Ay^2.

Part	Area	$\bar{y}$	$A\bar{y}$	$Igg=BD^3/12$	$A\bar{y}^2$	Iss
A	400	55	22000	3333	1210000	1213333
B	400	30	12000	53333	360000	413333
C	300	5	1500	22500	7500	30000
Total	1100		35500			1656666

The total 2nd moment of area is 1656666 mm^4 about the bottom. We require the answer about the centroid so we now use the parallel axis theorem to find the 2nd moment about the centroid of the whole section.

The centroid is 32.77 mm from the bottom edge.

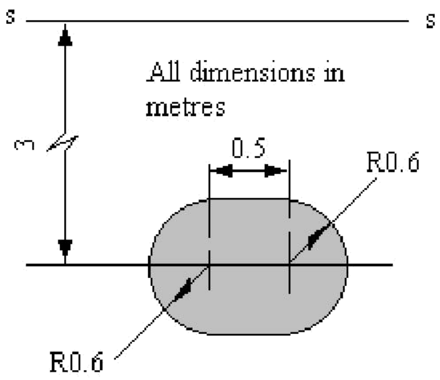
$$Igg = Iss - A\bar{y}^2$$

$$Igg = 1656666 - 1100 \times 32.77^2$$

$$Igg = 475405.8 \text{ mm}^4 = 475.4 \times 10^{-9} \text{ m}^4$$

Note $1 \text{ m}^4 = 10^{12} \text{ mm}^4$

EXAMPLE 7



SOLUTION

The shape is equivalent to a circle 1.2 m diameter and a rectangle 0.5 by 1.2

$\bar{y} = 3$ m

$$I_{gg} = \frac{\pi d^4}{64} + \frac{BD^3}{12} = \frac{\pi \times 1.2^4}{64} + \frac{0.5 \times 1.2^3}{12} = 0.1018 + 0.072 = 0.1738 \text{ m}^4$$

$$I_{ss} = I_{gg} + A\bar{y}^2 = 0.1738 + 1.731 \times 3^2 = 15.75 \text{ m}^4$$

$$\text{Area} = A = \frac{\pi d^2}{4} + BD = \frac{\pi \times 1.2^2}{4} + 0.5 \times 1.2 = 1.131 + 0.6 = 1.731 \text{ m}^2$$

ASSIGNMENT PROBLEMS

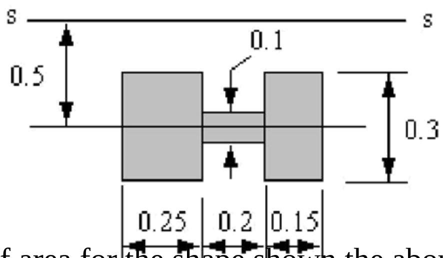
1. Find the second moment of area of a rectangle 3 m wide by 2 m deep about an axis parallel to the longer edge and 5 m from it. (218 m4).

2. Find the second moment of area of a rectangle 5 m wide by 2m deep about an axis parallel to the longer edge and 3 m from it. (163.33 m4).

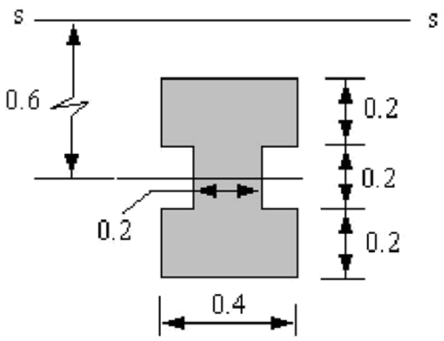
3. Find the second moment of area of a circle 2 m diameter about an axis 5 m from the centre. (79.3 m4).

4. Find the second moment of area of a circle 5 m diameter about an axis 4.5 m from the centre. (428.29 m4).

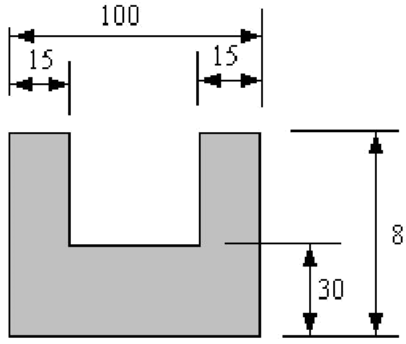
5. Find the 2nd moment of area for the shape shown the about the axis s – s. All the dimensions are in metres. (35.92×10^{-3} m^4)



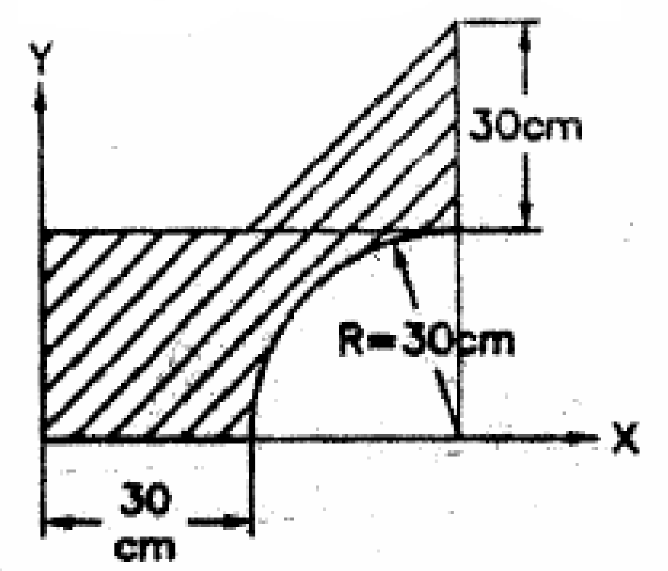
6. Find the 2nd moment of area for the shape shown the about the axis s – s. All the dimensions are in metres.. (79.33×10^{-3} m^4)



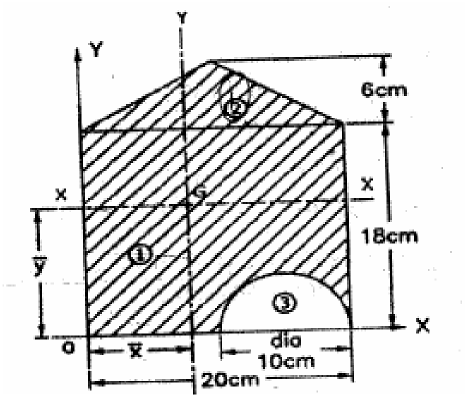
7. Find the position of the centroid for the shape shown and the 2nd moment of area about the bottom edge. (28.33 mm from the bottom and 2.138×10^{-6} m^4)



8. . Determine the co-ordinates of centroid of the shaded area shown in figure.

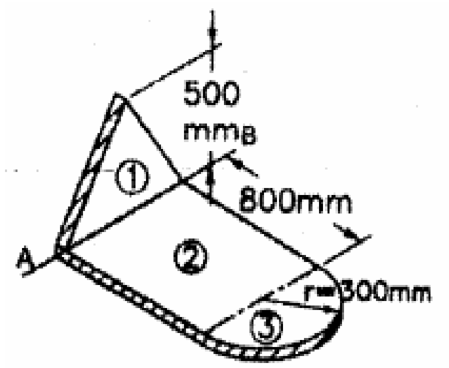


9. A Cylinder of height of 10 cm and radius of base 4 cm is placed under sphere of radius 4 cm such that they have a common vertical axis. If both of them are made of the same material, locate the centre of gravity of the combined unit.
10. Find the moment of inertia of the section shown in the figure about its horizontal centroidal axis.



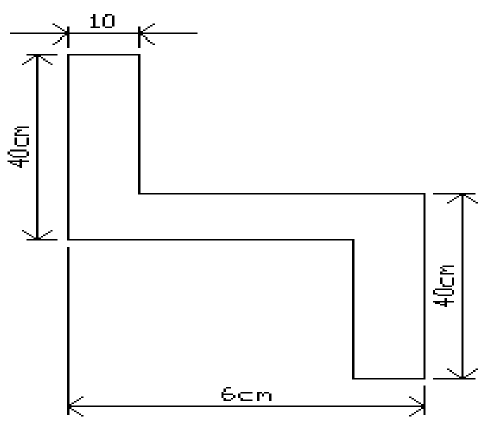
11. Calculate the mass moment of inertia of the plate shown in fig with respect to the axis AB. Thickness of the plate is 5mm and density of the material is 6500Kg/m

.

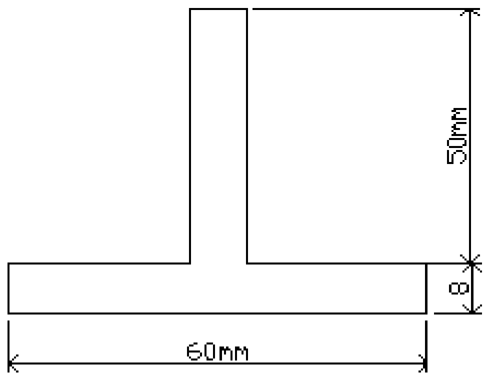


12. Derive expression form mass moment of inertia of prism along three axes.

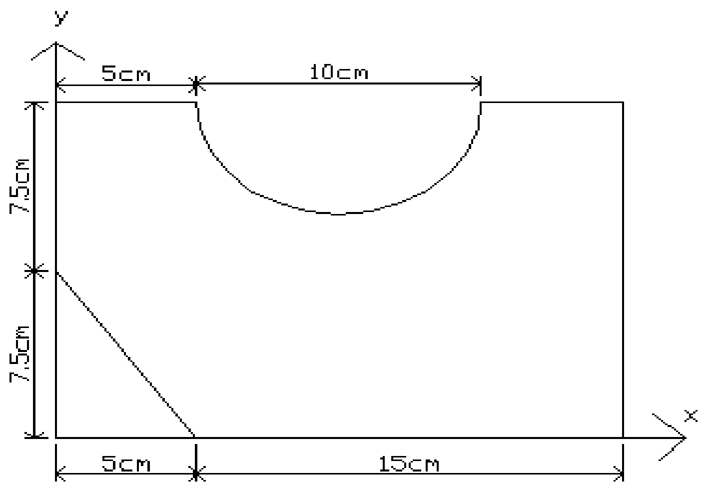
13. Determine Moment of Inertia about the co-ordinate axes of plane area shown in fig. Also find Polar Moment of Inertia.



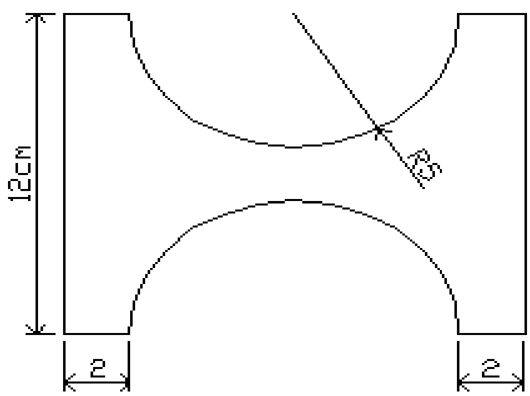
14. Determine the principal moments of inertia and find location of principal axes of surface shown in fig.



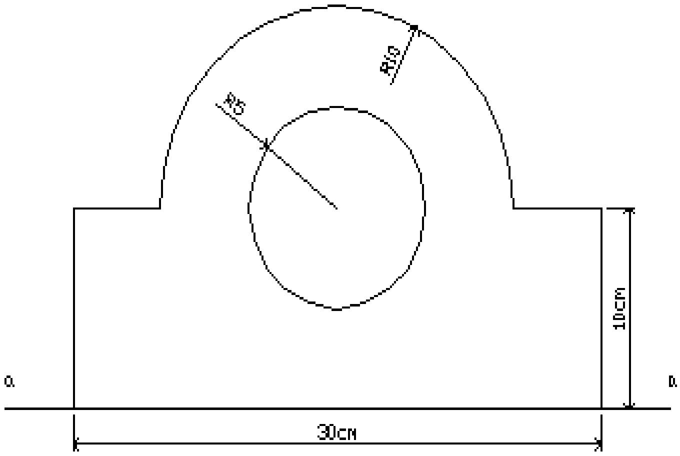
15. Find Moment of Inertia and radius of gyration of surface about x axis shown in fig. Also find MOI about centroidal x- axis.



16. Find the polar moment of inertia and polar radius of gyration of plane area about centroidal axes shown in fig.



17. Determine second moment of area about the centroidal XX axis and a-a axis of the surface shown in fig.



SCHOOL OF BUILDING AND ENVIRONMENT

DEPARTMENT OF CIVIL ENGINEERING

UNIT – IV - FRICTION– SCIA1101

IV. FRICTION

Frictional Force - Laws of Coulomb friction - Cone of friction - Angle of repose - Simple contact friction - Screw - Wedge - Ladder - Rolling resistance - Belt friction

FRICTION OF FORCE

Friction is the force that opposes the motion of an object. To stop a moving object, a force must act in the opposite direction to the direction of motion. For instance, if a book is pushed across a desk, the book will move. The force of the push moves the book. As the book slides across the desk, it slows down and stops moving. The force that opposes the motion of an object is called **friction**. Figure 1 shows a typical sketch for friction on a body.

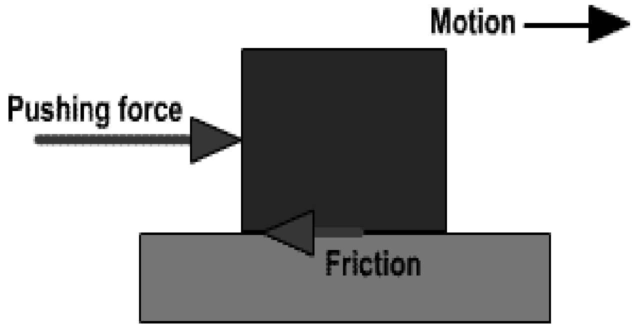


Figure 1 Friction on a body

Example: **A block of weight is placed on a rough horizontal plane surface and force P is applied on the horizontal such that the block tends to move.**

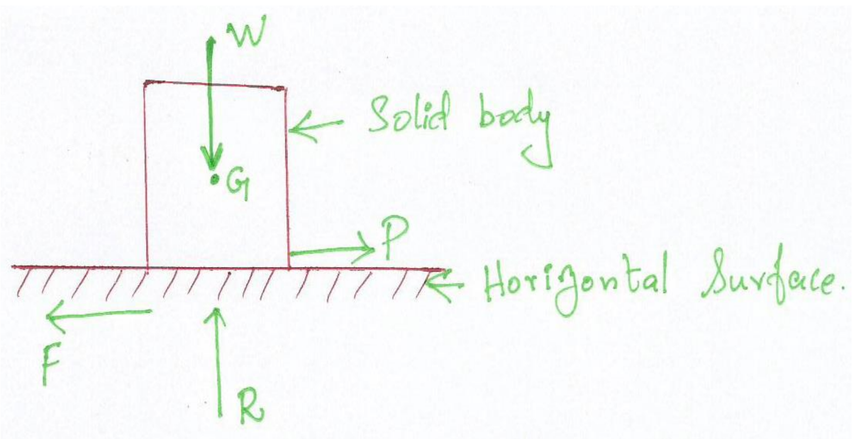


Figure 2 Solid object on a horizontal surface

Let W be the weight of a body acting through centre of gravity downward

G is the centre of gravity or Gravitational Force

R be the normal reaction of the body acting through the centre of gravity and

P is the force acting on the body through centre of gravity and parallel to horizontal surface.

If P is small, the body will not move as the force of friction acting on a body in the direction opposite to P will be more than P. but if the magnitude of P goes on increasing, a stage comes, when the solid body is on the point of motion. At this stage, the force of friction acting on the body is called *limiting force of friction*. It is denoted by F.

Resolving the forces on the body vertical and horizontal, we get,

$\sum H = 0$	**$\sum V = 0$**
$-F+P = 0$	**$-W+R = 0$**
Therefore, P = F	**Therefore, R = W**

If the magnitude of P is further increased the body will start moving. The force of Friction, acting on the body when the body is moving, is called as *Kinetic Friction*.

COEFFICIENT OF FRICTION

The coefficient of friction is a number which represents the friction between two surfaces. Between two equal surfaces, the coefficient of friction will be the same. The symbol usually used for the coefficient of friction is m

The maximum frictional force (when a body is sliding or is in limiting equilibrium) is equal to the coefficient of friction × the normal reaction force.

$$F = \mu R$$

Where μ is the coefficient of friction and R is the normal reaction force.

This frictional force, F, will act parallel to the surfaces in contact and in a direction to oppose the motion that is taking/ trying to take place as given in Figure 3.

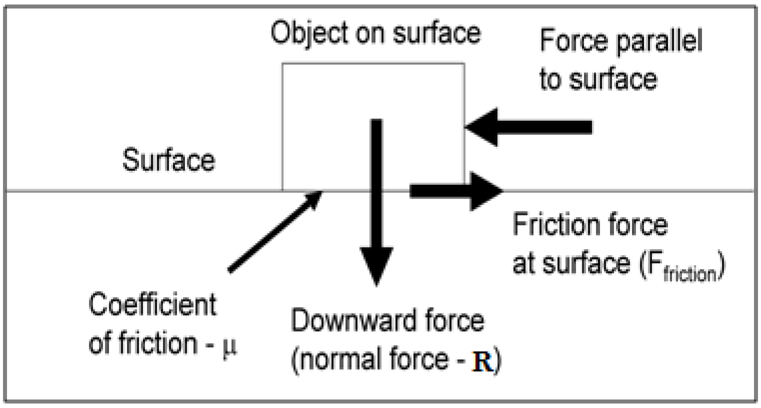


Figure 3 Coefficient of Friction

ANGLE OF FRICTION:

The angle made by the resultant of the normal reaction (R) and the limiting force of friction (F) with the normal reaction is called as angle of friction.

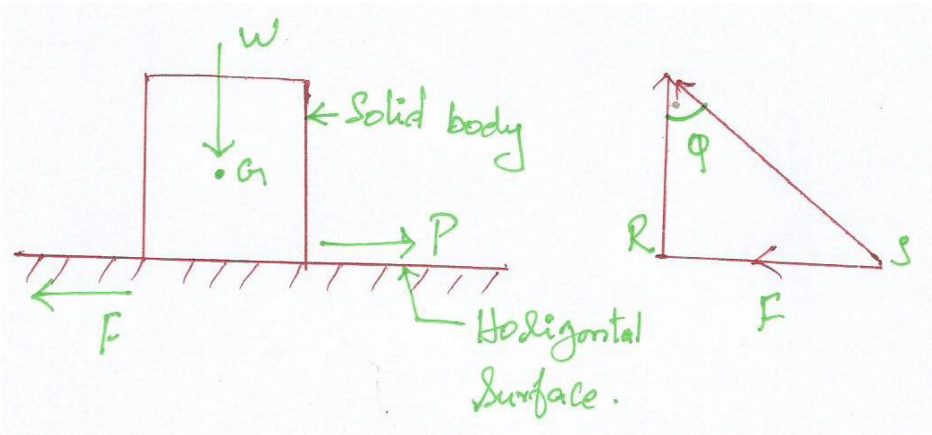


Figure 4 Angle of Friction

From Figure 4, S represents the resultant of R and F

φ represents angle between S and R.

Hence, tan φ = F/R = μR/R (Since F=μR)

Therefore, tan φ = μ

Thus the tangent angle of friction is equal to coefficient of friction.

Example: **A block of weight is placed on a rough horizontal plane surface and force P is applied at an angle θ with the horizontal such that the block tends to move.**

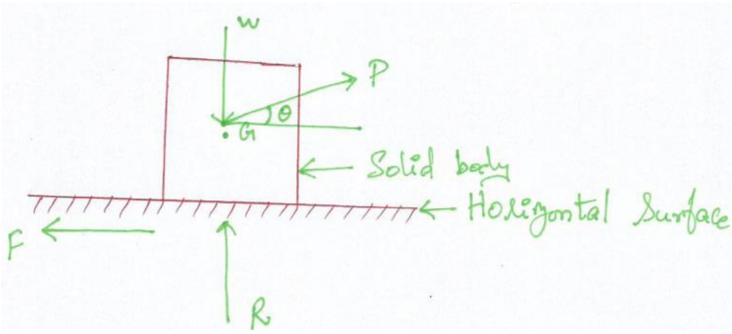


Figure 5 Force applied at an angle

Let R be Normal reaction

μ be the coefficient of friction

F be the Force of Friction = μR

In this case the R will not be equal to the weight of the body. The R is obtained by resolving the forces on block horizontally and vertically. The forces P is resolved in two components i.e. P cos θ in horizontal direction and P sin θ in vertical direction.

$\sum H=0$ $-F+P\cos\theta=0$

Therefore, $F = P\cos\theta$ or $\mu R = P\cos\theta$

$\sum V=0$ $R+P\sin\theta-W = 0$

Therefore, $R = W-P\sin\theta$

From the above equations it is clear that R is not equal to W and the values of W, P and θ are known, R can be obtained which is used to determine μ.

$$\boxed{\mu = F/R}$$

Note:

i. **F always equal to μR**

ii. **R always not equal to W**

CONE OF FRICTION:

The right circular cone with vertex at the point of contact of the two bodies, axis in the direction of R and the semi-vertical angle is α. The inverted cone with semi central angle α equal to limiting frictional angle α, is called cone of friction.

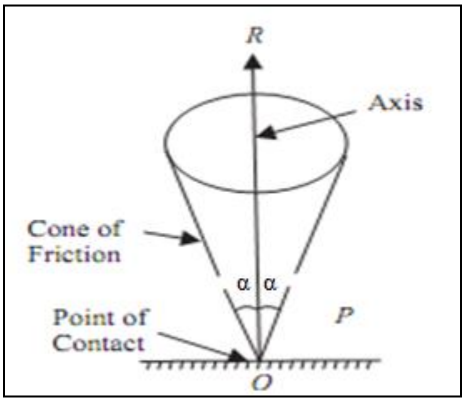


Figure6 Cone of Friction

ANGLE OF REPOSE

It is the angle of inclination (α) of the plane to the horizontal, at which the body just begins to move down the plane. A little consideration will show that the body will begin to move down the plane, if the angle of inclination (α) of the plane is equal to the angle of friction (ϕ). From Fig. 1.18, we find that

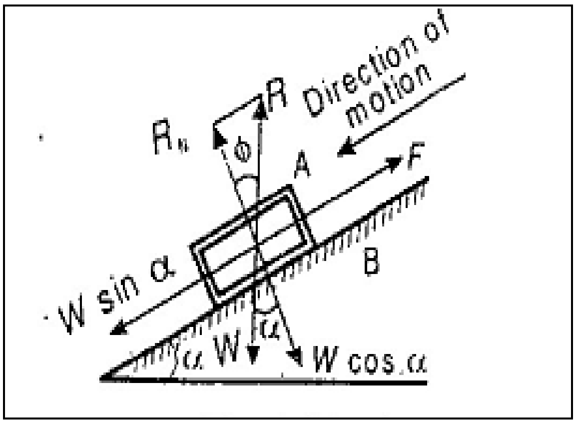


Figure 7 Angle of repose

W sin α = F = μ R = μ W cos α

tan α = μ

Therefore, **tan α = F/R**

Angle of repose is defined as the minimum angle made by an inclined plane with the horizontal such that an object placed on the inclined surface just begins to slide.

- **Relation between Angle of Friction and Angle of Repose**

Let us consider a body of mass 'W' resting on a plane.

Also, consider when the plane makes 'α' angle with the horizontal; the body just begins to move.

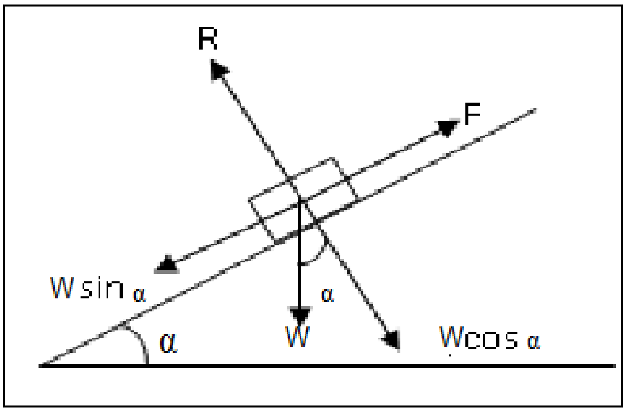


Figure 8 **Relation between φ and a**

Let 'R' be the normal reaction of the body and 'F' be the frictional force. Now α is the inclination of the plane with the horizontal.

Here,

$W \sin \alpha = F = \mu R = \mu W \cos \alpha$

$\tan \alpha = \mu = \tan \varphi$

Therefore, $\alpha = \varphi$

Angle of Friction, φ = Angle of repose, α

LAWS OF SOLID FRICTION or LAWS OF COLUMB FRICTION

1. The force of friction acts in opposite direction in which surface is having tendency to move.
2. The force of friction is equal to the force applied to the surface, so long as surface is at rest.
3. When the surface is on the point of motion, the force of friction is said to be maximum and this maximum frictional force is called as limiting friction force.
4. The limiting frictional force bears a constant ratio to the normal reaction between two surfaces.
5. The limiting frictional force does not depend upon the shape and area of the surface in contact.
6. The friction of force is independent of velocity of sliding.

1. TYPES OF FRICTION

There are four types of friction namely

i. Static friction
ii. Kinetic friction
iii. Rolling friction
iv. Fluid friction

i. Static Friction

Static friction comes into play when a body is forced to move along a surface but movement does not start. The magnitude of static friction remains equal to the applied external force and the direction is always opposite to the direction of motion. The magnitude of static friction depends upon **μ_s** (coefficient of static friction) and R (normal reaction of the body).

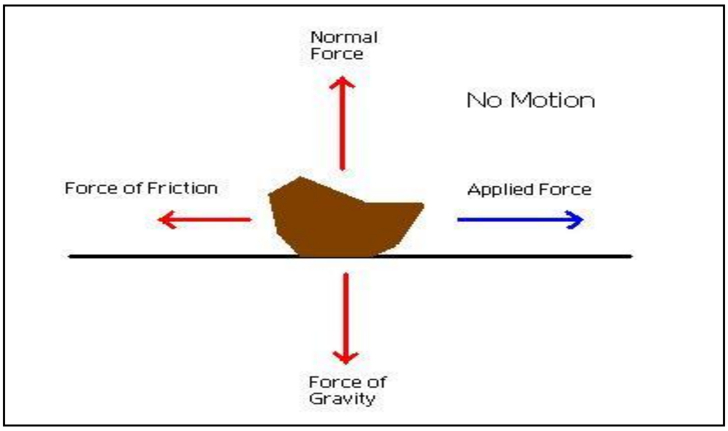


Figure 9 Static Friction

ii. Kinetic Friction

Kinetic friction denoted as μk comes into play when a body just starts moving along a surface. When external applied force is sufficient to move a body along a surface then the force which opposes this motion is called as kinetic frictional force.

Magnitude of kinetic frictional force $F_k = \mu_k R$

Where μ_k is coefficient of kinetic frictional force and N is the net normal reaction on the body. The magnitude of kinetic frictional force is always less than magnitude of static frictional force. When value of applied net external force F is more than F_k then body moves with a net acceleration and when these forces are equal then body moves with a constant velocity.

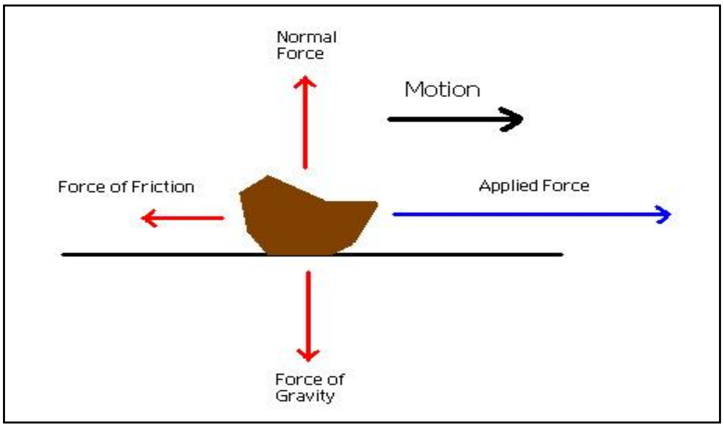


Figure 10 Kinetic Friction

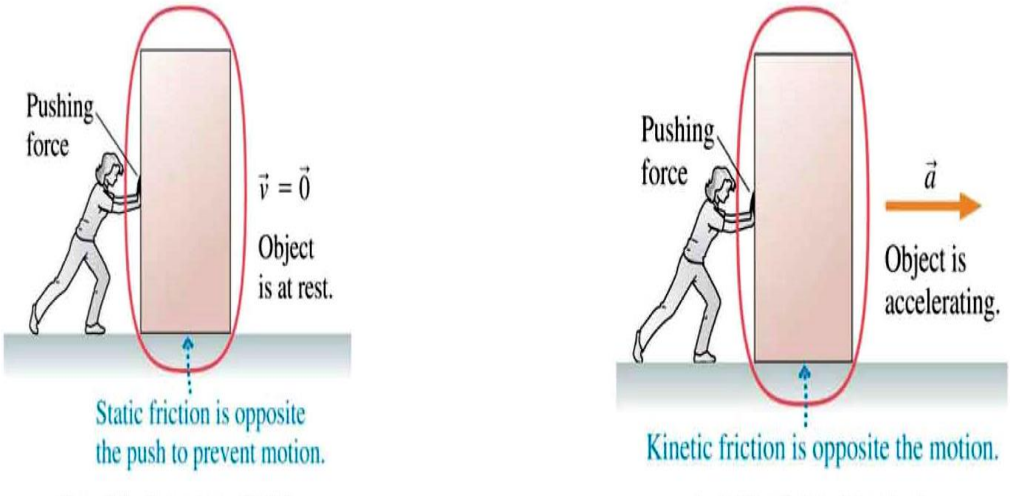


Figure 11 Combined Static and Dynamic Friction

iii. Rolling Friction

Rolling frictional force is a force that slows down the motion of a rolling object. Basically it is a combination of various types of frictional forces at point of contact of wheel and ground or surface. When a hard object moves along a hard surface then static and molecular friction force retards its motion. When soft object moves over a hard surface then its distortion makes it slow down.

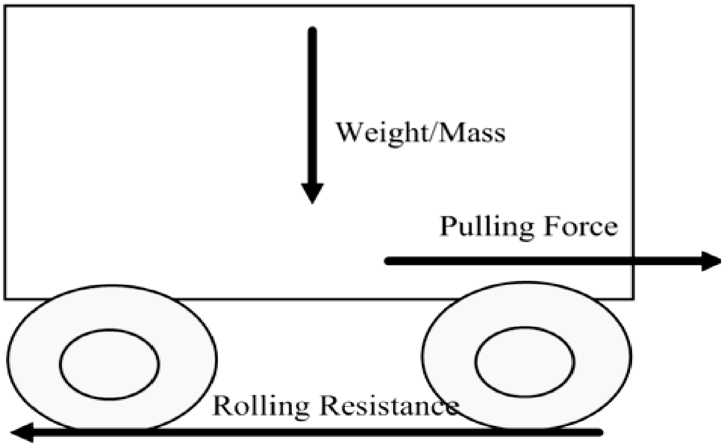


Figure 12 Rolling Friction

iv. Fluid Friction

When a body moves in a fluid or in air then there exists a resistive force which slows down the motion of the body, known as fluid frictional force. A freely falling skydiver feels a drag force due to air which acts in the upward direction or in a direction opposite to skydiver's motion. The magnitude of this drag force increases with increment in the downward velocity of skydiver. At a particular point of time the value of this drag force becomes equal to the driving force and skydiver falls with a constant velocity.

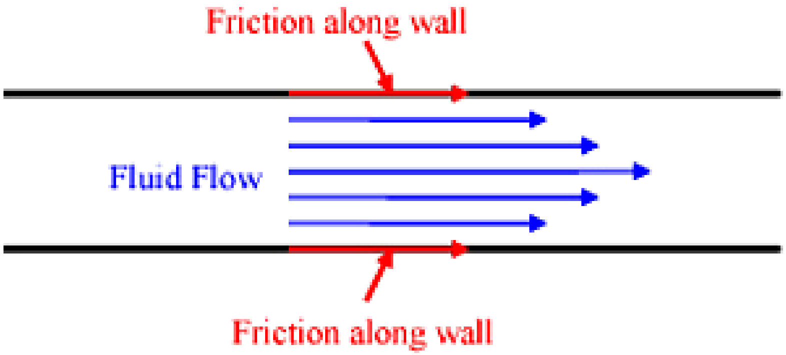


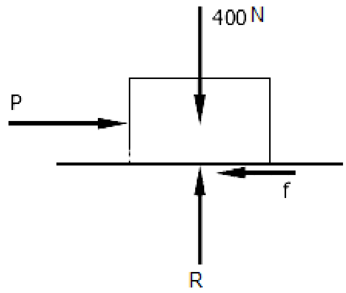
Figure 13 Fluid Friction

***Problems on Friction*:**

***Problem No. 1*.** A 400 N block is resting on a rough horizontal surface for which the coefficient of friction is 0.40. Determine the force P required to cause motion to impend if applied to the block (a) horizontally or (b) downward at 30° with the horizontal. (c) What minimum force is required to start motion?

Solution

Case (a) - Force is applied horizontally



ΣV=0

N=400 N

F=μR=0.40 x 400

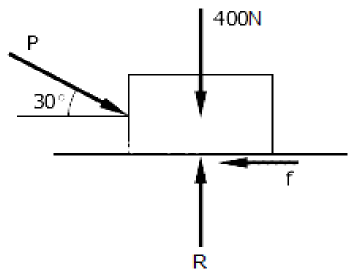
F=160 N

ΣH=0

P=F

P=160 N

Case (b) - Downward force at 30° from the horizontal



ΣV=0

N=400+Psin30°

N=400+0.5P

F=μR=0.40 x (400+0.5P)

F=160+0.2P

ΣH=0

Pcos30°=f

$P\cos 30^{\circ}=160+0.2P$

$0.666P=160$

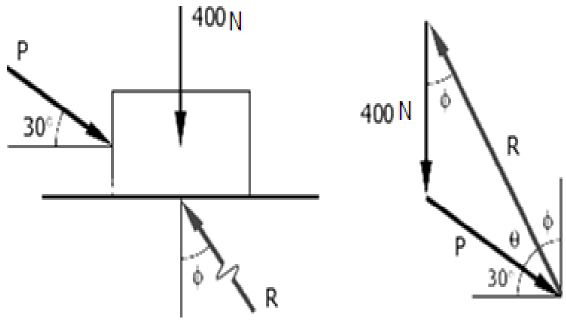
$P=240.23$ N

Another Solution for Case (b)

$\tan\phi=\mu$

$\tan\phi=0.40$

$\phi=21.80^{\circ}$

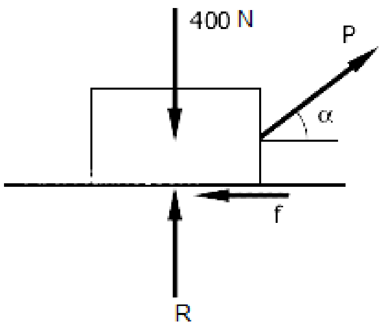


$P\sin\phi=400\sin\theta$

$P\sin 21.80^{\circ}=400\sin 38.20^{\circ}$

$P=240.21$ N

Case (c) - Minimum force required to cause impending motion



$\Sigma V=0$

$N=400-P\sin\alpha$

$F=\mu R=0.40(400-P\sin\alpha)$

$F=160-0.40Psin\alpha$

$\Sigma H=0$

$Pcos\alpha=f$

$Pcos\alpha=160-0.40Psin\alpha$

$Pcos\alpha+0.40Psin\alpha=160$

$(cos\alpha+0.40sin\alpha)P=160$

$P=160cos\alpha+0.40sin\alpha$

To minimize P, differentiate then equate to zero

$dPd\alpha=-160(-sin\alpha+0.40cos\alpha)(cos\alpha+0.40sin\alpha)2=0$

$sin\alpha-0.40cos\alpha=0$

$sin\alpha=0.40cos\alpha$

$tan\alpha=0.40$

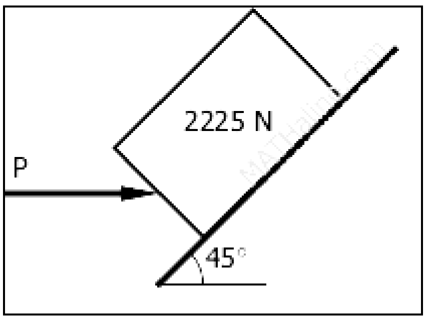
$\alpha=21.80^\circ$

Minimum value of P

$Pmin=160cos21.80^\circ+0.40sin21.80^\circ$

$Pmin=148.56$ N

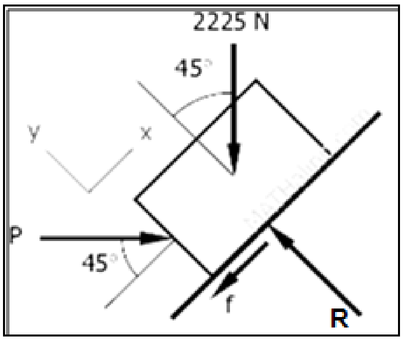
Problem No.2 The 2225-N block as in the give figure is in contact with 45° incline. The coefficient of static friction is 0.25. Compute the value of the horizontal force P necessary to (a) just start the block up the incline or (b) just prevent motion down the incline. (c) If P = 1780 N, what is the amount and direction of the friction force?



Solution

Case (a) – Force P to just start the block to move up the incline

The force P is pushing the block up the incline. The push is hard enough to overcome the maximum allowable friction causing an impending upward motion.



$\Sigma V=0$

$R=2225\cos45°+P\sin45°$

$R=1573.31+0.7071P$

$F=\mu R=0.25(1573.31+0.7071P)$

$F=393.33+0.1768P$

$\Sigma H=0$

$P\cos45°=f+2225\sin45°$

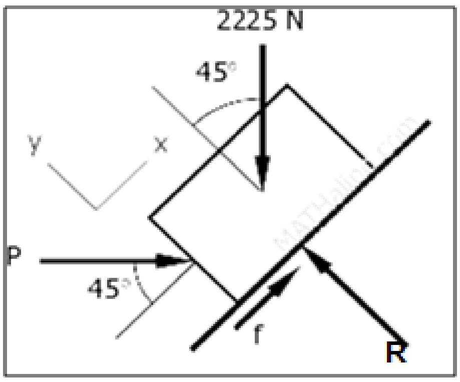
$P\cos45°=(393.33+0.1768P)+2225\sin45°$

$0.5303P=1966.64$

$P=3708.55\ N$

Case (b) – Force P to just prevent the block to slide down the incline

In this case, the force P is not pushing the block upward, it simply supports the block not to slide downward. Therefore, the total force that prevents the block from sliding down the plane is the sum of the component of P parallel to the incline and the upward friction force.



$\Sigma V=0$

$R=2225\cos45°+P\sin45°$

$R=1573.31+0.7071P$

$F=\mu R=0.25(1573.31+0.7071P)$

$F=393.33+0.1768P$

$\Sigma H=0$

$P\cos45°+f=2225\sin45°$

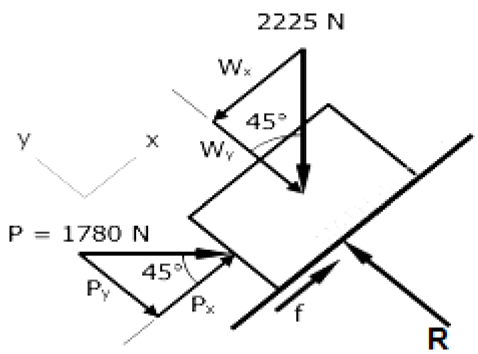
$P\cos45°+(393.33+0.1768P)=2225\sin45°$

$0.8839P=1179.98$

$P=1335$ N

Case (c) – Force P = 1780 N

If $P_x = W_x$, there will be no friction under the block. If $P_x > W_x$, friction is going downward to help W_x balance the P_x. If $P_x < W_x$, friction is going upward to help P_x balance the W_x. In this problem, the maximum available friction is not utilized by the system.



$Wx=2225\sin45°=1573.31$ N

$Px=1780\cos45°=1258.65$ N

$W_x > P_x$, thus, f is upward

.

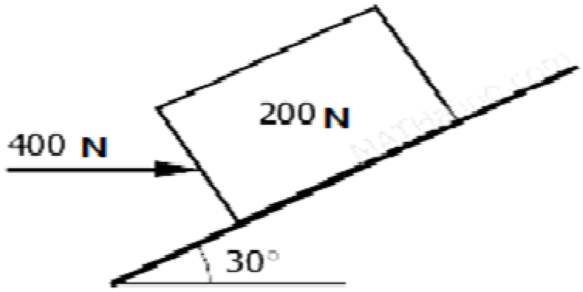
$\Sigma H=0$

F+Px=Wx

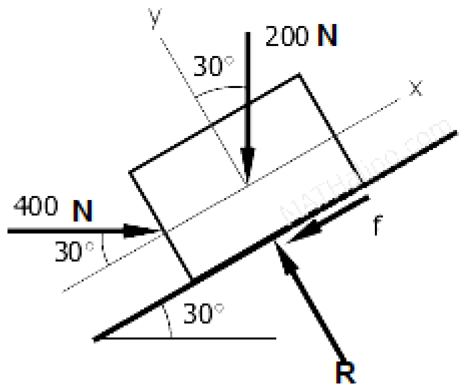
F+1258.65=1573.31

F=314.66 N upward

Problem No.3 The 200-N block as shown has impending motion up the plane caused by the horizontal force of 400 N. Determine the coefficient of static friction between the contact surfaces.



Solution



ΣV=0

N=400sin30°+200cos30°

N=373.20 N

ΣH=0

f+200sin30°=400cos30°

f=246.41 N

f=μR

246.41=μ x (373.20)

$\mu=0.66$

2. LADDER FRCITION

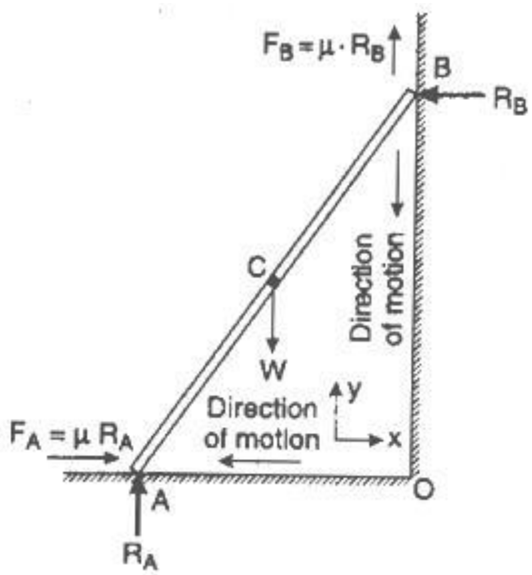
A commonly used ladder has two uprights of bamboo or iron, connected by a number of parallel bars to provide steps. These bars are called as rungs. As shown in figure, a ladder AB rests with its one end B against a vertical wall OB and the other end A on a horizontal plane OA.

In case of slip, the end B will move downwards and A outwards. In this situation of ladder is under the action of following forces.

(1) Normal reactions R_A and R_B

(2) Frictional forces caused by R_A and R_B.

(3) Weight of the ladder = W.



The necessary and sufficient conditions of equilibrium for the ladder are:

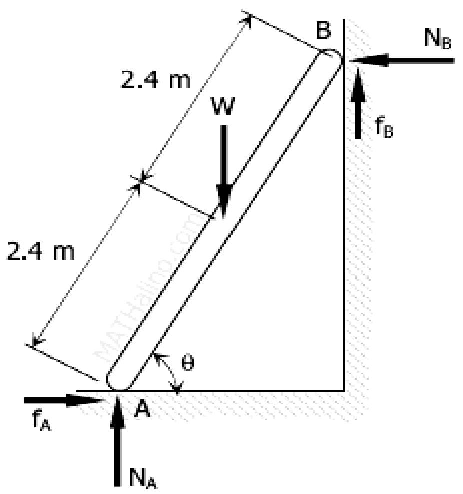
$\Sigma F_x = 0$ …(1)

$\Sigma F_y = 0$ …(2)

$\Sigma M = 0$ …(3)

Problems on Ladder Friction:

Problem No.1 A uniform ladder 4.8 m ft long and weighing W N is placed with one end on the ground and the other against a vertical wall. The angle of friction at all contact surfaces is 20°. Find the minimum value of the angle θ at which the ladder can be inclined with the horizontal before slipping occurs.



Solution

Coefficient of friction

$\mu = \tan\phi = \tan 20°$

$\mu = 0.364$

Friction forces at each end of the ladder

$F_A = \mu N_A = 0.364 N_A$

$F_B = \mu N_B = 0.364 N_A$

$\Sigma H = 0$

$N_B = F_A$

$N_B = 0.364 N_A$

$\Sigma V = 0$

$N_A + F_B = W$

$N_A + 0.364 N_A = W$

$N_A + 0.364(0.364 N_A) = W$

$1.1325 N_A = W$

N_A=0.883W

Thus,

F_A =0.364(0.883W)

F_A=0.3214W

ΣM_B=0

W x (2.4cosθ)+ F_A (4.8sinθ) = N_A (4.8cosθ)

Wcosθ+2 F_A sinθ = 2$N_A$cosθ

W+2 F_A tanθ = 2N_A

W+2 x (0.3214W) tanθ = 2 x (0.883W)

1+0.6428 tanθ = 1.766

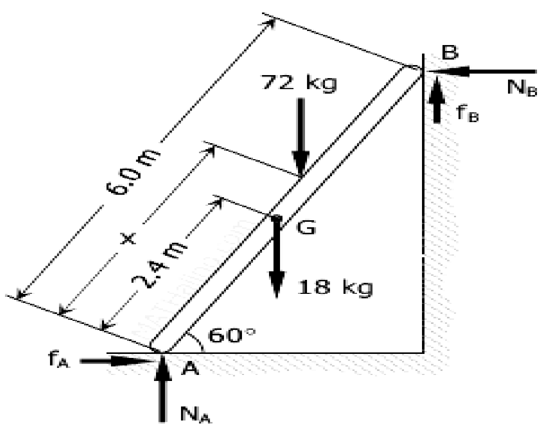
0.6428 tanθ = 0.766

Tanθ = 1.191 661 481

θ=50°

Problem No.2 A ladder 6 m long has a mass of 18 kg and its center of gravity is 2.4 m from the bottom. The ladder is placed against a vertical wall so that it makes an angle of 60° with the ground. How far up the ladder can a 72-kg man climb before the ladder is on the verge of slipping? The angle of friction at all contact surfaces is 15°.

Solution



Coefficient of friction

μ=tanϕ

μ=tan15°

Amount of friction at contact surfaces

$F_A = \mu N_A = N_A \tan 15°$

$F_B = \mu N_B = N_B \tan 15°$

$\Sigma V = 0$

$N_A + F_B = 18 + 72$

$N_A = 90 - F_B$

$N_A = 90 - N_B \tan 15°$

$\Sigma H = 0$

$F_A = N_B$

$N_A \tan 15° = N_B$

$(90 - N_B \tan 15°) \tan 15° = N_B$

$90 \tan 15° - N_B \tan 215° = N_B$

$90 \tan 15° = N_B + N_B \tan 215°$

$N_B (1 + \tan 215°) = 90 \tan 15°$

$N_B = 90 \tan 15° 1 + \tan 215°$

$N_B = 22.5$ kg

$F_B = 22.5 \tan 15°$

$F_B = 6.03$ kg

$\Sigma M_A = 0$

$N_B (6 \sin 60°) + F_B (6 \cos 60°) = 18(2.4 \cos 60°) + 72(x \cos 60°)$

$N_B (6 \tan 60°) + 6 F_B = 18(2.4) + 72x$

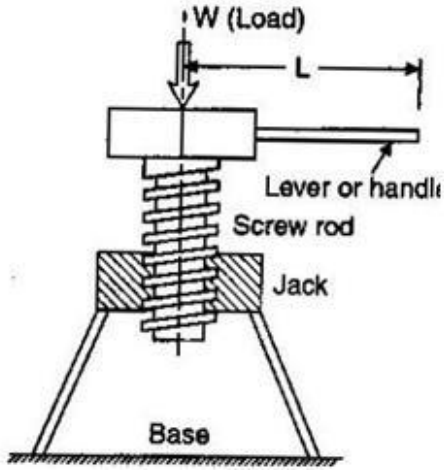
$6(22.5) \tan 60° + 6(6.03) = 43.2 + 72x$

$72x = 226.81$

$x = 3.15$ m

3. SCREW FRICTION

A screw jack is a device which is used to raise heavy loads by an small effort. It consists of a square threaded screw fitted into an internally threaded collar of a jack. The load W is placed on the screw and the effort Ph is applied horizontally at the end of lever arm L.



With one rotation of the lever the weight is lifted through a distance equal to the pitch. In case of multi-threaded screws the actual pitch is *np*.

From figure when a screw is given one revolution, it will move up by pitch p axially, and horizontally by πd_m (d_m is the mean diameter of the threads). Therefore, inclination of the threads can be calculated by,

$\tan \alpha = P / \pi d_m$

where α is the helix angle.

Problems on screw Friction

Problem No.1 The pitch of a single threaded screw jack is 6 mm and its mean diameter is 60 mm. If μ is 0.1, determine the force required at the end of lever 250 mm long measure from the axis of screw to a) raise a 50 kN load b) lower the same load

Solution

Helix angle of the thread = Angle of the inclined plane

Tan α = Pitch/Circumference = $p/2\pi r$

$= 0.006/2\pi(0.03) = 0.32$

Therefore, $\alpha = 1.823^\circ$

Friction angle, $\tan \varphi = \mu = 0.1$

Therefore, $\varphi = 5.71^\circ$

Case (a) Raise the load

$P_1 = Wr/2 (\tan (\varphi+\alpha)$

$= ((50)\ (0.03))*(\tan(5.71+1.823))/0.25$

$P_1 = 0.7934$ KN

Case (b) Lower the load

P_1 = Wr/2 (tan (φ-α)

= ((50) (0.03))*(tan(5.71-1.823))/0.25

P_1 = 0.4076 KN

Problem No.2 A screw thread of a screw jack has a mean diameter of 10 cm and a pitch of 1.25 cm. the μ between the screw and iots nut housing is 0.25. determine the force that must be applied at the end of a 50 cm lever arm to raise a mass of 5000 kg. is the device self locking? Also determine its efficiency.

<u>Solution</u>

Helix angle = Inclined Plane angle

Tan α = Pitch/Circumference = p/2πr

= (0.0125)/(2π(0.05)

Therefore, **α = 2.28°**

Friction angle, tan φ = μ = 0.25

Therefore, **φ = 14.04°**

From the above values it is clear that **φ>α. Therefore, it is proved that the device is self locking.**

<u>Efficiency of Screw</u>

Work output for one revolution of screw = Weight lifted x Pitch of the same

= W x p = W x 2πr tan α

Force, F = Effort at the end of the handle, P_1 =[Wr (tan φ+α)]/L

= [(49050 x 0.05) x tan (14.04+2.28)]/0.5

Therefore, **F = 1436.04 N**

Work input for one revolution = Effort x Distance moved by the effort in one revolution

= [Wr (tan φ+α)]/2πL

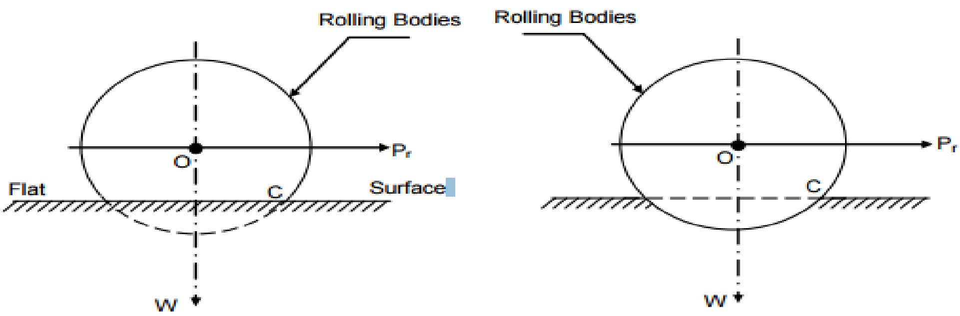
Efficiency, η =[W2πr tan α] /[W2πr tan (φ+α)] = tan α/ tan (φ+α) [Cancelling W2πr]

= tan 2.28/tan(14.04+2.28) = 0.136

Therefore, **η = 13.6%**

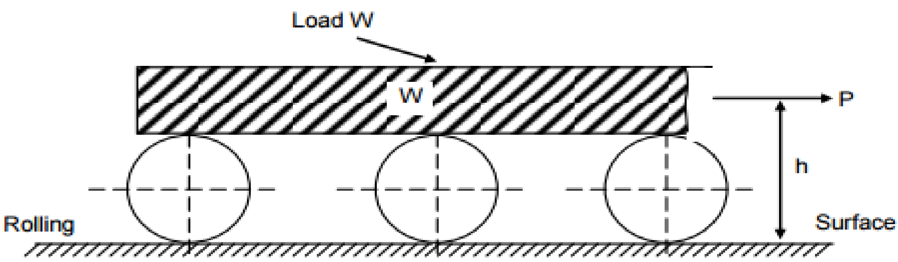
4. ROLLING FRICTION

The frictional resistance arises only when there is relative motion between the two connecting surfaces. When there is no relative motion between the connecting surfaces or stated plainly when one surface does not slide over the other question of occurrence of frictional resistance or frictional force does not arise. Journal Oil Hole Shaft Bearing 73 When a wheel rolls over a flat surface, there is a line contact between the two surfaces, Friction parallel to the central axis of the cylinder. On the other hand when a spherical body rolls over a flat surface, there is a point contact between the two. In both the above mentioned cases there is no relative motion of slip between the line or point of contact on the flat surface because of the rolling motion. If while rolling of a wheel or that of a spherical body on the flat surface there is no deformation of depression of either of the two under the load, it is said to be a case pure rolling. In practice it is not possible to have pure rolling and it can only be approached. How-so-ever hard the material be, either the rolling body will be deformed as happens in case of a car or cycle tyre, indicated in Figure 2.17(a) or the flat surface gets depressed or deformed. When the road roller passes over unsettled road or kacha road. The surface is depressed. (a) Rolling Body Deformed (b) Flat Surface Deformed



(a) Rolling Body Deformed **(b) Flat Surface Deformed**

Rolling Friction At times when a load or a heavy machine or its part is to be shifted from one place to another place, for a short distance, and no suitable mechanical lifting device is available, the same is placed on a few rollers in the form of short pieces of circular bars or pipes and comparatively with a less force the load is moved.



The rollers, roll and the one which becomes free at the rear side is again placed in the front of the load and so on. Because of the reduced friction, it requires less force. Figure 2.18 : Ball and Roller Bearing When a shaft revolves in a bush bearing, there is sliding motion between the journal and the bearing surface, resulting in loss of power due to friction. If between the journal and the bearing surface, balls of rollers are provided, instead of sliding motion, rolling motion will take place. To reduce the coefficient of rolling friction the balls or rollers are made of chromium steel or chrome-nickel steel and they are further heat treated with a view to make them more hard. They are finally ground and polished with high precision. Such an arrangement as mentioned above is provided in Ball and Roller bearings.

5. BELT FRICTION

A belt drive is a device with belt and pulley arrangement, which is used for transmitting power from one end to other end, applying brakes, lifting a load. In all these examples, the frictional forces developed between the belt and its contact surfaces of pulley is smooth, then the bolt will slip over the pulley, belt tension on either side are equal and hence, no power can be transmitted.

For the tensions T_1 and T_2 in two parts of belt or rope slipping around the cylindrical body shape the following formula is valid

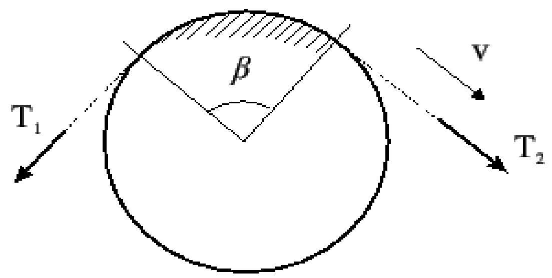


Figure Belt friction

$$T_2/T_1 = e^{\beta\mu}$$

where e is the base of natural logarithm

μ is the coefficient of kinetic friction

β is the angle of contact

Note:

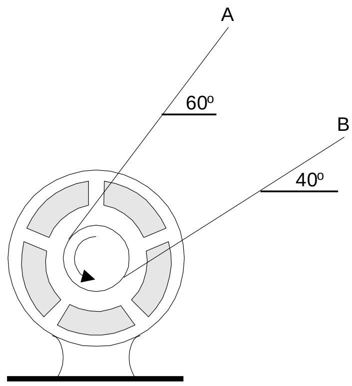
a. The angle of contact must be expressed in radians.
b. The tension in the belt will vary throughout the contact surface
c. The difference of T_1 and T_2 is responsible for torque and power transmission.
d. Torque = $(T_2 - T_1)$ x (radius of the shaft)

e. Power transmitted = = (T_2 -T_1) x (Velocity of belt)

f. The angle β may be larger then 2π. If a rope is wrapped *n* times around a post, $\beta = 2\pi n$.

Problems on Belt Friction

Problem No.1 A flat belt is used to transmit the 30 ft lb torque developed by an electric motor.

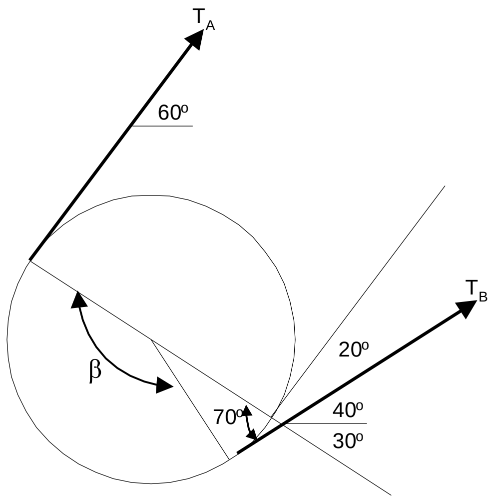


The drum in contact with the belt has a diameter of 8 in and $\mu_S = 0.30$.

Find: Determine the minimum allowable value of the tension in each part of the belt if the belt is not to slip.

$T_2 = T_A \qquad \beta = 160^\circ$

$T_1 = T_B$



$$\frac{T_2}{T_1} = e^{\mu_s \beta}$$

$$\frac{T_A}{T_B} = e^{0.3(8\pi/9)}$$

$$T_A = 2.311\,T_B$$

$$M = (T_A - T_B)r$$

$$30 = (T_A - T_B)\left(\frac{8\text{ in}}{2}\right)\left(\frac{1\text{ ft}}{12\text{ in}}\right)$$

$$T_A - T_B = 90$$

$$T_A = T_B + 90$$

$$2.311T_B = T_B + 90$$

$$T_B = 68.6\text{ lbs}$$

$$T_A = 158.6\text{ lbs}$$

Problem No.2 For the given above problem, using a V-belt with $\alpha = 36^\circ$ determine minimum allowable tension

$$\frac{T_2}{T_1} = e^{\mu_s \beta / \sin(\alpha/2)}$$

$$\frac{T_A}{T_B} = e^{0.3(8\pi/9)/\sin 18^\circ}$$

$$T_A = 15.045\, T_B$$

$$M = (T_A - T_B)r$$

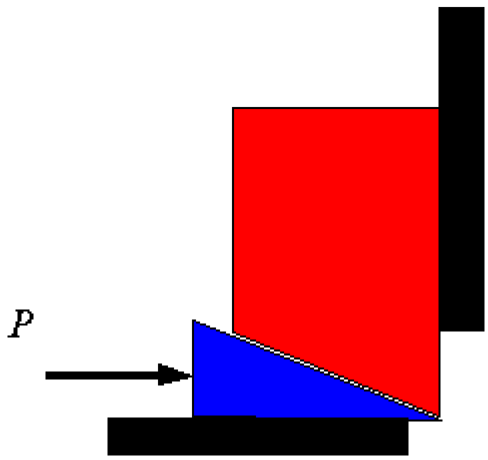
$$30 = (T_A - T_B)\left(\frac{4}{12}\right)$$

$$T_A - T_B = 90$$

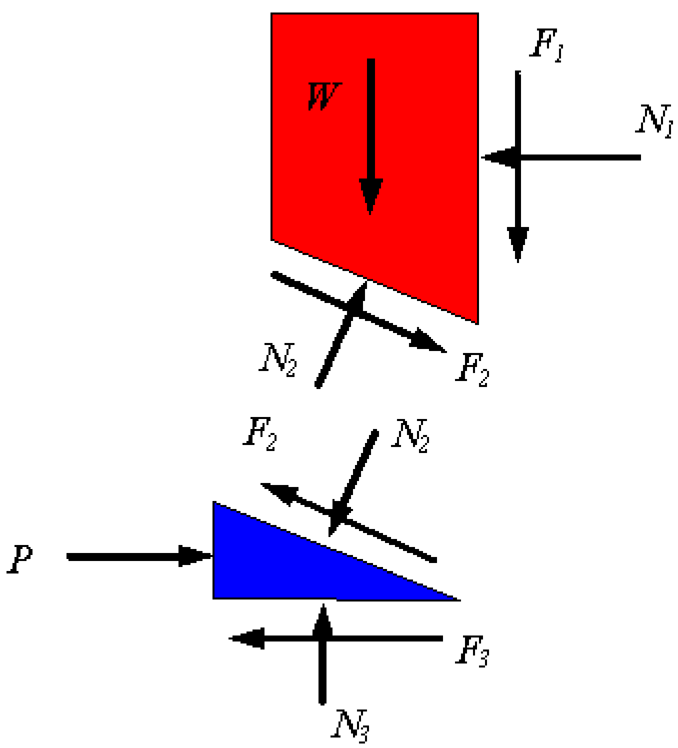
$$T_A = T_B + 90$$

12. WEDGE FRICTION

A wedge is in general a triangular object which is placed between two objects to either hold them in place or is used to move one relative to the other. For example, the following shows a wedge under a block that is supported by the wall.



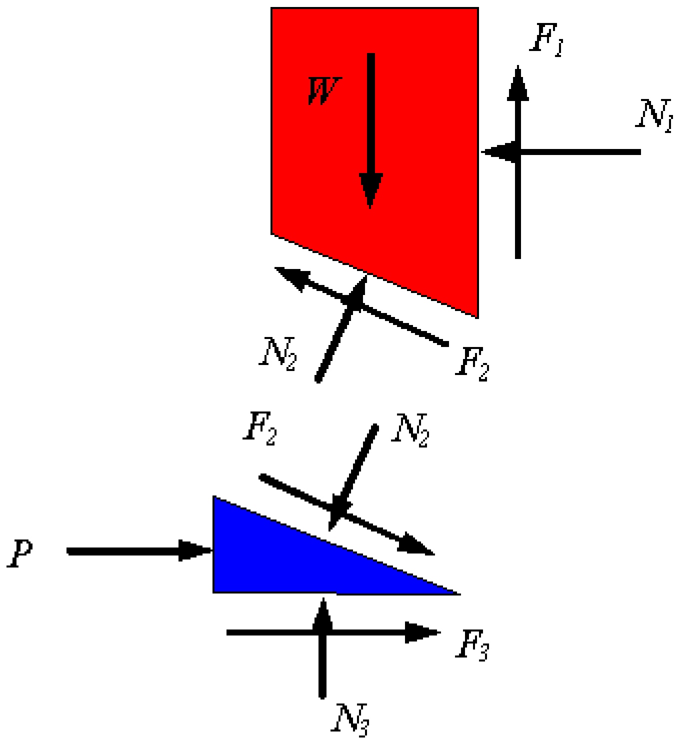
If the force P is large enough to push the wedge forward, then the block will rise and the following is an appropriate free-body diagram. Note that for the wedge to move one needs to have slip on all three surfaces. The direction of the friction force on each surface will oppose the slipping.



Since before the wedge can move each surface must overcome the resistance to slipping, one can assume that

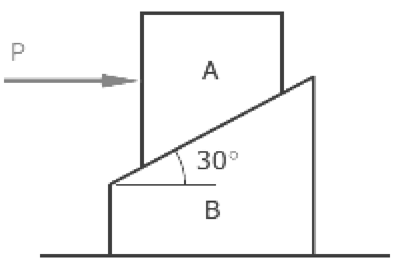
$$F_1 = \mu N_1$$
$$F_2 = \mu N_2$$
$$F_3 = \mu N_3$$

These equations and the equations of equilibrium are combined to solve the problem. If the force *P* is not large enough to hold the top block from coming down, then the wedge will be pushed to the left and the appropriate free-body diagram is the following. Note that the only change is the direction of the frictional forces. A similar analysis to the above yield the solution to the problem.



Problems on Wedge Friction

Problem No.1 Determine the minimum weight of block B that will keep it at rest while a force P starts blocks A up the incline surface of B. The weight of A is 100 N and the angle of friction for all surfaces in contact is 15°.



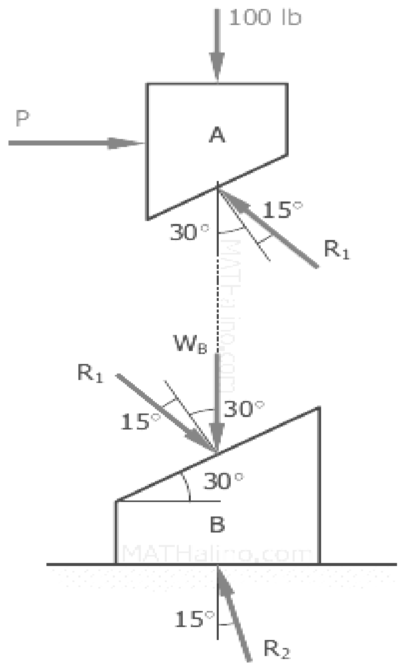
Solution

From the FBD of block A

$\Sigma V=0$

$R_1\cos45° = 100$

$R_1 = 141.42$ N



From the FBD of block B

$\Sigma H = 0$

$R_2\sin15° = R_1\sin45°$

$R_2\sin15° = 141.42\sin45°$

$R_2 = 386.37$ N

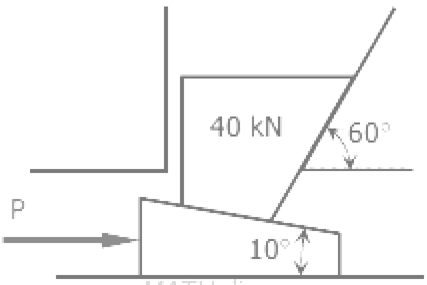
$\Sigma V = 0$

$W_B + R_1\cos45° = R_2\cos15°$

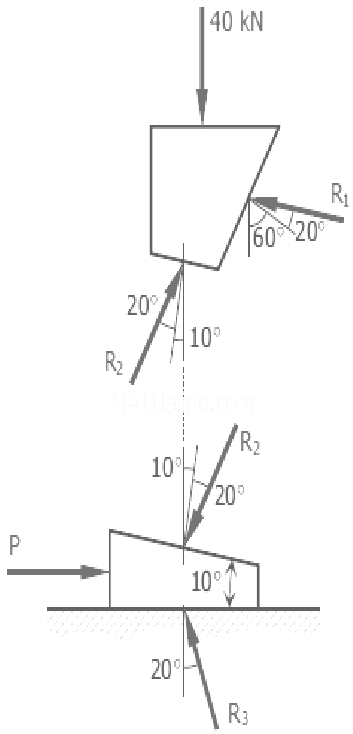
$W_B + 141.42\cos45° = 386.37\cos15°$

$W_B = 273.20$ N

Problem No.2 Determine the value of P just sufficient to start the 10° wedge under the 40-kN block. The angle of friction is 20° for all contact surfaces.



Solution



From the FBD of 40 kN block

$\Sigma H = 0$

$R_1\sin80° = R_2\sin30°$

$R_1 = R_2\sin30°\sin80°$

$R_1 = 0.5077R_2$

$\Sigma V = 0$

$R_2\cos30° + R_1\cos80° = 40$

$R_2\cos 30° + (0.5077R_2)\cos 80° = 40$

$0.9542R_2 = 40$

$\mathbf{R_2 = 41.92\ kN}$

From the FBD of lower block

$\Sigma V = 0$

$R_3\cos 20° = R_2\cos 30°$

$R_3\cos 20° = 41.92\cos 30°$

$\mathbf{R_3 = 38.634\ kN}$

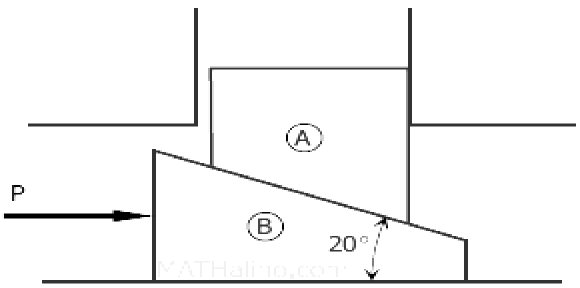
$\Sigma H = 0$

$P = R_2\sin 30° + R_3\sin 20°$

$P = 41.92\sin 30° + 38.634\sin 20°$

$\mathbf{P = 34.174\ kN}$

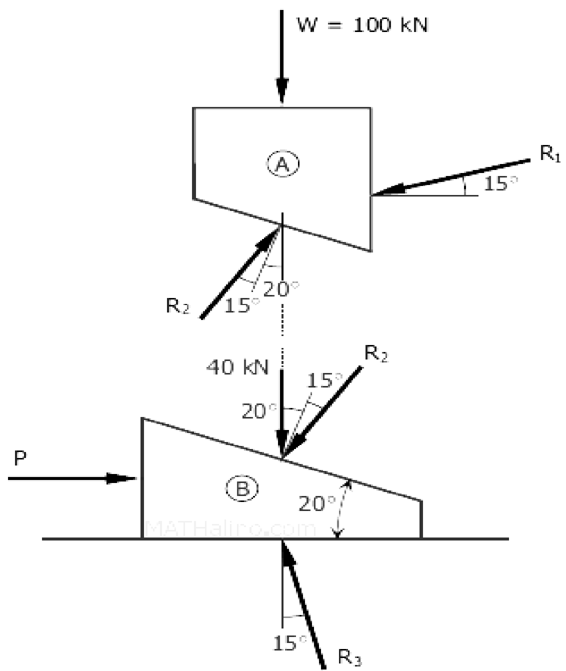
Problem No.3 The block A in Fig. P-539 supports a load W = 100 kN and is to be raised by forcing the wedge B under it. The angle of friction for all surfaces in contact is f = 15°. If the wedge had a weight of 40 kN, what value of P would be required (a) to start the wedge under the block and (b) to pull the wedge out from under the block?



<u>Solution</u>

Part (a): P to start the wedge under block A

From the FBD of block A



$\Sigma H=0$

$R_1\cos15°=R_2\sin35°$

$R_1=0.5938R_2$

$\Sigma V=0$

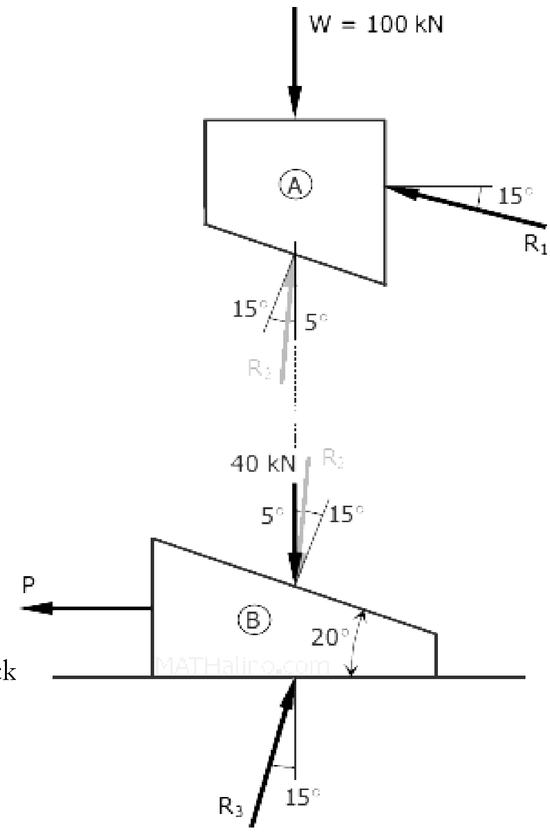
$R_2\cos35°=R_1\sin15°+100$

$R_2\cos35°=(0.5938R_2)\sin15°+100$

$0.6655R_2=100$

$R_2=150.27$ kN

From FBD of block B



$\Sigma V=0$

$R_3\cos15°=R_2\cos35°+40$

$R_3\cos15°=150.27\cos35°+40$

$R_3=168.85$ kN

$\Sigma H=0$

$P=R_2\sin35°+R_3\sin15°$

$P=150.27\sin35°+168.85\sin15°$

$P=129.89$ kN

Part (b): P to pull the wedge out from under the block
From FBD of block A

$\Sigma H=0$

$R_1\cos15°=R_2\sin5°$

$R_1=0.0902R_2$

$\Sigma V=0$

$R_2\cos5°+R_1\sin15°=100$

$R_2\cos5°+(0.0902R_2)\sin15°=100$

$1.0195R_2=100$

$R_2=98.08$ kN

From FBD of block B

$\Sigma V=0$

$R_3\cos15°=R_2\cos5°+40$

$R_3\cos15°=98.08\cos5°+40$

$R_3=142.57$ kN

$\Sigma H=0$

$P+R_2\sin5°=R_3\sin15°$

$P+98.08\sin5°=142.57\sin15°$

$P=28.35$ kN

SCHOOL OF BUILDING AND ENVIRONMENT

DEPARTMENT OF CIVIL ENGINEERING

UNIT – V - DYNAMICS OF PARTICLES – SCIA1101

V. Dynamics of Particles

Displacement, Velocity and acceleration their relationship - Relative motion - Curvilinear motion - Newton's Law - D'Alembert's Principle, Work Energy Equation - Impulse and Momentum - Impact of elastic bodies, Translation and rotation of rigid bodies - General plane motion.

INTRODUCTION

The dynamics of particles deals with the study of forces acting on a body and its effects, when the body is in motion. It is further divided into Kinematics and kinetics.

Kinematics – The study of motion of body without considering the forces which cause the motion of the body.

Kinetics – The study of motion of body with considering the external forces which cause the motion of the body.

<u>*Plane motion*</u> – If a particle has no size but mass it is considered to have only plane motion, not rotation. In this chapter the study motion of particles with only plane motion is taken without considering force that cause motion i.e., Kinematics.

The plane motion of the body can be sub divided into two types

(i) Rectilinear motion
(ii) Curvilinear motion

1. RECTILINEAR MOTION *(Straight Line Motion)* - It is the motion of the particle along a straight line.

Example: A car moving on a straight road
A stone falls vertically downwards
A ball thrown vertically upwards

This deals with the relationship among displacement, velocity, acceleration and time for a moving particle. The rectilinear motion is of two types as Uniform acceleration and Variable acceleration.

Displacement –The displacement of a moving particle is the change in its position, during which the particle remain in motion. It is the vector quantity, i.e., it has both magnitude and direction. The SI unit for displacement is the metre (m).

Velocity – The rate of change of displacement is velocity. It is the ratio between distances travelled in particular direction to the time taken. It is also a vector quantity, i.e., it has both magnitude and direction. The SI unit for velocity is the metre/second (m/sec) or kilometer/hour (km/h)

Acceleration – The rate of change of velocity is acceleration. It is the ratio between changes in velocity to the time taken. The change in velocity means the difference between final velocity and initial velocity. It is also a vector quantity. The SI unit for acceleration is the metre/second^2 (m/sec^2).

Retardation – The negative acceleration is retardation. It occurs when final velocity is less than initial velocity ($v<u$).

Speed – The distance travelled by a particle or a body along its path per unit time. It is a scalar quantity, i.e., it has only magnitude. The SI unit for speed is the metre/second (m/sec) or kilometer/hour (km/h)

RELATIVE MOTION

A body is said to be in motion if it changes its position with respect to the surroundings, taken as fixed. This type of motion is known as the individual motion of the body. An example of relative motion is how the sun appears to move across the sky, when the earth is actually spinning and causing that apparent motion. Usually, we consider motion with respect to the ground or the Earth. Within the Universe there is no real fixed point. The basis for Einstein's Theory of Relativity is that all motion is relative to what we define as a fixed point.

Relative velocity – Basic concept

Let's consider two motors A and B are moving on a road in same direction moving in uniform speed. Let the uniform velocities of motors A and B be u m/sec and v m/sec respectively (assume $v > u$)

Now, a person standing on the road looks at the motor A and finds that it is going at a speed of u m/sec. Similarly, looks at motor B and finds it is going at a speed of v m/sec separately. But for the driver of motor A, the motor B seems to move faster than him at the rate of only (v - u) m/sec. i.e., the motor A is imagined to be at ret or, the driver of motor A forgets his own motion.

Relative velocity of B with respect to A is (v-u). It is denoted by $V_{B/A}$

$$\therefore \mathbf{V_{B/A} = V_B - V_A = (v - u)\ m/sec}$$

Similarly for the driver of motor B, the motor A seems to move slower (assume $u < v$) than him at the rate of only (u - v) m/sec. i.e., the motor B is imagined to be at ret or, the driver of motor B forgets his own motion.

Relative velocity of A with respect to B is (v - u). It is denoted by $V_{A/B}$

$$\therefore \mathbf{V_{A/B} = V_A - V_B = (u - v)\ m/sec}$$

PROBLEM

Example1. The car A travels at a speed of 30 m/ sec and car B travels at a speed of 20 m/ sec in the same direction. Determine, i) the velocity of car A relative to car B ii) the velocity of car B, relative to car A

Given data

V_A = 30 m/se

V_B = 20 m/sec

Same direction

Solution

Let the cars A and B, travels in the same direction, say towards right.

Now, let's use the sign convention, the RHS velocity is taken as positive, and the LHS velocity is taken as negative. Hence, V_A = 30 m/sec and V_B = 20 m/sec.

Velocity of car A relative to car B

$V_{A/B} = V_A - V_B = 30 - 20 = 10$ m/sec (→) (since due to positive)

Velocity of car B relative to car A

$V_{B/A} = V_B - V_A = 20 - 30 = - 10$ m/sec (←) (since due to negative)

Example2. The car A travels at a speed of 30 m/ sec and car B travels at a speed of 20 m/ sec in the opposite direction. Determine, i) the velocity of car A relative to car B ii) the velocity of car B, relative to car A

Given data

V_A = 30 m/se

V_B = - 20 m/sec (- due to LHS)

Opposite direction

<u>Solution</u>

Let the cars A and B, travels in the opposite direction, say A towards right and towards left.

Velocity of car A relative to car B

$V_{A/B} = V_A - V_B = 30 - (-20) = 50$ m/sec (→) (since due to positive)

Velocity of car B relative to car A

$V_{B/A} = V_B - V_A = - 20 - 30 = - 50$ m/sec (←) (since due to negative)

MATHEMATICAL EXPRESSION FOR VELOCITY AND ACCELERATION

(i) Velocity, $v = ds/dt$

(ii) Acceleration, $a = d^2s/dt^2$

Where, s – distance travelled by a particle in a straight line.

t – time taken by the particle to travel the distance 's'

Equation of motion in straight line

Let, u – initial velocity (m/sec)

v – Final velocity (m/sec)

s – Distance travelled by a particle (m)

t – Time taken by the particle to change from u to v (second)

a – acceleration of the particle (m/sec^2)

$$v = u + at$$

$$s = ut + ½$$

$$(at^2)\ v^2 = u^2$$

Note: 1) If a body starts from rest, its initial velocity is zero i.e., u=0

2) If a body comes to rest, its final velocity is zero i.e., v=0

PROBLEMS

Example1. A car is moving with a velocity of 20 m/sec. the car is brought to rest by applying brakes in 6 seconds. Find i) retardation ii) distance travelled by the car after applying brakes.

Given data

u = 20 m/s

v = 0 (car is brought to rest)

t = 6 sec

Solution

i) Retardation or negative acceleration

Using equation of motion, v = u+at

$0 = 20 + (a*6)$

$a = -3.33\ m/sec^2$

Retardation = 3.33 m/sec^2

ii) Distance travelled

Using equation of motion, $s = ut + 1/2\ (at^2)$

$= (20*6) + 1/2(3.33*6^2)$

$= 60\ m$

Distance, s = 60 m

Example2. A train starts from rest and attains a velocity of 45 kmph in 2 minutes, with uniform acceleration. Calculate i) acceleration ii) distance travelled and iii) time required to reach a velocity of 36n kmph.

Given data

Initial velocity, u = 0 (train starts from rest)

Final velocity, v = 45 kmph = 12.5 m/sec

Time taken , t = 2 minutes = 120 seconds

Solution

i) Acceleration, a

Using equation of motion, $v = u + at$

$A = 0.104\ m/sec^2$

ii) Distance travelled in 2 minutes, s

Using equation of motion, $s = ut + 1/2\ (at^2)$

$S = 748.8\ m$

iii) Time required to attain velocity of 36 kmph

u = 0

v = 36 kmph = 10 m/sec

Using equation of motion, $v = u + at$

$t = 96.15\ sec$

Example3. A thief's car had a start with an acceleration of 2 m/sec^2**.** A police's car came after 5 seconds and continued to chase the thief's car with a uniform velocity of 20 m/sec. Find the time taken in which the police car will overtake the thief's car?

Given data

Initial velocity of thief's car = 0

Acceleration of thief's car = 2 m/sec^2

Uniform velocity of police van = 20 m/sec

Police's car came after 5 seconds of the start of thief's car.

Solution

Let us consider that the police's car takes't' seconds to overtake thief's car. Now, the cars are taken separately to solve.

Motion of thief's car

u = 0

a = 2 m/sec^2

t = (t+5)

Using equation of motion, s = ut+1/2 (at^2) = (t+5)^2----------------------- (1)

Motion of police's car

The police's car is moving with an uniform velocity of 20 m/sec.

Therefore, distance travelled by the police's car, from starting point of thief's car and to overtake it

Take, s = uniform velocity * time taken

= 20*t = 20t.................................(2)

The police car overtakes the thief's car. Hence, the distances travelled by both the cars should be equal.

Therefore, equate (1) and (2)

(t+5)^2 = 20t

t^2+25+10t = 20t

t^2+25-10t = 0

Using arithmetical equation, $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ where, a = 1, b = 10 and c = 25

The t is found as 5 seconds.

Conclusion - The time taken by police's car to overtake thief's car is 5 seconds.

2. CURVILINEAR MOTION - It is the motion of the particle along a curved path. It has two dimensions.

Example: A stone thrown into the air at an angle

Throwing paper airplanes in air

There are two systems involved in curvilinear motion. They are

(i) Cartesian systems (rectangular coordinates)

(ii) Polar system (radial coordinates)

CARTESIAN SYSTEMS

It is a rectangular coordinate system which has the horizontal component in X-axis and vertical component in Y-axis.

Horizontal component of velocity, $V_x = dx/dt$

Vertical component of velocity, $V_y = dy/dt$

Therefore, resultant velocity of a particle, $V = \sqrt{(V_x^2+V_y^2)}$

Angle of inclination of velocity with X-axis, $\alpha=\tan^{-1}(V_y/V_x)$

Acceleration of a particle along X-axis, $a_x = d^2x/dt^2$

Acceleration of a particle along Y-axis, $a_y = d^2y/dt^2$

Resultant acceleration of a particle, $a = \sqrt{(a_x^2+a_y^2)}$

Angle of inclination of acceleration with X-axis, $\varphi = \tan^{-1}(a_y/a_x)$

PROBLEMS

Example1. The portion of a particle along a curved path is given by the equations $x=t^2+8t+4$ and $y=t^3+3t^2+8t+4$. Find the i) initial velocity, u ii) velocity of the particle at t=2 sec iii) acceleration of the particle at t=0 and iv) acceleration of the particle at t= 2 sec.

Given data

$x=t^2+8t+4$

$y=t^3+3t^2+8t+4$

Solution

Horizontal component of velocity , $V_x = dx/dt = d(t^2+8t+4)/dt = 2t+8$-------------- (1)

Vertical component of velocity, $V_y = dy/dt = d(t^3+3t^2+8t+4)/dt = 3t^2+6t+8$------- (2)

Acceleration of a particle along X-axis, $a_x = d^2x/dt^2 = d(2t+8)/dt = 2$ ---------------(3)

Acceleration of a particle along Y-axis, $a_y = d^2y/dt^2 = d(3t^2+6t+8)/dt = 6t+6$------ (4)

i) <u>Initial velocity, u</u>

Put t = 0 in equation (1) and (2)

$V_x = 2t+8$

Now, $\mathbf{V_x = 8}$ **m/sec**

$V_y = 3t^2+6t+8$

Now, $\mathbf{V_y = 8}$ **m/sec**

Therefore, resultant velocity of a particle, $V = \sqrt{(V_x^2+V_y^2)}$

$= \sqrt{(8^2+8^2)}$

V= 11.31 m/sec

Angle of inclination of velocity with X-axis, $\alpha = \tan^{-1}(V_y/V_x)$

$= \tan^{-1}(8/8)$

$\boldsymbol{\alpha = 45^\circ}$

ii) <u>Velocity at t= 2 sec</u>

Put t = 2 seconds in equation (1) and (2)

$V_x = 2t+8$

Now, $\mathbf{V_x = 12}$ **m/sec**

$V_y = 3t^2+6t+8$

Now, $\mathbf{V_y = 32}$ **m/sec**

Therefore, resultant velocity of a particle, $V = \sqrt{(V_x^2+V_y^2)}$

$= \sqrt{(12^2+32^2)}$

V= 34.17 m/sec

Angle of inclination of velocity with X-axis, $\alpha = \tan^{-1}(V_y/V_x)$

$= \tan^{-1}(32/12)$

α = 69.4°

iii) Acceleration at t=0

Put t = 0 in equation (3) and (4)

Acceleration of a particle along X-axis, $a_x = d^2x/dt^2 = 2$ m/sec^2

Acceleration of a particle along Y-axis, $a_y = d^2y/dt^2 = 6t+6 = 6$ m/sec^2

Resultant acceleration of a particle, $a = \sqrt{(a_x^2+a_y^2)} = \sqrt{(2^2+6^2)} = 6.34$ m/sec^2

Angle of inclination of acceleration with X-axis, $\varphi = \tan^{-1}(a_y/a_x) = \tan^{-1}(6/2) = 71.56°$

iv) Acceleration at t = 2 sec

Put t = 2 sec in equation (3) and (4)

Acceleration of a particle along X-axis, $a_x = d^2x/dt^2 = 2$ m/sec^2

Acceleration of a particle along Y-axis, $a_y = d^2y/dt^2 = 6t+6 = 18$ m/sec^2

Resultant acceleration of a particle, $a = \sqrt{(a_x^2+a_y^2)} = \sqrt{(2^2+18^2)} = 18.11$ m/sec^2

Angle of inclination of acceleration with X-axis, $\varphi = \tan^{-1}(a_y/a_x) = \tan^{-1}(18/2) = 83.66°$

PROJECTILES

The projectile is an example of curvilinear motion of a particle in plane motion. The motion of a particle is neither vertical nor horizontal, but inclined to the horizontal plane

Definitions

Projectile – A particle projected in space at an angle to the horizontal plane.

Angle of projection means the angle to the horizontal at which the projectile is projected. It is denoted by a.

Velocity of projectile means the velocity with which the projectile is thrown into space. It is denoted by u (m/sec)

Trajectory means the path described by the projectile.

Time of flight is the total time taken by the projectile from the instant of projection up to the projectile hits the plane again.

Range is the distance along the plane between the point of projection and the point at which the projectile hits the plane at the end of its journey.

Path of the Projectile

The *horizontal distance* travelled by the projectile in any time t.

X = Velocity * Time taken

Therefore, $X = u \cos \alpha t$

Or

$t = X / u \cos \alpha$

.

$Y = \tan\alpha X – ½ (g^2 X/u^2 \cos^2\alpha)$

Similarly for *vertical distance*,

From the equation of the trajectory, it is clear that the two variables of projectile motion are initial velocity (u) and the angle of projection (a) to arrive standard results of projectile motion.

Time of flight (T) and time taken to reach highest point (t):

$T = 2 u \sin \alpha / g$

$t = u \sin \alpha / g$

Maximum height attained:

$h_{max} = u^2 \sin^2\alpha / 2g$

Horizontal range:

$R = u^2 \sin 2\alpha / g$

PROBLEMS

Example1. A particle is projected with an initial velocity of 60 m/sec, at an angle of 75˚ with the horizontal. Determine i) the maximum height attained by the particle ii) horizontal range of the particle iii) time taken by the particle to reach highest point iv) time of flight

Given data

Initial velocity, u = 60 m/sec

Angle of projection, α = 75˚

Solution

i) the maximum height attained by the particle

$h_{max} = u^2 \sin^2\alpha / 2g = 171.19$ m (take $g = 9.81$ m/sec^2)

ii) horizontal range

$R = u^2 \sin 2\alpha / g = 183.48$ m

iii) time taken to reach highest point

$t = u \sin \alpha / g = 5.9$ sec

iv) time of flight

$T = 2 u \sin \alpha / g = 11.8$ sec

Example2. A particle is projected with an initial velocity of 12 m/sec at an angle α with the horizontal. After sometime the position of the particle is observed by its x and y distances of 6m and 4 m respectively from the point of projection. Find the angle of projection?

Given data

Initial velocity, u = 12 m/sec

Horizontal distance, x = 6 m

Vertical distance, y = 4 m

Solution

If the coordinate points on the projectile path are given, then use equation of trajectory.

Equation of path of projectile (trajectory)

$Y = \tan \alpha \, X – ½ (g^2X/u^2 \cos^2\alpha)$

Put u = 12 m/sec, X = 6 m and Y = 4 m

Take g = 9.81 m/sec^2

We get,

$4 = 6 \tan\alpha - (1.226/\cos^2\alpha)$

$1.226 \tan^2\alpha - 6 \tan\alpha + 5.226 = 0$

Using arithmetical equation, $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$ where, a = 1.226, b = 6 and c = 5.226

Therefore, $\tan\alpha = -6\pm\sqrt{(6^2 - (4*1.226*5.226)}/(2*1.226)$

$\boldsymbol{\alpha = 75.1^\circ}$ **or** $\boldsymbol{53.06^\circ}$

Important definitions on kinetics

a) Mass - a fundamental measure of the amount of matter in the object. It is denoted by 'm'. The SI unit of mass is Kilograms (Kg). It's a scalar quantity.

b) Weight - The weight of an object is defined as the force of gravity on the object and may be calculated as the mass times the acceleration of gravity, w = mg. Since the weight is a force, its SI unit is the Newton.

Weight = mass * acceleration due to gravity

W = mg

c) Momentum - Momentum can be defined as "mass in motion." All objects have mass; so if an object is moving, then it has momentum - it has its mass in motion. It depends upon the variables mass and velocity. In terms of an equation, the momentum of an object is equal to the mass of the object times the velocity of the object. Its SI unit is $kg.m/sec^2$

Momentum = mass • velocity

M = mv

LAWS OF MOTION

When a particle / body is at rest, or moving in a straight line (rectilinear motion) or in a curved line (curvilinear motion), the particle / body obeys certain laws of motion. These laws are called Newton's law of motion. These laws are also called the principles of motion, or principles of Dynamics.

First Law

Every body continues to be in its state of rest or of uniform motion in a straight line unless and until it is acted upon some external force to change that state. It is also called ***the law of inertia***, and consists of the following two parts:

1. A body at rest continues in the same state, unless acted upon by some external force. It appears to be self-evident, as a train at rest on a level track will not move unless pulled by an engine. Similarly, a book lying on a table remains at rest, unless it is lifted or pushed.
2. A body moving with a uniform velocity continues its state of uniform motion in a straight line, unless it is compelled by some external force to change its state. It cannot be exemplified because it is, practically, impossible to get rid of the forces acting on a body.

Second Law

The rate of change of momentum of a moving body is directly proportional to the impressed force and takes place in the direction of the force applied.

The change of momentum = final momentum – initial momentum

= mv – mu = m (v – u)

The rate of change of momentum = change of momentum / time taken

= m (v – u) / t = m*a (since (v – u) / t = a)

Basically, to increase the velocity of the moving body from u to v, there must be some external force to cause this change. Let that external force be 'P'.

As per the law, the external force 'F' is directly proportional to the rate of change of momentum i.e., $F \propto ma \rightarrow F = k * ma$ where, k is the constant of proportionality.

But for a moving body, k and m are constants, and hence it states that, the force acting on the body is directly proportional to the acceleration of the body. From this we can conclude that,

1. For a given body, greater force produces greater acceleration and the lesser force produces the lesser acceleration.
2. The acceleration is zero, if there is no external force on the body which results in u = v.

To find the value of constant 'k' in equation F = k*ma

We know that, 1 N = 1 kg * 1 m/sec^2

That is, the unit force (N) is a force, which produce unit acceleration (1 m/sec^2) on an unit mass (1 kg) hence, by substituting F = 1; m = 1 and a = 1. We get

$$F = ma$$

Example1. A body of mass 4 kg is moving with a velocity of 2 m/sec and when certain force is applied, it attains a velocity of 8 m/sec in 6 seconds?

Given data

Mass, m = 4 kg

Initial velocity, u = 2 m/sec

Final velocity, v = 8 m/sec

Time, t = 6 sec

Solution

Acceleration, a = v - u / t = 8-2/6 = 1 m/sec^2

Let, 'P' be the force applied to cause this acceleration.

P = ma = 4*1 = 4 N

Example2. A body of mass 4 kg is at rest. What force should be applied to move it to a distance f 2 m in 4 seconds?

Given data

Mass, m = 4 kg

Distance, s = 12 m

Time taken, t = 4 sec

Initial velocity, u = 0

Solution

Using the equation, $s = ut + ½ at^2$

$$12 = 0 + 8a$$

Therefore, a = 12/8 m/sec^2

The force required to move, P = m*a = 4*(12/8) = 6 N

Therefore, P = 6 N

4. D’ALEMBERT’S PRINCIPLE

It states, “If a rigid body is acted upon by a system of forces, this system may be reduced to a single resultant force whose magnitude, direction and the line of action may be found out by the methods of graphic statics.”

We know that, that force acting on a body.

$$P = ma \text{ (i)}$$

Where, m = mass of the body, and

a = Acceleration of the body.

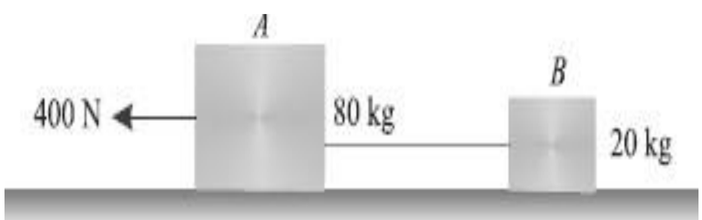
The equation (i) may also be written as:

$$P - ma = 0 \text{-----(ii)}$$

It may be noted that equation (i) is the equation of dynamics whereas the equation (ii) is the equation of statics. The equation (ii) is also known as the equation of dynamic equilibrium under the action of the real force P. This principle is known as D' Alembert’s principle.

PROBLEMS

Example1.Two bodies A and B of mass 80 kg and 20 kg are connected by a thread and move along a rough horizontal plane under the action of a force 400 N applied to the first body of mass 80 kg as shown in Figure. The coefficient of friction between the sliding surfaces of the bodies and the plane is 0.3.Determine the acceleration of the two bodies and the tension in the thread, using D' Alembert’s principle.



Given data

Mass of body A (m_1) = 80 kg

Mass of the body B (m_2) = 20 kg

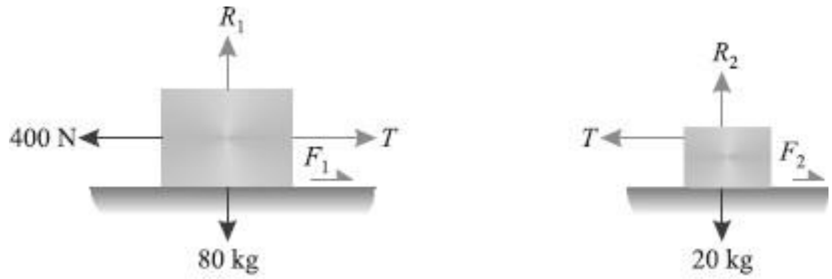
Force applied on first body (P) = 400 N and

Coefficient of friction (μ) = 0.3

Solution

Let a = Acceleration of the bodies, and

T = Tension in the thread.



(*a*) *Body A* (*b*) *Body B*

Consider the body A. The forces acting on it are:

400 N forces (acting towards left)

Mass of the body = 80 kg (acting downwards)

Reaction R_1 = 80 × 9.8 = 784 N (acting upwards)

Force of friction, $F_1 = \mu R_1$ = 0.3 × 784 = 235.2 N (acting towards right)

Tension in the thread = T (acting towards right).

∴ Resultant horizontal force, $P_1 = 400 - T - F_1 = 400 - T - 235.2$

$= 164.8 - T$ (acting towards left)

We know that force causing acceleration to the body A; $\rightarrow m_1 a = 80\, a$

And according to D' Alembert's principle $P_1 - m_1 a = 0 \rightarrow 1\,64.8 - T - 80\, a = 0$

∴ $T = 164.8 - 80a$-----------(*i*)

Now consider the body B. The forces acting on it are:

Tension in the thread = T (acting towards left)

Mass of the body = 20 kg (acting downwards)

Reaction R_2 = 20 × 9.8 = 196 N (acting upwards)

Force of friction, $F_2 = \mu R_2$ = 0.3 × 196 = 58.8 N (acting towards right)

∴ Resulting horizontal force, $P_2 = T - F_2 = T - 58.8$

We know that force causing acceleration to the body $B \rightarrow m_2 a = 20\, a$

And according to D' Alembert's principle $P2 - m2a = 0 \rightarrow (T - 58.8) - 20\, a = 0$

∴ $T = 58.8 + 20\, a$----------- (*ii*)

Now equating the two values of *T* from equation (*i*) and (*ii*),

$$164.8 - 80\,a = 58.8 + 20\,a$$

$$100\,a = 106$$

$$a = 106/100$$

$$\therefore \mathbf{a = 1.06\ m/sec^2}$$

Tension in the thread

Substituting the value of *a* in equation (*ii*)

$$T = 58.8 + (20 \times 1.06)$$

$$\therefore\ \mathbf{T = 80\ N}$$

Third Law

To every action, there is always an equal and opposite reaction

This law appears to be self-evident as when a bullet is fired from a gun, the bullet moves out with a great velocity, and the reaction of the bullet, in the opposite direction, gives an unpleasant shock to the man holding the gun. Similarly, when a swimmer tries to swim, he pushes the water backwards and the reaction of the water pushes the swimmer forward.

Example: When a bullet is fired from a gun, the opposite reaction of the bullet is known as the recoil of gun.

Let M = Mass of the gun,

V = Velocity of the gun with which it recoils,

m = mass of the bullet, and

v = Velocity of the bullet after explosion.

∴ Momentum of the bullet after explosion = mv (i)

Momentum of the gun = MV ------------------------(ii)

Equating the equations (i) and (ii), MV = mv

This relation is popularly known as *Law of Conservation of Momentum.*

PROBLEMS

Example1. A machine gun of mass 25 kg fires a bullet of mass 30 gram with a velocity of 250 m/s. Find the velocity with which the machine gun will recoil?

Given data

Mass of the machine gun (M) = 25 kg

Mass of the bullet (m) = 30 g = 0.03kg and

Velocity of firing (v) = 250 m/s.

Solution

Let V = Velocity with which the machine gun will recoil.

We know that $MV = mv$

$25 \times v = 0.03 \times 250 = 7.5 \rightarrow v = 7.5 / 25$

∴ v = 0.3 m/s

Example2. A bullet of mass 20 g is fired horizontally with a velocity of 300 m/s, from a gun carried in a carriage; which together with the gun has mass of 100 kg. The resistance to sliding of the carriage over the ice on which it rests is 20 N. Find (a) velocity with which the gun will recoil, (b) distance, in which it comes to rest, and (c) time taken to do so.

Given data

Mass of the bullet (m) = 20 g = 0.02 kg

Velocity of bullet (v) = 300 m/s

Mass of the carriage with gun (M) = 100 kg and

Resistance to sliding (F) = 20 N

Solution

(a) Velocity, with which the gun will recoil

Let V = velocity with which the gun will recoil.

We know that $MV = mv$

$100 \times V = 0.02 \times 300 = 6 \rightarrow V = 6 / 100 = 0.06$ m/s

∴ V = 0.06 m/sec

(b) Distance, in which the gun comes to rest

Now consider motion of the gun. In this case, initial velocity (u) = 0.06 m/s and final velocity, $v = 0$ (because it comes to rest)

Let a = Retardation of the gun, and

s = Distance in which the gun comes to rest.

We know that resisting force to sliding of carriage (F)

$$20 = Ma = 100\, a \rightarrow a = 20 / 100$$

$$\therefore \mathbf{a = 0.2\ m/sec^2}$$

We also know that $v^2 = u^2 - 2as$ (Minus sign due to retardation)

$$0 = (0.06)2 - 2 \times 0.2\, s$$

$$= 0.0036 - 0.4\, s \rightarrow s = 0.0036 / 0.4 = 0.009 \text{ m or } 9 \text{ mm}$$

$$\therefore \mathbf{s = 9\ mm}$$

(c) Time taken by the gun in coming to rest

Let t = Time taken by the gun in coming to rest.

We know that final velocity of the gun (v)

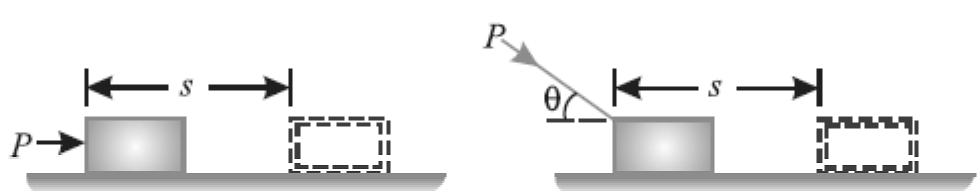
$0 = u + at = 0.06 - 0.2\, t$ (Minus sign due to retardation)

$t = 0.06 / 0.2$ $\therefore \mathbf{t = 0.3}$ **seconds**

WORK ENERGY EQUATION

Work

Whenever a force acts on a body, and the body undergoes some displacement, then work is said to be done. *e.g.*, if a force P, acting on a body, causes it to move through a distance s as shown in Figure (a).



(a) Body moving in the direction of force (b) Body not moving in the direction of force

Then work done by the force P = Force × Distance = $P \times s$

Work done by the force = P*S

Sometimes, the force P does not act in the direction of motion of the body, or in other words, the body does not move in the direction of the force as shown in Figure (*b*).

Then work done by the force P = Component of the force in the direction of motion × Distance

$= P \cos \theta \times s$

Work done by the force = P cos θ*S

In SI system of units, force is in Newton and the distance is in meters.

∴ Unit of work is (Newton * meter) = 1 Nm = 1 joule

∴ In SI system of units, unit of work is joule

PROBLEMS

Example1.A horse pulling a cart exerts a steady horizontal pull of 300 N and walks at the rate of 4·5 kmph. How much work is done by the horse in 5 minutes?

Given data

Pull (*i.e.* force) = 300 N

Velocity (v) = 4·5 kmph. = 75 m/ min and

Time, t = 5 min.

Solution

We know that distance travelled in 5 minutes

$s = 75 \times 5 = 375$ m → **∴ s = 375 m**

Work done by the horse, W = Force × Distance

$= 300 \times 375 = 112\ 500$ N-m = 112.5 kN-m

∴ W = 112.5 kJ

Example2. A spring is stretched by 50 mm by the application of a force. Find the work done, if the force required to stretch 1 mm of the spring is 10 N.

Given data

Spring stretched by the application of force (*s*) = 50 mm

Stretching of spring = 1 mm and force = 10 N

Solution

We know that force required stretching the spring by 50 mm = 10 × 50 = 500 N

∴ Average force = 500 / 2 = 250 N

Work done = Average force × Distance = 250 × 50 = 12 500 N-mm = 12.5 N-m

∴ Work done = 12.5 J

Power

The power may be defined as the rate of doing work.

∴ Power = work done / time

= (Force * Distance) / Time

Or

∴ Power = Force * (Distance/Time)

= Force * Velocity

In SI systems of units, unit of work is Newton metre, and the unit of time is seconds.

Unit of power = Nm / Seconds = 1 watt

∴ In SI systems, unit of power is watt

Energy

The energy may be defined as the capacity to do work. It exists in many forms i.e., mechanical, electrical chemical, heat, light etc. the energy is the capacity to do work. Since the energy of a body is measured by the work it can do, therefore the units of energy will be the same as those of the work. Therefore, the SI system of unit of work is joule.

In the study of mechanics, we are concerned only with mechanical energy. Mechanical energy is classified into two types.

1. **Potential energy. 2. Kinetic energy.**

Potential energy

It is the energy possessed by a body, for doing work, by virtue of its position.

Example1. A body, raised to some height above the ground level, possesses some potential energy; because it can do some work by falling on the earth's surface.

Example2. Compressed air also possesses potential energy; because it can do some work in expanding, to the volume it would occupy at atmospheric pressure.

Example3. A compressed spring also possesses potential energy; because it can do some work in recovering to its original shape.

Now consider a body of mass (m) raise through a height (h) above the datum level. We know that work done in raising the body = Weight × Distance = $(mg)\ h = mgh$

Potential Energy, P.E = mg *h

PROBLEM

Example1. A man of mass 60 kg dives vertically downwards into a swimming pool from a tower of height 20 m. He was found to go down in water by 2 m and then started rising. Find the average resistance of the water. Neglect the air resistance.

Given data

Mass of the man (m) = 60 kg and

Height of the tower (h) = 20 m

Solution

Let P = Average resistance of the water

We know that potential energy of the man before jumping

P.E = mg*h = 60 × 9.8 × 20 = 11 760 N-m---------------(i)

Work done by the average resistance of water = Average resistance of water × Depth of water

= P × 2 = 2 P N-m--------(ii)

Since the total potential energy of the man is used in the work done by the water, therefore equating equations (i) and (ii),

→ 11 760 = 2 P → P = 11760 / 2 ∴ **P = 5880 N**

Kinetic energy

It is the energy, possessed by a body, for doing work by virtue of its mass and velocity of motion. Now consider a body, which has been brought to rest by a uniform retardation due to the applied force.

Let m = Mass of the body

u = Initial velocity of the body

P = Force applied on the body to bring it to rest,

a = Constant retardation, and

s = Distance travelled by the body before coming to rest.

Since the body is brought to rest, therefore its final velocity, $v = 0$ and

Work done, W = Force × Distance = $P \times s$ ---------- (i)

Now substituting value of ($P = m.a$) in equation (i),

$W = ma \times s = mas$------------(ii)

We know that $v^2 = u^2 - 2\,as$ (Minus sign due to retardation)

$\therefore 2as = u^2$ (since, $v = 0$)

Now substituting the value of ($a.s$) in equation (ii) and replacing work done with kinetic energy,

$$\mathbf{K.E} = \boldsymbol{mu^2/2}$$

In most of the cases, the initial velocity is taken as v (instead of u), therefore kinetic energy,

$$\mathbf{K.E} = \boldsymbol{mv^2/2}$$

Kinetic Energy, K.E = ½ (mv^2)

PROBLEM

Example1. A truck of mass 15 tones travelling at 1.6 m/s impacts with a buffer spring, which compresses 1.25 mm per kN. Find the maximum compression of the spring?

Given data

Mass of the truck (m) = 15 t

Velocity of the truck (v) = 1.6 m/s and

Buffer spring constant (k) = 1.25 mm/ kN

Solution

Let x = Maximum compression of the spring in mm.

We know that kinetic energy of the truck = $mv^2/2$ = $(15*1.6^2) / 2$ = 19.2 = 19200 kN - mm

Kinetic Energy, K.E = 19200 kN-mm -------------- (i)

Compressive load = x / 1.25 = 0.8 x kN

Work done in compressing the spring = Average compressive load × Displacement

= (0.8 x / 2) * x = 0.4 x^2 ------------------- (ii)

Since the entire kinetic energy of the truck is used to compress the spring therefore equating equations (i) and (ii),

19 200 = 0.4 $x^2 \rightarrow x^2$ = 19200 / 0.4

= 48000

∴ **x = 219 mm**

Work Energy Equation

The equation of motion in one-dimension (taking the variable to be *x*, and the force to be *F*) is

$$m\frac{d^2x}{dt^2} = F(x)$$

Let us again eliminate time from the left-hand using the technique used above

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{dx}{dt}\frac{d}{dx}\left(\frac{dx}{dt}\right) = v\frac{dv}{dx}$$

To get

$$mv\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}mv^2\right) = F(x)$$

On integration this equation gives

$$\frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = \int_{x_i}^{x_f} F(x)dx$$

where x_i and x_f refer to the initial and final positions, and v_i and v_f to the initial and final velocities, respectively. We now interpret this result. We define the kinetic energy of a particle of mass *m*and velocity *v* to be

$$Kinetic\ energy = \frac{1}{2}mv^2$$

and the work done in moving from one position to the other as the integral given above

$$Work\ done = \int F(x)dx$$

With these definitions the equation derived above tells us that work done on a particle changes its kinetic energy by an equal amount; this known as the *work-energy theorem.*

IMPULSE AND MOMENTUM

Impulse

The impulse of a constant force F is defined as the product of the force and the time t for which it acts. The SI unit of linear impulse is ***N.sec*** --(i)

$$\text{Impulse} = Ft$$

The effect of the impulse on a body can be found using equation (i) where, a is acceleration, u and v are initial and final velocities respectively and t is time.

$$v = u + at$$

So

$$mat = m(v-u)$$

$$F = ma$$

$$Ft = m(v-u) = \text{change in momentum} \quad \text{.................(ii)}$$

So we can say that

$$\text{Impulse of a constant force} = Ft = \text{change in momentum produced}$$

Impulse is a vector quantity and has the sane units as momentum, Ns or kg m/s. The impulse of a variable force can be defined by the integral

$$\text{Impulse} = \int_0^t F\,dt$$ Where, t is the time for which F acts.

By Newton's 2nd law

$$F = ma = m\frac{dv}{dt}$$

So impulse can also be written

$$\text{Impulse} = \int_0^t m\frac{dv}{dt}dt$$
$$= \int_u^v m\,dv$$
$$= \left[mv\right]_u^v$$ Which for a constant mass

$$\text{Impulse} = m(v-u)$$

In summary

$$\text{Impulse} = \int_0^t F\,dt = \text{change in momentum produced} \quad \text{..................(iii)}$$

Impulsive force

Suppose the force F is very large and acts for a very short time. During this time the distance moved is very small and under normal analysis would be ignored. Under these condition the only effect of the force can be measured is the impulse, or change I momentum - the force is called an impulsive force.

In theory this force should be infinitely large and the time of action infinitely small. Some applications where the conditions are approached are collision of snooker balls, a hammer hitting a nail or the impact of a bullet on a target.

PROBLEMS

Example1. A nail of mass 0.02 kg is driven into a fixed wooden block, its initial speed is 30 m/s and it is brought to rest in 5ms. Find a) the impulse b) value of the force (assume this constant) on the nail.

Given data

Mass, m = 0.02 kg

Velocity, v = 30 m/sec

Initial velocity, u = 0

Time, t = 5 minutes

Solution

Using the equation,

$$\begin{aligned} \text{Impulse} &= \text{change in momentum of the nail} \\ &= 0.02(30 - 0) \\ &= 0.6\ Ns \end{aligned}$$

$$\text{Impulse} = Ft$$

$$F = \frac{\text{Impulse}}{t} = \frac{0.6}{0.005} = 120\ N$$

Momentum

The quantity of motion possessed by the moving body is called momentum. It is the product of mass and velocity.

Momentum = Mass * Velocity

i.e., M = mv

Where, m is mass I kilogram

v = velocity in m/sec

M = Momentum in kg.m/sec

→ mv = (w/g) * v

The SI unit of momentum is also ***N.sec***

Impulse – Momentum equation

The impulse – Momentum equation is also derived from the Newton’s secong law,

F = ma = m * (dv/dt) i.e., F dt = m dv

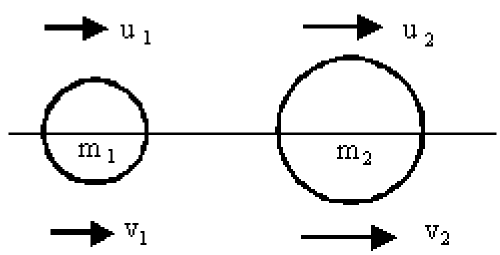
As derived in the impulse, the term $\int_0^t F\,dt$ is called impulse and m (v-u) is called the change of momentum, i.e., Final momentum – Initial momentum.

Impulse = m (v-u) = (W/g) * (v-u)

Impact of elastic bodies

In the last section the bodies were assumed to stay together after impact. An elastic body is one which tends to return to its original shape after impact. When two elastic bodies collide, they rebound after collision. An example is the collision of two snooker balls.

If the bodies are travelling along the same straight line before impact, then the collision is called a direct collision. This is the only type of collision considered here.



Direct collision of two elastic spheres

Consider the two elastic spheres as shown in figure. By the principle of conservation of linear momentum

$$\text{Momentum befor impact} = \text{Momentum after impact}$$

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

Where the *u*’s are the velocities before collision and the *v*’s, the velocities after.

When the spheres are inelastic v_1 and v_2 are equal as we saw in the last section. For elastic bodies v_1 and v_2 depend on the elastic properties of the bodies. A measure of the elasticity is the **coefficient of restitution** *e*, for direct collision this is defined as

$$e = -\left(\frac{v_1 - v_2}{u_1 - u_2}\right)$$

This equation is the result of experiments performed by Newton. The values of *e* in practice vary from between 0 and 1. For inelastic bodies *e* = 0, for completely elastic *e* = 1. In this latter case no energy is lost in the collision.

PROBLEMS

Example1. A body of mass 2kg moving with speed 5m/s collides directly with another of mass 3 kg moving in the same direction. The coefficient of restitution is 2/3. Find the velocities after collision.

Solution

$$\text{Momentum befor impact} = \text{Momentum after impact}$$

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$

$$2 \times 5 + 3 \times 4 = 2v_1 + 3v_2$$

$$22 = 2v_1 + 3v_2 \quad \text{.................(i)}$$

From equation

$$e = -\left(\frac{v_1 - v_2}{u_1 - u_2}\right)$$

$$\frac{2}{3} = -\left(\frac{v_1 - v_2}{5 - 4}\right)$$

$$-2 = 3v_1 - 3v_2 \quad \text{.......................(ii)}$$

Adding [i] and [ii] gives

$$20 = 5v_1$$

$$v_1 = 4\,m/s$$

And by [i]

$$22 = 8 + 3v_2$$

$$v_2 = \frac{14}{3}\,m/s$$

Example2.A railway wagon has mass 15 tones and is moving at 1.0 m/s. It collides with a second wagon of mass 20 tones moving in the opposite direction at 0.5m/s. After the collision the second wagon has changed its speed to 0.4m/s in the opposite direction as before the collision. Find i) the velocity of the 15 tones wagon after the collision ii) the coefficient of restitution and iii) the loss in kinetic energy.

Solution

$$\text{Momentum befor impact} = \text{Momentum after impact}$$
$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2$$
$$15000\times 1.0 - 20000\times 0.5 = 15000v_1 + 20000\times 0.4$$
$$-3000 = 15000v_1$$
$$v_1 = -0.2m/s$$

The negative sign means it has change direction of travel.

Coefficient of restitution is

$$e = -\left(\frac{v_1 - v_2}{u_1 - u_2}\right)$$
$$e = -\left(\frac{(-0.2) - 0.4}{1.0 - (-0.5)}\right)$$
$$e = 0.4$$

$$\text{kinetic energy before impact} = \frac{1}{2}15000\times 1.0^2 + \frac{1}{2}20000\times 0.5^2$$
$$= 10000\,J$$

$$\text{kinetic energy after impact} = \frac{1}{2}15000\times 0.2^2 + \frac{1}{2}20000\times 0.4^2$$
$$= 1900\,J$$

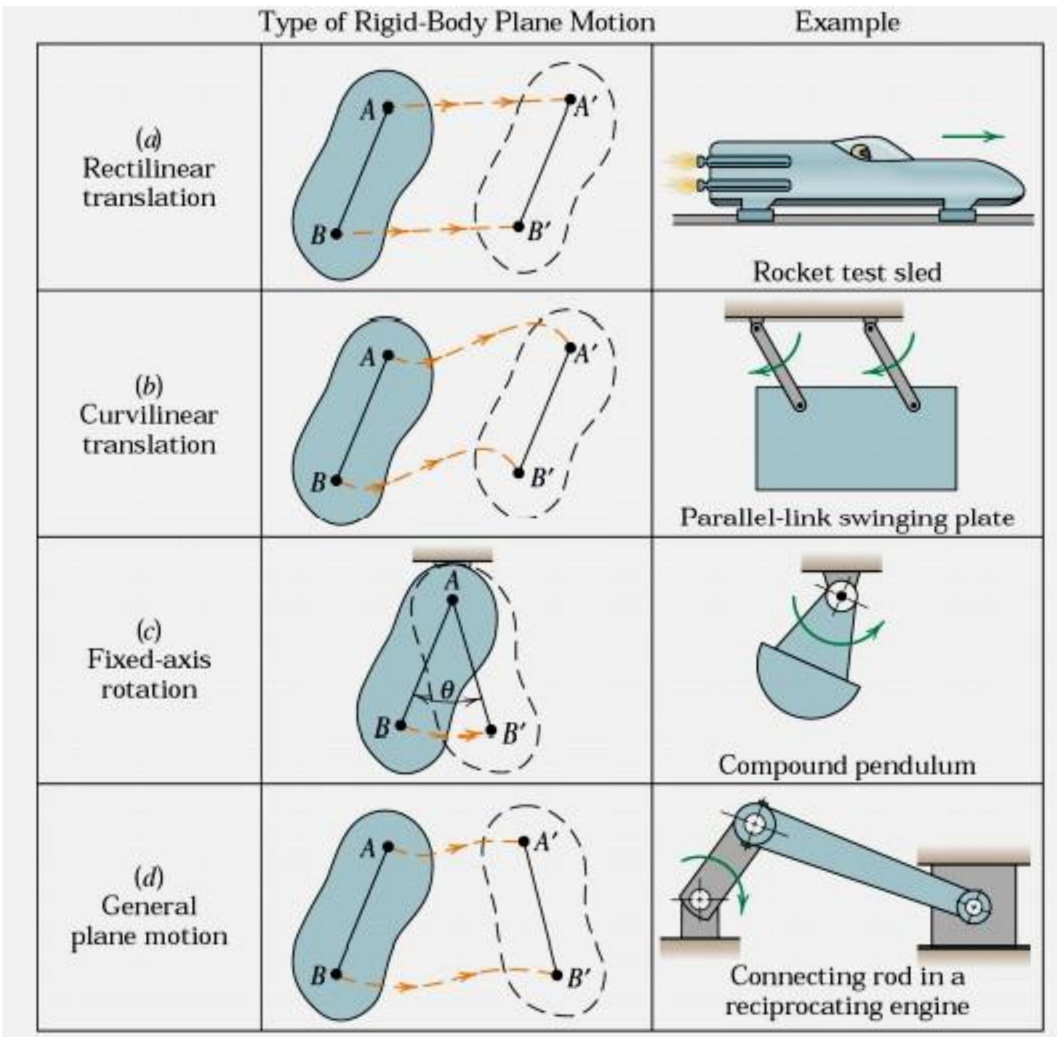
$$\text{loss of kinetic energy} = 10000 - 1900 = 8100\,J$$

Translation, Rotation of rigid bodies and General plane motion

Introduction

Forces acting of rigid bodies can be also separated in two groups: (a) The *external forces* represent the action of other bodies on the rigid body under consideration; (b) The *internal forces* are the forces which hold together the particles forming the rigid body. Only external forces can impart to the rigid body a motion of translation or rotation or both.

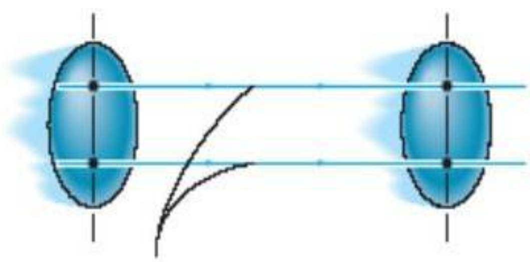
In kinematics the types of motion are *TRANSLATION*, *ROTATION* about a fixed axis and *GENERAL PLANE MOTION*.



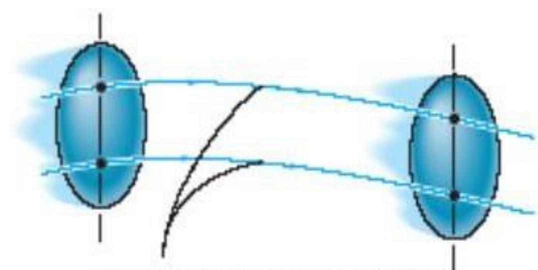
TRANSLATION

A motion is said to be a translation if any straight line inside the body keeps the same direction during the movement. It occurs if every line segment on the body remains parallel to its original direction during the motion

All the particles forming the body move along parallel paths. If these paths are straight lines, the motion is said a ***rectilinear translation***; if the paths are curved lines, the motion is a ***curvilinear motion*** as given below in figure.



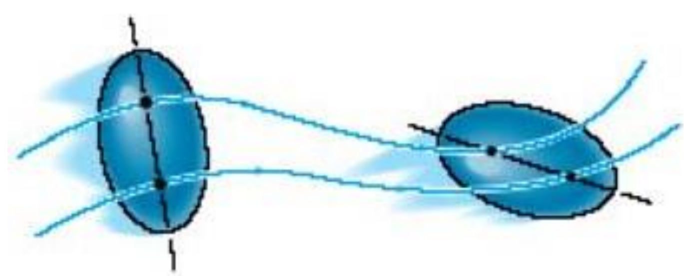
Path of rectilinear translation



Path of curvilinear translation

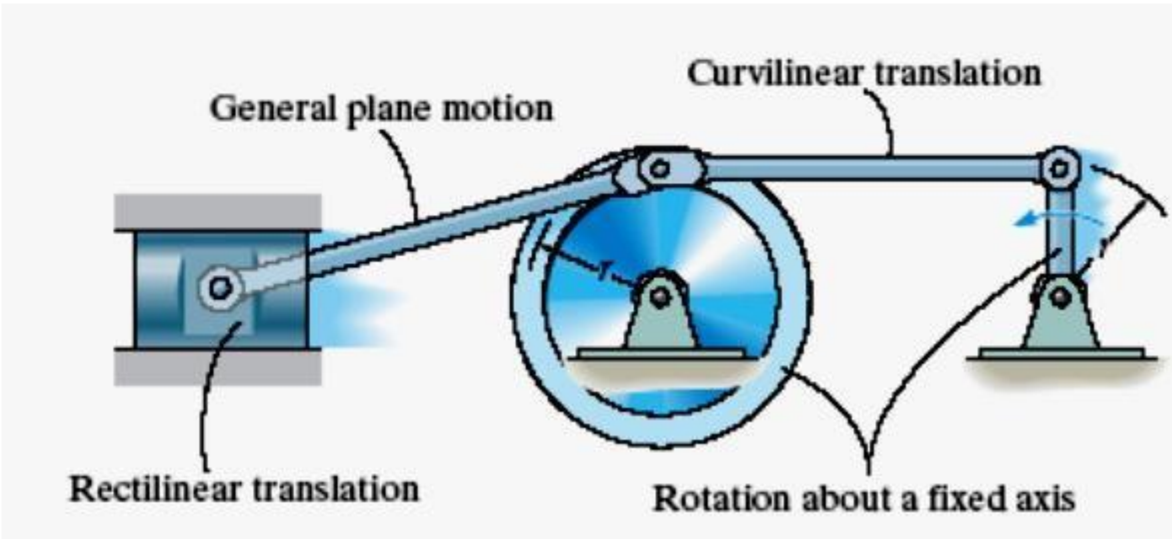
GENERAL PLANE MOTION

Any plane motion which is neither a translation nor a rotation is referred as a general plane motion. Plan motion is that in which all the particles of the body move in parallel planes. Translation occurs within a plane and rotation occurs about an axis perpendicular to this plane.



General plane motion

An example of bodies undergoing the three types of motion is shown in this mechanism. The wheel and crank undergo rotation about a fixed axis. In this case, both axes of rotation are at the location of the pins and perpendicular to the plane of the figure. The piston undergoes rectilinear translation since it is constrained to slide in a straight line. The connecting rod undergoes curvilinear translation, since it will remain horizontal as it moves along a circular path. The connecting rod undergoes general plane motion, as it will both translate and rotate.



ROTATION

Some bodies like pulley, shafts, and flywheels have motion of rotation (*i.e.*, angular motion) which takes place about the geometric axis of the body. The angular velocity of a body is always expressed in terms of revolutions described in one minute, *e.g.*, if at an instant the angular velocity of rotating body in N r.p.m. (*i.e*. revolutions per min) the corresponding angular velocity ω (in rad) may be found out as discussed below:

$$1 \text{ revolution/min} = 2\pi \text{ rad/min}$$

$$\therefore \text{ N revolutions/min} = 2\pi N \text{ rad/min}$$

$$\text{Angular velocity } \omega = 2\,\pi\, N \text{ rad/min}$$

$$\omega = \boldsymbol{2\,\pi\, N / 60 \text{ rad/sec}}$$

Important Terms

The following terms, which will be frequently used in this chapter, should be clearly understood at this stage:

Angular velocity - It is the rate of change of angular displacement of a body, and is expressed in r.p.m. (revolutions per minute) or in radian per second. It is, usually, denoted by ω (omega).

Angular acceleration - It is the rate of change of angular velocity and is expressed in radian per second per second (rad/s^2) and is usually, denoted by α. It may be constant or variable.

Angular displacement - It is the total angle, through which a body has rotated, and is usually denoted by θ. If a body is rotating with a uniform angular velocity (ω) then in t seconds, the angular displacement is $\boldsymbol{\theta = \omega * t}$

Motion of rotation under constant angular acceleration

Consider a particle, rotating about its axis.

Let ω_0 = Initial angular velocity,

ω = Final angular velocity,

t = Time (in seconds) taken by the particle to change its velocity from ω_0 to ω.

α = Constant angular acceleration in rad/s^2, and

θ =Total angular displacement in radians.

Since in t seconds, the angular velocity of the particle has increased steadily from ω_0 to ω at the rate of α rad/s^2, therefore

$$\omega = \omega_0 + \alpha t \quad \ldots(i)$$

and average angular velocity

$$= \frac{\omega_0 + \omega}{2}$$

We know that the total angular displacement,

$$\theta = \text{Average velocity} \times \text{Time} = \left(\frac{\omega_0 + \omega}{2}\right) \times t \quad \ldots(ii)$$

Substituting the value of ω from equation (i),

$$\theta = \frac{\omega_0 + (\omega_0 + \alpha t)}{2} \times t = \frac{2\omega_0 + \alpha t}{2} \times t = \omega_0 t + \frac{1}{2}\alpha t^2 \quad \ldots(iii)$$

and from equation (i), we find that

$$t = \frac{\omega - \omega_0}{\alpha}$$

Substituting this value of t in equation (ii),

$$\theta = \left(\frac{\omega_0 + \omega}{2}\right) \times \left(\frac{\omega - \omega_0}{\alpha}\right) = \frac{\omega^2 - \omega_0^2}{2\alpha}$$

$$\therefore \quad \omega^2 = \omega_0^2 + 2\alpha\theta \quad \ldots(iv)$$

Relation between linear motion and angular motion

PROBLEMS

Example1. A flywheel starts from rest and revolves with an acceleration of 0.5 rad/ sec^2. What will be its angular velocity and angular displacement after 10 seconds?

Given data

Initial angular velocity ($\omega 0$) = 0 (becasue it starts from rest)

Angular acceleration (α) = 0.5 rad/sec^2 and

Time (t) = 10 sec.

Solution

Angular velocity of the flywheel

We know that angular velocity of the flywheel,

$\omega = \omega 0 + \alpha t = 0 + (0.5 \times 10) = 5$ rad/sec

Angular displacement of the flywheel

We also know that angular displacement of the flywheel,

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 = (0 \times 10) + \left[\frac{1}{2} \times 0.5 \times (10)^2\right] = 25 \text{ rad}$$

Example2. A wheel rotates for 5 seconds with a constant angular acceleration and describes during this time 100 radians. It then rotates with a constant angular velocity and during the next five seconds describes 80 radians. Find the initial angular velocity and the angular acceleration.

Given data

Time (t) = 5 sec and

Angular displacement (θ) = 100 rad

Solution

Initial angular velocity

Let ω_0 = Initial angular velocity in rad/s,

α = Angular acceleration in rad/s^2, and

ω = Angular velocity after 5 s in rad/s.

First of all, consider the angular motion of the wheel with constant acceleration for 5 seconds.

We know that angular displacement (θ),

$$100 = \omega_0 t + \frac{1}{2}\alpha t^2 = \omega_0 \times 5 + \frac{1}{2} \times \alpha (5)^2 = 5\omega_0 + 12.5\alpha$$

$$\therefore \quad 40 = 2\omega_0 + 5\alpha \qquad \ldots(i)$$

and final velocity, $\omega = \omega_0 + \alpha t = \omega_0 + \alpha \times 5 = \omega_0 + 5\alpha$

Now consider the angular motion of the wheel with a constant angular velocityof $(\omega_0 + 5\alpha)$ for 5 seconds and describe 80 radians. We know that the angular displacement,

$$80 = 5(\omega_0 + 5\alpha)$$

or

$$16 = \omega_0 + 5\alpha \qquad \ldots(ii)$$

Subtracting equation (*ii*) from (*i*),

$$24 = \omega_0 \quad \text{or} \quad \omega_0 = 24 \text{ rad/s} \quad \textbf{Ans.}$$

Angular acceleration

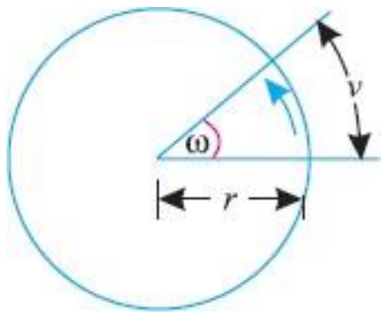
Substituting this value of ω_0 in equation (*ii*),

$$16 = 24 + 5\alpha \quad \text{or} \quad \alpha = \frac{16 - 24}{5} = -1.6 \text{ rad/s}^2 \quad \textbf{Ans.}$$

...(Minus sign means retardation)

Linear (Or Tangential) Velocity of a Rotating Body

Consider a body **rotating about its axis** as shown in Figure.



Let ω = Angular velocity of the body in rad/s,

r = Radius of the circular path in meters, and

v = Linear velocity of the particle on the periphery in m/s.

After one second, the particle will move v meters along the circular path and the angular displacement will be ω rad.

We know that length of arc = Radius of arc × Angle subtended in rad.

$$\therefore\ v = r\,\omega$$

PROBLEMS

Example1. A wheel of 1.2 m diameter starts from rest and is accelerated at the rate of 0.8 rad/s2 . Find the linear velocity of a point on its periphery after 5 seconds.

Given data

Diameter of wheel = 1.2 m or radius (r) = 0.6 m

Initial angular velocity (ω0) = 0 (becasue, it starts from rest)

Angular acceleration (α) = 0.8 rad/s2 and

Time (t) = 5 s

Solution

We know that angular velocity of the wheel after 5 seconds,

$\omega = \omega 0 + \alpha t = 0 + (0.8 \times 5) = 4$ rad/s

∴ Linear velocity of the point on the periphery of the wheel,

$v = r\omega = 0.6 \times 4 = 2.4$ m/s

Example2. A pulley 2 m in diameter is keyed to a shaft which makes 240 r.p.m. Find the linear velocity of a particle on the periphery of the pulley.

Given data

Diameter of pulley = 2 m or radius (r) = 1 m and

Angular frequency (N) = 240 r.p.m.

Solution

We know that angular velocity of the pulley,

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 240}{60} = 25.1 \text{ rad/s}$$

$\therefore$ Linear velocity of the particle on the periphery of the pulley,

$$v = r\omega = 1 \times 25.1 = 25.1 \text{ m/s}$$

Linear (Or Tangential) Acceleration of a Rotating Body

Consider a body rotating about its axis with a constant angular (as well as linear) acceleration. We know that linear acceleration,

$$a = \frac{dv}{dt} = \frac{d}{dt}(v) \qquad \ldots(i)$$

We also know that in motion of rotation, the linear velocity,

$$v = r\omega$$

Now substituting the value of v in equation (i),

$$a = \frac{d}{dt}(r\omega) = r\frac{d\omega}{dt} = r\alpha$$

Where α = Angular acceleration in rad/sec^2 and is equal to $d\omega/dt$.

PROBLEMS

Exapmle1. A car is moving at 72 kmph. If the wheels are 75 cm diameter, find the angular velocity of the tyre about its axis. If the car comes to rest in a distance of 20 meters, under a uniform retardation, find angular retardation of the wheels.

Given data

Linear velocity (v) = 72 kmph = 20 m/s

Diameter of wheel (d) = 75 cm or radius (r) = 0.375 m and

Distance travelled by the car (s) = 20 m.

Solution

Angular retardation of the wheel

We know that the angular velocity of the wheel,

$$\omega = \frac{v}{r} = \frac{20}{0.375} = 53.3 \text{ rad/sec}$$

Let a = Linear retardation of the wheel.

We know that $v^2 = u^2 + 2as$

$\therefore$ $0 = (20)^2 + 2 \times a \times 20 = 400 + 40a$

or $a = -\dfrac{400}{40} = -10 \text{ m/sec}^2$ ...(Minus sign indicates retardation)

We also know that the angular retardation of the wheel,

$$\alpha = \frac{a}{r} = \frac{-10}{0.375} = -26.7 \text{ rad/sec}^2$$

...(Minus sign indicates retardation)

Example2. The equation for angular displacement of a body moving on a circular path is given by θ = 2t3 + 0.5 where θ is in rad and t in sec. Find angular velocity, displacement and acceleration after 2 sec.

Given data

Equation for angular displacement θ = 2t3 + 0.5------(i)

Solution

Angular displacement after 2 seconds

Substituting $t = 2$ in equation (i),

$$\theta = 2\,(2)^3 + 0.5 = 16.5 \text{ rad}$$

Angular velocity after 2 seconds

Differentiating both sides equation (i) with respect to t,

$$\frac{d\theta}{dt} = 6\,t^2 \qquad \text{...}(ii)$$

velocity, $\omega = 6\,t^2$ ...(iii)

Substituting $t = 2$ in equation (iii),

$$\omega = 6\,(2)^2 = 24 \text{ rad/sec}$$

Angular acceleration after 2 seconds

Differentiating both sides of equation (iii) with respect to t,

$$\frac{d\omega}{dt} = 12t \text{ or Acceleration } \alpha = 12t$$

Now substituting $t = 2$ in above equation,

$$\alpha = 12 \times 2 = 24 \text{ rad/sec}^2$$

Example3. The equation for angular displacement of a particle, moving in a circular path (radius 200 m) is given by $\theta = 18t + 3t^2 - 2t^3$ where θ is the angular displacement at the end of t sec. Find (i) angular velocity and acceleration at start, (ii) time when the particle reaches its maximum angular velocity; and (iii) maximum angular velocity of the particle.

Given data

Equation for angular displacement $\theta = 18t + 3t2 - 2t^3$ ------------- (i)

Solution

(i) Angular velocity and acceleration at start

Differentiating both sides of equation (i) with respect to t, $d\theta / dt = 18 + 6t - 6t^2$

i.e. angular velocity, $\omega = 18 + 6t - 6t^2$ --------------(ii)

Substituting t = 0 in equation (ii),

$$\omega = 18 + 0 - 0 = 18 \text{ rad/s}$$

Differentiating both sides of equation (ii) with respect to t, $d\omega / dt = 6 - 12t$

i.e. angular acceleration, $\alpha = 6 - 12t$ ---------------- (iii)

Now substituting t = 0 in equation (iii),

$$\boldsymbol{\alpha = 6\ \text{rad/s}^2}$$

(ii) Time when the particle reaches maximum angular velocity

For maximum angular velocity, take equation (iii) and equate it to zero

$6 - 12t = 0$ or $t = 6 / 12$

$$\mathbf{t = 0.5\ seconds.}$$

(iii) Maximum angular velocity of the particle

The maximum angular velocity of the particle may now be found out by substituting t = 0.5 in equation (ii),

$$\omega_{max} = 18 + (6 \times 0.5) - 6\,(0.5)^2$$

$$\boldsymbol{\omega_{max} = 19.5\ \text{rad/s}}$$

ASSIGNMENT PROBLEMS - UNIT V

1. A car starts from rest with uniform acceleration of 0.6 m/sec^2. A second car B starts from the same point after 10 seconds. The car B follows the same route with an acceleration of 1.2 m/sec^2. Determine the time necessary to overcome the car A, and the distance covered when B passes A?
2. A motor is moving with a uniform acceleration covers a distance of 20 m in 4 seconds. Find the uniform acceleration of the motor.
3. A car starts from rest and uniformly accelerated to speed of 20 kmph over a distance of 200 m. calculate the acceleration and time taken. If further acceleration raises the speed to 50 kmph in 8 seconds, find the acceleration and further distance moved?
4. The motion of a particle is specified by the equations $x = 5t+0.75t^2$ and $y = 4t+0.6t^2$ where x and y are in meters and t is in seconds. Determine i) the path of the particle ii) velocity of the particle after 4 sec iii) acceleration of the particle after 2 sec.
5. The motion of a body moving on a curved path is given by the equations $x = 4 \sin 3t$ and $y = 4 \cos 3t$. Find the velocity and acceleration after 2 seconds.
6. A ball is thrown straight up from the top of a 64 foot tall building with an initial speed of 48 feet per second. The height of the ball as a function of time can be modeled by the function $h(t) = -16t^2 + 48t + 64$. How long will it take for the ball to hit the ground?
7. A projectile is aimed at a mark on the horizontal plane through the point of projection and falls 12 m short when the angle of projection is 15˚. When it is tried again it over shoots the mark by 24 m when the angle of projection is 45˚. Find the correct angle of projectile to hit the mark. Velocity of projection is constant in all the cases.
8. Two balls are projected from the same point in directions inclined at 60˚ and 30˚ to the horizontal. If they attain the same maximum height, what is the ratio of their velocities of projection?
9. A vehicle, of mass 500 kg, is moving with a velocity of 25 m/s. A force of 200 N acts on it for 2 minutes. Find the velocity of the vehicle: (1) when the force acts in the direction of motion, and (2) when the force acts in the opposite direction of the motion.
10. A constant retarding force of 50 N is applied to a body of mass 20 kg moving initially with a velocity of 15 m/s. How long the body will take to stop?

11. A stone is thrown vertically upward with a velocity of 29.4 m/sec. From the top of a tower of 49m height. Calculate (i) Time required for the stone to reach the ground (ii) Maximum height to which the stone will rise in its flight.
12. A car starts from, rest moves on a curved road of 300 m radius and accelerated at as constant tangential acceleration to 1 m/s^2. Find the time and distance when total acceleration becomes 2 m/s^2.
13. A trains starts from rest, moves with uniform acceleration for the first kilometer and attains the maximum velocity. With this velocity it moves for the next 6 kilometers and then comes to rest with uniform retardation. If the total journey is 9 kilometers and it takes 6 minutes for the train to complete it, calculate the maximum velocity attained.
14. A wheel, starting form rest, is accelerated at the rate of 5 rad/s^2 for a period of 10 seconds. It is then made to stop in the next 5 seconds by applying brakes. Find (a) maximum velocity attained by the wheel, and (b) total angle turned by the wheel.
15. A wheel rotates for 5 seconds with a constant angular acceleration and describes 80 radians. It then rotates with a constant angular velocity in the next 5 seconds and describes 100 radians. Find the initial angular velocity and angular acceleration of the wheel.
16. The relation between the angle of rotation (θ) in radians and time (t) in seconds of a rotating body is given by the equation. $\theta = 2t^3 + 3t^2 + 10$. Find displacement, angular velocity and angular acceleration after 4 seconds.