A Guide to Schillinger's Theory of Rhythm

Second Edition

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Frans Absil

2015

Colophon

This book is a guide to Book I *The Theory of Rhythm* from *The Schillinger System of Musical Composition*. It follows the chapters from the original book. Rhythm creation techniques will be discussed in detail and musical examples will be presented.

This document was created using the public domain LaTeX computer typesetting program. Diagrams were created using the LaTeX picture environment. Musical examples were created using the MakeMusic Finale 2014 music notation software. Score examples were imported into the document as PDF file format figures, using the `graphicx` package. The navigation links (printed in blue) in the PDF file (use a PDF reader) were created using the `hyperref` package from the LaTeX distribution.

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A Guide to Schillinger's Theory of Rhythm, Second Edition, Frans Absil

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Contents

List of Figures

Preface

This is the second edition of the book *A Guide to Schillinger's Theory of Rhythm*. This document describes a system for creating musical rhythm. Combining a mathematical basis with evolutionary growth processes provides various techniques for creating rhythmical patterns. The content will be useful for composers and arrangers that need a toolset for creating rhythms when they want to create alternatives to regular, short patterns and repetitive loops. These techniques are most useful for writing film, dance and computer game music.

A book about rhythm

The *Schillinger System of Musical Composition* [3] was published originally in 1946 and as a reprint in a 1640 pages two-volume book set in 1978. The books were compiled by the editors from source material by Joseph Schillinger (1895–1943) and from student notes. The system consists of 12 books on various aspects of musical composition. These books are not part of the standard literature; they are in general not being considered as serious studies in musicology. The mathematical notation and number theory approach did not help spreading the information. However, for a certain period they were considered as a toolbox for composers and arrangers in radio, film, dance and theatre music. The numerous techniques in these books will develop craftmanship.

The system begins with a book on the *Theory of Rhythm*. Rhythm is coordinated musical time; the theory of rhythm provides techniques for generating basic patterns, how to develop and vary these patterns using the analogy of natural growth processes, and how to apply the resultant rhythms to multiple parts in a musical score. The book also contains musings about rhythmical styles and evolution, and there are predictons about the emergence of future rhythm families. Book I from the Schillinger System consists of 14 chapters; this *Guide to Schillinger's Theory of Rhythm* closely follows the structure from the book and covers the chapter subjects using mathematical, graphical and musical notation.[1]

Why did I write this guide to the theory of rhythm? First, hardopies of the Schillinger books are hard to find; the volumes are out of print. Secondly, they are hard to read, due to the confusing mathematical notation, the exhaustive presentation of combinations and permutations of integer number sets.[2] Finally, some techniques definitely needed more examples to clarify the application. Although the guide started as a summary of the Schillinger

[1] Regarding the theory of rhythm there is a companion publication called *Encyclopedia of Rhythms* [4]. It is a reference volume with single and double staff notation of all possible rhythmical patterns, created within the Schillinger System.

[2] In the pre-computer era (the 1930s and 1940s) the many pages with numbers may have served as a handbook with lookup tables.

book on rhythm, the second edition of this guide is significantly longer than the original text with its 95 pages.

So this guide is both a replacement for and an addition to the original document. The mathematical and graphical notation were somewhat adapted, many new musical notation examples were created. The techniques and examples are explained in great detail, to the best of my understanding. Application tips are provided for various musical styles. Website statistics are evidence of the demand for such a book. The PDF version of the first edition received a significant number of monthly hits. The first edition was incomplete, covering only half of the chapters from the original Schillinger book on rhythm. Therefore the time seemed appropriate to complete this work and publish a second edition. Studying the techniques from the book will help the composer and arranger in creating, developing and applying rhythms. If you feel threatened by the danger of two-measure, repetitive rhythmical loops, then this guide may trigger your creativity and provide more than sufficient alternatives for creating interesting rhythms, that still contain homogeneity and consistency. When inspiration is failing, try this recipe book to overcome your writer's block.

Document update history

Here is an overview of the document history:

Vs. 1.1, September 2006: Chapter 1 to 4 completed (text plus figures), Chapter 6 text only.

Vs. 1.2, March 2011: Chapter 5 added, additions to Chapter 6, minor edits to other chapters. Total: 37 pages.

Second Edition, May 2015, Vs. 2.1: Chapters 7–14 added. Layout and styling updates. Additional examples and figures in Chapters 2–6. Errors corrected. Total: 159 pages in the full version.

Revision 2.2, September 2015: Errors corrected and additions, based on reader suggestions (mainly in Chapter 1, see the overview of the terminology, and in Chapter 7, text and figures). Mathematical notation improved for consistency and easier reading. Full version total: 167 pages.

Chapter 1

Introduction

The *Theory of Rhythm* from the *Schillinger System of Musical Composition* [3] deals with temporal aspects of music. Rhythm is coordinated use of music time through note attacks and durations. In the field of musical rhythm there are not many textbooks. However, here's a number of suggestions for books that discuss rhythm and temporal aspects of music: see [1, 2].

This introductory chapter will provide a structural overview of the content of this guide and familiarize the reader with the integer number approach to rhythm notation. The original Schillinger *Theory of Rhythm* contains 14 chapters that describe rhythmical aspects and techniques on different levels. Studying the many techniques from these chapters carries the risk of losing the total picture. Therefore we start with an overview that is not part of the original book.

1.1 Document structure overview

The overview of the document structure is shown in a diagram in Figure 1.1. In Schillinger's *Theory of Rhythm* three levels may be discerned:

1. There is the lowest level, the *source* level where rhythmical patterns are generated. There are several techniques for creating interference patterns from two or three clocks or metronomes. These metronome clocks will be ticking at constant time intervals, but variable speeds may also be used. The resultant rhythmical pattern from these *generators*, i.e., an *attack-duration group* is the output at this level.

2. Using the generator source pattern, there are various techniques that yield *variation* and *development*. Subdivision of the original rhythm attack-duration series, grouping patterns into measures at a specified time signature or meter, multiplying the resultant by a set of coefficients, all these approaches will create homogeneous variation and a rhythmical *continuity* at an intermediate level.

3. At the highest level there is the *application* of a rhythmical resultant to a musical instrument or a number of parts in a score. Patterns may be combined in parallel, distributed into a *simultaneity* over multiple staves. Then there is also the *evolution* of rhythms: from the basic patterns *families* of rhythms will evolve with coherent and unique characteristics. This brings along the aspect of rhythmical *style*.

In the diagram the chapters are positioned on these levels. Most chapters remain on a single level, but some of them will cover aspects on multiple levels. So, a clear separation is not possible, but the positioning will indicate the focus of the chapter. The diagram will help the reader to keep the big picture, while studying the detailed techniques in the book.

1.2 Terminology

In this book about rhythm we will use much of the original nomenclature from the *Schillinger System of Musical Composition* books. An overview of the most frequently used terminology, symbols and equivalent meaning in other music literature may help in reading the document and understanding the fundamental concepts. The list below may serve as a glossary of terms.

Generator. A *generator* is a rhythm source, creating note attacks at a fixed, constant time interval. Symbol: $A, B, \ldots$, with time intervals $a\Delta t, b\Delta t, \ldots$, respectively. The perfect example of a rhythm generator is a metronome, ticking at a specific Beats-Per-Minute (BPM) setting.

Attack. The *attack* is the beginning of a note, i.e., a new event in a rhythmical sequence. Symbol: a.[1] Note that often the note duration is implicit when using the term attack. So the rhythm $3 + 1 + 2 + 1 + 1 + 4$ consists of 6 attacks, or $N_a = 6$. In most chapters we will see attack groups in the time domain; however, they may also be applied as a pitch distribution series.

Resultant. The *resultant* is the output of a rhythm generation process, such as interference. Symbol r. The resultant consists of an attack-duration series, i.e., a number of note events with durations. It is a pattern in the time domain. So we will write $r = 3 + 1 + 2 + 2 + 1 + 3$ or $r = 3\Delta t + \Delta t + 2\Delta t + 2\Delta t + \Delta t + 3\Delta t$, an attack-duration series with 6 elements and a total duration of $T_r = 12\Delta t$, i.e., 12 time units.

Time unit. The *time unit* is the smallest division in the time domain. Symbol: Δt. Note durations and rhythms are expressed as integer multiples of this time unit. So a note duration may be $3\Delta t$, a single measure may contain $12\Delta t$ time units, and a rhythm written in short as $3+1+2+1+1+4$ consists of six notes with durations $3\Delta t + 1\Delta t + 2\Delta t + 1\Delta t + 1\Delta t + 4\Delta t$.

Periodicity. Many rhythmical phenomena have a *periodic* character; they will repeat after a certain, constant period of time. E.g., the rhythm $2 + 1 + 1 + 2 + 1 + 1 + 2 + 1 + 1$ is repeating after 4 time units. We may write this also as $(2+1+1)+(2+1+1)+(2+1+1)$ which helps in recognizing the periodic nature. In this example the period is $4\Delta t$; it is 4 time units long.

Interference. Rhythm is often the result of a combination of two or more periodic processes, running simultaneously, but each with a different period. Think of two metronomes, each with a different BPM setting. The combination of time events from the separate processes is called *interference*.

[1]Notation ambiguity cannot be totally avoided in this book. Here we have an example of such a double meaning. The symbol a represents both the note attack and the time interval setting of the rhythm generator A. The context should resolve the ambiguity.

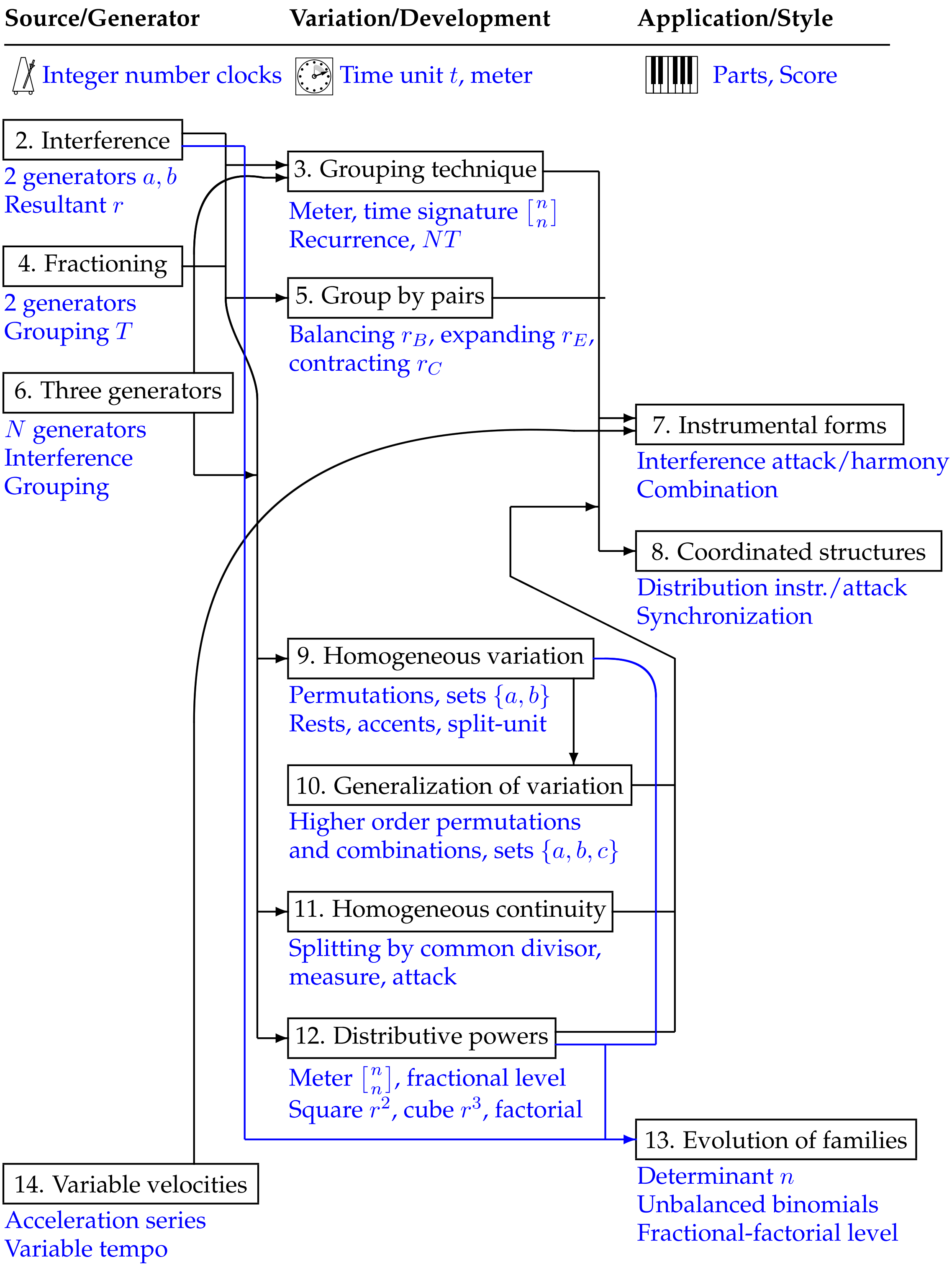

Figure 1.1: Overview of the book structure

Synchronization. The *synchronization* process is closely related to the *interference* phenomenon. Two or more periodic processes (clocks, metronomes) start at a certain time instant with a simultaneous first time event (an attack). When the duration of two periods have a simple ratio, then after a while both phenomena will return to the beginning of the pattern. For example, one rhythmic pattern has a duration of 3 time units, the other has the period $2\Delta t$, then after 6 time units both patterns return to the start of a new period. The first pattern will have played twice, the other pattern three times.

Grouping. Rhythms are *grouped* into fixed duration units, i.e., measures of a certain duration. Grouping is relevant for the notation of music, the counting of time units by musicians or in DAW software.

Measure, bar. A *measure* or *bar* of music (synonyms) is a rhythm time division unit. Measures are the result of regular grouping into units with duration T_M.

Meter, time signature. The *meter* and *time signature* are synonyms, indicating the subdivision of a measure into smaller units. The notation is $\left[{n \atop m}\right]$, with n and m two integer numbers. The lower integer m indicates the time unit Δt within the measure, n is the total number of time units in a single measure. So we have $T_M = nm = n\Delta t$. Examples are the $\begin{bmatrix} 4 \\ 4 \end{bmatrix}$ time signature, with the quarter note time unit and four beats to the measure, the $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ Waltz, and the $\begin{bmatrix} 6 \\ 8 \end{bmatrix}$ meter with six units of 8th notes in a bar.

Recurrence. The result of synchronization is the *recurrence* of the combined periodic phenomena after a certain time interval. Symbol: T_r, T_R.[2] See the example under synchronization, which repeats after $T_R = 6\Delta t$. Often we will be calculating the recurrence of a certain rhythm in combination with grouping into a meter. E.g., the resultant $r = 2 + 1$ with duration $T_r = 3\Delta t$, when grouped into a meter of two time units $T_M = 2\Delta t$ (one bar contains two time units) will have recurrence after $T_R = 6\Delta t = 3T_M$, i.e., after three full meaures.

Instrumental form. An *instrumental form* is a pattern in the pitch domain. An attack pattern is mapped onto an ordered set of pitches, either a melody (a sequence of pitches) or a harmonic structure (simultaneous pitches). In this book about rhythm an instrumental form is created by applying an attack-duration group (time domain) to a given pitch-attack distribution pattern (pitch domain). This may happen on a single staff or on multiple staves in a musical score.

Some of the terminology may seem puzzling now, but should become clear when studying the subsequent chapters in this book. A graphical representation of the technical terms is shown in diagram in Fig. 1.2.

1.3 Notation system

In his *Theory of Rhythm* Schillinger introduces alternative systems of rhythmic pattern notation: numbers, graphs and musical notes. This document will also use all three notation

[2] The symbol T_r is also the length of a resultant, an attack-duration series. Creating a rhythmic resultant from a binary synchronization process is also a recurrent process.

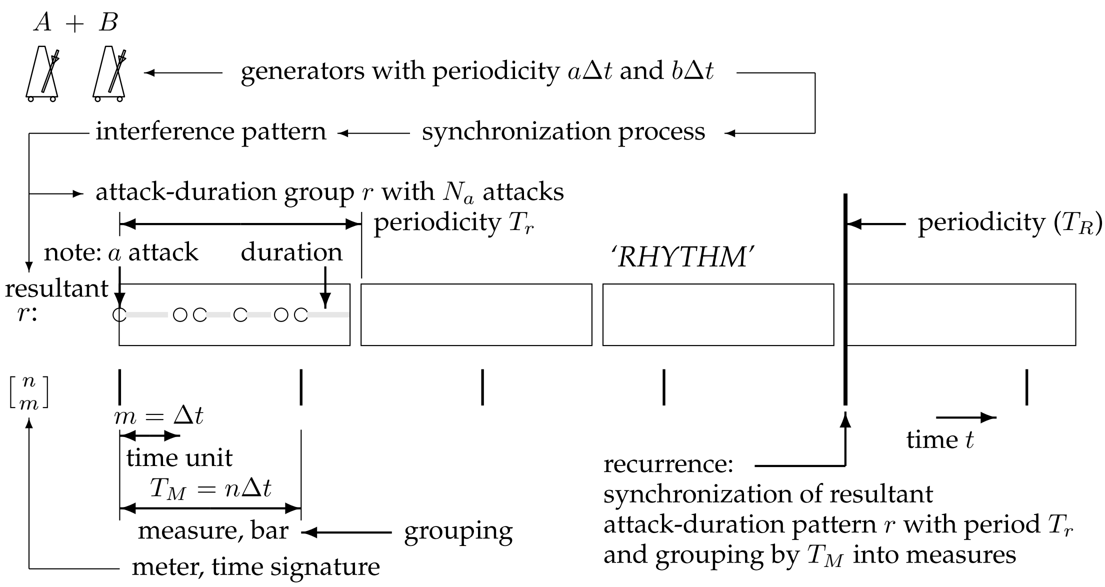

Figure 1.2: Overview of rhythm terminology in diagram.

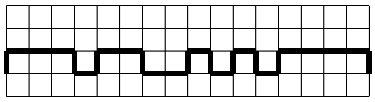

Figure 1.3: Plotting the rhythm $3+1+2+2+1+1+1+1+4$ on graph paper according to the Schillinger System books. A vertical line segment indicates an attack. The note duration is represented by the horizontal line segments.

systems. In the original book the usefulness of graphs is stressed, because of the visualization aspect.

1.3.1 Graphing music

In the book the analogy between acoustic waveforms (periodic patterns, displaying the sound amplitude vs. time) and durations (stressed accents) is used to introduce the square wave graphical notation of musical attacks. So the Schillinger books use the graphic representation of a rhythm, as shown in Fig. 1.3. On a regular grid the duration each note is represented by a horizontal line segment.

Here we will use the analogy of *clocks* or *metronomes* ticking (short pulses of sound) at regular intervals, as shown in Fig. 1.4. The *time instants* of the ticking will be represented as symbols (circles) in a diagram or as numbers. Most of the techniques (the mathematical processes, the arithmetic) will be done in integer number calculations; don't be afraid, this is all very simple and the analogy of the ticking metronomes will help to understand the concepts and results.

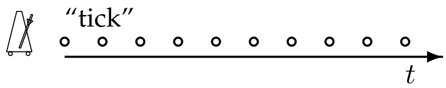

Figure 1.4: Plotting the time instants of a ticking metronome along a time axis

1.3.2 Integer number representation of ticking metronomes

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1.3.3 Forms of periodicity

Uniform periodicity can be achieved with a single metronome generating pulses at a constant rate. This is called *monomial periodicity*, since it is determined by a single coefficient. The pulses are generated at discrete time intervals Δt, i.e., the smallest rhythmical time unit. Attacks will occur at multiples of this smallest time unit. We can write the time instant of the i-th attack t_i (from a total series of N attacks or pulses) in mathematical form as follows

$$t_i = (i-1)\Delta t, \quad i = 1, 2, \ldots, N. \tag{1.1}$$

Note that the first attack occurs at $t = 0$ (and not $t = 1\,\Delta t$). That may seem a bit odd, but later we will see that this makes the arithmetic a lot easier to understand. We may represent the whole series of ticks as a *vector* $\vec{t}$ (note the small arrow over the symbol t) and therefore we write

$$\vec{t} = [0\ 1\ 2\ \ldots\ (N-1)] \cdot \Delta t. \tag{1.2}$$

Let us look at an example of uniform periodicity with ticking metronomes.

Example 1.1

Monomial periodicity: attack series.
The idea of attack series will be illustrated by considering three ticking metronomes, each ticking at a fixed time interval Δt.

- Consider metronome A ticking five times at intervals of one time unit (e.g., 1 second intervals). Then we have $N = 5$ and $\Delta t = 1$ and the series of attacks is written as[3]

  $$\vec{t}_A = [0\ 1\ 2\ 3\ 4].$$

  The tick pattern is sketched in Fig. 1.5. This is the *four-on-the-floor* kick drum pattern in electronic dance music (EDM).

- Another metronome B, generating 11 ($N = 11$) pulses at three time unit intervals ($\Delta t = 3$) will yield an attack series

  $$\vec{t}_B = [0\ 1\ 2\ \ldots\ 10] \cdot 3 = [0\ 3\ 6\ \ldots\ 30].$$

  The tick pattern is shown in the figure above the A metronome. This could be the pizzicato contrabass pattern on the downbeat of every measure in a waltz.

- As a last example we will consider metronome C generating N pulses at n time unit intervals. The number N has no specific value, and this in general means that we have a very long or infinitely long sequence of ticks. In that case the attack series is

  $$\vec{t}_C = [0\ 1\ 2\ \ldots\ N-1] \cdot n = [0\ n\ 2n\ \ldots\ (N-1)n].$$

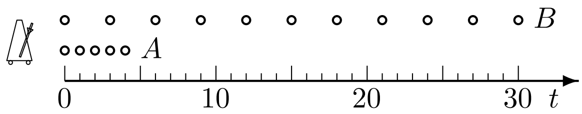

Figure 1.5: Two metronomes A and B ticking at different time intervals $\Delta t_A = 1$ and $\Delta t_B = 3$ time units. Tickmarks on the time axis represent the time instants.

Note that we have only indicated the time instant of the beginning of a musical event, i.e., an attack (e.g., the staccato tones from a xylophone). The notation in the Schillinger book uses $+$-signs between the elements in an attack series, because the numbers in the series there also indicate the *duration* of the attacks. The duration of an attack a is the time interval between two ticks, i.e., $a = t_{i+1} - t_i = \Delta t$.[4] As an analogy, consider an electronic keyboard, where you would press a specific key at the abovementioned time instant and keep the key depressed, until the next time event occurs. The result obviously is a series of repeated pitches with a specific duration, i.e., an attack-duration group. If the duration series is meant we will use the series with the $+$-signs between the terms.

Example 1.2

Monomial periodicity: duration series.
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[3] In general, the ticking interval for a generator (metronome) A is written as $a\Delta t$ with reference to a common smallest time unit Δt. In this and part of other chapters we write for brevity $\Delta t_A = a\Delta t$.

[4] Setting the note length, i.e., the duration, equal to the ticking interval implies a legato playing. There are no rests or pauses between the metronome ticks, such as in staccato playing.

Chapter 2

Interference of periodicities

This chapter introduces a basic technique for generating *attack series*; in the Schillinger text these are also called an *attack group*. The technique is based in the interference pattern that results when two clocks or metronomes tick at a different constant time interval. These attack series may then be grouped using a specified number of time units per measure.

2.1 Binary synchronization

Suppose there are two clocks or metronomes A and B ticking at different constant time 2
intervals $\Delta t_A = a\Delta t$ and $\Delta t_B = b\Delta t$, where a and b are integer numbers and Δt is the musical time unit (e.g., a quarter note $\frac{1}{4}$ or an 8th note $\frac{1}{8}$). We assume that metronome B is ticking faster than metronome A, and therefore $\Delta t_B < \Delta t_A$. Metronome B is called the *minor generator*, metronome A is called the *major generator*.

Starting these two clocks at the same time instant will yield an attack pattern, called *the resultant* r. The attack pattern will repeat after $T_r = ab\Delta t$ time units. The process of combining the two metronomes, the two monomial periodicities, is called *interference* and since there are two clocks we call this *binary synchronization*. When metronome A and B produce an attack series, the interference process is notated as $a \div b$ and the resultant attack-duration series is written as $r_{a \div b}$.

We determine the resultant time series by finding the combination (in mathematical terms, the *union*) of the two attack series $\vec{t}_A$ and $\vec{t}_B$, i.e.,

$$\vec{t}_r = \vec{t}_A \cup \vec{t}_B. \tag{2.1}$$

Given the attack series $\vec{t}_r$ we determine the duration series r by listing the difference between two consecutive terms in the attack series

$$r_i = t_{r,i+1} - t_{r,i}. \tag{2.2}$$

2.1.1 Uniform binary synchronization

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The figure is included in the full version

Figure 2.1: Uniform binary synchronization

Table 2.1: List of generator combinations for non-uniform binary synchronization. The first number is the major generator value a, the second number is the minor generator value b

Major	Interference group (combination of generators)						
3:	$3 \div 2$						
4:	$(2 \div 1)$	$4 \div 3$					
5:	$5 \div 2$	$5 \div 3$	$5 \div 4$				
6:	$(3 \div 1)$	$(2 \div 1)$	$(3 \div 2)$	$6 \div 5$			
7:	$7 \div 2$	$7 \div 3$	$7 \div 4$	$7 \div 5$	$7 \div 6$		
8:	$(4 \div 1)$	$8 \div 3$	$(2 \div 1)$	$8 \div 5$	$(4 \div 3)$	$8 \div 7$	
9:	$9 \div 2$	$(3 \div 1)$	$9 \div 4$	$9 \div 5$	$(3 \div 2)$	$9 \div 7$	$9 \div 8$

2.1.2 Non-uniform binary synchronization

In the case of *non-uniform binary synchronization* we have two metronomes A and B with $\Delta t_A > \Delta t_B$ (both an integer number) and $\Delta t_B \neq 1$. The resultant attack series is repeated 2 
after $T_r = \Delta t_r = \Delta t_A \Delta t_B = ab\Delta t$ time units and the attack-duration group r will contain non-equal durations.

On [3] p. 14 in the Schillinger book we find the list of practical combinations of generators, here reproduced (somewhat re-arranged and complemented) as Table 2.1. We note that only the lower left triangular region of the table is filled; this is due to the fact that we must have $\Delta t_A > \Delta t_B$. Then there is a number of terms between brackets, e.g., $(2 \div 1)$. These are either uniform synchronization cases (where the minor generator value $b = 1$), or combinations that can be reduced to simpler ratios using the common denominator. For example for $\Delta t_A = 4, (a = 4)$ and $\Delta t_B = 2, (b = 2)$ we may write

$$a \div b = 4 \div 2 = (2 \cdot 2) \div (1 \cdot 2) = 2 \div 1,$$

and for $\Delta t_A = 6, (a = 6)$ and $\Delta t_B = 4, (b = 4)$ we may write

$$a \div b = 6 \div 4 = (3 \cdot 2) \div (2 \cdot 2) = 3 \div 2.$$

Now we will consider the combinations listed in the table and determine the resultant r for each pair (a, b). These will be shown in graphical and in musical notation. The musical notation will be shown in Chapter 3, when the aspect of grouping has been discussed in more detail.

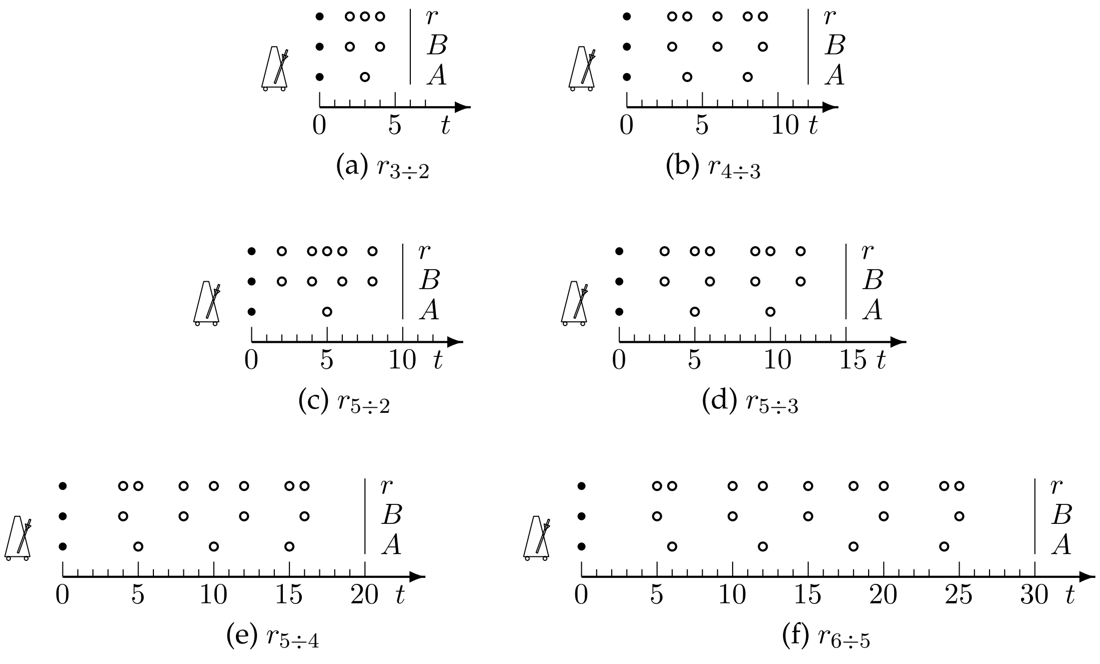
Interference generators: 3 and 2

For $\Delta t_A = 3$ and $\Delta t_B = 2$ and using Eq. 1.2 we get the attack series

$$\vec{t}_A = [0\ 3], \quad \vec{t}_B = [0\ 2\ 4],$$

repeating itself after $T_r = ab\Delta t = 3 \cdot 2 = 6$ time units. We determine the resultant from the combination of these two attack series, Eq. 2.1, which yields the following attack and duration series

$$\vec{t}_r = [0\ 2\ 3\ 4], \quad r = 2 + 1 + 1 + 2,$$



(a) $r_{3\div2}$ (b) $r_{4\div3}$ (c) $r_{5\div2}$ (d) $r_{5\div3}$ (e) $r_{5\div4}$ (f) $r_{6\div5}$

Figure 2.2: Non-uniform binary synchronization. The resultant $r_{a\div b}$ of two generators A and B that tick at different time intervals. a): $\Delta t_A = 3, \Delta t_B = 2$, b): $\Delta t_A = 4, \Delta t_B = 3$, c): $\Delta t_A = 5, \Delta t_B = 2$, d): $\Delta t_A = 5, \Delta t_B = 3$, e): $\Delta t_A = 5, \Delta t_B = 4$, f): $\Delta t_A = 6, \Delta t_B = 5$ time units. Accented attacks are indicated by closed circles.

i.e., four attacks ($N_a = 4$) and two note duration values. The non-uniform generator interference process is illustrated in Fig. 2.2.a. The diagram shows the attacks as a repeating series of dots and circles along a time axis t. The top row shows the resultant r, obtained from the union of the two generator attack series.

Interference generators: 4 and 3

For $\Delta t_A = 4$ and $\Delta t_B = 3$ and using Eq. 1.2 we get the attack series

$$\vec{t_A} = [0\ 4\ 8], \quad \vec{t_B} = [0\ 3\ 6\ 9],$$

with recurrence after $T_r = 4 \cdot 3 = 12$ time units. We determine the resultant from the combination of these two attack series, Eq. 2.1, which yields the following attack and duration series

$$\vec{t_r} = [0\ 3\ 4\ 6\ 8\ 9], \quad r = 3 + 1 + 2 + 2 + 1 + 3,$$

i.e., six attacks ($N_a = 6$) and three note duration values. The synchronization resultant r is shown in Fig. 2.2.b. When we discuss grouping in Chapter 3, we wll see that the total pattern duration $T_r = 12\Delta t$ is convenient, since the number can be divided by 2, 3, 4 and 6.

Interference generators: 5 and 2

With $\Delta t_A = 5$ and $\Delta t_B = 2$ as input into Eq. 1.2 we get the attack series

$$\vec{t_A} = [0\ 5], \quad \vec{t_B} = [0\ 2\ 4\ 6\ 8],$$

with total pattern length $T_r = 5 \cdot 2 = 10$ time units. We determine the resultant from the union of these two attack series, Eq. 2.1, which yields the following attack and duration series

$$\vec{t}_r = [0\ 2\ 4\ 5\ 6\ 8], \quad r = 2 + 2 + 1 + 1 + 2 + 2,$$

i.e., $N_a = 6$ and two note duration values. The resulting pattern is shown in Fig. 2.2.c.

Interference generators: 5 and 3

With $\Delta t_A = 5$ and $\Delta t_B = 3$ Eq. 1.2 the two attack series becomes

$$\vec{t}_A = [0\ 5\ 10], \quad \vec{t}_B = [0\ 3\ 6\ 9\ 12],$$

with recurrence after $T_r = 5 \cdot 3 = 15$ time units. The union of these two attack series, Eq. 2.1 yields the attack and duration series

$$\vec{t}_r = [0\ 3\ 5\ 6\ 9\ 10\ 12], \quad r = 3 + 2 + 1 + 3 + 1 + 2 + 3,$$

i.e., $N_a = 7$ and three note duration values. The resultant rhythm is shown in Fig. 2.2.d.

Interference generators: 5 and 4

The final case for this major generator is $\Delta t_A = 5$ and $\Delta t_B = 4$. Using Eq. 1.2 leads to the attack series

$$\vec{t}_A = [0\ 5\ 10\ 15], \quad \vec{t}_B = [0\ 4\ 8\ 12\ 16],$$

with total pattern duration $T_r = 5 \cdot 4 = 20$ time units. With Eq. 2.1 we find the following attack and duration series

$$\vec{t}_r = [0\ 4\ 5\ 8\ 10\ 12\ 15\ 16], \quad r = 4 + 1 + 3 + 2 + 2 + 3 + 1 + 4,$$

i.e., $N_a = 8$ and four note duration values. The resulting pattern is shown in Fig. 2.2.e. The major generator $a = 5$ was combined with three values of the minor generator $b = \{2, 3, 4\}$. Comparing these cases we note that the latter number is determining the range of duration values and the number of attacks in the series; the resultant series contains more elements when b increases and consists of duration values between $1\Delta t$ and $b\Delta t$.

Interference generators: 6 and 5

For $\Delta t_A = 6$ and $\Delta t_B = 5$ and using Eq. 1.2 we get the attack series

$$\vec{t}_A = [0\ 6\ 12\ 18\ 24], \quad \vec{t}_B = [0\ 5\ 10\ 15\ 20\ 25],$$

repeating itself after $T_r = 6 \cdot 5 = 30$ time units. We determine the resultant from the union of these two attack series, using Eq. 2.1, which yields the following attack and duration series

$$\vec{t}_r = [0\ 5\ 6\ 10\ 12\ 15\ 18\ 20\ 24\ 25], \quad r = 5 + 1 + 4 + 2 + 3 + 3 + 2 + 4 + 1 + 5,$$

i.e., $N_a = 10$ and five note duration values. The resultant rhythm is shown in Fig. 2.2.f.

Interference generators: 7 and 2

When $\Delta t_A = 7$ and $\Delta t_B = 2$, Eq. 1.2 yields the attack series

$$\vec{t_A} = [0\ 7], \quad \vec{t_B} = [0\ 2\ 4\ 6\ 8\ 10\ 12],$$

with recurrence after $T_r = 7 \cdot 2 = 14$ time units. The resultant from Eq. 2.1 leads to the following attack and duration series

$$\vec{t_r} = [0\ 2\ 4\ 6\ 7\ 8\ 10\ 12], \quad r = 2+2+2+1+1+2+2+2,$$

i.e., $N_a = 8$ and two note duration values.

Interference generators: 7 and 3

With $\Delta t_A = 7$ and $\Delta t_B = 3$ as input into Eq. 1.2 the attack series is

$$\vec{t_A} = [0\ 7\ 14], \quad \vec{t_B} = [0\ 3\ 6\ 9\ 12\ 15\ 18],$$

repeating after $T_r = 7 \cdot 3 = 21$ time units. From the combination of these two attack series, using Eq. 2.1, the resultant attack and duration series becomes

$$\vec{t_r} = [0\ 3\ 6\ 7\ 9\ 12\ 14\ 15\ 18], \quad r = 3+3+1+2+3+2+1+3+3,$$

i.e., $N_a = 9$ and three note duration values.

Interference generators: 7 and 4

For $\Delta t_A = 7$ and $\Delta t_B = 4$ Eq. 1.2 produces the attack series

$$\vec{t_A} = [0\ 7\ 14\ 21], \quad \vec{t_B} = [0\ 4\ 8\ 12\ 16\ 20\ 24],$$

repeating itself after $T_r = 7 \cdot 4 = 28$ time units. Determine the resultant from the combination of these two attack series, Eq. 2.1 and find the attack and duration series

$$\vec{t_r} = [0\ 4\ 7\ 8\ 12\ 14\ 16\ 20\ 21\ 24], \quad r = 4+3+1+4+2+2+4+1+3+4,$$

i.e., $N_a = 10$ and four note duration values.

Interference generators: 7 and 5

With $\Delta t_A = 7$ and $\Delta t_B = 5$ and using Eq. 1.2 the two attack series are

$$\vec{t_A} = [0\ 7\ 14\ 21\ 28], \quad \vec{t_B} = [0\ 5\ 10\ 15\ 20\ 25\ 30],$$

with recurrence after $T_r = 7 \cdot 5 = 35$ time units. The union of both according to Eq. 2.1 gives the resultant attack and duration series

$$\vec{t_r} = [0\ 5\ 7\ 10\ 14\ 15\ 20\ 21\ 25\ 28\ 30], \quad r = 5+2+3+4+1+5+1+4+3+2+5,$$

i.e., $N_a = 11$ and five note duration values.

Interference generators: 7 and 6

The final case for this major generator is $\Delta t_A = 7$ and $\Delta t_B = 6$ time units. Using Eq. 1.2 the pair of attack series is

$$\vec{t}_A = [0\ 7\ 14\ 21\ 28\ 35], \quad \vec{t}_B = [0\ 6\ 12\ 18\ 24\ 30\ 36],$$

with total pattern duration $T_r = 7 \cdot 6 = 42$ time units. We determine the resultant from the combination of these two attack series, Eq. 2.1, which yields the following attack and duration series

$$\vec{t}_r = [0\ 6\ 7\ 12\ 14\ 18\ 21\ 24\ 28\ 30\ 35\ 36], \quad r = 6+1+5+2+4+3+3+4+2+5+1+6,$$

i.e., $N_a = 12$ and six note duration values. Major generator $a = 7$ enables five combinations $b = \{2, 3, 4, 5, 6\}$. Again note the dependence of the resultant pattern length and the range of note durations on the value of the minor generator b.

Interference generators: 8 and 3

For $\Delta t_A = 8$ and $\Delta t_B = 3$ and using Eq. 1.2 we get the attack series

$$\vec{t}_A = [0\ 8\ 16], \quad \vec{t}_B = [0\ 3\ 6\ 9\ 12\ 15\ 18\ 21],$$

repeating itself after $T_r = 8 \cdot 3 = 24$ time units. We determine the resultant from the combination of these two attack series, Eq. 2.1, which yields the following attack and duration series

$$\vec{t}_r = [0\ 3\ 6\ 8\ 9\ 12\ 15\ 16\ 18\ 21], \quad r = 3+3+2+1+3+3+1+2+3+3,$$

i.e., $N_a = 10$ and three note duration values. The pattern duration $T_r = 24\Delta$ provides many grouping options, since the number 24 is divisible by 2, 3, 4., 6, 8 and 12.

Interference generators: 8 and 5

When $\Delta t_A = 8$ and $\Delta t_B = 5$, Eq. 1.2 produces the two attack series

$$\vec{t}_A = [0\ 8\ 16\ 24\ 32], \quad \vec{t}_B = [0\ 5\ 10\ 15\ 20\ 25\ 30\ 35],$$

repeating itself after $T_r = 8 \cdot 5 = 40$ time units. We determine the resultant from the combination of these two attack series, Eq. 2.1, which yields the following attack and duration series

$$\vec{t}_r = [0\ 5\ 8\ 10\ 15\ 16\ 20\ 24\ 25\ 30\ 32\ 35], \quad r = 5+3+2+5+1+4+4+1+5+2+3+5,$$

i.e., $N_a = 12$ and five note duration values.

Interference generators: 8 and 7

The third and final case for major generator $a = 8$ is $\Delta t_A = 8$ and $\Delta t_B = 7$ and using Eq. 1.2 we get the attack series

$$\vec{t}_A = [0\ 8\ 16\ 24\ 32\ 40\ 48], \quad \vec{t}_B = [0\ 7\ 14\ 21\ 28\ 35\ 42\ 49],$$

with recurrence after $T_r = 8 \cdot 7 = 56$ time units. Ths is a pattern of considerable length. Determine the resultant from the combination of these two attack series, Eq. 2.1, and find the following attack and duration series

$$\vec{t}_r = [0\ 7\ 814\ 16\ 21\ 24\ 28\ 32\ 35\ 40\ 42\ 48\ 49], \quad r = 7{+}1{+}6{+}2{+}5{+}3{+}4{+}4{+}3{+}5{+}2{+}6{+}1{+}7,$$

i.e., $N_a = 14$ and seven note duration values.

Interference generators: 9 and 2

The last practical major generator value is $a = 9$, with five possible combinations for non-uniform binary synchronization. For $\Delta t_A = 9$ and $\Delta t_B = 2$ and using Eq. 1.2 we get the attack series

$$\vec{t}_A = [0\ 9], \quad \vec{t}_B = [0\ 2\ 4\ 6\ 8\ 10\ 12\ 14\ 16],$$

repeating itself after $T_r = 9 \cdot 2 = 18$ time units. The resultant from the union of these two attack series according to Eq. 2.1 consists of the attack and duration series

$$\begin{aligned} \vec{t}_r &= [0\ 2\ 4\ 6\ 8\ 9\ 10\ 12\ 14\ 16], \\ r &= 2 + 2 + 2 + 2 + 1 + 1 + 2 + 2 + 2 + 2, \end{aligned}$$

i.e., $N_a = 10$ and two note duration values.

Interference generators: 9 and 4

With $\Delta t_A = 9$ and $\Delta t_B = 4$ Eq. 1.2 produces the attack series

$$\vec{t}_A = [0\ 9\ 18\ 27], \quad \vec{t}_B = [0\ 4\ 8\ 12\ 16\ 20\ 24\ 28\ 32],$$

with recurrence after $T_r = 9 \cdot 4 = 36$ time units. The grouping options are many, since 36 allows division by 2, 3, 4, 6, 9 and 12. Determine the combination of these two attack series, Eq. 2.1 and find the resultant attack and duration series

$$\begin{aligned} \vec{t}_r &= [0\ 4\ 8\ 9\ 12\ 16\ 18\ 20\ 24\ 27\ 28\ 32], \\ r &= 4 + 4 + 1 + 3 + 4 + 2 + 2 + 4 + 3 + 1 + 4 + 4, \end{aligned}$$

i.e., $N_a = 12$ and four note duration values.

Interference generators: 9 and 5

Enter $\Delta t_A = 9$ and $\Delta t_B = 5$ into Eq. 1.2 and find the pair of attack series

$$\vec{t}_A = [0\ 9\ 18\ 27\ 36], \quad \vec{t}_B = [0\ 5\ 10\ 15\ 20\ 25\ 30\ 35\ 40],$$

with total pattern duration $T_r = 9 \cdot 5 = 45$ time units. With Eq. 2.1 the resultant attack and duration series is

$$\begin{aligned}\vec{t_r} &= [0\ 5\ 9\ 10\ 15\ 18\ 20\ 25\ 27\ 30\ 35\ 36\ 40],\\ r &= 5+4+1+5+3+2+5+2+3+5+1+4+5,\end{aligned}$$

i.e., $N_a = 13$ and five note duration values.

Interference generators: 9 and 7

For $\Delta t_A = 9$ and $\Delta t_B = 7$ and using Eq. 1.2 the generator attack series are

$$\vec{t_A} = [0\ 9\ 18\ 27\ 36\ 45\ 54], \quad \vec{t_B} = [0\ 7\ 14\ 21\ 28\ 35\ 42\ 49\ 56],$$

repeating itself after $T_r = 9 \cdot 7 = 63$ time units. The combination of these two attack series according to Eq. 2.1 yields the attack and duration series

$$\begin{aligned}\vec{t_r} &= [0\ 7\ 9\ 14\ 18\ 21\ 27\ 28\ 35\ 36\ 42\ 45\ 49\ 54\ 56]\\ r &= 7+2+5+4+3+6+1+7+1+6+3+4+5+2+7,\end{aligned}$$

i.e., $N_a = 15$ and seven note duration values.

Interference generators: 9 and 8

Finally, when $\Delta t_A = 9$ and $\Delta t_B = 8$ using Eq. 1.2 produces the attack series

$$\vec{t_A} = [0\ 9\ 18\ 27\ 36\ 45\ 54\ 63], \quad \vec{t_B} = [0\ 8\ 16\ 24\ 32\ 40\ 48\ 56\ 64],$$

repeating itself after $T_r = 9 \cdot 8 = 72$ time units. Although this is a long pattern, it allows many regular groupings, since 72 can be divided by 3, 4, 6, 8, 9 and 12.

The union of these generator attack series, Eq. 2.1, leads to the resultant attack and duration series

$$\begin{aligned}\vec{t_r} &= [0\ 8\ 9\ 16\ 18\ 24\ 27\ 32\ 36\ 40\ 45\ 48\ 54\ 56\ 63\ 64]\\ r &= 8+1+7+2+6+3+5+4+4+5+3+6+2+7+1+8,\end{aligned}$$

i.e., $N_a = 16$ and eight note duration values. The five combinations for major generator $a = 9$ once again show the effect of the value of the minor generator b on total pattern length, number of attacks and range of durations in the resultant series.

Considering all cases for larger values of the major generator, say $a > 5$, there is also a sense of *balance* within the resultant; this is mainly determined by the value of the ratio of the longest to shortest duration. For example, take the $(a, b) = (9, 8)$ generator combination, this ratio is $8:1$. This implies that as the minor generator value b increases, the patterns become more unbalanced. Another determining factor for internal balance is the succession of two duration values with a large difference, such as $5+1, 8+1, 7+2$.

2.1.3 Overview of non-uniform binary synchronization and fractioning resultants

Table 2.2 presents an overview of the most important characteristics of the non-uniform binary synchronization attack-duration groups. For each combination of generators $\{a, b\}$ the resultant pattern r has total length T_r, and consists of N_a attacks. Also the distribution of the note durations is shown: e.g., $N_{2\Delta t}$ is the number of notes in the pattern with length 2 time units.

Use this table when looking for an attack-duration series with specific characteristics. The second half of the table presents these characteristics for the resultants r obtained through the fractioning technique, discussed in Chapter 4.

2.2 Grouping

Grouping is the selection of the *meter* or *time signature*; the resultant will be divided into a number of *measures* with each measure containing a fixed number of time units. In the case of binary synchronization there are three options for grouping, discussed in the following subsections.

Note that the next chapter, Ch. 3, is completely devoted to the aspect of grouping. In addition, note that apart from the three options mentioned here, Schillinger also considers what he calls *alien measure grouping* (see [3], Ch. 7, p. 33).

2.2.1 Grouping by the common product

Table 2.2: Overview of the characteristics of the resultants r for non-uniform binary synchronization and fractioning attack-duration groups. For combinations of two generators $\{a, b\}$ the total duration of the pattern T_r, the number of atacks in the series N_a and the distribution of the note duration values $N_{n\Delta t}$ are listed.

Resultant			**Note duration distribution**							
$r_{a \div b}$	T_r	N_a	$N_{\Delta t}$	$N_{2\Delta t}$	$N_{3\Delta t}$	$N_{4\Delta t}$	$N_{5\Delta t}$	$N_{6\Delta t}$	$N_{7\Delta t}$	$N_{8\Delta t}$
Non-uniform binary synchronization										
$3 \div 2$	6	4	2	2						
$4 \div 3$	12	6	2	2	2					
$5 \div 2$	10	6	2	4						
$5 \div 3$	15	7	2	2	3					
$5 \div 4$	20	8	2	2	2	2				
$6 \div 5$	30	10	2	2	2	2	2			
$7 \div 2$	14	8	2	6						
$7 \div 3$	21	9	2	2	5					
$7 \div 4$	28	10	2	2	2	4				
$7 \div 5$	35	11	2	2	2	2	3			
$7 \div 6$	42	12	2	2	2	2	2	2		
$8 \div 3$	24	10	2	2	6					
$8 \div 5$	40	12	2	2	2	2	4			
$8 \div 7$	56	14	2	2	2	2	2	2	2	
$9 \div 2$	18	10	2	8						
$9 \div 4$	36	12	2	2	2	6				
$9 \div 5$	45	13	2	2	2	2	5			
$9 \div 7$	63	15	2	2	2	2	2	2	3	
$9 \div 8$	72	16	2	2	2	2	2	2	2	2
$r_{\underline{a \div b}}$	T_r	N_a	$N_{\Delta t}$	$N_{2\Delta t}$	$N_{3\Delta t}$	$N_{4\Delta t}$	$N_{5\Delta t}$	$N_{6\Delta t}$	$N_{7\Delta t}$	$N_{8\Delta t}$
Fractioning										
$\underline{3 \div 2}$	9	7	5	2						
$\underline{4 \div 3}$	16	10	6	2	2					
$\underline{5 \div 2}$	25	21	4	21						
$\underline{5 \div 3}$	25	17	19	4	2					
$\underline{5 \div 4}$	25	13	7	2	2	2				
$\underline{6 \div 5}$	36	16	8	2	2	2	2			
$\underline{7 \div 3}$	49	37	29	4	4					
$\underline{7 \div 4}$	49	31	21	4	4	2				
$\underline{7 \div 5}$	49	25	9	12	2		2			
$\underline{7 \div 6}$	49	19	9	2	2	2	2	2		

2.2.2 Superimposition of a

Superimposition of a means that the major generator A will determine the meter. A single measure (bar) will have the duration

$$T_M = \Delta t_A = a\,\Delta t. \tag{2.3}$$

The selection of the grouping time unit T_M is independent of the generator synchronization process; it is a new degree of freedom. There is relation to the rhythm generators through the use of the factors a and b. But, as this section shows, we have several alternatives with measures of longer or shorter duration. Refer to the diagram in Fig. 1.2 to see the relation between the attack-duration group (the rhythmic pattern) and the grouping process.

2.2.3 Superimposition of b

Superimposition of b means that the minor generator B will determine the time signature

$$T_M = \Delta t_B = b\,\Delta t. \tag{2.4}$$

Now, obviously there are less time units in a measure, compared to the grouping by the major generator.

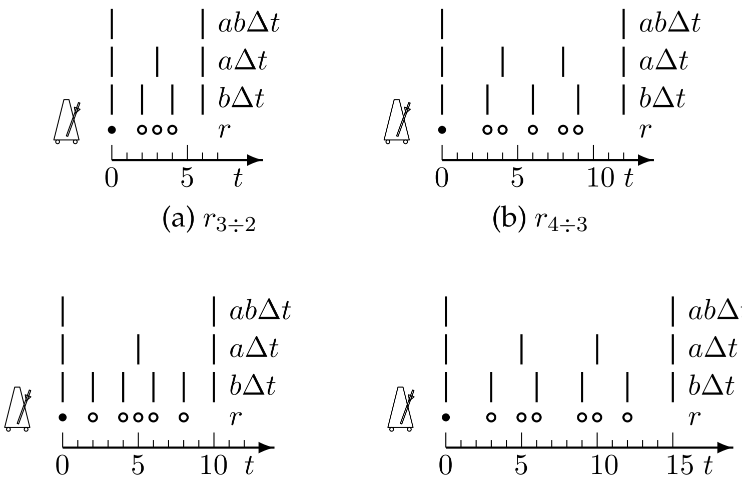
2.2.4 Alien measure grouping

In the case of non-uniform binary synchronization the potential of alien measure grouping is determined by the set of divisors of the total pattern length T_r. For example, the combination $(a, b) = (6, 5)$ has total duration $T_r = 30\Delta t$ time units and can be factorized into $30 = 1\times 30 = 2 \times 15 = 3 \times 10 = 5 \times 6$. The last factorization pair enables regular grouping and time signature, such as $\left[\frac{5}{4}\right]$ and $\left[\frac{6}{8}\right]$ meter. The first factorization (30) is impractical. The second and third factorization pairs yield alien groupings and time signatures, such as the regular $\left[\frac{2}{4}\right]$ and $\left[\frac{3}{8}\right]$ meter, and and the irregular $\left[\frac{10}{8}\right]$ and $\left[\frac{15}{8}\right]$ meters.

Example 2.1

Grouping and superimposition.
The grouping and superimposition process is illustrated in Fig. 2.3. Four non-uniform binary synchronization cases $r_{a \div b}$ are considered, each with the three possible grouping mechanisms: grouping by the common product $ab\Delta t$, superimposition of the major generator $a\Delta t$, and superimposition of the minor generator $b\Delta t$. The synchronization process is discusssed in Section 2.1.2 and the resultants were already shown in Fig. 2.2.

- Case 1: $r_{3\div 2}$. This non-uniform binary synchronization pattern leads to three possible time signatures, determined by the grouping approach: grouping by $T_M = ab\Delta t = 6\Delta t$ and meter $\left[\frac{6}{8}\right]$, superimposition of $T_M = a\Delta t = 3\Delta t$ with meter $\left[\frac{3}{4}\right]$ (waltz), and superimposition of $T_M = b\Delta t = 2\Delta t$ and meter $\left[\frac{2}{4}\right]$. The meter for these three alternatives is determined by the value of the time unit; here we have chosen $t = \frac{1}{8}, \frac{1}{4}$ and $\frac{1}{4}$, respectively (8th note and quarter note).



(a) $r_{3\div2}$ (b) $r_{4\div3}$

(c) $r_{5\div2}$ (d) $r_{5\div3}$

Figure 2.3: Grouping and superimposition for non-uniform binary synchronization attack series $r_{a\div b}$. Four cases are shown. a): $r_{3\div2}$, b): $r_{4\div3}$, c): $r_{5\div2}$, d): $r_{5\div3}$. For each case three groupings are shown: grouping by the common product $ab\Delta t$, superimposition of the major generator $a\Delta t$, and superimposition of the minor generator $b\Delta t$. This yields three meters or time signatures per case.

- Case 2: $r_{4\div3}$. This yields three groupings and time signature proposals: grouping by $T_M = ab\Delta t = 12\Delta t$ and time signature $\left[\begin{smallmatrix}12\\8\end{smallmatrix}\right]$, superimposition of $T_M = a\Delta t = 4\Delta t$ with time signature $\left[\begin{smallmatrix}4\\4\end{smallmatrix}\right]$, and superimposition of $T_M = b\Delta t = 3\Delta t$, e.g., $\left[\begin{smallmatrix}3\\4\end{smallmatrix}\right]$ meter.
- Case 3: $r_{5\div2}$. With time unit $t = \frac{1}{8}$ the three groupings and potential time signatures are: grouping by $T_M = ab\Delta t = 10\Delta t$ and $\left[\begin{smallmatrix}5\\4\end{smallmatrix}\right]$, superimposition of $T_M = a\Delta t = 5\Delta t$ and $\left[\begin{smallmatrix}5\\4\end{smallmatrix}\right]$, and superimposition of $T_M = b\Delta t = 2\Delta t$ and $\left[\begin{smallmatrix}2\\2\end{smallmatrix}\right]$. Note the irregular meter $\left[\begin{smallmatrix}5\\4\end{smallmatrix}\right]$ when grouping by the major generator a.
- Case 4: $r_{5\div3}$. And finally, the three groupings and time signatures are: grouping by $T_M = ab\Delta t = 15\Delta t$ and $\left[\begin{smallmatrix}15\\8\end{smallmatrix}\right]$, superimposition of $T_M = a\Delta t = 5\Delta t$ and $\left[\begin{smallmatrix}5\\4\end{smallmatrix}\right]$, and superimposition of $T_M = b\Delta t = 3\Delta t$ and $\left[\begin{smallmatrix}3\\8\end{smallmatrix}\right]$.

The example shows that synchronizaton and grouping are independent processes. For each pair of generators we obtain many options for selecting a time signature (duple, triple, irregular meter). In musical staff notation this will be further illustrated in Chapter 3.

2.3 Characteristics of the resultant

Example 2.2

The natural nucleus of a musical score.
This example is included in the full version of the book.

The figure is included in the full version

Figure 2.4: The natural nucleus of a musical score

Chapter 3

The techniques of grouping

This chapter focusses on the *grouping* of the resultant $r_{a \div b}$. An overview in musical notation is presented for the practical *time signatures* listed in Table 2.1 for the non-uniform binary generator combinations in the synchronization process.[1] We briefly repeat the grouping options from Section 2.2 of the interference resultant $r_{a \div b}$ (a is the major, b the minor generator for binary synchronization). Here we will consider the regular grouping options only:

1. Grouping by the common product ab. A single measure contains ab time units, i.e., $T_M = ab\Delta t$, and this wll determine the meter. Use this time signature only for reasonable values of ab (the practical limit is $ab < 15$).

2. Grouping by the major generator a (previously called superimposition by a). The fragment contains b measures with length $T_M = a\Delta t$, and we will obtain the rhythmic effect of *syncopation*.

3. Grouping by the minor generator b. Now we get a measures with length $T_M = b\Delta t$.

The grouping of the 19 combinations from Table 2.1 is shown in musical notation in Fig. 3.1 and 3.2. We use the time signatures from [3], p. 14 in Schillinger's book.

Verify that grouping by the common product is only shown for values $ab < 15$; time signatures for larger values will hamper reading of the musical notation. Note the rhythmic symmetry about the middle of the series of durations. The following pairs of generators generate two note lengths with ratio 2:1 (half note and quarter note, or quarter note and 8th note): 3:2, 5:2 and 7:2. We see three note duration values for the pairs: 4:3, 5:3 and 7:3. The range of note durations therefore is determined by the minor generator b, as was already discussed in Section 2.1.2.

The differences in durations (short-long) are maximum at either beginning or end of the series, with more even durations in the middle (this was already noted in Section 2.1.2). For grouping by either the major or the minor generator, check the number of measures and note that there are no slurred notes across bar lines.

[1] Time signature and meter are considered synonyms in this text, as should was stated in the introduction in Section 1.2.

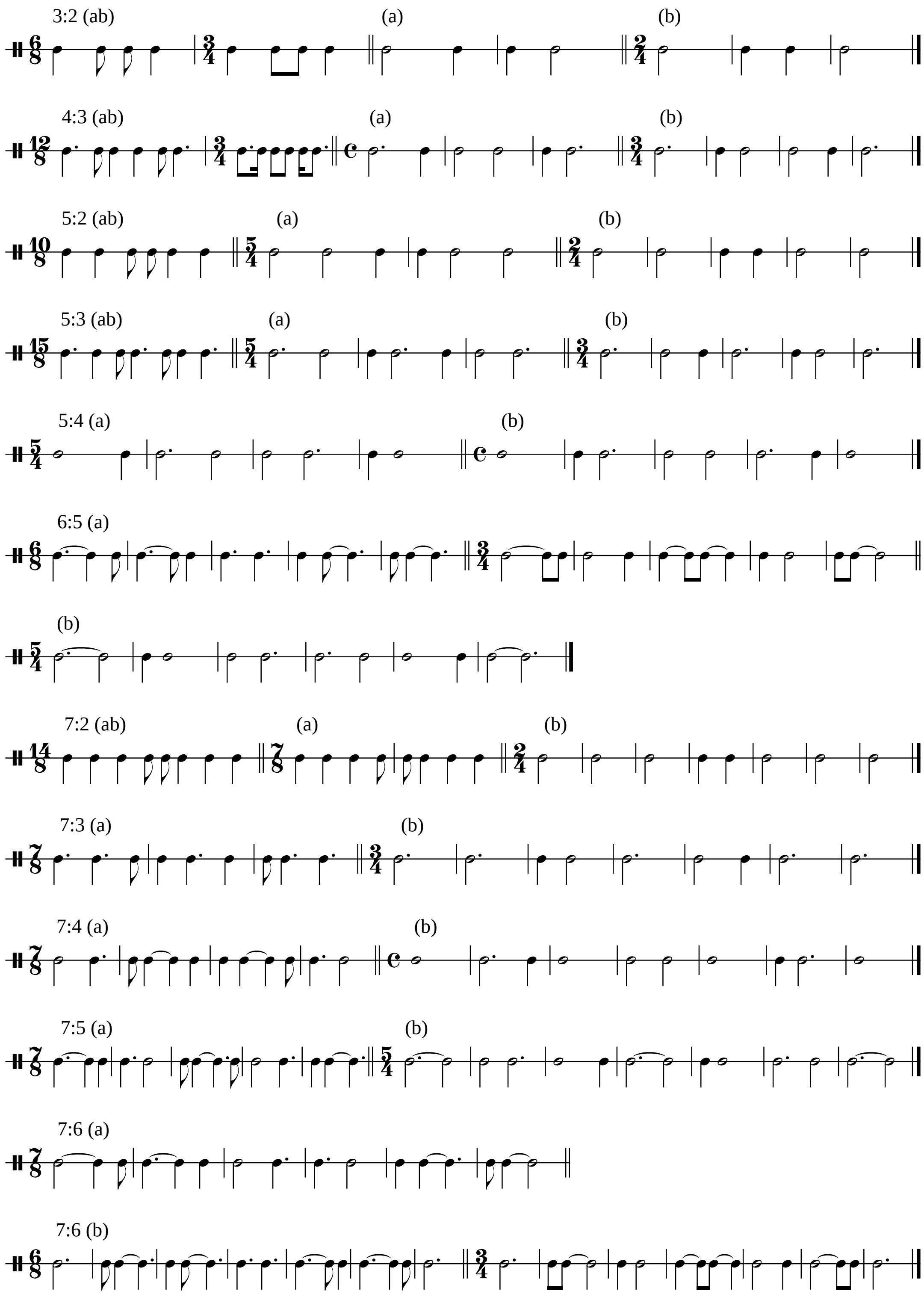


Figure 3.1: Grouping of non-uniform interference patterns

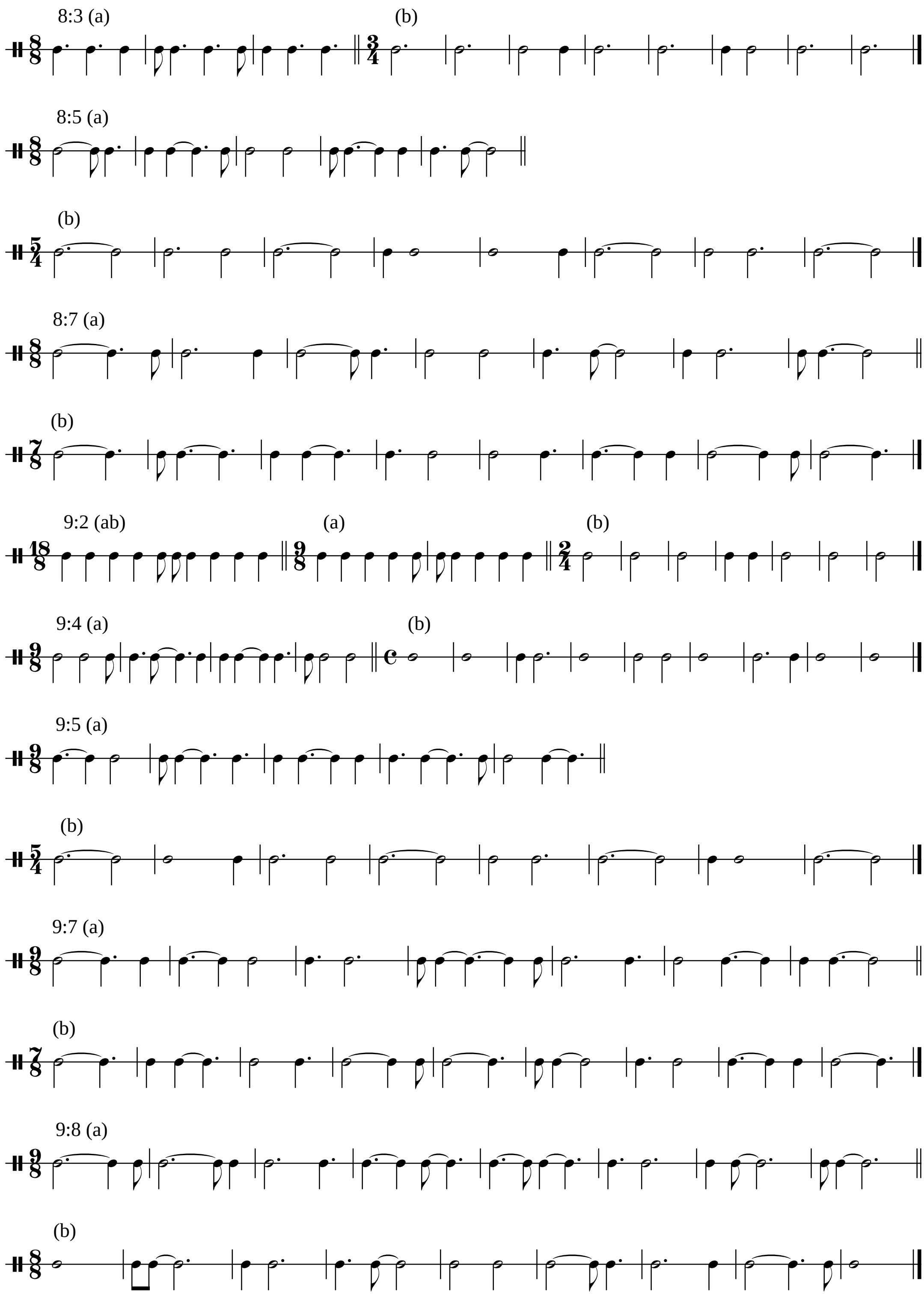


Figure 3.2: Grouping of non-uniform interference patterns (cont'd)

Chapter 4

The techniques of fractioning

This chapter introduces another basic technique for generating attack series with two generators a and b, called fractioning and notated as $\underline{a \div b}$. The interference and the grouping process are discussed.

4.1 The process of fractioning

The process of *fractioning* is also based on combining two generators, the major generator a and the minor generator b (two integer numbers, with $a > b$). Again, we will use the analogy of ticking clocks or metronomes. For the fractioning technique we will use one metronome of type A, ticking at intervals of Δt_A time units, and using N_b metronomes of type B, each generating a ticks at a time interval of Δt_B time units, with

$$N_b = a - b + 1. \tag{4.1}$$

We will synchronize these metronomes by starting the first B metronome at the same time instant as metronome A. At each subsequent tick of metronome A we start another metronome B, until all N_b metronomes are ticking. The resultant attack series $\vec{t_r}$ is determined by the combination, the union, of all metronomes (compare this with Eq. 2.1 for binary synchronization)

$$\vec{t_r} = \vec{t}_A \cup \vec{t}_{B_1} \cup \vec{t}_{B_2} \cup \ldots \cup \vec{t}_{B_{N_b}}, \tag{4.2}$$

and the duration series will repeat itself after $T_r = (\Delta t_A)^2$ time units.

The combinations for fractioning are listed in Table 2.1; here we will consider a number of examples and determine the resultant $r_{\underline{a \div b}}$. Not all combinations from the table will be considered here. See the example in Section 4.2 for the musical notation. A summary of the characteristics of the fractioning patterns is found in the lower half of Table 2.2.

4.1.1 Fractioning group: 3 and 2

For $a = 3$ and $b = 2$, the interference pattern $\underline{3 \div 2}$ will repeat after $T_r = a^2 = 9$ time units. We determine the number of B minor generators using Eq. 4.1; this yields $N_b = 3 - 2 + 1 = 2$. Each B clock will generate $a = 3$ ticks. The process is shown in Fig. 4.1.a. The attack series for the A and B generators are

$$\begin{aligned}\vec{t}_A &= [0\ 3\ 6],\\ \vec{t}_{B_1} &= [0\ 2\ 4],\\ \vec{t}_{B_2} &= [3\ 5\ 7].\end{aligned}$$

Applying Eq. 4.2 the resultant attack and duration series are

$$\begin{aligned}\vec{t}_r &= [0\ 2\ 3\ 4\ 5\ 6\ 7],\\ r &= 2+1+1+1+1+1+2,\end{aligned}$$

i.e., seven attacks ($N_a = 7$) and two note duration values (in length ratio 2:1). Note the symmetry of the attack pattern about the centre.

4.1.2 Fractioning group: 4 and 3

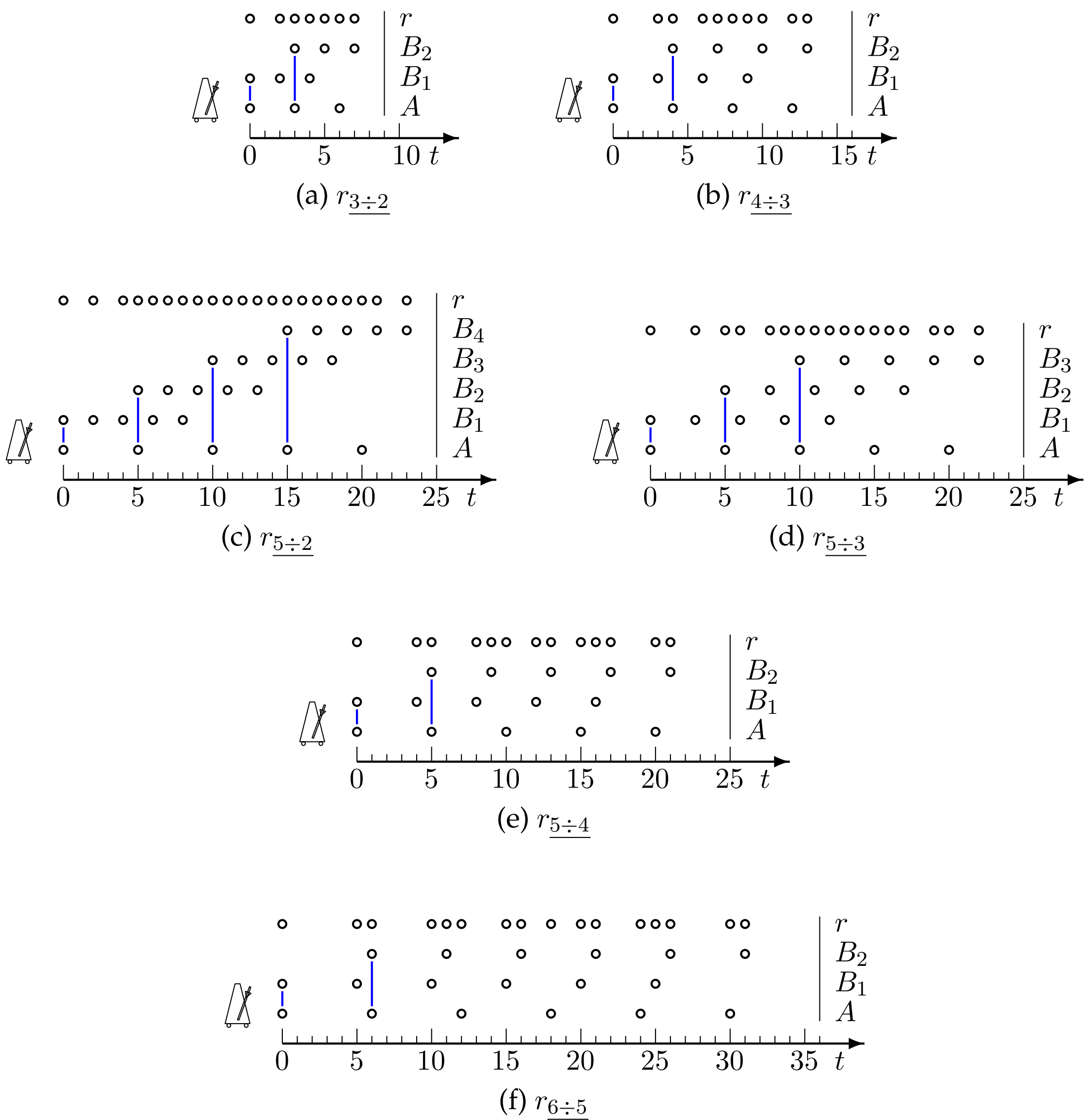


Figure 4.1: Fractioning. The resultant $r_{\underline{a\div b}}$ (top row) of two generators A and B that tick at different time intervals. a): $\Delta t_A = 3, \Delta t_B = 2$, b): $\Delta t_A = 4, \Delta t_B = 3$, c): $\Delta t_A = 5, \Delta t_B = 2$, d): $\Delta t_A = 5, \Delta t_B = 3$, e): $\Delta t_A = 5, \Delta t_B = 4$, f): $\Delta t_A = 6, \Delta t_B = 5$ time units.

4.1.3 Fractioning group: 5 and 2

For $a = 5$ and $b = 2$, the interference pattern $\underline{5 \div 2}$ will repeat after $T_r = a^2 = 25$ time units. We determine the number of B minor generators using Eq. 4.1; this yields $N_b = 5 - 2 + 1 = 4$. Each B clock will generate $a = 5$ ticks. The attack series for the A and B generators are

$$\begin{aligned}
\vec{t}_A &= [0\ 5\ 10\ 15\ 20], \\
\vec{t}_{B_1} &= [0\ 2\ 4\ 6\ 8], \\
\vec{t}_{B_2} &= [5\ 7\ 9\ 11\ 13], \\
\vec{t}_{B_3} &= [10\ 12\ 14\ 16\ 18], \\
\vec{t}_{B_4} &= [15\ 17\ 19\ 21\ 23],
\end{aligned}$$

as shown in Fig. 4.1.c.

Applying Eq. 4.2 the resultant attack and duration series are

$$\begin{aligned}
\vec{t}_r &= [0\ 2\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11\ 12\ 13\ 14\ 15\ 16\ 17\ 18\ 19\ 20\ 21\ 23], \\
r &= 2+2+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+1+2+2,
\end{aligned}$$

i.e., $N_a = 21$ and two different note durations. Again the note length shows the tendency to increase from the pattern centre towards both ends.

4.1.4 Fractioning group: 5 and 3

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4.1.5 Fractioning group: 5 and 4

For $a = 5$ and $b = 4$, the interference pattern $\underline{5 \div 4}$ will repeat after $T_r = a^2 = 25$ time units. We determine the number of B minor generators using Eq. 4.1; this yields $N_b = 5 - 4 + 1 = 2$. Each B clock will generate $a = 5$ ticks. The attack series for the A and B generators are

$$\begin{aligned} \vec{t}_A &= [0\ 5\ 10\ 15\ 20], \\ \vec{t}_{B_1} &= [0\ 4\ 8\ 12\ 16], \\ \vec{t}_{B_2} &= [5\ 9\ 13\ 17\ 21], \end{aligned}$$

as shown in Fig. 4.1.e.

Applying Eq. 4.2 the resultant attack and duration series are

$$\begin{aligned} \vec{t}_r &= [0\ 4\ 5\ 8\ 9\ 10\ 12\ 13\ 15\ 16\ 17\ 20\ 21], \\ r &= 4+1+3+1+1+2+1+2+1+1+3+1+4, \end{aligned}$$

i.e., $N_a = 13$ and four different note durations.

In Section 2.1.2 we already discussed the effect of the value of the minor generator b on the resultant r; this effect is also present in the fractioning process. Compare the patterns for $r_{\underline{5 \div 2}}$ to $r_{\underline{5 \div 4}}$ and note that the range of note durations is determined by the value of b.

4.1.6 Fractioning group: 6 and 5

For $a = 6$ and $b = 5$, the interference pattern $\underline{6 \div 5}$ will repeat after $a^2 = 36$ time units and we determine the number of B minor generators using Eq. 4.1; this yields $N_b = 6 - 5 + 1 = 2$. Each B clock will generate $a = 6$ ticks. The attack series for the A and B generators are

$$\begin{aligned} \vec{t}_A &= [0\ 6\ 12\ 18\ 24\ 30], \\ \vec{t}_{B_1} &= [0\ 5\ 10\ 15\ 20\ 25], \\ \vec{t}_{B_2} &= [6\ 11\ 16\ 21\ 26\ 31], \end{aligned}$$

as shown in Fig. 4.1.f.

Applying Eq. 4.2 the resultant attack and duration series are

$$\begin{aligned} \vec{t}_r &= [0\ 5\ 6\ 10\ 11\ 12\ 15\ 16\ 18\ 20\ 21\ 24\ 25\ 26\ 30\ 31], \\ r &= 5+1+4+1+1+3+1+2+2+1+3+1+1+4+1+5, \end{aligned}$$

i.e., $N_a = 16$ and five note durations.

4.1.7 Fractioning group: 7 and 3

The figure is included in the full version

Figure 4.2: Fractioning (cont'd)

4.1.8 Fractioning group: 7 and 4

For $a = 7$ and $b = 4$, the interference pattern $\underline{7 \div 4}$ will repeat after $T_r = a^2 = 49$ time units. We determine the number of B minor generators using Eq. 4.1; this yields $N_b = 7-4+1 = 4$. Each B clock will generate $a = 7$ ticks. The attack series for the A and B generators are

$$\begin{aligned}
\vec{t}_A &= [0\ 7\ 14\ 21\ 28\ 35\ 42], \\
\vec{t}_{B_1} &= [0\ 4\ 8\ 12\ 16\ 20\ 24], \\
\vec{t}_{B_2} &= [7\ 11\ 15\ 19\ 23\ 27\ 31], \\
\vec{t}_{B_3} &= [14\ 18\ 22\ 26\ 30\ 34\ 38], \\
\vec{t}_{B_4} &= [21\ 25\ 29\ 33\ 37\ 41\ 45],
\end{aligned}$$

as shown in Fig. 4.2.b.

Applying Eq. 4.2 the resultant attack and duration series are

$$\begin{aligned}
\vec{t}_r &= [0\ 4\ 7\ 8\ 11\ 12\ 14\ 15\ 16\ 18\ 19\ 20\ 21\ 22\ 23\ 24 \\
&\quad 25\ 26\ 27\ 28\ 29\ 30\ 31\ 33\ 34\ 35\ 37\ 38\ 41\ 42\ 45], \\
r &= 4+3+1+3+1+2+1+1+2+1+1+1+1+1+1+ \\
&\quad 1+1+1+1+1+1+1+2+1+1+2+1+3+1+3+4,
\end{aligned}$$

i.e., $N_a = 31$ and four different note lengths. Here also the pattern centre contains 13 notes with unit duration; as a standalone pattern this yields rhythmic monotony.

4.1.9 Fractioning group: 7 and 5

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4.1.10 Fractioning group: 7 and 6

For $a = 7$ and $b = 6$, the interference pattern $\underline{7 \div 5}$ will repeat after $T_r = a^2 = 49$ time units. We determine the number of B minor generators using Eq. 4.1; this yields $N_b = 7-6+1 = 2$. Each B clock will generate $a = 7$ ticks. The attack series for the A and B generators are

$$\begin{aligned}
\vec{t}_A &= [0\ 7\ 14\ 21\ 28\ 35\ 42], \\
\vec{t}_{B_1} &= [0\ 6\ 12\ 18\ 26\ 32\ 38], \\
\vec{t}_{B_2} &= [7\ 13\ 19\ 25\ 31\ 37\ 43],
\end{aligned}$$

as shown in Fig. 4.2.d.

Applying Eq. 4.2 the resultant attack and duration series are

$$\begin{aligned}
\vec{t}_r &= [0\ 6\ 7\ 12\ 13\ 14\ 18\ 19\ 21\ 24 \\
&\quad 25\ 28\ 30\ 31\ 35\ 36\ 37\ 42\ 43], \\
r &= 6+1+5+1+1+4+1+2+3+ \\
&\quad 1+3+2+1+4+1+1+5+1+6,
\end{aligned}$$

i.e., $N_a = 19$ and six different note lengths.

4.2 Grouping

Grouping is the selection of the *meter* or *time signature*; the resultant will be divided into a number of *measures* with each measure containing T_M time units.

In the case of fractioning there are again three options for grouping:

1. The resultant attack pattern repeats itself after $T_r = (\Delta t_A)^2$ time units. So, grouping by a^2 implies
$$T_M = (\Delta t_A)^2 = a^2\,\Delta t. \tag{4.3}$$
The fractioning pattern consists of a single measure. Like the case for binary synchronization (see Section 2.2), there is an upper limit to this grouping technique: do not group the resultant by a^2 when $T_M > 15\Delta t$.

2. Superimposition of a means that the major generator A will determine the time signature
$$T_M = \Delta t_A = a\,\Delta t. \tag{4.4}$$
The pattern length now is a measures.

3. Superimposition of b means that the minor generator B will determine the time signature
$$T_M = \Delta t_B = b\,\Delta t. \tag{4.5}$$
This yields syncopated rhythms for the fractioning attack series. When repeating the grouped resultant attack series until stopping at a complete measure, we have achieved recurrence, i.e., the attack series is completed at the end of a full measure, just before the bar line. The pattern now contains a higher total number of measures, compared to the previous case (superposition of a).

Example 4.1

Grouping of fractioning patterns.
Several groupings of the resultant fractioning patterns discussed in Section 4.1 are shown in musical notation in Fig. 4.3. The smallest time unit is either the 8th or the 4th note duration ($t = \frac{1}{8}$ or $\frac{1}{4}$).

- Case 1: $r_{\underline{3 \div 2}}$. Note that the grouping by either a^2 ($T_M = a^2 \Delta t$) or a ($T_M = a\Delta t$) again leads to symmetrical attack patterns. Grouping the resultant by b ($T_M = b\Delta t$) leads to syncopated rhythm patterns; the end of each resultant pattern is indicated by the breathing sign (′).
- Case 2: $r_{\underline{5 \div 2}}$. Only the superimposition by a ($T_M = a\Delta t$) is shown. The pattern is $a = 5$ measures long.
- For Case 3: $r_{\underline{5 \div 3}}$, Case 4: $r_{\underline{5 \div 4}}$ and Case 5: $r_{\underline{6 \div 5}}$ there is grouping by either the major generator a ($T_M = a\Delta t$) or the minor generator b ($T_M = b\Delta t$).

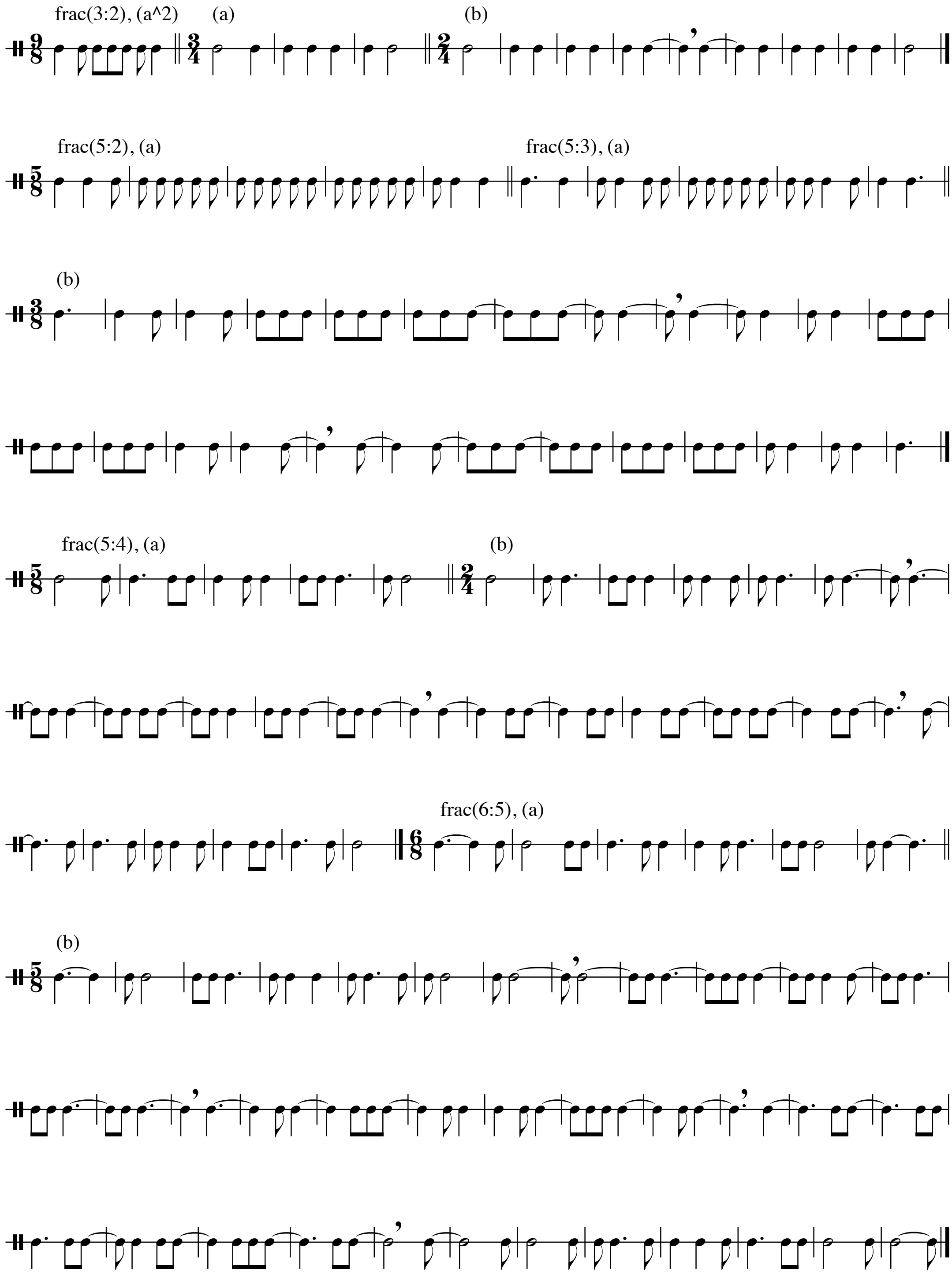

Figure 4.3: Grouping of fractioning patterns

Chapter 5

Composition of groups by pairs

In this chapter we will see the process of composing rhythmic resultants in pairs. Based on the interference of two generators A and B with different tick intervals $a\Delta t$ and $b\Delta b$ pairs of *adjacent groups* will be formed. The two members in the group are generated with the techniques described in Section 2.1 (binary synchronization) and Section 4.1 (fractioning). Binary synchronization yields the resultant $r_{a\div b}$, fractioning generates the resultant $r_{\underline{a\div b}}$.

The pairing of resultants is a method to generate a longer rhythmic continuity. The unifying element is the set of two generators A and B, that determine the note durations in the adjacent group. There are three approaches to the combination of these resultants, labeled as *balance*, *expansion*, and *contraction*. For all three approaches we will demonstrate measure grouping by a time units only.

5.1 Balancing adjacent groups

A balanced pairing is given by

$$r_B(a,b) = r_{\underline{a\div b}} + r_{a\div b} + a(a-b). \tag{5.1}$$

In fact we are juxtaposing three sub-patterns with length a^2, ab and $a(a-b)$ time units, respectively. The total duration of this balanced adjacent group therefore is

$$T_r = a^2 + ab + a(a-b) = a^2 + ab + a^2 - ab = 2a^2,$$

balancing the first half of the pattern with an equally long second half. The result is that the duration symmetry about the pattern centre will be disturbed; this may be considered an improvement to the rhythm. There is also less obvious risk of short note rhythmical monotony in the pattern centre.

The balancing resultant sounds unnatural when $a \geq 2b$, as is the case in generator combinations such as $(a,b) = (5,2)$ or $(9,4)$. In that case a balanced pairing is achieved by using

$$r_B(a > mb) = r_{\underline{a\div b}} + m\, r_{a\div b} + (a^2 - mab), \tag{5.2}$$

where m is an integer number determined by the next lower integer rounding of the ratio $a^2/(ab)$. This can be explained as follows: the total duration of the balanced adjacent group must be $T_r = 2a^2$ time units. The first half is the fractioning resultant $r_{\underline{a\div b}}$; therefore we will determine the number of times that the entire non-uniform binary synchronization resultant $r_{a\div b}$ fits into the second half; this is the integer number m. The remaining duration $(a^2 - mab)$ in the second half is the last element in this rhythmical pattern.

Example 5.1

Balancing resultant: $r_B(a,b)$.
For the following examples we will present the duration series, using Eq. 5.1 or Eq. 5.2, and the results from Section 2.1.2 (interference through non-uniform binary synchronization) and Section 4.1 (fractioning). The resultants are shown in musical notation in Fig. 5.1.

- Generator time units $a = 3, b = 2$.

$$\begin{aligned} r_B(3,2) &= r_{3\div 2} + r_{\underline{3\div 2}} + 3(3-2) \\ &= (2+1+1+2) + (2+1+1+1+1+1+2) + 3, \end{aligned}$$

  i.e., a total of 12 attacks ($N_a = 12$), repeating after $T_r = 2a^2 = 2 \times 9 = 18$ time units. Looking at the pattern we may note that the second half suggests a variation of the first group, with some developmental character and concluding with a sustained note. The note durations in all elements but the last are the same, i.e., $\{\Delta t, 2\Delta t\}$, supporting the homogeneity of the adjacent group.

- Generator time units $a = 3, b = 2$. The ratio $a/b = 3/2 = 1.5$; therefore the adjacent group pair is determined with Eq. 5.1.

$$\begin{aligned} r_B(4,3) &= r_{4\div 3} + r_{\underline{4\div 3}} + 4(4-3) \\ &= (3+1+2+2+1+3) + \\ &\quad +(3+1+2+1+1+1+1+2+1+3) + 4, \end{aligned}$$

  i.e., a total of 17 attacks ($N_a = 17$), repeating after $T_r = 2a^2 = 2 \times 16 = 32$ time units.

- Generator time units $a = 5, b = 3$.

$$\begin{aligned} r_B(5,3) &= r_{\underline{5\div 3}} + r_{5\div 3} + +5(5-3) \\ &= (3+2+1+2+1+1+1+1+1+1+ \\ &\quad +1+1+1+1+2+1+2+3) + \\ &\quad +(3+2+1+3+1+2+3) + 10, \end{aligned}$$

  i.e., $N_a = 25$, repeating after $T_r = 2a^2 = 2 \times 25 = 50$ time units. Since the ratio $a/b = 5/3 = 1.67$ we begin to notice the unbalanced long last element in the pattern with duration $10\Delta t$.

- Generator time units $a = 5, b = 2$. Since the ratio $a/b = 5/2 = 2.5 > 2$, we will generate the adjacent group with Eq. 5.2, the alternative approach to balancing the adjacent groups. Determine the value of m from the ratio $a^2/(ab) = 25/(5 \times 2) = 25/10 = 2.5$; rounding this to the next lower integer number yields $m = 2$, a double statement of the two-generator interference pattern. Therefore the balancing pattern is

$$\begin{aligned} r_B(5,2) &= \underline{r_{5\div 2}} + 2r_{5\div 2} + (25 - 2 \times 10) \\ &= (2+2+1+1+1+1+1+1+1+1+1+ \\ &\quad +1+1+1+1+1+1+1+1+2+2)+ \\ &\quad +(2+2+1+1+2+2)+(2+2+1+1+2+2)+5, \end{aligned}$$

i.e.,$N_a = 35$ repeating after $T_r = 2a^2 = 2 \times 25 = 50$ time units. The length of the sustained note at the end is acceptable, compared to the maximum duration in the synchronization resultant: this ratio is $5/2 = 2.5$.

- Finally, we demonstrate the generator time units $a = 7, b = 3$. The ratio $a/b = 7/3 = 2.335 > 2$, leading to the application of Eq. 5.2. Determine the value of m from the ratio $a^2/(ab) = 49/(7 \times 3) = 49/21 = 2.33$; rounding this to the next lower integer number yields $m = 2$. There fore the balanced adjacent pair rhythmical pattern is

$$\begin{aligned} r_B(7,3) &= \underline{r_{7\div 3}} + 2r_{7\div 3} + (49 - 2 \times 21) \\ &= (3+3+1+2+1+3+1+1+2+2+1+ \\ &\quad +1+1+1+1+1+1+1+1+1+1+1+ \\ &\quad +1+2+2+1+1+3+1+2+1+3+3)+ \\ &\quad +(3+3+1+2+3+2+1+3+3)+ \\ &\quad +(3+3+1+2+3+2+1+3+3)+7, \end{aligned}$$

i.e., $N_a = 52$, repeating after $T_r = 2a^2 = 2 \times 49 = 98$ time units. The sustained note at the end has a duration of 7 time units.

5.2 Expanding adjacent groups

Example 5.2

Expanding resultant: $r_E(a,b)$**.**
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5.3 Contracting adjacent groups

A contracting pairing of adjacent groups is given by

$$r_C(a,b) = r_{\underline{a\div b}} + r_{a\div b}. \tag{5.3}$$

The total length is $T_r = a^2 + ab$, with the fractioning resultant in the first part longer than the binary synchronization resultant. This aspect leads to the contracting character of the adjacent group. It is a kind of mirrored counterpart of the expanding group.

Example 5.3

Contracting resultant: $r_C(a,b)$.

For the following examples we will present the duration series, using Eq. 5.3, and the results from Section 2.1.2 and Section 4.1. The resultants are shown in musical notation in Fig. 5.1.

- Generator time units $a = 4, b = 3$.

$$\begin{aligned} r_C(4,3) &= r_{\underline{4\div 3}} + r_{4\div 3} \\ &= (3+1+2+1+1+1+1+2+1+3)+ \\ &\quad (3+1+2+2+1+3), \end{aligned}$$

  i.e., $N_a = 17$, repeating after $T_r = a^2 + ab = 16 + 4 \times 3 = 16 + 12 = 28$ time units. The shorter second part suggests a rhythmical summary of the fractioning resultant in the first part. The series of single time unit attacks in the centre of the fractioning group has been eliminated in the second group.

- Generator time units $a = 5, b = 3$.

$$\begin{aligned} r_C(5,3) &= r_{\underline{5\div 3}} + r_{5\div 3} \\ &= (3+2+1+2+1+1+1+1+1+ \\ &\quad +1+1+1+1+2+1+2+3)+ \\ &= (3+2+1+3+1+2+3), \end{aligned}$$

  i.e., a resultant containing 24 attacks ($N_a = 24$) and repeating after $T_r = a^2 + ab = 25 + 5 \times 3 = 25 + 15 = 40$ time units. The length ratio of the fractioning to the interference part is $a/b = 5/3 = 1.67$. Again, the second group is commenting on the fractioning pattern, skipping the single time unit attacks in the centre and sort of replacing it by a single attack wth duration $3\Delta t$.

Note from the examples, that:

- The second group of the resultant pair always is a varied version of the first group. This creates both homogeneity and variation.
- In balancing (r_B) and contracting (r_C) the pair, there is more rhythmic activity, i.e., shorter note durations in the first half of the resultant. The balanced pairing concludes with a sustained, long duration note.

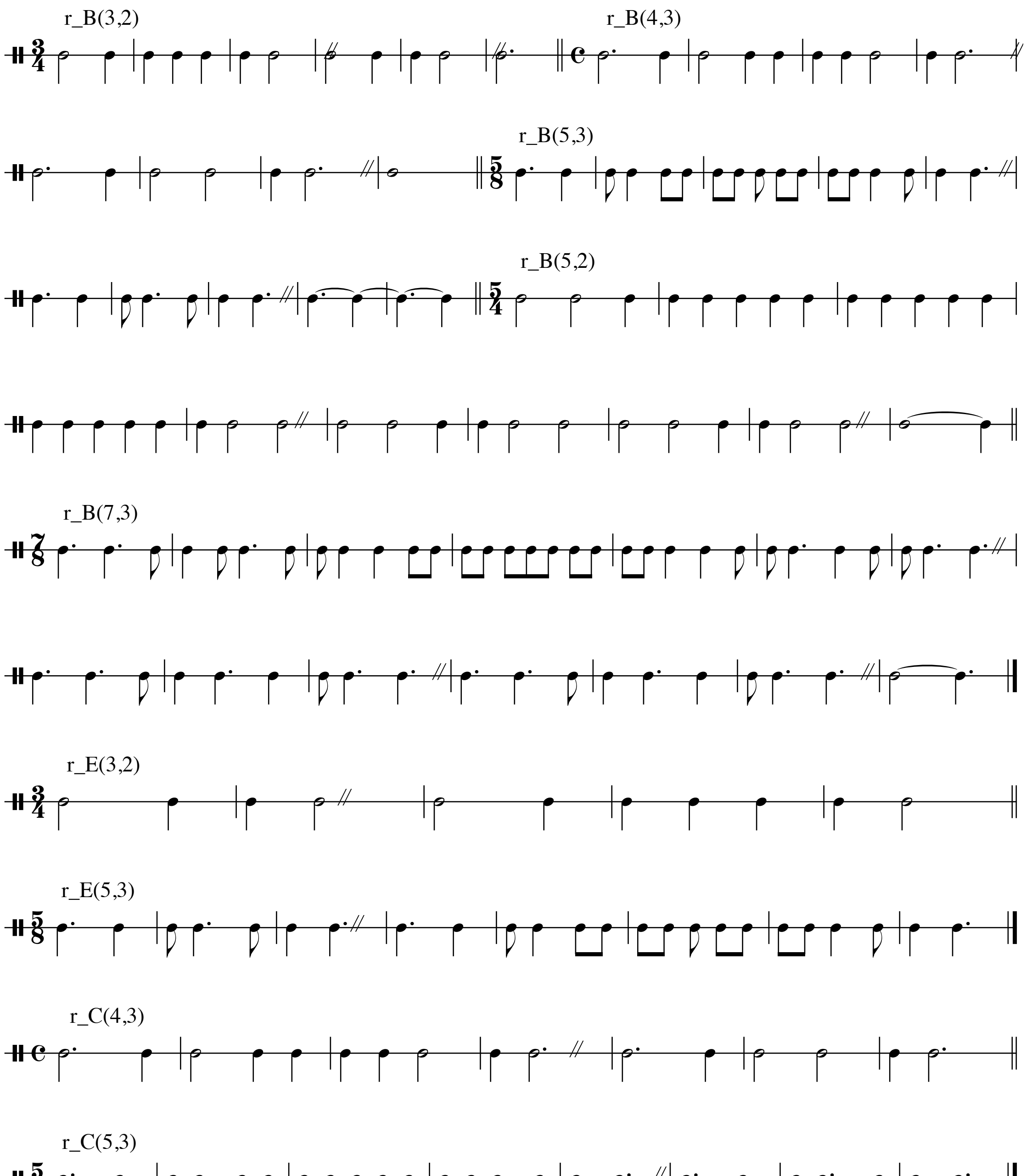

Figure 5.1: Balancing (r_B), expanding (r_E) and contracting (r_C) a pair of adjacent groups. These groups are based on A and B generator combinations, ticking at $(a, b)\Delta t$ time intervals.

- When the time interval difference between the two generators becomes large, the regular balanced adjacent group pairing (r_B) has unnatural characteristics, such as long series of equal short notes in the first half and a very long closing note. This was noted by Schillinger and corrected by the special case of balancing adjacent pairs.
- The expanding pair (r_E) has more rhythmic activity (shorter notes) in the second half of the resultant.

Chapter 6

Utilization of three or more generators

This chapter introduces a basic technique for generating attack series, by considering the interference pattern of more than two clocks or metronomes that each tick at a different interval. These attack series may then be grouped using different numbers of time units per measure.

The three generators will form a *family of rhythms* when they are based on the same series of growth, also called *summation series* or *Fibonacci series*, shown in Table 6.1.[1]

We will limit the combinations to practical sizes, and consider only the combinations discussed in the next section.

6.1 The technique of synchronization of three generators

Now there are three clocks or metronomes $A,\ B$ and C ticking at different regular time 3
intervals $\Delta t_A,\ \Delta t_B$ and Δt_C. We assume that metronome C is ticking faster than metronome B, and metronome B is ticking faster than metronome A. So we have $\Delta t_C < \Delta t_B < \Delta t_A$.

We will derive two resultants: r and the alternative r'. The interference pattern will repeat after $T_r = \Delta t_C \cdot \Delta t_B \cdot \Delta t_A$ time units. The resultant r is determined analogous to the case of binary synchronization (see Section 2.1.2), by finding the combination, the union, of three attack series

$$\vec{t_r} = \vec{t_A} \cup \vec{t_B} \cup \vec{t_C}. \tag{6.1}$$

The alternative resultant r' is determined by synchronising the *complementary factors*: this is a set of three alternative generators, but now the metronomes tick at intervals of $\Delta t_B \cdot \Delta t_C$ (the complementary factor of generator A), $\Delta t_A \cdot \Delta t_C$ (the complementary factor of generator B), and $\Delta t_A \cdot \Delta t_B$ (the complementary factor of generator C).

We will see in the examples that resultant r yields series of more attacks and shorter duration, whereas the alternative r', creates rhythmic patterns with less attacks and longer durations. Both the r and r' pattern have the same length.

6.1.1 Interference group: 5, 3 and 2

For $\Delta t_A = 5,\ \Delta t_B = 3$ and $\Delta t_C = 2$ time units and using Eq. 1.2 we get the attack series

$$\vec{t_A} = [0\ 5\ 10\ \ldots\ 25], \quad \vec{t_B} = [0\ 3\ 6\ \ldots\ 27], \quad \vec{t_C} = [0\ 2\ 4 \ldots\ 28],$$

[1] The summation series will return in Chapter 13, where the evolution of rhythm families is discussed.

Table 6.1: The summation series serving musical purposes. Each row in the table is a Fibonacci summation series with the third and higher column number element being the sum of the two previous elements.

1	2	3	5	8	13	...
1	3	4	7	11	18	...
1	4	5	9	14	23	...

repeating itself after $T_r = 2 \cdot 3 \cdot 5 = 30$ time units. We determine the resultant r from the combination of these attack series, the union according to Eq. 6.1, which yields the following attack and duration series

$$\begin{aligned} \vec{t}_r &= [0\ 2\ 3\ 4\ 5\ 6\ 8\ 9\ 10\ 12\ 14\ 15\ 16\ 18\ 20\ 21\ 22\ 24\ 25\ 26\ 27\ 28], \\ r &= 2+1+1+1+1+2+1+1+2+2+1+ \\ &\quad 1+2+2+1+1+2+1+1+1+1+2. \end{aligned}$$

The resultant number of attacks is 22 ($N_a = 22$), with two different note duration values. The attack series is shown in Fig. 6.1.a. The resultant will be written as $r_{5\div3\div2}$ (descending order) or $r_{2\div3\div5}$ (ascending order).

The resultant r' is obtained with the complementary clocks, with $\Delta t_{A'} = 6$, $\Delta t_{B'} = 10$ and $\Delta t_{C'} = 15$ and using Eq. 1.2 we get the attack series

$$\vec{t}_{A'} = [0\ 6\ 12 \ldots\ 24], \quad \vec{t}_{B'} = [0\ 10\ 20], \quad \vec{t}_{C'} = [0\ 15],$$

repeating itself after $T_r = 30$ time units and shown in Fig. 6.1.b. The combination of these attack series yields the following attack and duration series

$$\begin{aligned} \vec{t}_{r'} &= [0\ 6\ 10\ 12\ 15\ 18\ 20\ 24], \\ r' &= 6+4+2+3+3+2+4+6. \end{aligned}$$

This is a series of eight attacks ($N_a = 8$), with four different note duration values.

6.1.2 Interference group: 7, 4 and 3

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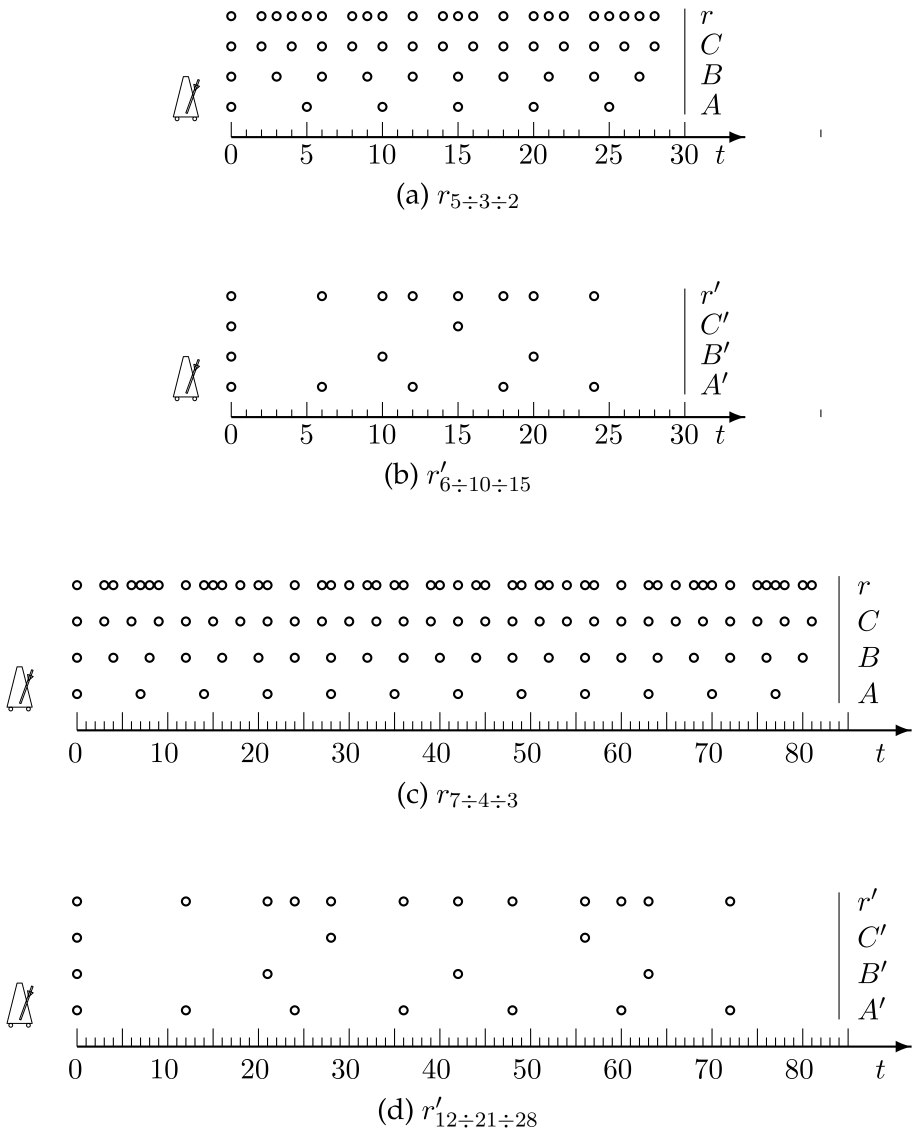

Figure 6.1: Synchronization of three generators. The resultant $r_{a\div b\div c}$ and the alternative complement resultant r' of three generators A, B and C that tick at different time intervals. a): $\Delta t_A = 5, \Delta t_B = 3, \Delta t_C = 2$, b): $\Delta t_{A'} = 6, \Delta t_{B'} = 10, \Delta t_{C'} = 15$, c) $\Delta t_A = 7, \Delta t_B = 4, \Delta t_C = 3$, d) $\Delta t_{A'} = 12, \Delta t_{B'} = 21, \Delta t_{C'} = 28$ time units.

6.2 Grouping

The previous section discussed the generation of a rhythm using three generators: A, B, and C. The synchronization process yields two alternative attack-duration groups with resultant r or r'.

These rhythms can be grouped into measures in six alternative ways. The length of a single measure is now based on either the generator time intervals $\{a, b, c\}$ or the complementary factors $\{bc, ac, ab\}$. This means that a single measure may contain either $T_M = \Delta t_A,\ \Delta t_B,\ \Delta t_C,\ \Delta t_B \times \Delta t_C,\ \Delta t_A \times \Delta t_C$, or $T_M = \Delta t_A \times \Delta t_B$ time units. As before we will limit the grouping to a practical maximum limit of $T_M \leq 15\Delta t$.

For the interference group $5 \div 3 \div 2$ this yields grouping by either $T_M = 2,\ 3,\ 5,\ 6,\ 10$ or 15 time units. The result is shown in musical notation in Fig. 6.2. For the interference group $7 \div 4 \div 3$ this yields grouping by either $T_M = 3,\ 4,\ 7$ or 12 time units, and the result is shown in Fig. 6.3.

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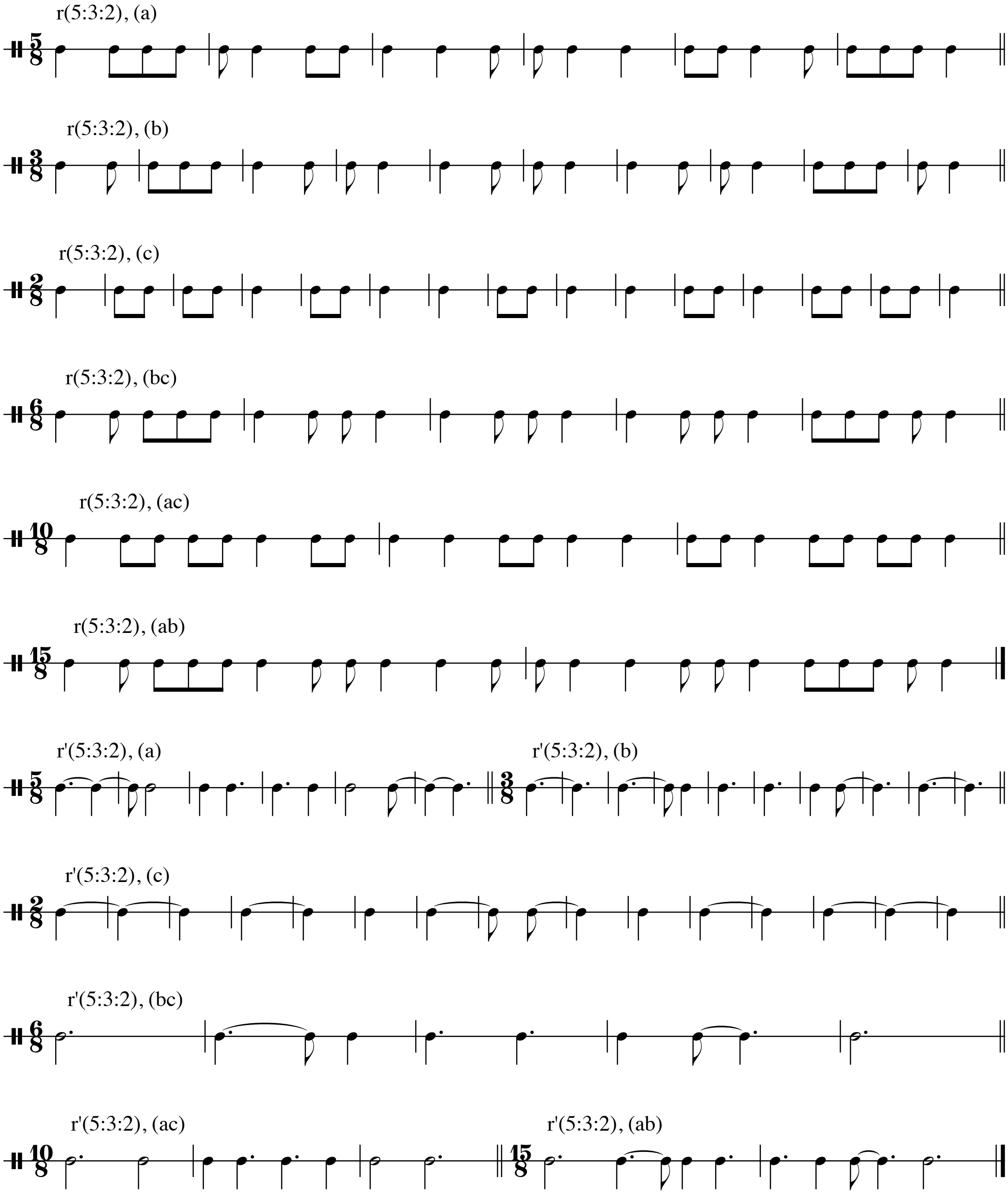

Figure 6.2: Grouping of the three-generator interference group $5 \div 3 \div 2$. Time signatures are determined by the (complementary) generator time intervals $T = \{a, b, c, bc, ac, ab\}\Delta t$ (6 options), with a practical limit $T_M < 15\Delta t$.

The figure is included in the full version

Figure 6.3: Grouping of the three-generator interference group $3 \div 4 \div 7$

Chapter 7

Resultants applied to instrumental forms

This chapter discusses the application of a rhythmic resultant to a given pitch sequence (application to melody) or to a given set of simultaneous pitches (application to harmony). The harmonic application is limited to the coordinated attack patterns of pitch subsets from a single static chord structure.

The techniques are based on a given time rhythm, i.e., a resultant set of attack-durations, obtained through one of the techniques from the previous or following chapters, and a given instrumental rhythm. An instrumental rhythm is a pre-determined *ordered set* of pitches (melody) or a given sequence of simultaneous pitches (elements, subsets from a given harmony).[1]

The process in this chapter involves interference of the time rhythm with the instrumental sequence pattern. The result of this process is a *pitch-attack series* with N_{pa} elements and a specific distribution over multiple staves in a musical score. This result is called an *instrumental form*, a subject to which Schillinger devotes an entire book (i.e., Book 8) in Volume 2 of [3].

7.1 Instrumental rhythm

The essential step in this approach is that we are determinining the synchronization pattern of an attack series (with N_a elements and note durations) on a single staff with an ordered set of pitches N_p. The resulting pattern is an instrumental part from a musical score.

The synchronization process depends on the ratio of number of attacks to number of pitches N_a/N_p, or its inverse N_a/N_p. When there are integer ratios, such as $N_a/N_p = 1, 2, 3$ we find short recurrent pitch-attack series N_{pa}, where the pitch pattern fits an integer number of times into one attack-duration series r. The consequences for achieving recurrence through synchronization are illustrated in Fig. 7.1. The diagram shows a pitch pattern, consisting of two pitches, i.e., $N_p = 2$. There are three examples of attack-duration patterns that lead to integer attack to pitch ratios, $N_a = 2, 4, 6$, and two rhythm resultants r with non-integer ratios, $N_a = 3, 5$. The integer ratio cases yield recurrence after a few repeats of

[1] Note that this chapter introduces processes in the pitch domain. In previous chapters all operations took place in the time domain only, such as the interference of generators or the synchronization of an attack-duration pattern with a grouping scheme.

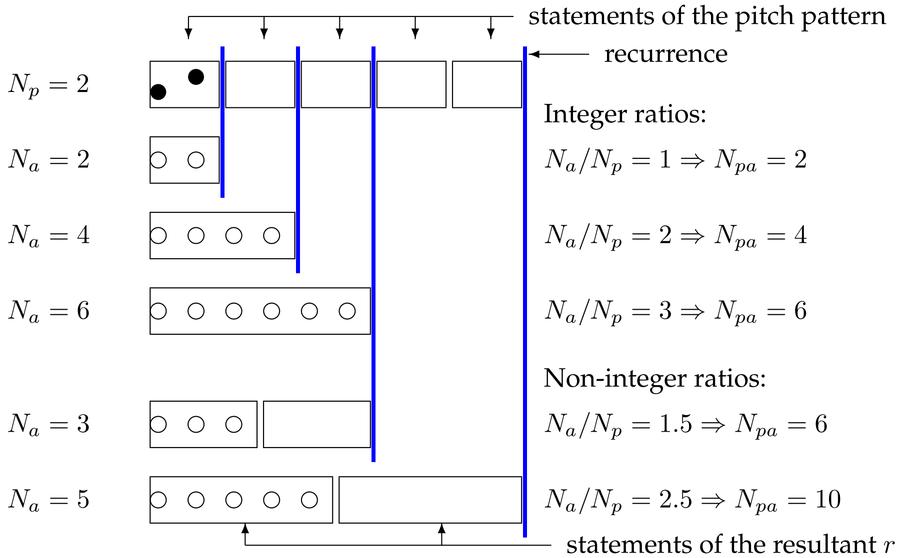

Figure 7.1: The dependence of the instrumental rhythm on the number of elements in the pitch N_p and attack series N_a. Closed circles represent the ordered pitch series, open circles represent rhythmic attacks (no durations indicated)

the pitch distribution pattern; for $N_a = N_p$ there is the perfect match after a single statement, and $N_{pa} = 2$. The non-integer ratios require three or more repeats of the pitch pattern $N_{pa} \geq 6$. The diagram illustrates the process for the case of more attacks than pitches, i.e., $N_a \geq N_p$. In principle we may have to deal with the reverse situation, where there are more pitches than attacks, but the approach remains the same. Note durations are irrelevant for this synchronization process; note lengths only become relevant when we add the grouping into measures process or want to write out the rhythm.

Let's demonstrate the pitch-attack synchronization with an example in musical notation.

Example 7.1

Create an instrumental rhythm from interference of an attack-duration series with an ordered pitch set.

For the following examples we will apply various duration series to a given ordered pitch set. The duration series were created using the techniques of (non)-uniform binary synchronization from Section 2.1.2, interference of three generators (Section 6.1) and the composition of groups by pairs, shown in Chapter 5. Various groupings (see Chapter 3) will yield the accompanying time signature and the total number of measures. The results are shown in musical notation in Fig. 7.2.

- The given two-pitch set $N_p = 2$ is shown in m. 1. First there is interference of a constant duration, i.e., uniform binary synchronization time series $\Delta t = 1$ unit in m. 2. For a grouping at time signature $\left[\frac{4}{4}\right]$ this yields the characteristic march bass pattern in the lower octaves; each measure contains two statements of the pitch distribution pattern since we have $N_a/N_p = 2$. Then there are two cases of two-generator patterns, the interference resultant $r_{3\div 2}$ with $N_a = 4$ and the fractioning $r_{\underline{5\div 4}}$ with $N_a = 13$. The latter would require two statements of r to achieve recurrence at $N_{pa} = 26$; here we took the liberty of opening and ending on the same tonic pitch c with $N_{pa} = N_a = 13$. Finally we apply the three-generator interference resultant $r_{5\div 3\div 2}$ and $N_{pa} = N_a = 8 = 4N_p$, four statements of the pitch pattern.

- The three-pitch set $N_p = 3$ is shown in m. 17. The constant duration pattern yields an arpeggio with $N_{pa} = 6$. This example also shows various groupings (see the meter in m. 20-21 and 28–32), the interference pattern $r_{5\div 2}$ with $N_a = 6$ and thus $N_{pa} = 6$, and a version with contracting adjacent groups $r_C(4,3)$ (see Section 5.3). This attack-duration group has $N_a = 16$ attacks; we disregard the strict approach and instead create a more musical alternative, stressing the root of the G major triad.
- The four-pitch example $N_p = 4$ starts in m. 33 and contains an example of fractioning $r_{\underline{6\div 5}}$ with $N_a = 16$ and thus four statements of the pitch-distribution pattern $N_{pa} = 16 = 4N_P$. The balancing adjacent groups $r_B(4,3)$ pattern with $N_a = 17$ (see Section 5.1) encompasses 4 statements of the pitch pattern plus a repeat of the opening pitch. The three-generator alternative resultant $r'(5\div 3\div 2)$ with $N_{pa} = N_a = 8 = 2N_P$ is demonstrated for two complementary factor groupings at $T_M = 10\Delta t$ and $T_M = 15\Delta t$.

What these examples demonstrate is that with careful design, combining appropriate pitch-distribution and attack-duration patterns we are able to obtain short patterns, with a *riff* character. Sometimes we use the freedom to deviate from the strict calculus in order to obtain musically sensible results.

7.2 Applying the principles of interference to harmony

Example 7.2

Interference applied to harmony.
This example is included in the full version of the book.

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Figure 7.2: Instrumental rhythm example for a 2-, 3- and 4-part ordered pitch set (see m. 1, 17 and 33). Several attack patterns are considered for interference with these pitches: uniform binary synchronization with constant duration t, synchronization of two or three generators, fractioning and composition of balancing groups by pairs. Note the various meter groupings (see the time signature).

The figure is included in the full version

Figure 7.3: Interference applied to harmony example

7.3 Combination of instrumental form techniques

There are several extensions to the two basic approaches described in the previous sections.

7.3.1 Extension 1: Doubling on multiple staves

This approach yields a score with more than two staves. Typically, the upper harmony staff will be doubled at the higher octave and the rhythmic resultant will now be distributed over three (or more) staves. There will be single bass part, since in general there the doubling (into the lower octave) does not make sense and will confuse the overall feeling of rhythm. The attacks are then distributed over the total number of staves.

Example 7.3

Interference to harmony: doubling on multiple staves.
Note the three-staff system, with the middle staff harmony copied into the upper staff at the higher octave.

- A six-part harmony $H(6p)$ with root, fifth and third in the bass $N_l = 3$ and therefore $N_p = 9$ for the F chord, i.e., 9 distributed attack in the pitch domain. Three resultants are applied, the 9:1 uniform binary, the $r_{7:3}$ with $N_{pa} = N_p = N_a = 9$ and the $r_{9 \div 4}$ non-uniform binary synchronization pattern with $N_a = 12$, see Fig. 7.4. The $\left[\begin{smallmatrix}2\\4\end{smallmatrix}\right]$ meter for grouping $r_{9 \div 4}$ (m. 5–31) leads to interference with three statements of the attack series, before there is recurrence on the downbeat. Thus $N_{pa} = 36 = 3N_a = 4N_p$. Note the constant half note durations in the middle layer; this coincidence is caused by the deliberately selected generator combination (or coincidentally, that depends on the composer's intention).
- A seven-part harmony $H(7p)$ with doubled root and fifth in the bass $N_l = 3$, corresponding to $N_p = 5$ for the $Am^{6/9}$ chord, see Fig. 7.5. The attack distribution is chosen freely here: the attack series starts with the middle harmony layer.[2] The bass notes are on the offbeats, with the root doubled at the lower octave. There is $r_{5 \div 1}$ uniform binary with a single statement, $r_{9 \div 7}$ non-uniform binary synchronization with $N_a = 15$, and $r_{\underline{4 \div 3}}$ fractioning with $N_a = 10$. The $\left[\begin{smallmatrix}3\\8\end{smallmatrix}\right]$ time signature for the 9:7 synchronization leads to three attack pattern statements before achieving recurrence, thus $N_{pa} = 15 = N_a = 3N_p$. The $r_{\underline{4 \div 3}}$ fractioning pattern at time signature $\left[\begin{smallmatrix}4\\4\end{smallmatrix}\right]$ yields $N_{pa} = 10 = N_a = 2N_p$.

---

[2] This freedom of creating pitch distributions is in fact the subject of the following chapter. Here we demonstrate a simple form of that generalization to get a feel for what is to come.

Figure 7.4: Interference applied to harmony. Doubling on multiple staves, Case 1: interference between the chord structure with $H(6p), N_p = 9$ and either binary synchronization or fractioning rhythmic resultants r. The upper staff with harmony parts is doubled at the higher octave.

Figure 7.5: Interference applied to harmony. doubling on multiple staves, Case 2: interference between the chord structure with $H(7p), N_p = 5$ and either binary synchronization or fractioning rhythmic resultants r. The upper staff with harmony parts is doubled at the higher octave.

7.3.2 Extension 2: Interference between time and instrumental groups

Example 7.4

Interference to harmony: interference with time signature.
This example is included in the full version of the book.

The figure is included in the full version

Figure 7.6: Interference between time and instrumental groups

Example 7.5

Interference to harmony: time-shifted variant of a single resultant,
This example is included in the full version of the book.

7.3.3 Extension 3: The combination of techniques

A full score with multiple staves is created from a single harmonic structure by combining various techniques from this chapter. E.g., a single staff instrumental rhythm may be combined with one or more instrumental forms of interference applied to harmony. The resulting score looks begins to show similarities with the example of the *natural nucleus*, discussed in Section 2.3.

Example 7.6

Instrumental form: Combination of techniques.
In Fig. 7.7 there is an instrumental rhythm (top), a doubled harmony layer (middle) and a given source bass-harmony staff pair (bottom). The chord structure is $H(6p)$, the Am^{add4} chord, with root and fifth in the bass part ($N_l = 2$). We will create three instrumental forms from this source, using a combination of techniques and patterns.

- In m. 2–4, there are constant durations in the lower and middle layer: the bass-harmony lower layer has a 4:1 uniform binary synchronization pattern with one-measure duration, thus $N_p = N_a = 4$. The middle layer uses $\Delta t = 3t$ constant duration, while the upper instrumental rhythm uses the $r_{9\div2}$ non-uniform binary synchronization with $N_a = 10$ applied to three pitches in the ordered subset $\{d, e, g\}$ from the harmony layer $N_p = 3$. The horizontal bracket above the staff shows the length of a single statement (where we have not reached full recurrence). Note the different time scales in each layer, with the shortest notes in the top layer. The middle layer acts as a sustained harmonic background. Since it is doubling in the same octave as the bottom layer, we must inspect the attack series for simultaneous attacks, cross-rhythms, and use a different instrumentation, in order to make them audibly discernable as separate elements.
- The second instrumental form in m. 5–8 uses a balanced adjacent pair grouping $r_B(4, 3)$ with $N_a = 17$ in the lower layer. The middle layer has non-uniform binary synchronization 3:2, while the top layer uses fractioning $r_{\underline{4\div3}}$ with $N_a = 10$ to create an instrumental rhythm. The top layer with the shortest time unit (16th notes) repeats three times and is a sort of ostinato riff pattern. The middle layer has two statements at half note triplet time unit. Again, no full recurrence is achieved in this example; see the return of the root a at the end in the bottom staff.
- The third instrumental form in m. 9–14 demonstrates maximum activity in the lower layer (8th note time unit in $\left[\begin{smallmatrix}6\\8\end{smallmatrix}\right]$ time signature). The rhythm is $r_{\underline{3\div2}}$ with $N_a = 7$; there is no full recurrence, note the root a at the start and end of the pattern. The middle layer shows a time-shifted alternative resultant for the three-generator pattern $r'_{2\div3\div5}$ with $N_a = 8$. The top layer uses a balanced adjacent pair group $r_B(3, 2)$ with $N_a = 12$.

Figure 7.7: Instrumental form created through combination of techniques. The six-part chord structure $H(6p)$ is used in three layers: the bass-harmony lower layer (bottom), the doubled harmony layer (middle) and the instrumental rhythm (top). Three instrumental forms are shown.

With this third extension, the combination of techniques, we may (have to) apply creativity and artistic taste to generate or combine patterns that occasionally and locally deviate from the strict procedures described in the first half of this chapter. The degrees of freedom in pitch, time and grouping domain provide many options for creating complex rhythmic assemblies from a simple source.

Chapter 8

Coordination of time structures

In this chapter the aspect of *coordinated time*, i.e., *rhythm* is discussed further. Schillinger here splits coordinated time into two application cases: *simultaneity* for multiple parts in a score and *continuity* in a single part. The latter case was discussed in Chapter 7, Section 7.1, where the interference of a time series over either a single staff with a number of pitches was discussed (yielding an instrumental rhythm) or over a set of staves with a given chord structure. In that chapter also a number of extensions were presented, including doubling of harmony parts. However, the attacks then were moving one-by-one between the staves.

In this chapter there is a generalization of that principle to the case of multiple staves. There will be two independent resultants at work. One has to do with the distribution of attacks over the staves in a score, the other with the coordination of rhythm over the entire score. And, as usual, there is the grouping degree of freedom, where multiple time signatures may be selected. All this leads to a multitude of cases that will generate shorter or longer examples with coordinated time.

The section titles below are copied from the Schillinger volumes [3]; they may be puzzling at first sight, but they will become clear as we interpret them in terms of what we have learned in earlier chapters.

8.1 Distribution of a duration group through instrumental and attack groups

This section shows the principle of multiple independent generators that are applied to the general case of a set of staves in a score. In the case of a single staff, the number of pitches on that staff is relevant.[1] The time structure coordination process is illustrated in Fig. 8.1. The upper diagram shows the single staff case with an ordered set of pitches, the lower diagram shows the general case of multiple instruments in a score.

The coordination of time structures involves three subprocesses, three recurrence steps:

1. *Attack distribution recurrence*. This implies the synchronization of the pitch attack distribution group r_p with N_p attacks with the number of instruments N_I in the score. In general $N_p > N_I$ and the instruments will play more than one attack in a single pattern statement. The lower diagram shows four instruments $N_I = 4$ and three subgroups

[1] Remember that attack groups and attack series are synonyms. The section titles in this chapter from the Schillinger book use the attack groups alternative.

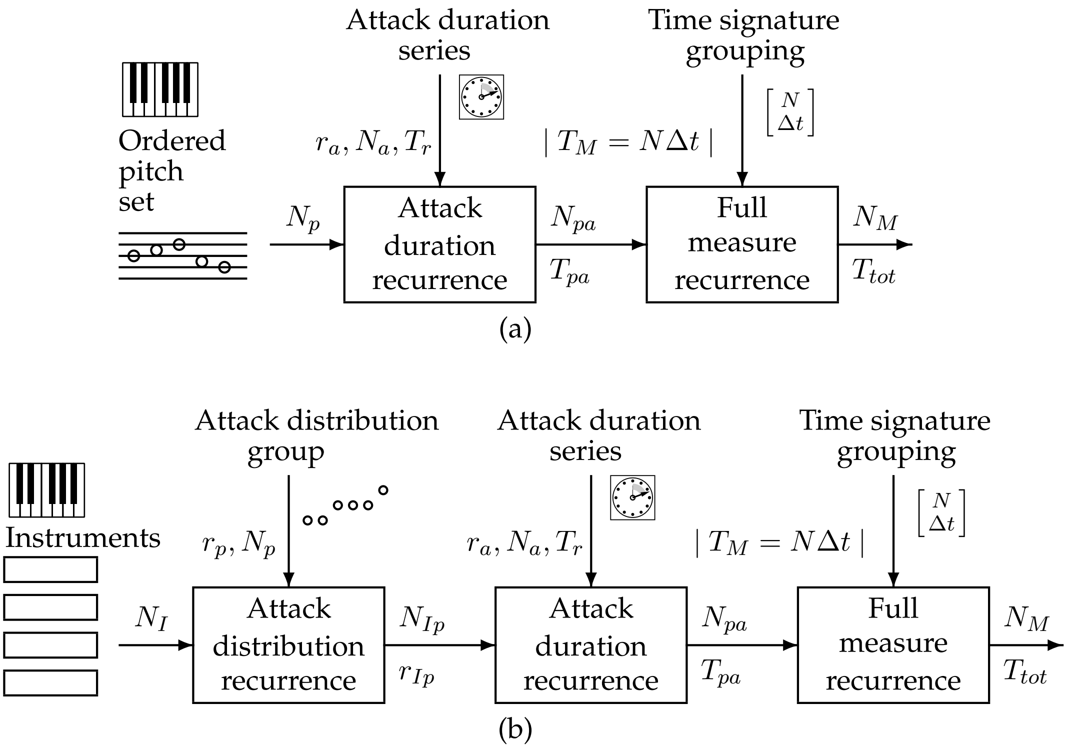

Figure 8.1: The process of coordination of time structures through interference and recurrence. a): Synchronization of an attack-duration group for a single staff with ordered pitch set, b): Synchronization of an instrument group with and attack group for multiple instruments in a score. The final subprocess for both cases is the grouping into measures for a specific meter.

of attacks $r_p = 2a + 3a + a, N_p = 6$. Recurrence requires more than one statement of the pitch attack distribution group. In general $N_{Ip} = N_1 N_p$ with $N_1 > 1$. In the sketch we have $N_I = 4$ and thus $N_{Ip} = 4N_p = 24$ attacks in a specific pitch order r_{Ip}. This corresponds to $N_1 = 4$, four statements of the pitch attack distribution pattern.

2. *Attack-duration recurrence.* In this step the rhythm will be determined from the attack-duration series r_a with N_a elements and a total duration of T_r time units. We synchronize the N_{Ip} element pitch attack group with the N_a note duration series. This also in general will imply multiple statements of both, i.e., $N_2 r_a$ and $N_3 r_{Ip}$, before there is recurrence. This determines the total set of durations which now has $N_{pa} = N_2 N_a$ elements and the total duration $T_{pa} = N_2 T_r$ time units.

3. In the *time signature grouping* process we choose the time unit Δt and a meter $\left[\begin{smallmatrix} N \\ \Delta t \end{smallmatrix}\right]$. The duration of a single measure is $T_M = N\Delta t$. We synchronize the N_{pa} duration pattern with the measure length until we find recurrence after N_M full measures; this may require multiple statements $N_4 N_{pa}$. The total duration thence is $T_{tot} = N_M T_M = N_4 T_{pa}$.

In the case of the single staff with ordered pitches in Fig. 8.1.a the first subprocess is missing. Instead we synchronize the attack-duration series r_t with the N_p elements pitch set. The entire process is strongly dependent on a careful selection of the attack distribution group r_a, the attack duration series r_t and the time signature grouping $\left[\begin{smallmatrix} N \\ t \end{smallmatrix}\right]$. There is some calculus involved. When the integer values N_1, N_2, N_3 or N_4 become large numbers, we will have very long patterns with lots of offbeat and afterbeat attacks. The potential of achieving interesting non-standard rhythmic patterns then has to be balanced against the overall length until the pattern repeats.[2]

8.2 Synchronization of an attack group with a duration group

Example 8.1

Synchronize an attack group with a duration series.
This example is included in the full version of the book.

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[2]Creating the examples in the following sections meant careful design, some trial-and-error, and lots of number checking.

The figure is included in the full version

Figure 8.2: Synchronize an attack group with a duration series

8.3 Distribution of a synchronized duration group through the final duration group

This section is revisiting the aspect of grouping and using a specific meter. It refers to the last recurrence subprocess in Fig. 8.1. The *final duration group* is the number of time units in the time signature, i.e., a single measure contains $T = Nt$ time units.[3] For example the $\begin{bmatrix}4\\4\end{bmatrix}$ meter is equivalent to $T = 4t$ with a quarter note ($t = \frac{1}{4}$) time unit, $\begin{bmatrix}6\\8\end{bmatrix}$ meter implies $T = 6t = 2 \times 3t$ with 8th note ($t = \frac{1}{8}$) time units.

Grouping options were already discussed in Chapter 3. There is nothing new in this section, except for Schillinger introducing the notion of the final duration group. Example 8.1 shows the effect of two final duration groups (two meters) for each case.

8.4 Synchronization of an instrumental group with an attack group

This is the case of a score with multiple instruments on separate staves, which will require the three steps sketched in Fig. 8.1.b. There is more freedom now, as we need a resultant r_p (with a total of N_p attacks) to determine the distribution of attacks over the number of instruments N_I in the score and a second resultant r_a (an N_a element attack-duration group) to coordinate the rhythm over the entire score. And finally, the time signature grouping selection will determine the number of measures until recurrence.

The double use of the term *attack group* in this context is confusing. On a single staff it has been used to control the time domain, in the case of a score it is used also to distribute attacks over instruments on separate staves. Hopefully, the examples below will clarify this issue.

Schillinger demonstrated this process with single pitches on each staff in order to concentrate on the aspect of rhythm. In the example here there are also multiple pitches on the staff; that might better illustrate how the coordinated time techniques may be applied to generate musical scores.

Example 8.2

Synchronization of a three-layer instrumental group with an attack and duration group.

The example is based on an extended dominant chord $G_7^{\flat 9/13}$ distributed over three instruments in a score ($N_I = 3$). From bottom to top the layers contain one (I_1: root), two (I_2: third and dominant 7th) and three (I_3: the extensions plus doubled third) pitches. The synchronization is demonstrated for two cases.

- Case 1: the pitch distribution attack series is $r_p = 3a + 2a$ ($N_p = 5$), i.e., three attacks in one layer followed by two attacks in the next layer. Recurrence requires three iterations of the attack group: $(I_1 + I_2) + (I_3 + I_1) + (I_2 + I_3)$ with a total of $N_{Ip} = 15$ attacks, as shown in Fig. 8.3.a. We demonstrate three possible attack-duration synchronizations, see Fig. 8.4:[4]

[3] Here we write $t = \Delta t$, indicating the time unit for rhythmic division.

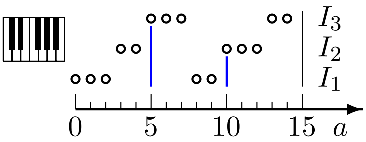
[4] Horizontal brackets in the score indicate the first statement of the attack distribution or the duration group.

1. Uniform distribution t, grouping by $T_M = 5t$, i.e., $\left[\frac{5}{4}\right]$. The time unit is a quarter note $t = \frac{1}{4}$. All attacks will be one time unit long, and the example is three measures long (m. 2–4).
2. Non-uniform duration series $r_a = 3t + t + 2t + t + t$; note that this is not the resultant of a two-generator mechanism, but a traditional rhythm. The number of elements in the time series is equal to the number of elements in the attack series $N_a = N_p$. The time unit is $t = \frac{1}{16}$, grouped by $T = 2 \times 4t = 8t$, yielding a $\left[\frac{2}{4}\right]$ meter. Again, there is recurrence after three measures (m. 5–7).
3. The duration series is the fractioning resultant $r_{a(4\div3)} = 3t + t + 2t + t + t + t + t + 2t + t + 3t$, with $N_a = 10$ and total duration $T_r = 16t$. Synchronization of attack and duration series requires three statements of the resultant $N_2 = 3$, and $N_{pa} = 2N_{Ip} = 30$, total duration $T_{tot} = 3T_r = 48t$. Grouping with $T_M = 8t$ at time unit $t = \frac{1}{8}$ yields a $\left[\frac{4}{4}\right]$ time signature and a total of six measures (m. 8–13).

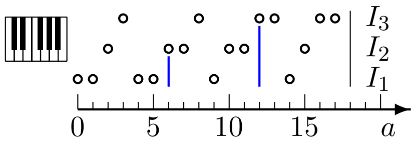
- Case 2: the pitch distribution attack group now is generated through non-uniform binary synchronization, i.e., $r_{p(3\div2)}$. This yields $\vec{p} = 2a + a + a + 2a$, $N_p = 6$, see Fig. 8.3.b. Once again, we need three attack group iterations to achieve recurrence: $(I_1+I_2+I_3+I_1)+(I_2+I_3+I_1+I_2)+(I_3+I_1+I_2+I_3)$ with a total of $N_{Ip} = 18$ attacks. Three attack-duration synchronization results are shown in Fig. 8.5.

  1. Uniform distribution t, time unit $t = \frac{1}{8}$, grouping by $T_M = 6t$, i.e., $\left[\frac{6}{8}\right]$, see m. 1–3.
  2. Non-uniform binary synchronization with $r_{a(3\div2)} = 2t + t + t + 2t$, $N_a = 4$ with total duration $T_r = 6t$. Attack-duration recurrence requires $N_{pa} = 2N_{Ip} = 36$, i.e., nine statemens of r_a with a total duration of $T_{pa} = 54t$. Grouping by $T_M = 9t$ at the time unit $t = \frac{1}{8}$ then leads to a six-measure pattern, m. 4–9, with $N_M = 6, T_{tot} = T_{pa}$.
  3. Finally, we use an expanding adjacent groups resultant $r_E(5,3)$ with $N_a = 24$ and total duration $T_r = 40t$ (see Section 5.2). Recurrence is achieved after $N_{pa} = 4N_{Ip} = 72$, i.e., three statemens of $r_E(5,3)$ and a total duration of $T_{pa} = 120t$. We group at $T_M = 8t$, time unit $t = \frac{1}{8}$, time signature $\left[\frac{4}{4}\right]$ into a total of $N_M = 15$ measures, see m. 10-24. This example demonstrates that we may extend the duration of notes, see e.g., the sustained chords in the upper layer in m. 10–11 and 14. The rhythm is determined by the attacks, the beginning of the notes. Since the entire example is based on a single given chord, there will not be a harmonic clash when we extend the note durations.

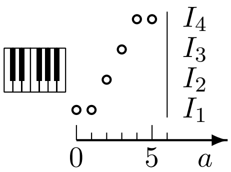
Note that the order of the attack subgroups has always been from the instrument on the bottom staff in the score to the upper staff instrument, or equivalently, from layer 1 to layer N_I. In the example this also implies from lower to higher pitches. This, however, is another degree of freedom. In a different staff order or with a modified attack distribution group



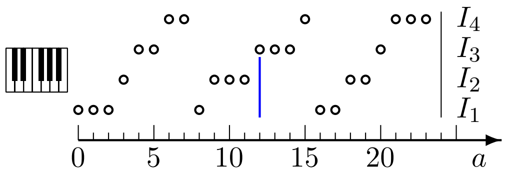
(a) $r_p = 3a + 2a, N_I = 3$



(b) $r_{p(3\div 2)}, N_I = 3$



(c) $r_{p(3\div 2)}, N_I = 4$



(d) $r_{p(4\div 3)}, N_I = 4$

Figure 8.3: Pitch attack group distribution over multiple instruments in a score. a): $r_p = 3a + 2a$, $N_I = 3$, b): $r_{p(3\div 2)}, N_I = 3$, c): $r_{p(3\div 2)}, N_I = 4$, d): $r_{p(4\div 3)}, N_I = 4$.

Figure 8.4: Synchronization of a three-layer instrumental group with an attack group. The score distributes an extended $G_7^{\flat 9/13}$ over three instruments. Case 1 is based on the pitch attack distribution group $3a + 2a, N_p = 5$. Shown are a uniform (m. 2–4), a non-uniform (m. 5–7) duration series and a fractioning attack-duration pattern $r_{\underline{4\div 3}}$ (m. 8–13).

Figure 8.5: Synchronization of a three-layer instrumental group with an attack group (cont'd). The three-layer $G_7^{\flat 9/13}$ with pitch attack distribution Case 2 ($r_p = 3 \div 2$) is synchronized with a uniform (m. 1–3), a non-uniform binary synchronization $r_{3\div 2}$ (m. 4–9) and an expanded group pairing $r_E(5,3)$ (m. 10–24) attack-duration series.

(that skips or swaps certain instruments) the coordination process would have resulted in a different recurrence pattern. The time structure coordination technique and the subprocesses would have remained identical, though.

The next example demonstrates how the same principle may be applied to multiple layers for a single instrument; either a four-note arpeggio pattern for a monophonic instrument or a chord voicing for a polyphonic instrument as is the case here.

Example 8.3

Synchronization of a four-layer instrument with an attack and duration group.
This example is included in the full version of the book.

The figure is included in the full version

Figure 8.6: Synchronization of a four-layer instrument with an attack group

Chapter 9

Homogeneous simultaneity and continuity (variations)

This chapter introduces the concept of *variation* for short patterns. The variation technique is used to prevent monotony in the case of repetition of shorter patterns with just a few attacks. We have seen in the previous chapters how sets of generators with higher values for the periodicity (slowly ticking metronomes) may generate long attack-duration patterns (see Chapters 2 to 6). Interference between instrument or score attack groups, duration series and time signature grouping also may lead to long patterns before recurrence takes place (see Chapter 7 and 8.) Limited musical perception memory will disguise the repeats in these cases.

For shorter patterns, typically between two and five elements in the attack-duration group, we may apply the technique of *permutation* in order to create variation. The original pattern plus the set of variations may then be used either in series to create *continuity* or in parallel on multiples staves to create *simultaneity*. The examples will also demonstrate the combination of both continuity and simultaneity. The permutation approach will be applied to various attack group parameters such as duration (note length), note vs. rests, dynamics (accented notes), split-unit groups and groups in general (higher order permutations). The use of a simple starting attack pattern as the basis for the variations guarantuees the homogeneous character of the result.

9.1 Variation through circular and general permutations

Consider an ordered set with N elements; e.g., abc is a set with $N = 3$ elements, where a is the first, b the second and c the third element. There are several options for changing the element order: these are called *permutations*.

We may discern two types of permutations for a set containing N different (non-equal) elements:

1. the set has $N! = 1 \times 2 \times \ldots \times N$ *general permutations*. This set of general permutations contains all possible orderings of the elements in the original set.

2. the set has N *circular permutations*. Imagine positioning the elements on a disk, starting at the top; then rotate the disk by one element in clockwise direction and the last element appears on top. This is the first circular permutation. Repeat the rotation process

until the first element returns on top. Alternatively, we may change the direction and move in counterclockwise direction to obtain the circular permutations in a different order.

The set size of possible permutations will change when the original set contains a number of equal elements, but let's first consider the case for dissimilar elements.

9.1.1 The set of permutations

Permutations of a set of two elements

The two-element set is $\{a, b\}$, $N = 2$. The mathematics tell us that there are $N! = 2! = 1 \times 2 = 2$ general and $N = 2$ circular permutations. Well, here they are:

$$\{ab, ba\},$$

where the general and the circular permutations are identical, since we can only swap the two elements.[1]

Example 9.1

Permutation of a two-element set.
The two-element group permutation is shown in musical notation in Fig. 9.1 m. 1–2, where it is applied to the attack-duration group $r = 3t + t, a = 3t, b = t$.[2] The time unit is the quarter note $t = \frac{1}{4}$, the grouping time signature is $\left[\begin{smallmatrix}4\\4\end{smallmatrix}\right]$.

Permutations of a set of three elements

The number of permutations increases for $N = 3$, i.e., the set $\{a, b, c\}$. There are $N! = 3! = 1 \times 2 \times 3 = 6$ general permutations

$$\{abc, acb, bac, bca, cab, cba\}$$

and $N = 3$ circular permutations

$$\{abc, bca, cab\}.$$

Note that the circular permutations are included in the set of general permutations.

Example 9.2

Circular and general permutations of a three-element set.
See the three-element group permutations in rhythmic musical notation in Fig. 9.1 m. 3–11, where it is applied to the attack-duration group $r = 3t + t + 2t$, i.e., $a = 3t, b = t, c = 2t$. First the three circular permutations $\{abc, bca, cab\}$ are shown (m. 3–5), then there are the six general permutations. The time unit is $t = \frac{1}{8}$ note, the grouping time signature is $\left[\begin{smallmatrix}6\\8\end{smallmatrix}\right]$.

[1] Note that the possible permutations are represented as a comma-separated superset of ordered sets. Later we will see how this superset is used in music as either a continuity or simultaneity.

[2] Note that we have used the symbol a once again. Now it represents the duration of the first element in an attack-duration group.

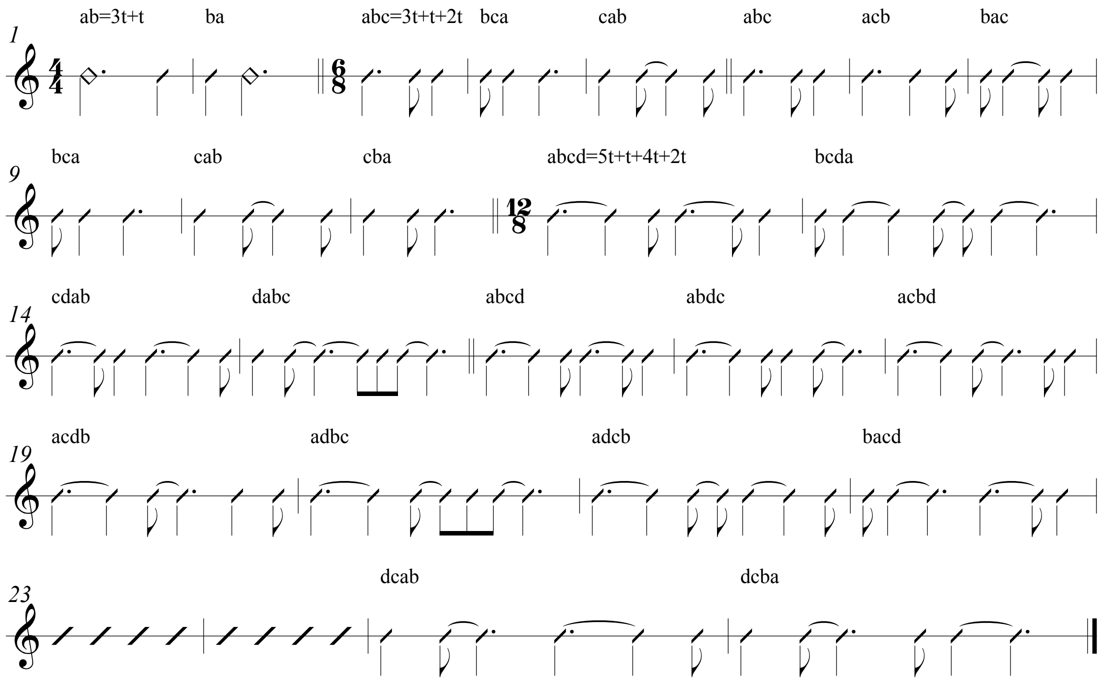


Figure 9.1: Variation through permutations. Circular and general permutations are shown in rhythmic notation for a two-element ($N = 2$, m. 1–2), three-element ($N = 3$, m. 3–11) and four-element attack-duration group ($N = 4$, m. 12–26). For the four-element set not all general permutations are shown, as indicated by the slashes.

Permutations of a set of four elements

Using the same approach for a set of $N = 4$ elements $\{a, b, c, d\}$ yields $N! = 4! = 1 \times 2 \times 3 \times 4 = 24$ general permutations. Here they are:

$$\begin{array}{llllllll} \{abcd, & abdc, & acbd, & acdb, & adbc, & adcb, & bacd, & badc, \\ bcad, & bcda, & bdac, & bdca, & cabd, & cadb, & cbad, & cbda, \\ cdab, & cdba, & dabc, & dacb, & dbac, & dbca, & dcab, & dcba\}. \end{array}$$

There are $N = 4$ circular permutations

$$\{abcd, bcda, cdab, dabc\}.$$

It will be obvious that for larger sets the number of general permutations increases exponentially. For the next larger set, with $N \geq 5$, there are 120 general permutations. Using the complete permutation set in a rhythm is most unlikely. I tis more likely to use a specific subset in a musical application.

Example 9.3

Circular and general permutations of a four-element set.
The four-element group permutations in musical notation are shown in Fig. 9.1 m. 12–27, where it is applied to the attack-duration group $r = 5t + t + 4t + 2t, a = 5t, b = t, c = 4t, d = 2t$. The four circular permutations $\{abcd, bcda, cdab, dabc\}$ are shown in m. 12–15, then the incomplete set of 24 general permutations is shown. The time unit is $t = \frac{1}{8}$ note, the grouping time signature is $\left[\frac{12}{8}\right]$.

Permutations of a set containing equal elements

Until now all set elements were different. Let's consider how the number of permutations decreases when there are equal elements in the set. For $N = 3$ this implies two equal elements $N_e = 2$, i.e., $\{a, a, b\}$. Now the reduced set of permutations becomes

$$\{aab, aba, baa\},$$

with the number of permutations given by

$$\frac{N!}{N_e!(N - N_e)!} = \frac{3!}{2!\,1!} = \frac{6}{2} = 3. \tag{9.1}$$

Next case is three equal elements, $N_e = 3$ in a set with $N = 4$ elements: $\{a, a, a, b\}$. The number of permutations is

$$\frac{N!}{N_e!(N - N_e)!} = \frac{4!}{3!\,1!} = \frac{24}{6} = 4,$$

i.e.,

$$\{aaab, aaba, abaa, baaa\}.$$

When there are two equal elements $\{a, a, b, c\}$ in this $N = 4$ set, i.e., $N_a = 2, N_b = 1, N_c = 1$, we find

$$\frac{N!}{N_a!N_b!N_c!} = \frac{4!}{2!\,1!\,1!} = \frac{24}{2} = 12, \tag{9.2}$$

permutations, i.e.,

$$\{aabc, aacb, abac, abca, acba, baca, bcaa, cbaa, acab, caab, baac, caba\}.$$

For the case two equal subsets $\{a, a, b, b\}$, i.e., $N = 4, N_a = 2, N_b = 2$ there are

$$\frac{N!}{N_a!N_b!} = \frac{4!}{2!\,2!} = \frac{24}{4} = 6,$$

permutations, i.e.,

$$\{aabb, abab, abba, baba, bbaa, baab\}.$$

9.1.2 Permutations in continuity and simultaneity

9.2 Permutations applied to musical parameters

Now that we have seen the process of obtaining the permutation sets and the creation of a time continuity or a vertical distribution over multiple parts, let's decide to what musical parameter we will apply the pattern. Mostly these patterns are either a two-element attack-duration series, called a *binomial*, or a three-element series, called a *trinomial*.[3] In a binomial the periodicity of the pattern is determined by two coefficients, in a trinomial by three coefficients (we encountered the monomial periodicity at the beginning of this book, see Section 1.3.3).

Schillinger introduces a number of new aspects in this chapter. Here is his application proposal:

Durations: Permutation of an attack-duration series r_a is the predictable application after reading the previous chapters. Suppose a two-element duration series, e.g., the binomial $r_a = 2t + t$, the continuity based on permutations yields $r_c = (2t + t) + (t + 2t)$ or $r_c = (t + 2t) + (2t + t)$. However, there are other parameters that may undergo permutation and variation.

Rests: When the duration series consists of a combination of note attacks and rests, we may also apply permutations. Suppose we have the trinomial $r_a = t + 3t + 2\bar{r}$, where $\bar{r}$ indicates a rest of a certain number of time units, we may create a continuity based on general permutations. For example $r_c = (t+3t+2\bar{r})+(3t+t+2\bar{r})+(2\bar{r}+3t+t)+(t+2\bar{r}+3t)$, a continuity of four permutations. A possible simultaneity based on circular permutations would be

$$r_s = \begin{pmatrix} (t+3t+2\bar{r}) + (2\bar{r}+t+3t) + (3t+2\bar{r}+t) \\ (2\bar{r}+t+3t) + (3t+2\bar{r}+t) + (t+3t+2\bar{r}) \\ (3t+2\bar{r}+t) + (t+3t+2\bar{r}) + (2\bar{r}+t+3t) \end{pmatrix}.$$

Accents: Suppose there are accented notes in the duration series. These are equivalent to rests in the pattern. Suppose we have the trinomial $r_a = \hat{t} + t + 2t$, where $\hat{t}$ indicates the accented note, we may create a continuity based on circular permutations. For example $r_c = (2t+\hat{t}+t)+(\hat{t}+t+2t)+(t+2t+\hat{t})$. A possible four-part combination of continuity and simultaneity based on general permutations would be

$$r_{cs} = \begin{pmatrix} (\hat{t}+t+2t) + (\hat{t}+2t+t) + (t+\hat{t}+2t) \\ (t+\hat{t}+2t) + (\hat{t}+t+2t) + (2t+t+\hat{t}) \\ (\hat{t}+2t+t) + (2t+t+\hat{t}) + (t+2t+\hat{t}) \\ (2t+t+\hat{t}) + (t+2t+\hat{t}) + (\hat{t}+t+2t) \end{pmatrix}.$$

Note that the number of parts $N_p = 4$ here is greater than the number of permutations in the continuity. Some parts may have simultaneous rests, but each part is playing an independent rhythm. By now it should be obvious that we have a wealth of possibilities to create rhythmic multi-part scores.

Split-unit groups: This is another new technique for creating variation. A given resultant attack-duration series may be split into one or more subgroups. These subgroups with

[3] The labels binomial and trinomial might have been introduced earlier in this text, but adhering to the original Schillinger book, they are being used from here on regularly.

their elements will be treated as constant, but we probably will have more options for permutations than before. Let's illustrate that with the simplest example, where the original resultant is the binomial $r = 2t + 2t$. This pattern has no permutations. Now, split one element into smaller time units: $r = (t + t) + 2t$. The two subgroups can now be permutated, yielding the set $\{(t + t) + 2t, 2t + (t + t)\}$ and this can be used for a continuity or simultaneity

$$r_c = [(t+t)+2t] + [2t+(t+t)] \quad \text{or} \quad r_{cs} = \begin{pmatrix} (t+t)+2t & + & 2t+(t+t) \\ 2t+(t+t) & + & (t+t)+2t \end{pmatrix}$$

Another example is the trinomial $r = 3t + 2t + 3t$, with the first element split into subgroups. There we have the options $r = (2t + t) + 2t + 3t$ or $r = (t + t + t) + 2t + 3t$. With these split-unit groups we may create a continuity such as $r_c = [(2t + t) + 2t + 3t] + [2t + 3t + (2t + t)] + [3t + (2t + t) + 2t]$ or a simultaneity

$$r_s = \begin{pmatrix} (t+t+t)+2t+3t \\ 2t+(t+t+t)+3t \\ 3t+2t+(t+t+t) \end{pmatrix}.$$

Groups in general: Schillinger now introduces permutation at a higher aggregation level. Implicitly we have already encountered that option in Section 9.1.2, where the three-part combination of continuity and simultaneity was shown. The simplest possible option here is the two-attack binomial resultant $r = 2t + t$ (total duration $T = 3t$). The two permutations in continuity are $a_c = (2t + t) + (t + 2t)$ and $b_c = (t + 2t) + (2t + t)$. Permutation of this higher level two-element set $\{a, b\}$ yields $r_{c,1} = a_c b_c = [(2t + t) + (t+2t)]+[(t+2t)+(2t+t)]$ and $r_{c,2} = b_c a_c = [(t+2t)+(2t+t)]+[(2t+t)+(t+2t)]$ with total duration $T' = 12t$.[4] And thus we may create a continuity $r_c = (r_{c,1}r_{c,2}) + (r_{c,2}r_{c,1})$ with a total duration of $T" = 24t$ or a simultaneity

$$r_s = \begin{pmatrix} r_{c,1}r_{c,2} & = a_c b_c + b_c a_c \\ r_{c,2}r_{c,1} & = b_c a_c + a_c b_c \end{pmatrix}$$

with total duration $T''' = 12t$. We have achieved a *self-scaling property* that may be repeated forever at higher aggregation levels. This variation technique will be further discussed in Chapter 10.

After all this mathematics it is appropriate to demonstrate the application of permutations in musical notation.

9.3 Application to attack-duration groups

We will apply the permutation technique to two- (binomial), three- (trinomial) and four-element attack-duration groups. The permutations may affect any of the five musical parameters listed in Section 9.2. The figures will group the examples in musical notation on two, three or four staves in a score.

[4]The notation of permutations used pairs ab. In this application section we will use that notation as an abbreviation for $r = ab = at + bt$. The same holds for the higher level continuity $r_c = r_{c,1}r_{c,2}$.

Example 9.4

Permutation of an attack group with two elements.
This example is based on two-element ($N = 2$) attack-duration groups with resultant $r = ab$.

- The first case uses the binomial $r = 2t + t, a = 2t, b = t$ and is shown in rhythm notation in Fig. 9.2 m. 1–8 as a combination of continuity ($r_c = ab$) and simultaneity $r_s = \begin{bmatrix} a \\ b \end{bmatrix}$. Permutation is applied to various musical parameters:

  1. The *note duration* permutations are $ab = (2t + t) + (t + 2t)$ and $ba = (t + 2t) + (2t + t)$, with time unit $t = \frac{1}{4}$ and $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ meter. This is shown as a continuity $r_c = ab$ on the upper staff in m. 1–2. The simultaneity in m. 1 is $r_s = \begin{bmatrix} a \\ b \end{bmatrix}$, the combination in m. 1–2 is $r_{cs} = \begin{bmatrix} ab \\ ba \end{bmatrix}$.
  2. When the second attack is replaced by a *rest*, the pattern becomes $r = 2t + \bar{r}$.[5] Measure 3 shows the continuity and simultaneity r_{cs} at time unit $t = \frac{1}{8}$ and grouping $\begin{bmatrix} 6 \\ 8 \end{bmatrix}$.
  3. The *accent* on the second attack $b = \hat{t}$ is demonstrated in m. 4.[6] Accented and regular attacks will sound simultaneously.
  4. In order to demonstrate the *split-unit group* approach, we consider the resultant $r = 2t + 2t, a = 2t, b = 2t$. Since both elements are equal there is no permutation, hence no potential for creating variation. By splitting the first element into two smaller units $a = (t + t)$ we obtain non-equal elements. The r_{cs} permutation is shown in m. 5–6 at grouping $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$. Remember that we keep the two smaller units together as a constant subgroup; therefore the permutation is $t + 2t + t$ is not allowed here (that would be a three-element attack group).
  5. Finally, in m. 7–8 we see the higher order permutation, that comes with the *groups in general* approach. The time unit is $t = \frac{1}{4}$, the grouping at $\begin{bmatrix} 6 \\ 4 \end{bmatrix}$ meter. The upper staff shows the continuity $r_{c,u} = r_{c,1} r_{c,2} = abba = [(2t + t) + (t + 2t)] + [(t + 2t) + (2t + t)]$, with $r_{c,1} = ab = (2t + t) + (t + 2t)$ and $r_{c,2} = ba = (t + 2t) + (2t + t)$. The lower staff contains $r_{c,l} = r_{c,2} r_{c,1} = baab$, and the score has the combination $r_{cs} = \begin{bmatrix} abba \\ baab \end{bmatrix}$.

- In Fig. 9.3, m. 1–2 the binomial attack-duration group $r = 3t + t$ ($N = 2, a = 3t, b = t$) is distributed over three staves, with pitches based on the extended tonic chord G_9^6, voiced in perfect fourths. There are two pitches per instrument, with the higher pitch assigned to the a-element, and the other to b.

[5] The rest with duration t is is indicated in the score as the text label [t] at the top, and is equivalent to $\bar{r}$ in the text. This alternative notation is simpler and faster in the music notation software.
[6] The accent label above the staff is written as t', and is equivalent to $\hat{t}$ in the text.

The distribution is based on the scheme

$$r_{cs} = \begin{pmatrix} ab & + & ba & & & & \\ ba & + & ab & + & ab & + & ba \\ & & & + & ba & + & ab \end{pmatrix}.$$

Here we have deliberately used more freedom in creating a distributed rhythm.

Example 9.5

Permutation of an attack group with three elements.

This example is included in the full version of the book.

This section is included in the full version of the book.
Order the E-book from the webstore at:
https://www.fransabsil.nl/htm/rhythmbk.htm

Figure 9.2: Variation through permutation of attack-duration groups. Two-, three- and four-element duration groups ($N = 2, 3, 4$) are assigned to two staves.

Figure 9.3: Variation through permutation of attack-duration groups. Two-, three- and four-element duration groups ($N = 2, 3, 4$) are distributed over three staves in the score.

The figure is included in the full version

Figure 9.4: Variation through permutation of attack-duration groups, four staves

Example 9.6

Permutation of an attack group with four elements.
In this case the resultant attack-duration group consists of four elements $abcd$, $N = 4$. There are 24 general permutations; using all of these will create very long patterns, so it is more likely to use either a subset from the general permutations, or use the four circular permutations. We will apply these alternatives to different musical parameters.

- In Fig. 9.2, m. 30–33 there is the four-element group $r = 4t + 2t + 3t + 3t$, time unit $t = \frac{1}{8}$, time grouping at $\left[\frac{12}{8}\right]$. So $a = 4t, b = 2t, c = 3t, d = 3t$. For the varied pattern we use the four circular permutations, according to the two-staff scheme

$$r_{cs} = \begin{pmatrix} abcd & + & bcda & + & cdab & + & dabc \\ bcda & + & cdab & + & dabc & + & abcd \end{pmatrix},$$

  which yields a four-measure pattern, shown in rhythmical notation.
- The previous example is modified in m. 34–37 in the sense that we introduce a rest for the first element $a = 4\bar{r}$, the second element attack is accented $b = 2\hat{t}$ and the fourth element becomes a split-unit group $d = (2t + t)$ with non-equal units. The extended dominant chord $D_7^{9/\text{sus4}}$ is distributed over a lower layer with three pitches and attack group $r_{p,l} = a(p_1) + a(p_2) + a(p_3) + a(p_2)$. Again, this example was constructed with $N_{p,l} = N_a = 4$. The upper layer plays three-part chords. Homogeneous variation was based on the four circular permutations using the scheme from the previous example. The result might be used as a rhythmic groove in Afro-Cuban style Latin music.
- In Fig. 9.3, m. 22–25, there is the four-element group $r = abac = t + (\frac{1}{3}t + \frac{1}{3}t + \frac{1}{3}t) + t + \bar{r}$, with a triplet elements b, time unit $t = \frac{1}{4}$ and grouping at $\left[\frac{4}{4}\right]$. This attack-duration group is distributed over three staves, the chord structure is $Am_7^{9/11}$. The example uses a continuity of four circular permutations $r_c = abac + caba + acab + baca$. This application demonstrates two effects: first there is the division of a time unit into triplets, and then there is the variable density setting within the staff and between staves. This implies that at any time instant there are between two and four pitches sounding. Each staff uses different pitches for the a- and b-element.
- Distribution over a four-staff score in demonstrated in Fig. 9.4. In m. 7–12 there is the attack-duration group $r = abab = 2t + (\bar{r} + t) + 2t + (\bar{r} + t)$, with a split-unit b-element. This is a typical kick drum pattern in Rock music. Since both a and b occur twice, or, $N_a = N_b = 2$, there are

$$\frac{N!}{N_a!\,N_b!} = \frac{4!}{2!\,2!} = \frac{24}{4} = 6$$

  general permutations. These are shown as a continuity in the lower staff $r_c = abab + abba + aabb + baab + baba + bbaa$; note the position of the syncopated and afterbeat 8th notes. The upper parts imitate this pattern, yielding a variable density setting that combines more and more permutations in parallel as a simultaneity.

- In the same figure, in m. 13–14, we find the attack-duration group with split-unit elements $r = abcc = 2t + (t + t) + (\bar{r} + \hat{t}) + (\bar{r} + t)$, time unit $t = \frac{1}{8}$, grouped at meter $\left[\begin{smallmatrix}4\\4\end{smallmatrix}\right]$. Ignore the accent on the first c-element, which is not considered a separate element for this example. We use the four circular permutations as a combination of four-part simultaneity and two patterns in continuity, using the scheme

$$r_{cs} = \begin{pmatrix} abcc & + & cabc \\ bcaa & + & abcc \\ ccab & + & bcca \\ cabc & + & ccab \end{pmatrix}.$$

  Note how in this syncopated pattern the accents are moving through the parts; this has the flavour of a Latin music percussion section. In fact, the rhythm in the first staff $r_c = abcc + cabc$ is a variation of the *cascara* pattern for timbales.

- The final four-element group in this example is in m. 15–16, with the attack-duration group $r = abbb = \bar{r}+t+t+t$. Since $N_b = 3$, there are $N!/(N_a!\,N_b!) = 4!/(1!\,3!) = 24/6 = 4$ general permutations, which are also the circular permutations. These are shown as a continuity $r_c = abbb + babb + bbab + bbba$, with two pitches in each staff. Together these form a minor 6-pitch scale $d - e - f - g - a - d$, giving the setting a minimal music cell flavour.

Example 9.7

Permutation of an attack group with five elements.
This example is included in the full version of the book.

Chapter 10

Generalization of variation techniques

This chapter elaborates the concept of *higher order permutations* as a generalization of variation techniques. The technique was in fact introduced at the first higher level in Section 9.2, where it was called *Groups in general*. Here we will investigate the sets of permutations that arise when we extend the technique to higher levels.

We will learn that the number of possiblities grows exponentially, when we perform higher order permutations. Therefore, in this chapter we will discuss only simple cases for the two- and three-element basic groups.

10.1 Higher order permutations of a two-element group

The starting point is the core two-element group at the lowest level $L_0 : \{a_0, b_0\} = \{a, b\}$, where $\{a, b\}$ might be an attack-duration group, e.g., the binomial $\{a, b\} = \{3t, t\}$ with resultant $r = 3t + t$. The core elements may contain notes, rests, accented notes, or split-unit groups (see the list of attributes in Section 9.2).

Combining the elements at the first application level yields a continuity r_c with two permutations $r_c(L_1) = \{a_1, b_1\} = \{a + b, b + a\}$. What happens when we extend this approach to higher levels is illustrated in Fig. 10.1.

The number of elements in the n-th layer $N_{e,n} = 2^n$, creating elongated but unique rhythms at the higher level. The original attack-duration group $a + b$ returns as a subgroup at each level, but in different combinations with its retrograde, i.e., the time-reversed pattern $b + a$. Higher order permutations of a two-element group will now be illustrated with an example in rhythmical notation.

Example 10.1

Higher order permutations of a two-element group.
This example is based on two-element attack-duration groups at level $L_0 : \{a, b\}$. Figure 10.2 shows four staves, that represent the higher order levels $L_1, \ldots, L_4$.

- The first case uses the two-element attack-duration pattern $\{a, b\}$, $a = 3t, b = t$ with time unit $t = \frac{1}{8}$ and is shown in Fig. 10.2 m. 1–6. The first higher order permutation, level L_1 consists of the two possible permutations $a+b = 3t+t$ and $b + a = t + 3t$ (the figure uses the abbreviate notation $ab = a + b$; the measures with rests separate the patterns). The order of the elements is

Level L_n: $a_n = a_{n-1} + b_{n-1} = \dots$ $\quad$ $b_n = b_{n-1} + a_{n-1} = \dots$

$\vdots$ $\quad$ $\vdots$

Level L_4: $a_4 = a_3 + b_3 = (abbabaab) + \dots$ $\quad$ $b_4 = b_3 + a_3 = (baababba) + \dots$

Level L_3: $a_3 = a_2 + b_2 = (abba) + (baab)$ $\quad$ $b_3 = b_2 + a_2 = (baab) + (abba)$

Level L_2: $a_2 = a_1 + b_1 = (ab) + (ba)$ $\quad$ $b_2 = b_1 + a_1 = (ba) + (ab)$

Level L_1: $a_1 = a_0 + b_0 = (ab)$ $\quad$ $b_1 = b_0 + a_0 = (ba)$

Level L_0: $a_0 = a,\ b_0 = b$

Figure 10.1: Higher order permutations of a two-element group. Continuity $r_c(L_i)$ at level $L_i, i = 1, \dots, 4$ for the two-element group $\{a, b\}$ at the lowest level L_0. Abbreviated notation: $(ab) = a + b$; the order is relevant for the rhythm.

relevant. At level L_2 the permutations of the L_1 patterns are $L2 : \{(a + b) + (b + a), (b + a) + (a + b)\}$. Grouped at meter $\begin{bmatrix}4\\4\end{bmatrix}$ this leads to one-measure patterns. Permutation of these two patterns produces the set $L_3 : \{[(a+b)+(b+a)]+[(b+a)+(a+b)], [(b+a)+(a+b)]+[(a+b)+(b+a)]\}$, a pair of two-measure patterns. Finally, at level L_4 only one pattern is shown $r_c = [(a+b)+(b+a)+(b+a)+(a+b)]+[(b+a)+(a+b)+(a+b)+(b+a)]$. The self-scaling property is visible with the original a- and b-element returning in various positions and groupings along the continuity r_c.

- The second case is shown in Fig. 10.2 m. 7–15 and is based on a two-element attack-duration group $L_0 : \{a, b\}$ with the split-unit group $a = (2t + t + t)$ and the accented note $b = 2\hat{t}$ at time unit $t = \frac{1}{8}$.[1] Again, the process is demonstrated as higher order permutations at levels $L_1, \dots, L_4$. At the first level this yields the set $L_1 : \{a+b = (2t+t+t)+2\hat{t},\ b+a = 2\hat{t}+(2t+t+t)\}$, each a one-measure pattern at $\begin{bmatrix}3\\4\end{bmatrix}$ grouping.
- The third example is shown in m. 16–21 for $L_0 : \{a, b\}$ with two split-unit groups $a = (4t + t + t)$ and $b = (2t + 2\bar{r} + 2\hat{t})$, the latter containing a rest and an accented note. The time unit is $t = \frac{1}{16}$, grouped at $\begin{bmatrix}12\\8\end{bmatrix}$. At L_4 the first permutation $L_4 : r_c = [(a + b + b + a) + (b + a + a + b)] + [(b + a + a + b) + (a + b + b + a)]$ is a four-measure pattern. At time unit $t = \frac{1}{8}$ (the augmentation variation) and $\begin{bmatrix}3\\4\end{bmatrix}$ time signature this pattern could be used as a waltz rhythmical background.

[1] Once again, the notation in the score is slightly different from the text. An accented note is notated as t' in the score and as $\hat{t}$ in the text. The rest is notated as r in the score and corresponds to $\bar{r}$ in the text.

Figure 10.2: Higher order permutations of a two-element group $\{a, b\}$: example in rhythm notation.

10.2 Higher order permutations of a three-element group

The starting point now is a three-element group $L_0 : \{a_0, b_0, c_0\} = \{a, b, c\}$ and resultant $r = at + bt + ct$, where each element may consist of a note, rest, accented attack, or split-unit group. We will see that the higher order permutations contain a wealth of rhythmic variations. In fact the number will become so large, that Schillinger in his book only considers the second higher order level L_2. He also writes the possible cases now as *combinations* of elements, where the order is irrelevant. For each combination the number of permutations N_p is indicated.

We will consider the two possible cases of combinations by two or by three elements at the first level L_1.

10.2.1 Combinations by two elements at the first higher order level

Example 10.2

Higher order permutations of a three-element group: combinations by two elements.

This example is included in the full version of the book.

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The figure is included in the full version

Figure 10.3: Higher order permutations of a three-element group: combinations by two elements

10.2.2 Combinations by three elements at the first higher order level

The three-element combination at the first higher order level is $L_1 : \{abc\}$. This combination has six permutations: $\{a_1, b_1, c_1, d_1, e_1, f_1\} = \{a+b+c,\ a+c+b,\ b+a+c,\ b+c+a,\ c+a+b,\ c+b+a\}$. At the second higher order level L_2 we may use combinations of two, three, four, five and six elements from the first higher order level.

Level 2: Combination by two elements

The set of combinations of two elements at the second order level L_2 contains 15 elements

$$\begin{array}{lllll}
a_1+b_1 & b_1+c_1 & c_1+d_1 & d_1+e_1 & e_1+f_1 \\
a_1+c_1 & b_1+d_1 & c_1+e_1 & d_1+f_1 & \\
a_1+d_1 & b_1+e_1 & c_1+f_1 & & \\
a_1+e_1 & b_1+f_1 & & & \\
a_1+f_1 & & & &
\end{array}$$

The first combination $L2 : a_1 + b_1 = (a + b + c) + (a + c + b)$ is a rhythmic pattern with six elements. All elements occur twice in the pattern; at this level there obviously do not exist combinations with an incomplete set of source elements. Each combination consists of two subgroups, implying two permutations. Therefore, the total number of cases is $15 \times 2 = 30$.

Level 2: Combination by three elements

The combination of three element from level L_1 yields the following set at the next higher level L_2

$$\begin{array}{llll}
a_1+b_1+c_1 & b_1+c_1+d_1 & c_1+d_1+e_1 & d_1+e_1+f_1 \\
a_1+b_1+d_1 & b_1+c_1+e_1 & c_1+d_1+f_1 & \\
a_1+b_1+e_1 & b_1+c_1+f_1 & c_1+e_1+f_1 & \\
a_1+b_1+f_1 & b_1+d_1+e_1 & & \\
a_1+c_1+d_1 & b_1+d_1+f_1 & & \\
a_1+c_1+e_1 & b_1+e_1+f_1 & & \\
a_1+c_1+f_1 & & & \\
a_1+d_1+e_1 & & & \\
a_1+d_1+f_1 & & & \\
a_1+e_1+f_1 & & &
\end{array}$$

The first combination $L2 : a_1+b_1+c_1 = (a+b+c)+(a+c+b)+(b+a+c)$ is a rhythmic pattern with nine elements. The three subgroups in each combination yield six permutations. The total number now is $20 \times 6 = 120$ cases.

Level 2: Combination by four elements

Using a combination of four elements at level L_2 produces the set of combinations

$$\begin{array}{lll}
a_1 + b_1 + c_1 + d_1 & b_1 + c_1 + d_1 + e_1 & c_1 + d_1 + e_1 + f_1 \\
a_1 + b_1 + c_1 + e_1 & b_1 + c_1 + d_1 + f_1 & \\
a_1 + b_1 + c_1 + f_1 & b_1 + c_1 + e_1 + f_1 & \\
a_1 + b_1 + d_1 + e_1 & b_1 + d_1 + e_1 + f_1 & \\
a_1 + b_1 + d_1 + f_1 & & \\
a_1 + b_1 + e_1 + f_1 & & \\
a_1 + c_1 + d_1 + e_1 & & \\
a_1 + c_1 + d_1 + f_1 & & \\
a_1 + c_1 + e_1 + f_1 & & \\
a_1 + d_1 + e_1 + f_1 & &
\end{array}$$

The first combination $L2 : a_1+b_1+c_1+d_1 = (a+b+c)+(a+c+b)+(b+a+c)+(b+c+a)$ is a rhythmic pattern with 12 elements. The four subgroups in each combination yield $4! = 24$ permutations. The total number therefore is $15 \times 24 = 360$ cases.

Level 2: Combination by five elements

With a five elements combined the full set at level L_2 is

$$\begin{array}{ll}
a_1 + b_1 + c_1 + d_1 + e_1 & b_1 + c_1 + d_1 + e_1 + f_1 \\
a_1 + b_1 + c_1 + d_1 + f_1 & \\
a_1 + b_1 + c_1 + e_1 + f_1 & \\
a_1 + b_1 + d_1 + e_1 + f_1 & \\
a_1 + c_1 + d_1 + e_1 + f_1 &
\end{array}$$

The first combination $L2 : a_1 + b_1 + c_1 + d_1 + e_1 = (a + b + c) + (a + c + b) + (b + a + c) + (b + c + a) + (c + a + b)$ is a rhythmic pattern with 15 elements. Each combination has five subgroups, i.e., $5! = 120$ permutations. The total number of cases therefore is $6 \times 120 = 720$.

Level 2: Combination by six elements

Finally, we may combine all lower level elements into a single combination al level L_2 : $a_1+b_1+c_1+d_1+e_1+f_1 = (a+b+c)+(a+c+b)+(b+a+c)+(b+c+a)+(c+a+b)+(c+b+a)$, i.e., a rhythmic pattern with 18 elements and 720 permutations.

Example 10.3

Higher order permutations of a three-element group: combinations by three elements.
This example demonstrates the combinations by three elements, derived from a three-element attack-duration group with resultant $r = abc$ at level L_0. The single staff notation with three pitches is shown in Fig. 10.4.

- The first case uses the three-element set $L_0 := \{a, b, c\} = \{4t, 3t, t\}$ at time unit $t = \frac{1}{8}$, see m. 1. All six permutations of the first level L_1 combination by three elements $\{a_1, b_1, c_1, d_1, e_1, f_1\} = \{abc,\ acb,\ bac,\ bca,\ cab\ cba\}$ are shown in m. 2–7. With grouping at $\begin{bmatrix}4\\4\end{bmatrix}$ some or these permutations yield syncopated or afterbeat patterns. A combination by two elements at the next higher level is $L2 : a_1 + c_1 = (a + b + c) + (b + a + c)$, which has two permutations. These are combined into a continuity at level $L_3 : r_c = (a_1 + c_1) + (c_1 + c_1)$ in m. 8–11.

At level L_2 there are many possible combinations by three elements. The three-element subset with c_1 as the first element is $L2 : \{c_1 + d_1 + e_1,\ c_1 + d_1 + f_1,\ c_1 + e_1 + f_1\}$ and shown in m. 12–20. A continuity at the next higher level L_3, based on the three-element combination $L2 : b_1 + c_1 + d_1$ in the three circular permutations, is $L_3 : (b_1 + c_1 + d_1) + (c_1 + d_1 + b_1) + (d_1 + b_1 + c_1)$, see m. 21–29.

- The second case is based on the trinomial $L_0 := \{a, b, c\} = \{(2t + \frac{1}{2}t + \frac{1}{2}t), (\bar{r} + \hat{t}), t\}$ with two split-unit groups.[2] The time unit $t = \frac{1}{8}$ and the L_0 resultant is shown in Fig. 10.4 m. 30. The six three-element combinations at level L_1 are shown in m. 31–36. Like the first case, here the grouping at meter $\begin{bmatrix}6\\8\end{bmatrix}$, leads to syncopated and afterbeat rhythmic patterns.

  From the 24 possible combinations by four elements at level L_2 the complete subset with $b_1 = (a + c + b)$ as first element is $\{b_1 + c_1 + d_1 + e_1,\ b_1 + c_1 + d_1 + f_1,\ b_1 + c_1 + e_1 + f_1,\ b_1 + d_1 + e_1 + f_1\}$ and is shown in m. 37–52. For the five circular permutations for the combination by five elements $L_2 : a_1 + b_1 + c_1 + e_1 + f_1$, see m. 53–77. Finally, the retrograde version of the only possible combination by six elements $L_2 : f_1 + e_1 + d_1 + c_1 + b_1 + a_1$ is shown in m. 78–83.

In summary, combining two- and three-element attack-duration groups with notes, rests, accented notes and split-unit groups opens up a huge potential for creating rhythmic continuities r_c, when higher order permutations are used. These have the *self-scaling property*, i.e., the original attack-duration group will return at the higher level in varied context.[3] This means homogeneity and variation in one approach, while preventing repetition and monotony.

Application Tip:

This technique will be most useful in film, video and game music, when an atmosphere, a mood must be underscored. Especially the latter application domain, writing background music for a specific level in a game requires looking for means to combine consistency with variation. Short-time repetitions must be prevented, because these become an obvious distraction from the immersion feeling and will bore the player to digital death. Well, here the higher order permutations of a basic rhythm may prove to be a useful technique in the composer's toolbox.

[2] The label in the score does not indicate the splitting of the 8th note into two 16th notes.

[3] A nice analogy is the composition of the two helical strands in the DNA molecule. The genetic information is determined by the sequences of the four elementary nucleotides labeled G,A,T and C.

Figure 10.4: Higher order permutations of a three-element group $\{a, b, c\}$: combinations by three elements.

Chapter 11

Composition of homogeneous rhythmic continuity

In this chapter Schillinger returns to the combination of simultaneity and continuity, that was introduced in Section 9.1.2. In Chapter 9 that concept was applied at the lowest level attack-duration group, by considering the set of possible permutations, and applying it to different musical attributes (notes, rests, accents, etc).

In Chapter 10 the starting point was a small attack-duration group with two or three elements. A longer continuity r_c was created from this core cell by considering higher order permutations, i.e., at higher aggregation levels. In this chapter the process follows the reverse path: the starting point is a longer series of attack-durations and we will apply *splitting* in order to achieve smaller units. These smaller groups then serve as the basis for a growth process, both in the time domain (continuity) and in a parallel distribution over multiple parts in a score (simultaneity).

Splitting as another mechanism for generating growth in simultaneity and continuity may be done:

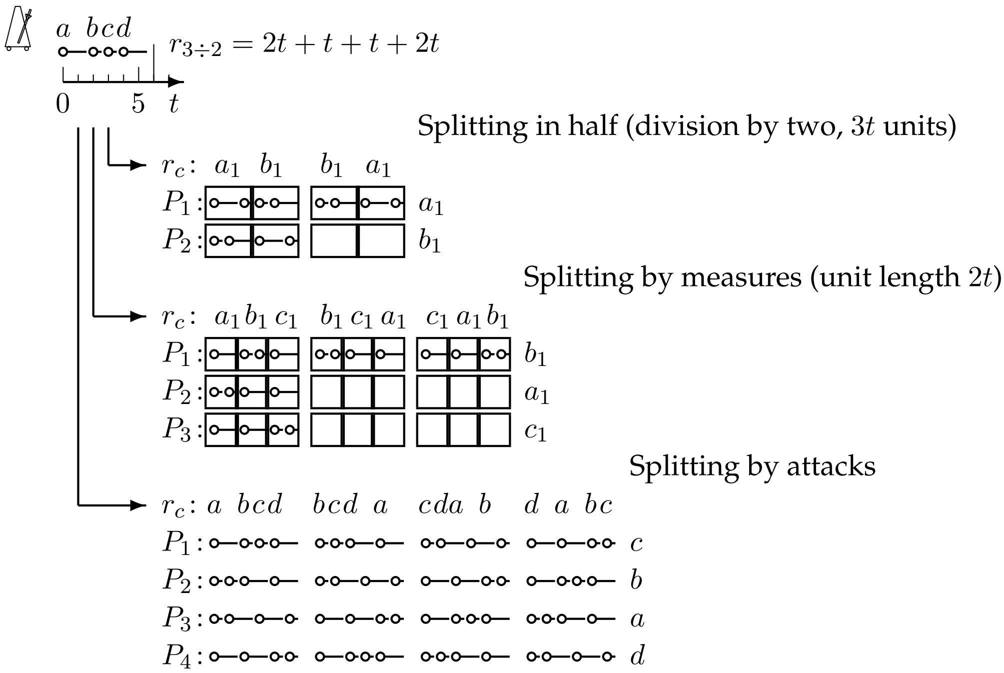
- through the simplest divisor. This yields multi-measure units and the minumum quantity of material to be evolved.
- at the measure level. This is an intermediate case.
- at the individual attack level. This leads to the maximum quantity of source material for evolving.

The splitting process alternatives are shown in diagram in Fig 11.1. These three approaches are covered in more detail in the next sections.

11.1 Splitting through the simplest divisor

Remember from Chapter 2, 4 and 6 that interference of two or three generators and the process of fractioning all lead to symmetrical patterns in the sense that the time-reversed (retrograde) rhythmical pattern is equal to the original series.

This means that for an even number of attacks we can split the pattern at the centre, and obtain two smaller attack-duration units that can be combined in simultaneity (i.e., parallel distribution over two staves) and continuity (i.e., juxtaposed permutations).



Vertical: simultaneity (parts P). Horizontal: continuity r_c

Figure 11.1: Composition of a homogeneous rhythmic continuity by splitting an attack-duration group. The non-uniform binary synchronization pattern with interference attack-duration group $r_{3\div 2} = 2t + t + t + 2t$ (4 attack elements $\{a.b.c.d\}$), is split into two halves, per measure and by individual attacks. This creates different continuities and multiple staff simultaneities.

We will illustrate the three splitting approaches graphically in Fig. 11.1 with the simples case possible. It is the non-uniform binary synchronization pattern $r_{3\div 2} = 2t + 1t + 1t + 2t$, presented in Section 2.1.2. It has four attacks and the total duration is $T_t = 6t$; therefore it is an unrealistic case. The division into two halves creates two units $\{a_1, b_1\} = \{(2t+t), (t+2t)\}$; these can be permutated to create a continuity $r_c = (a_1+b_1)+(b_1+a_1)$ with total duration $T_t = 12t$, and combined in a two-part ($2P$) simultaneity. The splitting by measure leads to three units $\{a_1, b_1, c_1\} = \{2t, (t+t), 2t\}$ with a length each of $2t$. Now the circular permutation process creates a continuity $r_c = (a_1 + b_1 + c_1) + (b_1 + c_1 + a_1) + (c_1 + a_1 + b_1)$ with total duration $T_t = 18t$, and combined in a three-part ($3P$) simultaneity. Finally, splitting by individual attacks using the four original units $\{a, b, c, d\} = \{2t, t, t, 2t\}$ creates a continuity $r_c = (a + b + c + d) + \ldots + (d + a + b + c)$ with total duration $T_t = 24t$, and a four-layer ($4P$) simultaneity.

A more realistic case is shown in the next example.

Example 11.1

Composition of homogeneous rhythmic continuity: splitting through the simplest divisor.

The approach of splitting the original attack-duration group through the simplest divisor is shown in Fig. 11.2 in rhythmic notation. Three cases will be discussed; the first and last are splitting the rhythmic pattern in half, the other divides it into three equally long parts.

- The source attack-duration group is the fractioning pattern $r_c = \underline{6 \div 5} = 5t + t + 4t + \bar{r} + t + 3t + t + 2t + 2t + t + 3t + t + \hat{t} + 4t + t + 5t$. This pattern has 16 attacks and a total duration of $T_t = 36t$ at time unit $t = \frac{1}{8}$. The pattern has been slightly modified by introducing a rest and an accented note. The simplest divisor is 2, splitting the group into two equal parts, the units $a_1 = (5t+t+4t+\bar{r}+t+3t+t+2t)$ and $b_1 = (2t+t+3t+t+\hat{t}+4t+t+5t)$. These may be combined into a continuity $r_c = (a_1 + b_1) + (b_1 + a_1)$, using the two permutations. This is shown in m. 1–12 at grouping by meter $\left[\begin{smallmatrix}6\\8\end{smallmatrix}\right]$. Combination of both units as a simultaneity creates a score with two staves.
- The second case is based on the non-uniform binary synchronization pattern $r_{9\div4} = 4t + 4\bar{r} + t + 3t + 4t + 2t + 2\hat{t} + 4t + 3t + t + 4t + 4t$. This group has 12 attacks and a total duration of $T_t = 36t$; one attack has been replaced with a rest and there is an accented note. The example in m. 13–39 uses time unit $t = \frac{1}{4}$ and grouping at $\left[\begin{smallmatrix}4\\4\end{smallmatrix}\right]$. Here we may divide into either two or three equal parts; the example demonstrates the latter splitting option. The result consists of three smaller units $\{a_1, b_1, c_1\} = \{(4t + 4\bar{r} + t + 3t), (4t + 2t + 2\hat{t} + 4t), (3t + t + 4t + 4t)\}$, each with duration $12t$, that can be used in continuity and simultaneity. The three-part score shows the vertical distribution of the continuity $r_c = (a_1 + b_1 + c_1) + (b_1 + c_1 + a_1) + (c_1 + a_1 + b_1)$, a $3 \times 9 = 27$ measures long pattern achieved through circular permutation.
- In Fig. 11.3, m. 1–4 the source pattern is the three-generator interference pattern $r'_{5\div3\div2} = 6t + 4t + 2t + 3t + 3t + 2t + 4t + 6\hat{t}$, consisting of 8 attack and a total duration of $T_t = 30t$. Time unit is $t = \frac{1}{8}$, the grouping is at meter $\left[\begin{smallmatrix}6\\8\end{smallmatrix}\right]$. Division by two leads yields to smaller units that are no longer an integer number of full measures: both $a_1 = 6t + 4t + 2t + 3t$ and $b_1 = 3t + 2t + 4t + 6\hat{t}$ have a duration of $T = 15t$, corresponding to $2\frac{1}{2}$ measures. The example shows the simultaneity in two parts

$$r_s = \begin{pmatrix} a_1 + b_1 & = & (6t + 4t + 2t + 3t) + (3t + 2t + 4t + 6\hat{t}) \\ b_1 + a_1 & = & (3t + 2t + 4t + 6\hat{t}) + (6t + 4t + 2t + 3t) \end{pmatrix}.$$

Figure 11.2: Homogeneous rhythmic continuity by splitting. Example 1, splitting through the simplest divisor. Division by two is applied to the fractioning pattern $r_{\underline{6 \div 5}}$, division by three to the non-uniform binary synchronization group $r_{9 \div 4}$.

The figure is included in the full version

Figure 11.3: Homogeneous rhythmic continuity by splitting. Example 2

When both an odd number of attacks and the total duration of the attack-duration group are an integer multiple of three, than the simplest divisor obviously is three. Looking at the examples in the previous chapters, there are not many combinations of generators that will satisfy this condition. One such case was illustrated in the example above. All cases consisted of smaller scale units that were longer than one measure.

11.2 Splitting through individual measures

Example 11.2

Composition of homogeneous rhythmic continuity: splitting by measures.
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11.3 Splitting through the individual attacks

At the smallest scale each attack may be considered a unit; we are splitting the original rhythmic group by individual attacks. This provides the maximum amount of material for the growth process of creating a homogeneous continuity (through permutation) and simultaneity (by distribution over multiple parts). Let's return to our previous example and see what the result will be when we split by attacks.

Example 11.3

Composition of homogeneous rhythmic continuity: splitting by attacks.
The result of splitting the three-generator interference pattern $r'_{5 \div 3 \div 2}$ by individual attacks is shown in Fig. 11.3, m. 22–32.

- We have modified the number of accented notes and introduced a rest: $r'_{5 \div 3 \div 2} = 6t + 4t + 2\bar{r} + 3t + 3\hat{t} + 2t + 4t + 6\hat{t}$.
- Splitting by individual attacks yields 8 units, that may be combined into a continuity r_c or an 8-part simultaneity r_s. Only the first and last permutation are shown in rhythmical notation.
- Each part is playing a different rhythm; the combination in the score contains a number of parts with many syncopated notes.

Chapter 12

Distributive powers

This chapter covers the application of *distributive powers* in rhythm. It is divided into two sections: the first has an analytical character. It discusses the possible power series that may be used either for the subdivision of rhythmical units within the measure or for the length of musical phrases. The second section is a technique for writing continuities and counterthemes applying distributed powers to a given attack-duration group.

12.1 Continuity of harmonic contrasts

The first application of power series in rhythmical continuity is to the grouping meter $\left[\begin{smallmatrix} n \\ n \end{smallmatrix}\right]$, where n is called the rhythm *determinant*. An overview of such series is shown in Fig. 12.1 for determinants $n = 2, \ldots, 9$. Taking powers of the denominator value $1/n$ determines the subdivision into shorter measure segments of the basic time unit $n = t$; Schillinger calls this *fractional continuity*. In the figure this subdivision is shown to the left of the centre as $1/n^k$, $k = 1, 2, 3, \ldots$ (k is also an integer number). So, for example, taking the most familiar grouping $\left[\begin{smallmatrix} n \\ n \end{smallmatrix}\right]=\begin{bmatrix} 2 \\ 2 \end{bmatrix}$, the subdivision of the measure is into two smaller units $1/n^1 = \frac{1}{2}$, the next smaller is $1/n^2 = \frac{1}{4}$, and so forth until the 4th power $1/n^4 = \frac{1}{16}$.

Applying the power series n^k to the numerator term yields the number of measures used in a pattern. Schillinger labels this as *factorial continuity*. These are the values shown in the figure to the right of the centre. For the meter $\left[\begin{smallmatrix} n \\ n \end{smallmatrix}\right]=\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ the result is a set of patterns with length $n^1 = 2, n^2 = 4, n^3 = 8$ and $n^4 = 16$ measures. The power series stop at both ends when the resulting values are either too large or small for application in music.

Schillinger then discusses which power series have been used in music. The most frequently used power series is the binary system: multiplication and division by 2; doubling the number of measures in a musical phrase and halving the note duration. The former leads to regular phrases of either 2, 4, 8 or 16 measures in classical music. This power series also covers the $n^5 = 32$ measure *chorus* in popular songs. Division by powers of two yields the familiar whole note – half note – quarter note – $\frac{1}{8}$ note – $\frac{1}{32}$ note durations. Already the powers of three are less frequently used: there are the triple division $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 9 \\ 8 \end{bmatrix}$ meter and the triplet subdivision $(\frac{1}{3})^1$. A rhythmical phrase with a length of three or nine measures is used most infrequently. Division by and multiplication by 4 are similar to the 2^2 power term. In some traditional folk music there is the occasional division by 5 and 7. However asymmetric

$$
\begin{array}{ccccccccccc}
\cdots & \frac{1}{n^4} & \frac{1}{n^3} & \frac{1}{n^2} & \frac{1}{n} & \frac{n}{n} & n & n^2 & n^3 & n^4 & \cdots \\
\cdots & \frac{1}{16} & \frac{1}{8} & \frac{1}{4} & \frac{1}{2} & \frac{2}{2} & 2 & 4 & 8 & 16 & \cdots \\
 & \cdots & \frac{1}{27} & \frac{1}{9} & \frac{1}{3} & \frac{3}{3} & 3 & 9 & 27 & \cdots & \\
 & \cdots & \frac{1}{64} & \frac{1}{16} & \frac{1}{4} & \frac{4}{4} & 4 & 16 & 64 & \cdots & \\
 & & \cdots & \frac{1}{25} & \frac{1}{5} & \frac{5}{5} & 5 & 25 & \cdots & & \\
 & & \cdots & \frac{1}{36} & \frac{1}{6} & \frac{6}{6} & 6 & 36 & \cdots & & \\
 & & \cdots & \frac{1}{49} & \frac{1}{7} & \frac{7}{7} & 7 & 49 & \cdots & & \\
 & & \cdots & \frac{1}{64} & \frac{1}{8} & \frac{8}{8} & 8 & 64 & \cdots & & \\
 & & \cdots & \frac{1}{81} & \frac{1}{9} & \frac{9}{9} & 9 & 81 & \cdots & &
\end{array}
$$

← Fractional continuity (within the measure) | Factorial continuity (groups of measures) →

Figure 12.1: Overview of rhythmical power series for creating fractional (division by powers of the determinant $1/n^k$) and factorial continuity (multiplication by powers n^k). Each row represents a power series family for a specific determinant n.

phrases are still fairly rare in music.[1]

There are also some hybrid power series. For example in $\begin{bmatrix} 6 \\ 8 \end{bmatrix}$ there is first division by 2 ($\frac{6}{8} = \frac{3}{8} + \frac{3}{8}$), then by three ($\frac{3}{8} = \frac{1}{8} + \frac{1}{8} + \frac{1}{8}$), then by two ($\frac{1}{8} = \frac{1}{16} + \frac{1}{16}$). The 12-measure Blues chorus, with its three subphrases of four measures each, is a multiplication of first by 4, then by 3 ($12 = 3 \times 4$). Schillinger blames the limited use of the power series in fractional and factorial continuitues on music notation schemes and indicates some potential for the future. In the text he claims that the determinant $\begin{bmatrix} 9 \\ 9 \end{bmatrix}$ is in the making. He points to *swing* and *shuffle* music as the hybrid of $\begin{bmatrix} 8 \\ 8 \end{bmatrix}$ and $\begin{bmatrix} 9 \\ 9 \end{bmatrix}$, a binary division followed by a triple division.[2]

12.2 Composition of rhythmic counterthemes by means of distributive powers

In this section there is a technique of applying the distributive power approach to an attack-duration group. Although the word *counterthemes* in the section title suggests that the approach is meant for creating a simultaneity, it may equally well be used for creating a rhythmic continuity.

Distributive powers will be used first for squaring an attack-duration group. Then the third power, the cube is explained. Finally, this approach is generalized to higher order powers.

[1] In classical music there are examples of phrase lengths of odd numbers, such as three, five or seven measures. Look for irregular phrases in the compositions by Johannes Brahms. In popular music there is the seven-measure A-phrase in *Yesterday*, the Paul McCartney song.

[2] Swing music was a current popular style at the writing of the Schillinger volumes. Now, almost a century later use your own analytical skills to determine which power series are dominating the popular music domain.

12.2.1 The square of an attack-duration group

An attack-duration group with resultant r, a binomial, trinomial or multinomial, with total duration $T = nt$ (n is the determinant) may be *squared*, written as r^2. The multinomial rhythmical pattern has a periodicity determined by multiple coefficients, typically more than three.[3] The length of the square r^2 is $T^2 = (nt)^2$ time units. However, squaring in the distributive power approach requires that we keep all the separate multiplication terms. This may seem a little confusing, but will become clear when we present the numerical examples.

The square of a binomial

The source is the binomial attack-duration group $r = a + b$. Equivalently this is written as $r = at + bt$ with the sum $a + b = n$ the determinant of the group. Squaring this group in the distributive power approach means

$$\begin{aligned} r^2 &= (a+b)^2 = (a+b)(a+b) \\ &= a(a+b) + b(a+b) = (a^2+ab) + (ba+b^2) = a^2 + ab + ba + b^2. \end{aligned} \tag{12.1}$$

It is essential to keep the separate terms; the square of the binomial leads to a group with four attacks.[4] Musically, it means that we have obtained two timescaled copies of the original pattern. Let's look at the simplest possible binomial example.

Example 12.1

The square of a binomial: simplest case

The binomial attack-duration group is $r = 2t + t$ with length $T = 3t$. We write the distributive square using Eq. 12.1: $r^2 = (2t+t)^2 = (2+1)(2t+t) = 2(2t+t) + 1(2t+t) = (4t+2t) + (2t+t) = 4t + 2t + 2t + t$ with four attacks. Note that we have used the distributive coefficients $\{2, 1\}$ for the multiplication without the t time unit. The total length is now $T^2 = 9t = 6t + 3t$, two subgroups where the first is twice the length of the second group. This scaling effect is the consequence of the ratio $(a/b) = (2/1) = 2$ of the elements in the binomial.

This effect of the distributive power approach s another *self-scaling property*. In Chapter 9 we discovered this property when using the *groups in general* approach in Section 9.2. There it had to do with the return of smaller scale rhythmic cells at a higher aggregation level. Here it is the duration pattern, that returns on a different timescale, depending on the power series.

The binomial has two permutations; therefore the square in our example may be either $r^2 = (2t+t)^2 = (4t+2t) + (2t+t)$ or $r^2 = (t+2t)^2 = (t+2t) + (2t+4t)$. Such a continuity is great for creating either a rhythmical acceleration or for the reverse, going from a busy rhythm to longer durations and calming down, e.g., closing with a fermata (see Chapter 14 for variable velocity patterns and acceleration series).

An overview of practical binomials for squaring is shown in the second column in Table 12.1. The first column shows the determinant n, the sum of the two elements in the binomial attack-duration group.

[3] Until now, we have almost exclusively encountered binomials and trinomials.

[4] Do not add the $ab + ba = 2ab$ terms, as is taught in calculus courses, because then we lose one attack.

Table 12.1: Practical distributive powers, i.e., square and cube, applied to binomials $a + b$ and trinomials $a+b+c$ with determinant n. Each binomial pair $\{a, b\}$ has two permutations. Trinomials have either three or six permutations. Combinations in brackets, such as $(3+2+1)$, are not mentioned in Schillinger's book.

Determinant	**Binomial**		**Trinomial**	
(Sum) n	**Square**	**Cube**	**Square**	**Cube**
$\left[\begin{smallmatrix} n \\ n \end{smallmatrix}\right]$	$(a+b)^2$	$(a+b)^3$	$(a+b+c)^2$	$(a+b+c)^3$
3	2+1	2+1		
4	3+1	3+1	2+1+1	2+1+1
5	3+2	3+2	2+2+1	2+2+1
	4+1		3+1+1	3+1+1
6	5+1		(3+2+1)	3+2+1
			4+1+1	
7	4+3		3+2+2	3+2+2
	5+2		3+3+1	
	6+1		(4+2+1)	
			5+1+1	
8	5+3		3+3+2	3+3+2
	7+1		(4+3+1)	
			(5+2+1)	
			6+1+1	
9	5+4		(4+3+2)	
	7+2		4+4+1	
	8+1		5+2+2	
			(5+3+1)	
			(6+2+1)	
			7+1+1	

The figure is included in the full version

Figure 12.2: Distributive powers. The square of a binomial

All binomial squares are shown in diagram in Fig. 12.2. The figure shows one permutation only as a *continuity*, with the greatest number as the a-element, i.e., $a > b$. Blue lines in the diagrams indicate the division of the four attacks into two subgroups. The scaling effect is obvious; compare the $r^2 = (5+3)^2$ with $r^2 = (7+1)^2$, both combinations with determinant $n = 8$. The latter suggests an enormous rhythmical acceleration, whereas the former is a smoother, more balanced change in durations.

Example 12.2

The square of a binomial: various cases.
We will perform the calculations for a number of two-element attack-duration groups with resultant $r = (at + bt)$. All cases are shown in diagram in Fig. 12.2, the musical notation is presented in Fig 12.3:

- Case 1, determinant $n = 4$: $r = 3t + t$ $(a = 3, b = 1, T = 4t)$. The distributive second power is calculated with Eq. 12.1 and yields the square

  $$r^2 = (3+1)(3t+t) = 3(3t+t) + 1(3t+t) = (9t+3t) + (3t+t).$$

  This four-attack pattern with total duration $T_t = T^2 = 16t$ consists of two subgroups with durations $12t, 4t$, respectively. This is suitable for a $\left[\begin{smallmatrix}4\\4\end{smallmatrix}\right]$ grouping. The other permutation is $r = t + 3t$, which has the square $r^2 = (t+3t) + (3t+9t)$. Both permutations are shown in the upper staves in m. 1–4 of Fig. 12.3.
- Case 2, determinant $n = 5$: $r = 4t + t$ $(a = 4, b = 1, T = 5t)$. The square of r is

  $$r^2 = (4+1)(4t+t) = 4(4t+t) + 1(4t+t) = (16t+4t) + (4t+t).$$

  Compared to the first case we note the increased difference in length between the two subgroups; the ratio of longest to shortest attack now is $16t/1t = 16$. The total duration $T_t = 25t$, i.e., 5 measures at $\left[\begin{smallmatrix}5\\4\end{smallmatrix}\right]$ time signature, as shown in m. 10–14. Recurrence at meter $\left[\begin{smallmatrix}3\\4\end{smallmatrix}\right]$ or $\left[\begin{smallmatrix}4\\4\end{smallmatrix}\right]$ requires three or four statements of the pattern, respectively.
- Case 3, determinant $n = 7$: $r = 4t + 3t$ $(a = 4, b = 3, T = 7t)$. The square of this binomial is

  $$r^2 = (4+3)(4t+3t) = 4(4t+3t) + 3(4t+3t) = (16t+12t) + (12t+9t).$$

  The two subgroups are now more in balance $28t + 21t, T_t = 7^2t = 49t$ and the longest to shortest ratio is $16t/9t = 1.77$. The determinant $n = 7$ requires irregular meter such as $\left[\begin{smallmatrix}7\\8\end{smallmatrix}\right]$ in folk music. This is illustrated in two permutations in Fig. 12.3 m, 15–21.
- Case 4, determinant $n = 9$: $r = 7t + 2t$ $(a = 7, b = 2, T = 9t)$. The square of this binomial is

  $$r^2 = (7+2)(7t+2t) = 7(7t+2t) + 2(7t+2t) = (49t+14t) + (14t+4t).$$

  The total duration is $T_t = 81t$, a 27-measure pattern in $\left[\begin{smallmatrix}3\\4\end{smallmatrix}\right]$, as shown in m. 22–48.

The original group may also be combined with the square in a *simultaneity*. In order to achieve synchronization the original group durations must be multiplied with the determinant n. This leads to a timescaled, augmented version of the original pattern nr in the top part and the square r^2 in the lower part

$$r_s = \begin{pmatrix} P_1 : & nr & = & n(a+b) = na + nb \\ P_2 : & r^2 & = & (a^2+ab)+(ba+b^2) \end{pmatrix}. \tag{12.2}$$

Both now have the duration $T = n^2 t$ time units. The combination in simultaneity leads to the style known as *isorhythm*, used in the *Ars Nova* period in Medieval music. A rhythmic pattern sounds in parallel with its augmented version r^a (in the augmentation the durations are multiplied) or with the diminution (durations halved or in some other fraction). As a combination various permutations may be used in each part; for the binomial there are four possibilities; this potential was discussed in Section 9.2.

However, there is another candidate for synchronization with the square. Remember the fractioning pattern from Chapter 4. The length of the fractioning pattern $r_{\underline{a \div b}}$ is $T = a^2$, the square of the major generator value. This enables us to the combination of the square $r_1 = (a+b)^2$ and determinant $n = a + b$ with the fractioning pattern $r_2 = r_{\underline{n \div m}}$.

Combining the three approaches in simultaneity leads to a three-part score

$$r_s = \begin{pmatrix} P_1 : & nr & = & n(a+b) = na + nb \\ P_2 : & r^2 & = & (a^2+ab)+(ba+b^2) \\ P_3 : & r_{\underline{n \div m}} & & \end{pmatrix}. \tag{12.3}$$

We may use any subset from this set in any vertical order in the score; there are three subsets of two parts $\{(P_1, P_2), (P_1, P_3), (P_2, P_3)\}$ in two possible vertical orderings, or six vertical distributions of the three parts.

Example 12.3

Combination of the square of a binomial with the original and the fractioning pattern.

This example is included in the full version of the book.

Figure 12.3: Distributive powers. Example: the square of a binomial $r^2 = (at + bt)^2$, as permutations and combinations in simultaneity. Note the combinations with the augmented original r^a and the fractioning pattern $r_{\underline{n \div m}}$ (top).

The figure is included in the full version

Figure 12.4: Combination of the square of a binomial with the original and the fractioning pattern

The square of a trinomial

Taking the distributive second power of a trinomial $r = (a + b + c)$ yields the square

$$\begin{aligned} r^2 &= (a+b+c)^2 = (a+b+c)(a+b+c) \\ &= a(a+b+c) + b(a+b+c) + c(a+b+c) \\ &= (a^2+ab+ac) + (ba+b^2+bc) + (ca+cb+c^2) \\ &= a^2+ab+ac+ba+b^2+bc+ca+cb+c^2. \end{aligned} \tag{12.4}$$

For a three-element group the square contains nine attacks. Practical trinomials for the distributive second power are shown in the fourth column of Table 12.1. In the Schillinger book there are only trinomials with a doubled element, such as $(2 + 2 + 1)^2$; the possible combinations with three different elements, such as $(4 + 2 + 1)^2$ are discarded. The explanation for this is given in Chapter 13, where he discusses families of rhythms that are the result of an evolutionary process. Figure 12.5 shows the graphical representation of the square of a subset of these trinomials.

The original trinomial is now reproduced at either two or three different timescales, depending on the ratios a/b and a/c. Only one permutation is displayed, with the coefficients in decreasing order, i.e., with $a > b > c$ Each combination with two equal elements has three permutations, e.g., $(5 + 2 + 2)^2, (2 + 5 + 2)^2, (2 + 2 + 5)^2$, trinomials with three different elements have 6 permutations.

Example 12.4

The square of a trinomial.

We will perform the calculations for a number of three-element attack-duration groups with $r = (at + bt + ct)$ and determinant $n = a + b + c$. We will select a number of cases, shown in diagram in Fig. 12.5 and calculate the results using Eq. 12.4:

- Case 1, determinant $n = 6$: $r = 4t + t + t$ $(a = 4, b = c = 1, T = 6t)$, with two equal elements. The distributive second power yields the square

  $$\begin{aligned} r^2 &= (4+1+1)(4t+t+t) = 4(4t+t+t) + 1(4t+t+t) + 1(4t+t+t) \\ &= (16t+4t+4t) + (4t+t+t) + (4t+t+t). \end{aligned}$$

  This nine-attack pattern with total duration $T_t = T^2 = 36t$ contains three subgroups with durations $24t, 6t, 6t$, respectively. There are three permutations of the subgroups: these are $\{(16t+4t+4t)+(4t+t+t)+(4t+t+t), (t+4t+t)+(4t+16t+4t)+(t+4t+t), (t+t+4t)+(t+t+4t)+(4t+4t+16t)\}$, shown as a simultaneity in Fig. 12.6, m. 1–9 and applied to an extended Dm_7^{11} chord. The pattern will group into 12 measures at $\left[\frac{3}{8}\right]$ time signature or nine measures in $\left[\frac{4}{4}\right]$ meter.

- Case 2, determinant $n = 6$: $r = 3t + 2t + t$ $(a = 3, b = 2, c = 1, T = 6t)$, with three different elements. The square of r is

  $$r^2 = (3+2+1)(3t+2t+t)$$

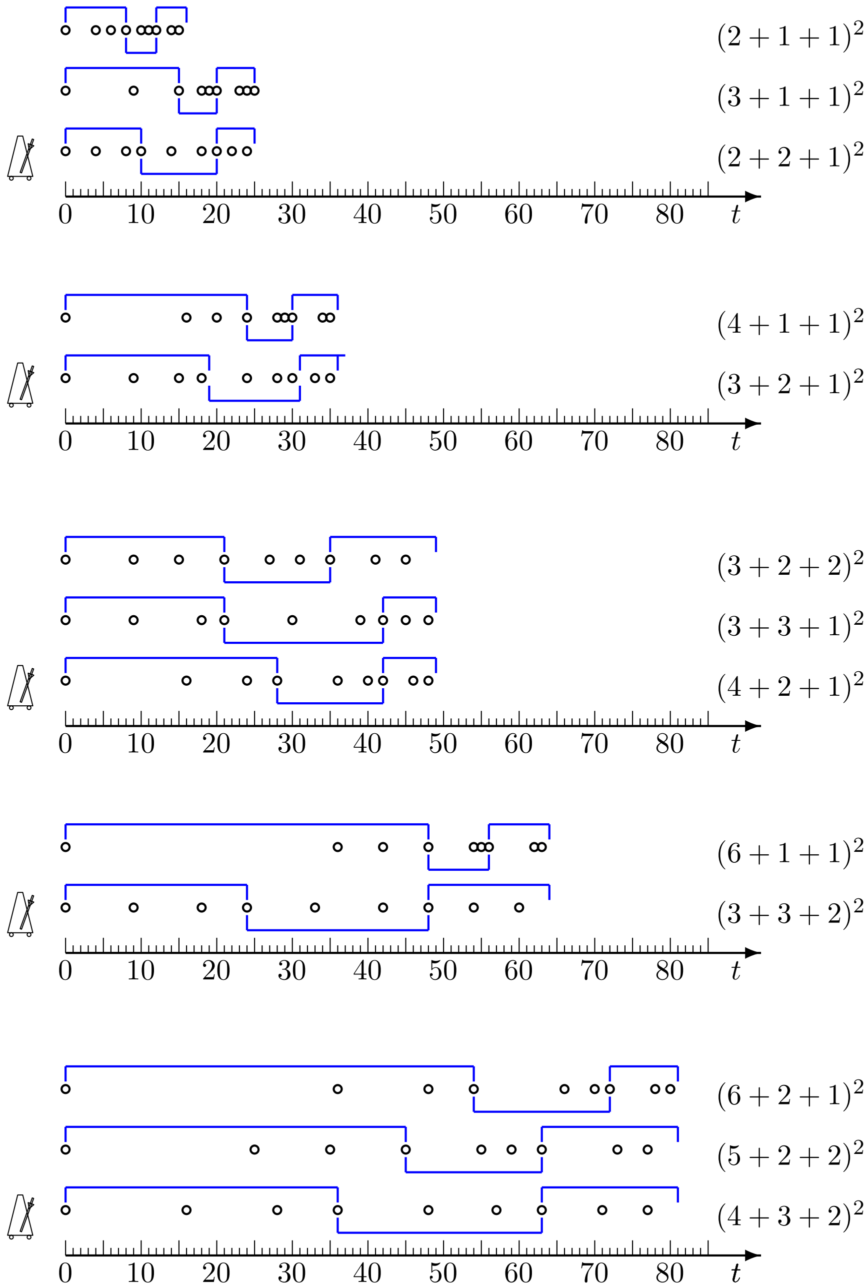

Figure 12.5: Distributive powers. The square of a trinomial $r^2 = (at + bt + ct)^2$ with determinant $n = a + b + c$; trinomials with two equal elements have three permutations, trinomials with three different elements have six permutations.

$$\begin{aligned} &= 3(3t+2t+t)+2(3t+2t+t)+1(3t+2t+t) \\ &= (9t+6t+3t)+(6t+4t+2t)+(3t+2t+t). \end{aligned}$$

The three subgroups may be used in six permutations; the three circular permutations are shown in Fig. 12.6, m. 10–21, applied to a G_9^6 chord. Note that all permutations have a simultaneous attack in m. 10 and 18. The total duration $T_t = 36t$, corresponding to 12 measures in $\left[\frac{3}{4}\right]$.

- Case 3, determinant $n = 8$: $r = 3t + 3t + 2t$ ($a = b = 3, c = 2, T = 8t$), a familiar rhythm pattern in Latin music , e.g., in the rumba, and in popular music at time unit $t = \frac{1}{8}$. The square of this trinomial is

$$\begin{aligned} r^2 &= (3+3+2)(3t+3t+2t) \\ &= 3(3t+3t+2t)+3(3t+3t+2t)+2(3t+3t+2t) \\ &= (9t+9t+6t)+(9t+9t+6t)+(6t+6t+4t). \end{aligned}$$

The three subgroups have lengths $24t + 24t + 16t, T_t = 8^2t = 64t$, leading to an 8 measure pattern in $\left[\frac{8}{8}\right]$ grouping. The three permutations for $r = 3t + 3\hat{t} + 2t$ with an accented note are shown in rhythmical notation in m. 22–29; note the syncopations and afterbeats with occasional simultaneous attacks.

Also the square of the trinomial can be used as a continuity $r_c = r^2$ or as a simultaneity with the augmented version of the original group $nr = n(a + b + c) = na + nb + nc$ or with the fractioning pattern $r_{\underline{n \div m}}$.

Example 12.5

Combination of the square of a trinomial with the original and the fractioning pattern.

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Figure 12.6: Distributive powers. Example: the square of a trinomial $r^2 = (at + bt + ct)^2$ with determinant $n = a + b + c$.

The figure is included in the full version

Figure 12.7: Combination of the square of a trinomial with the original and the fractioning pattern

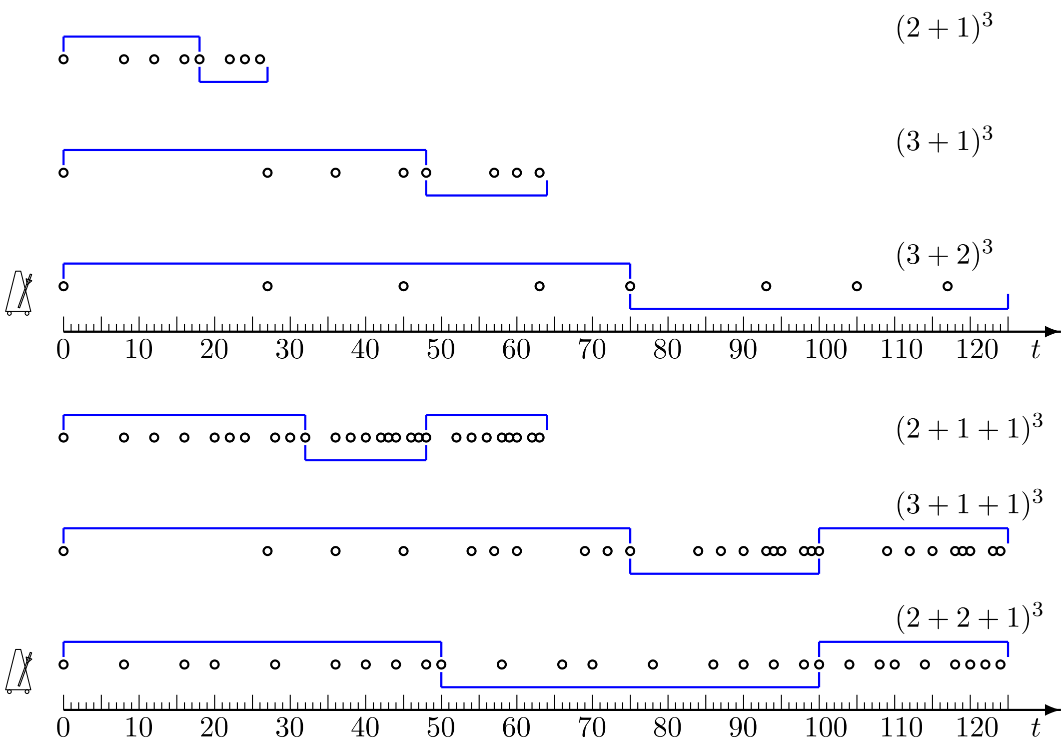

Figure 12.8: Distributive powers. The cube of some binomials $r^3 = (at + bt)^3$ and trinomials $r^3 = (at + bt + ct)^3$. The cube of a binomial consists of eight attacks in two subgroups, the cube of the trinomial contains three subgroups with nine attacks each.

12.2.2 The cube of an attack-duration group

The distributed third power of an attack-duration group is the *cube* of the resultant r^3. The procedure is the same as for the square; after completing the distributive second power we repeat the multiplication process once more. This is an iterative procedure.

The cube of a binomial

The cube of the binomial $r = (a + b)$ with total duration $T = n$ is

$$\begin{aligned} r^3 &= (a+b)^3 = (a+b)(a+b)^2 = (a+b)(a^2+ab+ba+b^2) \\ &= (a^3+a^2b+aba+ab^2)+(ba^2+bab+b^2a+b^3) \\ &= (a^3+a^2b+a^2b+ab^2)+(ba^2+ab^2+b^2a+b^3), \end{aligned} \tag{12.5}$$

a series of 8 attacks with total duration n^3. The calculation for the easiest case $r = 2t + t$ yields $r^3 = (2t + t)^3 = (2 + 1)(4t + 2t + 2t + t) = (8t + 4t + 4t + 2t) + (4t + 2t + 2t + t)$. The three practical binomials for the cube are shown in the third column of Table 12.1, some examples are shown graphically in the upper half of Fig. 12.8.

Also the cube leads to a continuity with two subgroups of four attacks and two permutations. Obviously for determinants $n \geq 3$ the total duration of the cube explodes; the example

$(3t + 2t)^2$ already leads to a total duration of $(5^3)t = 125t$ time units. Therefore the table has no entries beyond $n = 5$.

Example 12.6

The cube of a binomial.

We will perform the calculations for a number of two-element attack-duration groups with $r = (at + bt)$. All practical cases for cubing are shown in diagram in Fig. 12.8:

- Case 1, determinant $n = 3$: $r = 2t + t$ $(a = 2, b = 1, T = 3t)$. The distributive third power according to Eq. 12.5 yields the cube

$$\begin{aligned} r^2 &= (2+1)(2t+t) = 2(2t+t) + 1(2t+t) = (4t+2t) + (2t+t) \\ r^3 &= (2+1)(2t+t)^2 = 2(4t+2t+2t+t) + 1(4t+2t+2t+t) \\ &= (8t+4t+4t+2t) + (4t+2t+2t+t). \end{aligned}$$

  This accelerating eight-attack pattern with total duration $T_t = T^3 = 3^3t = 27t$ consists of two subgroups with durations $18t, 9t$, respectively. This implies 9 measures in $\begin{bmatrix} 3 \\ 8 \end{bmatrix}$ grouping. The other permutation is $r = t + 2t$ with the cube $r^3 = (t + 2t + 2t + 4t) + (2t + 4t + 4t + 8t)$. Both are shown in Fig. 12.9, m. 1–9, applied to an Am_7 chord. There is also the distribution of the duration group over the two-part attack patterns (see the technique in Section 8.1).

- Case 2, determinant $n = 5$: $r = 3t + 2t$ $(a = 3, b = 2, T = 5t)$. The cube of this binomial is

$$\begin{aligned} r^3 &= (3+2)(3t+2t)^2 = 3(9t+6t+6t+4t) + 2(9t+6t+6t+4t) \\ &= (27t+18t+18t+12t) + (18t+12t+12t+8t). \end{aligned}$$

  For the square see Example 12.3 Case 2. The two subgroups are now more in balance $75t + 50t, T_t = 5^3t = 125t$ and the longest to shortest note duration ratio is $27t/8t = 3.37$. The determinant $n = 5$ requires irregular meter such as $\begin{bmatrix} 5 \\ 8 \end{bmatrix}$ in folk music, yielding a 25-measure rhythmical pattern. The cube is demonstrated in two simultaneous permutations in a pair of staves for the pattern $r = 3t + 2\bar{r}$ in m. 10–34, where we see sustained 2- and 3-part chords, interspersed with long rests.

- Case 3, determinant $n = 4$: $r = 3t + t$ $(a = 3, b = 1, T = 4t)$. The cube of r is

$$\begin{aligned} r^3 &= (3+1)(3t+t)^2 = 3(9t+3t+3t+t) + 1(9t+3t+3t+t) \\ &= (27t+9t+9t+3t) + (9t+3t+3t+t). \end{aligned}$$

  For the square see Example 12.2 Case 1. The ratio of longest to shortest attack now is $27t/1t = 27$. The total duration $T_t = 64t$, with subgroups of lenght $48t$ and $16t$, respectively. This corresponds to 8 measures in $\begin{bmatrix} 4 \\ 4 \end{bmatrix}$ grouping and time unit $t = \frac{1}{8}$. This is shown in the upper staff of Fig. 12.9,

m. 35–42. Here the cube is applied to an attack-duration group $r = 3t + \bar{r}$, containing a rest (see Section 9.2 for the technique of using accented notes and rests) and an Em_7^{11} chord. Two permutations are shown on the upper staff.

Creating a simultaneity now opens up a number of possibilties for combining the original group, the square, the cube and the fractioning pattern. Each of these must be properly scaled, i.e., multiplied with the a power of the determinant n to achieve synchronization through augmentation. The possibilities may be written as a four-part score

$$r_s \begin{pmatrix} P_1: & n^2 r & = & n^2(a+b) \\ P_2: & nr^2 & = & n(a+b)^2 \\ P_3: & r^3 & = & (a+b)^3 \\ P_4: & nr_{\underline{n \div m}} & & \end{pmatrix}. \tag{12.6}$$

Remember that each part has a number of permutations, increasing the potential for combination even further.

Example 12.7

Combination of the cube of a binomial with the original, square and the fractioning pattern.
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Figure 12.9: Distributive powers. Example: the cube of a binomial $r^3 = (at + bt)^3$ and trinomial $r^3 = (at + bt + ct)^3$. Permutations and combinations in simultaneity.

The figure is included in the full version

Figure 12.10: Combination of the cube of a binomial with the original, square and the fractioning pattern

The cube of a trinomial

The cube of the trinomial $r = (a + b + c)$ is

$$
\begin{aligned}
r^3 &= (a+b+c)^3 = (a+b+c)(a+b+c)^2 \\
&= a(a^2 + ab + ac + ba + b^2 + bc + ca + cb + c^2) + \\
&\quad b(a^2 + ab + ac + ba + b^2 + bc + ca + cb + c^2) + \\
&\quad c(a^2 + ab + ac + ba + b^2 + bc + ca + cb + c^2) \\
&= (a^3 + a^2b + a^2c + aba + ab^2 + abc + aca + acb + ac^2) + \\
&\quad (ba^2 + bab + bac + b^2a + b^3 + b^2c + bca + bcb + bc^2) + \\
&\quad (ca^2 + cab + cac + cba + cb^2 + cbc + c^2a + c^2b + c^3) \\
&= (a^3 + a^2b + a^2c + a^2b + ab^2 + abc + a^2c + acb + ac^2) + \\
&\quad (ba^2 + b^2a + bac + b^2a + b^3 + b^2c + bca + b^2c + bc^2) + \\
&\quad (ca^2 + cab + c^2a + cba + cb^2 + c^2b + c^2a + c^2b + c^3). \qquad (12.7)
\end{aligned}
$$

There are 27 attacks, subdivided into three subgroups, whose durations depend on the ratios a/b and a/c. Practical trinomials for the distributive third power are shown in the last column of Fig. 12.1. This is a small set of 6 trinomials, of which 5 have two equal elements. Some examples are shown graphically in the lower half of Fig. 12.8. These may have three or six permutations. Again, note the total pattern duration of the cube as a continuity.

Example 12.8

The cube of a trinomial.
We will perform the calculations for the cube of the trinomials $r = (at + bt + ct)$ shown in diagram in the lower half of Fig. 12.8. The three cases are:

- Case 1, determinant $n = 4$: $r = 2t + t + t$ ($a = 2, b = c = 1$). The distributive power approac used iteratively yields the third power, i.e., the cube

$$
\begin{aligned}
r^2 &= (2+1+1)(2t+t+t) \\
&= (4t+2t+2t) + (2t+t+t) + (2t+t+t) \\
r^3 &= r(r^2) = (2+1+1)(2t+t+t)^2 = \\
&= (2+1+1)[(4t+2t+2t) + (2t+t+t) + (2t+t+t)] \\
&= [(8t+4t+4t) + (4t+2t+2t) + (4t+2t+2t)] \\
&\quad +[(4t+2t+2t) + (2t+t+t) + (2t+t+t)] \\
&\quad +[(4t+2t+2t) + (2t+t+t) + (2t+t+t)].
\end{aligned}
$$

The 27-attack pattern with total duration $T_t = T^3 = 4^3t = 64t$ consists of three subgroups with durations $32t, 16t, 16t$, respectively. This implies 8 measures in $\left[\begin{smallmatrix}4\\4\end{smallmatrix}\right]$ grouping at time unit $t = \frac{1}{8}$, as illustrated in Fig. 12.9, m. 43–53. Here the original resultant contains a rest and an accented note $r = 2t + \bar{r} + \hat{t}$ and there is an attack group distribution over an Em chord. The arpeggio patterns in the three upper staves demonstrate the three permutations in simultaneity, combined with a percussion rhythm in the lower staff, based on the augmented fractioning pattern $r^a_{\underline{n \div m}} = nr_{\underline{n \div m}} = 4r_{\underline{4 \div 3}}$. Note how busier rhythms are moving through the upper parts against a more regular and steady lower part.

- Case 2, determinant $n = 5$: $r = 3t + t + t$ $(a = 3, b = c = 1)$. The cube of r is calculated in two steps

$$\begin{aligned}
r^2 &= (3+1+1)(3t+t+t) \\
&= (9t+3t+3t)+(3t+t+t)+(3t+t+t) \\
r^3 &= r(r^2) = (3+1+1)(3t+t+t)^2 = \\
&= (3+1+1)[(9t+3t+3t)+(3t+t+t)+(3t+t+t)] \\
&= [(27t+9t+9t)+(9t+3t+3t)+(9t+3t+3t)] \\
&\quad +[(9t+3t+3t)+(3t+t+t)+(3t+t+t)] \\
&\quad +[(9t+3t+3t)+(3t+t+t)+(3t+t+t)].
\end{aligned}$$

The total duration of this 27-attack group $T_t = 5^3t = 125t$, divided into three subgroups $75t + 25t + 25t = 125t$. This leads to a visible unbalance when comparing the length of the first with the other subgroups. This case is not shown in musical notation.

- Case 3, determinant $n = 5$: $r = 2t + 2t + t$ $(a = b = 2, c = 1)$. The cube of this binomial is

$$\begin{aligned}
r^2 &= (2+2+1)(2t+2t+t) \\
&= (4t+4t+2t)+(4t+4t+2t)+(2t+2t+t) \\
r^3 &= r(r^2) = (2+2+1)(2t+2t+t)^2 = \\
&= (2+2+1)[(4t+4t+2t)+(4t+4t+2t)+(2t+2t+t)] \\
&= [(8t+8t+4t)+(8t+8t+4t)+(4t+4t+2t)] \\
&\quad +[(8t+8t+4t)+(8t+8t+4t)+(4t+4t+2t)] \\
&\quad +[(4t+4t+2t)+(4t+4t+2t)+(2t+2t+t)].
\end{aligned}$$

The three subgroups add up to $50t + 50t + 25t = 125t$, with an improved balance between the subgroups.

In a synchronized simultaneity we may once again combine the cube $P_1 : r^3 = (a+b+c)^3$ with the timescaled version of the original group $P_2 : n^2r = n^2(a+b+c)$ and the augmented square $P_3 : nr^2 = n(a+b)^2$ or the fractioning pattern $P_4 : nr_{\underline{n \div m}}$, in any permutation.

12.2.3 The generalization of all powers

The process of taking the distributive power of a binomial, trinomial or multinomial may be extended to higher powers, beyond the cube (third power). In general this implies the N-th power $r^N = (a + b + \ldots)^N$. In practice the calculations are done in an iterative process, written as

$$r^N = r(r^{N-1}). \tag{12.8}$$

As we have seen in the caluculations the cube of a binomial involves the distributive multiplication process applied to the square, i.e., $r^3 = (at + bt)^3 = r(r^2) = (a + b)(at + bt)^2$. This approach is repeated at higher order powers: $r^4 = r(r^3)$, etc.

These may yield continuities and simultaneities after proper timescaling. The essential point is that all the higher order powers are derived from a single source attack-duration group r; this acts as a binding element stylewise and will create a homogeneous continuity or simultaneity.

Chapter 13

Evolution of rhythm styles (families)

This chapter is about *families* of rhythms, as they can be derived from a specific determinant. It connects the previous chapters. Chapter 9 and 10 covered the technique of variation through permutation on the smaller timescale and at higher aggregation levels. Chapter 11 showed the technique of splitting as another approach to small-scale variation. Then Chapter 12 explained a mechanism for creating longer multi-measure rhythmical patterns using distributive powers.

In this chapter there is the concept of the *evolution* of rhythms from a common starting point, given by the determinant n or meter grouping $\left[\begin{smallmatrix} n \\ n \end{smallmatrix}\right]$. The starting point of the evolution is the subdivision of a group with determinant $n = r$ into a binomial with a major generator a and minor generator b, such that $r = a + b$.

Starting from this binomial we can proceed with the subdivision approach, i.e., evolution on the fractional level. Using distributive powers there is the potential for evolution on the larger scale, i.e., on the factorial level. The evolution is determined by a sequence of the two processes *permutation* and *synchronization*, as illustrated in Fig. 13.1.

On the fractional level we write the two possible permutations of the binomial $\{a+b, b+a\}$, as illustrated by the two attacks on the right at the ouput of the permutation process block. Synchronization yields three attacks. Then there is a test to check for uniform distribution with all attacks of equal duration. We stop the process when the ultimate subdivision into equal time units has been reached. Otherwise we iterate the process for the new trinomial or multinomial.

On the factorial level there is a similar approach. We raise the binomial to the distributive second power, i.e., the square $r^2 = (a + b)^2$. This leads to four attacks, to which we then also apply iterative permutation and synchronization until the uniform distribution is reached.

The integer numbers for the new elements we obtain along this process establish a *family of rhythms*, that carry specific characteristics. We will now consider this evolution process on the fractional and factorial level for practical determinants between $n = 2, \ldots, 9$. We will use a subscript to indicate the generations in the evolution process: r_0 is the parent pattern, which yields the first generation of children with resultant r_1 after one iteration cycle of permutation and synchronization.

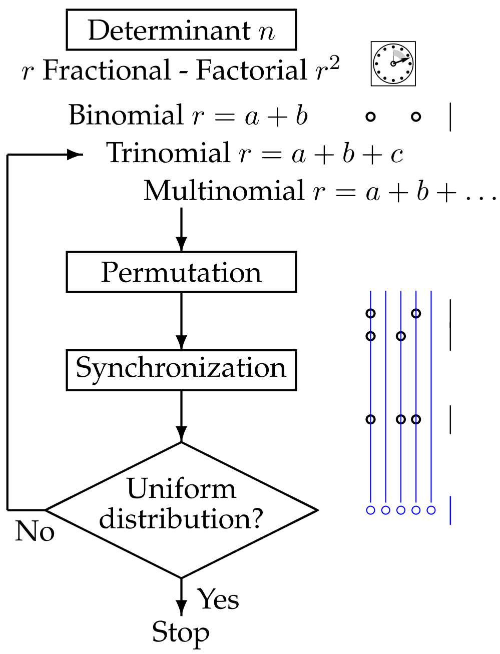

Figure 13.1: Iterative process of permutation and synchronization. The initial binomial pair $\{a, b\}$ with determinant n will produce generations of a rhythm family on either the fractional or the factorial level.

13.1 Evolution on the fractional level

Subdividing the determinant $n = 2$ leads to the only possible binomial $r_0 = a + b = 1 + 1$, as shown in Fig. 13.2. This is a uniform distribution $a = b = 1$; there are no permutations of this rhythmical pattern. There is no evolutionary process and we have the only member of this family; this is a trivial case.

The permutation-synchronization process starts to make sense at determinant $n = 3$. There is one possible subdivision into the binomial $r_0 = a + b = 2 + 1$, where $a = 2$ is the major generator and $b = 1$ the minor generator. When we synchronize the two permutations we reach uniform distribution $r_1 = 1 + 1 + 1$ and stop the evolution.

For determinant $n = 4$ there is one non-uniform distribution $r_0 = a + b = 3 + 1$, with major generator $a = 3$ (see Table 2.1 in Section 2.1.2 for the set of unique combinations for a given major generator). As Fig. 13.2 shows (top right), the synchronization of the two permutations yields the next generation trinomial $r_1 = 1 + 2 + 1$, which is a non-uniform pattern with two equal elements ($a_1 = 2, b_1 = 1, r_1 = b + a + b$). Therefore we enter the second iteration of the evolution process; now permutation and synchronization reach uniformity $r_2 = 1 + 1 + 1 + 1$. This is another family of rhythms.

Determinant $n = 5$ is the first to have two possible subdivisions into binomials $r_0 = a + b = \{3 + 2, 4 + 1\}$. For both cases the lower half of the figure illustrates the evolution process: $r_0 = 3 + 2 \rightarrow r_1 = 2 + 1 + 2 \rightarrow r_2 = 1 + 1 + 1 + 1 + 1$ and $r_0 = 4 + 1 \rightarrow r_1 = 1 + 3 + 1 \rightarrow r_2 = 1 + 1 + 1 + 1 + 1$. We may discern the general characteristic of the first generation resultant: it will always be a trinomial with two equal elements, specific for that family. This is the reason why Table 12.1 in Schillinger's book only listed these trinomials

$\begin{bmatrix}2\\2\end{bmatrix}$	$\begin{bmatrix}3\\3\end{bmatrix}$	$\begin{bmatrix}4\\4\end{bmatrix}$	Determinant
$\circ\circ \mid 1+1$	$\circ\ \circ \mid 2+1$	$\circ\quad\circ \mid 3+1$	r_0 Binomial $a+b$
	$\circ\circ\ \ \mid 1+2$	$\circ\circ\quad \mid 1+3$	Permutation
			↓ Synchronize
	$\circ\circ\circ \mid 1+1+1$	$\circ\circ\ \circ \mid 1+2+1$	r_1 Trinomial
		$\circ\ \circ\circ \mid 2+1+1$	Permutations
		$\circ\circ\circ\ \ \mid 1+1+2$	⋮
			↓ Synchronize
		$\circ\circ\circ\circ \mid 1+1+1+1$	Uniform distribution
$n=2$	$n=3$	$n=4$	

$\begin{bmatrix}5\\5\end{bmatrix}$	$\begin{bmatrix}5\\5\end{bmatrix}$	Determinant
$\circ\quad\circ\quad \mid 3+2$	$\circ\qquad\circ \mid 4+1$	r_0 Binomial $a+b$
$\circ\ \ \circ\qquad \mid 2+3$	$\circ\circ\qquad \mid 1+4$	Permutation
		↓ Synchronize
$\circ\ \ \circ\circ\quad \mid 2+1+2$	$\circ\circ\quad\circ \mid 1+3+1$	r_1 Trinomial
$\circ\circ\ \ \circ\quad \mid 1+2+2$	$\circ\quad\circ\circ \mid 3+1+1$	Permutations
$\circ\ \ \circ\ \ \circ \mid 2+2+1$	$\circ\circ\circ\quad \mid 1+1+3$	⋮
		↓ Synchronize
$\circ\circ\circ\circ\circ \mid 1+1+1+1+1$	$\circ\circ\circ\circ\circ \mid 1+1+1+1+1$	Uniform distribution
$n=5$	$n=5$	

Figure 13.2: Fractional evolution of rhythm families through iterations of permutation and synchronization. Determinants $n = 2, 3, 4, 5$. The parent generation is the binomial $r_0 = a+b$, the i-th child generation is labeled r_i.

and discarded the trinomials with three different elements. In his theory of rhythm, the latter are standalone patterns and not part of an evolving family.

Determinant $n = 6$ has one non-uniform binomial $r_0 = 5 + 1$, see Fig. 13.3. There are two generations of non-uniform descendants $r_1 = 1 + 4 + 1$ and the quintinomial, five-element $r_2 = 1 + 1 + 2 + 1 + 1$. Uniform duration distribution is reached at child generation r_3. Determinant $n = 7$ also leads to three generations, but now there are three parent binomials $r_0 = \{4 + 3, 5 + 2, 6 + 1\}$. The rhythmical asymmetry increases (i.e., the ratio of longest to shortest duration) towards the last binomial, which might be characterized as most unbalanced.

The figure is included in the full version

Figure 13.3: Fractional evolution of rhythm families. Determinants $n = 6, 7$

The rhythm families of determinants $n = 2, \ldots, 7$ are shown in musical notation in Fig. 13.4. The parent generation r_0, the starting binomial is shown on the top staff. The second staff from the top shows the retrograde permutation, the third staff has the result of the first synchronization step. In case of a non-uniform child generation r_1, the three permutations are shown on the grouped staves 3 to 5 (remember that two elements are equal and therefore the set of permutations is $3!/2! = 3$). The five permutations of the second generation quintinomial r_2 are shown on grouped staves 6–10. The uniform distribution after the last synchronization process is shown on the bottom staff.

Also for the determinants $n = 8$ and $n = 9$ the iterative process stops at the third generation, as shown in Fig. 13.5. The former has two parent binomials $r_0 = \{5 + 3, 7 + 1\}$, the latter has three starting non-uniform binomials $9 = \{5 + 4, 7 + 2, 8 + 1\}$. Note how all 2nd generation quintinomials have either 4 equal elements, $r_2 = b + b + a + b + b$ or the pattern $r_2 = b + a + b + a + b$ (two equal larger values a and three equal smaller values b). These rhythmical patterns are shown in musical notation in Fig. 13.6.

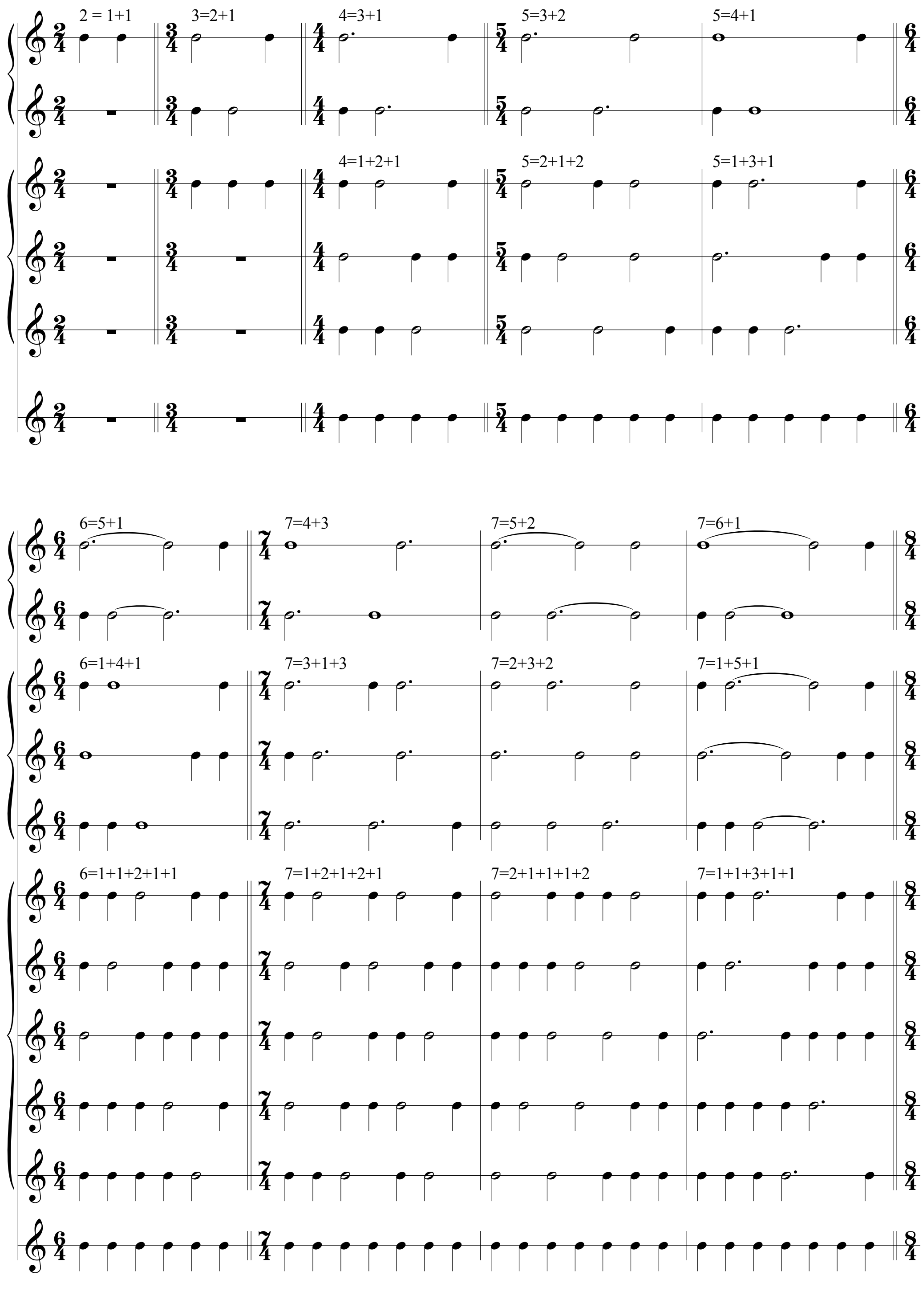

Figure 13.4: Fractional division of rhythm families. Determinants $n = 2, 3, 4, 5, 6, 7$. The parent generation binomial combinations $r_0 = a + b$ are shown in the top staff. Lower staves illustrate the results of the iterative permutation-synchronization process.

	$\begin{bmatrix}8\\8\end{bmatrix}$		$\begin{bmatrix}8\\8\end{bmatrix}$	Determinant
∘ ∘	$\mid 5+3$	∘ ∘	$\mid 7+1$	r_0 Binomial $a+b$
∘ ∘	$\mid 3+5$	∘∘	$\mid 1+7$	Permutation
				↓ Synchronize
∘ ∘ ∘	$\mid 3+2+3$	∘∘ ∘	$\mid 1+6+1$	r_1 Trinomial
∘ ∘ ∘	$\mid 2+3+3$	∘ ∘∘	$\mid 6+1+1$	Permutations
∘ ∘ ∘	$\mid 3+3+2$	∘∘∘	$\mid 1+1+6$	⋮
				↓ Synchronize
∘ ∘∘ ∘∘	$\mid 2+1+2+1+2$	∘∘∘ ∘∘	$\mid 1+1+4+1+1$	r_2 Quintinomial
∘∘ ∘∘ ∘	$\mid 1+2+1+2+2$	∘∘ ∘∘∘	$\mid 1+4+1+1+1$	Permutations
∘ ∘∘ ∘ ∘	$\mid 2+1+2+2+1$	∘ ∘∘∘∘	$\mid 4+1+1+1+1$	⋮
∘∘ ∘ ∘∘	$\mid 1+2+2+1+2$	∘∘∘∘∘	$\mid 1+1+1+1+4$	⋮
∘ ∘ ∘∘ ∘	$\mid 2+2+1+2+1$	∘∘∘∘ ∘	$\mid 1+1+1+4+1$	⋮
				↓ Synchronize
∘∘∘∘∘∘∘∘	$\mid 1+\ldots+1$	∘∘∘∘∘∘∘∘	$\mid 1+\ldots+1$	Uniform distribution
$n=8$		$n=8$		

	$\begin{bmatrix}9\\9\end{bmatrix}$		$\begin{bmatrix}9\\9\end{bmatrix}$		$\begin{bmatrix}9\\9\end{bmatrix}$
∘ ∘	$\mid 5+4$	∘ ∘	$\mid 7+2$	∘ ∘	$\mid 8+1$
∘ ∘	$\mid 4+5$	∘ ∘	$\mid 2+7$	∘∘	$\mid 1+8$
∘ ∘∘	$\mid 4+1+4$	∘ ∘ ∘	$\mid 2+5+2$	∘∘ ∘	$\mid 1+7+1$
∘∘ ∘	$\mid 1+4+4$	∘ ∘ ∘	$\mid 5+2+2$	∘ ∘∘	$\mid 7+1+1$
∘ ∘ ∘	$\mid 4+4+1$	∘ ∘ ∘	$\mid 2+2+5$	∘∘∘	$\mid 1+1+7$
∘∘ ∘∘ ∘	$\mid 1+3+1+3+1$	∘ ∘ ∘∘ ∘	$\mid 2+2+1+2+2$	∘∘∘ ∘∘	$\mid 1+1+5+1+1$
∘ ∘∘ ∘∘	$\mid 3+1+3+1+1$	∘ ∘∘ ∘ ∘	$\mid 2+1+2+2+2$	∘∘ ∘∘∘	$\mid 1+5+1+1+1$
∘∘ ∘∘∘	$\mid 1+3+1+1+3$	∘∘ ∘ ∘ ∘	$\mid 1+2+2+2+2$	∘∘ ∘∘∘	$\mid 5+1+1+1+1$
∘ ∘∘∘ ∘	$\mid 3+1+1+3+1$	∘ ∘ ∘ ∘ ∘	$\mid 2+2+2+2+1$	∘∘∘ ∘∘	$\mid 1+1+1+1+5$
∘∘∘ ∘∘	$\mid 1+1+3+1+3$	∘ ∘ ∘ ∘∘	$\mid 2+2+2+1+2$	∘∘∘∘ ∘	$\mid 1+1+1+5+1$
∘∘∘∘∘∘∘∘∘	$\mid 1+\ldots+1$	∘∘∘∘∘∘∘∘∘	$\mid 1+\ldots+1$	∘∘∘∘∘∘∘∘∘	$\mid 1+\ldots+1$
$n=9$		$n=9$		$n=9$	

Figure 13.5: Fractional evolution of rhythm families. Determinants $n=8,9$

The figure is included in the full version

Figure 13.6: Fractional division of rhythm families. Determinants $n = 8, 9$

13.2 Evolution on the factorial level

The evolutionary process may also be applied on the factorial level by using distributive powers. These powers may be applied to the parent generation binomial, i.e., the square $r_0^2 = (a+b)^2$ or to the trinomial from the first generation of children $r_1^2 = (b+a+b)^2$.

After calculating the distributive power the resulting longer pattern will then undergo the iterative process of permutation and synchronization, shown in diagram in Fig. 13.1. For the simplest case, the square of a binomial, this is shown in numbers in Fig. 13.7 for determinants $n = 3, 4$ and one binomial for $n = 5$, $r_0^2 = (3+2)^2$. The patterns have duration n^2 and the first generation r_1 has four attacks (see Section 12.2.1 for the calculations involved). The resultant has the general characteristic $r_1 = a + b + b + c$, with twelve permutations. For $n = 2$ and $n = 3$ the synchronization process yields uniform distribution at this child generation r_1, and the evolution stops. For $n = 5$ the next generation is the 11 attack group $r_2 = 4+2+3+1+2+1+2+1+3+2+4$, with numerous permutations.

The same process may be illustrated for the evolution of the cube of a binomial, the square of a trinomial but the number of permutations will be beyond handling and limit practical musical applications. Although the resulting child generation patterns are unique to the family it is questionable whether the listener will recognize or experience this characteristic.

Schillinger in his book illustrates the evolution process with examples of rhythmical patterns in classical and popular music and once again tries to prove the potential and rising importance of the determinants $n = 6, 9$, through the subdivision and grouping into $\begin{bmatrix} 6 \\ 6 \end{bmatrix}$ and $\begin{bmatrix} 9 \\ 9 \end{bmatrix}$ meters. This aspect of musical style analysis was already discussed in Section 12.1. He provides an analysis of swing music as evidence of the emergence of triple-division, albeit in a hybrid form in $\begin{bmatrix} 4 \\ 4 \end{bmatrix}$, where one measure contains $4 \times (\frac{2}{8} + \frac{1}{8})$ (swing) or $4 \times (\frac{1}{8} + \frac{1}{8} + \frac{1}{8})$ (shuffle) subgroups at the fractional level.

$(2+1)^2$

∘ ∘ ∘ ∘ $|\, 4+2+2+1$
∘ ∘ ∘∘ $|\, 4+2+1+2$
∘ ∘∘ ∘ $|\, 4+1+2+2$
∘ ∘ ∘ ∘ $|\, 2+4+2+1$
⋮ 12 permutations
∘∘ ∘ ∘ $|\, 1+2+2+4$
$\downarrow$ synchronize
∘∘∘∘∘∘∘∘∘ $|\, 1+\ldots+1$
$n=3$

$(3+1)^2$

∘ ∘ ∘ ∘ $|\, 9+3+3+1$
∘ ∘ ∘∘ $|\, 9+3+1+3$
∘ ∘∘ ∘ $|\, 9+1+3+3$
∘ ∘ ∘ ∘ $|\, 3+9+3+1$
⋮ 12 permutations
∘∘ ∘ ∘ $|\, 1+3+3+9$
$\downarrow$ synchronize
∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘ $|\, 1+\ldots+1$
$n=4$

$(3+2)^2$

∘ ∘ ∘ ∘ $|\, 9+6+6+4$
∘ ∘ ∘ ∘ $|\, 9+6+4+6$
∘ ∘ ∘ ∘ $|\, 9+4+6+6$
∘ ∘ ∘ ∘ $|\, 6+9+6+4$
⋮ 12 permutations
∘ ∘ ∘ ∘ $|\, 4+6+6+9$
$\downarrow$ synchronize
∘ ∘ ∘ ∘∘ ∘∘ ∘∘ ∘ ∘ $|\, 4+2+3+1+2+1+2+1+3+2+4$
⋮ 69300 permutations
∘ ∘ ∘ ∘ ∘∘ ∘∘ ∘∘ ∘ $|\, 4+4+2+3+1+2+1+2+1+3+2$
$\downarrow$ synchronize
∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘∘ $|\, 1+\ldots+1$
$n=5$

Figure 13.7: Factorial evolution of rhythm families. Power series: square, applied to a binomial $a+b$. Three cases are shown for determinants $n=2,3$.

Chapter 14

Rhythms of variable velocities

This chapter is about rhythmical patterns with an *accelerando* or *ritardando* character. The use of *acceleration series* is the mechanism for creating the feeling of speeding up or slowing down. Schillinger calls these *variable velocity* rhythms, which is a confusing term in our MIDI era where note velocity refers to the speed at which a key is depressed on a MIDI input controller.

Essentially, acceleration series are affecting the duration of a sequence of attacks over a continuous, steady beat. Gradually increasing the note duration creates more and more widely spread attacks, with a feeling of slowing down. Reversing such a series creates an accelerando. Schillinger illustrates a number of variable time musical phenomena related to these acceleration series or the variable ratios between longer and shorter notes.

14.1 Acceleration series

Acceleration series are the mathematical equivalent of a natural growth process. Rhythms created with techniques from the previous chapters, such as interference (see Chapter 2) or fractioning of two generators (see Chapter 4) yield rhythms with a limited set of durations in ordered sequences. Depending on the determinant n they yield attack patterns with durations of $1t, 2t, 3t, \ldots$ time units in a characteristic ordered series, the resultant r (remember the retrograde symmetry property with longer notes at both ends). Acceleration series, on the other hand, keep growing in duration; each new term in the series has a longer duration than its predecessor.

There are various types of mathematical series of integers, that may used to model or represent a (natural) growth process. What characterizes these series is the *speed of growth*, it is the amount of duration growth, i.e., the difference in duration between two consecutive terms in the series.

A number of acceleration series are listed in Table 14.1. These are all infinite series of integer numbers n_i, where i is the index of the i-th element in the series. The mathematical formula will sometimes apply from the third element in the series onward, i.e., $i > 2$. Not all series start at initial value $n_0 = 1$; see the third and fourth example of the *geometrical progressions*. The *natural harmonic series* contains increasing integer numbers at constant speed, i.e., $n_i - n_{i-1} = 1$. The *power series* have the highest growth rate, a characteristic that we may recognize from Chapter 12, where distributive powers were used to create variation with longer rhythmical patterns. The *summation series* group is also known as *Fibonacci series*

Table 14.1: Acceleration series. The first column has the name of the series, the second column shows the mathematical formula for creating the infinite series, and the last column presents the result in integer numbers. n_i is the integer value of the i-th element in the series.

Type	$n_i, i = (1), 2, \ldots, \infty$	Series element values
Natural harmonic series	$n_i = n_{i-1} + 1$	$1, 2, 3, 4, 5, 6, 7, \ldots$
Arithmical progressions	$n_i = n_{i-1} + m, m = 2, 3, \ldots$	$1, 3, 5, 7, 9, \ldots (m = 2)$
		$1, 4, 7, 10, 13, \ldots (m = 3)$
	$n_i = n_{i-1} + (i - 1)$	$1, 2, 4, 7, 11, 16, 22, \ldots$
Geometrical progressions	$n_i = m\, n_{i-1}, m = 2, 3, \ldots$	$1, 2, 4, 8, 16, \ldots (m = 2)$
		$1, 3, 9, 27, 81, \ldots (m = 3)$
		$3, 6, 12, 24, 48, \ldots (m = 2)$
		$2, 6, 18, 54, 162, \ldots (m = 3)$
Power series	$n_i = m^i, m = 2, 3, \ldots$	$2, 4, 8, 16, 42 \ldots (m = 2)$
		$3, 9, 27, 81, \ldots (m = 3)$
		$5, 25, 125, 625, \ldots (m = 5)$
Summation series	$n_i = n_{i-1} + n_{i-2}$	$1, 2, 3, 5, 8, 13, 21, \ldots$
		$1, 3, 4, 7, 11, 18, 29, \ldots$
		$1, 4, 5, 9, 14, 23, 37, \ldots$
Prime number series		$1, 2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots$

and is a growth series that may be observed in nature; the distance between branches and leaves on plants, the growth of snail shells, the uncurling of a fern spiral, etc. We already encountered these natural growth series in Chapter 6, where we looked at three-generator combinations. The Fibonacci series has another property: for very large values of i, approaching infinity $i \to \infty$, the ratio of two consecutive numbers in the series becomes the *Golden Section*, also known as the golden ratio

$$\left[\frac{n_i}{n_{i-1}}\right]_{i\to\infty} = 1.618033988\ldots.$$

This number is used in architecture and in musical form, where it determines the ratio between geometrical dimensions and section lengths. This may be an intuitive or deliberate choice by the composer.

The acceleration series are shown graphically in Fig. 14.1. This representation provides a better view on the speed of growth of each series; the power series have the strongest acceleration effect. Vertical lines in these diagrams indicate options for regular grouping by either 3 or 4 time units into multiple full measures. These series are also shown in musical notation, see Fig. 14.2. In m. 1–10 the time unit is $t = \frac{1}{8}$ and the first five or six terms in the series are shown.

The figure is included in the full version

Figure 14.1: Acceleration series

The figure is included in the full version

Figure 14.2: Growth series in musical notation

14.2 Application of acceleration series

The use of an acceleration series may be at either a continuous steady beat or in a variable tempo phrase.

14.2.1 Acceleration in uniform groups

First we consider the case of a monomial, a single-element generator with time unit t. This leads to a uniform, steady beat attack pattern as we have seen in Section 2.1.1. Based on this time unit we select an acceleration series (or the reverse) to create a variable velocity effect at some point in the music.

Note that all the acceleration series consist of integer numbers. The reason for this is twofold: at the time of writing of the Schillinger's books the personal computer did not exist and calculating the numbers had to be doable on paper.[1] The other reason is the use of the series over a continuous steady beat; the time unit remains constant and in musical notation there is the grouping at the measure level, using a fixed time signature $\begin{bmatrix} n \\ n \end{bmatrix}$.

Choosing the appropriate growth series and the number of elements in the series allows grouping at full measures. For example, adding the first six terms in the summation series yields the sum $1 + 2 + 3 + 5 + 8 + 13 = 32$, which leads to 8 measures at meter $\begin{bmatrix} 4 \\ 4 \end{bmatrix}$ and time unit $t = \frac{1}{4}$, a regular phrase length. The natural harmonic series also has this property $1+2+3+4+5+6+7+8 = 36$, which yields 9 measures at $\begin{bmatrix} 4 \\ 4 \end{bmatrix}$ time signature and $t = \frac{1}{4}$, or 12 measures at $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ or three measures at $\begin{bmatrix} 12 \\ 8 \end{bmatrix}$ and $t = \frac{1}{8}$. The first four prime numbers add up to $1 + 2 + 3 + 5 + 7 = 18$. The power series also have this grouping property. All these grouping examples lead to a written-out ritardando. These grouping options into full measures are indicated in Fig. 14.1 as blue vertical lines.

Application Tip:

In Schillinger's book the potential of this technique in film music, stage and dance productions is mentioned. In such productions certain events in a cue or scene may have to be timed to occur on specific beats at a constant musical tempo, while at the same time this sequence has to suggest speeding up or slowing down. Selecting the appropriate acceleration series may offer a solution in these situations.[2]

Acceleration and ritardando may also be combined, distributing these contrasting effects over the parts in a score. The combination can be used for a climax effect.

Example 14.1

Combination of written-out acceleration and deceleration.

The summation series $r = t + 2t + 3t + 5t + 8t + 13t$ with total duration

[1]There obviously also exist growth series based on rational, non-integer numbers. An example of these is the exponential growth pattern in the *Risset rhythm*. This is a clever rhythm, named after French composer Jean-Claude Risset, that suggests a continuous acceleration or slowing down over a steady tempo. The effect is used in Electronic Dance Music (EDM).

[2]The alternative is a cue at variable tempo, using a *click track*. That is great for recording the music, but is more difficult in a live performance situation. In such cases the acceleration series over a steady tempo makes the job easier for the conductor and musicians; synchronization of the attacks is simpler to achieve.

$T = 32t$ is used to demonstrate the combination of a written-out deceleration with an acceleration, see Fig. 14.2, m. 11–14. The tempo is constant, the variable speed impression is a consequence of the changing note duration. This four-measure fragment might conclude a musical phrase and prepare for a transition. The slowing down in the upper staff might start after a number of steady shorter notes, preferably stressing the time unit. A series of four 8th notes just before this phrase makes the deceleration effect more noticeable.

14.2.2 Acceleration in non-uniform groups

Example 14.2

Acceleration in non-uniform groups.
This example is included in the full version of the book.

14.2.3 Rubato

The *rubato* is a case where the tempo is changing freely, with local accelerations and decelerations, while on the longer timescale there still is a steady tempo. Schillinger identifies the rubato as a mechanism with a general tendency to deviate in the performance from a balanced binomial towards a more unbalanced ratio.

This means that in a case of two equal attack-durations one of the pair is elongated by an amount Δt, where this difference is compensated in the second attack. For example, when there is a sequence of two equal 8th notes at time unit $t = \frac{1}{8}$ the rubato implies

$$r = (t + t) \to (1 + \Delta)t + (1 - \Delta)t \quad \text{or} \quad (1 - \Delta)t + (1 + \Delta)t.$$

The value of Δt will differ from pair to pair. A limit situation could be where $\Delta t = \frac{1}{2}t$, which means that the two equal 8th notes have become a dotted 8th-16th note pair.

A similar phenomenon is observed in the interpretation of swing music which is notated as (pairs of) equal, steady 8th note patterns. The interpretation leads to an elongation of the first 8th note of each pair (the onbeat note) with accompanying shorter second note (the afterbeat note). The elongation Δt in the swing style interpretation is tempo-dependent, with in inversely proportional relationship

$$\Delta t \propto \frac{1}{BPM},$$

where BPM is the tempo in beats-per-minute. At slower tempos, i.e., moderate swing, $\Delta t \to \frac{1}{16}$ corresponding to the dotted 8th-16th pair $r = \frac{3}{4}t + \frac{1}{4}t$. This is also known as the *bounce* swing pattern. At medium swing tempos this leads to the familiar triplet interpretation where $\Delta t = \frac{1}{6}t, r = \frac{2}{3}t + \frac{1}{3}t$. At fast, up-tempo swing $\Delta \to 0$, with almost equal 8th notes. The difference with the rubato is that the swing elongation is constant and applied to every pair, whereas the rubato is flexible and variable on the local timescale.

14.2.4 Fermata (hold)

Chapter 15

Conclusion

We have come to the conclusion of this detailed guide to Schillinger's *Theory of Rhythm*. The previous 14 chapters have adhered to the naming and sectioning of the original book. They have provided a collection of techniques for creating and modifying rhythmical patterns on different timescales.

We have seen a number of source mechanisms, using interference or fractioning of either two or three uniform generators, 'ticking' at constant intervals. These mechanisms create resultants, i.e., attack-duration groups with multiple elements, ranging from non-uniform binomials to long series of multinomials with variable note duration.

Some chapters have focused on variation techniques. Permutations and splitting were introduced to create homogeneous variability locally on the small timescale; it implies reordering the elements of an attack-duration group. Grouping by pairs and distributive powers enabled the creation of longer rhythmical patterns from a given source pattern. Rhythm creation at longer timescales involved juxtaposing several variants in a continuity.

The application to music was also covered: grouping attack-duration patterns into regular divisions based on the time signature, i.e., the meter. Thus the phenomenon of recurrence, the repeat of a rhythmical pattern at the full measure on the downbeat, introduced itself. The distribution of a rhythm using a specific, ordered attack sequence for a single instrument was discussed. We also saw the distribution of attacks over multiple parts. The combination of several variants in parallel generates a simultaneity, a multi-part score. As an obvious reminder of the potential of the techniques there were demonstrations of the application of resultants to musical attributes suchs as accents, rests and split-unit groups.

Some chapters considered the evolution of rhythm families on the subdivision, fractional level (within the measure) and on the longer timescale, factorial level (multiple measures). Schillinger provides style analysis and made predictions about the future prominence of certain rhythm families and meters.

This guide used simple mathematics with integer numbers, graphical representations and musical notation to illustrate the techniques and the various rhythmical aspects. Quite a number of examples were created, that do not appear in the original Schillinger book. Application tips, comments and suggestions were interspersed in the text.

Studying these techniques will create a toolbox for the composer and arranger. Mastering the content of his guide to the theory of rhythm will undoubtedly add to the craftmanship and skills of the creative musician. There is no need to limit yourself to short repetitive rhythmical loops, because you did not know how to deviate from the trodden path and find interesting, yet coherent and homogeneous alternatives. Play and experiment with the techniques in moments of lacking inspiration. Most likely, some great idea will pop-up while doing rhythmical sketches. The result might be someting that is or is not covered in this book; finding a personal style and be creative is the end goal anyway. May this book help you on your way.

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Index