Euclidean plane and its relatives

A minimalist introduction

Anton Petrunin

Contents

References

Introduction

This book is meant to be rigorous, conservative, elementary, and minimalist. At the same time, it includes about the maximum what students can absorb in one semester.

Approximately one-third of the material used to be covered in high school, but not anymore.

The present book is based on the courses given by the author at the Pennsylvania State University as an introduction to the foundations of geometry. The lectures were oriented to sophomore and senior university students. These students already had a calculus course. In particular, they are familiar with real numbers and continuity. It makes it possible to cover the material faster and in a more rigorous way than it could be done in high school.

Prerequisite

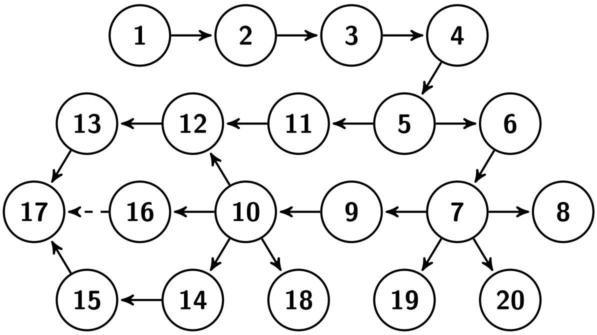
The students should be familiar with the following topics:

- ◇ Elementary set theory: $\in$, $\cup$, $\cap$, $\setminus$, $\subset$, $\times$.
- ◇ Real numbers: intervals, inequalities, algebraic identities.
- ◇ Limits, continuous functions, and the intermediate value theorem.
- ◇ Standard functions: absolute value, natural logarithm, exponential function. Occasionally, trigonometric functions are used, but these parts can be ignored.
- ◇ Chapter 14 uses basic vector algebra.
- ◇ To read Chapter 16, it is better to have some previous experience with the *scalar product*, also known as *dot product*.
- ◇ To read Chapter 18, it is better to have some previous experience with complex numbers.

Overview

We use the so-called *metric approach* introduced by Birkhoff. It means that we define the Euclidean plane as a *metric space* that satisfies a list of properties (*axioms*). This way we minimize the tedious parts which are unavoidable in the more classical Hilbert's approach. At the same time, the students have a chance to learn basic geometry of metric spaces.

Here is a dependency graph of the chapters.



In (1) we give all the definitions necessary to formulate the axioms; it includes metric space, lines, angle measure, continuous maps, and congruent triangles.

Further, we do Euclidean geometry: (2) Axioms and immediate corollaries; (3) Half-planes and continuity; (4) Congruent triangles; (5) Circles, motions, perpendicular lines; (6) Similar triangles and (7) Parallel lines — these are the first two chapters where we use Axiom V, an equivalent of Euclid's parallel postulate. In (8) we give the most classical theorem of triangle geometry; this chapter is included mainly as an illustration.

In the following two chapters we discuss the geometry of circles on the Euclidean plane: (9) Inscribed angles; (10) Inversion. It will be used to construct the model of the hyperbolic plane.

Further, we discuss non-Euclidean geometry: (11) Neutral geometry — geometry without the parallel postulate; (12) Conformal disc model — this is a construction of the hyperbolic plane, an example of a neutral plane which is not Euclidean. In (13) we discuss geometry of the constructed hyperbolic plane — this is the highest point in the book.

In the remaining chapters, we discuss some additional topics: (14) Affine geometry; (15) Projective geometry; (16) Spherical geometry; (17) Projective model of the hyperbolic plane; (18) Complex coordinates; (19) Geometric constructions; (20) Area. The proofs in these chapters are not completely rigorous.

We encourage the use of visual assignments at the author's website.

Disclaimer

It is impossible to find the original reference to most of the theorems discussed here, so I do not even try to. Most of the proofs discussed in the book already appeared in Euclid's Elements.

Recommended books

- ◇ Byrne's Euclid [7] — a colored version of the first six books of Euclid's Elements edited by Oliver Byrne.
- ◇ Kiselev's textbook [10] — a classical book for school students; it should help if you have trouble following this book.
- ◇ Hadamard's book [9] — an encyclopedia of elementary geometry originally written for school teachers.
- ◇ Prasolov's book [15] is perfect to master your problem-solving skills.
- ◇ Akopyan's book [1] — a collection of problems formulated in figures.
- ◇ Methodologically my lectures were very close to Sharygin's textbook [17]. This is the greatest textbook in geometry for school students, I recommend it to anyone who can read Russian.

Acknowledgments

Let me thank Thomas Barthelme, Matthew Chao, Svetlana Katok, Alexander Lytchak, Alexei Novikov, and Lukeria Petrunina for useful suggestions and correcting misprints.

This work was partially supported by NSF grants DMS-0103957, DMS-0406482, DMS-0905138, DMS-1309340, DMS-2005279, and Simons Foundation grants 245094, 584781.

Chapter 1

Preliminaries

What is the axiomatic approach?

In the axiomatic approach, one defines the plane as anything that satisfies a given list of properties. These properties are called *axioms*. The axiomatic system for the theory is like the rules for a game. Once the axiom system is fixed, a statement is considered to be true if it follows from the axioms and nothing else is considered to be true.

The formulations of the first axioms were not rigorous at all. For example, Euclid described a *line* as *breadthless length* and a *straight line* as a line that *lies evenly with the points on itself*. On the other hand, these formulations were sufficiently clear so that one mathematician could understand the other.

The best way to understand an axiomatic system is to make one by yourself. Look around and choose a physical model of the Euclidean plane; imagine an infinite and perfect surface of a chalkboard. Now try to collect the key observations about this model. Assume for now that we have an intuitive understanding of such notions as *line* and *point*.

(i) We can measure distances between points.

(ii) We can draw a unique line that passes thru two given points.

(iii) We can measure angles.

(iv) If we rotate or shift we will not see the difference.

(v) If we change the scale we will not see the difference.

These observations are good to start with. Further we will develop the language to reformulate them rigorously.

What is a model?

The Euclidean plane can be defined rigorously the following way:

Define a point *in the Euclidean plane as a pair of real numbers* (x, y) *and define the* distance *between the two points* (x_1, y_1) *and* (x_2, y_2) *by the following formula:*

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

That is it! We gave a *numerical model* of the Euclidean plane; it builds the Euclidean plane from real numbers while the latter is assumed to be known.

Shortness is the main advantage of the model approach, but it is not intuitively clear why we define points and the distances this way.

On the other hand, the observations made in the previous section are intuitively obvious — this is the main advantage of the axiomatic approach.

Another advantage lies in the fact that the axiomatic approach is easily adjustable. For example, we may remove one axiom from the list, or exchange it to another axiom. We will do such modifications in Chapter 11 and further.

Metric spaces

The notion of metric space provides a rigorous way to say: *"we can measure distances between points"*. That is, instead of (i) on page 9, we can say *"Euclidean plane is a metric space"*.

1.1. Definition. *Let* $\mathcal{X}$ *be a nonempty set and* d *be a function that returns a real number* $d(A, B)$ *for any pair* $A, B \in \mathcal{X}$*. Then* d *is called* metric *on* $\mathcal{X}$ *if for any* $A, B, C \in \mathcal{X}$*, the following conditions are satisfied:*

(a) *Positiveness:*

$$d(A, B) \geqslant 0.$$

(b) $A = B$ *if and only if*

$$d(A, B) = 0.$$

(c) *Symmetry:*

$$d(A, B) = d(B, A).$$

(d) *Triangle inequality:*

$$d(A, C) \leqslant d(A, B) + d(B, C).$$

A metric space *is a set with a metric on it. More formally, a metric space is a pair* $(\mathcal{X}, d)$ *where* $\mathcal{X}$ *is a set and* d *is a metric on* $\mathcal{X}$*.*

The elements of $\mathcal{X}$ *are called* points *of the metric space. Given two points* $A, B \in \mathcal{X}$*, the value* $d(A, B)$ *is called* distance *from* A *to* B*.*

Examples

- ⋄ *Discrete metric.* Let $\mathcal{X}$ be an arbitrary set. For any $A, B \in \mathcal{X}$, set $d(A, B) = 0$ if $A = B$ and $d(A, B) = 1$ otherwise. The metric d is called *discrete metric* on $\mathcal{X}$.
- ⋄ *Real line.* Set of all real numbers ($\mathbb{R}$) with metric d defined by

$$d(A, B) := |A - B|.$$

1.2. Exercise. *Show that $d(A, B) = |A - B|^2$ is* not *a metric on $\mathbb{R}$.*

- ⋄ *Metrics on the plane.* Suppose that $\mathbb{R}^2$ denotes the set of all pairs (x, y) of real numbers. Assume $A = (x_A, y_A)$ and $B = (x_B, y_B)$. Consider the following metrics on $\mathbb{R}^2$:
  - ∘ *Euclidean metric,* denoted by d_2, and defined as

$$d_2(A, B) = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}.$$

  - ∘ *Manhattan metric,* denoted by d_1 and defined as

$$d_1(A, B) = |x_A - x_B| + |y_A - y_B|.$$

  - ∘ *Maximum metric,* denoted by d_∞ and defined as

$$d_\infty(A, B) = \max\{|x_A - x_B|, |y_A - y_B|\}.$$

1.3. Exercise. *Prove that the following functions are metrics on $\mathbb{R}^2$: (a) d_1; (b) d_2; (c) d_∞.*

Shortcut for distance

Most of the time, we study only one metric on space. Therefore, we will not need to name the metric each time.

Given a metric space $\mathcal{X}$, the distance between points A and B will be further denoted by

$$AB \quad \text{or} \quad d_{\mathcal{X}}(A, B);$$

the latter is used only if we need to emphasize that A and B are points of the metric space $\mathcal{X}$.

For example, the triangle inequality can be written as

$$AC \leqslant AB + BC.$$

For the multiplication, we will always use "$\cdot$", so AB could not be confused with $A \cdot B$.

1.4. Exercise. *Show that the inequality*

$$AB + PQ \leqslant AP + AQ + BP + PQ$$

holds for any four points A, B, P, Q in a metric space.

Isometries, motions, and lines

In this section, we define *lines* in a metric space. Once it is done the sentence *"We can draw a unique line that passes thru two given points."* becomes rigorous; see (ii) on page 9.

Recall that a map $f\colon \mathcal{X} \to \mathcal{Y}$ is a *bijection* if it gives an exact pairing of the elements of two sets. Equivalently, $f\colon \mathcal{X} \to \mathcal{Y}$ is a bijection if it has an *inverse*; that is, a map $g\colon \mathcal{Y} \to \mathcal{X}$ such that $g(f(A)) = A$ for any $A \in \mathcal{X}$ and $f(g(B)) = B$ for any $B \in \mathcal{Y}$.

Let $\mathcal{X}$ and $\mathcal{Y}$ be two metric spaces and $d_\mathcal{X}$, $d_\mathcal{Y}$ be their metrics. A map

$$f\colon \mathcal{X} \to \mathcal{Y}$$

is called *distance-preserving* if

$$d_\mathcal{Y}(f(A), f(B)) = d_\mathcal{X}(A, B)$$

for any $A, B \in \mathcal{X}$.

A bijective distance-preserving map is called an *isometry*.

Two metric spaces are called *isometric* if there exists an isometry from one to the other.

The isometry from a metric space to itself is also called a *motion* of the space.

1.5. Exercise. *Show that any distance-preserving map is* injective*; that is, if $f\colon \mathcal{X} \to \mathcal{Y}$ is a distance-preserving map, then $f(A) \neq f(B)$ for any pair of distinct points $A, B \in \mathcal{X}$.*

1.6. Exercise. *Show that if $f\colon \mathbb{R} \to \mathbb{R}$ is a motion of the real line, then either (a) $f(x) = f(0) + x$ for any $x \in \mathbb{R}$, or (b) $f(x) = f(0) - x$ for any $x \in \mathbb{R}$.*

1.7. Exercise. *Prove that $(\mathbb{R}^2, d_1)$ is isometric to $(\mathbb{R}^2, d_\infty)$.*

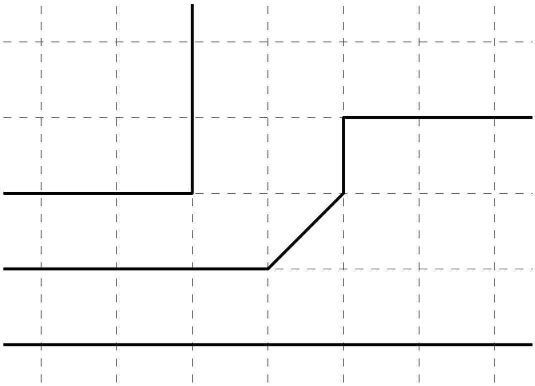
1.8. Advanced exercise. *Describe all the motions of the Manhattan plane, defined on page 11.*

If $\mathcal{X}$ is a metric space and $\mathcal{Y}$ is a subset of $\mathcal{X}$, then a metric on $\mathcal{Y}$ can be obtained by restricting the metric from $\mathcal{X}$. In other words, the distance between two points of $\mathcal{Y}$ is defined to be the distance between these points in $\mathcal{X}$. This way any subset of a metric space can be also considered as a metric space.

1.9. Definition. *A subset ℓ of metric space is called a* line*, if it is isometric to the real line.*

A triple of points that lie on one line is called *collinear*. Note that if A, B, and C are collinear, $AC \geqslant AB$ and $AC \geqslant BC$, then $AC = AB + BC$.

Some metric spaces have no lines; for example, discrete metrics. The picture shows examples of lines on the Manhattan plane $(\mathbb{R}^2, d_1)$.



1.10. Exercise. *Consider the graph* $y = |x|$ *in* $\mathbb{R}^2$*. In which of the following spaces (a)* $(\mathbb{R}^2, d_1)$*, (b)* $(\mathbb{R}^2, d_2)$*, (c)* $(\mathbb{R}^2, d_\infty)$ *does it form a line? Why?*

1.11. Exercise. *Show that any motion maps a line to a line.*

Half-lines and segments

Assume there is a line ℓ passing thru two distinct points P and Q. In this case, we might denote ℓ as (PQ). There might be more than one line thru P and Q, but if we write (PQ) we assume that we made a choice of such line.

We will denote by $[PQ)$ the *half-line* that starts at P and contains Q. Formally speaking, $[PQ)$ is a subset of (PQ) which corresponds to $[0, \infty)$ under an isometry $f\colon (PQ) \to \mathbb{R}$ such that $f(P) = 0$ and $f(Q) > 0$.

The subset of line (PQ) between P and Q is called the *segment* between P and Q and denoted by $[PQ]$. Formally, the segment can be defined as the intersection of two half-lines: $[PQ] = [PQ) \cap [QP)$.

1.12. Exercise. *Show that*

(a) if $X \in [PQ)$*, then* $QX = |PX - PQ|$*;*

(b) if $X \in [PQ]$*, then* $QX + XQ = PQ$*.*

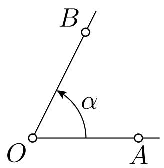
Angles

Our next goal is to introduce *angles* and *angle measures*; after that, the statement *"we can measure angles"* will become rigorous; see (iii) on page 9.

An ordered pair of half-lines that start at the same point is called an *angle*. The angle AOB (also denoted by $\angle AOB$) is the pair of half-lines $[OA)$ and $[OB)$. In this case, the point O is called the *vertex* of the angle.

Intuitively, the angle measure tells how much one has to rotate the first half-line counterclockwise, so it gets the position of the second half-line of the angle. The full turn is assumed to be $2\cdot\pi$; it corresponds to the angle measure in radians.[1]

The angle measure of $\angle AOB$ is denoted by $\measuredangle AOB$; it is a real number in the interval $(-\pi, \pi]$.



The notations $\angle AOB$ and $\measuredangle AOB$ look similar; they also have close but different meanings which better not be confused. For example, the equality $\angle AOB = \angle A'O'B'$ means that $[OA) = [O'A')$ and $[OB) = [O'B')$; in particular, $O = O'$. On the other hand, the equality $\measuredangle AOB = \measuredangle A'O'B'$ means only equality of two real numbers; in this case, O may be distinct from O'.

Here is the first property of angle measure which will become a part of the axiom.

Given a half-line $[OA)$ and $\alpha \in (-\pi, \pi]$ there is a unique half-line $[OB)$ such that $\measuredangle AOB = \alpha$.

Reals modulo $2\cdot\pi$

Consider three half-lines starting from the same point, $[OA)$, $[OB)$, and $[OC)$. They make three angles AOB, BOC, and AOC, so the value $\measuredangle AOC$ should coincide with the sum $\measuredangle AOB + \measuredangle BOC$ up to full rotation. This property will be expressed by the formula

$$\measuredangle AOB + \measuredangle BOC \equiv \measuredangle AOC,$$

where "$\equiv$" is a new notation which we are about to introduce. The last identity will become a part of the axioms.

We will write $\alpha \equiv \beta \pmod{2\cdot\pi}$, or briefly

$$\alpha \equiv \beta$$

[1] For a while you may think that π is a positive real number that measures the size of a half turn in certain units. Its concrete value $\pi \approx 3.14$ will not be important for a long time.

if $\alpha = \beta + 2\cdot\pi\cdot n$ for some integer n. In this case, we say

"α is equal to β modulo $2\cdot\pi$".

For example,

$$-\pi \equiv \pi \equiv 3\cdot\pi \quad \text{and} \quad \tfrac{1}{2}\cdot\pi \equiv -\tfrac{3}{2}\cdot\pi.$$

The introduced relation "$\equiv$" behaves as an equality sign, but

$$\dots \equiv \alpha - 2\cdot\pi \equiv \alpha \equiv \alpha + 2\cdot\pi \equiv \alpha + 4\cdot\pi \equiv \dots;$$

that is, if the angle measures differ by full turn, then they are considered to be the same.

With "$\equiv$", we can do addition, subtraction, and multiplication with integer numbers without getting into trouble. That is, if

$$\alpha \equiv \beta \quad \text{and} \quad \alpha' \equiv \beta',$$

then

$$\alpha + \alpha' \equiv \beta + \beta', \quad \alpha - \alpha' \equiv \beta - \beta' \quad \text{and} \quad n\cdot\alpha \equiv n\cdot\beta$$

for any integer n. But "$\equiv$" does not in general respect multiplication with non-integer numbers; for example,

$$\pi \equiv -\pi \quad \text{but} \quad \tfrac{1}{2}\cdot\pi \not\equiv -\tfrac{1}{2}\cdot\pi.$$

1.13. Exercise. *Show that $2\cdot\alpha \equiv 0$ if and only if $\alpha \equiv 0$ or $\alpha \equiv \pi$.*

Continuity

The angle measure is also assumed to be continuous. Namely, the following property of angle measure will become a part of the axioms:

The function

$$\measuredangle\colon (A, O, B) \mapsto \measuredangle AOB$$

is continuous at any triple of points (A, O, B) such that $O \ne A$ and $O \ne B$ and $\measuredangle AOB \ne \pi$.

To explain this property, we need to extend the notion of *continuity* to the functions between metric spaces. The definition is a straightforward generalization of the standard definition for real-to-real functions.

Further, let $\mathcal{X}$ and $\mathcal{Y}$ be two metric spaces, and $d_{\mathcal{X}}$, $d_{\mathcal{Y}}$ be their metrics.

A map $f\colon \mathcal{X} \to \mathcal{Y}$ is called *continuous* at point $A \in \mathcal{X}$ if for any $\varepsilon > 0$ there is $\delta > 0$, such that

$$d_{\mathcal{X}}(A, A') < \delta \quad \Rightarrow \quad d_{\mathcal{Y}}(f(A), f(A')) < \varepsilon.$$

(Informally it means that sufficiently small changes of A result in arbitrarily small changes of $f(A)$.)

A map $f\colon \mathcal{X} \to \mathcal{Y}$ is called *continuous* if it is continuous at every point $A \in \mathcal{X}$.

One may define a continuous map of several variables the same way. Assume $f(A, B, C)$ is a function that returns a point in the space $\mathcal{Y}$ for a triple of points (A, B, C) in the space $\mathcal{X}$. The map f might be defined only for some triples in $\mathcal{X}$.

Assume $f(A, B, C)$ is defined. Then, we say that f is continuous at the triple (A, B, C) if for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$d_{\mathcal{Y}}(f(A, B, C), f(A', B', C')) < \varepsilon.$$

if $d_{\mathcal{X}}(A, A') < \delta$, $d_{\mathcal{X}}(B, B') < \delta$, and $d_{\mathcal{X}}(C, C') < \delta$.

1.14. Exercise. *Let $\mathcal{X}$ be a metric space.*

(a) Let $A \in \mathcal{X}$ be a fixed point. Show that the function

$$f(B) := d_{\mathcal{X}}(A, B)$$

is continuous at any point B.

(b) Show that $d_{\mathcal{X}}(A, B)$ is continuous at any pair $A, B \in \mathcal{X}$.

1.15. Exercise. *Let $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{Z}$ be metric spaces. Assume that the functions $f\colon \mathcal{X} \to \mathcal{Y}$ and $g\colon \mathcal{Y} \to \mathcal{Z}$ are continuous at any point, and $h = g \circ f$ is their composition; that is, $h(A) = g(f(A))$ for any $A \in \mathcal{X}$. Show that $h\colon \mathcal{X} \to \mathcal{Z}$ is continuous at any point.*

1.16. Exercise. *Show that any distance-preserving map is continuous at any point.*

Congruent triangles

Our next goal is to give a rigorous meaning for (iv) on page 9. To do this, we introduce the notion of *congruent triangles* so instead of *"if we rotate or shift we will not see the difference"* we say that for triangles, the side-angle-side congruence holds; that is, two triangles are congruent if they have two pairs of equal sides and the same angle measure between these sides.

An *ordered* triple of distinct points in a metric space $\mathcal{X}$, say A, B, C, is called a *triangle* ABC (briefly $\triangle ABC$). Note that the triangles ABC and ACB are considered as different.

Two triangles $A'B'C'$ and ABC are called *congruent* (it can be written as $\triangle A'B'C' \cong \triangle ABC$) if there is a motion $f\colon \mathcal{X} \to \mathcal{X}$ such that

$$A' = f(A), \quad B' = f(B) \quad \text{and} \quad C' = f(C).$$

Let $\mathcal{X}$ be a metric space, and $f, g\colon \mathcal{X} \to \mathcal{X}$ be two motions. Note that the inverse $f^{-1} : \mathcal{X} \to \mathcal{X}$, as well as the composition $f \circ g : \mathcal{X} \to \mathcal{X}$, are also motions.

It follows that "$\cong$" is an *equivalence relation*; that is, any triangle congruent to itself, and the following two conditions hold:

◇ If $\triangle A'B'C' \cong \triangle ABC$, then $\triangle ABC \cong \triangle A'B'C'$.

◇ If $\triangle A''B''C'' \cong \triangle A'B'C'$ and $\triangle A'B'C' \cong \triangle ABC$, then

$$\triangle A''B''C'' \cong \triangle ABC.$$

Note that if $\triangle A'B'C' \cong \triangle ABC$, then $AB = A'B'$, $BC = B'C'$ and $CA = C'A'$.

For a discrete metric, as well as some other metrics, the converse also holds. The following example shows that it does not hold in the Manhattan plane:

Example. Consider three points $A = (0, 1)$, $B = (1, 0)$, and $C = (-1, 0)$ on the Manhattan plane $(\mathbb{R}^2, d_1)$. Note that

$$d_1(A, B) = d_1(A, C) = d_1(B, C) = 2.$$

On one hand,

$$\triangle ABC \cong \triangle ACB.$$

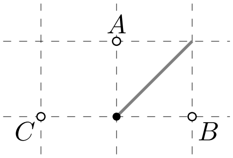
Indeed, the map $(x, y) \mapsto (-x, y)$ is a motion of $(\mathbb{R}^2, d_1)$ that sends $A \mapsto A$, $B \mapsto C$, and $C \mapsto B$.

On the other hand,

$$\triangle ABC \ncong \triangle BCA.$$

Indeed, arguing by contradiction, assume that $\triangle ABC \cong \triangle BCA$; that is, there is a motion f of $(\mathbb{R}^2, d_1)$ that sends $A \mapsto B$, $B \mapsto C$, and $C \mapsto A$.

We say that M is a midpoint of A and B if



$$d_1(A, M) = d_1(B, M) = \tfrac{1}{2} \cdot d_1(A, B).$$

Note that a point M is a midpoint of A and B if and only if $f(M)$ is a midpoint of B and C.

The set of midpoints for A and B is infinite, it contains all points (t, t) for $t \in [0, 1]$ (it is the gray segment on the picture above). On the other hand, the midpoint for B and C is unique (it is the black point on the picture). Thus, the map f cannot be bijective — a contradiction.

Chapter 2

Axioms

A system of axioms appears already in Euclid's "Elements" — the most successful and influential textbook ever written.

The systematic study of geometries as axiomatic systems was triggered by the discovery of non-Euclidean geometry. The branch of mathematics, emerging this way, is called "Foundations of geometry".

The most popular system of axioms was proposed in 1899 by David Hilbert. This is also the first rigorous system by modern standards. It contains twenty axioms in five groups, six "primitive notions", and three "primitive terms"; these are not defined in terms of previously defined concepts.

Later many different systems were proposed. It is worth mentioning the system of Alexandr Alexandrov [2] which is very intuitive and elementary, the system of Friedrich Bachmann [3] based on the concept of symmetry, and the system of Alfred Tarski [18] — a minimalist system designed for analysis using mathematical logic.

We will use another system which is very close to the one proposed by George Birkhoff [5]. This system is based on the *key observations* (i)–(v) listed on page 9. The axioms use the notions of metric space, lines, angles, triangles, equalities modulo $2\cdot\pi$ ($\equiv$), the continuity of maps between metric spaces, and the congruence of triangles ($\cong$). All this is discussed in the preliminaries.

Our system is built upon metric spaces. In particular, we use real numbers as a building block. By that reason our approach is not purely axiomatic — we build the theory upon something else; it resembles a model-based introduction to Euclidean geometry discussed on page 10. We used this approach to minimize the tedious parts which are unavoidable in purely axiomatic foundations.

The axioms

I. The *Euclidean plane* is a metric space with at least two points.

II. There is one and only one line, that contains any two given distinct points P and Q in the Euclidean plane.

III. Any angle AOB in the Euclidean plane defines a real number in the interval $(-\pi,\pi]$. This number is called the *angle measure of* $\angle AOB$ and denoted by $\measuredangle AOB$. It satisfies the following conditions:

(a) Given a half-line $[OA)$ and $\alpha \in (-\pi,\pi]$, there is a unique half-line $[OB)$, such that $\measuredangle AOB = \alpha$.

(b) For any points A, B, and C, distinct from O we have

$$\measuredangle AOB + \measuredangle BOC \equiv \measuredangle AOC.$$

(c) The function

$$\measuredangle\colon (A,O,B) \mapsto \measuredangle AOB$$

is continuous at any triple of points (A,O,B), such that $O \neq A$ and $O \neq B$ and $\measuredangle AOB \neq \pi$.

IV. In the Euclidean plane, we have $\triangle ABC \cong \triangle A'B'C'$ if and only if

$$A'B' = AB, \quad A'C' = AC, \quad \text{and} \quad \measuredangle C'A'B' = \pm\measuredangle CAB.$$

V. If for two triangles ABC, $AB'C'$ in the Euclidean plane and for $k > 0$ we have

$$\begin{aligned} B' &\in [AB), & C' &\in [AC), \\ AB' &= k\cdot AB, & AC' &= k\cdot AC, \end{aligned}$$

then

$$B'C' = k\cdot BC, \quad \measuredangle ABC = \measuredangle AB'C', \quad \measuredangle ACB = \measuredangle AC'B'.$$

From now on, we can use no information about the Euclidean plane that does not follow from the five axioms above.

2.1. Exercise. *Show that there are (a) an infinite set of points, (b) an infinite set of lines on the plane.*

Lines and half-lines

2.2. Proposition.[✓] *Any two distinct lines intersect at most at one point.*

Proof. Assume that two lines ℓ and m intersect at two distinct points P and Q. Applying Axiom II, we get that $\ell = m$. □

2.3. Exercise. *Suppose $A' \in [OA)$ and $A' \neq O$. Show that*

$$[OA) = [OA').$$

2.4. Proposition.[✓] *Given $r \geqslant 0$ and a half-line $[OA)$ there is a unique $A' \in [OA)$ such that $OA' = r$.*

Proof. According to the definition of half-line, there is an isometry

$$f\colon [OA) \to [0, \infty),$$

such that $f(O) = 0$. By the definition of isometry, $OA' = f(A')$ for any $A' \in [OA)$. Thus, $OA' = r$ if and only if $f(A') = r$.

Since isometry has to be bijective, the statement follows. □

Zero angle

2.5. Proposition.[✓] *$\measuredangle AOA = 0$ for any $A \neq O$.*

Proof. According to Axiom IIIb,

$$\measuredangle AOA + \measuredangle AOA \equiv \measuredangle AOA.$$

Subtract $\measuredangle AOA$ from both sides, we get that $\measuredangle AOA \equiv 0$.

By Axiom III, $-\pi < \measuredangle AOA \leqslant \pi$; therefore $\measuredangle AOA = 0$. □

2.6. Exercise. *Assume $\measuredangle AOB = 0$. Show that $[OA) = [OB)$.*

2.7. Proposition.[✓] *For any A and B distinct from O, we have*

$$\measuredangle AOB \equiv -\measuredangle BOA.$$

Proof. According to Axiom IIIb,

$$\measuredangle AOB + \measuredangle BOA \equiv \measuredangle AOA$$

By Proposition 2.5, $\measuredangle AOA = 0$. Hence the result. □

[✓] A statement marked with "✓" if Axiom V was not used in its proof. Ignore this mark for a while; it will be important in Chapter 11, see page 81.

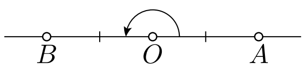
Straight angle

If $\measuredangle AOB = \pi$, we say that $\measuredangle AOB$ is a *straight angle*. Note that by Proposition 2.7, if $\angle AOB$ is straight, then so is $\angle BOA$.

We say that the point O *lies between* points A and B, if $O \neq A$, $O \neq B$, and $O \in [AB]$.

2.8. Theorem.✓ *The angle AOB is straight if and only if O* lies between *A and B.*

Proof. By Proposition 2.4, we may assume that $OA = OB = 1$.



"If" part. Assume O lies between A and B. Set $\alpha = \measuredangle AOB$.

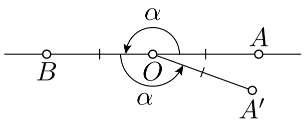
Applying Axiom IIIa, we get a half-line $[OA')$ such that $\alpha = \measuredangle BOA'$. By Proposition 2.4, we can assume that $OA' = 1$. According to Axiom IV,

$$\triangle AOB \cong \triangle BOA'.$$

Suppose that f denotes the corresponding motion of the plane; that is, f is a motion such that $f(A) = B$, $f(O) = O$, and $f(B) = A'$.

Then

$$(A'B) = f((AB)) \ni f(O) = O.$$



Therefore, both lines (AB) and $(A'B)$ contain B and O. By Axiom II, $(AB) = (A'B)$.

By the definition of the line, (AB) contains exactly two points A and B on distance 1 from O. Since $OA' = 1$ and $A' \neq B$, we get that $A = A'$.

By Axiom IIIb and Proposition 2.5, we get that

$$\begin{aligned} 2\cdot\alpha &= \measuredangle AOB + \measuredangle BOA' = \\ &= \measuredangle AOB + \measuredangle BOA \equiv \\ &\equiv \measuredangle AOA = \\ &= 0 \end{aligned}$$

Therefore, by Exercise 1.13, α is either 0 or π.

Since $[OA) \neq [OB)$, we have that $\alpha \neq 0$, see Exercise 2.6. Therefore, $\alpha = \pi$.

"Only if" part. Suppose that $\measuredangle AOB = \pi$. Consider the line (OA) and choose a point B' on (OA) so that O lies between A and B'.

From above, we have that $\measuredangle AOB' = \pi$. Applying Axiom IIIa, we get that $[OB) = [OB')$. In particular, O lies between A and B. □

A triangle ABC is called *degenerate* if A, B, and C lie on one line. The following corollary is just a reformulation of Theorem 2.8.

2.9. Corollary.✓ *A triangle is degenerate if and only if one of its angles is equal to π or 0. Moreover, in a degenerate triangle, the angle measures are 0, 0, and π.*

2.10. Exercise. *Show that three distinct points A, O, and B lie on one line if and only if*

$$2\cdot\measuredangle AOB \equiv 0.$$

2.11. Exercise. *Let A, B and C be three points distinct from O. Show that B, O and C lie on one line if and only if*

$$2\cdot\measuredangle AOB \equiv 2\cdot\measuredangle AOC.$$

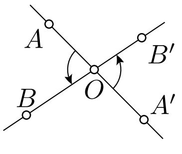
2.12. Exercise. *Show that there is a nondegenerate triangle.*

Vertical angles

A pair of angles AOB and $A'OB'$ is called *vertical* if the point O lies between A and A' and between B and B' at the same time.

2.13. Proposition.✓ *The vertical angles have equal measures.*

Proof. Assume that the angles AOB and $A'OB'$ are vertical. Note that $\angle AOA'$ and $\angle BOB'$ are straight. Therefore, $\measuredangle AOA' = \measuredangle BOB' = \pi$.



It follows that

$$0 = \measuredangle AOA' - \measuredangle BOB' \equiv$$
$$\equiv \measuredangle AOB + \measuredangle BOA' - \measuredangle BOA' - \measuredangle A'OB' \equiv$$
$$\equiv \measuredangle AOB - \measuredangle A'OB'.$$

Since $-\pi < \measuredangle AOB \leqslant \pi$ and $-\pi < \measuredangle A'OB' \leqslant \pi$, we get that $\measuredangle AOB = \measuredangle A'OB'$. □

2.14. Exercise. *Assume O is the midpoint for both segments $[AB]$ and $[CD]$. Prove that $AC = BD$.*

Chapter 3

Half-planes

This chapter contains long proofs of intuitively evident statements. It is okay to skip it, but make sure you know definitions of positive/negative angles and that your intuition agrees with 3.7, 3.9, 3.10, 3.12, and 3.17.

Sign of an angle

The positive and negative angles can be visualized as *counterclockwise* and *clockwise* directions; formally, they are defined the following way:

- ◇ The angle AOB is called *positive* if $0 < \measuredangle AOB < \pi$;
- ◇ The angle AOB is called *negative* if $\measuredangle AOB < 0$.

Note that according to the above definitions the straight angle, as well as the zero angle, are neither positive nor negative.

3.1. Exercise. *Show that $\angle AOB$ is positive if and only if $\angle BOA$ is negative.*

3.2. Lemma. *Let $\angle AOB$ be straight. Then $\angle AOX$ is positive if and only if $\angle BOX$ is negative.*

Proof. Set $\alpha = \measuredangle AOX$ and $\beta = \measuredangle BOX$. Since $\angle AOB$ is straight,

❶ $$\alpha - \beta \equiv \pi.$$

It follows that $\alpha = \pi \Leftrightarrow \beta = 0$ and $\alpha = 0 \Leftrightarrow \beta = \pi$. In these two cases, the sign of $\angle AOX$ and $\angle BOX$ are undefined.

In the remaining cases we have that $|\alpha| < \pi$ and $|\beta| < \pi$. If α and β have the same sign, then $|\alpha - \beta| < \pi$; the latter contradicts ❶. Hence the statement follows. □

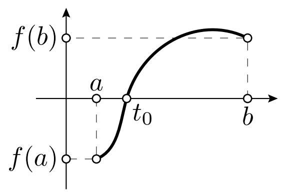
3.3. Exercise. *Assume that the angles ABC and $A'B'C'$ have the same sign and*

$$2\cdot\measuredangle ABC \equiv 2\cdot\measuredangle A'B'C'.$$

Show that $\measuredangle ABC = \measuredangle A'B'C'$.

Intermediate value theorem

3.4. Intermediate value theorem. *Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function. Assume $f(a)$ and $f(b)$ have opposite signs, then $f(t_0) = 0$ for some $t_0 \in [a,b]$.*



The intermediate value theorem is assumed to be known; it should be covered in any calculus course. We will use only the following corollary:

3.5. Corollary.✓ *Assume that for any $t \in [0,1]$ we have three points in the plane O_t, A_t, and B_t, such that*

(a) Each function $t \mapsto O_t$, $t \mapsto A_t$, and $t \mapsto B_t$ is continuous.

(b) For any $t \in [0,1]$, the points O_t, A_t, and B_t do not lie on one line.

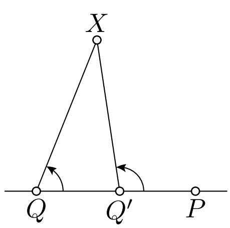
Then $\angle A_0O_0B_0$ and $\angle A_1O_1B_1$ have the same sign.

Proof. Consider the function $f(t) = \measuredangle A_tO_tB_t$.

Since the points O_t, A_t, and B_t do not lie on one line, Theorem 2.8 implies that $f(t) = \measuredangle A_tO_tB_t \neq 0$ nor π for any $t \in [0,1]$.

Therefore, by Axiom IIIc and Exercise 1.15, f is a continuous function. By the intermediate value theorem, $f(0)$ and $f(1)$ have the same sign; hence the result follows. □

Same sign lemmas



3.6. Lemma.✓ *Assume $Q' \in [PQ)$ and $Q' \neq P$. Then for any $X \notin (PQ)$ the angles PQX and $PQ'X$ have the same sign.*

Proof. By Proposition 2.4, for any $t \in [0,1]$ there is a unique point $Q_t \in [PQ)$ such that

$$PQ_t = (1-t)\cdot PQ + t\cdot PQ'.$$

Note that the map $t \mapsto Q_t$ is continuous,

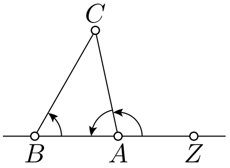
$$Q_0 = Q, \qquad\qquad Q_1 = Q'$$

and for any $t \in [0,1]$, we have that $P \neq Q_t$.

Applying Corollary 3.5, for $P_t = P$, Q_t, and $X_t = X$, we get that $\angle PQX$ has the same sign as $\angle PQ'X$. □

3.7. Signs of angles of a triangle.✓ *In arbitrary nondegenerate triangle ABC, the angles ABC, BCA, and CAB have the same sign.*

Proof. Choose a point $Z \in (AB)$ so that A lies between B and Z.



According to Lemma 3.6, the angles ZBC and ZAC have the same sign.

Note that $\measuredangle ABC = \measuredangle ZBC$ and

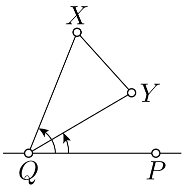
$$\measuredangle ZAC + \measuredangle CAB \equiv \pi.$$

Therefore, $\angle CAB$ has the same sign as $\angle ZAC$ which in turn has the same sign as $\measuredangle ABC = \measuredangle ZBC$.

Repeating the same argument for $\angle BCA$ and $\angle CAB$, we get the result. □

3.8. Lemma.✓ *Assume $[XY]$ does not intersect (PQ), then the angles PQX and PQY have the same sign.*

The proof is nearly identical to the one above.



Proof. According to Proposition 2.4, for any $t \in [0,1]$ there is a point $X_t \in [XY]$, such that

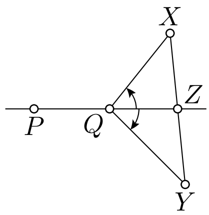
$$XX_t = t{\cdot}XY.$$

Note that the map $t \mapsto X_t$ is continuous. Moreover, $X_0 = X$, $X_1 = Y$, and $X_t \notin (QP)$ for any $t \in [0,1]$.

Applying Corollary 3.5, for $P_t = P$, $Q_t = Q$, and X_t, we get that $\angle PQX$ has the same sign as $\angle PQY$. □

Half-planes

3.9. Proposition. *Assume $X, Y \notin (PQ)$. Then the angles PQX and PQY have the same sign if and only if $[XY]$ does not intersect (PQ).*



Proof. The if-part follows from Lemma 3.8.

Assume $[XY]$ intersects (PQ); suppose that Z denotes the point of intersection. Without loss of generality, we can assume $Z \neq P$.

Note that Z lies between X and Y. According to Lemma 3.2, $\angle PZX$ and $\angle PZY$ have opposite signs. It proves the statement if $Z = Q$.

If $Z \neq Q$, then $\angle ZQX$ and $\angle QZX$ have opposite signs by 3.7. In the same way, we get that $\angle ZQY$ and $\angle QZY$ have opposite signs.

If Q lies between Z and P, then by Lemma 3.2 two pairs of angles $\angle PQX$, $\angle ZQX$ and $\angle PQY$, $\angle ZQY$ have opposite signs. It follows that $\angle PQX$ and $\angle PQY$ have opposite signs as required.

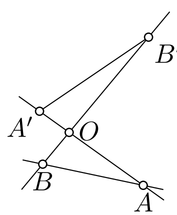
In the remaining case $[QZ) = [QP)$ and therefore $\angle PQX = \angle ZQX$ and $\angle PQY = \angle ZQY$. Therefore again $\angle PQX$ and $\angle PQY$ have opposite signs as required. □

3.10. Corollary.✓ *The complement of a line (PQ) in the plane can be presented in a unique way as a union of two disjoint subsets called* half-planes *such that*

(a) Two points $X, Y \notin (PQ)$ lie in the same half-plane if and only if the angles PQX and PQY have the same sign.

(b) Two points $X, Y \notin (PQ)$ lie in the same half-plane if and only if $[XY]$ does not intersect (PQ).

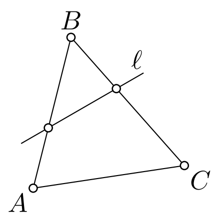
We say that X and Y lie on *one side of* (PQ) if they lie in one of the half-planes of (PQ) and we say that P and Q lie on the *opposite sides of* ℓ if they lie in the different half-planes of ℓ.



3.11. Exercise. *Suppose that the angles AOB and $A'OB'$ are vertical and $B \notin (OA)$. Show that the line (AB) does not intersect the segment $[A'B']$.*

Consider the triangle ABC. The segments $[AB]$, $[BC]$, and $[CA]$ are called *sides of the triangle.*

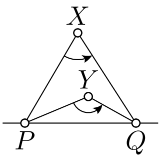
3.12. Pasch's theorem.✓ *Assume line ℓ does not pass thru any vertex of a triangle. Then it intersects either two or zero sides of the triangle.*



Proof. Assume that line ℓ intersects side $[AB]$ of the triangle ABC and does not pass thru A, B, and C.

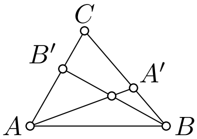
By Corollary 3.10, the vertices A and B lie on opposite sides of ℓ.

The vertex C may lie on the same side with A and on the opposite side with B or the other way around. By Corollary 3.10, in the first case, ℓ intersects side $[BC]$ and does not intersect $[AC]$; in the second case, ℓ intersects side $[AC]$ and does not intersect $[BC]$. Hence the statement follows. □



3.13. Exercise. *Show that two points $X, Y \notin (PQ)$ lie on the same side of (PQ) if and only if the angles PXQ and PYQ have the same sign.*

3.14. Exercise. *Let $\triangle ABC$ be a nondegenerate triangle, $A' \in [BC]$ and $B' \in [AC]$. Show that the segments $[AA']$ and $[BB']$ intersect.*



3.15. Exercise. *Assume that points X and Y lie on opposite sides of the line (PQ). Show that the half-line $[PX)$ does not intersect $[QY)$.*

3.16. Advanced exercise. *Note that the following quantity*

$$\tilde{\measuredangle} ABC = \begin{cases} \pi & \text{if } \measuredangle ABC = \pi \\ -\measuredangle ABC & \text{if } \measuredangle ABC < \pi \end{cases}$$

can serve as the angle measure; that is, the axioms hold if one exchanges $\measuredangle$ to $\tilde{\measuredangle}$ everywhere.

Show that $\measuredangle$ and $\tilde{\measuredangle}$ are the only possible angle measures on the plane.

Show that without Axiom IIIc, this is no longer true.

Triangle with the given sides

Consider the triangle ABC. Set

$$a = BC, \qquad b = CA, \qquad c = AB.$$

Without loss of generality, we may assume that

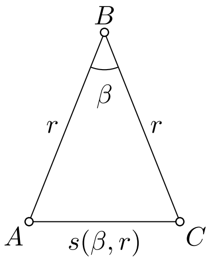
$$a \leqslant b \leqslant c.$$

Then all three triangle inequalities for $\triangle ABC$ hold if and only if

$$c \leqslant a + b.$$

The following theorem states that this is the only restriction on a, b, and c.

3.17. Theorem.✓ *Assume that $0 < a \leqslant b \leqslant c \leqslant a + b$. Then there is a triangle with sides a, b, and c; that is, there is $\triangle ABC$ such that $a = BC$, $b = CA$, and $c = AB$.*



The proof is given at the end of the section.

Assume $r > 0$ and $\pi > \beta > 0$. Consider the triangle ABC such that $AB = BC = r$ and $\measuredangle ABC = \beta$. The existence of such a triangle follows from Axiom IIIa and Proposition 2.4.

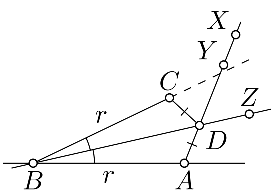
Note that according to Axiom IV, the values β and r define the triangle ABC up to the congruence. In particular, the distance AC depends only on β and r. Set

$$s(\beta, r) := AC.$$

3.18. Proposition.✓ *Given $r > 0$ and $\varepsilon > 0$, there is $\delta > 0$ such that*

$$0 < \beta < \delta \quad \Longrightarrow \quad s(r, \beta) < \varepsilon.$$

Proof. Fix two points A and B such that $AB = r$.



Choose a point X such that $\measuredangle ABX$ is positive. Let $Y \in [AX)$ be the point such that $AY = \frac{\varepsilon}{2}$; it exists by Proposition 2.4.

Note that X and Y lie on the same side of (AB); therefore, $\measuredangle ABY$ is positive. Set $\delta = \measuredangle ABY$.

Suppose $0 < \beta < \delta$; by Axiom IIIa, we can choose C so that $\measuredangle ABC = \beta$ and $BC = r$. Further we can choose a half-line $[BZ)$ such that $\measuredangle ABZ = \frac{1}{2}\cdot\beta$.

Note that A and Y lie on opposite sides of (BZ) and $\measuredangle ABZ \equiv -\measuredangle CBZ$. In particular, (BZ) intersects $[AY]$; denote by D the point of intersection.

Since D lies between A and Y, we have that $AD < AY$.

Since D lies on (BZ) we have that $\measuredangle ABD \equiv -\measuredangle CBD$. By Axiom IV, $\triangle ABD \cong \triangle CBD$. It follows that

$$\begin{aligned} s(r, \beta) = AC &\leqslant \\ &\leqslant AD + DC = \\ &= 2\cdot AD < \\ &< 2\cdot AY = \\ &= \varepsilon \end{aligned}$$

□

3.19. Corollary.✓ *Fix a real number $r > 0$ and two distinct points A and B. Then for any real number $\beta \in [0, \pi]$, there is a unique point C_β*

such that $BC_\beta = r$ *and* $\measuredangle ABC_\beta = \beta$. *Moreover,* $\beta \mapsto C_\beta$ *is a continuous map from* $[0, \pi]$ *to the plane.*

Proof. The existence and uniqueness of C_β follow from Axiom IIIa and Proposition 2.4.

Note that if $\beta_1 \neq \beta_2$, then

$$C_{\beta_1}C_{\beta_2} = s(r, |\beta_1 - \beta_2|).$$

By Proposition 3.18, the map $\beta \mapsto C_\beta$ is continuous. □

Proof of Theorem 3.17. Fix the points A and B such that $AB = c$. Given $\beta \in [0, \pi]$, suppose that C_β denotes the point in the plane such that $BC_\beta = a$ and $\measuredangle ABC = \beta$.

According to Corollary 3.19, the map $\beta \mapsto C_\beta$ is continuous. Therefore, the function $b(\beta) = AC_\beta$ is continuous (formally, it follows from Exercise 1.14 and Exercise 1.15).

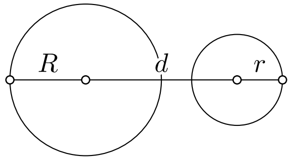
Note that $b(0) = c - a$ and $b(\pi) = c + a$. Since $c - a \leqslant b \leqslant c + a$, by the intermediate value theorem (3.4) there is $\beta_0 \in [0, \pi]$ such that $b(\beta_0) = b$, hence the result. □

Given a positive real number r and a point O, the set Γ of all points on distance r from O is called a *circle* with *radius* r and *center* O.

3.20. Exercise. *Show that two circles intersect if and only if*

$$|R - r| \leqslant d \leqslant R + r,$$

where R *and* r *denote their radiuses, and* d *— the distance between their centers.*



Chapter 4

Congruent triangles

Side-angle-side condition

Our next goal is to give conditions that guarantee congruence of two triangles.

One of such conditions is given in Axiom IV; it states that if two pairs of sides of two triangles are equal, and the included angles are equal up to sign, then the triangles are congruent. This axiom is also called *side-angle-side congruence condition*, or briefly, *SAS congruence condition*.

Angle-side-angle condition

4.1. ASA condition.✓ *Assume that*

$$AB = A'B', \qquad \measuredangle ABC = \pm\measuredangle A'B'C', \qquad \measuredangle CAB = \pm\measuredangle C'A'B'$$

and $\triangle A'B'C'$ *is nondegenerate. Then*

$$\triangle ABC \cong \triangle A'B'C'.$$

Note that for degenerate triangles the statement does not hold. For example, consider one triangle with sides 1, 4, 5 and the other with sides 2, 3, 5.

Proof. According to Theorem 3.7, either

❶ $$\begin{aligned} \measuredangle ABC &= \measuredangle A'B'C', \\ \measuredangle CAB &= \measuredangle C'A'B' \end{aligned}$$

or

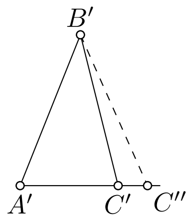
❷ $$\measuredangle ABC = -\measuredangle A'B'C', \quad \measuredangle CAB = -\measuredangle C'A'B'.$$

Further we assume that ❶ holds; the case ❷ is analogous.

Let C'' be the point on the half-line $[A'C')$ such that $A'C'' = AC$.

By Axiom IV, $\triangle A'B'C'' \cong \triangle ABC$. Applying Axiom IV again, we get that

$$\measuredangle A'B'C'' = \measuredangle ABC = \measuredangle A'B'C'.$$



By Axiom IIIa, $[B'C') = [BC'')$. Hence C'' lies on $(B'C')$ as well as on $(A'C')$.

Since $\triangle A'B'C'$ is not degenerate, $(A'C')$ is distinct from $(B'C')$. Applying Axiom II, we get that $C'' = C'$.

Therefore, $\triangle A'B'C' = \triangle A'B'C'' \cong \triangle ABC$. □

Isosceles triangles

A triangle with two equal sides is called *isosceles*; the remaining side is called the *base*.

4.2. Theorem.✓ *Assume $\triangle ABC$ is an isosceles triangle with the base $[AB]$. Then*

$$\measuredangle ABC \equiv -\measuredangle BAC.$$

Moreover, the converse holds if $\triangle ABC$ is nondegenerate.

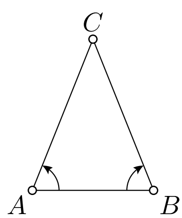
The following proof is due to Pappus of Alexandria.

Proof. Note that

$$CA = CB, \quad CB = CA, \quad \measuredangle ACB \equiv -\measuredangle BCA.$$

By Axiom IV,

$$\triangle CAB \cong \triangle CBA.$$



Applying the theorem on the signs of angles of triangles (3.7) and Axiom IV again, we get that

$$\measuredangle BAC \equiv -\measuredangle ABC.$$

To prove the converse, we assume that $\measuredangle CAB \equiv -\measuredangle CBA$. By ASA condition (4.1), $\triangle CAB \cong \triangle CBA$. Therefore, $CA = CB$. □

A triangle with three equal sides is called *equilateral*.

4.3. Exercise. *Let $\triangle ABC$ be an equilateral triangle. Show that*

$$\measuredangle ABC = \measuredangle BCA = \measuredangle CAB.$$

Side-side-side condition

4.4. SSS condition.✓ *$\triangle ABC \cong \triangle A'B'C'$ if*

$$A'B' = AB, \quad B'C' = BC \quad \textit{and} \quad C'A' = CA.$$

Note that this condition is valid for degenerate triangles as well.

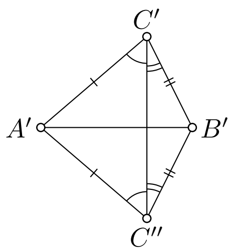
Proof. Choose C'' so that $A'C'' = A'C'$ and $\measuredangle B'A'C'' = \measuredangle BAC$. According to Axiom IV,

$$\triangle A'B'C'' \cong \triangle ABC.$$

It will suffice to prove that

❸ $$\triangle A'B'C' \cong \triangle A'B'C''.$$

The condition ❸ trivially holds if $C'' = C'$. Thus, it remains to consider the case $C'' \neq C'$.



Clearly, the corresponding sides of $\triangle A'B'C'$ and $\triangle A'B'C''$ are equal. Hence the triangles $\triangle C'A'C''$ and $\triangle C'B'C''$ are isosceles. By Theorem 4.2, we have

$$\measuredangle A'C''C' \equiv -\measuredangle A'C'C'',$$
$$\measuredangle C'C''B' \equiv -\measuredangle C''C'B'.$$

Adding them, we get that

$$\measuredangle A'C''B' \equiv -\measuredangle A'C'B'.$$

Applying Axiom IV again, we get ❸. □

4.5. Corollary.✓ *If $AB + BC = AC$, then $B \in [AC]$.*

Proof. Since $AB + BC = AC$, we can choose $B' \in [AC]$ such that $AB = AB'$; observe that $BC = B'C$.

We may assume that $AB > 0$ and $BC > 0$; otherwise, $A = B$ or $B = C$, and the statement follows. In this case, $\measuredangle AB'C = \pi$.

By SSS,

$$\triangle ABC \cong \triangle AB'C.$$

Therefore $\measuredangle ABC = \pi$. By Theorem 2.8, B lies between A and C. □

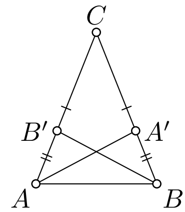
4.6. Advanced exercise. *Let M be the midpoint of the side $[AB]$ of $\triangle ABC$ and M' be the midpoint of the side $[A'B']$ of $\triangle A'B'C'$. Assume $C'A' = CA$, $C'B' = CB$, and $C'M' = CM$. Prove that*

$$\triangle A'B'C' \cong \triangle ABC.$$

4.7. Exercise. *Let $\triangle ABC$ be an isosceles triangle with the base $[AB]$. Suppose that $CA' = CB'$ for some points $A' \in [BC]$ and $B' \in [AC]$. Show that*

(a) $\triangle AA'C \cong \triangle BB'C$;

(b) $\triangle ABB' \cong \triangle BAA'$.



4.8. Exercise. *Let $\triangle ABC$ be a nondegenerate triangle and let f be a motion of the plane such that*

$$f(A) = A, \quad f(B) = B, \quad \text{and} \quad f(C) = C.$$

Show that f is the identity map; that is, $f(X) = X$ for any point X on the plane.

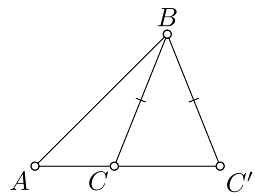
On side-side-angle and side-angle-angle

In each of the conditions SAS, ASA, and SSS we specify three corresponding parts of the triangles. Let us discuss other triples of corresponding parts.

The first triple is called *side-side-angle*, or briefly SSA; it specifies two sides and a non-included angle. This condition is not sufficient for congruence; that is, there are two nondegenerate triangles ABC and $A'B'C'$ such that

$$AB = A'B', \quad BC = B'C', \quad \measuredangle BAC \equiv \pm\measuredangle B'A'C',$$

but $\triangle ABC \ncong \triangle A'B'C'$ and $AC \neq A'C'$.



We will not use this negative statement in the sequel and therefore there is no need to prove it formally. An example can be guessed from the diagram.

The second triple is *side-angle-angle*, or briefly SAA; it specifies one side and two angles one of which is opposite to the side. This provides a congruence condition; that is, if one of the triangles ABC or $A'B'C'$ is nondegenerate and $AB = A'B'$, $\measuredangle ABC \equiv \pm\measuredangle A'B'C'$, $\measuredangle BCA \equiv \pm\measuredangle B'C'A'$, then $\triangle ABC \cong \triangle A'B'C'$.

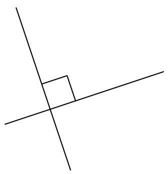
The SAA condition will not be used directly in the sequel. One proof of this condition can be obtained from ASA and the theorem on the sum of angles of a triangle proved below (see 7.12). For a more direct proof, see Exercise 11.6.

Another triple is called *angle-angle-angle*, or briefly AAA; by Axiom V, it is not a congruence condition in the Euclidean plane, but in the hyperbolic plane it is; see 13.8.

Chapter 5

Perpendicular lines

Right, acute and obtuse angles



- ⋄ If $|\measuredangle AOB| = \frac{\pi}{2}$, we say that $\angle AOB$ is *right*;
- ⋄ If $|\measuredangle AOB| < \frac{\pi}{2}$, we say that $\angle AOB$ is *acute*;
- ⋄ If $|\measuredangle AOB| > \frac{\pi}{2}$, we say that $\angle AOB$ is *obtuse*.

On the diagrams, the right angles will be marked with a little square, as shown.

If $\angle AOB$ is right, we say also that $[OA)$ is *perpendicular* to $[OB)$; it will be written as $[OA) \perp [OB)$.

From Theorem 2.8, it follows that two lines (OA) and (OB) are appropriately called *perpendicular*, if $[OA) \perp [OB)$. In this case, we also write $(OA) \perp (OB)$.

5.1. Exercise. *Assume point O lies between A and B and $X \neq O$. Show that $\angle XOA$ is acute if and only if $\angle XOB$ is obtuse.*

Perpendicular bisector

Assume M is the midpoint of the segment $[AB]$; that is, $M \in (AB)$ and $AM = MB$.

The line ℓ that passes thru M and perpendicular to (AB), is called the *perpendicular bisector* to the segment $[AB]$.

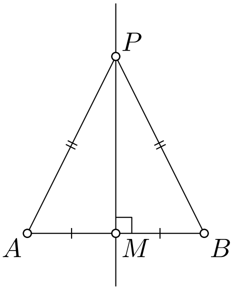
5.2. Theorem.✓ *Given distinct points A and B, all points equidistant from A and B and no others lie on the perpendicular bisector to $[AB]$.*

Proof. Let M be the midpoint of $[AB]$.

Assume $PA = PB$ and $P \neq M$. According to SSS (4.4), $\triangle AMP \cong \triangle BMP$. Hence

$$\measuredangle AMP = \pm\measuredangle BMP.$$

Since $A \neq B$, we have "$-$" in the above formula. Further,



$$\begin{aligned}\pi = \measuredangle AMB &\equiv \\ &\equiv \measuredangle AMP + \measuredangle PMB \equiv \\ &\equiv 2\cdot\measuredangle AMP.\end{aligned}$$

That is, $\measuredangle AMP = \pm\frac{\pi}{2}$. Therefore, P lies on the perpendicular bisector.

To prove the converse, suppose P is any point on the perpendicular bisector to $[AB]$ and $P \neq M$. Then $\measuredangle AMP = \pm\frac{\pi}{2}$, $\measuredangle BMP = \pm\frac{\pi}{2}$ and $AM = BM$. By SAS, $\triangle AMP \cong \triangle BMP$; in particular, $AP = BP$. □

5.3. Exercise. *Let ℓ be the perpendicular bisector to the segment $[AB]$ and X be an arbitrary point on the plane.*

Show that $AX < BX$ if and only if X and A lie on the same side from ℓ.

5.4. Exercise. *Let ABC be a nondegenerate triangle. Show that*

$$AC > BC \iff |\measuredangle ABC| > |\measuredangle CAB|.$$

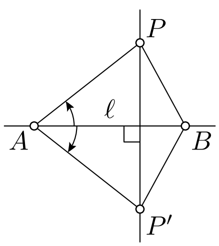
Uniqueness of a perpendicular

5.5. Theorem.✓ *There is one and only one line that passes thru a given point P and is perpendicular to a given line ℓ.*

According to the above theorem, there is a unique point $Q \in \ell$ such that $(QP) \perp \ell$. This point Q is called the *foot point* of P on ℓ.

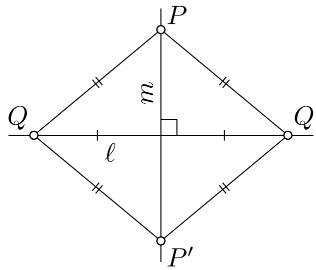
Proof. If $P \in \ell$, then both existence and uniqueness follow from Axiom III.

Existence for $P \notin \ell$. Let A and B be two distinct points of ℓ. Choose P' so that $AP' = AP$ and $\measuredangle BAP' \equiv -\measuredangle BAP$. According to Axiom IV, $\triangle AP'B \cong \triangle APB$. In particular, $AP = AP'$ and $BP = BP'$.



According to Theorem 5.2, A and B lie on the perpendicular bisector to $[PP']$. In particular, $(PP') \perp (AB) = \ell$.

Uniqueness for $P \notin \ell$. From above we can choose a point P' in such a way that ℓ forms the perpendicular bisector to $[PP']$.



Assume $m \perp \ell$ and $m \ni P$. Then m is a perpendicular bisector to some segment $[QQ']$ of ℓ; in particular, $PQ = PQ'$.

Since ℓ is the perpendicular bisector to $[PP']$, we get that $PQ = P'Q$ and $PQ' = P'Q'$. Therefore,

$$P'Q = PQ = PQ' = P'Q'.$$

By Theorem 5.2, P' lies on the perpendicular bisector to $[QQ']$, which is m. By Axiom II, $m = (PP')$. □

Reflection across a line

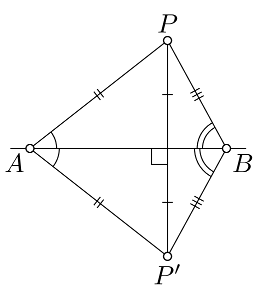
Assume the point P and the line (AB) are given. To find the *reflection* P' of P across (AB), one drops a perpendicular from P onto (AB), and continues it to the same distance on the other side.

According to Theorem 5.5, P' is uniquely determined by P.

Note that $P = P'$ if and only if $P \in (AB)$.

5.6. Proposition.✓ *Assume P' is a reflection of the point P across (AB). Then $AP' = AP$ and if $A \neq P$, then $\measuredangle BAP' \equiv -\measuredangle BAP$.*

Proof. Note that if $P \in (AB)$, then $P = P'$. By Corollary 2.9, $\measuredangle BAP = 0$ or π. Hence the statement follows.



If $P \notin (AB)$, then $P' \neq P$. By the construction of P', the line (AB) is a perpendicular bisector of $[PP']$. Therefore, according to Theorem 5.2, $AP' = AP$ and $BP' = BP$. In particular, $\triangle ABP' \cong \triangle ABP$. Therefore, $\measuredangle BAP' = \pm\measuredangle BAP$.

Since $P' \neq P$ and $AP' = AP$, we get that $\measuredangle BAP' \neq \measuredangle BAP$. That is, we are left with the case

$$\measuredangle BAP' = -\measuredangle BAP.$$

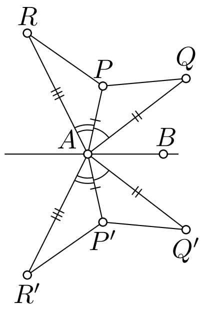
□

5.7. Corollary.✓ *The reflection across a line is a motion of the plane. Moreover, if $\triangle P'Q'R'$ is the reflection of $\triangle PQR$, then*

$$\measuredangle Q'P'R' \equiv -\measuredangle QPR.$$

Proof. Note that the composition of two reflections across the same line is the identity map. In particular, any reflection is a bijection.

Fix a line (AB) and two points P and Q; denote their reflections across (AB) by P' and Q'. Let us show that



❶ $$P'Q' = PQ;$$

that is, the reflection is distance-preserving,

Without loss of generality, we may assume that the points P and Q are distinct from A and B. By Proposition 5.6, we get that

$$\measuredangle BAP' \equiv -\measuredangle BAP, \qquad \measuredangle BAQ' \equiv -\measuredangle BAQ,$$
$$AP' = AP, \qquad AQ' = AQ.$$

It follows that

❷ $$\measuredangle P'AQ' \equiv -\measuredangle PAQ.$$

By SAS, $\triangle P'AQ' \cong \triangle PAQ$ and ❶ follows. Moreover, we also get that

$$\measuredangle AP'Q' \equiv \pm\measuredangle APQ.$$

From ❷ and the theorem on the signs of angles of triangles (3.7) we get

❸ $$\measuredangle AP'Q' \equiv -\measuredangle APQ.$$

Repeating the same argument for a pair of points P and R, we get that

❹ $$\measuredangle AP'R' \equiv -\measuredangle APR.$$

Subtracting ❹ from ❸, we get that

$$\measuredangle Q'P'R' \equiv -\measuredangle QPR.$$ □

5.8. Exercise. *Show that any motion of the plane can be presented as a composition of at most three reflections across lines.*

The motions of the plane can be divided into two types, *direct* and *indirect*. The motion f is direct if

$$\measuredangle Q'P'R' = \measuredangle QPR$$

for any $\triangle PQR$ and $P' = f(P)$, $Q' = f(Q)$ and $R' = f(R)$; if instead we have

$$\measuredangle Q'P'R' \equiv -\measuredangle QPR$$

for any $\triangle PQR$, then the motion f is called indirect.

Indeed, by Corollary 5.7, any reflection is an indirect motion. Applying the exercise above, any motion is a composition of reflections. If the number of reflections is odd then the composition indirect; if the number is even, then the motion is direct.

5.9. Exercise. *Let X and Y be the reflections of P across the lines (AB) and (BC) respectively. Show that*

$$\measuredangle XBY \equiv 2\cdot\measuredangle ABC.$$

Perpendicular is shortest

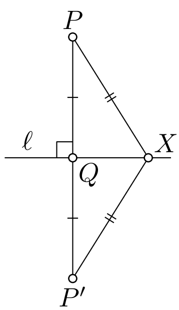
5.10. Lemma.✓ *Assume Q is the foot point of P on the line ℓ. Then the inequality*

$$PX > PQ$$

holds for any point X on ℓ distinct from Q.

If P, Q, and ℓ are as above, then PQ is called the *distance from P to ℓ*.

Proof. If $P \in \ell$, then the result follows since $PQ = 0$. Further we assume that $P \notin \ell$.



Let P' be the reflection of P across the line ℓ. Note that Q is the midpoint of $[PP']$ and ℓ is the perpendicular bisector of $[PP']$. Therefore

$$PX = P'X \quad \text{and} \quad PQ = P'Q = \tfrac{1}{2}\cdot PP'$$

Note that ℓ meets $[PP']$ only at the point Q. Therefore, $X \notin [PP']$; by the triangle inequality and Corollary 4.5,

$$PX + P'X > PP'$$

and hence the result: $PX > PQ$. □

5.11. Exercise. *Assume $\angle ABC$ is right or obtuse. Show that*

$$AC > AB.$$

5.12. Exercise. *Suppose that $\triangle ABC$ has a right angle at C. Show that for any $X \in [AC]$ the distance from X to (AB) is smaller than AB.*

Circles

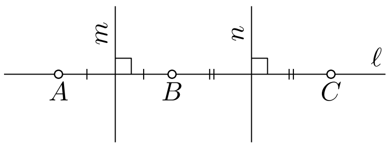
Recall that a circle with radius r and center O is the set of all points on distance r from O. We say that a point P lies *inside* of the circle if $OP < r$; if $OP > r$, we say that P lies *outside* of the circle.

5.13. Exercise. *Let* Γ *be a circle and* $P \notin \Gamma$*. Assume a line* ℓ *is passing thru the point* P *and intersects* Γ *at two distinct points,* X *and* Y*. Show that* P *is inside* Γ *if and only if* P *lies between* X *and* Y*.*

A segment between two points on a circle is called a *chord* of the circle. A chord passing thru the center of the circle is called its *diameter*.

5.14. Exercise. *Assume two distinct circles* Γ *and* Γ' *have a common chord* $[AB]$*. Show that the line between centers of* Γ *and* Γ' *forms a perpendicular bisector to* $[AB]$*.*

5.15. Lemma.✓ *A line and a circle can have at most two points of intersection.*



Proof. Assume A, B, and C are distinct points that lie on a line ℓ and a circle Γ with the center O. Then $OA = OB = OC$; in particular, O lies on the perpendicular bisectors m and n to $[AB]$ and $[BC]$ respectively. Note that the midpoints of $[AB]$ and $[BC]$ are distinct. Therefore, m and n are distinct. The latter contradicts the uniqueness of the perpendicular (Theorem 5.5). □

5.16. Exercise. *Show that two distinct circles can have at most two points of intersection.*

In consequence of the above lemma, a line ℓ and a circle Γ might have 2, 1, or 0 points of intersections. In the first two cases, the line is called *secant* or *tangent* respectively; if P is the only point of intersection of ℓ and Γ, we say that ℓ *is tangent to* Γ *at* P.

Similarly, according to Exercise 5.16, two distinct circles might have 2, 1 or 0 points of intersections. If P is the only point of intersection of circles Γ and Γ', we say that Γ *is tangent to* Γ *at* P; we also assume that circle is tangent to itself at any of its points.

5.17. Lemma.✓ *Let* ℓ *be a line and* Γ *be a circle with the center* O*. Assume* P *is a common point of* ℓ *and* Γ*. Then* ℓ *is tangent to* Γ *at* P *if and only if* $(PO) \perp \ell$*.*

Proof. Let Q be the foot point of O on ℓ.

Assume $P \neq Q$. Let P' be the reflection of P across (OQ). Note that $P' \in \ell$ and (OQ) is the perpendicular bisector of $[PP']$. Therefore, $OP = OP'$. Hence $P, P' \in \Gamma \cap \ell$; that is, ℓ is secant to Γ.

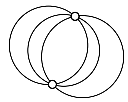
If $P = Q$, then according to Lemma 5.10, $OP < OX$ for any point $X \in \ell$ distinct from P. Hence P is the only point in the intersection $\Gamma \cap \ell$; that is, ℓ is tangent to Γ at P. □

5.18. Exercise. *Let Γ and Γ' be two distinct circles with centers at O and O' respectively. Assume Γ meets Γ' at a point P. Show that Γ is tangent to Γ' if and only if O, O', and P lie on one line.*

5.19. Exercise. *Let Γ and Γ' be two distinct circles with centers at O and O' and radiuses r and r'. Show that Γ is tangent to Γ' if and only if*

$$OO' = r + r' \quad \text{or} \quad OO' = |r - r'|.$$

5.20. Exercise. *Assume three circles have two points in common. Prove that their centers lie on one line.*



Geometric constructions

A *ruler-and-compass construction* in the plane is a construction of points, lines, and circles using only an idealized ruler and compass. These construction problems provide a valuable source of exercises in geometry, which we will use further in the book. In addition, Chapter 19 is devoted completely to the subject.

The idealized ruler can be used only to draw a line thru the given two points. The idealized compass can be used only to draw a circle with a given center and radius. That is, given three points A, B, and O we can draw the set of all points on distance AB from O. We may also mark new points in the plane, as well as on the constructed lines, circles, and their intersections (assuming that such points exist).

We can also look at the different sets of construction tools. For example, we may only use the ruler or we may invent a new tool, say a tool that produces a midpoint for any given two points.

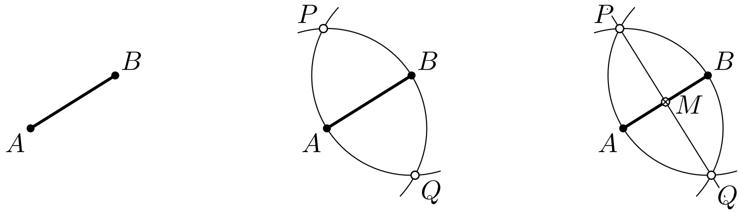
As an example, let us consider the following problem:

5.21. Construction of midpoint. *Construct the midpoint of the given segment $[AB]$.*

Construction.

1. Construct the circle with center A that is passing thru B. Construct the circle with center B that is passing thru A. Mark both points of intersection of these circles, label them with P and Q.

2. Draw the line (PQ). Mark the point M of intersection of (PQ) and $[AB]$; this is the midpoint.



Typically, you need to prove that the construction produces what was expected. Here is a proof for the example above.

Proof. According to Theorem 5.2, (PQ) is the perpendicular bisector to $[AB]$. Therefore, $M = (AB) \cap (PQ)$ is the midpoint of $[AB]$. □

5.22. Exercise. *Make a ruler-and-compass construction of a line thru a given point that is perpendicular to a given line.*

5.23. Exercise. *Make a ruler-and-compass construction of the center of a given circle.*

5.24. Exercise. *Make a ruler-and-compass construction of the lines tangent to a given circle that pass thru a given point.*

5.25. Exercise. *Given two circles* Γ_1 *and* Γ_2 *and a segment* $[AB]$ *make a ruler-and-compass construction of a circle with the radius* AB *that is tangent to each circle* Γ_1 *and* Γ_2*.*

Chapter 6

Similar triangles

Similar triangles

Two triangles $A'B'C'$ and ABC are called *similar* (briefly $\triangle A'B'C' \sim$ $\sim \triangle ABC$) if (1) their sides are proportional; that is,

❶ $$A'B' = k{\cdot}AB, \quad B'C' = k{\cdot}BC \quad \text{and} \quad C'A' = k{\cdot}CA$$

for some $k > 0$, and (2) the corresponding angles are equal up to sign:

❷ $$\begin{aligned} \measuredangle A'B'C' &= \pm\measuredangle ABC, \\ \measuredangle B'C'A' &= \pm\measuredangle BCA, \\ \measuredangle C'A'B' &= \pm\measuredangle CAB. \end{aligned}$$

Remarks.

- ⋄ According to 3.7, in the above three equalities, the signs can be assumed to be the same.
- ⋄ If $\triangle A'B'C' \sim \triangle ABC$ with $k = 1$ in ❶, then $\triangle A'B'C' \cong \triangle ABC$.
- ⋄ Note that "$\sim$" is an *equivalence relation*. That is,

  (i) $\triangle ABC \sim \triangle ABC$ for any $\triangle ABC$.

  (ii) If $\triangle A'B'C' \sim \triangle ABC$, then
  $$\triangle ABC \sim \triangle A'B'C'.$$

  (iii) If $\triangle A''B''C'' \sim \triangle A'B'C'$ and $\triangle A'B'C' \sim \triangle ABC$, then
  $$\triangle A''B''C'' \sim \triangle ABC.$$

Using the new notation "$\sim$", we can reformulate Axiom V:

6.1. Reformulation of Axiom V. *If for the two triangles $\triangle ABC$, $\triangle AB'C'$, and $k > 0$ we have $B' \in [AB)$, $C' \in [AC)$, $AB' = k{\cdot}AB$ and $AC' = k{\cdot}AC$, then $\triangle ABC \sim \triangle AB'C'$.*

In other words, the Axiom V provides a condition which guarantees that two triangles are similar. Let us formulate three more such *similarity conditions.*

6.2. Similarity conditions. *Two triangles $\triangle ABC$ and $\triangle A'B'C'$ are similar if one of the following conditions holds:*

(SAS) For some constant $k > 0$ we have

$$AB = k{\cdot}A'B', \quad AC = k{\cdot}A'C'$$

$$\text{and} \quad \measuredangle BAC = \pm\measuredangle B'A'C'.$$

(AA) The triangle $A'B'C'$ is nondegenerate and

$$\measuredangle ABC = \pm\measuredangle A'B'C', \quad \measuredangle BAC = \pm\measuredangle B'A'C'.$$

(SSS) For some constant $k > 0$ we have

$$AB = k{\cdot}A'B', \quad AC = k{\cdot}A'C', \quad CB = k{\cdot}C'B'.$$

Each of these conditions is proved by applying Axiom V with the SAS, ASA, and SSS congruence conditions respectively (see Axiom IV and the conditions 4.1, 4.4).

Proof. Set $k = \frac{AB}{A'B'}$. Choose points $B'' \in [A'B')$ and $C'' \in [A'C')$, so that $A'B'' = k{\cdot}A'B'$ and $A'C'' = k{\cdot}A'C'$. By Axiom V, $\triangle A'B'C' \sim \sim \triangle A'B''C''$.

Applying the SAS, ASA or SSS congruence condition, depending on the case, we get that $\triangle A'B''C'' \cong \triangle ABC$. Hence the result. □

A bijection $X \leftrightarrow X'$ from a plane to itself is called *angle preserving transformation* if

$$\measuredangle ABC = \measuredangle A'B'C'$$

for any triangle ABC and its image $\triangle A'B'C'$.

6.3. Exercise. *Show that any angle-preserving transformation of the plane multiplies all the distance by a fixed constant.*

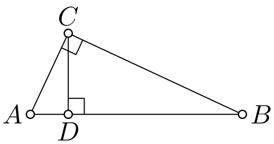
Pythagorean theorem

A triangle is called *right* if one of its angles is right. The side opposite the right angle is called the *hypotenuse*. The sides adjacent to the right angle are called *legs*.

6.4. Theorem. *Assume $\triangle ABC$ is a right triangle with the right angle at C. Then*

$$AC^2 + BC^2 = AB^2.$$

Proof. Let D be the foot point of C on (AB).



According to Lemma 5.10,

$$AD < AC < AB$$

and

$$BD < BC < AB.$$

Therefore, D lies between A and B; in particular,

❸ $$AD + BD = AB.$$

Note that by the AA similarity condition, we have

$$\triangle ADC \sim \triangle ACB \sim \triangle CDB.$$

In particular,

❹ $$\frac{AD}{AC} = \frac{AC}{AB} \quad \text{and} \quad \frac{BD}{BC} = \frac{BC}{BA}.$$

Let us rewrite the two identities in ❹:

$$AC^2 = AB{\cdot}AD \quad \text{and} \quad BC^2 = AB{\cdot}BD.$$

Summing up these two identities and applying ❸, we get that

$$AC^2 + BC^2 = AB{\cdot}(AD + BD) = AB^2.$$ □

The idea in the proof above appears in the Elements [8, X.33], but the proof given there [8, I.47] is different; it uses area method discussed in Chapter 20.

6.5. Exercise. *Assume A, B, C, and D are as in the proof above. Show that*

$$CD^2 = AD{\cdot}BD.$$

The following exercise is the converse to the Pythagorean theorem.

6.6. Exercise. *Assume that ABC is a triangle such that*

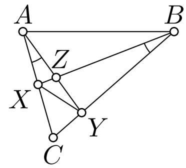
$$AC^2 + BC^2 = AB^2.$$

Prove that the angle at C is right.

Method of similar triangles

The proof of the Pythagorean theorem given above uses the *method of similar triangles.* To apply this method, one has to search for pairs of similar triangles and then use the proportionality of corresponding sides and/or equalities of corresponding angles. Finding such pairs might be tricky at first.

6.7. Exercise. *Let ABC be a nondegenerate triangle and the points X, Y, and Z as on the diagram. Assume $\measuredangle CAY \equiv \measuredangle XBC$. Find four pairs of similar triangles with these six points as the vertices and prove their similarity.*



Ptolemy's inequality

A *quadrangle* is defined as an ordered quadruple of distinct points in the plane. These 4 points are called *vertices.* The quadrangle $ABCD$ will be also denoted by $\square ABCD$.

Given a quadrangle $ABCD$, the four segments $[AB]$, $[BC]$, $[CD]$, and $[DA]$ are called *sides of* $\square ABCD$; the remaining two segments $[AC]$ and $[BD]$ are called *diagonals of* $\square ABCD$.

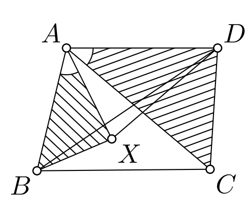
6.8. Ptolemy's inequality. *In any quadrangle, the product of diagonals cannot exceed the sum of the products of its opposite sides; that is,*

$$AC\cdot BD \leqslant AB\cdot CD + BC\cdot DA$$

for any $\square ABCD$.

We will present a classical proof of this inequality using the method of similar triangles with an additional construction. This proof is given as an illustration — it will not be used further in the sequel.

Proof. Consider the half-line $[AX)$ such that $\measuredangle BAX = \measuredangle CAD$. In this case, $\measuredangle XAD = \measuredangle BAC$ since adding $\measuredangle BAX$ or $\measuredangle CAD$ to the corresponding sides produces $\measuredangle BAD$. We can assume that



$$AX = \frac{AB}{AC}\cdot AD.$$

In this case, we have

$$\frac{AX}{AD} = \frac{AB}{AC}, \qquad \frac{AX}{AB} = \frac{AD}{AC}.$$

Hence

$$\triangle BAX \sim \triangle CAD, \qquad \triangle XAD \sim \triangle BAC.$$

Therefore

$$\frac{BX}{CD} = \frac{AB}{AC}, \qquad \frac{XD}{BC} = \frac{AD}{AC},$$

or, equivalently

$$AC \cdot BX = AB \cdot CD, \qquad AC \cdot XD = BC \cdot AD.$$

Adding these two equalities we get

$$AC \cdot (BX + XD) = AB \cdot CD + BC \cdot AD.$$

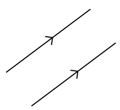
It remains to apply the triangle inequality, $BD \leqslant BX + XD$. □

Using the proof above together with Corollary 9.13, one can show that the equality holds only if the vertices A, B, C, and D appear on a line or a circle in the same cyclic order; see also 10.12 for another proof of the equality case. Exercise 18.2 below suggests another proof of Ptolemy's inequality using complex coordinates.

Chapter 7

Parallel lines

Parallel lines



In consequence of Axiom II, any two distinct lines ℓ and m have either one point in common or none. In the first case they are *intersecting* (briefly $\ell \nparallel m$); in the second case, ℓ and m are said to be *parallel* (briefly, $\ell \parallel m$); in addition, a line is always regarded as parallel to itself.

To emphasize that two lines on a diagram are parallel we will mark them with arrows of the same type.

7.1. Proposition.✓ *Let ℓ, m, and n be three lines. Assume that $n \perp m$ and $m \perp \ell$. Then $\ell \parallel n$.*

Proof. Assume the contrary; that is, $\ell \nparallel n$. Then there is a point, say Z, of intersection of ℓ and n. Then by Theorem 5.5, $\ell = n$. Since any line is parallel to itself, we have that $\ell \parallel n$ — a contradiction. □

7.2. Theorem. *For any point P and any line ℓ, there is a unique line m that passes thru P and is parallel to ℓ.*

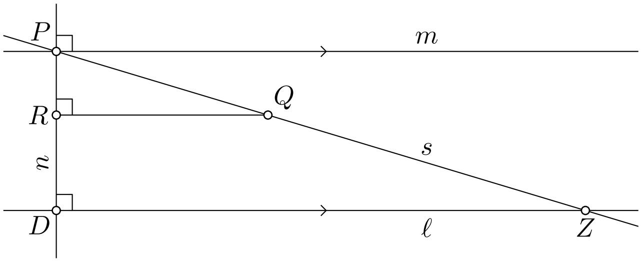
The above theorem has two parts, existence and uniqueness. In the proof of uniqueness, we will use the method of similar triangles.

Proof; existence. Apply Theorem 5.5 two times, first to construct the line n thru P that is perpendicular to ℓ, and second to construct the line n thru P that is perpendicular to m. Then apply Proposition 7.1.

Uniqueness. If $P \in \ell$, then $m = \ell$ by the definition of parallel lines. Further we assume $P \notin \ell$.

Let us construct the lines $n \ni P$ and $m \ni P$ as in the proof of existence, so $m \parallel \ell$.

Assume there is yet another line $s \ni P$ parallel to ℓ. Choose a point $Q \in s$ that lies with ℓ on the same side from m. Let R be the foot point of Q on n.



Let D be the point of intersection of n and ℓ. According to Proposition 7.1 $(QR) \parallel m$. Therefore, Q, R, and ℓ lie on the same side of m. In particular, $R \in [PD)$.

Choose $Z \in [PQ)$ such that

$$\frac{PZ}{PQ} = \frac{PD}{PR}.$$

By SAS similarity condition (or equivalently by Axiom V) we have that $\triangle RPQ \sim \triangle DPZ$; therefore $(ZD) \perp (PD)$. It follows that Z lies on ℓ and s — a contradiction. □

7.3. Corollary. *Assume ℓ, m, and n are lines such that $\ell \parallel m$ and $m \parallel n$. Then $\ell \parallel n$.*

Proof. Assume the contrary; that is, $\ell \nparallel n$. Then there is a point $P \in \ell \cap n$. By Theorem 7.2, $n = \ell$ — a contradiction. □

Note that from the definition, we have that $\ell \parallel m$ if and only if $m \parallel \ell$. Therefore, according to the above corollary, "$\parallel$" is an *equivalence relation.* That is, for any lines ℓ, m, and n the following conditions hold:

(i) $\ell \parallel \ell$;
(ii) if $\ell \parallel m$, then $m \parallel \ell$;
(iii) if $\ell \parallel m$ and $m \parallel n$, then $\ell \parallel n$.

7.4. Exercise. *Let k, ℓ, m, and n be lines such that $k \perp \ell$, $\ell \perp m$, and $m \perp n$. Show that $k \nparallel n$.*

7.5. Exercise. *Make a ruler-and-compass construction of a line thru a given point that is parallel to a given line.*

Reflection across a point

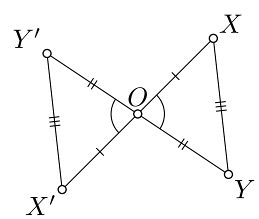
Fix a point O. If O is the midpoint of a line segment $[XX']$, then we say that X' is a reflection of X across the point O.

Note that the map $X \mapsto X'$ is uniquely defined; it is called a *reflection across* O. In this case, O is called the *center of reflection*. We assume that $O' = O$; that is, O is a reflection of itself across itself. If the reflection across O moves a set S to itself, then we say that S is *centrally symmetric* with respect to O.

Recall that any motion is either direct or indirect; that is, it either preserves or reverts the signs of angles (see page 37).

7.6. Proposition.✓ *Any reflection across a point is a direct motion.*

Proof. Observe that if X' is a reflection of X across O, then X is a reflection of X'. In other words, the composition of the reflection with itself is the identity map. In particular, any reflection across a point is a bijection.



Fix two points X and Y; let X' and Y' be their reflections across O. To check that the reflection is distance preserving, we need to show that $X'Y' = XY$.

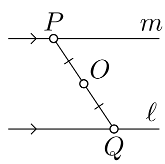
We may assume that X, Y, and O are distinct; otherwise, the statement is trivial. By definition of reflection across O, we have that $OX = OX'$, $OY = OY'$, and the angles XOY and $X'OY'$ are vertical; in particular, $\measuredangle XOY = \measuredangle X'OY'$. By SAS, $\triangle XOY \cong \triangle X'OY'$; therefore $X'Y' = XY$.

Finally, the reflection across O cannot be indirect since $\measuredangle XOY = \measuredangle X'OY'$; therefore it is a direct motion. ☐

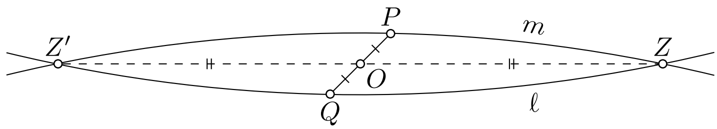
7.7. Exercise. *Suppose $\angle AOB$ is right. Show that the composition of reflections across the lines (OA) and (OB) is a reflection across O.*

Use this statement and Corollary 5.7 to build another proof of Proposition 7.6.

7.8. Theorem. *Let ℓ be a line, $Q \in \ell$, and P is an arbitrary point. Suppose O is the midpoint of $[PQ]$. Then a line m passing thru P is parallel to ℓ if and only if m is a reflection of ℓ across O.*



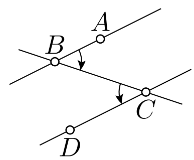
Proof; "if" part. Assume m is a reflection of ℓ across O. Suppose $\ell \nparallel m$; that is ℓ and m intersect at a single point Z. Denote by Z' be the reflection of Z across O.



Note that Z' lies on both lines ℓ and m. It follows that $Z' = Z$ or equivalently $Z = O$. In this case, $O \in \ell$ and therefore the reflection of ℓ across O is ℓ itself; that is, $\ell = m$ and in particular $\ell \parallel m$ — a contradiction.

"Only-if" part. Let ℓ' be the reflection of ℓ across O. According to the "if" part of the theorem, $\ell' \parallel \ell$. Note that both lines ℓ' and m pass thru P. By uniqueness of parallel lines (7.2), if $m \parallel \ell$, then $\ell' = m$; whence the statement follows. □

Transversal property



If the line t intersects each line ℓ and m at one point, then we say that t is a *transversal* to ℓ and m. For example, on the diagram, line (CB) is a transversal to (AB) and (CD).

7.9. Transversal property. *$(AB) \parallel (CD)$ if and only if*

❶ $$2\cdot(\measuredangle ABC + \measuredangle BCD) \equiv 0.$$

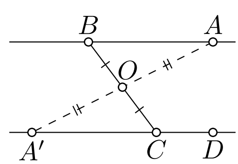
Equivalently

$$\measuredangle ABC + \measuredangle BCD \equiv 0 \quad or \quad \measuredangle ABC + \measuredangle BCD \equiv \pi.$$

Moreover, if $(AB) \ne (CD)$, then in the first case, A and D lie on opposite sides of (BC); in the second case, A and D lie on the same sides of (BC).

Proof; "only-if" part. Denote by O the midpoint of $[BC]$.

Assume $(AB) \parallel (CD)$. According to Theorem 7.8, (CD) is a reflection of (AB) across O.



Let A' be the reflection of A across O. Then $A' \in (CD)$ and by Proposition 7.6 we have that

❷ $$\measuredangle ABO = \measuredangle A'CO.$$

Note that

❸ $$\measuredangle ABO \equiv \measuredangle ABC, \qquad \measuredangle A'CO \equiv -\measuredangle BCA'.$$

Since A', C and D lie on one line, Exercise 2.11 implies that

❹ $$2{\cdot}\measuredangle BCD \equiv 2{\cdot}\measuredangle BCA'.$$

Finally note that ❷, ❸, and ❹ imply ❶. □

"If"-part. By Theorem 7.2 there is a unique line (CD) thru C that is parallel to (AB). From the "only-if" part we know that ❶ holds.

On the other hand, there is a *unique* line (CD) such that ❶ holds. Indeed, suppose there are two such lines (CD) and (CD'), then

$$2{\cdot}(\measuredangle ABC + \measuredangle BCD) \equiv 2{\cdot}(\measuredangle ABC + \measuredangle BCD') \equiv 0.$$

Therefore $2{\cdot}\measuredangle BCD \equiv 2{\cdot}\measuredangle BCD'$ and by Exercise 2.11, $D' \in (CD)$, or equivalently the line (CD) coincides with (CD').

Therefore if ❶ holds, then $(CD) \parallel (AB)$.

Finally, if $(AB) \neq (CD)$ and A and D lie on the opposite sides of (BC), then $\angle ABC$ and $\angle BCD$ have opposite signs. Therefore

$$-\pi < \measuredangle ABC + \measuredangle BCD < \pi.$$

Applying ❶, we get $\measuredangle ABC + \measuredangle BCD = 0$.

Similarly, if A and D lie on the same side of (BC), then $\angle ABC$ and $\angle BCD$ have the same sign. Therefore

$$0 < |\measuredangle ABC + \measuredangle BCD| < 2{\cdot}\pi$$

and ❶ implies that $\measuredangle ABC + \measuredangle BCD \equiv \pi$. □

7.10. Exercise. *Let $\triangle ABC$ be a nondegenerate triangle, and P lies between A and B. Suppose that a line ℓ passes thru P and is parallel to (AC). Show that ℓ crosses the side $[BC]$ at another point, say Q, and*

$$\triangle ABC \sim \triangle PBQ.$$

In particular,

$$\frac{PB}{AB} = \frac{QB}{CB}.$$

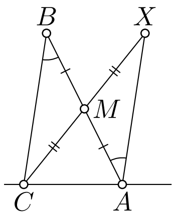
7.11. Exercise. *Trisect a given segment with a ruler and a compass.*

Angles of triangles

7.12. Theorem. *In any $\triangle ABC$, we have*

$$\measuredangle ABC + \measuredangle BCA + \measuredangle CAB \equiv \pi.$$

Proof. First note that if $\triangle ABC$ is degenerate, then the equality follows from Corollary 2.9. Further we assume that $\triangle ABC$ is nondegenerate.



Let X be the reflection of C across the midpoint M of $[AB]$. By Proposition 7.6 $\measuredangle BAX = \measuredangle ABC$. Note that (AX) is a reflection of (CB) across M; therefore by Theorem 7.8, $(AX) \parallel (CB)$.

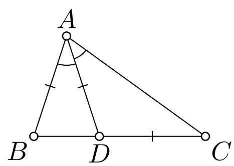
Since $[BM]$ and $[MX]$ do not intersect (CA), the points B, M, and X lie on the same side of (CA). Applying the transversal property for the transversal (CA) to (AX) and (CB), we get that

❺ $$\measuredangle BCA + \measuredangle CAX \equiv \pi.$$

Since $\measuredangle BAX = \measuredangle ABC$, we have

$$\measuredangle CAX \equiv \measuredangle CAB + \measuredangle ABC$$

The latter identity and ❺ imply the theorem. □



7.13. Exercise. *Let $\triangle ABC$ be a nondegenerate triangle. Assume there is a point $D \in [BC]$ such that*

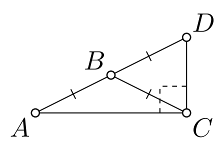
$$\measuredangle BAD \equiv \measuredangle DAC, \quad BA = AD = DC.$$

Find the angles of $\triangle ABC$.

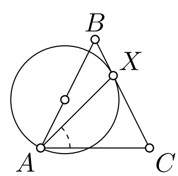
7.14. Exercise. *Show that*

$$|\measuredangle ABC| + |\measuredangle BCA| + |\measuredangle CAB| = \pi$$

for any $\triangle ABC$.



7.15. Exercise. *Let $\triangle ABC$ be an isosceles nondegenerate triangle with the base $[AC]$. Suppose D is a reflection of A across B. Show that $\angle ACD$ is right.*

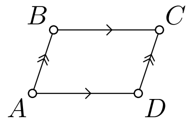


7.16. Exercise. *Let $\triangle ABC$ be an isosceles nondegenerate triangle with base $[AC]$. Assume that a circle is passing thru A, centered at a point on $[AB]$, and tangent to (BC) at the point X. Show that $\measuredangle CAX = \pm\frac{\pi}{4}$.*

7.17. Exercise. *Show that for any quadrangle $ABCD$, we have*

$$\measuredangle ABC + \measuredangle BCD + \measuredangle CDA + \measuredangle DAB \equiv 0.$$

Parallelograms



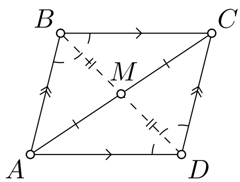
A quadrangle $ABCD$ in the Euclidean plane is called *nondegenerate* if no three points from A, B, C, D lie on one line.

A nondegenerate quadrangle is called a *parallelogram* if its opposite sides are parallel.

7.18. Lemma. *Any parallelogram is centrally symmetric with respect to a midpoint of one of its diagonals.*

In particular, if $\square ABCD$ is a parallelogram, then

(a) its diagonals $[AC]$ and $[BD]$ intersect each other at their midpoints;
(b) $\measuredangle ABC = \measuredangle CDA$;
(c) $AB = CD$.



Proof. Let $\square ABCD$ be a parallelogram. Denote by M the midpoint of $[AC]$.

Since $(AB) \parallel (CD)$, Theorem 7.8 implies that (CD) is a reflection of (AB) across M. In the same way, (BC) is a reflection of (DA) across M. Since $\square ABCD$ is nondegenerate, it follows that D is a reflection of B across M; in other words, M is the midpoint of $[BD]$.

The remaining statements follow since reflection across M is a direct motion of the plane (see 7.6). ☐

7.19. Exercise. *Assume $ABCD$ is a quadrangle such that*

$$AB = CD = BC = DA.$$

Show that $ABCD$ is a parallelogram.

A quadrangle as in the exercise above is called a *rhombus*.

A quadrangle $ABCD$ is called a *rectangle* if the angles ABC, BCD, CDA, and DAB are right. Note that according to the transversal property (7.9), any rectangle is a parallelogram.

A rectangle with equal sides is called a *square*.

7.20. Exercise. *Show that the parallelogram $ABCD$ is a rectangle if and only if $AC = BD$.*

7.21. Exercise. *Show that the parallelogram $ABCD$ is a rhombus if and only if $(AC) \perp (BD)$.*

Assume $\ell \parallel m$, and $X, Y \in m$. Let X' and Y' denote the foot points of X and Y on ℓ. Note that $\square XYY'X'$ is a rectangle. By Lemma 7.18, $XX' = YY'$. That is, any point on m lies at the same distance from ℓ. This distance is called the *distance between* ℓ and m.

Method of coordinates

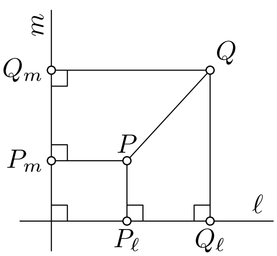
The following exercise is important; it shows that our axiomatic definition agrees with the model definition described on page 11.

7.22. Exercise. *Let ℓ and m be perpendicular lines in the Euclidean plane. Given a point P, let P_ℓ and P_m denote the foot points of P on ℓ and m respectively.*

(a) Show that for any $X \in \ell$ and $Y \in m$ there is a unique point P such that $P_\ell = X$ and $P_m = Y$.

(b) Show that $PQ^2 = P_\ell Q_\ell^2 + P_m Q_m^2$ for any pair of points P and Q.

(c) Conclude that the plane is isometric to $(\mathbb{R}^2, d_2)$; see page 11.



Once this exercise is solved, we can apply the method of coordinates to solve any problem in Euclidean plane geometry. This method is powerful and universal; it will be developed further in Chapter 18.

7.23. Exercise. *Use the Exercise 7.22 to give an alternative proof of Theorem 3.17 in the Euclidean plane.*

That is, prove that given the real numbers a, b, and c such that

$$0 < a \leqslant b \leqslant c \leqslant a + b,$$

there is a triangle ABC such that $a = BC$, $b = CA$, and $c = AB$.

7.24. Exercise. *Consider two distinct points $A = (x_A, y_A)$ and $B = (x_B, y_B)$ on the coordinate plane. Show that the perpendicular bisector to $[AB]$ is described by the equation*

$$2\cdot(x_B - x_A)\cdot x + 2\cdot(y_B - y_A)\cdot y = x_B^2 + y_B^2 - x_A^2 - y_B^2.$$

Conclude that line can be defined as a subset of the coordinate plane of the following type:

(a) Solutions of an equation $a\cdot x + b\cdot y = c$ for some constants a, b, and c such that $a \neq 0$ or $b \neq 0$.

(b) The set of points $(a\cdot t + c, b\cdot t + d)$ for some constants a, b, c, and d such that $a \neq 0$ or $b \neq 0$ and all $t \in \mathbb{R}$.

Apollonian circle

The exercises in this section are given as illustrations to the method of coordinates — it will not be used further in the sequel.

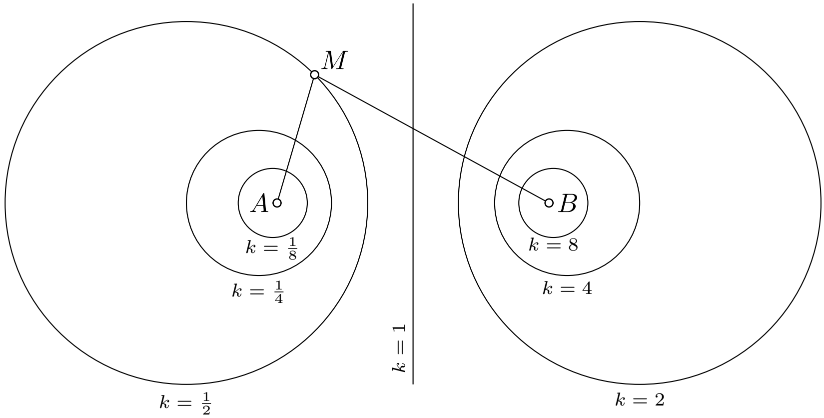
7.25. Exercise. *Show that for fixed real values a, b, and c the equation*

$$x^2 + y^2 + a{\cdot}x + b{\cdot}y + c = 0$$

describes a circle, one-point set or empty set.

Show that if it is a circle then it has center $(-\frac{a}{2}, -\frac{b}{2})$ and the radius $r = \frac{1}{2}{\cdot}\sqrt{a^2 + b^2 - 4{\cdot}c}$.

7.26. Exercise. *Use the previous exercise to show that given a positive real number $k \neq 1$, the locus of points M such that $AM = k{\cdot}BM$ for distinct points A and B is a circle.*



The circle in the exercise above is an example of the so-called *Apollonian circle with focuses A and B.* Few of these circles for different values k are shown on the diagram; for $k = 1$, it becomes the perpendicular bisector to $[AB]$.

7.27. Exercise. *Make a ruler-and-compass construction of an Apollonian circle with given focuses A and B thru a given point M.*

Chapter 8

Triangle geometry

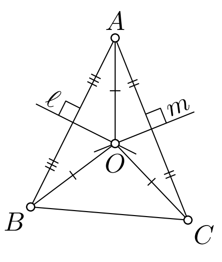
Triangle geometry is the study of the properties of triangles, including associated centers and circles.

We discuss the most basic results in triangle geometry, mostly to show that we have developed sufficient machinery to prove things.

Circumcircle and circumcenter

8.1. Theorem. *Perpendicular bisectors to the sides of any nondegenerate triangle intersect at one point.*

The point of intersection of the perpendicular bisectors is called *circumcenter*. It is the center of the *circumcircle* of the triangle; that is, a circle that passes thru all three vertices of the triangle. The circumcenter of the triangle is usually denoted by O.



Proof. Let $\triangle ABC$ be nondegenerate. Let ℓ and m be perpendicular bisectors to sides $[AB]$ and $[AC]$ respectively.

Assume ℓ and m intersect, let $O = \ell \cap m$.

Let us apply Theorem 5.2. Since $O \in \ell$, we have that $OA = OB$ and since $O \in m$, we have that $OA = OC$. It follows that $OB = OC$; that is, O lies on the perpendicular bisector to $[BC]$.

It remains to show that $\ell \nparallel m$; assume the contrary. Since $\ell \perp (AB)$ and $m \perp (AC)$, we get that $(AC) \parallel (AB)$ (see Exercise 7.4). Therefore, by Theorem 5.5, $(AC) = (AB)$; that is, $\triangle ABC$ is degenerate — a contradiction. □

8.2. Exercise. *There is a unique circle that passes thru the vertices of a given nondegenerate triangle in the Euclidean plane.*

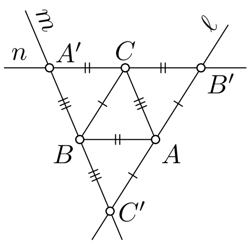
Altitudes and orthocenter

An *altitude* of a triangle is a line thru a vertex and perpendicular to the line containing the opposite side. The term *altitude* may also be used for the distance from the vertex to its foot point on the line containing the opposite side.

8.3. Theorem. *The three altitudes of any nondegenerate triangle intersect in a single point.*

The point of intersection of altitudes is called *orthocenter*; it is usually denoted by H.

Proof. Fix a nondegenerate triangle ABC. Consider three lines ℓ, m, and n such that



$$\begin{aligned} &\ell \parallel (BC), && m \parallel (CA), && n \parallel (AB), \\ &\ell \ni A, && m \ni B, && n \ni C. \end{aligned}$$

Since $\triangle ABC$ is nondegenerate, no pair of the lines ℓ, m, and n is parallel. Set

$$A' = m \cap n, \quad B' = n \cap \ell, \quad C' = \ell \cap m.$$

Note that $\square ABA'C$, $\square BCB'A$, and $\square CBC'A$ are parallelograms. Applying Lemma 7.18 we get that $\triangle ABC$ is the median triangle of $\triangle A'B'C'$; that is, A, B, and C are the midpoints of $[B'C']$, $[C'A']$, and $[A'B']$ respectively.

By Exercise 7.4, $(B'C') \parallel (BC)$, the altitude from A is perpendicular to $[B'C']$ and from above it bisects $[B'C']$.

Hence the altitudes of $\triangle ABC$ are also perpendicular bisectors of $\triangle A'B'C'$. Applying Theorem 8.1, we get that altitudes of $\triangle ABC$ intersect at one point. $\square$

8.4. Exercise. *Assume H is the orthocenter of an acute triangle ABC. Show that A is the orthocenter of $\triangle HBC$.*

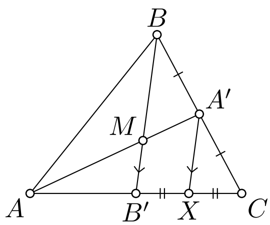
Medians and centroid

A median of a triangle is the segment joining a vertex to the midpoint of the opposing side.

8.5. Theorem. *The three medians of any nondegenerate triangle intersect in a single point. Moreover, the point of intersection divides each median in the ratio 2:1.*

The point of intersection of medians is called the *centroid* of the triangle; it is usually denoted by M. In the proof, we will apply exercises 3.14 and 7.10; their complete solutions are given in the hits.

Proof. Consider a nondegenerate triangle ABC. Let $[AA']$ and $[BB']$ be its medians. According to Exercise 3.14, $[AA']$ and $[BB']$ have a point of intersection; denote it by M.



Draw a line ℓ thru A' parallel to (BB'). Applying Exercise 7.10 for $\triangle BB'C$ and ℓ, we get that ℓ cross $[B'C]$ at some point X and

$$\frac{CX}{CB'} = \frac{CA'}{CB} = \frac{1}{2};$$

that is, X is the midpoint of $[CB']$.

Since B' is the midpoint of $[AC]$ and X is the midpoint of $[B'C]$, we get that

$$\frac{AB'}{AX} = \frac{2}{3}.$$

Applying Exercise 7.10 for $\triangle XA'A$ and the line (BB'), we get that

❶ $$\frac{AM}{AA'} = \frac{AB'}{AX} = \frac{2}{3};$$

that is, M divides $[AA']$ in the ratio 2:1.

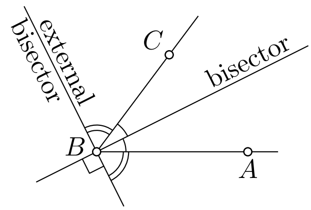
Note that ❶ uniquely defines M on $[AA']$. Repeating the same argument for medians $[AA']$ and $[CC']$, we get that they intersect at M as well, hence the result. □

8.6. Exercise. *Let $\square ABCD$ be a nondegenerate quadrangle and X, Y, V, W be the midpoints of its sides $[AB]$, $[BC]$, $[CD]$, and $[DA]$. Show that $\square XYVW$ is a parallelogram.*

Angle bisectors

If $\measuredangle ABX \equiv -\measuredangle CBX$, then we say that the line (BX) *bisects* $\angle ABC$, or the line (BX) is a *bisector* of $\angle ABC$. If $\measuredangle ABX \equiv \pi - \measuredangle CBX$, then the line (BX) is called the *external bisector* of $\angle ABC$.

If $\measuredangle ABA' = \pi$; that is, if B lies between A and A', then the bisector of $\angle ABC$ is the external bisector of $\angle A'BC$ and the other way around.



Note that the bisector and the external bisector are uniquely defined by the angle.

8.7. Exercise. *Show that for any angle, its bisector and external bisector are perpendicular.*

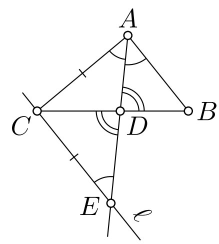
The bisectors of $\angle ABC$, $\angle BCA$, and $\angle CAB$ of a nondegenerate triangle ABC are called *bisectors of the triangle ABC* at vertices A, B, and C respectively.

8.8. Lemma. *Let $\triangle ABC$ be a nondegenerate triangle. Assume that the bisector at the vertex A intersects the side $[BC]$ at the point D. Then*

❷ $$\frac{AB}{AC} = \frac{DB}{DC}.$$

Proof. Let ℓ be a line passing thru C that is parallel to (AB). Note that $\ell \nparallel (AD)$; set

$$E = \ell \cap (AD).$$



Note also that B and C lie on opposite sides of (AD). Therefore, by the transversal property (7.9),

❸ $$\measuredangle BAD = \measuredangle CED.$$

Further, the angles ADB and EDC are vertical; in particular, by 2.13

$$\measuredangle ADB = \measuredangle EDC.$$

By the AA similarity condition, $\triangle ABD \sim \triangle ECD$. In particular,

❹ $$\frac{AB}{EC} = \frac{DB}{DC}.$$

Since (AD) bisects $\angle BAC$, we get that $\measuredangle BAD = \measuredangle DAC$. Together with ❸, it implies that $\measuredangle CEA = \measuredangle EAC$. By Theorem 4.2, $\triangle ACE$ is isosceles; that is,

$$EC = AC.$$

Together with ❹, it implies ❷. □

8.9. Exercise. *Formulate and prove an analog of Lemma 8.8 for the external bisector.*

Equidistant property

Recall that distance from a line ℓ to a point P is defined as the distance from P to its foot point on ℓ; see page 38.

8.10. Proposition.✓ *Assume $\triangle ABC$ is not degenerate. Then a point X lies on the bisector or external bisector of $\angle ABC$ if and only if X is equidistant from the lines (AB) and (BC).*

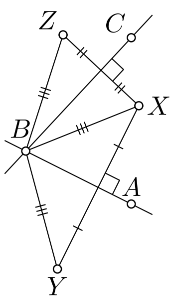
Proof. We can assume that X does not lie on the union of (AB) and (BC). Otherwise, the distance to one of the lines vanish; in this case, $X = B$ is the only point equidistant from the two lines.

Let Y and Z be the reflections of X across (AB) and (BC) respectively. Note that

$$Y \neq Z.$$

Otherwise, both lines (AB) and (BC) are perpendicular bisectors of $[XY]$, that is, $(AB) = (BC)$ which is impossible since $\triangle ABC$ is not degenerate.

By Proposition 5.6,



$$XB = YB = ZB.$$

Note that X is equidistant from (AB) and (BC) if and only if $XY = XZ$. Applying SSS and then SAS, we get that

$$XY = XZ.$$
$$\Updownarrow$$
$$\triangle BXY \cong \triangle BXZ.$$
$$\Updownarrow$$
$$\measuredangle XBY \equiv \pm\measuredangle BXZ.$$

Since $Y \neq Z$, we get that $\measuredangle XBY \neq \measuredangle BXZ$; therefore

❺ $$\measuredangle XBY = -\measuredangle BXZ.$$

By Proposition 5.6, A lies on the bisector of $\angle XBY$ and B lies on the bisector of $\angle XBZ$; that is,

$$2{\cdot}\measuredangle XBA \equiv \measuredangle XBY, \qquad 2{\cdot}\measuredangle XBC \equiv \measuredangle XBZ.$$

By ❺,

$$2{\cdot}\measuredangle XBA \equiv -2{\cdot}\measuredangle XBC.$$

The last identity means either

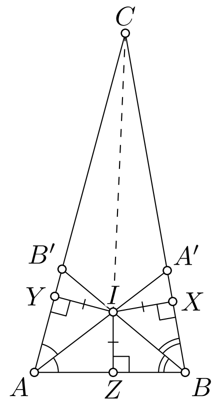
$$\measuredangle XBA + \measuredangle XBC \equiv 0 \quad \text{or} \quad \measuredangle XBA + \measuredangle XBC \equiv \pi,$$

and hence the result. $\square$

Incenter

8.11. Theorem.✓ *The angle bisectors of any nondegenerate triangle intersect at one point.*

The point of intersection of bisectors is called the *incenter* of the triangle; it is usually denoted by I. The point I lie at the same distance from each side. In particular, it is the center of a circle tangent to each side of the triangle. This circle is called the *incircle* and its radius is called the *inradius* of the triangle.



Proof. Let $\triangle ABC$ be a nondegenerate triangle.

Note that the points B and C lie on opposite sides of the bisector of $\angle BAC$. Hence this bisector intersects $[BC]$ at a point, say A'.

Analogously, there is $B' \in [AC]$ such that (BB') bisects $\angle ABC$.

Applying Pasch's theorem (3.12) twice for the triangles $AA'C$ and $BB'C$, we get that $[AA']$ and $[BB']$ intersect. Suppose that I denotes the point of intersection.

Let X, Y, and Z be the foot points of I on (BC), (CA), and (AB) respectively. Applying Proposition 8.10, we get that

$$IY = IZ = IX.$$

From the same lemma, we get that I lies on the bisector or on the exterior bisector of $\angle BCA$.

The line (CI) intersects $[BB']$, the points B and B' lie on opposite sides of (CI). Therefore, the angles ICB' and ICB have opposite signs. Note that $\angle ICA = \angle ICB'$. Therefore, (CI) cannot be the exterior bisector of $\angle BCA$. Hence the result. □

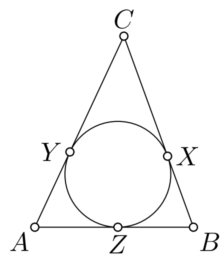
More exercises

8.12. Exercise. *Assume that an angle bisector of a nondegenerate triangle bisects the opposite side. Show that the triangle is isosceles.*

8.13. Exercise. *Assume that at one vertex of a nondegenerate triangle the bisector coincides with the altitude. Show that the triangle is isosceles.*

8.14. Exercise. *Assume sides $[BC]$, $[CA]$, and $[AB]$ of $\triangle ABC$ are tangent to the incircle at X, Y, and Z respectively. Show that*

$$AY = AZ = \tfrac{1}{2}\cdot(AB + AC - BC).$$

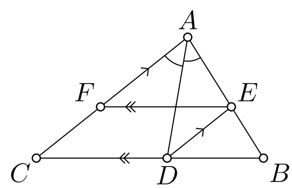


By the definition, the vertices of *orthic triangle* are the base points of the altitudes of the given triangle.

8.15. Exercise. *Prove that the orthocenter of an acute triangle coincides with the incenter of its orthic triangle.*

What should be an analog of this statement for an obtuse triangle?

8.16. Exercise. *Assume that the bisector at A of the triangle ABC intersects the side $[BC]$ at the point D; the line thru D and parallel to (CA) intersects (AB) at the point E; the line thru E and parallel to (BC) intersects (AC) at F. Show that $AE = FC$.*



Chapter 9

Inscribed angles

Angle between a tangent line and a chord

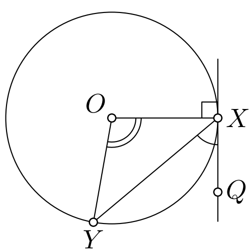
9.1. Theorem. *Let Γ be a circle with the center O. Assume the line (XQ) is tangent to Γ at X and $[XY]$ is a chord of Γ. Then*

➊ $$2\cdot\measuredangle QXY \equiv \measuredangle XOY.$$

Equivalently,

$$\measuredangle QXY \equiv \tfrac{1}{2}\cdot\measuredangle XOY \quad or \quad \measuredangle QXY \equiv \tfrac{1}{2}\cdot\measuredangle XOY + \pi.$$

Proof. Note that $\triangle XOY$ is isosceles. Therefore, $\measuredangle YXO = \measuredangle OYX$.



Applying Theorem 7.12 to $\triangle XOY$, we get

$$\begin{aligned}\pi &\equiv \measuredangle YXO + \measuredangle OYX + \measuredangle XOY \equiv \\ &\equiv 2\cdot\measuredangle YXO + \measuredangle XOY.\end{aligned}$$

By Lemma 5.17, $(OX) \perp (XQ)$. Therefore,

$$\measuredangle QXY + \measuredangle YXO \equiv \pm\tfrac{\pi}{2}.$$

Therefore,

$$2\cdot\measuredangle QXY \equiv \pi - 2\cdot\measuredangle YXO \equiv \measuredangle XOY.$$

□

Inscribed angle

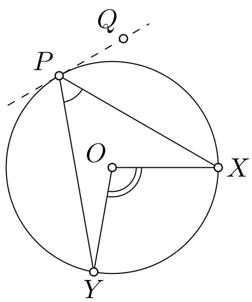
We say that a triangle is *inscribed* in the circle Γ if all its vertices lie on Γ.

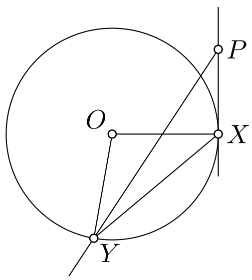
9.2. Theorem. *Let Γ be a circle with the center O, and X, Y be two distinct points on Γ. Then $\triangle XPY$ is inscribed in Γ if and only if*

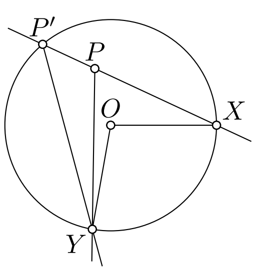
❷ $$2\cdot\measuredangle XPY \equiv \measuredangle XOY.$$

Equivalently, if and only if

$$\measuredangle XPY \equiv \tfrac{1}{2}\cdot\measuredangle XOY \quad \textit{or} \quad \measuredangle XPY \equiv \pi + \tfrac{1}{2}\cdot\measuredangle XOY.$$







Proof; the "only if" part. Let (PQ) be the tangent line to Γ at P. By Theorem 9.1,

$$2\cdot\measuredangle QPX \equiv \measuredangle POX, \quad 2\cdot\measuredangle QPY \equiv \measuredangle POY.$$

Subtracting one identity from the other, we get ❷.

"If" part. Assume that ❷ holds for some $P \notin \Gamma$. Note that $\measuredangle XOY \neq 0$. Therefore, $\measuredangle XPY \neq 0$ nor π; that is, $\triangle PXY$ is nondegenerate.

The line (PX) might be tangent to Γ at the point X or intersect Γ at another point; in the latter case, suppose that P' denotes this point of intersection.

In the first case, by Theorem 9.1, we have

$$2\cdot\measuredangle PXY \equiv \measuredangle XOY \equiv 2\cdot\measuredangle XPY.$$

Applying the transversal property (7.9), we get that $(XY) \parallel (PY)$, which is impossible since $\triangle PXY$ is nondegenerate.

In the second case, applying the "if" part and that P, X, and P' lie on one line (see Exercise 2.11) we get that

$$\begin{aligned} 2\cdot\measuredangle P'PY \equiv 2\cdot\measuredangle XPY \equiv \measuredangle XOY \equiv \\ \equiv 2\cdot\measuredangle XP'Y \equiv 2\cdot\measuredangle XP'P. \end{aligned}$$

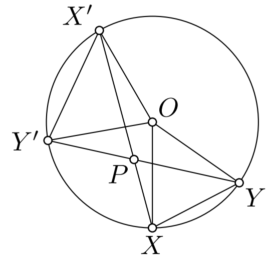
Again, by transversal property, $(PY) \parallel (P'Y)$, which is impossible since $\triangle PXY$ is nondegenerate. □

9.3. Exercise. *Let X, X', Y, and Y' be distinct points on the circle Γ. Assume (XX') meets (YY') at a point P. Show that*

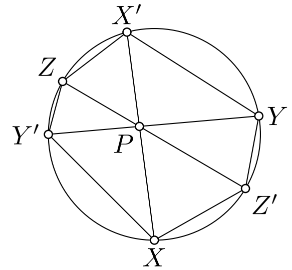
(a) $2{\cdot}\measuredangle XPY \equiv \measuredangle XOY + \measuredangle X'OY'$;

(b) $\triangle PXY \sim \triangle PY'X'$;

(c) $PX{\cdot}PX' = |OP^2 - r^2|$, *where O is the center and r is the radius of Γ.*

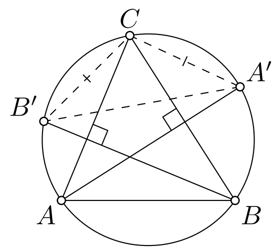


(The value $OP^2 - r^2$ is called the *power* of the point P with respect to the circle Γ. Part *(c)* of the exercise makes it a useful tool to study circles, but we are not going to consider it further in the book.)



9.4. Exercise. *Three chords $[XX']$, $[YY']$, and $[ZZ']$ of the circle Γ intersect at a point P. Show that*

$$XY'{\cdot}ZX'{\cdot}YZ' = X'Y{\cdot}Z'X{\cdot}Y'Z.$$



9.5. Exercise. *Let Γ be a circumcircle of an acute triangle ABC. Let A' and B' denote the second points of intersection of the altitudes from A and B with Γ. Show that $\triangle A'B'C$ is isosceles.*

9.6. Exercise. *Let $[XY]$ and $[X'Y']$ be two parallel chords of a circle. Show that $XX' = YY'$.*

9.7. Exercise. *Watch "Why is pi here? And why is it squared? A geometric answer to the Basel problem" by Grant Sanderson. (It is available on YouTube.)*

Prepare one question.

Points on a common circle

Recall that the diameter of a circle is a chord that passes thru the center. If $[XY]$ is the diameter of a circle with center O, then $\measuredangle XOY = \pi$. Hence Theorem 9.2 implies the following:

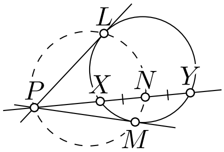
9.8. Corollary. *Suppose Γ is a circle with the diameter $[AB]$. A triangle ABC has a right angle at C if and only if $C \in \Gamma$.*

9.9. Exercise. *Given four points A, B, A', and B', construct a point Z such that both angles AZB and $A'ZB'$ are right.*

9.10. Exercise. *Let $\triangle ABC$ be a nondegenerate triangle, A' and B' be foot points of altitudes from A and B respectfully. Show that the four points A, B, A', and B' lie on one circle. What is the center of this circle?*

9.11. Exercise. *Assume a line ℓ, a circle with its center on ℓ and a point $P \notin \ell$ are given. Make a ruler-only construction of the perpendicular to ℓ from P.*

9.12. Exercise. *Suppose that lines ℓ, m, and n pass thru a point P; the lines ℓ and m are tangent to a circle Γ at L and M; the line n intersects Γ at two points X and Y. Let N be the midpoint of $[XY]$. Show that the points P, L, M, and N lie on one circle.*

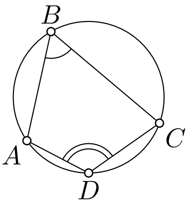


We say that a quadrangle $ABCD$ is *inscribed in circle* Γ if all the points A, B, C, and D lie on Γ.

9.13. Corollary. *A nondegenerate quadrangle $ABCD$ is inscribed in a circle if and only if*

$$2\cdot\measuredangle ABC \equiv 2\cdot\measuredangle ADC.$$

Proof. Since $\square ABCD$ is nondegenerate, so is $\triangle ABC$. Let O and Γ denote the circumcenter and circumcircle of $\triangle ABC$ (they exist by Exercise 8.2).



According to Theorem 9.2,

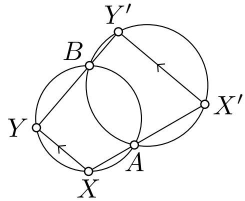
$$2\cdot\measuredangle ABC \equiv \measuredangle AOC.$$

From the same theorem, $D \in \Gamma$ if and only if

$$2\cdot\measuredangle ADC \equiv \measuredangle AOC,$$

hence the result. □

9.14. Exercise. *Let Γ and Γ' be two circles that intersect at two distinct points: A and B. Assume $[XY]$ and $[X'Y']$ are the chords of Γ and Γ' respectively, such that A lies between X and X' and B lies between Y and Y'. Show that $(XY) \parallel (X'Y')$.*



Method of additional circle

9.15. Problem. *Assume that two chords $[AA']$ and $[BB']$ intersect at the point P inside their circle. Let X be a point such that both angles XAA' and XBB' are right. Show that $(XP) \perp (A'B')$.*

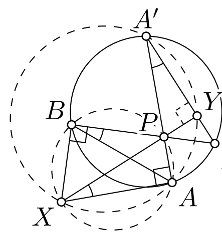
Solution. Set $Y = (A'B') \cap (XP)$.

Both angles XAA' and XBB' are right; therefore

$$2\cdot\measuredangle XAA' \equiv 2\cdot\measuredangle XBB'.$$

By Corollary 9.13, $\square XAPB$ is inscribed. Applying this theorem again we get that

$$2\cdot\measuredangle AXP \equiv 2\cdot\measuredangle ABP.$$



Since $\square ABA'B'$ is inscribed,

$$2\cdot\measuredangle ABB' \equiv 2\cdot\measuredangle AA'B'.$$

It follows that

$$2\cdot\measuredangle AXY \equiv 2\cdot\measuredangle AA'Y.$$

By the same theorem $\square XAYA'$ is inscribed, and therefore,

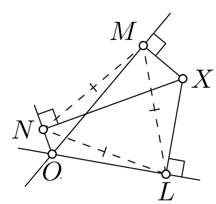
$$2\cdot\measuredangle XAA' \equiv 2\cdot\measuredangle XYA'.$$

Since $\angle XAA'$ is right, so is $\angle XYA'$. That is $(XP) \perp (A'B')$. □

9.16. Exercise. *Find an inaccuracy in the solution of Problem 9.15 and try to fix it.*

The method used in the solution is called the *method of additional circle*, since the circumcircles of the quadrangles $XAPB$ and $XAPB$ above can be considered as *additional constructions.*

9.17. Exercise. *Assume three lines ℓ, m, and n intersect at point O and form six equal angles at O. Let X be a point distinct from O. Let L, M, and N denote the foot points of perpendiculars from X on ℓ, m, and n respectively. Show that $\triangle LMN$ is equilateral.*



9.18. Advanced exercise. *Assume that a point P lies on the circumcircle of the triangle ABC. Show that three foot points of P on the lines (AB), (BC), and (CA) lie on one line. (This line is called the* Simson line *of P).*

Arcs of circlines

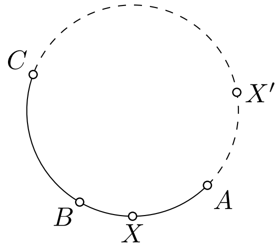
A subset of a circle bounded by two points is called a circle arc.

More precisely, suppose A, B, C are distinct points on a circle Γ. The *circle arc* ABC is the subset that includes the points A, C, as well as all the points on Γ that lie with B on the same side of (AC).

Points A and C are called *endpoints* of the circle arc ABC. There are precisely two circle arcs of Γ with the given endpoints; they are *opposite* to each other.

Suppose X be another point on Γ. By Corollary 9.13 we have that $2\cdot\measuredangle AXC \equiv 2\cdot\measuredangle ABC$; that is,

$$\measuredangle AXC \equiv \measuredangle ABC \quad \text{or} \quad \measuredangle AXC \equiv \measuredangle ABC + \pi.$$



Recall that X and B lie on the same side from (AC) if and only if $\angle AXC$ and $\angle ABC$ have the same sign (see Exercise 3.13). It follows that

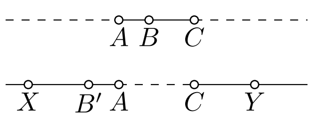
- X lies on the arc ABC if and only if

$$\measuredangle AXC \equiv \measuredangle ABC;$$

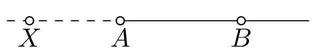
- X lies on the arc opposite to ABC if

$$\measuredangle AXC \equiv \measuredangle ABC + \pi.$$

Note that a circle arc ABC is defined if $\triangle ABC$ is not degenerate. If $\triangle ABC$ is degenerate, then arc ABC is defined as a subset of line bounded by A and C that contains B.



More precisely, if B lies between A and C, then the *arc* ABC is defined as the line segment $[AC]$. If B' lies on the extension of $[AC]$, then the arc $AB'C$ is defined as a union of disjoint half-lines $[AX)$ and $[CY)$ in (AC). In this case, the arcs ABC and $AB'C$ are called opposite to each other.



In addition, any half-line $[AB)$ will be regarded as an arc. If A lies between B and X, then $[AX)$ will be called opposite to $[AB)$. This degenerate arc has only one endpoint A.

It will be convenient to use the notion of *circline*, which means *circle or line*. For example, any arc is a subset of a circline; we also may use the term *circline arc* if we want to emphasize that the arc might be degenerate. Note that for any three distinct points A, B, and C there is a unique circline arc ABC.

The following statement summarizes the discussion above.

9.19. Proposition. *Let ABC be a circline arc and X be a point distinct from A and C. Then*

(a) X lies on the arc ABC if and only if

$$\measuredangle AXC = \measuredangle ABC;$$

(b) X lies on the arc opposite to ABC if and only if

$$\measuredangle AXC \equiv \measuredangle ABC + \pi;$$

9.20. Exercise. *Given an acute triangle ABC make a compass-and-ruler construction of the point Z such that*

$$\measuredangle AZB = \measuredangle BZC = \measuredangle CZA = \pm\tfrac{2}{3}\cdot\pi$$

9.21. Exercise. *Suppose that point P lies on the circumcircle of an equilateral triangle ABC and $PA \leqslant PB \leqslant PC$. Show that $PA + PB = PC$.*

A quadrangle $ABCD$ is *inscribed* if all the points A, B, C, and D lie on a circline Γ. If the arcs ABC and ADC are opposite, then we say that the points A, B, C, and D appear on Γ in the same *cyclic order*.

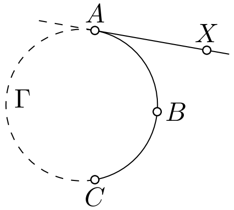
This definition makes it possible to formulate the following refinement of Corollary 9.13 which includes the degenerate quadrangles. It follows directly from 9.19.

9.22. Proposition. *A quadrangle $ABCD$ is inscribed in a circline if and only if*

$$\measuredangle ABC = \measuredangle ADC \quad \text{or} \quad \measuredangle ABC \equiv \measuredangle ADC + \pi.$$

Moreover, the second identity holds if and only if the points A, B, C, D appear on the circline in the same cyclic order.

Tangent half-lines



Suppose ABC is an arc of a circle Γ. A half-line $[AX)$ is called *tangent* to the arc ABC at A if the line (AX) is tangent to Γ, and the points X and B lie on the same side of the line (AC).

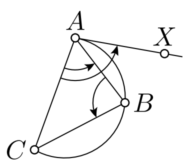
If the arc is formed by the line segment $[AC]$, then the half-line $[AC)$ is considered to be tangent at A. If the arc is formed by a union of two half-lines $[AX)$ and $[BY)$ in (AC), then the half-line $[AX)$ is considered to be tangent to the arc at A.

9.23. Proposition. *The half-line $[AX)$ is tangent to the arc ABC if and only if*

$$\measuredangle ABC + \measuredangle CAX \equiv \pi.$$

Proof. For a degenerate arc ABC, the statement is evident. Further we assume the arc ABC is nondegenerate.

Note that the tangent half-line to the arc ABC at A is uniquely defined. Further, there is a unique half-line $[AX)$ such that the equation in the proposition holds. Therefore it is sufficient to prove the "only-if" part.



If $[AX)$ is tangent to the arc ABC, then by 9.1 and 9.2, we get that

$$2\cdot\measuredangle ABC + 2\cdot\measuredangle CAX \equiv 0.$$

Therefore, either

$$\measuredangle ABC + \measuredangle CAX \equiv \pi, \quad \text{or} \quad \measuredangle ABC + \measuredangle CAX \equiv 0.$$

By the definition of tangent half-line, X and B lie on the same side of (AC). By 3.10 and 3.7, the angles CAX, CAB, and ABC have the same sign. In particular, $\measuredangle ABC + \measuredangle CAX \not\equiv 0$; that is, we are left with the case

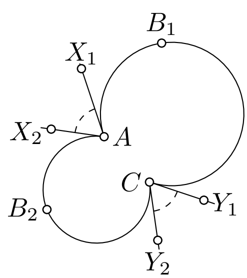
$$\measuredangle ABC + \measuredangle CAX \equiv \pi.$$ □

9.24. Exercise. *Show that there is a unique arc with endpoints at the given points A and C, that is tangent to the given half-line $[AX)$ at A.*

9.25. Exercise. *Let $[AX)$ be the tangent half-line to an arc ABC. Assume Y is a point on the arc ABC that is distinct from A. Show that $\measuredangle XAY \to 0$ as $AY \to 0$.*

9.26. Exercise. *Given two circle arcs AB_1C and AB_2C, let $[AX_1)$ and $[AX_2)$ be the half-lines tangent to the arcs AB_1C and AB_2C at A, and $[CY_1)$ and $[CY_2)$ be the half-lines tangent to the arcs AB_1C and AB_2C at C. Show that*

$$\measuredangle X_1AX_2 \equiv -\measuredangle Y_1CY_2.$$



Chapter 10

Inversion

Let Ω be the circle with center O and radius r. The *inversion* of a point P in Ω is the point $P' \in [OP)$ such that

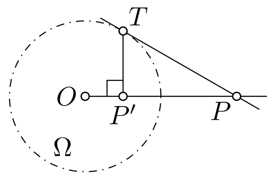
$$OP{\cdot}OP' = r^2.$$

In this case, the circle Ω will be called the *circle of inversion* and its center O is called the *center of inversion*.

The inverse of O is undefined.

Note that if P is inside Ω, then P' is outside and the other way around. Further, $P = P'$ if and only if $P \in \Omega$.

Note that the inversion maps P' back to P.



10.1. Exercise. *Let Ω be a circle centered at O. Suppose that a line (PT) is tangent to Ω at T. Let P' be the foot point of T on (OP).*

Show that P' is the inverse of P in Ω.

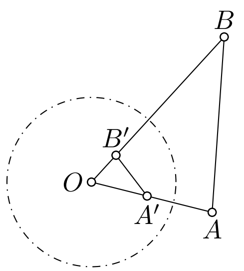
10.2. Lemma. *Let Γ be a circle with the center O. Assume A' and B' are the inverses of A and B in Γ. Then*

$$\triangle OAB \sim \triangle OB'A'.$$

Moreover,

❶
$$\begin{aligned}\measuredangle AOB &\equiv -\measuredangle B'OA',\\ \measuredangle OBA &\equiv -\measuredangle OA'B',\\ \measuredangle BAO &\equiv -\measuredangle A'B'O.\end{aligned}$$

Proof. Let r be the radius of the circle of the inversion.



By the definition of inversion,

$$OA{\cdot}OA' = OB{\cdot}OB' = r^2.$$

Therefore,

$$\frac{OA}{OB'} = \frac{OB}{OA'}.$$

Clearly,

❷ $$\measuredangle AOB = \measuredangle A'OB' \equiv -\measuredangle B'OA'.$$

From SAS, we get that

$$\triangle OAB \sim \triangle OB'A'.$$

Applying Theorem 3.7 and ❷, we get ❶. □

10.3. Exercise. *Let P' be the inverse of P in the circle Γ. Assume that $P \neq P'$. Show that the value $\frac{PX}{P'X}$ is the same for all $X \in \Gamma$.*

The converse to the exercise above also holds. Namely, given a positive real number $k \neq 1$ and two distinct points P and P' the locus of points X such that $\frac{PX}{P'X} = k$ forms a circle which is called the *Apollonian circle.* In this case, P' is inverse of P in the Apollonian circle.

10.4. Exercise. *Let A', B', and C' be the images of A, B, and C under the inversion in the incircle of $\triangle ABC$. Show that the incenter of $\triangle ABC$ is the orthocenter of $\triangle A'B'C'$.*

10.5. Exercise. *Make a ruler-and-compass construction of the inverse of a given point in a given circle.*

Cross-ratio

The following theorem gives some quantities expressed in distances or angles that do not change after inversion.

10.6. Theorem. *Let $ABCD$ and $A'B'C'D'$ be two quadrangles such that the points A', B', C', and D' are the inverses of A, B, C, and D respectively.*

Then

(a)

$$\frac{AB{\cdot}CD}{BC{\cdot}DA} = \frac{A'B'{\cdot}C'D'}{B'C'{\cdot}D'A'}.$$

(b)

$$\measuredangle ABC + \measuredangle CDA \equiv -(\measuredangle A'B'C' + \measuredangle C'D'A').$$

(c) If the quadrangle $ABCD$ is inscribed, then so is $\square A'B'C'D'$.

Proof; (a). Let O be the center of the inversion. According to Lemma 10.2, $\triangle AOB \sim \triangle B'OA'$. Therefore,

$$\frac{AB}{A'B'} = \frac{OA}{OB'}.$$

Analogously,

$$\frac{BC}{B'C'} = \frac{OC}{OB'}, \qquad \frac{CD}{C'D'} = \frac{OC}{OD'}, \qquad \frac{DA}{D'A'} = \frac{OA}{OD'}.$$

Therefore,

$$\frac{AB}{A'B'} \cdot \frac{B'C'}{BC} \cdot \frac{CD}{C'D'} \cdot \frac{D'A'}{DA} = \frac{OA}{OB'} \cdot \frac{OB'}{OC} \cdot \frac{OC}{OD'} \cdot \frac{OD'}{OA}.$$

Hence *(a)* follows.

(b). According to Lemma 10.2,

❸
$$\begin{aligned} \measuredangle ABO &\equiv -\measuredangle B'A'O, & \measuredangle OBC &\equiv -\measuredangle OC'B', \\ \measuredangle CDO &\equiv -\measuredangle D'C'O, & \measuredangle ODA &\equiv -\measuredangle OA'D'. \end{aligned}$$

By Axiom IIIb,

$$\begin{aligned} \measuredangle ABC &\equiv \measuredangle ABO + \measuredangle OBC, & \measuredangle D'C'B' &\equiv \measuredangle D'C'O + \measuredangle OC'B', \\ \measuredangle CDA &\equiv \measuredangle CDO + \measuredangle ODA, & \measuredangle B'A'D' &\equiv \measuredangle B'A'O + \measuredangle OA'D'. \end{aligned}$$

Therefore, summing the four identities in ❸, we get that

$$\measuredangle ABC + \measuredangle CDA \equiv -(\measuredangle D'C'B' + \measuredangle B'A'D').$$

Applying Axiom IIIb and Exercise 7.17, we get that

$$\begin{aligned} \measuredangle A'B'C' + \measuredangle C'D'A' &\equiv -(\measuredangle B'C'D' + \measuredangle D'A'B') \equiv \\ &\equiv \measuredangle D'C'B' + \measuredangle B'A'D'. \end{aligned}$$

Hence *(b)* follows.

(c). Follows by *(b)* and Corollary 9.13. $\square$

Inversive plane and circlines

Let Ω be a circle with the center O and the radius r. Consider the inversion in Ω.

Recall that the inverse of O is undefined. To deal with this problem it is useful to add to the plane an extra point; it will be called the *point at infinity*; we will denote it as ∞. We can assume that ∞ is inverse of O and the other way around.

The Euclidean plane with an added point at infinity is called the *inversive plane.*

We will always assume that any line and half-line contains ∞.

Recall that *circline* means *circle or line.* Therefore we may say *"if a circline contains ∞, then it is a line"* or *"a circline that does not contain ∞ is a circle".*

Note that according to Theorem 8.1, for any $\triangle ABC$ there is a unique circline that passes thru A, B, and C (if $\triangle ABC$ is degenerate, then this is a line and if not it is a circle).

10.7. Theorem. *In the inversive plane, inverse of a circline is a circline.*

Proof. Suppose that O denotes the center of the inversion and r its radius.

Let Γ be a circline. Choose three distinct points A, B, and C on Γ. (If $\triangle ABC$ is nondegenerate, then Γ is the circumcircle of $\triangle ABC$; if $\triangle ABC$ is degenerate, then Γ is the line passing thru A, B, and C.)

Let A', B', and C' denote the inverses of A, B, and C respectively. Let Γ' be the circline that passes thru A', B', and C'.

Assume D is a point of the inversive plane that is distinct from A, C, O, and ∞. Suppose that D' denotes the inverse of D.

By Theorem 10.6*c*, $D' \in \Gamma'$ if and only if $D \in \Gamma$.

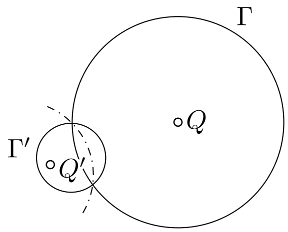
It remains to prove that $O \in \Gamma \Leftrightarrow \infty \in \Gamma'$ and $\infty \in \Gamma \Leftrightarrow O \in \Gamma'$. We will prove that

$$\infty \in \Gamma \quad \Longrightarrow \quad O \in \Gamma';$$

the remaining implications can be proved along the same lines.

If $\infty \in \Gamma$, then Γ is a line; or, equivalently, for any $\varepsilon > 0$, the circline Γ contains a point P such that $OP > r/\varepsilon$. For the inversion $P' \in \Gamma'$ of P, we have that $OP' = r^2/OP < r{\cdot}\varepsilon$. That is, the circline Γ' contains points arbitrarily close to O. It follows that $O \in \Gamma'$. ☐

10.8. Exercise. *Assume that the circle Γ' is the inverse of the circle Γ. Suppose that Q denotes the center of Γ and Q' denotes the inverse of Q. Show that Q' is not the center of Γ'.*



Assume that a *circumtool* is a geometric construction tool that produces a circline passing thru any three given points.

10.9. Exercise. *Show that with only a circumtool, it is impossible to construct the center of a given circle.*

10.10. Exercise. *Show that for any pair of tangent circles in the inversive plane, there is an inversion that sends them to a pair of parallel lines.*

10.11. Theorem. *Consider the inversion of the inversive plane in the circle Ω with the center O. Then*

(a) A line passing thru O is inverted into itself.

(b) A line not passing thru O is inverted into a circle that passes thru O, and the other way around.

(c) A circle not passing thru O is inverted into a circle not passing thru O.

Proof. In the proof, we use Theorem 10.7 without mentioning it.

(a). Note that if a line passes thru O, it contains both ∞ and O. Therefore, its inverse also contains ∞ and O. In particular, the image is a line passing thru O.

(b). Since any line ℓ passes thru ∞, its image ℓ' has to contain O. If the line does not contain O, then $\ell' \not\ni \infty$; that is, ℓ' is not a line. Therefore, ℓ' is a circle that passes thru O.

(c). If the circle Γ does not contain O, then its image Γ' does not contain ∞. Therefore, Γ' is a circle. Since $\Gamma \not\ni \infty$ we get that $\Gamma' \not\ni O$. Hence the result. □

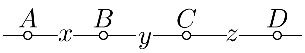
Method of inversion

Here is an application of inversion, which we include as an illustration; we will not use it further in the book.

10.12. Ptolemy's identity. *Let $ABCD$ be an inscribed quadrangle. Assume that the points A, B, C, and D appear on the circline in the same order. Then*

$$AB\cdot CD + BC\cdot DA = AC\cdot BD.$$

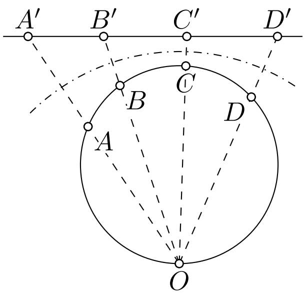
Proof. Assume the points A, B, C, D lie on one line in this order.



Set $x = AB$, $y = BC$, $z = CD$. Note that

$$x{\cdot}z + y{\cdot}(x + y + z) = (x + y){\cdot}(y + z).$$

Since $AC = x + y$, $BD = y + z$, and $DA = x + y + z$, it proves the identity.



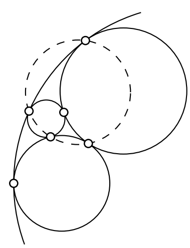
It remains to consider the case when the quadrangle $ABCD$ is inscribed in a circle, say Γ.

The identity can be rewritten as

$$\frac{AB{\cdot}DC}{BD{\cdot}CA} + \frac{BC{\cdot}AD}{CA{\cdot}DB} = 1.$$

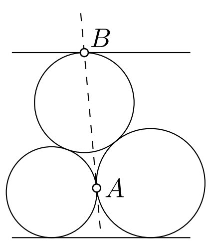
On the left-hand side we have two cross-ratios. According to Theorem 10.6a, the left-hand side does not change if we apply an inversion to each point.

Consider an inversion in a circle centered at a point O that lies on Γ between A and D. By Theorem 10.11, this inversion maps Γ to a line. This reduces the problem to the case when A, B, C, and D lie on one line, which was already considered. □



In the proof above, we rewrite Ptolemy's identity in a form that is invariant with respect to inversion and then apply an inversion which makes the statement evident. The solution of the following exercise is based on the same idea; one has to make a right choice of inversion.

10.13. Exercise. *Assume that four circles are mutually tangent to each other. Show that four (among six) of their points of tangency lie on one circline.*



10.14. Advanced exercise. *Assume that three circles tangent to each other and to two parallel lines as shown in the picture.*

Show that the line passing thru A and B is also tangent to two circles at A.

Perpendicular circles

Assume two circles Γ and Ω intersect at two points X and Y. Let ℓ and m be the tangent lines at X to Γ and Ω respectively. Analogously, ℓ' and m' be the tangent lines at Y to Γ and Ω.

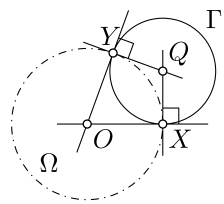
From Exercise 9.26, we get that $\ell \perp m$ if and only if $\ell' \perp m'$.

We say that the circle Γ is *perpendicular* to the circle Ω (briefly $\Gamma \perp \Omega$) if they intersect and the lines tangent to the circles at one point (and therefore, both points) of intersection are perpendicular.

Similarly, we say that the circle Γ is perpendicular to the line ℓ (briefly $\Gamma \perp \ell$) if $\Gamma \cap \ell \neq \varnothing$ and ℓ perpendicular to the tangent lines to Γ at one point (and therefore, both points) of intersection. According to Lemma 5.17, it happens only if the line ℓ passes thru the center of Γ.

Now we can talk about *perpendicular circlines.*

10.15. Theorem. *Assume Γ and Ω are distinct circles. Then $\Omega \perp \Gamma$ if and only if the circle Γ coincides with its inversion in Ω.*



Proof. Suppose that Γ' denotes the inverse of Γ.

"Only if" part. Let O be the center of Ω and Q be the center of Γ. Let X and Y denote the points of intersections of Γ and Ω. According to Lemma 5.17, $\Gamma \perp \Omega$ if and only if (OX) and (OY) are tangent to Γ.

Since $O \neq X$, Lemma 5.10 implies that O lies outside of Γ. By Theorem 10.11*c*, Γ' is a circle.

Note that Γ' is also tangent to (OX) and (OY) at X and Y respectively. It follows that X and Y are the foot points of the center of Γ' on (OX) and (OY). Therefore, both Γ' and Γ have the center Q. Finally, $\Gamma' = \Gamma$, since both circles pass thru X.

"If" part. Assume $\Gamma = \Gamma'$.

Since $\Gamma \neq \Omega$, there is a point P that lies on Γ, but not on Ω. Let P' be the inverse of P in Ω. Since $\Gamma = \Gamma'$, we have that $P' \in \Gamma$. In particular, the half-line $[OP)$ intersects Γ at two points. By Exercise 5.13, O lies outside of Γ.

As Γ has points inside and outside of Ω, the circles Γ and Ω intersect. The latter follows from Exercise 3.20.

Let X be a point of their intersection. We need to show that (OX) is tangent to Γ; that is, X is the only intersection point of (OX) and Γ.

Assume Z is another point of intersection of (OX) and Γ. Since O is outside of Γ, the point Z lies on the half-line $[OX)$.

Suppose that Z' denotes the inverse of Z in Ω. Clearly, the three points Z, Z', X lie on Γ and (OX). The latter contradicts Lemma 5.15. □

It is convenient to define the *inversion in the line ℓ* as the reflection across ℓ. This way we can talk about *inversion in an arbitrary circline.*

10.16. Corollary. *Let Ω and Γ be distinct circlines in the inversive plane. Then the inversion in Ω sends Γ to itself if and only if $\Omega \perp \Gamma$.*

Proof. By Theorem 10.15, it is sufficient to consider the case when Ω or Γ is a line.

Assume Ω is a line, so the inversion in Ω is a reflection. In this case, the statement follows from Corollary 5.7.

If Γ is a line, then the statement follows from Theorem 10.11. □

10.17. Corollary. *Let P and P' be two distinct points such that P' is the inverse of P in the circle Ω. Assume that the circline Γ passes thru P and P'. Then $\Gamma \perp \Omega$.*

Proof. Without loss of generality, we may assume that P is inside and P' is outside Ω. By Theorem 3.17, Γ intersects Ω. Suppose that A denotes a point of intersection.

Suppose that Γ' denotes the inverse of Γ. Since A is a self-inverse, the points A, P, and P' lie on Γ'. By Exercise 8.2, $\Gamma' = \Gamma$ and by Theorem 10.15, $\Gamma \perp \Omega$. □

10.18. Corollary. *Let P and Q be two distinct points inside the circle Ω. Then there is a unique circline Γ perpendicular to Ω that passes thru P and Q.*

Proof. Let P' be the inverse of the point P in the circle Ω. According to Corollary 10.17, the circline is passing thru P and Q is perpendicular to Ω if and only if it passes thru P'.

Note that P' lies outside of Ω. Therefore, the points P, P', and Q are distinct.

According to Exercise 8.2, there is a unique circline passing thru P, Q, and P'. Hence the result. □

10.19. Exercise. *Let P, Q, P', and Q' be points in the Euclidean plane. Assume P' and Q' are inverses of P and Q respectively. Show that the quadrangle $PQP'Q'$ is inscribed.*

10.20. Exercise. *Let Ω_1 and Ω_2 be two perpendicular circles with centers at O_1 and O_2 respectively. Show that the inverse of O_1 in Ω_2 coincides with the inverse of O_2 in Ω_1.*

10.21. Exercise. *Three distinct circles — Ω_1, Ω_2, and Ω_3, intersect at two points — A and B. Assume that a circle Γ is perpendicular to Ω_1 and Ω_2. Show that $\Gamma \perp \Omega_3$.*

Let us consider two new construction tools: the *circum-tool* that constructs a circline thru three given points, and the *inversion-tool* — a tool that constructs an inverse of a given point in a given circle.

10.22. Exercise. *Given two circles Ω_1, Ω_2 and a point P that does not lie on the circles, use only circum-tool and inversion-tool to construct a circline Γ that passes thru P, and perpendicular to both Ω_1 and Ω_2.*

10.23. Advanced exercise. *Given three disjoint circles Ω_1, Ω_2 and Ω_3, use only circum-tool and inversion-tool to construct a circline Γ that perpendicular to each circle Ω_1, Ω_2, and Ω_3.*

Think what to do if two of the circles intersect.

Angles after inversion

10.24. Proposition. *In the inversive plane, the inverse of an arc is an arc.*

Proof. Consider four distinct points A, B, C, and D; let A', B', C', and D' be their inverses. We need to show that D lies on the arc ABC if and only if D' lies on the arc $A'B'C'$. According to Proposition 9.19, the latter is equivalent to the following:

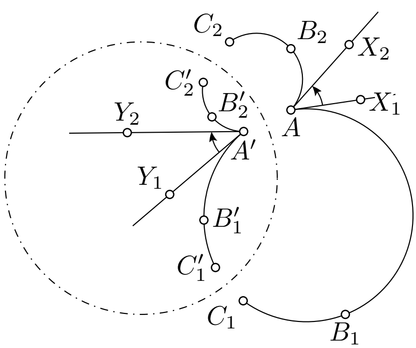
$$\measuredangle ADC = \measuredangle ABC \quad \Longleftrightarrow \quad \measuredangle A'D'C' = \measuredangle A'B'C'.$$

The latter follows from Theorem 10.6*b*. □

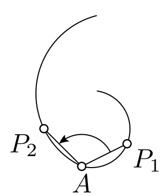
The following theorem states that the angle between arcs changes only its sign after the inversion.

10.25. Theorem. *Let AB_1C_1, AB_2C_2 be two arcs in the inversive plane, and the arcs $A'B_1'C_1'$, $A'B_2'C_2'$ be their inverses. Let $[AX_1)$ and $[AX_2)$ be the half-lines tangent to AB_1C_1 and AB_2C_2 at A, and $[A'Y_1)$ and $[A'Y_2)$ be the half-lines tangent to $A'B_1'C_1'$ and $A'B_2'C_2'$ at A'. Then*

$$\measuredangle X_1AX_2 \equiv -\measuredangle Y_1A'Y_2.$$



The *angle between arcs* can be defined as the angle between its tangent half-lines at the common endpoint. Therefore under inversion, the angles between arcs are preserved up to sign.



From Exercise 5.24, it follows that the angle between arcs with common endpoint A is the limit of $\measuredangle P_1AP_2$

where P_1 and P_2 are points approaching A along the corresponding arcs. This observation can be used to define the angle between a pair of curves emerging from one point. It turns out that under inversion, angles between curves are also preserved up to sign.

Proof. Applying to Proposition 9.23,

$$\begin{aligned}\measuredangle X_1AX_2 &\equiv \measuredangle X_1AC_1 + \measuredangle C_1AC_2 + \measuredangle C_2AX_2 \equiv \\ &\equiv (\pi - \measuredangle C_1B_1A) + \measuredangle C_1AC_2 + (\pi - \measuredangle AB_2C_2) \equiv \\ &\equiv -(\measuredangle C_1B_1A + \measuredangle AB_2C_2 + \measuredangle C_2AC_1) \equiv \\ &\equiv -(\measuredangle C_1B_1A + \measuredangle AB_2C_1) - (\measuredangle C_1B_2C_2 + \measuredangle C_2AC_1).\end{aligned}$$

In the same way, we get that

$$\measuredangle Y_1A'Y_2 \equiv -(\measuredangle C_1'B_1'A' + \measuredangle A'B_2'C_1') - (\measuredangle C_1'B_2'C_2' + \measuredangle C_2'A'C_1').$$

By Theorem 10.6*b*,

$$\begin{aligned}\measuredangle C_1B_1A + \measuredangle AB_2C_1 &\equiv -(\measuredangle C_1'B_1'A' + \measuredangle A'B_2'C_1'),\\ \measuredangle C_1B_2C_2 + \measuredangle C_2AC_1 &\equiv -(\measuredangle C_1'B_2'C_2' + \measuredangle C_2'A'C_1')\end{aligned}$$

and hence the result. □

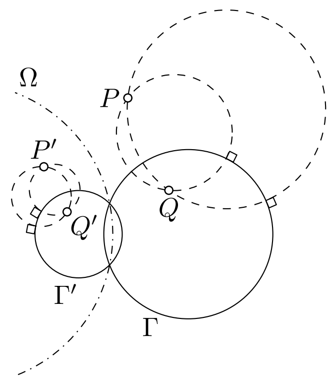
10.26. Corollary. *Let P be the inverse of point Q in a circle Γ. Assume that P', Q', and Γ' are the inverses of P, Q, and Γ in another circle Ω. Then P' is the inverse of Q' in Γ'.*

Proof. If $P = Q$, then $P' = Q' \in \Gamma'$. Therefore, P' is the inverse of Q' in Γ'.

It remains to consider the case $P \neq Q$. Let Δ_1 and Δ_2 be two distinct circles that intersect at P and Q. According to Corollary 10.17, $\Delta_1 \perp \Gamma$ and $\Delta_2 \perp \Gamma$.

Let Δ_1' and Δ_2' denote the inverses of Δ_1 and Δ_2 in Ω. Clearly, Δ_1' meets Δ_2' at P' and Q'.

By Theorem 10.25, $\Delta_1' \perp \Gamma'$ and $\Delta_2' \perp \perp \Gamma'$. By Corollary 10.16, P' is the inverse of Q' in Γ'. □



Chapter 11

Neutral plane

Let us remove Axiom V from our axiomatic system (see page 19). This way we define a new object called *neutral plane* or *absolute plane*. (In a neutral plane, the Axiom V may or may not hold.)

Clearly, any theorem in neutral geometry holds in Euclidean geometry. In other words, the Euclidean plane is an example of a neutral plane. In the next chapter, we will construct an example of a neutral plane that is not Euclidean.

In this book, the Axiom V was used starting from Chapter 6. Therefore all the statements before hold in neutral geometry.

It makes all the discussed results about half-planes, signs of angles, congruence conditions, perpendicular lines, and reflections true in neutral geometry. Recall that a statement above marked with "✓" (for example, "**Theorem.**✓") if it holds in any neutral plane and the same proof works.

Let us give an example of a theorem in neutral geometry that admits a simpler proof in Euclidean geometry.

11.1. Hypotenuse-leg congruence condition. *Assume that triangles ABC and $A'B'C'$ have right angles at C and C' respectively, $AB = A'B'$ and $AC = A'C'$. Then $\triangle ABC \cong \triangle A'B'C'$.*

Euclidean proof. By the Pythagorean theorem $BC = B'C'$. Then the statement follows from the SSS congruence condition. □

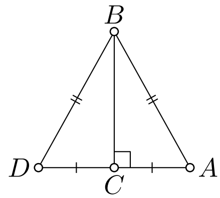
The proof of the Pythagorean theorem used properties of similar triangles, which in turn used Axiom V. Therefore this proof does not work in a neutral plane.

Neutral proof. Suppose that D denotes the reflection of A across (BC) and D' denotes the reflection of A' across $(B'C')$. Note that

$$AD = 2\cdot AC = 2\cdot A'C' = A'D', \qquad BD = BA = B'A' = B'D'.$$

By SSS congruence condition (4.4), we get that $\triangle ABD \cong \triangle A'B'D'$.

The statement follows since C is the midpoint of $[AD]$ and C' is the midpoint of $[A'D']$. □



11.2. Exercise. *Give a proof of Exercise 8.12 that works in the neutral plane.*

11.3. Exercise. *Let $ABCD$ be an inscribed quadrangle in the neutral plane. Show that*

$$\measuredangle ABC + \measuredangle CDA \equiv \measuredangle BCD + \measuredangle DAB.$$

Note that one cannot use Corollary 9.13 to solve the exercise above, since it uses Theorems 9.1 and 9.2, which in turn uses Theorem 7.12.

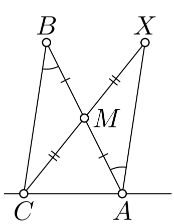
Two angles of a triangle

In this section, we will prove a weaker form of Theorem 7.12 which holds in any neutral plane.

11.4. Proposition. *Let $\triangle ABC$ be a nondegenerate triangle in the neutral plane. Then*

$$|\measuredangle CAB| + |\measuredangle ABC| < \pi.$$

Note that according to 3.7, the angles ABC, BCA, and CAB have the same sign. Therefore, in the Euclidean plane, the theorem follows immediately from Theorem 7.12.



Proof. Let X be the reflection of C across the midpoint M of $[AB]$. By Proposition 7.6 $\measuredangle BAX = \measuredangle ABC$ and therefore

$$\text{❶} \qquad \measuredangle CAX \equiv \measuredangle CAB + \measuredangle ABC.$$

Since $[BM]$ and $[MX]$ do not intersect (CA), the points B, M, and X lie on the same side of (CA). Therefore the angles CAB and CAX have the same sign. By 3.7, the angles CAB, ABC have the same sign; that is all angles in ❶ have the same sign.

Note that $\measuredangle CAX \not\equiv \pi$; otherwise, X would lie on (AC). Therefore the identity ❶ implies that

$$|\measuredangle CAB| + |\measuredangle ABC| = |\measuredangle CAX| < \pi.$$

□

11.5. Exercise. *Assume A, B, C, and D are points in a neutral plane such that*

$$2\cdot\measuredangle ABC + 2\cdot\measuredangle BCD \equiv 0.$$

Show that $(AB) \parallel (CD)$.

Note that one cannot apply the transversal property (7.9).

11.6. Exercise. *Prove the* side-angle-angle congruence condition *in the neutral geometry.*

In other words, let ABC and $A'B'C'$ be two triangles in a neutral plane; suppose that $\triangle A'B'C'$ is nondegenerate. Show that $\triangle ABC \cong \cong \triangle A'B'C'$ if

$$AB = A'B', \quad \measuredangle ABC = \pm\measuredangle A'B'C' \quad \text{and} \quad \measuredangle BCA = \pm\measuredangle B'C'A'.$$

Note that in the Euclidean plane, the above exercise follows from ASA and the theorem on the sum of angles of a triangle (7.12). However, Theorem 7.12 cannot be used here, since its proof uses Axiom V. Later (Theorem 13.7) we will show that Theorem 7.12 does not hold in a neutral plane.

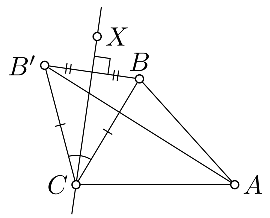
11.7. Exercise. *Assume that point D lies between the vertices A and B of $\triangle ABC$ in a neutral plane. Show that*

$$CD < CA \quad \text{or} \quad CD < CB.$$

Three angles of triangle

11.8. Proposition. *Let $\triangle ABC$ and $\triangle A'B'C'$ be two triangles in the neutral plane such that $AC = A'C'$ and $BC = B'C'$. Then*

$$AB < A'B' \quad \text{if and only if} \quad |\measuredangle ACB| < |\measuredangle A'C'B'|.$$



Proof. Without loss of generality, we may assume that $A = A'$, $C = C'$, and $\measuredangle ACB, \measuredangle ACB' \geqslant 0$. In this case, we need to show that

$$AB < AB' \iff \measuredangle ACB < \measuredangle ACB'.$$

Choose a point X so that

$$\measuredangle ACX = \tfrac{1}{2}\cdot(\measuredangle ACB + \measuredangle ACB').$$

Note that

- ⋄ (CX) bisects $\angle BCB'$.
- ⋄ (CX) is the perpendicular bisector of $[BB']$.
- ⋄ A and B lie on the same side of (CX) if and only if

$$\measuredangle ACB < \measuredangle ACB'.$$

From Exercise 5.3, A and B lie on the same side of (CX) if and only if $AB < AB'$. Hence the result. □

11.9. Theorem. *Let $\triangle ABC$ be a triangle in the neutral plane. Then*

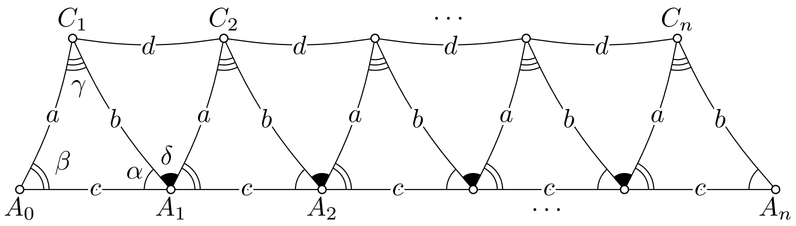
$$|\measuredangle ABC| + |\measuredangle BCA| + |\measuredangle CAB| \leqslant \pi.$$

The following proof is due to Legendre [12], earlier proofs were due to Saccheri [16] and Lambert [11].

Proof. Set

$$a = BC, \qquad b = CA, \qquad c = AB,$$
$$\alpha = \measuredangle CAB, \qquad \beta = \measuredangle ABC, \qquad \gamma = \measuredangle BCA.$$

Without loss of generality, we may assume that $\alpha, \beta, \gamma \geqslant 0$.



Fix a positive integer n. Consider the points A_0, $A_1, \dots, A_n$ on the half-line $[BA)$, such that $BA_i = i{\cdot}c$ for each i. (In particular, $A_0 = B$ and $A_1 = A$.) Let us construct the points C_1, $C_2, \dots, C_n$, so that $\measuredangle A_iA_{i-1}C_i = \beta$ and $A_{i-1}C_i = a$ for each i.

By SAS, we have constructed n congruent triangles

$$\triangle ABC = \triangle A_1A_0C_1 \cong \triangle A_2A_1C_2 \cong \dots \cong \triangle A_nA_{n-1}C_n.$$

Set $d = C_1C_2$ and $\delta = \measuredangle C_2A_1C_1$. Note that

❷ $$\alpha + \beta + \delta = \pi.$$

By Proposition 11.4, we get that $\delta \geqslant 0$.

By construction

$$\triangle A_1C_1C_2 \cong \triangle A_2C_2C_3 \cong \dots \cong \triangle A_{n-1}C_{n-1}C_n.$$

In particular, $C_iC_{i+1} = d$ for each i.

By repeated application of the triangle inequality, we get that

$$\begin{aligned} n{\cdot}c = A_0A_n \leqslant \\ \leqslant A_0C_1 + C_1C_2 + \dots + C_{n-1}C_n + C_nA_n = \\ = a + (n-1){\cdot}d + b. \end{aligned}$$

In particular,

$$c \leqslant d + \tfrac{1}{n}{\cdot}(a + b - d).$$

Since n is an arbitrary positive integer, the latter implies $c \leqslant d$. By Proposition 11.8, it is equivalent to

$$\gamma \leqslant \delta.$$

From ❷, the theorem follows. □

11.10. Exercise. *Let $ABCD$ be a quadrangle in the neutral plane. Suppose that the angles DAB and ABC are right. Show that $AB \leqslant CD$.*

Defect

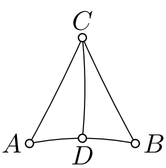
The *defect of triangle $\triangle ABC$* is defined as

$$\operatorname{defect}(\triangle ABC) := \pi - |\measuredangle ABC| - |\measuredangle BCA| - |\measuredangle CAB|.$$

Note that Theorem 11.9 states that the defect of any triangle in a neutral plane has to be nonnegative. According to Theorem 7.12, any triangle in the Euclidean plane has zero defect.

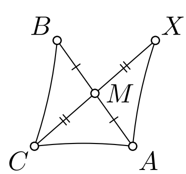
11.11. Exercise. *Let $\triangle ABC$ be a nondegenerate triangle in the neutral plane. Assume D lies between A and B. Show that*

$$\operatorname{defect}(\triangle ABC) = \operatorname{defect}(\triangle ADC) + \operatorname{defect}(\triangle DBC).$$



11.12. Exercise. *Let ABC be a nondegenerate triangle in the neutral plane. Suppose X is the reflection of C across the midpoint M of $[AB]$. Show that*

$$\operatorname{defect}(\triangle ABC) = \operatorname{defect}(\triangle AXC).$$



11.13. Exercise. *Suppose that $ABCD$ is a rectangle in a neutral plane; that is, $ABCD$ is a quadrangle with all right angles. Show that $AB = CD$.*

11.14. Advanced exercise. *Show that if a neutral plane has a rectangle, then all its triangles have zero defect.*

How to prove that something cannot be proved

Many attempts were made to prove that any theorem in Euclidean geometry holds in neutral geometry. The latter is equivalent to the statement that Axiom V is a *theorem* in neutral geometry.

Some of these attempts were accepted as proof for long periods, until a mistake was found.

There are many statements in neutral geometry that are *equivalent* to the Axiom V. It means that if we exchange the Axiom V to any of these statements, then we will obtain an equivalent axiomatic system.

The following theorem provides a short list of such statements. We are not going to prove it in the book.

11.15. Theorem. *A neutral plane is Euclidean if and only if one of the following equivalent conditions holds:*

(a) There is a line ℓ and a point $P \notin \ell$ such that there is only one line passing thru P and parallel to ℓ.
(b) Every nondegenerate triangle can be circumscribed.
(c) There exists a pair of distinct lines that lie at a bounded distance from each other.
(d) There is a triangle with an arbitrarily large inradius.
(e) There is a nondegenerate triangle with zero defect.
(f) There exists a quadrangle in which all the angles are right.

It is hard to imagine a neutral plane that does not satisfy some of the properties above. That is partly the reason for a large number of false proofs; each used one of such statements by accident.

Let us formulate the negation of *(a)* above as a new axiom; we label it h-V as a *hyperbolic version* of Axiom V on page 19.

> h-V. For any line ℓ and any point $P \notin \ell$ there are at least two lines that pass thru P and parallel to ℓ.

By Theorem 7.2, a neutral plane that satisfies Axiom h-V is not Euclidean. Moreover, according to Theorem 11.15 (which we do not prove) in any non-Euclidean neutral plane, Axiom h-V holds.

It opens a way to look for a proof by contradiction. Simply exchange Axiom V to Axiom h-V and start to prove theorems in the obtained axiomatic system. In the case if we arrive at a contradiction, we prove the Axiom V in a neutral plane. This idea was growing since the 5^{th} century; the most notable results were obtained by Saccheri in [16].

The system of axioms I–IV and h-V defines a new geometry which is now called *hyperbolic* or *Lobachevskian geometry*. The more this geometry was developed, it became more and more believable that there is no contradiction; that is, the system of axioms I–IV, and h-V is *consistent*. In fact, the following theorem holds true:

11.16. Theorem. *The hyperbolic geometry is consistent if and only if so is the Euclidean geometry.*

The claims that hyperbolic geometry has no contradiction can be found in private letters of Gauss, Schweikart, and Taurinus.[1] They all seem to be afraid to state it in public. For instance, in 1818 Gauss writes to Gerling:

> *...I am happy that you have the courage to express yourself as if you recognized the possibility that our parallels theory along with our entire geometry could be false. But the wasps whose nest you disturb will fly around your head.*

Lobachevsky came to the same conclusion independently. Unlike the others, he dared to state it in public (and in print; see [13]). That cost him serious troubles. A couple of years later, also independently, Bolyai published his work (see [6]).

It seems that Lobachevsky was the first who had a proof of Theorem 11.16 altho its formulation required rigorous axiomatics which wes not developed at his time. Later, Beltrami gave a cleaner proof of the "if" part of the theorem. It was done by modeling points, lines, distances, and angle measures of one geometry using some other objects in another geometry. The same idea was used earlier by Lobachevsky; in [14, §34] he modeled the Euclidean plane in the hyperbolic space.

The proof of Beltrami is the subject of the next chapter.

Curvature

In a letter from 1824 Gauss writes:

> *The assumption that the sum of the three angles is less than π leads to a curious geometry, quite different from ours but completely consistent, which I have developed to my entire satisfaction, so that I can solve every problem in it with the exception of a determination of a constant, which cannot be designated a priori. The greater one takes this constant, the nearer one comes to Euclidean geometry, and when it is chosen indefinitely large the two coincide. The theorems of*

[1] The oldest surviving letters were the Gauss letter to Gerling in 1816 and yet more convincing letter dated 1818 of Schweikart sent to Gauss via Gerling.

> *this geometry appear to be paradoxical and, to the uninitiated, absurd; but calm, steady reflection reveals that they contain nothing at all impossible. For example, the three angles of a triangle become as small as one wishes, if only the sides are taken large enuf; yet the area of the triangle can never exceed a definite limit, regardless how great the sides are taken, nor indeed can it ever reach it.*

In modern terminology, the constant that Gauss mentions can be expressed as $1/\sqrt{-k}$, where $k \leqslant 0$, is the so-called *curvature* of the neutral plane, which we are about to introduce.

The identity in Exercise 11.11 suggests that the defect of a triangle should be proportional to its area.[2]

In fact, for any neutral plane, there is a nonpositive real number k such that

$$k\cdot \operatorname{area}(\triangle ABC) + \operatorname{defect}(\triangle ABC) = 0$$

for any $\triangle ABC$. This number k is called the *curvature* of the plane.

For example, by Theorem 7.12, the Euclidean plane has zero curvature. By Theorem 11.9, the curvature of any neutral plane is nonpositive.

It turns out that up to isometry, the neutral plane is characterized by its curvature; that is, two neutral planes are isometric if and only if they have the same curvature.

In the next chapter, we will construct a *hyperbolic plane*; this is, an example of a neutral plane with curvature $k = -1$.

Any neutral planes, distinct from Euclidean, can be obtained by scaling the metric on the hyperbolic plane. Indeed, if we scale the metric by a positive factor c, the area changes by factor c^2, while the defect stays the same. Therefore, taking $c = \sqrt{-k}$, we can get the neutral plane of the given curvature $k < 0$. In other words, all the non-Euclidean neutral planes become identical if we use $r = 1/\sqrt{-k}$ as the unit of length.

In Chapter 16, we discuss spherical geometry. Altho spheres are not neutral planes, the spherical geometry is a close relative of Euclidean and hyperbolic geometries.

Nondegenerate spherical triangles have negative defects. Moreover, if R is the radius of the sphere, then

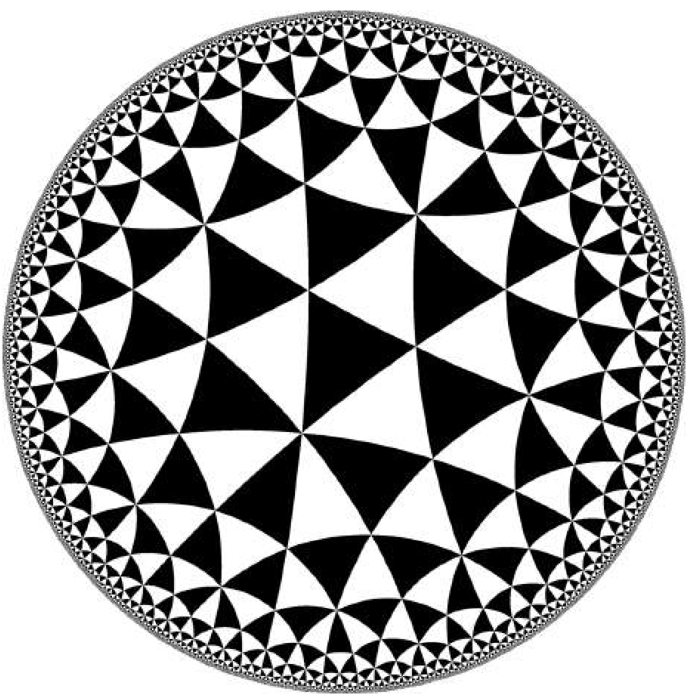
$$\tfrac{1}{R^2}\cdot \operatorname{area}(\triangle ABC) + \operatorname{defect}(\triangle ABC) = 0$$

for any spherical triangle ABC. In other words, the sphere of the radius R has the curvature $k = \tfrac{1}{R^2}$.

[2] The area in the neutral plane is discussed briefly in the end of Chapter 20, but the reader could also refer to an intuitive understanding of area measurement.

Chapter 12

Hyperbolic plane



In this chapter, we use inversive geometry to construct the model of a hyperbolic plane — a neutral plane that is not Euclidean.

Namely, we construct the so-called *conformal disc model* of the hyperbolic plane. This model was discovered by Beltrami [4]; it is often called the *Poincaré disc model*.

The figure above shows the conformal disc model of the hyperbolic plane which is cut into congruent triangles with angles $\frac{\pi}{3}$, $\frac{\pi}{3}$, and $\frac{\pi}{4}$.

Conformal disc model

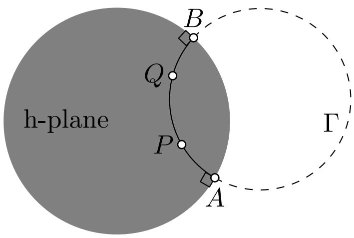
In this section, we give new names for some objects in the Euclidean plane which will represent lines, angle measures, and distances in the hyperbolic plane.

Hyperbolic plane. Let us fix a circle on the Euclidean plane and call it *absolute*. The set of points inside the absolute will be called the *hyperbolic plane* (or *h-plane*).

Note that the points on the absolute do *not* belong to the h-plane. The points in the h-plane will be also called *h-points*.

Often we will assume that the absolute is a unit circle.

Hyperbolic lines. The intersections of the h-plane with circlines perpendicular to the absolute are called *hyperbolic lines* or *h-lines*.



By Corollary 10.18, there is a unique h-line that passes thru the given two distinct h-points P and Q. This h-line will be denoted by $(PQ)_h$.

The arcs of hyperbolic lines will be called *hyperbolic segments* or *h-segments*. An h-segment with endpoints P and Q will be denoted by $[PQ]_h$.

The subset of an h-line on one side from a point will be called a *hyperbolic half-line* (or *h-half-line*). More precisely, an h-half-line is an intersection of the h-plane with arc perpendicular to the absolute that has exactly one of its endpoints in the h-plane. An h-half-line starting at P and passing thru Q will be denoted by $[PQ)_h$.

If Γ is the circline containing the h-line $(PQ)_h$, then the points of intersection of Γ with the absolute are called *ideal points* of $(PQ)_h$. (Note that the ideal points of an h-line do not belong to the h-line.)

An ordered triple of h-points, say (P, Q, R) will be called *h-triangle* PQR and denoted by $\triangle_h PQR$.

Let us point out, that so far an h-line $(PQ)_h$ is just a subset of the h-plane; below we will introduce h-distance and later we will show that $(PQ)_h$ is a line for the h-distance in the sense of the Definition 1.9.

12.1. Exercise. *Show that an h-line is uniquely determined by its ideal points.*

12.2. Exercise. *Show that an h-line is uniquely determined by one of its ideal points and one h-point on it.*

12.3. Exercise. *Show that the h-segment $[PQ]_h$ coincides with the Euclidean segment $[PQ]$ if and only if the line (PQ) passes thru the center of the absolute.*

Hyperbolic distance. Let P and Q be distinct h-points; let A and B denote the ideal points of $(PQ)_h$. Without loss of generality, we may assume that on the Euclidean circline containing the h-line $(PQ)_h$, the points A, P, Q, B appear in the same order.

Consider the function

$$\delta(P, Q) := \frac{AQ{\cdot}PB}{AP{\cdot}QB}.$$

Note that the right-hand side is a cross-ratio; by Theorem 10.6 it is invariant under inversion. Set $\delta(P, P) = 1$ for any h-point P. Let us define h-distance as the logarithm of δ; that is,

$$PQ_h := \ln[\delta(P, Q)].$$

The proof that PQ_h is a metric on the h-plane will be given later. For now, it is just a function that returns a real value PQ_h for any pair of h-points P and Q.

12.4. Exercise. *Let O be the center of the absolute and the h-points O, X, and Y lie on one h-line in the same order. Assume $OX = XY$. Prove that $OX_h < XY_h$.*

Hyperbolic angles. Consider three h-points P, Q, and R such that $P \neq Q$ and $R \neq Q$. The *hyperbolic angle PQR* (briefly $\angle_h PQR$) is an ordered pair of h-half-lines $[QP)_h$ and $[QR)_h$.

Let $[QX)$ and $[QY)$ be (Euclidean) half-lines that are tangent to $[QP]_h$ and $[QR]_h$ at Q. Then the *hyperbolic angle measure* (or *h-angle measure*) of $\angle_h PQR$ is denoted by $\measuredangle_h PQR$ and defined as $\measuredangle XQY$.

12.5. Exercise. *Let ℓ be an h-line and P be an h-point that does not lie on ℓ. Show that there is a unique h-line passing thru P and perpendicular to ℓ.*

Plan of the proof

We defined all the *h-notions* needed in the formulation of the axioms I–IV and h-V. It remains to show that all these axioms hold; this will be done by the end of this chapter.

Once we are done with the proofs, we get that the model provides an example of a neutral plane; in particular, Exercise 12.5 can be proved the same way as Theorem 5.5.

Most importantly we will prove the “if”-part of Theorem 11.16.

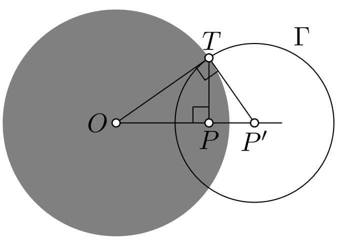
Indeed, any statement in hyperbolic geometry can be restated in the Euclidean plane using the introduced h-notions. Therefore, if the system of axioms I–IV, and h-V leads to a contradiction, then so does the system axioms I–V.

Auxiliary statements

One may compare the conformal model with a *telescope* — it makes it possible to *see* the h-plane from the Euclidean plane. Continuing this analogy further, we may say that the following lemma will be used to *aim* the telescope at any particular point in the h-plane.

12.6. Lemma. *Consider an h-plane with a unit circle as the absolute. Let O be the center of the absolute and P be another h-point. Suppose that P' denotes the inverse of P in the absolute.*

Then the circle Γ with the center P' and radius $\frac{\sqrt{1-OP^2}}{OP}$ is perpendicular to the absolute. Moreover, O is the inverse of P in Γ.



Proof. Follows by Exercise 10.20. □

Assume Γ is a circline that is perpendicular to the absolute. Consider the inversion $X \mapsto X'$ in Γ, or if Γ is a line, set $X \mapsto X'$ to be the reflection across Γ.

The following observation says that the map $X \mapsto X'$ respects all the notions introduced in the previous section. Together with the lemma above, it implies that in any problem that is formulated entirely *in h-terms* we can assume that a given h-point lies in the center of the absolute.

12.7. Main observation. *The map $X \mapsto X'$ described above is a bijection from the h-plane to itself. Moreover, for any h-points P, Q, R such that $P \neq Q$ and $Q \neq R$, the following conditions hold:*

(a) The h-line $(PQ)_h$, h-half-line $[PQ)_h$, and h-segment $[PQ]_h$ are transformed into $(P'Q')_h$, $[P'Q')_h$, and $[P'Q']_h$ respectively.

(b) $\delta(P',Q') = \delta(P,Q)$ and $P'Q'_h = PQ_h$.

(c) $\measuredangle_h P'Q'R' \equiv -\measuredangle_h PQR$.

It is instructive to compare this observation with Proposition 5.6.

Proof. According to Theorem 10.15, the map sends the absolute to itself. Note that the points on Γ do not move, it follows that points inside of the absolute remain inside after the mapping. Whence the $X \mapsto X'$ is a bijection from the h-plane to itself.

Part *(a)* follows from 10.7 and 10.25.

Part *(b)* follows from Theorem 10.6.

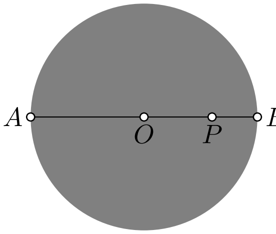
Part *(c)* follows from Theorem 10.25. □

12.8. Lemma. *Assume that the absolute is a unit circle centered at O. Given an h-point P, set $x = OP$ and $y = OP_h$. Then*

$$y = \ln \frac{1+x}{1-x} \qquad \textit{and} \qquad x = \frac{e^y - 1}{e^y + 1}.$$

Observe that according to lemma, $OP_h \to \infty$ as $OP \to 1$. That is if P approaches absolute in Euclidean sense, it escapes to infinity in the h-sense.

Proof. Note that the h-line $(OP)_h$ forms a diameter of the absolute. If A and B are the ideal points as in the definition of h-distance, then



$$\begin{aligned} OA &= OB = 1, \\ PA &= 1 + x, \\ PB &= 1 - x. \end{aligned}$$

In particular,

$$y = \ln \frac{AP \cdot BO}{PB \cdot OA} = \ln \frac{1+x}{1-x}.$$

Taking the exponential function of the left and the right-hand side and applying obvious algebra manipulations, we get that

$$x = \frac{e^y - 1}{e^y + 1}.$$

□

12.9. Lemma. *Assume the points P, Q, and R appear on one h-line in the same order. Then*

$$PQ_h + QR_h = PR_h.$$

Proof. Note that

$$PQ_h + QR_h = PR_h$$

is equivalent to

$$\text{❶} \qquad \delta(P,Q)\cdot\delta(Q,R) = \delta(P,R).$$

Let A and B be the ideal points of $(PQ)_h$. Without loss of generality, we can assume that the points A, P, Q, R, and B appear in the same order on the circline containing $(PQ)_h$. Then

$$\begin{aligned}\delta(P,Q)\cdot\delta(Q,R) &= \frac{AQ\cdot BP}{QB\cdot PA}\cdot\frac{AR\cdot BQ}{RB\cdot QA} = \\ &= \frac{AR\cdot BP}{RB\cdot PA} = \\ &= \delta(P,R).\end{aligned}$$

Hence ❶ follows. □

Let P be an h-point and $\rho > 0$. The set of all h-points Q such that $PQ_h = \rho$ is called an *h-circle* with the center P and the *h-radius* ρ.

12.10. Lemma. *Any h-circle is a Euclidean circle that lies completely in the h-plane.*

More precisely for any h-point P and $\rho \geqslant 0$ there is a $\hat\rho \geqslant 0$ and a point $\hat P$ such that

$$PQ_h = \rho \qquad \Longleftrightarrow \qquad \hat PQ = \hat\rho$$

for any h-point Q.

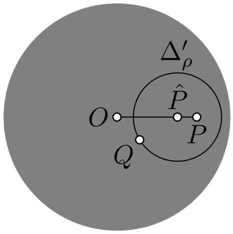
Moreover, if O is the center of the absolute, then

1. *$\hat O = O$ for any ρ and*
2. *$\hat P \in (OP)$ for any $P \neq O$.*

Proof. According to Lemma 12.8, $OQ_h = \rho$ if and only if

$$OQ = \hat\rho = \frac{e^\rho - 1}{e^\rho + 1}.$$

Therefore, the locus of h-points Q such that $OQ_h = \rho$ is a Euclidean circle, denote it by Δ_ρ.



If $P \neq O$, then by Lemma 12.6 and the main observation (12.7) there is an inversion that respects all h-notions and sends $O \mapsto P$.

Let Δ'_ρ be the inverse of Δ_ρ. Since the inversion preserves the h-distance, $PQ_h = \rho$ if and only if $Q \in \Delta'_\rho$.

According to Theorem 10.7, Δ'_ρ is a Euclidean circle. Let $\hat P$ and $\hat\rho$ denote the Euclidean center and radius of Δ'_ρ.

Finally, note that Δ'_ρ reflects to itself across (OP); that is, the center $\hat{P}$ lies on (OP). □

12.11. Exercise. *Assume P, $\hat{P}$, and O are as in the Lemma 12.10 and $P \neq O$. Show that $\hat{P} \in [OP]$.*

Axiom I

Evidently, the h-plane contains at least two points. Therefore, to show that Axiom I holds in the h-plane, we need to show that the h-distance defined on page 91 is a metric on h-plane; that is, the conditions *(a)–(d)* in Definition 1.1 hold for h-distance.

The following claim says that the h-distance meets the conditions *(a)* and *(b)*.

12.12. Claim. *Given the h-points P and Q, we have $PQ_h \geqslant 0$ and $PQ_h = 0$ if and only if $P = Q$.*

Proof. According to Lemma 12.6 and the main observation (12.7), we may assume that Q is the center of the absolute. In this case

$$\delta(Q,P) = \frac{1+QP}{1-QP} \geqslant 1$$

and therefore

$$QP_h = \ln[\delta(Q,P)] \geqslant 0.$$

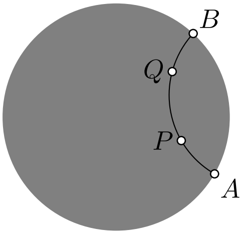
Moreover, the equalities hold if and only if $P = Q$. □

The following claim says that the h-distance meets Condition 1.1*c*.

12.13. Claim. *For any h-points P and Q, we have $PQ_h = QP_h$.*

Proof. Let A and B be ideal points of $(PQ)_h$ and A, P, Q, B appear on the circline containing $(PQ)_h$ in the same order.

Then



$$\begin{aligned} PQ_h &= \ln\frac{AQ{\cdot}BP}{QB{\cdot}PA} = \\ &= \ln\frac{BP{\cdot}AQ}{PA{\cdot}QB} = \\ &= QP_h. \end{aligned}$$

□

The following claim shows, in particular, that the triangle inequality (which is condition 1.1*d*) holds for *h*-distance.

12.14. Claim. *Given a triple of h-points P, Q, and R, we have*

$$PQ_h + QR_h \geqslant PR_h.$$

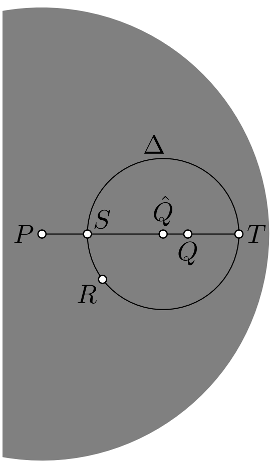
Moreover, the equality holds if and only if P, Q, and R lie on one h-line in the same order.

Proof. Without loss of generality, we may assume that P is the center of the absolute and $PQ_h \geqslant QR_h > 0$.

Suppose that Δ denotes the h-circle with the center Q and h-radius $\rho = QR_h$. Let S and T be the points of intersection of (PQ) and Δ.

By Lemma 12.9, $PQ_h \geqslant QR_h$. Therefore, we can assume that the points P, S, Q, and T appear on the h-line in the same order.

According to Lemma 12.10, Δ is a Euclidean circle; suppose that $\hat{Q}$ denotes its Euclidean center. Note that $\hat{Q}$ is the Euclidean midpoint of $[ST]$.



By the Euclidean triangle inequality,

$$PT = P\hat{Q} + \hat{Q}R \geqslant PR \qquad ❷$$

and the equality holds if and only if $T = R$.

By Lemma 12.8,

$$PT_h = \ln\frac{1+PT}{1-PT},$$
$$PR_h = \ln\frac{1+PR}{1-PR}.$$

Since the function $f(x) = \ln\frac{1+x}{1-x}$ is increasing for $x \in [0,1)$, inequality ❷ implies

$$PT_h \geqslant PR_h$$

and the equality holds if and only if $T = R$.

Finally, applying Lemma 12.9 again, we get that

$$PT_h = PQ_h + QR_h.$$

Hence the claim follows. □

Axiom II

Note that once the following claim is proved, Axiom II follows from Corollary 10.18.

12.15. Claim. *A subset of the h-plane is an h-line if and only if it forms a line for the h-distance in the sense of Definition 1.9.*

Proof. Let ℓ be an h-line. Applying the main observation (12.7) we can assume that ℓ contains the center of the absolute. In this case, ℓ is an intersection of a diameter of the absolute and the h-plane. Let A and B be the endpoints of the diameter.

Consider the map $\iota\colon \ell \to \mathbb{R}$ defined as

$$\iota(X) = \ln \frac{AX}{XB}.$$

Note that $\iota\colon \ell \to \mathbb{R}$ is a bijection.

Further, if $X, Y \in \ell$ and the points A, X, Y, and B appear on $[AB]$ in the same order, then

$$\iota(Y) - \iota(X) = \ln \frac{AY}{YB} - \ln \frac{AX}{XB} = \ln \frac{AY{\cdot}BX}{YB{\cdot}XB} = XY_h.$$

We proved that any h-line is a line for h-distance. The converse follows from Claim 12.14. □

Axiom III

Note that the first part of Axiom III follows directly from the definition of the h-angle measure defined on page 91. It remains to show that $\measuredangle_h$ satisfies the conditions IIIa, IIIb, and IIIc on page 19.

The following two claims say that $\measuredangle_h$ satisfies IIIa and IIIb.

12.16. Claim. *Given an h-half-line $[OP)_h$ and $\alpha \in (-\pi, \pi]$, there is a unique h-half-line $[OQ)_h$ such that $\measuredangle_h POQ = \alpha$.*

12.17. Claim. *For any h-points P, Q, and R distinct from an h-point O, we have*

$$\measuredangle_h POQ + \measuredangle_h QOR \equiv \measuredangle_h POR.$$

Proof of 12.16 and 12.17. Applying the main observation, we may assume that O is the center of the absolute. In this case, for any h-point $P \ne O$, the h-half-line $[OP)_h$ is the intersection of the Euclidean half-line $[OP)$ with h-plane. Hence the claims 12.16 and 12.17 follow from the axioms IIIa and IIIb of the Euclidean plane. □

The following claim says that $\measuredangle_h$ satisfies IIIc.

12.18. Claim. *The function*

$$\measuredangle_h\colon (P, Q, R) \mapsto \measuredangle_h PQR$$

is continuous at any triple of points (P, Q, R) *such that* $Q \neq P$, $Q \neq R$, *and* $\measuredangle_h PQR \neq \pi$.

Proof. Suppose that O denotes the center of the absolute. We can assume that Q is distinct from O.

Suppose that Z denotes the inverse of Q in the absolute; suppose that Γ denotes the circle perpendicular to the absolute and centered at Z. According to Lemma 12.6, the point O is the inverse of Q in Γ.

Let P' and R' denote the inversions in Γ of the points P and R respectively. Note that the point P' is completely determined by Q and P. Moreover, the map $(Q, P) \mapsto P'$ is continuous at any pair of points (Q, P) such that $Q \neq O$. The same is true for the map $(Q, R) \mapsto R'$

According to the main observation

$$\measuredangle_h PQR \equiv -\measuredangle_h P'OR'.$$

Since $\measuredangle_h P'OR' = \measuredangle P'OR'$ and the maps $(Q, P) \mapsto P'$, $(Q, R) \mapsto R'$ are continuous, the claim follows from the corresponding axiom of the Euclidean plane. □

Axiom IV

The following claim says that Axiom IV holds in the h-plane.

12.19. Claim. *In the h-plane, we have* $\triangle_h PQR \cong \triangle_h P'Q'R'$ *if and only if*

$$Q'P'_h = QP_h, \qquad Q'R'_h = QR_h \qquad \text{and} \qquad \measuredangle_h P'Q'R' = \pm\measuredangle PQR.$$

Proof. Applying the main observation, we can assume that Q and Q' coincide with the center of the absolute; in particular, $Q = Q'$. In this case,

$$\measuredangle P'QR' = \measuredangle_h P'QR' = \pm\measuredangle_h PQR = \pm\measuredangle PQR.$$

Since

$$QP_h = QP'_h \quad \text{and} \quad QR_h = QR'_h,$$

Lemma 12.8 implies that the same holds for the Euclidean distances; that is,

$$QP = QP' \quad \text{and} \quad QR = QR'.$$

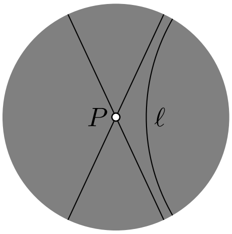
By SAS, there is a motion of the Euclidean plane that sends Q to itself, P to P', and R to R'.

Note that the center of the absolute is fixed by the corresponding motion. It follows that this motion gives also a motion of the h-plane; in particular, the h-triangles $\triangle_h PQR$ and $\triangle_h P'QR'$ are h-congruent. □

Axiom h-V

Finally, we need to check that the Axiom h-V on page 86 holds; that is, we need to prove the following claim.

12.20. Claim. *For any h-line ℓ and any h-point $P \notin \ell$ there are at least two h-lines that pass thru P and have no points of intersection with ℓ.*



Instead of proof. Applying the main observation we can assume that P is the center of the absolute.

The remaining part of the proof can be guessed from the picture. □

12.21. Exercise. *Show that in the h-plane there are 3 mutually parallel h-lines such that any pair of these three lines lies on one side of the remaining h-line.*

Hyperbolic trigonometry

In this section, we give formulas for h-distance using *hyperbolic functions.* One of these formulas will be used in the proof of the hyperbolic Pythagorean theorem (13.13).

Recall that ch, sh, and th denote *hyperbolic cosine*, *hyperbolic sine*, and *hyperbolic tangent*; that is, the functions defined by

$$\operatorname{ch} x := \tfrac{e^x+e^{-x}}{2}, \quad \operatorname{sh} x := \tfrac{e^x-e^{-x}}{2},$$

$$\operatorname{th} x := \tfrac{\operatorname{sh} x}{\operatorname{ch} x}.$$

These hyperbolic functions are analogous to sine and cosine and tangent.

12.22. Exercise. *Prove the following identities:*

$$\operatorname{ch}' x = \operatorname{sh} x; \quad \operatorname{sh}' x = \operatorname{ch} x; \quad (\operatorname{ch} x)^2 - (\operatorname{sh} x)^2 = 1.$$

12.23. Double-argument identities. *The identities*

$$\operatorname{ch}(2\cdot x) = (\operatorname{ch} x)^2 + (\operatorname{sh} x)^2 \qquad \textit{and} \qquad \operatorname{sh}(2\cdot x) = 2\cdot \operatorname{sh} x\cdot \operatorname{ch} x$$

hold for any real value x.

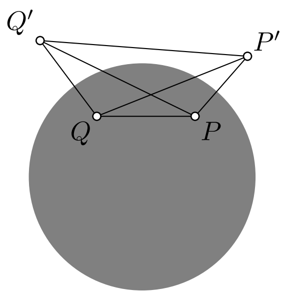
Proof.

$$\begin{aligned}(\operatorname{sh} x)^2 + (\operatorname{ch} x)^2 &= (\tfrac{e^x - e^{-x}}{2})^2 + (\tfrac{e^x + e^{-x}}{2})^2 = \\ &= \tfrac{e^{2\cdot x} + e^{-2\cdot x}}{2} = \\ &= \operatorname{ch}(2\cdot x); \\ 2\cdot \operatorname{sh} x \cdot \operatorname{ch} x &= 2\cdot(\tfrac{e^x - e^{-x}}{2})\cdot(\tfrac{e^x + e^{-x}}{2}) = \\ &= \tfrac{e^{2\cdot x} - e^{-2\cdot x}}{2} = \\ &= \operatorname{sh}(2\cdot x).\end{aligned}$$

□

12.24. Advanced exercise. *Let P and Q be two h-points distinct from the center of absolute. Denote by P' and Q' the inverses of P and Q in the absolute.*

Show that



(a) $\operatorname{ch}[\tfrac{1}{2}\cdot PQ_h] = \sqrt{\dfrac{PQ'\cdot P'Q}{PP'\cdot QQ'}}$;

(b) $\operatorname{sh}[\tfrac{1}{2}\cdot PQ_h] = \sqrt{\dfrac{PQ\cdot P'Q'}{PP'\cdot QQ'}}$;

(c) $\operatorname{th}[\tfrac{1}{2}\cdot PQ_h] = \sqrt{\dfrac{PQ\cdot P'Q'}{PQ'\cdot P'Q}}$;

(d) $\operatorname{ch} PQ_h = \dfrac{PQ\cdot P'Q' + PQ'\cdot P'Q}{PP'\cdot QQ'}$.

Chapter 13

Geometry of the h-plane

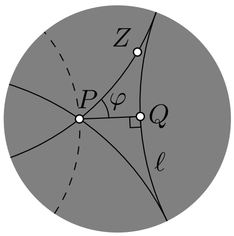
In this chapter, we study the geometry of the plane described by the conformal disc model. For briefness, this plane will be called the *h-plane*.

We can work with this model directly from inside of the Euclidean plane. We may also use the axioms of neutral geometry since they all hold in the h-plane; the latter proved in the previous chapter.

Angle of parallelism

Let P be a point off an h-line ℓ. Drop a perpendicular $(PQ)_h$ from P to ℓ; let Q be its foot point. Let φ be the smallest value such that the h-line $(PZ)_h$ with $|\measuredangle_h QPZ| = \varphi$ does not intersect ℓ.

The value φ is called the *angle of parallelism* of P to ℓ. Clearly, φ depends only on the h-distance $s = PQ_h$. Further, $\varphi(s) \to \pi/2$ as $s \to 0$, and $\varphi(s) \to 0$ as $s \to \infty$. (In Euclidean geometry, the angle of parallelism is identically equal to $\pi/2$.)



If ℓ, P, and Z are as above, then the h-line $m = (PZ)_h$ is called *asymptotically parallel* to ℓ. In other words, two h-lines are asymptotically parallel if they share one ideal point. (In hyperbolic geometry, the term *parallel lines* is often used for *asymptotically parallel lines*; we do not follow this convention.)

Given $P \notin \ell$, there are exactly two asymptotically parallel lines thru P to ℓ; the remaining parallel lines are called *ultra parallel.*

On the diagram, the two solid h-lines passing thru P are asymptotically parallel to ℓ; the dashed h-line is ultra parallel to ℓ.

13.1. Exercise. *Show that two distinct h-lines ℓ and m are ultraparallel if and only if they have a* common perpendicular*; that is, there is an h-line n such that $n \perp \ell$ and $n \perp m$.*

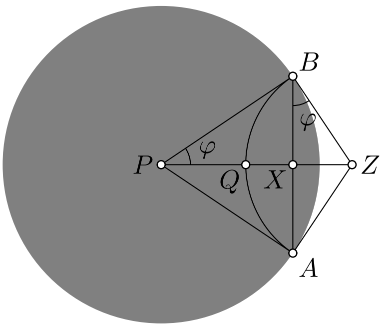
13.2. Proposition. *Let Q be the foot point of P on h-line ℓ. Then*

$$PQ_h = \tfrac{1}{2}\cdot \ln \frac{1+\cos\varphi}{1-\cos\varphi},$$

where φ is the angle of parallelism of P to ℓ.

In particular, if $P \notin \ell$ and $\beta = |\measuredangle_h XPY|$ for some points $X, Y \in \ell$, then

$$PQ_h < \tfrac{1}{2}\cdot \ln \frac{1+\cos\frac{\beta}{2}}{1-\cos\frac{\beta}{2}}.$$



Proof. Applying a motion of the h-plane if necessary, we may assume P is the center of the absolute. Then the h-lines thru P are the intersections of Euclidean lines with the h-plane.

Let A and B denote the ideal points of ℓ. Without loss of generality, we may assume that $\angle APB$ is positive. In this case,

$$\varphi = \measuredangle QPB = \measuredangle APQ = \tfrac{1}{2}\cdot\measuredangle APB.$$

Let Z be the center of the circle Γ containing the h-line ℓ. Set X to be the point of intersection of the Euclidean segment $[AB]$ and the line (PQ).

Note that, $PX = \cos\varphi$. Therefore, by Lemma 12.8,

$$PX_h = \ln \tfrac{1+\cos\varphi}{1-\cos\varphi}.$$

Note that both angles PBZ and BXZ are right. Since the angle PZB is shared, $\triangle ZBX \sim \triangle ZPB$. In particular,

$$ZX\cdot ZP = ZB^2;$$

that is, X is the inverse of P in Γ.

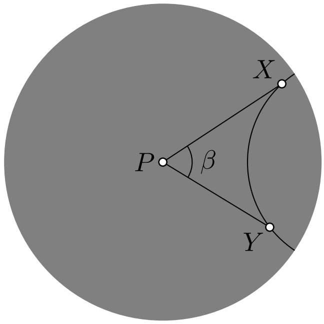
The inversion in Γ is the reflection of the h-plane across ℓ. Therefore

$$\begin{aligned} PQ_h &= QX_h = \\ &= \tfrac{1}{2}\cdot PX_h = \\ &= \tfrac{1}{2}\cdot \ln \tfrac{1+\cos\varphi}{1-\cos\varphi}. \end{aligned}$$

The last statement follows since $\varphi > \frac{\beta}{2}$ and the function

$$\varphi \mapsto \tfrac{1}{2}\cdot \ln \tfrac{1+\cos\varphi}{1-\cos\varphi}$$

is decreasing in the interval $(0, \frac{\pi}{2}]$. □



13.3. Exercise. *Let ABC be an equilateral h-triangle with side 100. Show that*

$$|\measuredangle_h ABC| < \frac{1}{10\,000\,000\,000}.$$

Inradius of h-triangle

13.4. Theorem. *The inradius of any h-triangle is less than $\frac{1}{2}\cdot\ln 3$.*

Proof. Let I and r be the h-incenter and h-inradius of $\triangle_h XYZ$.

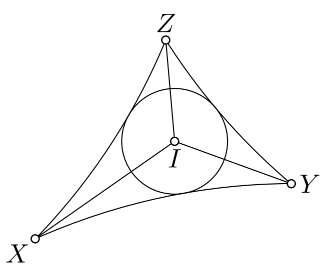
Note that the h-angles XIY, YIZ and ZIX have the same sign. Without loss of generality, we can assume that all of them are positive and therefore

$$\measuredangle_h XIY + \measuredangle_h YIZ + \measuredangle_h ZIX = 2\cdot\pi$$

We can assume that $\measuredangle_h XIY \geqslant \frac{2}{3}\cdot\pi$; if not relabel X, Y, and Z.

Since r is the h-distance from I to $(XY)_h$, Proposition 13.2 implies that

$$\begin{aligned} r &< \tfrac{1}{2}\cdot \ln \tfrac{1+\cos\frac{\pi}{3}}{1-\cos\frac{\pi}{3}} = \\ &= \tfrac{1}{2}\cdot \ln \frac{1+\frac{1}{2}}{1-\frac{1}{2}} = \\ &= \tfrac{1}{2}\cdot \ln 3. \end{aligned}$$



□

13.5. Exercise. *Let $\square_h ABCD$ be a quadrangle in the h-plane such that the h-angles at A, B, and C are right and $AB_h = BC_h$. Find the optimal upper bound for AB_h.*

Circles, horocycles, and equidistants

Note that according to Lemma 12.10, any h-circle is a Euclidean circle that lies completely in the h-plane. Further, any h-line is an intersection of the h-plane with the circle perpendicular to the absolute.

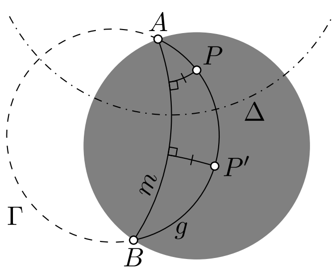
In this section, we will describe the h-geometric meaning of the intersections of the other circles with the h-plane.

You will see that all these intersections have a *perfectly round shape* in the h-plane.

One may think of these curves as trajectories of a car with a fixed position of the steering wheel. In the Euclidean plane, this way you either run along a circle or along a line.

In the hyperbolic plane, the picture is different. If you turn the steering wheel to the far right, you will run along a circle. If you turn it less, at a certain position of the wheel, you will never come back to the same point, but the path will be different from the line. If you turn the wheel further a bit, you start to run along a path that stays at some fixed distance from an h-line.

Equidistants of h-lines. Consider the h-plane with the absolute Ω. Assume a circle Γ intersects Ω in two distinct points, A and B. Suppose that g denotes the intersection of Γ with the h-plane.



Let us draw an h-line m with the ideal points A and B. According to Exercise 12.1, m is uniquely defined.

Consider any h-line ℓ perpendicular to m; let Δ be the circle containing ℓ.

Note that $\Delta \perp \Gamma$. Indeed, according to Corollary 10.16, m and Ω invert to themselves in Δ. It follows that A is the inverse of B in Δ. Finally, by Corollary 10.17, we get that $\Delta \perp \Gamma$.

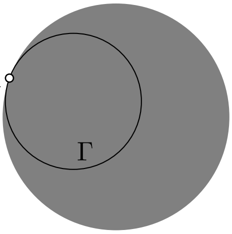
Therefore, inversion in Δ sends both m and g to themselves. For any two points $P', P \in g$ there is a choice of ℓ and Δ as above such that P' is the inverse of P in Δ. By the main observation (12.7) the inversion in Δ is a motion of the h-plane. Therefore, all points of g lie at the same distance from m.

In other words, g is the set of points that lie at a fixed h-distance and on the same side of m.

Such a curve g is called *equidistant* to h-line m. In Euclidean geometry, the equidistant from a line is a line; apparently, in hyperbolic geometry, the picture is different.

Horocycles. If the circle Γ touches the absolute from inside at one point A, then the complement $h = \Gamma \backslash \{A\}$ lies in the h-plane. This set is called a *horocycle*. It also has a perfectly round shape in the sense described above.

The shape of a horocycle is between shapes of circles and equidistants to h-lines. A horocycle might be considered as a limit of circles thru a fixed point with the centers running to infinity along a line. The same horocycle is a limit of equidistants thru a fixed point to sequence of h-lines that runs to infinity.



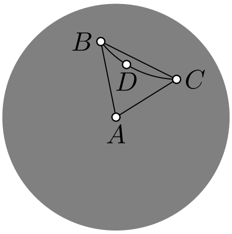
Since any three points lie on a circline, we have that any nondegenerate h-triangle is inscribed in an h-circle, horocycle, or equidistant.

13.6. Exercise. *Find the leg of an isosceles right h-triangle inscribed in a horocycle.*

Hyperbolic triangles

13.7. Theorem. *Any nondegenerate hyperbolic triangle has a positive defect.*

Proof. Fix an h-triangle ABC. According to Theorem 11.9,



$$\text{❶} \qquad \operatorname{defect}(\triangle_h ABC) \geqslant 0.$$

It remains to show that in the case of equality, $\triangle_h ABC$ degenerates.

Without loss of generality, we may assume that A is the center of the absolute; in this case, $\measuredangle_h CAB = \measuredangle CAB$. Yet we may assume that

$$\measuredangle_h CAB, \quad \measuredangle_h ABC, \quad \measuredangle_h BCA, \quad \measuredangle ABC, \quad \measuredangle BCA \geqslant 0.$$

Let D be an arbitrary point in $[CB]_h$ distinct from B and C. From Proposition 9.23, we have

$$\measuredangle ABC - \measuredangle_h ABC \equiv \pi - \measuredangle CDB \equiv \measuredangle BCA - \measuredangle_h BCA.$$

From Exercise 7.14, we get that

$$\operatorname{defect}(\triangle_h ABC) = 2\cdot(\pi - \measuredangle CDB).$$

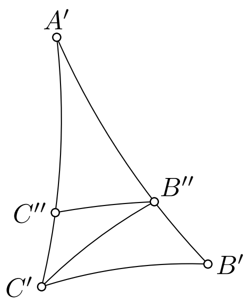
Therefore, if we have equality in ❶, then $\measuredangle CDB = \pi$. In particular, the h-segment $[BC]_h$ coincides with the Euclidean segment $[BC]$. By

Exercise 12.3, the latter can happen only if the h-line $(BC)_h$ passes thru the center of the absolute (A); that is, if $\triangle_h ABC$ degenerates. □

The following theorem states, in particular, that nondegenerate hyperbolic triangles are congruent if their corresponding angles are equal. In particular, in hyperbolic geometry, similar triangles have to be congruent.

13.8. AAA congruence condition. *Two nondegenerate h-triangles ABC and $A'B'C'$ are congruent if $\measuredangle_h ABC = \pm\measuredangle_h A'B'C'$, $\measuredangle_h BCA = \pm\measuredangle_h B'C'A'$ and $\measuredangle_h CAB = \pm\measuredangle_h C'A'B'$.*

Proof. Note that if $AB_h = A'B'_h$, then the theorem follows from ASA.



Assume the contrary. Without loss of generality, we may assume that $AB_h < A'B'_h$. Therefore, we can choose the point $B'' \in [A'B']_h$ such that $A'B''_h = AB_h$.

Choose an h-half-line $[B''X)$ so that

$$\measuredangle_h A'B''X = \measuredangle_h A'B'C'.$$

According to Exercise 11.5, $(B''X)_h \parallel (B'C')_h$.

By Pasch's theorem (3.12), $(B''X)_h$ intersects $[A'C']_h$. Suppose that C'' denotes the point of intersection.

According to ASA, $\triangle_h ABC \cong \triangle_h A'B''C''$; in particular,

❷ $$\mathrm{defect}(\triangle_h ABC) = \mathrm{defect}(\triangle_h A'B''C'').$$

Applying Exercise 11.11 twice, we get that

❸ $$\begin{aligned}\mathrm{defect}(\triangle_h A'B'C') = \mathrm{defect}(\triangle_h A'B''C'')&+\\ + \mathrm{defect}(\triangle_h B''C''C') &+ \mathrm{defect}(\triangle_h B''C'B').\end{aligned}$$

By Theorem 13.7, all the defects have to be positive. Therefore

$$\mathrm{defect}(\triangle_h A'B'C') > \mathrm{defect}(\triangle_h ABC).$$

On the other hand,

$$\begin{aligned}\mathrm{defect}(\triangle_h A'B'C') &= |\measuredangle_h A'B'C'| + |\measuredangle_h B'C'A'| + |\measuredangle_h C'A'B'| =\\ &= |\measuredangle_h ABC| + |\measuredangle_h BCA| + |\measuredangle_h CAB| =\\ &= \mathrm{defect}(\triangle_h ABC)\end{aligned}$$

— a contradiction. □

Recall that a bijection from an h-plane to itself is called *angle preserving* if

$$\measuredangle_h ABC = \measuredangle_h A'B'C'$$

for any $\triangle_h ABC$ and its image $\triangle_h A'B'C'$.

13.9. Exercise. *Show that any angle-preserving transformation of the h-plane is a motion.*

Conformal interpretation

Let us give another interpretation of the h-distance.

13.10. Lemma. *Consider the h-plane with the unit circle centered at O as the absolute. Fix a point P and let Q be another point in the h-plane. Set $x = PQ$ and $y = PQ_h$. Then*

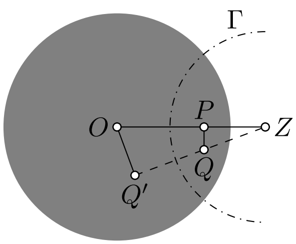
$$\lim_{x\to 0}\frac{y}{x} = \frac{2}{1 - OP^2}.$$

The above formula tells us that the h-distance from P to a nearby point Q is almost proportional to the Euclidean distance with the coefficient $\frac{2}{1-OP^2}$. The value $\lambda(P) = \frac{2}{1-OP^2}$ is called the *conformal factor* of the h-metric.

The value $\frac{1}{\lambda(P)} = \frac{1}{2}\cdot(1 - OP^2)$ can be interpreted as the *speed limit* at the given point P. In this case, the h-distance is the minimal time needed to travel from one point of the h-plane to another point.

Proof. If $P = O$, then by Lemma 12.8

❹ $$\frac{y}{x} = \frac{\ln\frac{1+x}{1-x}}{x} \to 2$$



as $x \to 0$.

If $P \ne O$, let Z denotes the inverse of P in the absolute. Suppose that Γ denotes the circle with the center Z perpendicular to the absolute.

According to the main observation (12.7) and Lemma 12.6, the inversion in Γ is a motion of the h-plane which sends P to O. In particular, if Q' denotes the inverse of Q in Γ, then $OQ'_h = PQ_h$.

Set $x' = OQ'$. According to Lemma 10.2,

$$\frac{x'}{x} = \frac{OZ}{ZQ}.$$

Since Z is the inverse of P in the absolute, we have that $PO\cdot OZ = 1$. Therefore,

$$\frac{x'}{x} \to \frac{OZ}{ZP} = \frac{1}{1 - OP^2}$$

as $x \to 0$.

According to ❹, $\frac{y}{x'} \to 2$ as $x' \to 0$. Therefore

$$\frac{y}{x} = \frac{y}{x'} \cdot \frac{x'}{x} \to \frac{2}{1 - OP^2}$$

as $x \to 0$. □

Here is an application of the lemma above.

13.11. Proposition. *The circumference of an h-circle of the h-radius r is*

$$2 \cdot \pi \cdot \operatorname{sh} r,$$

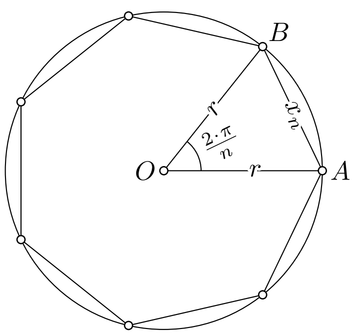
where $\operatorname{sh} r$ *denotes the* hyperbolic sine *of r; that is,*

$$\operatorname{sh} r := \frac{e^r - e^{-r}}{2}.$$

Before we proceed with the proof, let us discuss the same problem in the Euclidean plane.

The circumference of a circle in the Euclidean plane can be defined as the limit of perimeters of regular n-gons inscribed in the circle as $n \to \infty$.

Namely, let us fix $r > 0$. Given a positive integer n, consider $\triangle AOB$ such that $\measuredangle AOB = \frac{2 \cdot \pi}{n}$ and $OA = OB = r$. Set $x_n = AB$. Note that x_n is the side of a regular n-gon inscribed in the circle of radius r. Therefore, the perimeter of the n-gon is $n \cdot x_n$.



The circumference of the circle with the radius r might be defined as the limit

❺ $$\lim_{n \to \infty} n \cdot x_n = 2 \cdot \pi \cdot r.$$

(This limit can be taken as the definition of π.)

In the following proof, we repeat the same construction in the h-plane.

Proof. Without loss of generality, we can assume that the center O of the circle is the center of the absolute.

By Lemma 12.8, the h-circle with the h-radius r is the Euclidean circle with the center O and the radius

$$a = \frac{e^r - 1}{e^r + 1}.$$

Let x_n and y_n denote the side lengths of the regular n-gons inscribed in the circle in the Euclidean and hyperbolic plane respectively.

Note that $x_n \to 0$ as $n \to \infty$. By Lemma 13.10,

$$\lim_{n\to\infty} \frac{y_n}{x_n} = \frac{2}{1-a^2}.$$

Applying ❺, we get that the circumference of the h-circle can be found the following way:

$$\begin{aligned}\lim_{n\to\infty} n{\cdot}y_n &= \frac{2}{1-a^2}\cdot \lim_{n\to\infty} n{\cdot}x_n = \\ &= \frac{4{\cdot}\pi{\cdot}a}{1-a^2} = \\ &= \frac{4{\cdot}\pi{\cdot}\left(\frac{e^r-1}{e^r+1}\right)}{1-\left(\frac{e^r-1}{e^r+1}\right)^2} = \\ &= 2{\cdot}\pi{\cdot}\frac{e^r-e^{-r}}{2} = \\ &= 2{\cdot}\pi{\cdot}\operatorname{sh} r.\end{aligned}$$

□

13.12. Exercise. *Let* $\operatorname{circum}_h(r)$ *denote the circumference of the h-circle of the h-radius* r*. Show that*

$$\operatorname{circum}_h(r+1) > 2{\cdot}\operatorname{circum}_h(r)$$

for all $r > 0$*.*

Pythagorean theorem

Recall that ch denotes *hyperbolic cosine*; that is, the function defined by

$$\operatorname{ch} x := \tfrac{e^x+e^{-x}}{2}.$$

13.13. Hyperbolic Pythagorean theorem. *Assume that* ACB *is an h-triangle with right angle at* C*. Set*

$$a = BC_h, \quad b = CA_h \quad \textit{and} \quad c = AB_h.$$

Then

❻ $$\operatorname{ch} c = \operatorname{ch} a{\cdot}\operatorname{ch} b.$$

The formula ❻ will be proved by means of direct calculations. Before giving the proof, let us discuss the limit cases of this formula.

Note that $\operatorname{ch} x$ can be written using the Taylor expansion

$$\operatorname{ch} x = 1 + \tfrac{1}{2}\cdot x^2 + \tfrac{1}{24}\cdot x^4 + \dots.$$

It follows that if a, b, and c are small, then

$$\begin{aligned} 1 + \tfrac{1}{2}\cdot c^2 \approx \operatorname{ch} c = \operatorname{ch} a\cdot \operatorname{ch} b &\approx \\ &\approx (1 + \tfrac{1}{2}\cdot a^2)\cdot(1 + \tfrac{1}{2}\cdot b^2) \approx \\ &\approx 1 + \tfrac{1}{2}\cdot(a^2 + b^2). \end{aligned}$$

In other words, the original Pythagorean theorem (6.4) is a limit case of the hyperbolic Pythagorean theorem for small triangles.

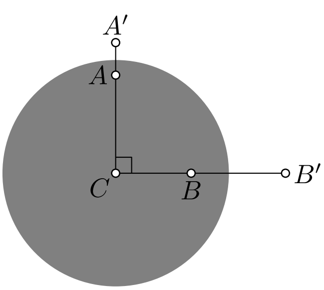
For large a and b the terms e^{-a}, e^{-b}, and $e^{-a-b+\ln 2}$ are neglectable. In this case, we have the following approximations:

$$\begin{aligned} \operatorname{ch} a\cdot \operatorname{ch} b \approx \tfrac{e^a}{2}\cdot\tfrac{e^b}{2} &= \\ &= \frac{e^{a+b-\ln 2}}{2} \approx \\ &\approx \operatorname{ch}(a + b - \ln 2). \end{aligned}$$

Therefore $c \approx a + b - \ln 2$.

13.14. Exercise. *Assume that ACB is an h-triangle with right angle at C. Set $a = BC_h$, $b = CA_h$, and $c = AB_h$. Show that*

$$c + \ln 2 > a + b.$$



In the proof of the hyperbolic Pythagorean theorem, we use the following formula from Exercise 12.24:

$$\operatorname{ch} AB_h = \frac{AB\cdot A'B' + AB'\cdot A'B}{AA'\cdot BB'},$$

here A, B are h-points distinct from the center of absolute and A', B' are their inversions in the absolute. This formula is derived in the hints.

Proof of 13.13. We assume that absolute is a unit circle. By the main observation (12.7) we can assume that C is the center of absolute. Let A' and B' denote the inverses of A and B in the absolute.

Set $x = BC$, $y = AC$. By Lemma 12.8

$$a = \ln \tfrac{1+x}{1-x}, \qquad b = \ln \tfrac{1+y}{1-y}.$$

Therefore

❼

$$\operatorname{ch} a = \tfrac{1}{2}\cdot\left(\tfrac{1+x}{1-x} + \tfrac{1-x}{1+x}\right) = \frac{1+x^2}{1-x^2}, \qquad \operatorname{ch} b = \tfrac{1}{2}\cdot\left(\tfrac{1+y}{1-y} + \tfrac{1-y}{1+y}\right) = \frac{1+y^2}{1-y^2}.$$

Note that

$$B'C = \tfrac{1}{x}, \qquad A'C = \tfrac{1}{y}.$$

Therefore

$$BB' = \tfrac{1}{x} - x, \qquad AA' = \tfrac{1}{y} - y.$$

Since the triangles ABC, $A'BC$, $AB'C$, $A'B'C$ are right, the original Pythagorean theorem (6.4) implies

$$AB = \sqrt{x^2 + y^2}, \qquad AB' = \sqrt{\tfrac{1}{x^2} + y^2},$$
$$A'B = \sqrt{x^2 + \tfrac{1}{y^2}}, \qquad A'B' = \sqrt{\tfrac{1}{x^2} + \tfrac{1}{y^2}}.$$

According to Exercise 12.24,

❽

$$\begin{aligned}\operatorname{ch} c &= \frac{AB\cdot A'B' + AB'\cdot A'B}{AA'\cdot BB'} = \\ &= \frac{\sqrt{x^2+y^2}\cdot\sqrt{\tfrac{1}{x^2}+\tfrac{1}{y^2}} + \sqrt{\tfrac{1}{x^2}+y^2}\cdot\sqrt{x^2+\tfrac{1}{y^2}}}{(\tfrac{1}{y}-y)\cdot(\tfrac{1}{x}-x)} = \\ &= \frac{x^2+y^2+1+x^2\cdot y^2}{(1-y^2)\cdot(1-x^2)} = \\ &= \frac{1+x^2}{1-x^2}\cdot\frac{1+y^2}{1-y^2}.\end{aligned}$$

Finally note that ❼ and ❽ imply ❻. □

Chapter 14

Affine geometry

Affine transformations

Affine geometry studies the so-called *incidence structure* of the Euclidean plane. The incidence structure sees only which points lie on which lines and nothing else; it does not directly see distances, angle measures, and many other things.

A bijection from the Euclidean plane to itself is called *affine transformation* if it maps lines to lines; that is, the image of any line is a line. So we can say that affine geometry studies the properties of the Euclidean plane preserved under affine transformations.

14.1. Exercise. *Show that an affine transformation of the Euclidean plane sends any pair of parallel lines to a pair of parallel lines.*

The observation below follows since the lines are defined using the metric only.

14.2. Observation. *Any motion of the Euclidean plane is an affine transformation.*

The following exercise provides more general examples of affine transformations.

14.3. Exercise. *The following maps of a coordinate plane to itself are affine transformations:*

(a) Shear map defined by $(x,y)\mapsto(x+k\cdot y,y)$ *for a constant* k.
(b) Scaling defined by $(x,y)\mapsto(a\cdot x,a\cdot y)$ *for a constant* $a\neq 0$.
(c) x*-scaling and* y*-scaling defined respectively by*

$$(x,y)\mapsto(a\cdot x,y),\quad \text{and}\quad (x,y)\mapsto(x,a\cdot y)$$

for a constant $a\neq 0$.

(d) A transformation defined by

$$(x, y) \mapsto (a{\cdot}x + b{\cdot}y + r, c{\cdot}x + d{\cdot}y + s)$$

for constants a, b, c, d, r, z such that the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is invertible.

From the fundamental theorem of affine geometry (14.11), it will follow that any affine transformation can be written in the form *(d)*.

Recall that points are *collinear* if they lie on one line.

14.4. Exercise. *Suppose $P \mapsto P'$ is a bijection of the Euclidean plane that maps collinear triples of points to collinear triples. Show that $P \mapsto P'$ maps noncollinear triples to noncollinear.*

Conclude that $P \mapsto P'$ is an affine transformation.

Constructions

Let us consider geometric constructions with a ruler and a *parallel tool*; the latter makes it possible to draw a line thru a given point parallel to a given line. By Exercisers 14.1, any construction with these two tools is invariant with respect to affine transformations. For example, to solve the following exercise, it is sufficient to prove that the midpoint of a given segment can be constructed with a ruler and a parallel tool.

14.5. Exercise. *Let M be the midpoint of segment $[AB]$ in the Euclidean plane. Assume that an affine transformation sends the points A, B, and M to A', B', and M' respectively. Show that M' is the midpoint of $[A'B']$.*

The following exercise will be used in the proof of Proposition 14.10.

14.6. Exercise. *Assume that points with coordinates $(0,0)$, $(1,0)$, $(a,0)$, and $(b,0)$ are given. Using a ruler and a parallel tool, construct points with coordinates $(a{\cdot}b, 0)$ and $(a+b, 0)$.*

14.7. Exercise. *Use ruler and parallel tool to construct the center of a given circle.*

Note that the shear map (described in 14.3a) can change angles between lines almost arbitrarily. This observation can be used to prove the impossibility of some constructions; here is one example:

14.8. Exercise. *Show that with a ruler and a parallel tool one cannot construct a line perpendicular to a given line.*

Fundamental theorem of affine geometry

Further we assume knowledge of vector algebra; namely, multiplication by a real number, addition, and the parallelogram rule.

14.9. Exercise. *Show that affine transformations map parallelograms to parallelograms. Conclude that if $P \mapsto P'$ is an affine transformation, then*

$$\overrightarrow{XY} = \overrightarrow{AB}, \quad \textit{if and only if} \quad \overrightarrow{X'Y'} = \overrightarrow{A'B'}.$$

14.10. Proposition. *Let $P \mapsto P'$ be an affine transformation of the Euclidean plane. Then, for any triple of points O, X, P, we have*

❶ $$\overrightarrow{OP} = t{\cdot}\overrightarrow{OX} \quad \textit{if and only if} \quad \overrightarrow{O'P'} = t{\cdot}\overrightarrow{O'X'}.$$

Proof. Observe that the affine transformations described in Exercise 14.3, as well as all motions, satisfy the condition ❶. Therefore a given affine transformation $P \mapsto P'$ satisfies ❶ if and only if its composition with motions and scalings satisfies ❶.

Applying this observation, we can reduce the problem to its partial case. Namely, we may assume that $O' = O$, $X' = X$, the point O is the origin of a coordinate system, and X has coordinates $(1, 0)$.

In this case, $\overrightarrow{OP} = t{\cdot}\overrightarrow{OX}$ if and only if $P = (t, 0)$. Since O and X are fixed, the transformation maps the x-axis to itself. That is, $P' = (f(t), 0)$ for some function $t \mapsto f(t)$, or, equivalently, $\overrightarrow{O'P'} = f(t){\cdot}\overrightarrow{O'X'}$. It remains to show that

❷ $$f(t) = t$$

for any t.

Since $O' = O$ and $X' = X$, we get that $f(0) = 0$ and $f(1) = 1$. Further, according to Exercise 14.6, we have that $f(x{\cdot}y) = f(x){\cdot}f(y)$ and $f(x + y) = f(x) + f(y)$ for any $x, y \in \mathbb{R}$. By the algebraic lemma (proved below, see 14.14), these conditions imply ❷. □

14.11. Fundamental theorem of affine geometry. *Suppose an affine transformation maps a nondegenerate triangle OXY to a triangle $O'X'Y'$. Then $\triangle O'X'Y'$ is nondegenerate, and*

$$\overrightarrow{OP} = x{\cdot}\overrightarrow{OX} + y{\cdot}\overrightarrow{OY} \quad \textit{if and only if} \quad \overrightarrow{O'P'} = x{\cdot}\overrightarrow{O'X'} + y{\cdot}\overrightarrow{O'Y'}.$$

Proof. Since an affine transformation maps lines to lines, the triangle $O'X'Y'$ is nondegenerate.

If $x=0$ or $y=0$, then the second statement follows directly from the proposition. Otherwise, consider points V and W defined by

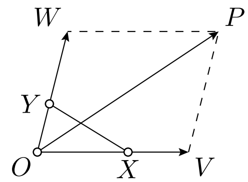
$$\overrightarrow{OV}=x\cdot\overrightarrow{OX},\quad \overrightarrow{OW}=y\cdot\overrightarrow{OY}.$$

By the proposition,

$$\overrightarrow{O'V'}=x\cdot\overrightarrow{O'X'},\quad \overrightarrow{O'W'}=y\cdot\overrightarrow{O'Y'}.$$

Note that

$$\overrightarrow{OP}=\overrightarrow{OV}+\overrightarrow{OW},$$



or, equivalently, $\square OVPW$ is a parallelogram. According to Exercise 14.1, $\square O'V'P'W'$ is a parallelogram as well. Therefore

$$\begin{aligned}\overrightarrow{O'P'}&=\overrightarrow{O'V'}+\overrightarrow{O'W'}=\\ &=x\cdot\overrightarrow{O'X'}+y\cdot\overrightarrow{O'Y'}.\end{aligned}$$

□

14.12. Exercise. *Show that any affine transformation is continuous.*

The following exercise provides the converse to Exercise 14.3*d*.

14.13. Exercise. *Show that any affine transformation can be written in coordinates as*

$$(x,y)\mapsto(a\cdot x+b\cdot y+r, c\cdot x+d\cdot y+s)$$

for constants a,b,c,d,r,s such that the matrix $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ is invertible.

Algebraic lemma

The following lemma was used in the proof of Proposition 14.10.

14.14. Lemma. *Assume $f\colon\mathbb{R}\to\mathbb{R}$ is a function such that for any $x,y\in\mathbb{R}$ we have*

(a) $f(1)=1$,
(b) $f(x+y)=f(x)+f(y)$,
(c) $f(x\cdot y)=f(x)\cdot f(y)$.

Then f is the identity function; that is, $f(x)=x$ for any $x\in\mathbb{R}$.

Note that we do not assume that f is continuous.

A function f satisfying these three conditions is called a *field automorphism*. Therefore, the lemma states that the identity function is the only automorphism of the field of real numbers. For the field of complex numbers, the conjugation $z \mapsto \bar{z}$ (see page 144) gives an example of a nontrivial automorphism.

Proof. By *(b)* we have

$$f(0) + f(1) = f(0+1).$$

By *(a)*,

$$f(0) + 1 = 1;$$

whence

❸ $$f(0) = 0.$$

Applying *(b)* again, we get that

$$0 = f(0) = f(x) + f(-x).$$

Therefore,

❹ $$f(-x) = -f(x) \quad \text{for any} \quad x \in \mathbb{R}.$$

Applying *(b)* recurrently, we get that

$$\begin{aligned} f(2) &= f(1) + f(1) = 1 + 1 = 2; \\ f(3) &= f(2) + f(1) = 2 + 1 = 3; \\ &\dots \end{aligned}$$

Together with ❹, the latter implies that

$$f(n) = n \quad \text{for any integer} \quad n.$$

By *(c)*

$$f(m) = f(\tfrac{m}{n}) {\cdot} f(n).$$

Therefore

❺ $$f(\tfrac{m}{n}) = \tfrac{m}{n}$$

for any rational number $\frac{m}{n}$.

Assume $a \geqslant 0$. Then the equation $x{\cdot}x = a$ has a real solution $x = \sqrt{a}$. Therefore, $[f(\sqrt{a})]^2 = f(\sqrt{a}){\cdot}f(\sqrt{a}) = f(a)$. Hence $f(a) \geqslant 0$. That is,

❻ $$a \geqslant 0 \quad \Longrightarrow \quad f(a) \geqslant 0.$$

Applying ❹, we also get

❼ $$a \leqslant 0 \quad \Longrightarrow \quad f(a) \leqslant 0.$$

Now assume $f(a) \neq a$ for some $a \in \mathbb{R}$. Then there is a rational number $\frac{m}{n}$ that lies between a and $f(a)$; that is, the numbers

$$x = a - \tfrac{m}{n} \quad \text{and} \quad y = f(a) - \tfrac{m}{n}$$

have opposite signs.

By ❺,

$$\begin{aligned} y + \tfrac{m}{n} &= f(a) = \\ &= f(x + \tfrac{m}{n}) = \\ &= f(x) + f(\tfrac{m}{n}) = \\ &= f(x) + \tfrac{m}{n}; \end{aligned}$$

that is, $f(x) = y$. By ❻ and ❼ the values x and y cannot have opposite signs — a contradiction. □

On inversive transformations

Recall that the inversive plane is the Euclidean plane with an added point at infinity, denoted by ∞. We assume that every line passes thru ∞. Recall that the term *circline* stands for *circle or line*.

An *inversive transformation* is a bijection from the inversive plane to itself that sends circlines to circlines. *Inversive geometry* studies the *circline incidence structure* of the inversive plane (it sees which points lie on which circlines and nothing else).

14.15. Theorem. *A map from the inversive plane to itself is an inversive transformation if and only if it can be presented as a composition of inversions and reflections.*

Exercise 18.16 gives another description of inversive transformations by means of complex coordinates.

Proof. Evidently, reflection is an inversive transformation — it maps lines to lines and circles to circles. According to Theorem 10.7, any inversion is an inversive transformation as well. Therefore, the same holds for any composition of inversions and reflections.

To prove the converse, fix an inversive transformation α.

Assume $\alpha(\infty) = \infty$. Recall that any circline passing thru ∞ is a line. If follows that α maps lines to lines; that is, α is an affine transformation that also maps circles to circles.

Note that any motion or scaling (defined in Exercise 14.3*b*) are affine transformations that map circles to circles. Composing α with motions and scalings, we can obtain another affine transformation α' that maps a given unit circle Γ to itself. By Exercise 14.7, α' fixes the center, say O, of the circle Γ.

Set $P' = \alpha'(P)$. It follows that if $OP = 1$, then $OP' = 1$. By Proposition 14.10, $OP = OP'$ for any point P. Finally, by Exercise 14.9, we have that if $\overrightarrow{XY} = \overrightarrow{OP}$, then $\overrightarrow{X'Y'} = \overrightarrow{O'P'}$. It follows that $XY = X'Y'$ for any points X and Y; that is, α' is a motion.

Summarizing the discussion above, α is a composition of motions and scalings. Observe that any scaling is a composition of two inversions in concentric circles. Recall that any motion is a composition of reflections (see Exercise 5.8). Whence α is a composition of inversions and reflections.

In the remaining case $\alpha(\infty) \neq \infty$, set $P = \alpha(\infty)$. Consider an inversion β in a circle with center P and set $\gamma = \beta \circ \alpha$. Note that $\beta(P) = \infty$; therefore, $\gamma(\infty) = \infty$. Since α and β are inversive, so is γ. From above we get that γ is a composition of reflections and inversions. Since β is self-inverse, we get $\alpha = \beta \circ \gamma$; therefore α is a composition of reflections and inversions as well. □

14.16. Exercise. *Show that inversive transformations preserve the angle between arcs up to sign.*

More precisely, assume $A'B_1'C_1'$, $A'B_2'C_2'$ are the images of two arcs AB_1C_1, AB_2C_2 under an inversive transformation. Let α and α' denote the angle between the tangent half-lines to AB_1C_1 and AB_2C_2 at A and the angle between the tangent half-lines to $A'B_1'C_1'$ and $A'B_2'C_2'$ at A' respectively. Then

$$\alpha' = \pm\alpha.$$

14.17. Exercise. *Show that any reflection can be presented as a composition of three inversions.*

The exercise above implies a stronger version of Theorem 14.15; namely, *any inversive transformation is a composition of inversions* — no reflections needed.

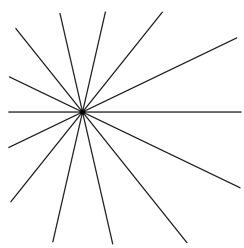
Chapter 15

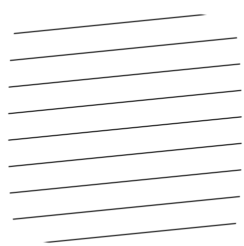
Projective geometry

Projective completion

In the Euclidean plane, two distinct lines might have one or zero points of intersection (in the latter case the lines are parallel). Our aim is to extend the Euclidean plane by ideal points so that any two distinct lines will have exactly one point of intersection.

A collection of lines in the Euclidean plane is called *concurrent* if they all intersect at a single point or all of them pairwise parallel. A maximal set of concurrent lines in the plane is called a *pencil*. There are two types of pencils: *central pencils* contain all lines passing thru a fixed point called the *center of the pencil* and *parallel pencils* contain pairwise parallel lines.





Each point in the Euclidean plane uniquely defines a central pencil with the center in it. Note that any two lines completely determine the pencil containing both.

Let us add one *ideal point* for each parallel pencil, and assume that all these ideal points lie on one *ideal line*. We also assume that the ideal line belongs to each parallel pencil.

We obtain the so-called *real projective plane* (or *projective completion* of the original plane). It comes with an incidence structure — we say that three points lie on one line if the corresponding pencils contain a common line. Projective geometry studies this incidence structure.

Let us describe points of projective completion in coordinates. A parallel pencil contains the ideal line and the lines $y = m{\cdot}x + b$ with fixed slope m; if $m = \infty$, we assume that the lines are given by equations $x = a$. Therefore the real projective plane contains every point (x, y) in the coordinate plane plus the ideal line containing one ideal point P_m for every slope $m \in \mathbb{R} \cup \{\infty\}$.

Euclidean space

Let us repeat the construction of metric d_2 (page 11) in space.

Suppose that $\mathbb{R}^3$ denotes the set of all triples (x, y, z) of real numbers. Assume $A = (x_A, y_A, z_A)$ and $B = (x_B, y_B, z_B)$ are arbitrary points in $\mathbb{R}^3$. Define the metric on $\mathbb{R}^3$ the following way:

$$AB := \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2}.$$

The obtained metric space is called *Euclidean space*.

The subset of points in $\mathbb{R}^3$ is called *plane* if it can be described by an equation

$$a{\cdot}x + b{\cdot}y + c{\cdot}z + d = 0$$

for some constants a, b, c, and d such that at least one of the values a, b or c is distinct from zero.

It is straightforward to show the following:

- ◇ Any plane in the Euclidean space is isometric to the Euclidean plane.
- ◇ Any three points in the space lie on a plane.
- ◇ An intersection of two distinct planes (if it is nonempty) is a line in each of these planes.

These statements make it possible to generalize many notions and results from Euclidean plane geometry to the Euclidean space by applying plane geometry in the planes of the space.

Space model

Let us identify the Euclidean plane with a plane Π in the Euclidean space $\mathbb{R}^3$ that does not pass thru the origin O. Denote by $\hat{\Pi}$ the projective completion of Π.

Denote by Φ the set of all lines in the space thru O. Let us define a bijection $P \leftrightarrow \dot{P}$ between $\hat{\Pi}$ and Φ. If $P \in \Pi$, then take the line $\dot{P} = (OP)$; if P is an ideal point of $\hat{\Pi}$, so it is defined by a parallel pencil of lines, then take the line $\dot{P}$ thru O parallel to the lines in this pencil.

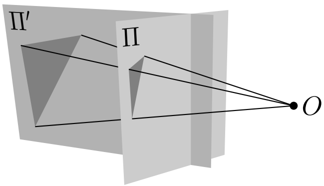
Further, denote by Ψ the set of all planes in the space thru O. In a similar fashion, we can define a bijection $\ell \leftrightarrow \dot{\ell}$ between lines in $\hat{\Pi}$ and Ψ. If a line ℓ is not ideal, then take the plane $\dot{\ell}$ that contains ℓ and O; if the line ℓ is ideal, then take $\dot{\ell}$ to be the plane thru O that is parallel to Π (that is, $\dot{\ell} \cap \Pi = \varnothing$).

15.1. Observation. *Let P and ℓ be a point and a line in the real projective plane. Then $P \in \ell$ if and only if $\dot{P} \subset \dot{\ell}$, where $\dot{P}$ and $\dot{\ell}$ denote the line and plane defined by the constructed bijections.*

Perspective projection

Consider two planes Π and Π' in the Euclidean space. Let O be a point that belongs neither to Π nor Π'.

A *perspective projection from* Π *to* Π' *with center* O maps a point $P \in \Pi$ to the intersection point $P' = \Pi' \cap (OP)$.



In general, perspective projection is not a bijection between the planes. Indeed, if the line (OP) is parallel to Π' (that is, if $(OP) \cap \Pi' = \varnothing$) then the perspective projection of $P \in \Pi$ is undefined. Also, if $(OP') \parallel \Pi$ for $P' \in \Pi'$, then the point P' is not an image of the perspective projection.

For example, suppose O is the origin of (x, y, z)-coordinate space, and the planes Π and Π' are given by the equations $z = 1$ and $x = 1$ respectively. Then the perspective projection from Π to Π' can be written in the coordinates as

$$(x, y, 1) \mapsto (1, \tfrac{y}{x}, \tfrac{1}{x}).$$

Indeed the coordinates have to be proportional; points on Π have unit z-coordinate, and points on Π' have unit z-coordinate.

The perspective projection maps one plane to another. However, we can identify the two planes by fixing a coordinate system in each. In this case, we get a partially defined map from the plane to itself. We will keep the name *perspective transformation* for such maps.

For the described perspective projection; we may get the map

❶ $$\beta\colon (x, y) \mapsto (\tfrac{1}{x}, \tfrac{y}{x}).$$

This map is undefined on the line $x = 0$. Also, points on this line are not images of points under perspective projection.

Denote by $\hat{\Pi}$ and $\hat{\Pi}'$ the projective completions of Π and Π' respectively. Note that the perspective projection is a restriction of composition of the two bijections $\hat{\Pi} \leftrightarrow \Phi \leftrightarrow \hat{\Pi}'$ constructed in the previous section.

By Observation 15.1, the perspective projection can be extended to a bijection $\hat{\Pi} \leftrightarrow \hat{\Pi}'$ that sends lines to lines.[1]

For example, to define an extension of the perspective projection β in ❶, we have to observe that

- ◇ The pencil of vertical lines $x = a$ is mapped to itself.
- ◇ The ideal points defined by pencils of lines $y = m{\cdot}x + b$ are mapped to the point $(0, m)$ and the other way around — point $(0, m)$ is mapped to the ideal point defined by the pencil of lines $y = m{\cdot}x + b$.

Projective transformations

A bijection from the real projective plane to itself that sends lines to lines is called *projective transformation.*

Note that any affine transformation defines a projective transformation on the corresponding real projective plane. We will call such projective transformations *affine*; these are projective transformations that send the ideal line to itself.

The extended perspective projection discussed in the previous section provides another source of examples of projective transformations.

15.2. Theorem. *Given a line ℓ in the real projective plane, there is a perspective projection that sends ℓ to the ideal line.*

Moreover, a perspective transformation is either affine or, in a suitable coordinate system, it can be written as a composition of the extension of perspective projection

$$\beta\colon (x, y) \mapsto (\tfrac{x}{y}, \tfrac{1}{y})$$

and an affine transformation.

Proof. We may choose an (x, y)-coordinate system such that the line ℓ is defined by equation $y = 0$. Then the extension of β gives the needed transformation.

Fix a projective transformation γ. If γ sends the ideal line to itself, then it has to be affine. It proves the theorem in this case.

Suppose γ sends the ideal line to a line ℓ. Choose a perspective projection β as above. The composition $\beta \circ \gamma$ sends the ideal line to itself. That is, $\gamma = \beta \circ \gamma$ is affine. Note that β is self-inverse; therefore $\alpha = \beta \circ \gamma$ — hence the result. □

[1]A similar story happened with inversion. An inversion is not defined at its center; moreover, the center is not an inverse of any point. To deal with this problem we passed to the inversive plane which is the Euclidean plane extended by one ideal point. The same strategy worked for perspective projection $\Pi \to \Pi'$, but this time we need to add an ideal line.

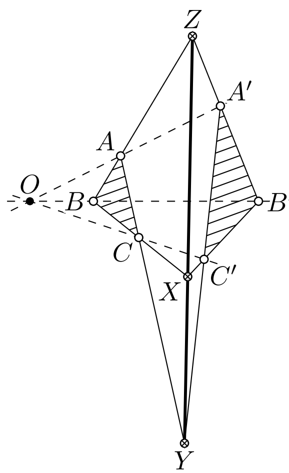
15.3. Exercise. *Let $P \mapsto P'$ be (a) an affine transformation, (b) the perspective projection defined by $(x, y) \mapsto (\frac{x}{y}, \frac{1}{y})$, or (c) arbitrary projective transformation. Suppose P_1, P_2, P_3, P_4 lie on one line. Show that*

$$\frac{P_1P_2 \cdot P_3P_4}{P_2P_3 \cdot P_4P_1} = \frac{P_1'P_2' \cdot P_3'P_4'}{P_2'P_3' \cdot P_4'P_1'};$$

that is, each of these maps preserves cross-ratio for quadruples of points on one line.

Moving points to infinity

Theorem 15.2 makes it possible to take any line in the projective plane and declare it to be ideal. In other words, we can choose a preferred affine plane by removing one line from the projective plane. This construction provides a method for solving problems in projective geometry which will be illustrated by the following classical example:



15.4. Desargues' theorem. *Consider three concurrent lines (AA'), (BB'), and (CC') in the real projective plane. Set*

$$X = (BC) \cap (B'C'),$$
$$Y = (CA) \cap (C'A'),$$
$$Z = (AB) \cap (A'B').$$

Then the points X, Y, and Z are collinear.

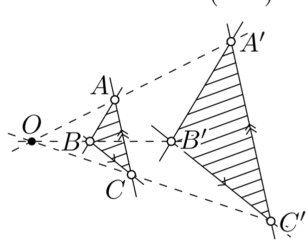
Proof. Without loss of generality, we may assume that the line (XY) is ideal. If not, apply a perspective projection that sends the line (XY) to the ideal line.

That is, we can assume that

$$(BC) \parallel (B'C') \quad \text{and} \quad (CA) \parallel (C'A')$$

and we need to show that

$$(AB) \parallel (A'B').$$



Assume that the lines (AA'), (BB'), and (CC') intersect at point O. Since $(BC) \parallel (B'C')$, the transversal property (7.9) implies that $\measuredangle OBC = \measuredangle OB'C'$ and $\measuredangle OCB = \measuredangle OC'B'$. By the AA similarity condition,

$\triangle OBC \sim \triangle OB'C'$. In particular,

$$\frac{OB}{OB'} = \frac{OC}{OC'}.$$

In the same way, we get that $\triangle OAC \sim \triangle OA'C'$ and

$$\frac{OA}{OA'} = \frac{OC}{OC'}.$$

Therefore,

$$\frac{OA}{OA'} = \frac{OB}{OB'}.$$

By the SAS similarity condition, we get that $\triangle OAB \sim \triangle OA'B'$; in particular, $\measuredangle OAB = \pm\measuredangle OA'B'$.

Note that $\measuredangle AOB = \measuredangle A'OB'$. Therefore,

$$\measuredangle OAB = \measuredangle OA'B'.$$

By the transversal property (7.9), we have $(AB) \parallel (A'B')$.

The case $(AA') \parallel (BB') \parallel (CC')$ is done similarly. In this case, the quadrangles $B'BCC'$ and $A'ACC'$ are parallelograms. Therefore,

$$BB' = CC' = AA'.$$

Hence $\square B'BAA'$ is a parallelogram and $(AB) \parallel (A'B')$. □

Here is another classical theorem of projective geometry.

15.5. Pappus' theorem. *Assume that two triples of points A, B, C, and A', B', C' are collinear. Suppose that points X, Y, Z are uniquely defined by*

$$X = (BC') \cap (B'C), \quad Y = (CA') \cap (C'A), \quad Z = (AB') \cap (A'B).$$

Then the points X, Y, Z are collinear.

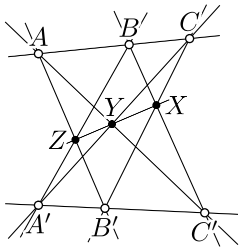
Pappus' theorem can be proved the same way as Desargues' theorem.

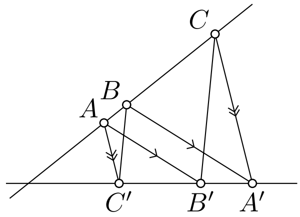
Idea of the proof. Applying a perspective projection, we can assume that Y and Z lie on the ideal line. It remains to show that X lies on the ideal line.

In other words, assuming that $(AB') \parallel (A'B)$ and $(AC') \parallel (A'C)$, we need to show that $(BC') \parallel (B'C)$.

15.6. Exercise. *Finish the proof of Pappus' theorem using the idea described above.*

The following exercise gives a partial converse to Pappus' theorem.





15.7. Exercise. *Given two triples of points A, B, C, and A', B', C', suppose distinct points X, Y, Z are uniquely defined by*

$$X = (BC') \cap (B'C), \quad Y = (CA') \cap (C'A), \quad Z = (AB') \cap (A'B).$$

Assume that the triples A, B, C, and X, Y, Z are collinear. Show that the triple A', B', C' is collinear.

15.8. Exercise. *Solve the following construction problem*

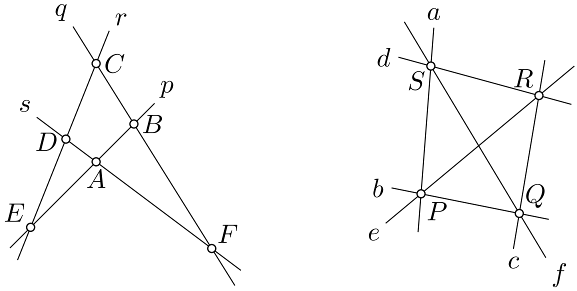
(a) *using Desargues' theorem;*

(b) *using Pappus' theorem.*

Problem. *Suppose a parallelogram and a line ℓ are given. Assume the line ℓ crosses all sides (or their extensions) of the parallelogram at different points are given. Construct another line parallel to ℓ with a ruler only.*

Duality

Assume that a bijection $P \leftrightarrow p$ between the set of lines and the set of points of a plane is given. That is, given a point P, we denote by p the



Dual configurations.

corresponding line; and the other way around, given a line ℓ we denote by L the corresponding point.

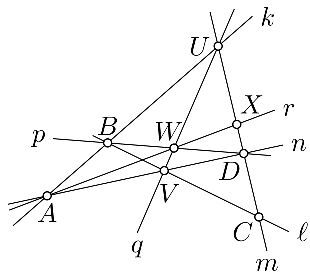
The bijection between points and lines is called *duality*[2] if

$$P \in \ell \quad \Longleftrightarrow \quad p \ni L.$$

for any point P and line ℓ.

15.9. Exercise. *Consider the configuration of lines and points on the diagram.*

Start with a generic quadrangle $KLMN$ and extend it to a dual diagram; label the lines and points using the convention described above.



15.10. Exercise. *Show that the Euclidean plane does not admit a duality.*

15.11. Theorem. *The real projective plane admits a duality.*

Proof. Consider a plane Π and a point $O \notin \Pi$ in the space; suppose that $\hat{\Pi}$ denotes the corresponding real projective plane.

Recall that Φ and Ψ denote the set of all lines and planes passing thru O. According to Observation 15.1, there are bijections $P \leftrightarrow \dot{P}$ between points of $\hat{\Pi}$ and Φ and $\ell \leftrightarrow \dot{\ell}$ between lines in $\hat{\Pi}$ and Ψ such that $P \in \ell$ if and only if $\dot{P} \subset \dot{\ell}$.

It remains to construct a bijection $\dot{\ell} \leftrightarrow \dot{L}$ between Φ and Ψ such that

❷ $$\dot{P} \subset \dot{\ell} \quad \Longleftrightarrow \quad \dot{p} \supset \dot{L}$$

for any two lines $\dot{P}$ and $\dot{L}$ passing thru O.

Set $\dot{\ell}$ to be the plane thru O that is perpendicular to $\dot{L}$. Note that both conditions ❷ are equivalent to $\dot{P} \perp \dot{L}$; hence the result follows. □

15.12. Exercise. *Consider the Euclidean plane with (x, y)-coordinates; suppose that O denotes the origin. Given a point $P \neq O$ with coordinates (a, b) consider the line p given by the equation $a{\cdot}x + b{\cdot}y = 1$.*

Show that the correspondence P to p extends to a duality of the real projective plane.

Which line corresponds to O?

Which point corresponds to the line $a{\cdot}x + b{\cdot}y = 0$?

Duality says that lines and points have the same rights in terms of incidence. It makes it possible to formulate an equivalent dual statement

[2] The standard definition of duality is more general; we consider a special case which is also called *polarity*.

to any statement in projective geometry. For example, the dual statement for "the points X, Y, and Z lie on one line ℓ" would be the "lines x, y, and z intersect at one point L". Let us formulate the dual statement for Desargues' theorem 15.4.

15.13. Dual Desargues' theorem. *Consider the collinear points X, Y, and Z. Assume that*

$$X = (BC) \cap (B'C'), \quad Y = (CA) \cap (C'A'), \quad Z = (AB) \cap (A'B').$$

Then the lines (AA'), (BB'), and (CC') are concurrent.

In this theorem, the points X, Y, and Z are dual to the lines (AA'), (BB'), and (CC') in the original formulation, and the other way around.

Once Desargues' theorem is proved, applying duality (15.11) we get the dual Desargues' theorem. Note that the dual Desargues' theorem is the converse to the original Desargues' theorem 15.4.

15.14. Exercise. *Formulate the dual Pappus' theorem (see 15.5).*

15.15. Exercise. *Solve the following construction problem*

(a) using dual Desargues' theorem;

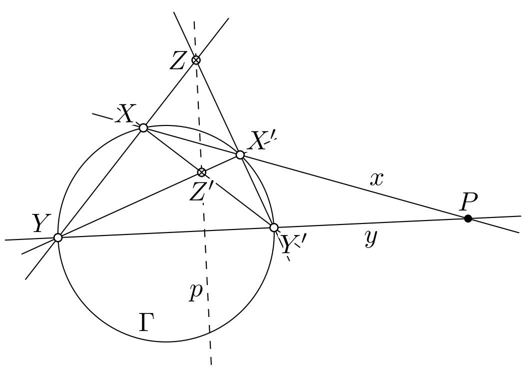
(b) using Pappus' theorem or its dual.

Problem. *Given two parallel lines, construct a third parallel line thru a given point with a ruler only.*

Construction of a polar

In this section, we describe a powerful trick that can be used in the constructions with ruler.

Assume Γ is a circle in the plane and $P \notin \Gamma$. Draw two lines x and



y thru P that intersect Γ at two pairs of points X, X' and Y, Y'. Let $Z = (XY)\cap(X'Y')$ and $Z' = (XY')\cap(X'Y)$. Consider the line $p = (ZZ')$.

15.16. Claim. *The constructed line $p = (ZZ')$ does not depend on the choice of the lines x and y.*

Moreover, $P \leftrightarrow p$ can be extended to a duality such that any point P on the circle Γ corresponds to a line p tangent to Γ at P.

We will not prove this claim, but the proof is not hard. If P lies outside of Γ, it can be done by moving P to infinity keeping Γ fixed as a set. If P lies inside of Γ, it can be done by moving P to the center of Γ. The existence of corresponding projective transformations follows from the idea in Exercise 16.6.

The line p is called the *polar* of the point P with respect to Γ.

The point P is called the *pole* of the line p with respect to Γ.

15.17. Exercise. *Revert the described construction. That is, given a circle Γ and a line p that is not tangent to Γ, construct a point P such that the described construction for P and Γ produces the line p.*

15.18. Exercise. *Let p be the polar line of point P with respect to the circle Γ. Assume that p intersects Γ at points V and W. Show that the lines (PV) and (PW) are tangent to Γ.*

Come up with a ruler-only construction of the tangent lines to the given circle Γ thru the given point $P \notin \Gamma$.

15.19. Exercise. *Assume two concentric circles Γ and Γ' are given. Construct the common center of Γ and Γ' with a ruler only.*

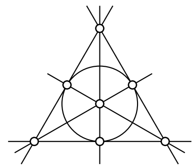
15.20. Exercise. *Assume a line ℓ and a circle Γ with its center O are given. Suppose $O \notin \ell$. Construct a perpendicular from O on ℓ with a ruler only.*

Axioms

Note that the real projective plane described above satisfies the following set of axioms:

p-I. Any two distinct points lie on a unique line.
p-II. Any two distinct lines pass thru a unique point.
p-III. There exist at least four points of which no three are collinear.

Let us take these three axioms as a definition of the *projective plane*; so the real projective plane discussed above becomes its particular example.



There is an example of a projective plane that contains exactly 3 points on each line. This is the so-called *Fano plane* which you can see on the diagram; it contains 7 points and 7 lines. This is an example of a *finite projective plane*; that is, a projective plane with finitely many points.

15.21. Exercise. *Show that any line in a projective plane contains at least three points.*

Consider the following dual analog of Axiom p-III:

p-III′. There exist at least four lines of which no three are concurrent.

15.22. Exercise. *Show that Axiom* p-III′ *is equivalent to Axiom* p-III. *That is,*

$$\text{p-I, p-II, and p-III imply p-III}',$$

and

$$\text{p-I, p-II, and p-III}' \text{ imply p-III.}$$

The exercise above shows that in the given axiomatic system, lines and points have the same rights. One can switch everywhere words "point" with "line", "pass thru" with "lies on", "collinear" with "concurrent" and we get an equivalent set of axioms — Axioms p-I and p-II convert into each other, and the same happens with the pair p-III and p-III′.

15.23. Exercise. *Assume that one of the lines in a finite projective plane contains exactly* $n+1$ *points.*

(a) *Show that each line contains exactly* $n+1$ *points.*
(b) *Show that the plane contains exactly* n^2+n+1 *points.*
(c) *Show that there is no projective plane with exactly 10 points.*
(d) *Show that in any finite projective plane the number of points coincides with the number of lines.*

The number n in the above exercise is called *order* of finite projective plane. For example, the Fano plane has order 2. Let us finish by stating a famous open problem in finite geometry.

15.24. Conjecture. *The order of any finite projective plane is a power of a prime number.*

Chapter 16

Spherical geometry

Spherical geometry studies the surface of a unit sphere. This geometry has applications in cartography, navigation, and astronomy.

The spherical geometry is a close relative of the Euclidean and hyperbolic geometries. Most of the theorems of hyperbolic geometry have spherical analogs, but spherical geometry is easier to visualize.

Euclidean space

Recall that Euclidean space is the set $\mathbb{R}^3$ of all triples (x, y, z) of real numbers such that the distance between a pair of points $A = (x_A, y_A, z_A)$ and $B = (x_B, y_B, z_B)$ is defined by the following formula:

$$AB := \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2}.$$

The planes in the space are defined as the set of solutions of

$$a{\cdot}x + b{\cdot}y + c{\cdot}z + d = 0$$

for real numbers a, b, c, and d such that at least one of the numbers a, b or c is not zero. Any plane in the Euclidean space is isometric to the Euclidean plane.

A sphere in space is the direct analog of a circle in the plane. Formally, *sphere* with center O and radius r is the set of points in the space that lie at the distance r from O.

Let A and B be two points on the unit sphere centered at O. The *spherical distance* from A to B (briefly AB_s) is defined as $|\measuredangle AOB|$.

In spherical geometry, the role of lines play the *great circles*; that is, the intersection of the sphere with a plane passing thru O.

Note that the great circles do not form lines in the sense of Definition 1.9. Also, any two distinct great circles intersect at two antipodal points. In particular, the sphere does not satisfy the axioms of the neutral plane.

Pythagorean theorem

Here is an analog of the Pythagorean theorems (6.4 and 13.13) in spherical geometry.

16.1. Spherical Pythagorean Theorem. *Let $\triangle_s ABC$ be a spherical triangle with a right angle at C. Set $a = BC_s$, $b = CA_s$, and $c = AB_s$. Then*

$$\cos c = \cos a \cdot \cos b.$$

In the proof, we will use the notion of the scalar product which we are about to discuss.

Let $v_A = (x_A, y_A, z_A)$ and $v_B = (x_B, y_B, z_B)$ denote the position vectors of points A and B. The *scalar product* of the two vectors v_A and v_B in $\mathbb{R}^3$ is defined as

❶ $$\langle v_A, v_B \rangle := x_A \cdot x_B + y_A \cdot y_B + z_A \cdot z_B.$$

Assume both vectors v_A and v_B are nonzero; suppose that φ denotes the angle measure between them. Then the scalar product can be expressed the following way:

❷ $$\langle v_A, v_B \rangle = |v_A| \cdot |v_B| \cdot \cos\varphi,$$

where

$$|v_A| = \sqrt{x_A^2 + y_A^2 + z_A^2}, \qquad |v_B| = \sqrt{x_B^2 + y_B^2 + z_B^2}.$$

Now, assume that the points A and B lie on the unit sphere Σ in $\mathbb{R}^3$ centered at the origin. In this case, $|v_A| = |v_B| = 1$. By ❷ we get that

❸ $$\cos AB_s = \langle v_A, v_B \rangle.$$

Proof of the spherical Pythagorean Theorem. Since the angle at C is right, we can choose the coordinates in $\mathbb{R}^3$ so that $v_C = (0, 0, 1)$, v_A lies in the xz-plane, so $v_A = (x_A, 0, z_A)$, and v_B lies in yz-plane, so $v_B = (0, y_B, z_B)$.

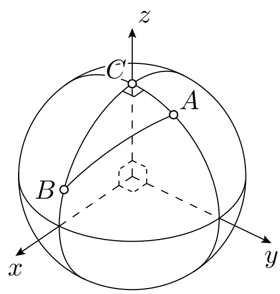
Applying, ❸, we get that

$$z_A = \langle v_C, v_A \rangle = \cos b,$$
$$z_B = \langle v_C, v_B \rangle = \cos a.$$

Applying, ❶ and ❸, we get that

$$\begin{aligned}\cos c &= \langle v_A, v_B\rangle = \\ &= x_A{\cdot}0 + 0{\cdot}y_B + z_A{\cdot}z_B = \\ &= \cos b{\cdot}\cos a.\end{aligned}$$

□



16.2. Exercise. *Show that if $\triangle_s ABC$ is a spherical triangle with a right angle at C, and $AC_s = BC_s = \frac{\pi}{4}$, then $AB_s = \frac{\pi}{3}$.*

Inversion of the space

The inversion in a sphere is defined the same way as we define the inversion in a circle.

Formally, let Σ be the sphere with the center O and radius r. The *inversion* in Σ of a point P is the point $P' \in [OP)$ such that

$$OP{\cdot}OP' = r^2.$$

In this case, the sphere Σ will be called the *sphere of inversion* and its center is called the *center of inversion*.

We also add ∞ to the space and assume that the center of inversion is mapped to ∞ and the other way around. The space $\mathbb{R}^3$ with the point ∞ will be called *inversive space*.

The inversion of the space has many properties of the inversion of the plane. Most important for us are the analogs of theorems 10.6, 10.7, 10.25 which can be summarized as follows:

16.3. Theorem. *The inversion in the sphere has the following properties:*

(a) Inversion maps a sphere or a plane into a sphere or a plane.

(b) Inversion maps a circle or a line into a circle or a line.

(c) Inversion preserves the cross-ratio; that is, if A', B', C', and D' are the inverses of the points A, B, C and D respectively, then

$$\frac{AB{\cdot}CD}{BC{\cdot}DA} = \frac{A'B'{\cdot}C'D'}{B'C'{\cdot}D'A'}.$$

(d) Inversion maps arcs into arcs.

(e) Inversion preserves the absolute value of the angle measure between tangent half-lines to the arcs.

We do not present the proofs here, but they nearly repeat the corresponding proofs in plane geometry. To prove *(a)*, you will need in addition the following lemma; its proof is left to the reader.

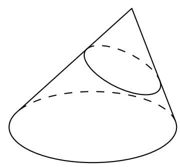
16.4. Lemma. *Let Σ be a subset of the Euclidean space that contains at least two points. Fix a point O in the space.*

Then Σ is a sphere if and only if for any plane Π passing thru O, the intersection $\Pi \cap \Sigma$ is either empty set, one point set, or a circle.

The following observation helps to reduce part *(b)* to part *(a)*.

16.5. Observation. *Any circle in the space is an intersection of two spheres.*

Let us define a *circular cone* as a set formed by line segments from a fixed point, called the *tip* of the cone, to all the points on a fixed circle, called the *base* of the cone; we always assume that the base does not lie in the same plane as the tip. We say that the cone is *right* if the center of the base circle is the foot point of the tip on the base plane; otherwise, we call it *oblique*.



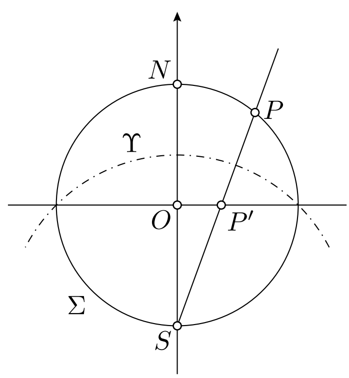
16.6. Exercise. *Let K be an oblique circular cone. Show that there is a plane Π that is not parallel to the base plane of K such that the intersection $\Pi \cap K$ is a circle.*

Stereographic projection

Consider the unit sphere Σ centered at the origin $(0,0,0)$. This sphere can be described by the equation $x^2+y^2+z^2 = 1$.

Suppose that Π denotes the xy-plane; it is defined by the equation $z = 0$. Clearly, Π runs thru the center of Σ.

Let $N = (0,0,1)$ and $S = (0,0,-1)$ denote the "north" and "south" poles of Σ; these are the points on the sphere that have extremal distances to Π. Suppose that Ω denotes the "equator" of Σ; it is the intersection $\Sigma \cap \Pi$.



The plane thru P, O, and S.

For any point $P \neq S$ on Σ, consider the line (SP) in the space. This line intersects Π in exactly one point, denoted by P'. Set $S' = \infty$.

The map $\xi_s\colon P \mapsto P'$ is called the *stereographic projection from Σ to Π with respect to the south pole*. The inverse of this map $\xi_s^{-1}\colon P' \mapsto P$ is called the *stereographic projection from* Π *to* Σ *with respect to the south pole.*

In the same way, one can define the *stereographic projections* ξ_n *and* ξ_n^{-1} *with respect to the north pole* N.

Note that $P = P'$ if and only if $P \in \Omega$.

Note that if Σ and Π are as above, then the composition of the stereographic projections $\xi_s : \Sigma \to \Pi$ and $\xi_s^{-1} : \Pi \to \Sigma$ are the restrictions to Σ and Π respectively of the inversion in the sphere Υ with the center S and radius $\sqrt{2}$.

From above and Theorem 16.3, it follows that the stereographic projection preserves the angles between arcs; more precisely *the absolute value of the angle measure* between arcs on the sphere.

This makes it particularly useful in cartography. A map of a big region of earth cannot be done on a constant scale, but using a stereographic projection, one can keep the angles between roads the same as on earth.

In the following exercises, we assume that Σ, Π, Υ, Ω, O, S, and N are as above.

16.7. Exercise. *Show that* $\xi_n \circ \xi_s^{-1}$*, the composition of stereographic projections from* Π *to* Σ *from* S*, and from* Σ *to* Π *from* N *is the inverse of the plane* Π *in* Ω*.*

16.8. Exercise. *Show that a stereographic projection* $\Sigma \to \Pi$ *sends the great circles to plane circlines that intersect* Ω *at opposite points.*

The following exercise is analogous to Lemma 13.10.

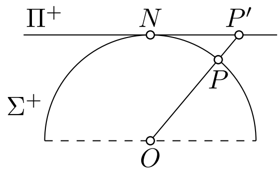
16.9. Exercise. *Fix a point* $P \in \Pi$ *and let* Q *be another point in* Π*. Let* P' *and* Q' *denote their stereographic projections to* Σ*. Set* $x = PQ$ *and* $y = P'Q'_s$*. Show that*

$$\lim_{x \to 0} \frac{y}{x} = \frac{2}{1 + OP^2}.$$

Central projection

The central projection is analogous to the projective model of hyperbolic plane which is discussed in Chapter 17.

Let Σ be the unit sphere centered at the origin which will be denoted by O. Suppose that Π^+ denotes the plane defined by the equation $z = 1$. This plane is parallel to the xy-plane and it passes thru the north pole $N = (0, 0, 1)$ of Σ.



Recall that the northern hemisphere of Σ, is the subset of points $(x,y,z)\in\Sigma$ such that $z>0$. The northern hemisphere will be denoted by Σ^+.

Given a point $P\in\Sigma^+$, consider the half-line $[OP)$. Suppose that P' denotes the intersection of $[OP)$ and Π^+. Note that if $P=(x,y,z)$, then $P'=(\frac{x}{z},\frac{y}{z},1)$. It follows that $P\leftrightarrow P'$ is a bijection between Σ^+ and Π^+.

The described bijection $\Sigma^+\leftrightarrow\Pi^+$ is called the *central projection* of the hemisphere Σ^+.

Note that the central projection sends the intersections of the great circles with Σ^+ to the lines in Π^+. The latter follows since the great circles are intersections of Σ with planes passing thru the origin as well as the lines in Π^+ are the intersection of Π^+ with these planes.

The following exercise is analogous to Exercise 17.5 in hyperbolic geometry.

16.10. Exercise. *Let $\triangle_s ABC$ be a nondegenerate spherical triangle. Assume that the plane Π^+ is parallel to the plane passing thru A, B, and C. Let A', B', and C' denote the central projections of A, B and C.*

- *(a) Show that the midpoints of $[A'B']$, $[B'C']$, and $[C'A']$ are central projections of the midpoints of $[AB]_s$, $[BC]_s$, and $[CA]_s$ respectively.*
- *(b) Use part (a) to show that the medians of a spherical triangle intersect at one point.*

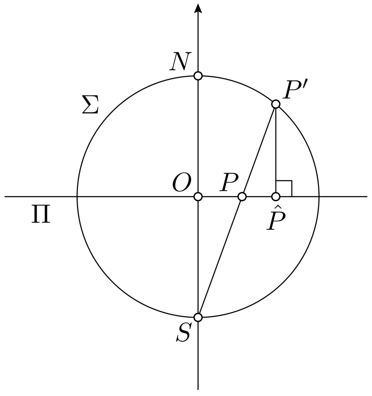
Chapter 17

Projective model

The *projective model* is another model of hyperbolic plane discovered by Beltrami; it is often called *Klein model*. The projective and conformal models are saying exactly the same thing but in two different languages. Some problems in hyperbolic geometry admit simpler proof using the projective model and others have simpler proof in the conformal model. Therefore, it is worth knowing both.

Special bijection on the h-plane

Consider the conformal disc model with the absolute at the unit circle Ω centered at O. Choose a coordinate system (x, y) on the plane with the origin at O, so the circle Ω is described by the equation $x^2 + y^2 = 1$.



The plane thru P, O, and S.

Let us think that our plane is the coordinate xy-plane in the Euclidean space; denote it by Π. Let Σ be the unit sphere centered at O; it is described by the equation

$$x^2 + y^2 + z^2 = 1.$$

Set $S = (0, 0, -1)$ and $N = (0, 0, 1)$; these are the south and north poles of Σ.

Consider stereographic projection $\Pi \to \Sigma$ from S; given point $P \in \Pi$ denote its image in Σ. Note that the h-plane is mapped to the *northern hemisphere*; that is, to the set of points (x, y, z) in Σ described by the inequality $z > 0$.

For a point $P'\in\Sigma$ consider its foot point $\hat P$ on Π; this is the closest point to P'.

Note that the composition $P\leftrightarrow P'\leftrightarrow\hat P$ of these two maps gives a bijection from the h-plane to itself. Further, note that $P=\hat P$ if and only if $P\in\Omega$ or $P=O$.

17.1. Exercise. *Suppose that $P\leftrightarrow\hat P$ is the bijection described above. Assume that P is a point of h-plane distinct from the center of absolute and Q is its inverse in the absolute. Show that the midpoint of $[PQ]$ is the inversion of $\hat P$ in the absolute.*

17.2. Lemma. *Let $(PQ)_h$ be an h-line with the ideal points A and B. Then $\hat P,\hat Q\in[AB]$.*

Moreover,

❶ $$\frac{A\hat Q\cdot B\hat P}{\hat QB\cdot\hat PA}=\left(\frac{AQ\cdot BP}{QB\cdot PA}\right)^2.$$

In particular, if A,P,Q,B appear in the same order, then

$$PQ_h=\tfrac12\cdot\ln\frac{A\hat Q\cdot B\hat P}{\hat QB\cdot\hat PA}.$$

Proof. Consider the stereographic projection $\Pi\to\Sigma$ from the south pole S. Note that it fixes A and B; denote by P' and Q' the images of P and Q;

According to Theorem 16.3c,

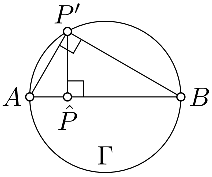
❷ $$\frac{AQ\cdot BP}{QB\cdot PA}=\frac{AQ'\cdot BP'}{Q'B\cdot P'A}.$$

By Theorem 16.3e, each circline in Π that is perpendicular to Ω is mapped to a circle in Σ that is still perpendicular to Ω. It follows that the stereographic projection sends $(PQ)_h$ to the intersection of the northern hemisphere of Σ with a plane perpendicular to Π.

Suppose that Λ denotes the plane; it contains the points A, B, P', $\hat P$ and the circle $\Gamma=\Sigma\cap\Lambda$. (It also contains Q' and $\hat Q$ but we will not use these points for a while.)

Note that

- ⋄ $A,B,P'\in\Gamma$,
- ⋄ $[AB]$ is a diameter of Γ,
- ⋄ $(AB)=\Pi\cap\Lambda$,
- ⋄ $\hat P\in[AB]$
- ⋄ $(P'\hat P)\perp(AB)$.



The plane Λ.

Since $[AB]$ is the diameter of Γ, by Corollary 9.8, the angle $AP'B$ is right. Hence $\triangle A\hat{P}P' \sim \triangle AP'B \sim \triangle P'\hat{P}B$. In particular

$$\frac{AP'}{BP'} = \frac{A\hat{P}}{P'\hat{P}} = \frac{P'\hat{P}}{B\hat{P}}.$$

Therefore

❸ $$\frac{A\hat{P}}{B\hat{P}} = \left(\frac{AP'}{BP'}\right)^2.$$

In the same way, we get that

❹ $$\frac{A\hat{Q}}{B\hat{Q}} = \left(\frac{AQ'}{BQ'}\right)^2.$$

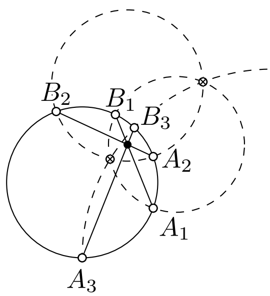
Finally, note that ❷+❸+❹ imply ❶.

The last statement follows from ❶ and the definition of h-distance. Indeed,

$$\begin{aligned} PQ_h &:= \ln \frac{AQ\cdot BP}{QB\cdot PA} = \\ &= \ln\left(\frac{A\hat{Q}\cdot B\hat{P}}{\hat{Q}B\cdot \hat{P}A}\right)^{\frac{1}{2}} = \\ &= \tfrac{1}{2}\cdot \ln \frac{A\hat{Q}\cdot B\hat{P}}{\hat{Q}B\cdot \hat{P}A}. \end{aligned}$$

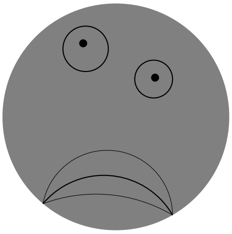
□

17.3. Exercise. *Let Γ_1, Γ_2, and Γ_3 be three circles perpendicular to the circle Ω. Let $[A_1B_1]$, $[A_2B_2]$, and $[A_3B_3]$ denote the common chords of Ω and Γ_1, Γ_2, Γ_3 respectively. Show that the chords $[A_1B_1]$, $[A_2B_2]$, and $[A_3B_3]$ intersect at one point inside Ω if and only if Γ_1, Γ_2, and Γ_3 intersect at two points.*

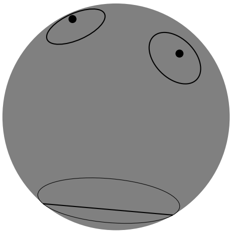


Projective model

The following picture illustrates the map $P \mapsto \hat{P}$ described in the previous section — if you take the picture on the left and apply the map $P \mapsto \hat{P}$, you get the picture on the right. The pictures are *conformal* and *projective models* of the hyperbolic plane respectively. The map $P \mapsto \hat{P}$ is a “translation” from one to another.



Conformal model



Projective model

In the projective model, things look different; some become simpler, other things become more complicated.

Lines. The h-lines in the projective model are chords of the absolute; more precisely, chords without its endpoints.

This observation can be used to transfer statements about lines and points from the Euclidean plane to the h-plane. As an example let us state a hyperbolic version of Pappus' theorem for h-plane.

17.4. Hyperbolic Pappus' theorem. *Assume that two triples of h-points A, B, C, and A', B', C' in the h-plane are h-collinear. Suppose that the h-points X, Y, and Z are defined by*

$$X = (BC')_h \cap (B'C)_h, \quad Y = (CA')_h \cap (C'A)_h, \quad Z = (AB')_h \cap (A'B)_h.$$

Then the points X, Y, Z are h-collinear.

In the projective model, this statement follows immediately from the original Pappus' theorem 15.5. The same can be done for Desargues' theorem 15.4. The same argument shows that the construction of a tangent line with a ruler-only described in Exercise15.18 works in the h-plane as well.

On the other hand, note that it is not at all easy to prove this statement using the conformal model.

Circles and equidistants. The h-circles and equidistants in the projective model are a certain type of ellipses and their open arcs.

It follows since the stereographic projection sends circles on the plane to circles on the unit sphere and the foot point projection of the circle back to the plane is an ellipse. (One may define *ellipse* as a foot point projection of a circle.)

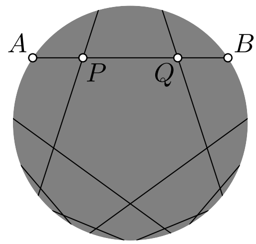
Distance. Consider a pair of h-points P and Q. Let A and B be the ideal points of the h-line in the projective model; that is, A and B are the intersections of the Euclidean line (PQ) with the absolute.

Then by Lemma 17.2,

❺ $$PQ_h = \tfrac{1}{2}\cdot\ln\frac{AQ\cdot BP}{QB\cdot PA},$$

assuming the points A, P, Q, B appear on the line in the same order.

Angles. The angle measures in the projective model are very different from the Euclidean angles and it is hard to figure out by looking on the picture. For example, all the intersecting h-lines on the picture are perpendicular.



There are two useful exceptions:

◇ If O is the center of the absolute, then

$$\measuredangle_h AOB = \measuredangle AOB.$$

◇ If O is the center of the absolute and $\measuredangle OAB = \pm\frac{\pi}{2}$, then

$$\measuredangle_h OAB = \measuredangle OAB = \pm\tfrac{\pi}{2}.$$

To find the angle measure in the projective model, you may apply a motion of the h-plane that moves the vertex of the angle to the center of the absolute; once it is done the hyperbolic and Euclidean angles have the same measure.

Motions. The motions of the h-plane in the conformal and projective models are relevant to inversive transformations and projective transformation in the same way. Namely:

◇ Inversive transformations that preserve the h-plane describe motions of the h-plane in the conformal model.

◇ Projective transformations that preserve h-plane describe motions of the h-plane in the projective model.[1]

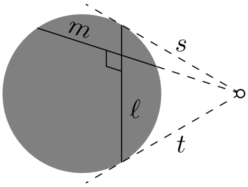
The following exercise is a hyperbolic analog of Exercise 16.10. This is the first example of a statement that admits an easier proof using the projective model.

17.5. Exercise. *Let P and Q be the points in h-plane that lie at the same distance from the center of the absolute. Observe that in the projective model, h-midpoint of $[PQ]_h$ coincides with the Euclidean midpoint of $[PQ]_h$.*

Conclude that if an h-triangle is inscribed in an h-circle, then its medians meet at one point.

Recall that an h-triangle might be also inscribed in a horocycle or an equidistant. Think how to prove the statement in this case.

[1] The idea described in the solution of Exercise 16.6 and in the sketch of proof of Theorem 19.13 can be used to construct many projective transformations of this type.



17.6. Exercise. *Let ℓ and m are h-lines in the projective model. Let s and t denote the Euclidean lines tangent to the absolute at the ideal points of ℓ. Show that if the lines s, t, and the extension of m intersect at one point, then ℓ and m are perpendicular h-lines.*

17.7. Exercise. *Use the projective model to derive the formula for angle of parallelism (Proposition 13.2).*

17.8. Exercise. *Use the projective model to find the inradius of the ideal triangle.*

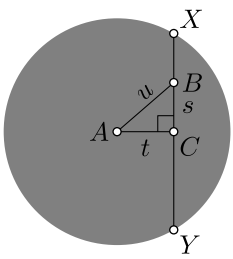
The projective model of h-plane can be used to give another proof of the hyperbolic Pythagorean theorem (13.13).

First, let us recall its statement:

❻ $$\operatorname{ch} c = \operatorname{ch} a \cdot \operatorname{ch} b,$$

where $a = BC_h$, $b = CA_h$, and $c = AB_h$ and $\triangle_h ACB$ is an h-triangle with a right angle at C.

Note that we can assume that A is the center of the absolute. Set $s = BC$, $t = CA$, $u = AB$. According to the Euclidean Pythagorean theorem (6.4), we have



❼ $$u^2 = s^2 + t^2.$$

It remains to express a, b, and c using s, u, and t and show that ❼ implies ❻.

17.9. Advanced exercise. *Finish the proof of hyperbolic Pythagorean theorem (13.13) indicated above.*

Bolyai's construction

Assume we need to construct a line thru P asymptotically parallel to the given line ℓ in the h-plane.

If A and B are ideal points of ℓ in the projective model, then we could simply draw the Euclidean line (PA). However, the ideal points do not lie in the h-plane; therefore there is no way to use them in the construction.

In the following construction we assume that you know a compass-and-ruler construction of the perpendicular line; see Exercise 5.22.

17.10. Bolyai's construction.

1. *Drop a perpendicular from P to ℓ; denote it by m. Let Q be the foot point of P on ℓ.*
2. *Erect a perpendicular from P to m; denote it by n.*
3. *Mark by R a point on ℓ distinct from Q.*
4. *Drop a perpendicular from R to n; denote it by k.*
5. *Draw the circle Γ with center P and the radius QR. Mark by T a point of intersection of Γ with k.*
6. *The line $(PT)_h$ is asymptotically parallel to ℓ.*

17.11. Exercise. *Explain what happens if one performs the Bolyai construction in the Euclidean plane.*

To prove that Bolyai's construction gives the asymptotically parallel line in the h-plane, it is sufficient to show the following:

17.12. Proposition. *Assume P, Q, R, S, T are points in h-plane such that*

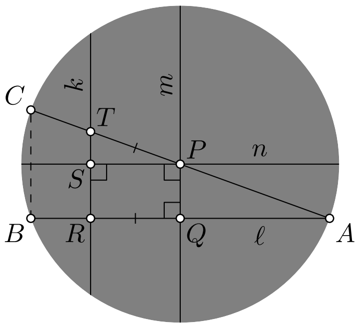
- ◇ $S \in (RT)_h$,
- ◇ $(PQ)_h \perp (QR)_h$,
- ◇ $(PS)_h \perp (PQ)_h$,
- ◇ $(RT)_h \perp (PS)_h$ *and*
- ◇ $(PT)_h$ *and* $(QR)_h$ *are asymptotically parallel.*

Then $QR_h = PT_h$.

Proof. We will use the projective model. Without loss of generality, we may assume that P is the center of the absolute. As it was noted on page 140, in this case, the corresponding Euclidean lines are also perpendicular; that is, $(PQ) \perp (QR)$, $(PS) \perp (PQ)$, and $(RT) \perp (PS)$.

Let A be the common ideal point of $(QR)_h$ and $(PT)_h$. Let B and C denote the remaining ideal points of $(QR)_h$ and $(PT)_h$ respectively.

Note that the Euclidean lines (PQ), (TR), and (CB) are parallel.



Therefore,

$$\triangle AQP \sim \triangle ART \sim \triangle ABC.$$

In particular,

$$\frac{AC}{AB} = \frac{AT}{AR} = \frac{AP}{AQ}.$$

It follows that

$$\frac{AT}{AR} = \frac{AP}{AQ} = \frac{BR}{CT} = \frac{BQ}{CP}.$$

In particular, $\frac{AT \cdot CP}{TC \cdot PA} = \frac{AR \cdot BQ}{RB \cdot QA}$. Applying the formula for h-distance ❺, we get that $QR_h = PT_h$. □

Chapter 18

Complex coordinates

In this chapter, we give an interpretation of inversive geometry using complex coordinates. The results of this chapter lead to a deeper understanding of both concepts.

Complex numbers

Informally, a complex number is a number that can be put in the form

➊ $$z = x + i{\cdot}y,$$

where x and y are real numbers and $i^2 = -1$.

The set of complex numbers will be further denoted by $\mathbb{C}$. If x, y, and z are as in ➊, then x is called the *real part* and y the *imaginary part* of the complex number z. Briefly, it is written as

$$x = \operatorname{Re} z \quad \text{and} \quad y = \operatorname{Im} z.$$

On the more formal level, a complex number is a pair of real numbers (x, y) with the addition and multiplication described below; the expression $x + i{\cdot}y$ is only a convenient way to write the pair (x, y).

➋ $$\begin{aligned}(x_1 + i{\cdot}y_1) + (x_2 + i{\cdot}y_2) &:= (x_1 + x_2) + i{\cdot}(y_1 + y_2);\\ (x_1 + i{\cdot}y_1){\cdot}(x_2 + i{\cdot}y_2) &:= (x_1{\cdot}x_2 - y_1{\cdot}y_2) + i{\cdot}(x_1{\cdot}y_2 + y_1{\cdot}x_2).\end{aligned}$$

Complex coordinates

Recall that one can think of the Euclidean plane as the set of all pairs of real numbers (x, y) equipped with the metric

$$AB = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2},$$

where $A = (x_A, y_A)$ and $B = (x_B, y_B)$.

One can pack the coordinates (x, y) of a point in one complex number $z = x + i{\cdot}y$. This way we get a one-to-one correspondence between points of the Euclidean plane and $\mathbb{C}$. Given a point $Z = (x, y)$, the complex number $z = x + i{\cdot}y$ is called the *complex coordinate* of Z.

Note that if O, E, and I are points in the plane with complex coordinates 0, 1, and i, then $\measuredangle EOI = \pm\frac{\pi}{2}$. Further, we assume that $\measuredangle EOI = \frac{\pi}{2}$; if not, one has to change the direction of the y-coordinate.

Conjugation and absolute value

Let $z = x + i{\cdot}y$; that is, z is a complex number with real part x and imaginary part y. If $y = 0$, we say that the complex number z is *real* and if $x = 0$ we say that z is *imaginary*. The set of points with real (imaginary) complex coordinates is a line in the plane, which is called *real* (respectively *imaginary*) line. The real line will be denoted as $\mathbb{R}$.

The complex number

$$\bar z := x - i{\cdot}y$$

is called the *complex conjugate* of $z = x + i{\cdot}y$. Let Z and $\bar Z$ be the points in the plane with the complex coordinates z and $\bar z$ respectively. Note that the point $\bar Z$ is the reflection of Z across the real line.

It is straightforward to check that

❸ $$x = \operatorname{Re} z = \frac{z + \bar z}{2}, \quad y = \operatorname{Im} z = \frac{z - \bar z}{i{\cdot}2}, \quad x^2 + y^2 = z{\cdot}\bar z.$$

The last formula in ❸ makes it possible to express the quotient $\frac{w}{z}$ of two complex numbers w and $z = x + i{\cdot}y$:

$$\frac{w}{z} = \tfrac{1}{z\cdot\bar z}{\cdot}w{\cdot}\bar z = \tfrac{1}{x^2+y^2}{\cdot}w{\cdot}\bar z.$$

Note that

$$\overline{z + w} = \bar z + \bar w, \qquad \overline{z - w} = \bar z - \bar w, \qquad \overline{z{\cdot}w} = \bar z{\cdot}\bar w, \qquad \overline{z/w} = \bar z/\bar w.$$

That is, the complex conjugation *respects* all the arithmetic operations.

The value

$$|z| := \sqrt{x^2 + y^2} = \sqrt{(x + i{\cdot}y){\cdot}(x - i{\cdot}y)} = \sqrt{z{\cdot}\bar z}$$

is called the *absolute value* of z. If $|z| = 1$, then z is called a *unit complex number*.

18.1. Exercise. *Show that* $|v{\cdot}w| = |v|{\cdot}|w|$ *for any* $v, w \in \mathbb{C}$.

Suppose that Z and W are points with complex coordinates z and w. Note that

❹ $$ZW = |z - w|.$$

The triangle inequality for the points with complex coordinates 0, v, and $v + w$ implies that

$$|v + w| \leqslant |v| + |w|$$

for any $v, w \in \mathbb{C}$; this inequality is also called *triangle inequality.*

18.2. Exercise. *Use the identity*

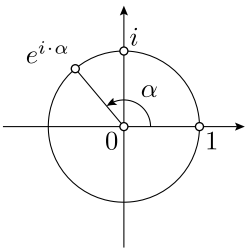
$$u\cdot(v - w) + v\cdot(w - u) + w\cdot(u - v) = 0$$

for $u, v, w \in \mathbb{C}$ and the triangle inequality to prove Ptolemy's inequality (6.8).

Euler's formula

Let α be a real number. The following identity is called *Euler's formula*:

❺ $$e^{i\cdot\alpha} = \cos\alpha + i\cdot\sin\alpha.$$



In particular, $e^{i\cdot\pi} = -1$ and $e^{i\cdot\frac{\pi}{2}} = i$.

Geometrically, Euler's formula means the following: Assume that O and E are the points with complex coordinates 0 and 1 respectively. Assume

$$OZ = 1 \quad \text{and} \quad \measuredangle EOZ \equiv \alpha,$$

then $e^{i\cdot\alpha}$ is the complex coordinate of Z. In particular, the complex coordinate of any point on the unit circle centered at O can be uniquely expressed as $e^{i\cdot\alpha}$ for some $\alpha \in (-\pi, \pi]$.

Why should you think that ❺ is true? The proof of Euler's identity depends on the way you define the exponential function. If you never had to apply the exponential function to an imaginary number, you may take the right-hand side in ❺ as the definition of the $e^{i\cdot\alpha}$.

In this case, formally nothing has to be proved, but it is better to check that $e^{i\cdot\alpha}$ satisfies familiar identities. Mainly,

$$e^{i\cdot\alpha}\cdot e^{i\cdot\beta} = e^{i\cdot(\alpha+\beta)}.$$

The latter can be proved using ❷ and the following trigonometric formulas, which we assume to be known:

$$\cos(\alpha+\beta) = \cos\alpha\cdot\cos\beta - \sin\alpha\cdot\sin\beta,$$
$$\sin(\alpha+\beta) = \sin\alpha\cdot\cos\beta + \cos\alpha\cdot\sin\beta.$$

If you know the power series for the sine, cosine, and exponential function, then the following might convince that the identity ❺ holds:

$$\begin{aligned} e^{i\cdot\alpha} &= 1 + i\cdot\alpha + \frac{(i\cdot\alpha)^2}{2!} + \frac{(i\cdot\alpha)^3}{3!} + \frac{(i\cdot\alpha)^4}{4!} + \frac{(i\cdot\alpha)^5}{5!} + \dots = \\ &= 1 + i\cdot\alpha - \frac{\alpha^2}{2!} - i\cdot\frac{\alpha^3}{3!} + \frac{\alpha^4}{4!} + i\cdot\frac{\alpha^5}{5!} - \dots = \\ &= \left(1 - \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \dots\right) + i\cdot\left(\alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \dots\right) = \\ &= \cos\alpha + i\cdot\sin\alpha. \end{aligned}$$

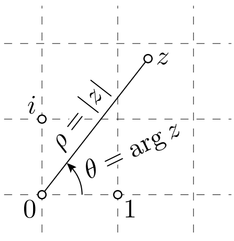
Argument and polar coordinates

As before, we assume that O and E are the points with complex coordinates 0 and 1 respectively.

Let Z be a point distinct from O. Set $\rho = OZ$ and $\theta = \measuredangle EOZ$. The pair (ρ, θ) is called the *polar coordinates* of Z.

If z is the complex coordinate of Z, then $\rho = |z|$. The value θ is called the *argument* of z (briefly, $\theta = \arg z$). In this case,

$$z = \rho\cdot e^{i\cdot\theta} = \rho\cdot(\cos\theta + i\cdot\sin\theta).$$



Note that

$$\arg(z\cdot w) \equiv \arg z + \arg w$$

and

$$\arg\tfrac{z}{w} \equiv \arg z - \arg w$$

if $z \neq 0$ and $w \neq 0$. In particular, if Z, V, W are points with complex coordinates z, v, and w respectively, then

❻
$$\begin{aligned} \measuredangle VZW &= \arg\left(\frac{w-z}{v-z}\right) \equiv \\ &\equiv \arg(w-z) - \arg(v-z) \end{aligned}$$

if $\measuredangle VZW$ is defined.

18.3. Exercise. *Use the formula ❻ to show that*

$$\measuredangle ZVW + \measuredangle VWZ + \measuredangle WZV \equiv \pi$$

for any $\triangle ZVW$ in the Euclidean plane.

18.4. Exercise. *Suppose that points O, E, V, W, and Z have complex coordinates 0, 1, v, w, and $z = v{\cdot}w$ respectively. Show that*

$$\triangle OEV \sim \triangle OWZ.$$

The following theorem is a reformulation of Corollary 9.13 which uses complex coordinates.

18.5. Theorem. *Let $\square UVWZ$ be a quadrangle and u, v, w, and z be the complex coordinates of its vertices. Then $\square UVWZ$ is inscribed if and only if the number*

$$\frac{(v-u){\cdot}(z-w)}{(v-w){\cdot}(z-u)}$$

is real.

The value $\frac{(v-u)\cdot(w-z)}{(v-w)\cdot(z-u)}$ is called the *complex cross-ratio* of u, w, v, and z; it will be denoted by $(u, w; v, z)$.

18.6. Exercise. *Observe that the complex number $z \neq 0$ is real if and only if $\arg z = 0$ or π; in other words, $2{\cdot}\arg z \equiv 0$.*

Use this observation to show that Theorem 18.5 is indeed a reformulation of Corollary 9.13.

Method of complex coordinates

The following problem illustrates the method of complex coordinates.

18.7. Problem. *Let $\triangle OPV$ and $\triangle OQW$ be isosceles right triangles such that*

$$\measuredangle VPO = \measuredangle OQW = \tfrac{\pi}{2}$$

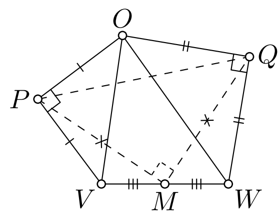
and M be the midpoint of $[VW]$. Assume P, Q, and M are distinct points. Show that $\triangle PMQ$ is an isosceles right triangle.

Solution. Choose the complex coordinates so that O is the origin; denote by v, w, p, q, m the complex coordinates of the remaining points respectively.

Since $\triangle OPV$ and $\triangle OQW$ are isosceles and $\measuredangle VPO = \measuredangle OQW = \frac{\pi}{2}$, ❹ and ❻ imply that

$$v - p = i{\cdot}p, \qquad\qquad q - w = i{\cdot}q.$$

Therefore



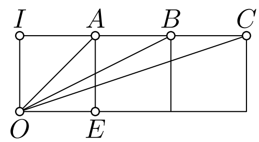
$$m = \tfrac{1}{2}\cdot(v+w) = \\ = \tfrac{1+i}{2}\cdot p + \tfrac{1-i}{2}\cdot q.$$

By straightforward computations, we get that

$$p - m = i\cdot(q-m).$$

In particular, $|p-m| = |q-m|$ and $\arg\frac{p-m}{q-m} = \frac{\pi}{2}$; that is, $PM = QM$ and $\measuredangle QMP = \frac{\pi}{2}$. □

18.8. Exercise. *Consider three squares with common sides as on the diagram. Use the method of complex coordinates to show that*

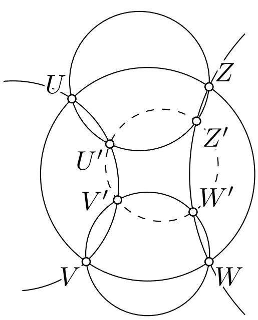


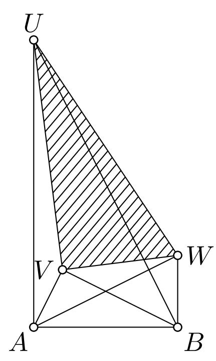
$$\measuredangle EOA + \measuredangle EOB + \measuredangle EOC = \pm\tfrac{\pi}{2}.$$

18.9. Exercise. *Check the following identity with six complex cross-ratios:*

$$(u,w;v,z)\cdot(u',w';v',z') = \frac{(v,w';v',w)\cdot(z,u';z',u)}{(u,v';u',v)\cdot(w,z';w',z)}.$$

Use it together with Theorem 18.5 to prove that if $\square UVWZ$, $\square UVV'U'$, $\square VWW'V'$, $\square WZZ'W'$, and $\square ZUU'Z'$ are inscribed, then $\square U'V'W'Z'$ is inscribed as well.





18.10. Exercise. *Suppose that points U, V and W lie on one side of line (AB) and $\triangle UAB \sim \triangle BVA \sim \triangle ABW$. Denote by a, b, u, v, and w the complex coordinates of A, B, U, V, and W respectively.*

(a) *Show that $\frac{u-a}{b-a} = \frac{b-v}{a-v} = \frac{a-b}{w-b} = \frac{u-v}{w-v}$.*

(b) *Conclude that $\triangle UAB \sim \triangle BVA \sim \triangle ABW \sim \triangle UVW$.*

Fractional linear transformations

18.11. Exercise. *Watch the video "Möbius transformations revealed" by Douglas Arnold and Jonathan Rogness. (It is available on YouTube.)*

The complex plane $\mathbb{C}$ extended by one ideal number ∞ is called the *extended complex plane*. It is denoted by $\hat{\mathbb{C}}$, so $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$

A *fractional linear transformation* or *Möbius transformation* of $\hat{\mathbb{C}}$ is a function of one complex variable z that can be written as

$$f(z) = \frac{a{\cdot}z + b}{c{\cdot}z + d},$$

where the coefficients a, b, c, d are complex numbers satisfying $a{\cdot}d - b{\cdot}c \neq 0$. (If $a{\cdot}d - b{\cdot}c = 0$ the function defined above is a constant and is not considered to be a fractional linear transformation.)

In case $c \neq 0$, we assume that

$$f(-d/c) = \infty \quad \text{and} \quad f(\infty) = a/c;$$

and if $c = 0$ we assume

$$f(\infty) = \infty.$$

Elementary transformations

The following three types of fractional linear transformations are called *elementary*:

1. $z \mapsto z + w$,
2. $z \mapsto w{\cdot}z$ for $w \neq 0$,
3. $z \mapsto \frac{1}{z}$.

The geometric interpretations. Suppose that O denotes the point with the complex coordinate 0.

The first map $z \mapsto z + w$, corresponds to the so-called *parallel translation* of the Euclidean plane, its geometric meaning should be evident.

The second map is called the *rotational homothety* with the center at O. That is, the point O maps to itself and any other point Z maps to a point Z' such that $OZ' = |w|{\cdot}OZ$ and $\measuredangle ZOZ' = \arg w$.

The third map can be described as a composition of the inversion in the unit circle centered at O and the reflection across $\mathbb{R}$ (the composition can be taken in any order). Indeed, $\arg z \equiv -\arg \frac{1}{z}$. Therefore,

$$\arg z = \arg(1/\bar{z});$$

that is, if the points Z and Z' have complex coordinates z and $1/\bar{z}$, then $Z' \in [OZ)$. Clearly, $OZ = |z|$ and $OZ' = |1/\bar{z}| = \frac{1}{|z|}$. Therefore, Z' is the inverse of Z in the unit circle centered at O.

Finally, $\frac{1}{z} = \overline{(1/\bar{z})}$ is the complex coordinate of the reflection of Z' across $\mathbb{R}$.

18.12. Proposition. *The map $f\colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a fractional linear transformation if and only if it can be expressed as a composition of elementary transformations.*

Proof; the "only if" part. Fix a fractional linear transformation

$$f(z) = \frac{a{\cdot}z + b}{c{\cdot}z + d}.$$

Assume $c \neq 0$. Then

$$\begin{aligned} f(z) &= \frac{a}{c} - \frac{a{\cdot}d - b{\cdot}c}{c{\cdot}(c{\cdot}z + d)} = \\ &= \frac{a}{c} - \frac{a{\cdot}d - b{\cdot}c}{c^2}{\cdot}\frac{1}{z + \frac{d}{c}}. \end{aligned}$$

That is,

❼ $$f(z) = f_4 \circ f_3 \circ f_2 \circ f_1(z),$$

where f_1, f_2, f_3, and f_4 are the following elementary transformations:

$$\begin{aligned} f_1(z) &= z + \tfrac{d}{c}, & f_2(z) &= \tfrac{1}{z}, \\ f_3(z) &= -\tfrac{a{\cdot}d - b{\cdot}c}{c^2}{\cdot}z, & f_4(z) &= z + \tfrac{a}{c}. \end{aligned}$$

If $c = 0$, then

$$f(z) = \frac{a{\cdot}z + b}{d}.$$

In this case, $f(z) = f_2 \circ f_1(z)$, where

$$f_1(z) = \tfrac{a}{d}{\cdot}z, \qquad f_2(z) = z + \tfrac{b}{d}.$$

"If" part. We need to show that by composing elementary transformations, we can only get fractional linear transformations. Note that it is sufficient to check that the composition of a fractional linear transformation

$$f(z) = \frac{a{\cdot}z + b}{c{\cdot}z + d}.$$

with any elementary transformation $z \mapsto z + w$, $z \mapsto w \cdot z$, and $z \mapsto \frac{1}{z}$ is a fractional linear transformation.

The latter is done by means of direct calculations.

$$\begin{aligned}\frac{a\cdot(z+w)+b}{c\cdot(z+w)+d} &= \frac{a\cdot z+(b+a\cdot w)}{c\cdot z+(d+c\cdot w)},\\ \frac{a\cdot(w\cdot z)+b}{c\cdot(w\cdot z)+d} &= \frac{(a\cdot w)\cdot z+b}{(c\cdot w)\cdot z+d},\\ \frac{a\cdot\frac{1}{z}+b}{c\cdot\frac{1}{z}+d} &= \frac{b\cdot z+a}{d\cdot z+c}.\end{aligned}$$

□

18.13. Corollary. *The image of a circline under a fractional linear transformation is a circline.*

Proof. By Proposition 18.12, it is sufficient to check that each elementary transformation sends a circline to a circline.

For the first and second elementary transformation, the latter is evident.

As it was noted above, the map $z \mapsto \frac{1}{z}$ is a composition of inversion and reflection. By Theorem 10.11, the inversion sends a circline to a circline. Hence the result. □

18.14. Exercise. *Show that the inverse of a fractional linear transformation is a fractional linear transformation.*

18.15. Exercise. *Given distinct values $z_0, z_1, z_\infty \in \hat{\mathbb{C}}$, construct a fractional linear transformation f such that*

$$f(z_0) = 0, \quad f(z_1) = 1 \quad \textit{and} \quad f(z_\infty) = \infty.$$

Show that such a transformation is unique.

18.16. Exercise. *Show that any inversion is a composition of the complex conjugation and a fractional linear transformation.*

Use Theorem 14.15 to conclude that any inversive transformation is either fractional linear transformation or a complex conjugate to a fractional linear transformation.

Complex cross-ratio

Let u, v, w, and z be four distinct complex numbers. Recall that the complex number

$$\frac{(u-w)\cdot(v-z)}{(v-w)\cdot(u-z)}$$

is called the *complex cross-ratio* of u, v, w, and z; it is denoted by $(u, v; w, z)$.

If one of the numbers u, v, w, z is ∞, then the complex cross-ratio has to be defined by taking the appropriate limit; in other words, we assume that $\frac{\infty}{\infty} = 1$. For example,

$$(u, v; w, \infty) = \frac{(u - w)}{(v - w)}.$$

Assume that U, V, W, and Z are the points with complex coordinates u, v, w, and z respectively. Note that

$$\begin{aligned}\frac{UW \cdot VZ}{VW \cdot UZ} &= |(u, v; w, z)|,\\ \measuredangle WUZ + \measuredangle ZVW &= \arg\frac{u - w}{u - z} + \arg\frac{v - z}{v - w} \equiv\\ &\equiv \arg(u, v; w, z).\end{aligned}$$

These equations make it possible to reformulate Theorem 10.6 using the complex coordinates the following way:

18.17. Theorem. *Let $UWVZ$ and $U'W'V'Z'$ be two quadrangles such that the points U', W', V', and Z' are inverses of U, W, V, and Z respectively. Assume u, w, v, z, u', w', v', and z' are the complex coordinates of U, W, V, Z, U', W', V', and Z' respectively.*

Then
$$(u', v'; w', z') = \overline{(u, v; w, z)}.$$

The following exercise is a generalization of the theorem above. It has a short solution using Proposition 18.12.

18.18. Exercise. *Show that complex cross-ratios are* invariant *under fractional linear transformations.*

That is, if a fractional linear transformation maps four distinct complex numbers u, v, w, z to complex numbers u', v', w', z' respectively, then

$$(u', v'; w', z') = (u, v; w, z).$$

Schwarz–Pick theorem

The following theorem shows that the metric in the conformal disc model naturally appears in other branches of mathematics. We do not give its proof, but it can be found in any textbook on geometric complex analysis.

Suppose that $\mathbb{D}$ denotes the unit disc in the complex plane centered at 0; that is, a complex number z belongs to $\mathbb{D}$ if and only if $|z| < 1$.

Let us use the disc $\mathbb{D}$ as an h-plane in the conformal disc model; the h-distance between $z, w \in \mathbb{D}$ will be denoted by $d_h(z, w)$; that is,

$$d_h(z, w) := ZW_h,$$

where Z and W are h-points with complex coordinates z and w respectively.

A function $f\colon \mathbb{D} \to \mathbb{C}$ is called *holomorphic* if for every $z \in \mathbb{D}$ there is a complex number s such that

$$f(z + w) = f(z) + s{\cdot}w + o(|w|).$$

In other words, f is *complex-differentiable* at any $z \in \mathbb{D}$. The complex number s is called the *derivative* of f at z, or briefly $s = f'(z)$.

18.19. Schwarz–Pick theorem. *Assume $f\colon \mathbb{D} \to \mathbb{D}$ is a holomorphic function. Then*

$$d_h(f(z), f(w)) \leqslant d_h(z, w)$$

for any $z, w \in \mathbb{D}$.

If the equality holds for one pair of distinct numbers $z, w \in \mathbb{D}$, then it holds for any pair. In this case, f is a fractional linear transformation as well as a motion of the h-plane.

18.20. Exercise. *Show that if a fractional linear transformation f appears in the equality case of Schwarz–Pick theorem, then it can be written as*

$$f(z) = \frac{v{\cdot}z + \bar{w}}{w{\cdot}z + \bar{v}}.$$

where v and w are complex constants such that $|v| > |w|$.

18.21. Exercise. *Recall that hyperbolic tangent* th *is defined on page 99. Show that*

$$\mathrm{th}[\tfrac{1}{2}{\cdot}d_h(z, w)] = \left|\frac{z - w}{1 - z{\cdot}\bar{w}}\right|.$$

Conclude that the inequality in Schwarz–Pick theorem can be rewritten as

$$\left|\frac{z' - w'}{1 - z'{\cdot}\bar{w}'}\right| \leqslant \left|\frac{z - w}{1 - z{\cdot}\bar{w}}\right|,$$

where $z' = f(z)$ and $w' = f(w)$.

18.22. Exercise. *Show that the Schwarz lemma stated below follows from Schwarz–Pick theorem.*

18.23. Schwarz lemma. *Let $f\colon \mathbb{D} \to \mathbb{D}$ be a holomorphic function and $f(0) = 0$. Then $|f(z)| \leqslant |z|$ for any $z \in \mathbb{D}$.*

Moreover, if equality holds for some $z \neq 0$, then there is a unit complex number u such that $f(z) = u{\cdot}z$ for any $z \in \mathbb{D}$.

Chapter 19

Geometric constructions

Geometric constructions have great pedagogical value as an introduction to mathematical proofs. The geometric constructions were introduced at the end of Chapter 5 and since that moment they were used everywhere in the book.

In this chapter, we briefly discuss classical geometric constructions.

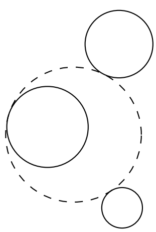
Classical problems

In this section, we list a couple of classical construction problems; each known for more than a thousand years.

The solutions of the following two problems are quite nontrivial.

19.1. Problem of Brahmagupta. *Construct an inscribed quadrangle with given sides.*

19.2. Problem of Apollonius. *Construct a circle that is tangent to three given circles.*



Several solutions of this problem based on different ideas are presented in [9]. The following exercise is a simplified version of the problem of Apollonius, which is still nontrivial.

19.3. Exercise. *Construct a circle that passes thru a given point and is tangent to two intersecting lines.*

The following three problems cannot be solved in principle; that is, the needed compass-and-ruler construction does not exist.

Doubling the cube. *Construct the side of a new cube that has the volume twice as big as the volume of a given cube.*

In other words, given a segment of the length a, one needs to construct a segment of length $\sqrt[3]{2}\cdot a$.

Squaring the circle. *Construct a square with the same area as a given circle.*

If r is the radius of the given circle, we need to construct a segment of length $\sqrt{\pi}\cdot r$.

Angle trisection. *Divide the given angle into three equal angles.*

In fact, there is no compass-and-ruler construction that trisects angle with measure $\frac{\pi}{3}$. The existence of such a construction would imply constructability of a regular 9-gon which is prohibited by the following famous result:

A *regular n-gon* inscribed in a circle with center O is a sequence of points $A_1 \dots A_n$ on the circle such that

$$\measuredangle A_n O A_1 = \measuredangle A_1 O A_2 = \dots = \measuredangle A_{n-1} O A_n = \pm\tfrac{2}{n}\cdot\pi.$$

The points $A_1, \dots, A_n$ are *vertices*, the segments $[A_1A_2], \dots, [A_nA_1]$ are *sides* and the remaining segments $[A_iA_j]$ are *diagonals* of the n-gon.

Construction of a regular n-gon, therefore, is reduced to the construction of an angle with size $\frac{2}{n}\cdot\pi$.

19.4. Gauss–Wantzel theorem. *A regular n-gon can be constructed with a ruler and a compass if and only if n is the product of a power of* 2 *and any number of distinct Fermat primes.*

A *Fermat prime* is a prime number of the form $2^k{+}1$ for some integer k. Only five Fermat primes are known today:

$$3, 5, 17, 257, 65537.$$

For example,

- ◇ one can construct a regular 340-gon since $340 = 2^2\cdot5\cdot17$ and 5, as well as 17, are Fermat primes;
- ◇ one cannot construct a regular 7-gon since 7 is not a Fermat prime;
- ◇ one cannot construct a regular 9-gon; altho $9 = 3\cdot3$ is a product of two Fermat primes, these primes are not distinct.

The impossibility of these constructions was proved only in the 19$^{\text{th}}$ century. The method used in the proofs is indicated in the next section.

Constructible numbers

In the classical compass-and-ruler constructions initial configuration can be completely described by a finite number of points; each line is defined by two points on it and each circle is described by its center and a point on it (equivalently, you may describe a circle by three points on it).

In the same way, the result of construction can be described by a finite collection of points.

We may always assume that the initial configuration has at least two points; if not add one or two points to the configuration. Moreover, applying a scaling to the whole plane, we can assume that the first two points in the initial configuration lie at distance 1 from each other.

In this case, we can choose a coordinate system, such that one of the initial points is the origin $(0,0)$ and yet another initial point has the coordinates $(1,0)$. In this coordinate system, the initial configuration of n points is described by $2\cdot n-4$ numbers — their coordinates $x_3, y_3, \dots$ $\dots, x_n, y_n$.

It turns out that the coordinates of any point constructed with a compass and ruler can be written thru the numbers $x_3, y_3, \dots, x_n, y_n$ using the four arithmetic operations "$+$", "$-$", "$\cdot$", "$/$" and the square root "$\sqrt{\ }$".

For example, assume we want to find the points $X_1 = (x_1, y_1)$ and $X_2 = (x_2, y_2)$ of the intersections of a line passing thru $A = (x_A, y_A)$ and $B = (x_B, y_B)$ and the circle with center $O = (x_O, y_O)$ that passes thru the point $W = (x_W, y_W)$. Let us write the equations of the circle and the line in the coordinates (x, y):

$$\begin{cases} (x-x_O)^2 + (y-y_O)^2 = (x_W-x_O)^2 + (y_W-y_O)^2, \\ \quad (x-x_A)\cdot(y_B-y_A) = (y-y_A)\cdot(x_B-x_A). \end{cases}$$

Note that coordinates (x_1, y_1) and (x_2, y_2) of the points X_1 and X_2 are solutions of this system. Expressing y from the second equation and substituting the result in the first one, gives us a quadratic equation in x, which can be solved using "$+$", "$-$", "$\cdot$", "$/$" and "$\sqrt{\ }$" only.

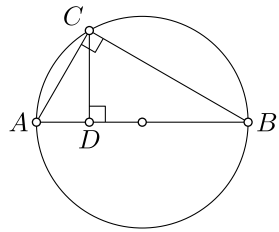
The same can be performed for the intersection of two circles. The intersection of two lines is even simpler; it is described as a solution of two linear equations and can be expressed using only four arithmetic operations; the square root "$\sqrt{\ }$" is not needed.

On the other hand, it is easy to make compass-and-ruler constructions that produce segments of the lengths $a + b$ and $a - b$ from two given segments of lengths $a > b$.

To perform "$\cdot$", "$/$" and "$\sqrt{\ }$" consider the following diagram: let $[AB]$ be a diameter of a circle; fix a point C on the circle and let D be the foot point of C on $[AB]$. By Corollary 9.8, the angle ACB is right. Therefore

$$\triangle ABC \sim \triangle ACD \sim \triangle CBD.$$

It follows that $AD\cdot BD = CD^2$.



Using this diagram, one should guess a solution to the following exercise.

19.5. Exercise. *Given two line segments with lengths a and b, give a ruler-and-compass construction of a segments with lengths $\frac{a^2}{b}$ and $\sqrt{a{\cdot}b}$.*

Taking 1 for a or b above, we can produce $\sqrt{a}$, a^2, $\frac{1}{b}$. Combining these constructions we can produce $a{\cdot}b = (\sqrt{a{\cdot}b})^2$, $\frac{a}{b} = a{\cdot}\frac{1}{b}$. In other words we produced a *compass-and-ruler calculator* which can do "+", "−", "·", "/", and "$\sqrt{\ }$".

The discussion above gives a sketch of the proof of the following theorem:

19.6. Theorem. *Assume that the initial configuration of geometric construction is given by the points $A_1 = (0, 0)$, $A_2 = (1, 0)$, $A_3 = (x_3, y_3), \dots$ $\dots, A_n = (x_n, y_n)$. Then a point $X = (x, y)$ can be constructed using a compass-and-ruler construction if and only if both coordinates x and y can be expressed from the integer numbers and x_3, y_3, x_4, $y_4, \dots, x_n, y_n$ using the arithmetic operations "+", "−", "·", "/", and the square root "$\sqrt{\ }$".*

The numbers that can be expressed from the given numbers using the arithmetic operations and the square root "$\sqrt{\ }$" are called *constructible*; if the list of given numbers is not given, then we can only use the integers.

The theorem above translates any compass-and-ruler construction problem into a purely algebraic language. For example:

- ⋄ The impossibility of a solution for doubling the cube problem states that $\sqrt[3]{2}$ is not a constructible number. That is $\sqrt[3]{2}$ cannot be expressed thru integers using "+", "−", "·", "/", and "$\sqrt{\ }$".
- ⋄ The impossibility of a solution for squaring the circle states that $\sqrt{\pi}$, or equivalently π, is not a constructible number.
- ⋄ The Gauss–Wantzel theorem says for which integers n the number $\cos\frac{2\cdot\pi}{n}$ is constructible.

Some of these statements might look evident, but rigorous proofs require some knowledge of abstract algebra (namely, field theory) which is out of the scope of this book.

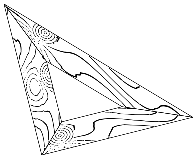
In the next section, we discuss similar but simpler examples of impossible constructions with an unusual tool.

19.7. Exercise.

(a) *Show that diagonal or regular pentagon is $\frac{1+\sqrt{5}}{2}$ times larger than its side.*

(b) *Use (a) to make a compass-and-ruler construction of a regular pentagon.*

Constructions with a set-square



A set-square is a construction tool shown in the picture — it can produce a line thru a given point that makes the angles $\frac{\pi}{2}$ or $\pm\frac{\pi}{4}$ to a given line and it can be also used as a ruler; that is, it can produce a line thru a given pair of points.

19.8. Exercise. *Trisect a given segment with a set-square.*

Let us consider set-square constructions. Following the same lines as in the previous section, we can define *set-square constructible numbers* and prove the following analog of Theorem 19.6:

19.9. Theorem. *Assume that the initial configuration of a geometric construction is given by the points $A_1 = (0,0)$, $A_2 = (1,0)$, $A_3 = (x_3, y_3), \dots, A_n = (x_n, y_n)$. Then a point $X = (x, y)$ can be constructed using a set-square construction if and only if both coordinates x and y can be expressed from the integer numbers and x_3, y_3, x_4, $y_4, \dots, x_n$, y_n using the arithmetic operations "$+$", "$-$", "$\cdot$", and "$/$" only.*

Let us apply this theorem to show the impossibility of some constructions with a set-square.

Note that if all the coordinates $x_3, y_3, \dots, x_n, y_n$ are rational numbers, then the theorem above implies that with a set-square, one can only construct the points with rational coordinates. A point with both rational coordinates is called *rational*, and if at least one of the coordinates is irrational, then the point is called *irrational*.

19.10. Exercise. *Show that an equilateral triangle in the Euclidean plane has at least one irrational point.*

Conclude that with a set-square, one cannot construct an equilateral triangle with a given base.

Verifications

Suppose we need to verify that a given configuration is defined by a certain property. Is it possible to do this task by a geometric construction with given tools? We assume that we can *verify* that two constructed points coincide.

Evidently, if a configuration is constructible, then it is verifiable — simply repeat the construction and check if the result is the same. Some nonconstuctable configurations are verifiable. For example, it does not pose a problem to verify that the given angle is trisected while it is impossible to trisect a given angle with ruler and compass. A regular 7-gon provides another example of that type — it is easy to verify, while Gauss–Wantzel theorem states that it is impossible to construct with ruler and compass.

Since we did not prove the impossibility of angle trisection and Gauss–Wantzel theorem, the following example might be more satisfactory. It is based on Exercise 19.10 which states that it is impossible to construct an equilateral with set-square only.

19.11. Exercise. *Make a set-square construction verifying that a given triangle is equilateral.*

This observation leads to a source of impossible constructions in a stronger sense — those that are even not verifiable.

The following example is closely related to Exercise 10.9. Recall that a *circumtool* produces a circle passing thru any given three points or a line if all three points lie on one line.

Problem. *Show that with a circumtool only, it is impossible to verify that a given point is the center of a given circle* Γ. *In particular, it is impossible to construct the center with a circumtool only.*

Remark. In geometric constructions, we allow to choose some free points, say any point on the plane, or a point on a constructed line, or a point that does not lie on a constructed line, or a point on a given line that does not lie on a given circle, and so on.

In principle, when you make such a free choice it is possible to make a right construction by accident. Nevertheless, we do not accept such a coincidence as true construction; we say that a construction produces the center if it produces it for any free choices.

Solution. Arguing by contradiction, assume we have a verifying construction.

Apply an inversion in a circle perpendicular to Γ to the whole construction. According to Corollary 10.16, the circle Γ maps to itself. Since the inversion sends a circline to a circline, we get that the whole construction is mapped to an equivalent construction; that is, a constriction with a different choice of free points.

According to Exercise 10.8, the inversion sends the center of Γ to another point. However, this construction claims that this another point is the center — a contradiction. □

A similar example of impossible constructions for a ruler and a parallel tool is given in Exercise 14.8.

Let us discuss another example for a ruler-only construction. Note that ruler-only constructions are invariant with respect to the projective transformations. In particular, to solve the following exercise, it is sufficient to construct a projective transformation that fixes two points and moves their midpoint.

19.12. Exercise. *Show that there is no ruler-only construction verifying that a given point is a midpoint of a given segment. In particular, it is impossible to construct the midpoint only with a ruler.*

The following theorem is a stronger version of the exercise above.

19.13. Theorem. *There is no ruler-only construction verifying that a given point is the center of a given circle. In particular, it is impossible to construct the center only with a ruler.*

The proof uses the construction in Exercise 16.6.

Sketch of the proof. The same argument as in the problem above shows that it is sufficient to construct a projective transformation that sends the given circle Γ to a circle Γ' such that the center of Γ' is not the image of the center of Γ.

Choose a circle Γ that lies in the plane Π in the Euclidean space. By Theorem 16.3, the inverse of a circle in a sphere is a circle or a line. Fix a sphere Σ with the center O so that the inversion Γ' of Γ is a circle and the plane Π' containing Γ' is not parallel to Π; any sphere Σ in a general position will do.

Let Z and Z' denote the centers of Γ and Γ'. Note that $Z' \notin (OZ)$. It follows that the perspective projection $\Pi \to \Pi'$ with center O sends Γ to Γ', but Z' is not the image of Z. □

Comparison of construction tools

We say that one set of tools is *stronger* than another if any configuration of points that can be constructed with the second set can be constructed with the first set as well. If in addition, there is a configuration constructible with the first set, but not constructible with the second, then we say that the first set is *strictly stronger* than the second. Otherwise (that is, if any configuration that can be constructed with the first set can be constructed with the second), we say that the sets of tools are *equivalent*.

For example, *Mohr–Mascheroni theorem* states that compass alone is equivalent to compass and ruler. Note that one may not construct a line with a compass, but since we consider only configurations of points we do not have to. One may think that a *line is constructed* if we construct two points on it.

For sure compass and ruler form a stronger set than compass alone. Therefore Mohr–Mascheroni theorem will follow once we solve the following two construction problems:

(i) Given four points X, Y, P, and Q, construct the intersection of the lines (XY) and (PQ) with compass only.

(ii) Given four points X, Y, and a circle Γ, construct the intersection of the lines (XY) and Γ with compass only.

Indeed, once we have these two constructions, we can do every step of a compass-and-ruler construction using a compass alone.

19.14. Exercise. *Compare the following set of tools: (a) a ruler and compass, (b) a set-square, and (c) a ruler and a parallel tool.*

Another classical example is the so-called *Poncelet–Steiner theorem*; it states that the set of compass and ruler is equivalent to ruler alone, provided that a single circle and its center are given.

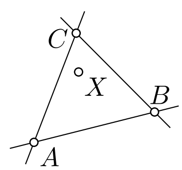
Chapter 20

Area

Area will be defined as a function satisfying certain conditions (page 164). The so-called *Lebesgue measure* gives an example of such a function. In particular, the existence Lebesgue measure implies the existence of an area function. This construction is included in any textbook in real analysis.

Based solely on its existence, we develop the concept of area with no cheating. We choose this approach since any rigorous introduction to area is tedious. We do not want to cheat and at the same time we do not want to waste your time; soon or later you will have to learn Lebesgue measure if it is not done already.

Solid triangles



We say that the point X lies *inside* a nondegenerate triangle ABC if the following three condition hold:

◇ A and X lie on the same side of the line (BC);

◇ B and X lie on the same side of the line (CA);

◇ C and X lie on the same side of the line (AB).

The set of all points inside $\triangle ABC$ and on its sides $[AB]$, $[BC]$, $[CA]$ will be called *solid triangle* ABC and denoted by $\blacktriangle ABC$.

20.1. Exercise. *Show that any solid triangle is* convex*; that is, for any pair of points* $X, Y \in \blacktriangle ABC$*, then the line segment* $[XY]$ *lies in* $\blacktriangle ABC$*.*

The notations $\triangle ABC$ and $\blacktriangle ABC$ look similar, they also have close but different meanings, which better not to confuse. Recall that $\triangle ABC$ is an ordered triple of distinct points (see page 16), while $\blacktriangle ABC$ is an infinite set of points.

In particular, $\blacktriangle ABC = \blacktriangle BAC$ for any triangle ABC. Indeed, any point that belongs to the set $\blacktriangle ABC$ also belongs to the set $\blacktriangle BAC$ and the other way around. On the other hand, $\triangle ABC \neq \triangle BAC$ simply because the ordered triple of points (A, B, C) is distinct from the ordered triple (B, A, C).

Note that $\blacktriangle ABC \cong \blacktriangle BAC$ even if $\triangle ABC \ncong \triangle BAC$, where congruence of the sets $\blacktriangle ABC$ and $\blacktriangle BAC$ is understood the following way:

20.2. Definition. *Two sets $\mathcal{S}$ and $\mathcal{T}$ in the plane are called* congruent *(briefly $\mathcal{S} \cong \mathcal{T}$) if $\mathcal{T} = f(\mathcal{S})$ for some motion f of the plane.*

If $\triangle ABC$ is not degenerate and

$$\blacktriangle ABC \cong \blacktriangle A'B'C',$$

then after relabeling the vertices of $\triangle ABC$ we will have

$$\triangle ABC \cong \triangle A'B'C'.$$

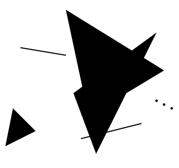
Indeed it is sufficient to show that if f is a motion that maps $\blacktriangle ABC$ to $\blacktriangle A'B'C'$, then f maps each vertex of $\triangle ABC$ to a vertex $\triangle A'B'C'$. The latter follows from the characterization of vertices of solid triangles given in the following exercise:

20.3. Exercise. *Let $\triangle ABC$ be nondegenerate and $X \in \blacktriangle ABC$. Show that X is a vertex of $\triangle ABC$ if and only if there is a line ℓ that intersects $\blacktriangle ABC$ at the single point X.*

Polygonal sets

Elementary set on the plane is a set of one of the following three types:

- ⋄ one-point set;
- ⋄ segment;
- ⋄ solid triangle.



A set in the plane is called *polygonal* if it can be presented as a union of a finite collection of elementary sets.

Note that according to this definition, the empty set $\varnothing$ is a polygonal set. Indeed, $\varnothing$ is a union of an empty collection of elementary sets.

A polygonal set is called *degenerate* if it can be presented as a union of a finite collection of one-point sets and segments.

If X and Y lie on opposite sides of the line (AB), then the union $\blacktriangle AXB \cup \blacktriangle BYA$ is a polygonal set which is called *solid quadrangle* $AXBY$ and denoted by $\blacksquare AXBY$. In particular, we can talk about *solid parallelograms*, *rectangles*, and *squares*.

Typically a polygonal set admits many presentations as a union of a finite collection of elementary sets. For example, if $\square AXBY$ is a parallelogram, then

$$\blacksquare AXBY = \blacktriangle AXB \cup \blacktriangle AYB = \blacktriangle XAY \cup \blacktriangle XBY.$$

20.4. Exercise. *Show that a solid square is not degenerate.*

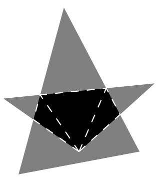
20.5. Exercise. *Show that a circle is not a polygonal set.*

20.6. Claim. *For any two polygonal sets $\mathcal{P}$ and $\mathcal{Q}$, the union $\mathcal{P} \cup \mathcal{Q}$, as well as the intersection $\mathcal{P} \cap \mathcal{Q}$, are also polygonal sets.*

A class of sets that is closed with respect to union and intersection is called a *ring of sets.* The claim above, therefore, states that polygonal sets in the plane form a ring of sets.

Informal proof. Let us present $\mathcal{P}$ and $\mathcal{Q}$ as a union of a finite collection of elementary sets $\mathcal{P}_1, \dots, \mathcal{P}_k$ and $\mathcal{Q}_1, \dots, \mathcal{Q}_n$ respectively.

Note that



$$\mathcal{P} \cup \mathcal{Q} = \mathcal{P}_1 \cup \dots \cup \mathcal{P}_k \cup \mathcal{Q}_1 \cup \dots \cup \mathcal{Q}_n.$$

Therefore, $\mathcal{P} \cup \mathcal{Q}$ is polygonal.

Note that $\mathcal{P} \cap \mathcal{Q}$ is the union of sets $\mathcal{P}_i \cap \mathcal{Q}_j$ for all i and j. Therefore, in order to show that $\mathcal{P} \cap \mathcal{Q}$ is polygonal, it is sufficient to show that each $\mathcal{P}_i \cap \mathcal{Q}_j$ is polygonal for any pair i, j.

The diagram should suggest an idea for the proof of the latter statement in case if $\mathcal{P}_i$ and $\mathcal{Q}_j$ are solid triangles. The other cases are simpler; a formal proof can be built on Exercise 20.1 □

Definition of area

Area is defined as a function $\mathcal{P} \mapsto \operatorname{area}\mathcal{P}$ that returns a nonnegative real number $\operatorname{area}\mathcal{P}$ for any polygonal set $\mathcal{P}$ and satisfying the following conditions:

(a) $\operatorname{area}\mathcal{K}_1 = 1$ *where $\mathcal{K}_1$ a solid square with unit side;*

(b) *the conditions*

$$\mathcal{P} \cong \mathcal{Q} \quad \Rightarrow \quad \operatorname{area}\mathcal{P} = \operatorname{area}\mathcal{Q};$$
$$\mathcal{P} \subset \mathcal{Q} \quad \Rightarrow \quad \operatorname{area}\mathcal{P} \leqslant \operatorname{area}\mathcal{Q};$$
$$\operatorname{area}\mathcal{P} + \operatorname{area}\mathcal{Q} = \operatorname{area}(\mathcal{P} \cup \mathcal{Q}) + \operatorname{area}(\mathcal{P} \cap \mathcal{Q})$$

hold for any two polygonal sets $\mathcal{P}$ and $\mathcal{Q}$.

The first condition is called *normalization*; essentially it says that a solid unit square is used as a unit to measure area. The three conditions in *(b)* are called *invariance*, *monotonicity*, and *additivity*.

The Lebesgue measure provides an example of area function; namely, if one takes area of $\mathcal{P}$ to be its Lebesgue measure, then the function $\mathcal{P} \mapsto \operatorname{area}\mathcal{P}$ satisfies the above conditions.

The construction of the Lebesgue measure can be found in any textbook on real analysis. We do not discuss it here.

If the reader is not familiar with the Lebesgue measure, then he should take the existence of area function as granted; it might be considered as an additional axiom altho it follows from the axioms I–V.

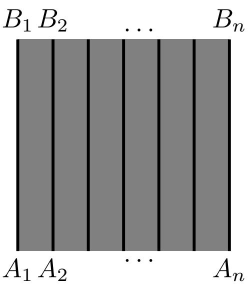
Vanishing area and subdivisions

20.7. Proposition. *Any one-point set, as well as any segment in the Euclidean plane, have vanishing area.*

Proof. Fix a line segment $[AB]$. Consider a sold square $\blacksquare ABCD$.

Note that given a positive integer n, there are n disjoint segments $[A_1B_1], \dots, [A_nB_n]$ in $\blacksquare ABCD$, such that each $[A_iB_i]$ is congruent to $[AB]$ in the sense of the Definition 20.2.

Applying invariance, additivity, and monotonicity of the area function, we get that



$$\begin{aligned} n\cdot \operatorname{area}[AB] &= \operatorname{area}([A_1B_1] \cup \dots \cup [A_nB_n]) \leqslant \\ &\leqslant \operatorname{area}(\blacksquare ABCD) \end{aligned}$$

That is,

$$\operatorname{area}[AB] \leqslant \tfrac{1}{n}\cdot \operatorname{area}(\blacksquare ABCD)$$

for any positive integer n. Therefore, $\operatorname{area}[AB] \leqslant 0$. On the other hand, by definition of area, $\operatorname{area}[AB] \geqslant 0$, hence

$$\operatorname{area}[AB] = 0.$$

For any one-point set $\{A\}$ we have that $\{A\} \subset [AB]$. Therefore,

$$0 \leqslant \operatorname{area}\{A\} \leqslant \operatorname{area}[AB] = 0.$$

Whence $\operatorname{area}\{A\} = 0$. ☐

20.8. Corollary. *Any degenerate polygonal set has vanishing area.*

Proof. Let $\mathcal{P}$ be a degenerate set, say

$$\mathcal{P} = [A_1B_1] \cup \dots \cup [A_nB_n] \cup \{C_1, \dots, C_k\}.$$

Since area is nonnegative by definition, applying additivity several times, we get that

$$\begin{aligned}\operatorname{area} \mathcal{P} \leqslant \operatorname{area}[A_1B_1] + \dots + \operatorname{area}[A_nB_n] + \\ + \operatorname{area}\{C_1\} + \dots + \operatorname{area}\{C_k\}.\end{aligned}$$

By Proposition 20.7, the right-hand side vanishes.

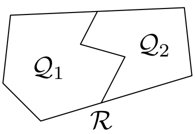
On the other hand, $\operatorname{area} \mathcal{P} \geqslant 0$, hence the result. □

We say that polygonal set $\mathcal{P}$ is *subdivided* into two polygonal sets $\mathcal{Q}_1, \dots, \mathcal{Q}_n$ if $\mathcal{P} = \mathcal{Q}_1 \cup \dots \cup \mathcal{Q}_n$ and the intersection $\mathcal{Q}_i \cap \mathcal{Q}_j$ is degenerate for any pair i and j. (Recall that according to Claim 20.6, the intersections $\mathcal{Q}_i \cap \mathcal{Q}_j$ are polygonal.)

20.9. Proposition. *Assume polygonal sets $\mathcal{P}$ is subdivided into polygonal sets $\mathcal{Q}_1, \dots, \mathcal{Q}_n$. Then*

$$\operatorname{area} \mathcal{P} = \operatorname{area} \mathcal{Q}_1 + \dots + \operatorname{area} \mathcal{Q}_n.$$

Proof. Assume $n = 2$; by additivity of area,



$$\operatorname{area} \mathcal{P} = \operatorname{area} \mathcal{Q}_1 + \operatorname{area} \mathcal{Q}_2 - \operatorname{area}(\mathcal{Q}_1 \cap \mathcal{Q}_2).$$

Since $\mathcal{Q}_1 \cap \mathcal{Q}_2$ is degenerate, by Corollary 20.8,

$$\operatorname{area}(\mathcal{Q}_1 \cap \mathcal{Q}_2) = 0.$$

Applying this formula a few times we get the general case. Indeed, if $\mathcal{P}$ is subdivided into $\mathcal{Q}_1, \dots, \mathcal{Q}_n$, then

$$\begin{aligned}\operatorname{area} \mathcal{P} &= \operatorname{area} \mathcal{Q}_1 + \operatorname{area}(\mathcal{Q}_2 \cup \dots \cup \mathcal{Q}_n) = \\ &= \operatorname{area} \mathcal{Q}_1 + \operatorname{area} \mathcal{Q}_2 + \operatorname{area}(\mathcal{Q}_3 \cup \dots \cup \mathcal{Q}_n) = \\ &\ \ \vdots \\ &= \operatorname{area} \mathcal{Q}_1 + \operatorname{area} \mathcal{Q}_2 + \dots + \operatorname{area} \mathcal{Q}_n.\end{aligned}$$ □

Remark. Two polygonal sets $\mathcal{P}$ and $\mathcal{P}'$ are called *equidecomposable* if they admit subdivisions into polygonal sets $\mathcal{Q}_1, \dots, \mathcal{Q}_n$ and $\mathcal{Q}'_1, \dots, \mathcal{Q}'_n$ such that $\mathcal{Q}_i \cong \mathcal{Q}'_i$ for each i.

According to the proposition, if $\mathcal{P}$ and $\mathcal{P}$ are equidecomposable, then $\operatorname{area} \mathcal{P} = \operatorname{area} \mathcal{P}'$. A converse to this statement also holds; namely, *if two nondegenerate polygonal sets have equal area, then they are equidecomposable.*

The last statement was proved by William Wallace, Farkas Bolyai and Paul Gerwien. The analogous statement in three dimensions, known as *Hilbert's third problem*, is false; it was proved by Max Dehn.

Area of solid rectangles

20.10. Theorem. *A solid rectangle with sides a and b has area $a{\cdot}b$.*

20.11. Algebraic lemma. *Assume that a function s returns a nonnegative real number $s(a,b)$ for any pair of positive real numbers (a,b) and it satisfies the following identities:*

$$\begin{aligned} s(1,1) &= 1;\\ s(a,b+c) &= s(a,b)+s(a,c)\\ s(a+b,c) &= s(a,c)+s(b,c) \end{aligned}$$

for any $a,b,c>0$. Then

$$s(a,b)=a{\cdot}b$$

for any $a,b>0$.

The proof is similar to the proof of Lemma 14.14.

Proof. Note that if $a>a'$ and $b>b'$ then

❶ $$s(a,b)\geqslant s(a',b').$$

Indeed, since s returns nonnegative numbers, we get that

$$\begin{aligned} s(a,b) &= s(a',b)+s(a-a',b)\geqslant\\ &\geqslant s(a',b)=\\ &\geqslant s(a',b')+s(a',b-b')\geqslant\\ &\geqslant s(a',b'). \end{aligned}$$

Applying the second and third identity few times we get that

$$s(a,m{\cdot}b)=s(m{\cdot}a,b)=m{\cdot}s(a,b)$$

for any positive integer m. Therefore

$$\begin{aligned} s(\tfrac{k}{l},\tfrac{m}{n}) &= k{\cdot}s(\tfrac{1}{l},\tfrac{m}{n})=\\ &= k{\cdot}m{\cdot}s(\tfrac{1}{l},\tfrac{1}{n})=\\ &= k{\cdot}m{\cdot}\tfrac{1}{l}{\cdot}s(1,\tfrac{1}{n})=\\ &= k{\cdot}m{\cdot}\tfrac{1}{l}{\cdot}\tfrac{1}{n}{\cdot}s(1,1)=\\ &= \tfrac{k}{l}{\cdot}\tfrac{m}{n} \end{aligned}$$

for any positive integers k, l, m, and n. That is, the needed identity holds for any pair of rational numbers $a=\frac{k}{l}$ and $b=\frac{m}{n}$.

Arguing by contradiction, assume $s(a,b) \neq a\cdot b$ for some pair of positive real numbers (a,b). We will consider two cases: $s(a,b) > a\cdot b$ and $s(a,b) < a\cdot b$.

If $s(a,b) > a\cdot b$, we can choose a positive integer n such that

❷ $$s(a,b) > (a+\tfrac{1}{n})\cdot(b+\tfrac{1}{n}).$$

Set $k = \lfloor a\cdot n\rfloor + 1$ and $m = \lfloor b\cdot n\rfloor + 1$; equivalently, k and m are positive integers such that

$$a < \tfrac{k}{n} \leqslant a+\tfrac{1}{n} \quad\text{and}\quad b < \tfrac{m}{n} \leqslant b+\tfrac{1}{n}.$$

By ❶, we get that

$$\begin{aligned} s(a,b) &\leqslant s(\tfrac{k}{n},\tfrac{m}{n}) = \\ &= \tfrac{k}{n}\cdot\tfrac{m}{n} \leqslant \\ &\leqslant (a+\tfrac{1}{n})\cdot(b+\tfrac{1}{n}), \end{aligned}$$

which contradicts ❷.

The case $s(a,b) < a\cdot b$ is similar. Fix a positive integer n such that $a > \tfrac{1}{n}$, $b > \tfrac{1}{n}$, and

❸ $$s(a,b) < (a-\tfrac{1}{n})\cdot(b-\tfrac{1}{n}).$$

Set $k = \lceil a\cdot n\rceil - 1$ and $m = \lceil b\cdot n\rceil - 1$; that is,

$$a > \tfrac{k}{n} \geqslant a-\tfrac{1}{n} \quad\text{and}\quad b > \tfrac{m}{n} \geqslant b-\tfrac{1}{n}.$$

Applying ❶ again, we get that

$$\begin{aligned} s(a,b) &\geqslant s(\tfrac{k}{n},\tfrac{m}{n}) = \\ &= \tfrac{k}{n}\cdot\tfrac{m}{n} \geqslant \\ &\geqslant (a-\tfrac{1}{n})\cdot(b-\tfrac{1}{n}), \end{aligned}$$

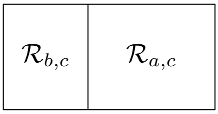
which contradicts ❸. □

Proof of Theorem 20.10. Suppose that $\mathcal{R}_{a,b}$ denotes the solid rectangle with sides a and b. Set

$$s(a,b) = \operatorname{area}\mathcal{R}_{a,b}.$$

By definition of area, $s(1,1) = \operatorname{area}(\mathcal{K}) = 1$. That is, the first identity in the algebraic lemma holds.

Note that the rectangle $\mathcal{R}_{a+b,c}$ can be subdivided into two rectangles congruent to $\mathcal{R}_{a,c}$ and $\mathcal{R}_{b,c}$. Therefore, by Proposition 20.9,



$\mathcal{R}_{a+b,c}$

$$\operatorname{area}\mathcal{R}_{a+b,c} = \operatorname{area}\mathcal{R}_{a,c} + \operatorname{area}\mathcal{R}_{b,c}$$

That is, the second identity in the algebraic lemma holds. The proof of the third identity is similar.

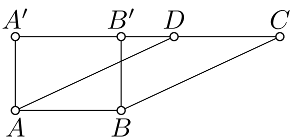
It remains to apply the algebraic lemma. □

Area of solid parallelograms

20.12. Proposition. *Let $\square ABCD$ be a parallelogram in the Euclidean plane, $a = AB$ and h be the distance between the lines (AB) and (CD). Then*

$$\operatorname{area}(\blacksquare ABCD) = a{\cdot}h.$$

Proof. Let A' and B' denote the foot points of A and B on the line (CD).



Note that $ABB'A'$ is a rectangle with sides a and h. By Proposition 20.10,

❹ $$\operatorname{area}(\blacksquare ABB'A') = h{\cdot}a.$$

Without loss of generality, we may assume that $\blacksquare ABCA'$ contains $\blacksquare ABCD$ and $\blacksquare ABB'A'$. In this case, $\blacksquare ABCA'$ admits two subdivisions:

$$\blacksquare ABCA' = \blacksquare ABCD \cup \blacktriangle AA'D = \blacksquare ABB'A' \cup \blacktriangle BB'C.$$

By Proposition 20.9,

❺ $$\begin{aligned}\operatorname{area}(\blacksquare ABCD) + \operatorname{area}(\blacktriangle AA'D) &= \\ = \operatorname{area}(\blacksquare ABB'A') + \operatorname{area}(\blacktriangle BB'C).\end{aligned}$$

Note that

❻ $$\triangle AA'D \cong \triangle BB'C.$$

Indeed, since the quadrangles $ABB'A'$ and $ABCD$ are parallelograms, by Lemma 7.18, we have that $AA' = BB'$, $AD = BC$, and $DC = AB = A'B'$. It follows that $A'D = B'C$. Applying the SSS congruence condition, we get ❻.

In particular,

❼ $$\operatorname{area}(\blacktriangle BB'C) = \operatorname{area}(\blacktriangle AA'D).$$

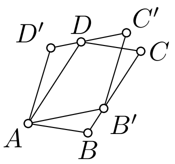
Subtracting ❼ from ❺, we get that

$$\text{area}(\blacksquare ABCD) = \text{area}(\blacksquare ABB'D).$$

It remains to apply ❹. □

20.13. Exercise. *Assume $\square ABCD$ and $\square AB'C'D'$ are two parallelograms such that $B' \in [BC]$ and $D \in$ $\in [C'D']$. Show that*

$$\text{area}(\blacksquare ABCD) = \text{area}(\blacksquare AB'C'D').$$



Area of solid triangles

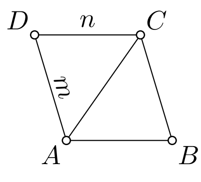
20.14. Theorem. *Let h_A be the altitude from A in $\triangle ABC$ and $a = BC$. Then*

$$\text{area}(\blacktriangle ABC) = \tfrac{1}{2}{\cdot}a{\cdot}h_A.$$

Proof. Draw the line m thru A that is parallel to (BC) and line n thru C parallel to (AB). Note that the lines m and n are not parallel; denote by D their point of intersection. By construction, $\square ABCD$ is a parallelogram.

Note that $\blacksquare ABCD$ admits a subdivision into $\blacktriangle ABC$ and $\blacktriangle CDA$. Therefore,

$$\text{area}(\blacksquare ABCD) = \text{area}(\blacktriangle ABC) + \text{area}(\blacktriangle CDA)$$



Since $\square ABCD$ is a parallelogram, Lemma 7.18 implies that

$$AB = CD \quad \text{and} \quad BC = DA.$$

Therefore, by the SSS congruence condition, we have $\triangle ABC \cong \triangle CDA$. In particular

$$\text{area}(\blacktriangle ABC) = \text{area}(\blacktriangle CDA).$$

From above and Proposition 20.12, we get that

$$\begin{aligned}\text{area}(\blacktriangle ABC) &= \tfrac{1}{2}{\cdot}\,\text{area}(\blacksquare ABCD) = \\ &= \tfrac{1}{2}{\cdot}h_A{\cdot}a\end{aligned}$$

□

20.15. Exercise. *Let h_A, h_B, and h_C denote the altitudes of $\triangle ABC$ from vertices A, B and C respectively. Note that from Theorem 20.14, it follows that*

$$h_A{\cdot}BC = h_B{\cdot}CA = h_C{\cdot}AB.$$

Give a proof of this statement without using Theorem 20.14.

20.16. Exercise. *Assume M lies inside the parallelogram $ABCD$; that is, M belongs to the solid parallelogram $\blacksquare ABCD$, but does not lie on its sides. Show that*

$$\operatorname{area}(\blacktriangle ABM) + \operatorname{area}(\blacktriangle CDM) = \tfrac{1}{2}\cdot \operatorname{area}(\blacksquare ABCD).$$

20.17. Exercise. *Assume that diagonals of a nondegenerate quadrangle $ABCD$ intersect at point M. Show that*

$$\operatorname{area}(\blacktriangle ABM)\cdot \operatorname{area}(\blacktriangle CDM) = \operatorname{area}(\blacktriangle BCM)\cdot \operatorname{area}(\blacktriangle DAM).$$

20.18. Exercise. *Let r be the inradius of $\triangle ABC$ and p be its* semiperimeter*; that is, $p = \frac{1}{2}\cdot(AB + BC + CA)$. Show that*

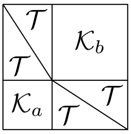
$$\operatorname{area}(\blacktriangle ABC) = p\cdot r.$$

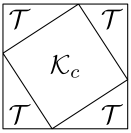
20.19. Exercise. *Show that any polygonal set admits a subdivision into a finite collection of solid triangles and a degenerate set. Conclude that for any polygonal set, its area is uniquely defined.*

Area method

In this section, we will give examples of slim proofs using the properties of area. Note that these proofs are not truly elementary since the price one pays to introduce the area function is high.

We start with the proof of the Pythagorean theorem. In the Elements of Euclid, the Pythagorean theorem was formulated as equality ❽ below and the proof used a similar technique.





Proof. We need to show that if a and b are legs and c is the hypotenuse of a right triangle, then

$$a^2 + b^2 = c^2.$$

Suppose that $\mathcal{T}$ denotes the right solid triangle with legs a and b and $\mathcal{K}_x$ the solid square with side x.

Let us construct two subdivisions of $\mathcal{K}_{a+b}$:

1. Subdivide $\mathcal{K}_{a+b}$ into two solid squares congruent to $\mathcal{K}_a$ and $\mathcal{K}_b$ and 4 solid triangles congruent to $\mathcal{T}$, see the first diagram.

2. Subdivide $\mathcal{K}_{a+b}$ into one solid square congruent to $\mathcal{K}_c$ and 4 solid right triangles congruent to $\mathcal{T}$, see the second diagram.

Applying Proposition 20.9 few times, we get that

$$\begin{aligned}\operatorname{area}\mathcal{K}_{a+b} &= \operatorname{area}\mathcal{K}_a + \operatorname{area}\mathcal{K}_b + 4\cdot \operatorname{area}\mathcal{T} = \\ &= \operatorname{area}\mathcal{K}_c + 4\cdot \operatorname{area}\mathcal{T}.\end{aligned}$$

Therefore,

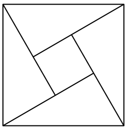
❽ $$\operatorname{area}\mathcal{K}_a + \operatorname{area}\mathcal{K}_b = \operatorname{area}\mathcal{K}_c.$$

By Theorem 20.10,

$$\operatorname{area}\mathcal{K}_x = x^2,$$

for any $x > 0$. Hence the statement follows. □

20.20. Exercise. *Build another proof of the Pythagorean theorem based on the diagram.*



(In the notations above it shows a subdivision of $\mathcal{K}_c$ into $\mathcal{K}_{a-b}$ and four copies of $\mathcal{T}$ if $a > b$.)

20.21. Exercise. *Show that the sum of distances from a point to the sides of an equilateral triangle is the same for all points inside the triangle.*

20.22. Claim. *Assume that two triangles ABC and $A'B'C'$ in the Euclidean plane have equal altitudes dropped from A and A' respectively. Then*

$$\frac{\operatorname{area}(\blacktriangle A'B'C')}{\operatorname{area}(\blacktriangle ABC)} = \frac{B'C'}{BC}.$$

In particular, the same identity holds if $A = A'$ and the bases $[BC]$ and $[B'C']$ lie on one line.

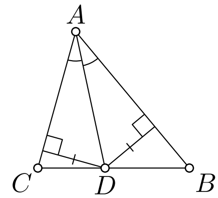
Proof. Let h be the altitude. By Theorem 20.14,

$$\frac{\operatorname{area}(\blacktriangle A'B'C')}{\operatorname{area}(\blacktriangle ABC)} = \frac{\frac{1}{2}\cdot h\cdot B'C'}{\frac{1}{2}\cdot h\cdot BC} = \frac{B'C'}{BC}.$$ □

Now let us show how to use this claim to prove Lemma 8.8. First, let us recall its statement:

Lemma 8.8. *If $\triangle ABC$ is nondegenerate and its angle bisector at A intersects $[BC]$ at the point D. Then*

$$\frac{AB}{AC} = \frac{DB}{DC}.$$



Proof. Applying Claim 20.22, we get that

$$\frac{\operatorname{area}(\blacktriangle ABD)}{\operatorname{area}(\blacktriangle ACD)} = \frac{BD}{CD}.$$

By Proposition 8.10 the triangles ABD and ACD have equal altitudes from D. Applying Claim 20.22 again, we get that

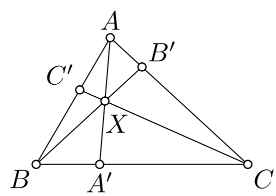
$$\frac{\operatorname{area}(\blacktriangle ABD)}{\operatorname{area}(\blacktriangle ACD)} = \frac{AB}{AC}$$

and hence the result. □

Suppose ABC is a nondegenerate triangle and A' lies between B and C. In this case, the line segment $[AA']$ is called *cevian*[1] of $\triangle ABC$ at A. The second statement in the following exercise is called *Ceva's theorem.*

20.23. Exercise. *Let ABC be a nondegenerate triangle. Suppose its cevians $[AA']$, $[BB']$ and $[CC']$ intersect at one point X. Show that*

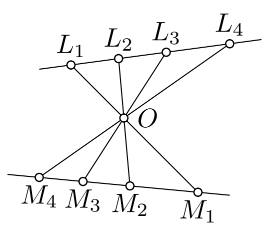
$$\frac{\operatorname{area}(\blacktriangle ABX)}{\operatorname{area}(\blacktriangle BCX)} = \frac{AB'}{B'C},$$
$$\frac{\operatorname{area}(\blacktriangle BCX)}{\operatorname{area}(\blacktriangle CAX)} = \frac{BC'}{C'A},$$
$$\frac{\operatorname{area}(\blacktriangle CAX)}{\operatorname{area}(\blacktriangle ABX)} = \frac{CA'}{A'B}.$$



Conclude that

$$\frac{AB'{\cdot}CA'{\cdot}BC'}{B'C{\cdot}A'B{\cdot}C'A} = 1.$$

20.24. Exercise. *Suppose that points L_1, L_2, L_3, L_4 lie on a line ℓ and points M_1, M_2, M_3, M_4 lie on a line m. Assume that the lines (L_1M_1), (L_2M_2), (L_3M_3), and (L_4M_4) pass thru a point O that does not lie on ℓ nor m.*



(a) Apply Claim 20.22 to show that

$$\frac{\operatorname{area}\blacktriangle OL_iL_j}{\operatorname{area}\blacktriangle OM_iM_j} = \frac{OL_i{\cdot}OL_j}{OM_i{\cdot}OM_j}$$

for any $i \neq j$.

---

[1]it is named after Giovanni Ceva and pronounced as *chevian*.

(b) Use (a) to prove that

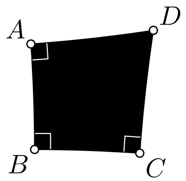
$$\frac{L_1L_2 \cdot L_3L_4}{L_2L_3 \cdot L_4L_1} = \frac{M_1M_2 \cdot M_3M_4}{M_2M_3 \cdot M_4M_1};$$

that is, the quadruples (L_1, L_2, L_3, L_4) *and* (M_1, M_2, M_3, M_4) *have the same cross-ratio.*

Area in the neutral planes and spheres

Area can be defined in the neutral planes and spheres. In the definition, the solid unit square $\mathcal{K}_1$ has to be exchanged to a fixed nondegenerate polygonal set $\mathcal{U}$. One has to make such a change for good reason — hyperbolic plane and sphere have no squares.

In this case, the set $\mathcal{U}$ plays the role of the unit measure for the area and changing $\mathcal{U}$ will require conversion of area units.



According to the standard convention, the set $\mathcal{U}$ is taken so that on small scales area behaves like area in the Euclidean plane. Say, if $\mathcal{K}_a$ denotes the solid quadrangle $\blacksquare ABCD$ with right angles at A, B, and C such that $AB = BC = a$, then we may assume that

$$\tfrac{1}{a^2} \cdot \operatorname{area} \mathcal{K}_a \to 1 \quad \text{as} \quad a \to 0.$$

This convention works equally well for spheres and neutral planes, including the Euclidean plane. In spherical geometry equivalently we may assume that if r is the radius of the sphere, then the area of the whole sphere is $4 \cdot \pi \cdot r^2$.

Recall that *defect of triangle* $\triangle ABC$ is defined as

$$\operatorname{defect}(\triangle ABC) := \pi - |\measuredangle ABC| - |\measuredangle BCA| - |\measuredangle CAB|.$$

It turns out that any neutral plane or sphere there is a real number k such that

❾ $$k \cdot \operatorname{area}(\blacktriangle ABC) + \operatorname{defect}(\triangle ABC) = 0$$

for any $\triangle ABC$.

This number k is called *curvature*; $k = 0$ for the Euclidean plane, $k = -1$ for the h-plane, $k = 1$ for the unit sphere, and $k = \frac{1}{r^2}$ for the sphere of radius r.

Since the angles of ideal triangle vanish, any ideal triangle in h-plane has area π. Similarly, in the unit sphere the area of equilateral triangle with right angles has area $\frac{\pi}{2}$; since whole sphere can be subdivided in eight such triangles, we get that the area of unit sphere is $4 \cdot \pi$.

The identity ❾ can be used as an alternative way to introduce area function; it works on spheres and all neutral planes, except for the Euclidean plane.

Quadrable sets

A set $\mathcal{S}$ in the plane is called *quadrable* if for any $\varepsilon > 0$ there are two polygonal sets $\mathcal{P}$ and $\mathcal{Q}$ such that

$$\mathcal{P} \subset \mathcal{S} \subset \mathcal{Q} \quad \text{and} \quad \operatorname{area}\mathcal{Q} - \operatorname{area}\mathcal{P} < \varepsilon.$$

If $\mathcal{S}$ is quadrable, its area can be defined as the necessarily unique real number $s = \operatorname{area}\mathcal{S}$ such that the inequality

$$\operatorname{area}\mathcal{Q} \leqslant s \leqslant \operatorname{area}\mathcal{P}$$

holds for any polygonal sets $\mathcal{P}$ and $\mathcal{Q}$ such that $\mathcal{P} \subset \mathcal{S} \subset \mathcal{Q}$.

20.25. Exercise. *Let $\mathcal{D}$ be the unit disc; that is, $\mathcal{D}$ is a set that contains the unit circle Γ and all the points inside Γ.*

Show that $\mathcal{D}$ is a quadrable set.

Since $\mathcal{D}$ is quadrable, the expression $\operatorname{area}\mathcal{D}$ makes sense and the constant π can be defined as $\pi = \operatorname{area}\mathcal{D}$.

It turns out that the class of quadrable sets is the largest class for which the area function satisfying the conditions on page 164 is uniquely defined.

If you do not require uniqueness, then there are ways to extend area function to all bounded sets. (A set in the plane is called *bounded* if it lies inside of a circle.) On the sphere and hyperbolic plane, there is no similar construction. If you wonder why, read about *doubling the ball*, a paradox of Felix Hausdorff, Stefan Banach, and Alfred Tarski.

Hints

1.2. Check the triangle inequality for $A = 0$, $B = 1$ and $C = 2$.

1.3. Only the triangle inequality requires a proof — the rest of the conditions in Definition 1.1 are evident. Let $A = (x_A, y_A)$, $B = (x_B, y_B)$, and $C = (x_C, y_C)$. Set

$$x_1 = x_B - x_A, \qquad y_1 = y_B - y_A,$$
$$x_2 = x_C - x_B, \qquad y_2 = y_C - y_B.$$

(a). The inequality

$$d_1(A, C) \leqslant d_1(A, B) + d_1(B, C)$$

can be written as

$$|x_1 + x_2| + |y_1 + y_2| \leqslant |x_1| + |y_1| + |x_2| + |y_2|.$$

The latter follows since $|x_1 + x_2| \leqslant |x_1| + |x_2|$ and $|y_1 + y_2| \leqslant |y_1| + |y_2|$.

(b). The inequality

❶ $$d_2(A, C) \leqslant d_2(A, B) + d_2(B, C)$$

can be written as

$$\sqrt{\left(x_1 + x_2\right)^2 + \left(y_1 + y_2\right)^2} \leqslant \sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2}.$$

Take the square of the left and the right-hand sides, simplify, take the square again and simplify again. You should get the following inequality:

$$0 \leqslant \left(x_1 \cdot y_2 - x_2 \cdot y_1\right)^2,$$

which is equivalent to ❶ and evidently true.

(c). The inequality

$$d_\infty(A, C) \leqslant d_\infty(A, B) + d_\infty(B, C)$$

can be written as

❷ $$\max\{|x_1 + x_2|, |y_1 + y_2|\} \leqslant \max\{|x_1|, |y_1|\} + \max\{|x_2|, |y_2|\}.$$

Without loss of generality, we may assume that

$$\max\{|x_1 + x_2|, |y_1 + y_2|\} = |x_1 + x_2|.$$

Further,

$$|x_1 + x_2| \leqslant |x_1| + |x_2| \leqslant \max\{|x_1|, |y_1|\} + \max\{|x_2|, |y_2|\}.$$

Hence ❷ follows.

1.4. Sum up four triangle inequalities.

1.5. If $A \neq B$, then $d_{\mathcal{X}}(A, B) > 0$. Since f is distance-preserving,

$$d_{\mathcal{Y}}(f(A), f(B)) = d_{\mathcal{X}}(A, B).$$

Therefore, $d_{\mathcal{Y}}(f(A), f(B)) > 0$; hence $f(A) \neq f(B)$.

1.6. Set $f(0) = a$ and $f(1) = b$. Note that $b = a + 1$ or $a - 1$. Moreover, $f(x) = a \pm x$ and at the same time, $f(x) = b \pm (x - 1)$ for any x.

If $b = a + 1$, it follows that $f(x) = a + x$ for any x.

In the same way, if $b = a - 1$, it follows that $f(x) = a - x$ for any x.

1.7. Show that the map $(x, y) \mapsto (x + y, x - y)$ is an isometry $(\mathbb{R}^2, d_1) \to (\mathbb{R}^2, d_\infty)$. That is, you need to check if this map is bijective and distance-preserving.

1.8. First prove that *two points $A = (x_A, y_A)$ and $B = (x_B, y_B)$ on the Manhattan plane have a unique midpoint if and only if $x_A = x_B$ or $y_A = y_B$*; compare with the example on page 17.

Then use the above statement to prove that any motion of the Manhattan plane can be written in one of the following two ways:

$$(x, y) \mapsto (\pm x + a, \pm y + b) \qquad \text{or} \qquad (x, y) \mapsto (\pm y + b, \pm x + a),$$

for some fixed real numbers a and b. (In each case we have 4 choices of signs, so for a fixed pair (a, b) we have 8 distinct motions.)

1.10. Assume three points A, B, and C lie on one line. Note that in this case one of the triangle inequalities with the points A, B, and C becomes an equality.

Set $A = (-1, 1)$, $B = (0, 0)$, and $C = (1, 1)$. Show that for d_1 and d_2 all the triangle inequalities with the points A, B, and C are strict. It follows that the graph is not a line.

For d_∞ show that $(x, |x|) \mapsto x$ gives the isometry of the graph to $\mathbb{R}$. Conclude that the graph is a line in $(\mathbb{R}^2, d_\infty)$.

1.11. Spell the definitions of line and motion.

1.12. Fix an isometry $f\colon (PQ) \to \mathbb{R}$ such that $f(P) = 0$ and $f(Q) = q > 0$.

Assume that $f(X) = x$. By the definition of the half-line $X \in [PQ)$ if and only if $x \geqslant 0$. Show that the latter holds if and only if $|x - q| = \big||x| - |q|\big|$. Hence *(a)* follows.

To prove*(b)*, observe that $X \in [PQ]$ if and only if $0 \leqslant x \leqslant q$. Show that the latter holds if and only if $|x - q| + |x| = |q|$.

1.13. The equation $2{\cdot}\alpha \equiv 0$ means that $2{\cdot}\alpha = 2{\cdot}k{\cdot}\pi$ for some integer k. Therefore, $\alpha = k{\cdot}\pi$ for some integer k.

Equivalently, $\alpha = 2{\cdot}n{\cdot}\pi$ or $\alpha = (2{\cdot}n+1){\cdot}\pi$ for some integer n. The first identity means that $\alpha \equiv 0$ and the second means that $\alpha \equiv \pi$.

1.14. *(a).* By the triangle inequality, $|f(A') - f(A)| \leqslant d(A', A)$. Therefore, we can take $\delta = \varepsilon$.

(b). By the triangle inequality,

$$|f(A',B') - f(A,B)| \leqslant |f(A',B') - f(A,B')| + |f(A,B') - f(A,B)| \leqslant \\ \leqslant d(A',A) + d(B',B).$$

Therefore, we can take $\delta = \frac{\varepsilon}{2}$.

1.15. Fix $A \in \mathcal{X}$ and $B \in \mathcal{Y}$ such that $f(A) = B$.

Fix $\varepsilon > 0$. Since g is continuous at B, there is a positive value δ_1 such that

$$d_{\mathcal{Z}}(g(B'), g(B)) < \varepsilon \quad \text{if} \quad d_{\mathcal{Y}}(B', B) < \delta_1.$$

Since f is continuous at A, there is $\delta_2 > 0$ such that

$$d_{\mathcal{Y}}(f(A'), f(A)) < \delta_1 \quad \text{if} \quad d_{\mathcal{X}}(A', A) < \delta_2.$$

Since $f(A) = B$, we get that

$$d_{\mathcal{Z}}(h(A'), h(A)) < \varepsilon \quad \text{if} \quad d_{\mathcal{X}}(A', A) < \delta_2.$$

Hence the result.

2.1. By Axiom I, there are at least two points in the plane. Therefore, by Axiom II, the plane contains a line. To prove *(a)*, it remains to note that a line is an infinite set of points. To prove *(b)* apply in addition Axiom III.

2.3. By Axiom II, $(OA) = (OA')$. Therefore, the statement boils down to the following:

Assume $f\colon \mathbb{R} \to \mathbb{R}$ is a motion of the line that sends $0 \mapsto 0$ and one positive number to a positive number, then f is an identity map.

The latter follows from 1.6.

2.6. By Proposition 2.5, $\measuredangle AOA = 0$. It remains to apply Axiom IIIa.

2.10. Apply Proposition 2.5, Theorem 2.8 and 1.13.

2.11. By Axiom IIIb, $2{\cdot}\measuredangle BOC \equiv 2{\cdot}\measuredangle AOC - 2{\cdot}\measuredangle AOB \equiv 0$. By 1.13, it implies that $\measuredangle BOC$ is either 0 or π. It remains to apply 2.6 and Theorem 2.8 respectively in these two cases.

2.12. Fix two points A and B provided by Axiom I.

Fix a real number $0 < \alpha < \pi$. By Axiom IIIa there is a point C such that $\measuredangle ABC = \alpha$. Use Proposition 2.2 to show that $\triangle ABC$ is nondegenerate.

2.14. Applying Proposition 2.13, we get that $\measuredangle AOC = \measuredangle BOD$. It remains to apply Axiom IV.

3.1. Set $\alpha = \measuredangle AOB$ and $\beta = \measuredangle BOA$. Note that $\alpha = \pi$ if and only if $\beta = \pi$. Otherwise, $\alpha = -\beta$. Hence the result.

3.3. Set $\alpha = \measuredangle ABC$, $\beta = \measuredangle A'B'C'$. Since $2\cdot\alpha \equiv 2\cdot\beta$, 1.13 implies that $\alpha \equiv \beta$ or $\alpha \equiv \beta+\pi$. In the latter case, the angles have opposite signs which is impossible.

Since $\alpha, \beta \in (-\pi, \pi]$, equality $\alpha \equiv \beta$ implies $\alpha = \beta$.

3.11. Note that O and A' lie on the same side of (AB). Analogously O and B' lie on the same side of (AB). Hence the result.

3.13. Apply Theorem 3.7 for $\triangle PQX$ and $\triangle PQY$ and then apply Corollary 3.10*a*.

3.14. We can assume that $A' \neq B, C$ and $B' \neq A, C$; otherwise, the statement trivially holds.

Note that (BB') does not intersect $[A'C]$. Applying Pasch's theorem (3.12) for $\triangle AA'C$ and (BB'), we get that (BB') intersects $[AA']$; denote the point of intersection by M.

In the same way, we get that (AA') intersects $[BB']$; that is, M lies on $[AA']$ and $[BB']$.

3.15. Assume that Z is the point of intersection.

Note that $Z \neq P$ and $Z \neq Q$. Therefore, $Z \notin (PQ)$.

Show that Z and X lie on one side of (PQ). Repeat the argument to show that Z and Y lie on one side of (PQ). It follows that X and Y lie on the same side of (PQ) — a contradiction.

3.20. The "only-if" part follows from the triangle inequality. To prove the "if" part, observe that Theorem 3.17 implies the existence of a triangle with sides r_1, r_2, and d. Use this triangle to show that there is a point X such that $O_1X = r_1$ and $O_2X = r_2$, where O_1 and O_2 are the centers of the corresponding circles.

4.3. Apply Theorem 4.2 twice.

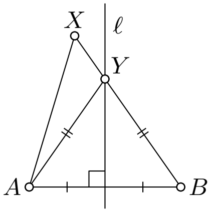
4.6. Consider the points D and D', such that M is the midpoint o $[CD]$ and M' is the midpoint of $[C'D']$. Show that $\triangle BCD \cong \triangle B'C'D'$ and use it to prove that $\triangle A'B'C' \cong$ $\cong \triangle ABC$.

4.7. *(a)* Apply SAS.

(b) Use *(a)* and apply SSS.

4.8. Without loss of generality, we may assume that X is distinct from A, B, and C. Set $f(X) = X'$; assume $X' \neq X$.

Note that $AX = AX'$, $BX = BX'$, and $CX = CX'$. By SSS we get that $\measuredangle ABX =$ $= \pm\measuredangle ABX'$. Since $X \neq X'$, we get that $\measuredangle ABX \equiv -\measuredangle ABX'$. In the same way, we get that $\measuredangle CBX \equiv -\measuredangle CBX'$. Subtracting these two identities from each other, we get that $\measuredangle ABC \equiv -\measuredangle ABC$. Conclude that $\measuredangle ABC = 0$ or π. That is, $\triangle ABC$ is degenerate — a contradiction.



5.1. By Axiom IIIb and Theorem 2.8, we have $\measuredangle XOA - \measuredangle XOB \equiv$ $\equiv \pi$. Since $|\measuredangle XOA|$, $|\measuredangle XOB| \leqslant \pi$, we get that $|\measuredangle XOA| +$ $+ |\measuredangle XOB| = \pi$. Hence the statement follows.

5.3. Assume X and A lie on the same side of ℓ.

Note that A and B lie on opposite sides of ℓ. Therefore, by Corollary 3.10, $[AX]$ does not intersect ℓ and $[BX]$ intersects ℓ; suppose that Y denotes the intersection point.

Note that $BX = AY + YX \geqslant AX$. Since $X \notin \ell$, by Theorem 5.2 we have $BX \neq BA$. Therefore $BX > AX$.

This way we proved the "if" part. To prove the "only if" part, you need to switch A and B and repeat the above argument.

5.4. Apply 5.3, Theorem 4.1, and 3.3.

5.8. Choose an arbitrary nondegenerate triangle ABC. Suppose that $\triangle\hat{A}\hat{B}\hat{C}$ denotes its image after the motion.

If $A \neq \hat{A}$, apply the reflection across the perpendicular bisector of $[A\hat{A}]$. This reflection sends A to $\hat{A}$. Let B' and C' denote the reflections of B and C respectively.

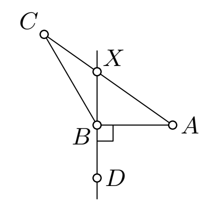
If $B' \neq \hat{B}$, apply the reflection across the perpendicular bisector of $[B'\hat{B}]$. This reflection sends B' to $\hat{B}$. Note that $\hat{A}\hat{B} = \hat{A}B'$; that is, $\hat{A}$ lies on the perpendicular bisector. Therefore, $\hat{A}$ reflects to itself. Suppose that C'' denotes the reflection of C'.

Finally, if $C'' \neq \hat{C}$, apply the reflection across $(\hat{A}\hat{B})$. Note that $\hat{A}\hat{C} = \hat{A}C''$ and $\hat{B}\hat{C} = \hat{B}C''$; that is, (AB) is the perpendicular bisector of $[C''\hat{C}]$. Therefore, this reflection sends C'' to $\hat{C}$.

Apply 4.8 to show that the composition of the constructed reflections coincides with the given motion.

5.9. Note that $\measuredangle XBA = \measuredangle ABP$, $\measuredangle PBC = \measuredangle CBY$. Therefore,

$$\measuredangle XBY \equiv \measuredangle XBP + \measuredangle PBY \equiv 2\cdot(\measuredangle ABP + \measuredangle PBC) \equiv 2\cdot\measuredangle ABC.$$

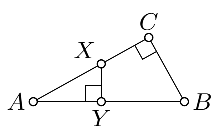


5.11. If $\angle ABC$ is right, the statement follows from Lemma 5.10. Therefore, we can assume that $\angle ABC$ is obtuse.

Draw a line (BD) perpendicular to (BA). Since $\angle ABC$ is obtuse, the angles DBA and DBC have opposite signs.

By Corollary 3.10, A and C lie on opposite sides of (BD). In particular, $[AC]$ intersects (BD) at a point; denote it by X.

Note that $AX < AC$ and by Lemma 5.10, $AB \leqslant AX$.



5.12. Let Y be the foot point of X on (AB). Apply Lemma 5.10 to show that $XY < AX \leqslant AC < AB$.

5.13. Let O be the center of the circle. Note that we can assume that $O \neq P$.

Assume P lies between X and Y. By 5.1, we can assume that $\angle OPX$ is right or obtuse. By 5.11, $OP < OX$; that is, P lies inside Γ.

If P does not lie between X and Y, we can assume that X lies between P and Y. Since $OX = OY$, 5.11 implies that $\angle OXY$ is acute. Therefore, $\angle OXP$ is obtuse. Applying 5.11 again we get that $OP > OX$; that is, P lies outside Γ.

5.14. Apply Theorem 5.2.

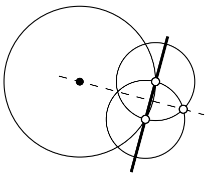
5.16. Use 5.14 and Theorem 5.5.

5.18. Let P' be the reflection of P across (OO'). Note that P' lies on both circles and $P' \neq P$ if and only if $P \notin (OO')$.

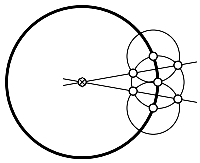
5.19. Apply 5.18.

5.20. Let A and B be the points of intersection. Note that the centers lie on the perpendicular bisector of the segment $[AB]$.

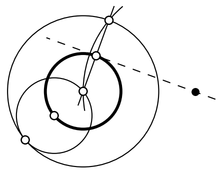
5.22–5.25. The given data is marked in bold.



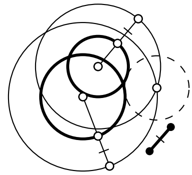
5.22



5.23



5.24



5.25

6.3. By the AA similarity condition, the transformation multiplies the sides of any nondegenerate triangle by some number that may depend on the triangle.

Note that for any two nondegenerate triangles that share one side this number is the same. Applying this observation to a chain of triangles leads to a solution.

6.5. Apply that $\triangle ADC \sim \triangle CDB$.

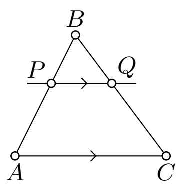
6.6. Apply the Pythagorean theorem (6.4) and the SSS congruence condition.

6.7. By the AA similarity condition (6.2), $\triangle AYC \sim \triangle BXC$. Conclude that $\frac{YC}{AC} = \frac{XC}{BC}$. Apply the SAS similarity condition to show that $\triangle ABC \sim \triangle YXC$.

Similarly, apply AA and equality of vertical angles to prove that $\triangle AZX \sim \triangle BZY$ and use SAS to show that $\triangle ABZ \sim \triangle YXZ$.

7.4. Apply Proposition 7.1 to show that $k \parallel m$. By Corollary 7.3, $k \parallel n \Rightarrow m \parallel n$. The latter contradicts that $m \perp n$.

7.5. Repeat the construction in 5.22 twice.



7.10. Since $\ell \parallel (AC)$, it cannot cross $[AC]$. By Pasch's theorem (3.12), ℓ has to cross another side of $\triangle ABC$. Therefore ℓ crosses $[BC]$; denote the point of intersection by Q.

Use the transversal property (7.9) to show that $\measuredangle BAC = \measuredangle BPQ$. The same argument shows that $\measuredangle ACB = \measuredangle PQB$; it remains to apply the AA similarity condition.

7.11. Assume we need to trisect segment $[AB]$. Construct a line $\ell \neq (AB)$ with four points A, C_1, C_2, C_3 such that C_1 and C_2 trisect $[AC_3]$. Draw the line (BC_3) and draw parallel lines thru C_1 and C_2. The points of intersections of these two lines with (AB) trisect the segment $[AB]$.

7.13. Apply twice Theorem 4.2 and twice Theorem 7.12.

7.14. If $\triangle ABC$ is degenerate, then one of the angle measures is π and the other two are 0. Hence the result.

Assume $\triangle ABC$ is nondegenerate. Set $\alpha = \measuredangle CAB$, $\beta = \measuredangle ABC$, and $\gamma = \measuredangle BCA$.

By Theorem 3.7, we may assume that $0 < \alpha, \beta, \gamma < \pi$. Therefore,

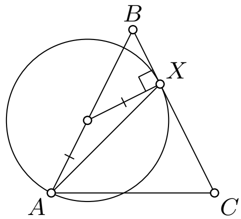
❶ $$0 < \alpha + \beta + \gamma < 3 \cdot \pi.$$

By Theorem 7.12,

❷ $$\alpha + \beta + \gamma \equiv \pi.$$

From ❶ and ❷ the result follows.

7.15. Apply twice Theorem 4.2 and once Theorem 7.12.



7.16. Suppose that O denotes the center of the circle.

Note that $\triangle AOX$ is isosceles and $\angle OXC$ is right. Applying 7.12 and 4.2 and simplifying, you should get $4 \cdot \measuredangle CAX \equiv \pi$.

Show that $\angle CAX$ has to be acute. It follows then that $\measuredangle CAX = \pm\frac{\pi}{4}$.

7.17. Apply Theorem 7.12 to $\triangle ABC$ and $\triangle BDA$.

7.19. Since $\triangle ABC$ is isosceles, $\measuredangle CAB = \measuredangle BCA$.

By SSS, $\triangle ABC \cong \triangle CDA$. Therefore, $\pm\measuredangle DCA = \measuredangle BCA = \measuredangle CAB$.

Since $D \neq C$, we get "$-$" in the last formula. Use the transversal property (7.9) to show that $(AB) \parallel (CD)$. Repeat the argument to show that $(AD) \parallel (BC)$

7.20. By Lemma 7.18 and SSS, $AC = BD$ if and only if $\measuredangle ABC = \pm\measuredangle BCD$. By the transversal property (7.9), $\measuredangle ABC + \measuredangle BCD \equiv \pi$.

Therefore, $AC = BD$ if and only if $\measuredangle ABC = \measuredangle BCD = \pm\frac{\pi}{2}$.

7.21. Fix a parallelogram $ABCD$. By Lemma 7.18, its diagonals $[AC]$ and $[BD]$ have a common midpoint; denote it by M.

Use SSS and Lemma 7.18 to show that

$$AB = CD \quad \Longleftrightarrow \quad \triangle AMB \cong \triangle AMD \quad \Longleftrightarrow \quad \measuredangle AMB = \pm\tfrac{\pi}{2}.$$

7.22. *(a).* Use the uniqueness of the parallel line (Theorem 7.2).

(b) Use Lemma 7.18 and the Pythagorean theorem (6.4).

7.23. Set $A = (0,0)$, $B = (c,0)$, and $C = (x,y)$. Clearly, $AB = c$, $AC^2 = x^2 + y^2$ and $BC^2 = (c-x)^2 + y^2$.

It remains to show that there is a pair of real numbers (x,y) that satisfy the following system of equations:

$$\begin{cases} b^2 = x^2 + y^2 \\ a^2 = (c-x)^2 + y^2 \end{cases}$$

if $0 < a \leqslant b \leqslant c \leqslant a + c$.

7.24. Note that $MA = MB$ if and only if

$$(x - x_A)^2 + (y - y_A)^2 = (x - x_B)^2 + (y - y_B)^2,$$

where $M = (x,y)$. To prove the first part, simplify this equation. For the remaining parts use that any line is a perpendicular bisector to some line segment.

7.25. Rewrite it the following way and think

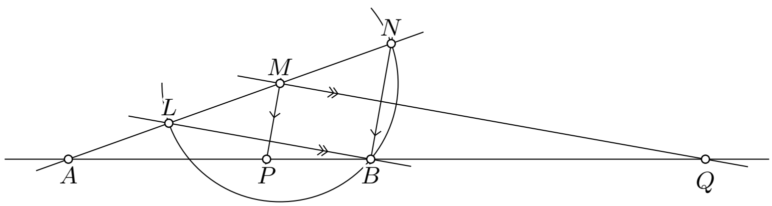
$$(x + \tfrac{a}{2})^2 + (y + \tfrac{b}{2})^2 = (\tfrac{a}{2})^2 + (\tfrac{b}{2})^2 - c.$$

7.26. We can choose the coordinates so that $B = (0, 0)$ and $A = (a, 0)$ for some $a > 0$. If $M = (x, y)$, then the equation $AM = k{\cdot}BM$ can be written in coordinates as

$$k^2{\cdot}(x^2 + y^2) = (x - a)^2 + y^2.$$

It remains to rewrite this equation as in 7.25.

7.27. Assume $M \notin (AB)$. Show and use that the points P and Q constructed on the following diagram lie on the Apollonian circle.



8.2. Apply Theorem 8.1 and Theorem 5.2.

8.4. Note that $(AC) \perp (BH)$ and $(BC) \perp (AH)$ and apply Theorem 8.3.

(Note that each of A, B, C, H is the orthocenter of the remaining three; such a quadruple of points A, B, C, H is called *orthocentric system.*)

8.6. Use the idea from the proof of Theorem 8.5 to show that $(XY) \parallel (AC) \parallel (VW)$ and $(XV) \parallel (BD) \parallel (YW)$.

8.7. Let (BX) and (BY) be the internal and external bisectors of $\angle ABC$. Then

$$2{\cdot}\measuredangle XBY \equiv 2{\cdot}\measuredangle XBA + 2{\cdot}\measuredangle ABY \equiv \measuredangle CBA + \pi + 2{\cdot}\measuredangle ABC \equiv \pi + \measuredangle CBC = \pi$$

and hence the result.

8.9. If E is the point of intersection of (BC) with the external bisector of $\angle BAC$, then $\frac{AB}{AC} = \frac{EB}{EC}$. It can be proved along the same lines as Lemma 8.8.

8.12. Apply Lemma 8.8. See also the solution of 11.2.

8.13. Apply ASA to the two triangles that the bisector cuts from the original triangle.

8.14. Let I be the incenter. By SAS, we get that $\triangle AIZ \cong \triangle AIY$. Therefore, $AY = AZ$. In the same way, we get that $BX = BZ$ and $CX = CY$. Hence the result.

8.15. Let $\triangle ABC$ be the given acute triangle and $\triangle A'B'C'$ be its orthic triangle. Note that $\triangle AA'C \sim \triangle BB'C$. Use it to show that $\triangle A'B'C \sim \triangle ABC$.

In the same way, we get that $\triangle AB'C' \sim \triangle ABC$. It follows that $\measuredangle A'B'C = \measuredangle AB'C'$. Conclude that (BB') bisects $\angle A'B'C'$.

If $\triangle ABC$ is obtuse, then its orthocenter coincides with one of the *excenters* of $\triangle ABC$; that is, the point of intersection of two external and one internal bisectors of $\triangle ABC$.

8.16. Apply 4.2, 7.9 and 7.18.

9.3. *(a).* Apply Theorem 9.2 for $\angle XX'Y$ and $\angle X'YY'$ and Theorem 7.12 for $\triangle PYX'$.

(b) If P is inside of Γ, then P lies between X and X' and between Y and Y'. In this case, $\angle XPY$ is vertical to $\angle X'PY'$. If P is outside of Γ then $[PX) = [PX')$ and $[PY) = [PY')$. In both cases we have that $\measuredangle XPY = \measuredangle X'PY'$.

Applying Theorem 9.2 and 2.11, we get that

$$2{\cdot}\measuredangle Y'X'P \equiv 2{\cdot}\measuredangle Y'X'X \equiv 2{\cdot}\measuredangle Y'YX \equiv 2{\cdot}\measuredangle PYX.$$

According to Theorem 3.7, $\angle Y'X'P$ and $\angle PYX$ have the same sign; therefore $\measuredangle Y'X'P = \measuredangle PYX$. It remains to apply the AA similarity condition.

(c) Apply *(b)* assuming $[YY']$ is the diameter of Γ.

9.4. Apply 9.3*b* three times.

9.5. Let X and Y be the foot points of the altitudes from A and B. Suppose that O denotes the circumcenter.

By AA condition, $\triangle AXC \sim \triangle BYC$. Thus

$$\measuredangle A'OC \equiv 2{\cdot}\measuredangle A'AC \equiv -2{\cdot}\measuredangle B'BC \equiv -\measuredangle B'OC.$$

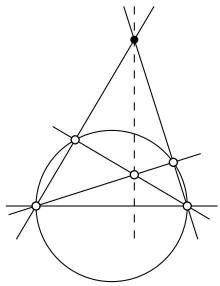
By SAS, $\triangle A'OC \cong \triangle B'OC$. Therefore, $A'C = B'C$.

9.9. Construct the circles Γ and Γ' on the diameters $[AB]$ and $[A'B']$ respectively. By Corollary 9.8, any point Z in the intersection $\Gamma \cap \Gamma'$ will do.

9.10. Note that $\measuredangle AA'B = \pm\frac{\pi}{2}$ and $\measuredangle AB'B = \pm\frac{\pi}{2}$. Then apply Corollary 9.13 to $\square AA'BB'$.

If O is the center of the circle, then $\measuredangle AOB \equiv 2{\cdot}\measuredangle AA'B \equiv \pi$. That is, O is the midpoint of $[AB]$.

9.11. Guess the construction from the diagram. To prove it, apply Theorem 8.3 and Corollary 9.8.



9.12. Denote by O the center of Γ. Use Corollary 9.8 to show that the points lie on the circle with diameter $[PO]$.

9.6. Apply the transversal property (7.9) and the theorem on inscribed angles (9.2).

9.14. Apply Corollary 9.13 twice for $\square ABYX$ and $\square ABY'X'$ and use the transversal property (7.9).

9.16. One needs to show that the lines $(A'B')$ and (XP) are not parallel; otherwise, the first line in the proof does not make sense.

In addition, we implicitly used the following identities:

$$2{\cdot}\measuredangle AXP \equiv 2{\cdot}\measuredangle AXY, \qquad 2{\cdot}\measuredangle ABP \equiv 2{\cdot}\measuredangle ABB', \qquad 2{\cdot}\measuredangle AA'B' \equiv 2{\cdot}\measuredangle AA'Y.$$

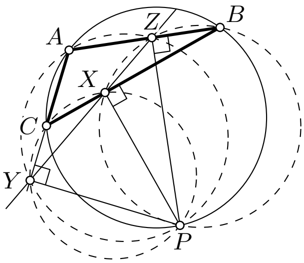
9.17. By Corollary 9.8, the points L, M, and N lie on the circle Γ with diameter $[OX]$. It remains to apply Theorem 9.2 for the circle Γ and two inscribed angles with vertex at O.

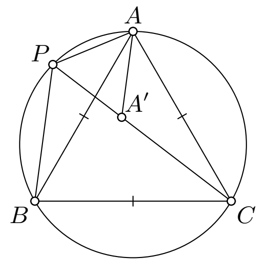
9.18. Let X, Y, and Z denote the foot points of P on (BC), (CA), and (AB) respectively.

Show that $\square AZPY$, $\square BXPZ$, $\square CYPX$, and $\square ABCP$ are inscribed. Use it to show that

$$2\cdot\measuredangle CXY \equiv 2\cdot\measuredangle CPY, \qquad 2\cdot\measuredangle BXZ \equiv 2\cdot\measuredangle BPZ,$$
$$2\cdot\measuredangle YAZ \equiv 2\cdot\measuredangle YPZ, \qquad 2\cdot\measuredangle CAB \equiv 2\cdot\measuredangle CPB.$$

Conclude that $2\cdot\measuredangle CXY \equiv 2\cdot\measuredangle BXZ$ and hence X, Y, and Z lie on one line.





9.21. Show that P lies on the arc opposite from ACB; conclude that $\measuredangle APC = \measuredangle CPB = \pm\frac{\pi}{3}$.

Choose a point $A' \in [PC]$ such that $PA' = PA$. Note that $\triangle APA'$ is equilateral. Prove and use that $\triangle AA'C \cong \triangle APB$.

9.24. If $C \in (AX)$, then the arc is the line segment $[AC]$ or the union of two half-lines in (AX) with vertices at A and C.

Assume $C \notin (AX)$. Let ℓ be the perpendicular line dropped from A to (AX) and m be the perpendicular bisector of $[AC]$.

Note that $\ell \nparallel m$; set $O = \ell \cap m$. Note that the circle with center O passing thru A is also passing thru C and tangent to (AX).

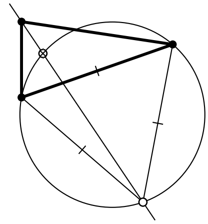
Note that one of the two arcs with endpoints A and C is tangent to $[AX)$.

The uniqueness follows from 9.23.

9.25. Use 9.23 and 7.12 to show that $\measuredangle XAY = \measuredangle ACY$. By Axiom IIIc, $\measuredangle ACY \to 0$ as $AY \to 0$; hence the result.

9.26. Apply Proposition 9.23 twice.

(Alternatively, apply Corollary 5.7 for the reflection across the perpendicular bisector of $[AC]$.)



9.20. Guess a construction from the diagram. To show that it produces the needed point, apply Theorem 9.2.

10.1. By Lemma 5.17, $\angle OTP'$ is right. Therefore, $\triangle OPT \sim \triangle OTP'$ and in particular $OP\cdot OP' = OT^2$ and hence the result.

10.3. Suppose that O denotes the center of Γ. Assume that $X, Y \in \Gamma$; in particular, $OX = OY$.

Note that the inversion sends X and Y to themselves. By Lemma 10.2,

$$\triangle OPX \sim \triangle OXP' \quad \text{and} \quad \triangle OPY \sim \triangle OYP'.$$

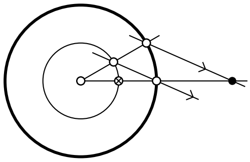
Therefore, $\frac{PX}{P'X} = \frac{OP}{OX} = \frac{OP}{OY} = \frac{PY}{P'Y}$ and hence the result.

10.4. By Lemma 10.2,

$$\begin{aligned} &\measuredangle IA'B' \equiv -\measuredangle IBA, \qquad &&\measuredangle IB'C' \equiv -\measuredangle ICB, \qquad &&\measuredangle IC'A' \equiv -\measuredangle IAC, \\ &\measuredangle IB'A' \equiv -\measuredangle IAB, &&\measuredangle IC'B' \equiv -\measuredangle IBC, &&\measuredangle IA'C' \equiv -\measuredangle ICA. \end{aligned}$$

It remains to apply the theorem on the sum of angles of triangle (7.12) to show that $(A'I) \perp (B'C')$, $(B'I) \perp (C'A')$ and $(C'I) \perp (B'A')$.

10.5. Guess the construction from the diagram (the two non-intersecting lines on the diagram are parallel).



10.8. First show that for any $r > 0$ and for any real numbers x, y distinct from 0,

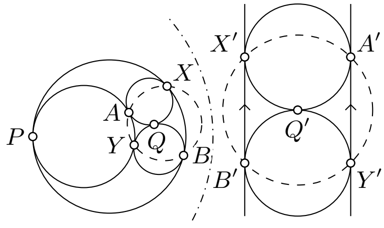
$$\frac{r^2}{(x+y)/2} = \left(\frac{r^2}{x} + \frac{r^2}{y}\right)/2$$

if and only if $x = y$.

Suppose that ℓ denotes the line passing thru Q, Q', and the center of the inversion O. Choose an isometry $\ell \to \mathbb{R}$ that sends O to 0; assume $x, y \in \mathbb{R}$ are the values of ℓ for the two points of intersection $\ell \cap \Gamma$; note that $x \neq y$. Assume r is the radius of the circle of inversion. Then the left-hand side above is the coordinate of Q' and the right-hand side is the coordinate of the center of Γ'.

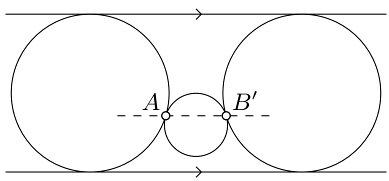
10.9. A solution is given on page 159.

10.10. Apply an inversion in a circle with the center at the only point of intersection of the circles; then use Theorem 10.11.



10.13. Label the points of tangency by X, Y, A, B, P, and Q as on the diagram. Apply an inversion with the center at P. Observe that the two circles that tangent at P become parallel lines and the remaining two circles are tangent to each other and these two parallel lines.

Note that the points of tangency A', B', X', and Y' with the parallel lines are vertices of a square; in particular, they lie on one circle. These points are images of A, B, X, and Y under the inversion. By Theorem 10.7, the points A, B, X, and Y also lie on one circline.



10.14. Apply the inversion in a circle with center A. Point A will go to infinity, the two circles tangent at A will become parallel lines and the two parallel lines will become circles tangent at A; see the diagram.

It remains to show that the dashed line (AB') is parallel to the other two lines.

10.19. Apply Theorem 10.6*b*, 7.17 and Theorem 9.2.

10.20. Suppose that T denotes a point of intersection of Ω_1 and Ω_2. Let P be the foot point of T on (O_1O_2). Show that $\triangle O_1PT \sim \triangle O_1TO_2 \sim \triangle TPO_2$. Conclude that P coincides with the inverses of O_1 in Ω_2 and of O_2 in Ω_1.

10.21. Since $\Gamma \perp \Omega_1$ and $\Gamma \perp \Omega_2$, Corollary 10.16 implies that the circles Ω_1 and Ω_2 are inverted in Γ to themselves. Conclude that A and B are inverses of each other.

Since $\Omega_3 \ni A, B$, Corollary 10.17 implies that $\Omega_3 \perp \Gamma$.

10.22. Let P_1 and P_2 be the inverses of P in Ω_1 and Ω_2. Apply Corollary 10.17 and Theorem 10.15 to show that a circline Γ that passes thru P, P_1, and P_3 is a solution.

10.23. All circles that perpendicular to Ω_1 and Ω_2 pass thru a fixed point P. Try to construct P.

If two of the circles intersect, try to apply Corollary 10.26.

11.2. Suppose that D denotes the midpoint of $[BC]$. Assume (AD) is the angle bisector at A.

Let $A' \in [AD)$ be the point distinct from A such that $AD = A'D$. Note that $\triangle CAD \cong \cong \triangle BA'D$. In particular, $\measuredangle BAA' = \measuredangle AA'B$. It remains to apply Theorem 4.2 for $\triangle ABA'$.

11.3. The statement is evident if A, B, C, and D lie on one line.

In the remaining case, suppose that O denotes the circumcenter. Apply theorem about isosceles triangle (4.2) to the triangles AOB, BOC, COD, DOA.

(Note that in the Euclidean plane the statement follows from Corollary 9.13 and 7.17, but one cannot use these statements in the neutral plane.)

11.5. Arguing by contradiction, assume $2\cdot(\measuredangle ABC + \measuredangle BCD) \equiv 0$, but $(AB) \nparallel (CD)$. Let Z be the point of intersection of (AB) and (CD).

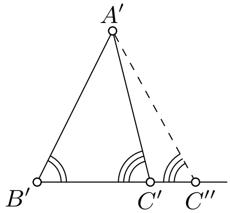
Note that $2\cdot\measuredangle ABC \equiv 2\cdot\measuredangle ZBC$, and $2\cdot\measuredangle BCD \equiv 2\cdot\measuredangle BCZ$.

Apply Proposition 11.4 to $\triangle ZBC$ and try to arrive at a contradiction.

11.6. Let $C'' \in [B'C')$ be the point such that $B'C'' = BC$.

Note that by SAS, $\triangle ABC \cong \triangle A'B'C''$. Conclude that $\measuredangle B'C'A' = \measuredangle B'C''A'$.

Therefore, it is sufficient to show that $C'' = C'$. If $C' \neq \neq C''$ apply Proposition 11.4 to $\triangle A'C'C''$ and try to arrive at a contradiction.



11.7. Use 5.4 and Proposition 11.4.

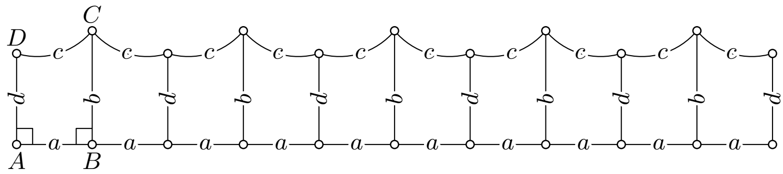
Alternatively, use the same argument as in the solution of 5.13.

11.10. Set $a = AB$, $b = BC$, $c = CD$ and $d = DA$; we need to show that $c \geqslant a$.

Mimic the proof of 11.9 for the shown fence made from copies of quadrangle $ABCD$.

11.11. Note that $|\measuredangle ADC| + |\measuredangle CDB| = \pi$. Then apply the definition of the defect.

11.12. Show that $\triangle AMX \cong \triangle BMC$. Apply 11.11 to $\triangle ABC$ and $\triangle AXC$.



11.13. Show that B and D lie on the opposite sides of (AC). Conclude that

$$\text{defect}(\triangle ABC) + \text{defect}(\triangle CDA) = 0.$$

Apply Theorem11.9 to show that

$$\text{defect}(\triangle ABC) = \text{defect}(\triangle CDA) = 0$$

Use it to show that $\measuredangle CAB = \measuredangle ACD$ and $\measuredangle ACB = \measuredangle CAD$. By ASA, $\triangle ABC \cong \triangle CDA$, and, in particular, $AB = CD$.

(Alternatively, you may apply 11.10.)

12.1. Let A and B be the ideal points of the h-line ℓ. Note that the center of the Euclidean circle containing ℓ lies at the intersection of the lines tangent to the absolute at the ideal points of ℓ.

12.2. Assume A is an ideal point of the h-line ℓ and $P \in \ell$. Suppose that P' denotes the inverse of P in the absolute. By Corollary 10.16, ℓ lies in the intersection of the h-plane and the (necessarily unique) circline passing thru P, A, and P'.

12.3. Let Ω and O denote the absolute and its center.

Let Γ be the circline containing $[PQ]_h$. Note that $[PQ]_h = [PQ]$ if and only if Γ is a line.

Suppose that P' denotes the inverse of P in Ω. Note that O, P, and P' lie on one line.

By the definition of h-line, $\Omega \perp \Gamma$. By Corollary 10.16, Γ passes thru P and P'. Therefore, Γ is a line if and only if it passes thru O.

12.4. Assume that the absolute is a unit circle.

Set $a = OX = OY$. Note that $0 < a < \frac{1}{2}$, $OX_h = \ln\frac{1+a}{1-a}$, and $XY_h = \ln\frac{(1+2\cdot a)\cdot(1-a)}{(1-2\cdot a)\cdot(1+a)}$. It remains to check that the inequalities

$$1 < \tfrac{1+a}{1-a} < \tfrac{(1+2\cdot a)\cdot(1-a)}{(1-2\cdot a)\cdot(1+a)}$$

hold if $0 < a < \frac{1}{2}$.

12.5. Spell the meaning of terms "perpendicular" and "h-line" and then apply 10.22.

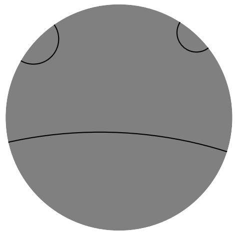
12.11. Let X and Y denote the points of intersections of (OP) and Δ'_ρ. Consider an isometry $(OP) \to \mathbb{R}$ such that O corresponds to 0. Let x, y, p, and $\hat{p}$ denote the real numbers corresponding to X, Y, P, and $\hat{P}$.

We can assume that $p > 0$ and $x < y$. Note that $\hat{p} = \frac{x+y}{2}$ and

$$\frac{(1+x)\cdot(1-p)}{(1-x)\cdot(1+p)} = \frac{(1+p)\cdot(1-y)}{(1-p)\cdot(1+y)}.$$

It remains to show that all this implies $0 < \hat{p} < p$.

12.21. Look at the diagram and think.



12.24. By Corollary 10.26 and Theorem 10.6, the right-hand sides in the identities survive under an inversion in a circle perpendicular to the absolute.

As usual, we assume that the absolute is a unit circle. Suppose that O denotes the h-midpoint of $[PQ]_h$. By the main observation (12.7) we can assume that O is the center of the absolute. In this case, O is also the Euclidean midpoint of $[PQ]$.[1]

Set $a = OP = OQ$; in this case, we have

$$PQ = 2{\cdot}a, \qquad PP' = QQ' = \tfrac{1}{a} - a,$$
$$P'Q' = 2{\cdot}\tfrac{1}{a}, \qquad PQ' = QP' = \tfrac{1}{a} + a.$$

and

$$PQ_h = \ln\tfrac{(1+a)^2}{(1-a)^2} = 2{\cdot}\ln\tfrac{1+a}{1-a}.$$

Therefore

$$\operatorname{ch}[\tfrac{1}{2}{\cdot}PQ_h] = \tfrac{1}{2}{\cdot}(\tfrac{1+a}{1-a} + \tfrac{1-a}{1+a}) = \qquad \sqrt{\frac{PQ'{\cdot}P'Q}{PP'{\cdot}QQ'}} = \frac{\frac{1}{a}+a}{\frac{1}{a}-a} =$$
$$= \tfrac{1+a^2}{1-a^2}; \qquad = \tfrac{1+a^2}{1-a^2}.$$

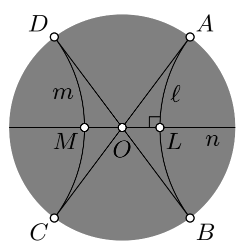
Hence the part *(a)* follows. Similarly,

$$\operatorname{sh}[\tfrac{1}{2}{\cdot}PQ_h] = \tfrac{1}{2}{\cdot}\left(\tfrac{1+a}{1-a} - \tfrac{1-a}{1+a}\right) = \qquad \sqrt{\frac{PQ{\cdot}P'Q'}{PP'{\cdot}QQ'}} = \frac{2}{\frac{1}{a}-a} =$$
$$= \tfrac{2\cdot a}{1-a^2}; \qquad = \tfrac{2\cdot a}{1-a^2}.$$

Hence the part *(b)* follows.

The parts *(c)* and *(d)* follow from *(a)*, *(b)*, the definition of hyperbolic tangent, and the double-argument identity for hyperbolic cosine, see 12.23.

13.1; *"only-if" part.* Suppose ℓ and m are ultraparallel; that is, they do not intersect and have distinct ideal points. Denote the ideal points by A, B, C, and D; we may assume that they appear on the absolute in the same order. In this case, the h-line with ideal points A and C intersects the h-line with ideal points B and D. Denote by O their point of intersection.



By 12.6, we can assume that O is the center of absolute. Note that ℓ is the reflection of m across O in the Euclidean sense.

Drop an h-perpendicular n from O to ℓ, and show that $n \perp m$.

"If" part. Suppose n is a common perpendicular. Denote by L and M its points of intersection with ℓ and m respectively. By 12.6, we can assume that the center of absolute

[1]Instead, we may move Q to the center of absolute. In this case, the derivations are simpler. But since $Q'Q = Q'P = Q'P' = \infty$, one has to justify that $\frac{\infty}{\infty} = 1$ every time.

O is the h-midpoint of L and M. Note that in this case ℓ is the reflection of m across O in the Euclidean sense. It follows that the ideal points of the h-lines ℓ and m are symmetric to each other. Therefore, if one pair of them coincides, then so is the other pair. By 12.1, $\ell = m$, which contradicts the assumption $\ell \neq m$.

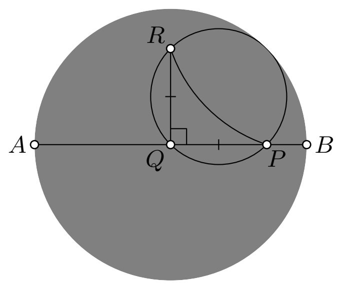
13.3. By the triangle inequality, the h-distance from B to $(AC)_h$ is at least 50. It remains to estimate $|\measuredangle_h ABC|$ using Proposition 13.2. The inequalities $\cos\varphi \leqslant 1 - \frac{1}{10}\cdot\varphi^2$ for $|\varphi| < \frac{\pi}{2}$ and $e^3 > 10$ should help to finish the proof.

13.5. Note that the angle of parallelism of B to $(CD)_h$ is bigger than $\frac{\pi}{4}$, and it converges to $\frac{\pi}{4}$ as $CD_h \to \infty$.

Applying Proposition 13.2, we get that

$$BC_h < \tfrac{1}{2}\cdot \ln \frac{1+\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}} = \ln\left(1+\sqrt{2}\right).$$

The right-hand side is the limit of BC_h if $CD_h \to \infty$. Therefore, $\ln\left(1+\sqrt{2}\right)$ is the optimal upper bound.



13.6. As usual, we assume that the absolute is a unit circle.

Let PQR be a hyperbolic triangle with a right angle at Q, such that $PQ_h = QR_h$ and the vertices P, Q, and R lie on a horocycle.

Without loss of generality, we may assume that Q is the center of the absolute. In this case, $\measuredangle_h PQR = \measuredangle PQR = \pm\frac{\pi}{2}$ and $PQ = QR$.

Note that Euclidean circle passing thru P, Q, and R is tangent to the absolute. Conclude that $PQ = \frac{1}{\sqrt{2}}$. Apply 12.8 to find PQ_h.

13.9. Apply AAA-congruence condition (13.8).

13.12. Apply Proposition 13.11. Use that the function $r \mapsto e^{-r}$ is decreasing and $e > 2$.

13.14. Apply the hyperbolic Pythagorean theorem and the definition of hyperbolic cosine. The following observations should help:

◇ $x \mapsto e^x$ is an increasing positive function.

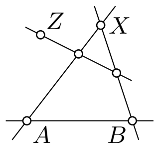
◇ By the triangle inequality, we have $-c \leqslant a - b$ and $-c \leqslant b - a$.

14.1. Assume the two distinct lines ℓ and m are mapped to the intersecting lines ℓ' and m'. Suppose that P' denotes their point of intersection.

Let P be the inverse image of P'. By the definition of affine map, it has to lie on both ℓ and m; that is, ℓ and m are intersecting. Hence the result.

14.3. In each case check that the map is a bijection and apply 7.24.

14.4. Choose a line (AB).

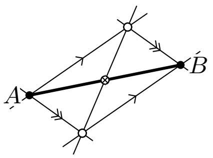


Assume that $X' \in (A'B')$ for some $X \notin (AB)$. Since $P \mapsto P'$ maps collinear points to collinear, the three lines (AB), (AX), and (BX) are mapped to $(A'B')$. Further, any line that connects a pair of points on these three lines is also mapped to $(A'B')$. Use it to show that the whole plane is mapped to $(A'B')$. The latter contradicts that the map is a bijection.

By assumption, if $X \in (AB)$, then $X' \in (A'B')$. From above the converse holds as well. Use it to prove the second statement.

14.5. According to the remark before the exercise, it is sufficient to construct the midpoint of $[AB]$ with a ruler and a parallel tool.

Guess a construction from the diagram.



14.6. Let O, E, A, and B denote the points with the coordinates $(0,0)$, $(1,0)$, $(a,0)$, and $(b,0)$ respectively.

To construct a point W with the coordinates $(0, a+b)$, try to construct two parallelograms $OAPQ$ and $BWPQ$.

To construct Z with coordinates $(0, a{\cdot}b)$ choose a line $(OE') \neq (OE)$ and try to construct the points $A' \in (OE')$ and $Z \in (OE)$ so that $\triangle OEE' \sim \triangle OAA'$ and $\triangle OE'B \sim$ $\sim \triangle OA'Z$.

14.7. Draw two parallel chords $[XX']$ and $[YY']$. Set $Z = (XY) \cap (X'Y')$ and $Z' =$ $= (XY') \cap (X'Y)$. Note that (ZZ') passes thru the center.

Repeat the same construction for another pair of parallel chords. The center lies in the intersection of the obtained lines.

14.8. Assume a construction produces two perpendicular lines. Apply a shear map that changes the angle between the lines (see 14.3*a*).

Note that it transforms the construction to the same construction for other free choices of points. Therefore, this construction does not produce perpendicular lines in general. (It might produce a perpendicular line only by a coincidence.)

14.9. Apply twice 14.1 and the parallelogram rule. (Note that the case if A, B, X, and Y are collinear requires extra work.)

14.12 and 14.13. Fix a coordinate system and apply the fundamental theorem of affine geometry (14.11) for the points $O = (0,0)$, $X = (1,0)$ and $Y = (0,1)$.

14.16. Apply 10.25 and 14.15.

14.17. Fix a line ℓ. Choose a circle Γ with its center not on ℓ. Let Ω be the inverse of ℓ in Γ; note that Ω is a circle.

Let ι_Γ and ι_Ω denote the inversions in Γ and Ω. Apply 10.26 to show that the composition $\iota_\Gamma \circ \iota_\Omega \circ \iota_\Gamma$ is the reflection across ℓ.

15.3. To prove *(a)*, apply Proposition 14.10.

To prove *(b)*, suppose $P_i = (x_i, y_i)$; show and use that

$$\frac{P_1P_2 \cdot P_3P_4}{P_2P_3 \cdot P_4P_1} = \left|\frac{(x_1 - x_2)\cdot(x_3 - x_4)}{(x_2 - x_3)\cdot(x_4 - x_1)}\right|$$

if all P_i lie on a horizontal line $y = b$, and

$$\frac{P_1P_2\cdot P_3P_4}{P_2P_3\cdot P_4P_1} = \left|\frac{(y_1 - y_2)\cdot(y_3 - y_4)}{(y_2 - y_3)\cdot(y_4 - y_1)}\right|$$

otherwise. (See 20.24 for another proof.)

To prove *(c)*, apply *(a)*, *(b)*, and Theorem 15.2.

15.6. Assume that (AB) meets $(A'B')$ at O. Since $(AB') \parallel (BA')$, we get that $\triangle OAB' \sim \sim \triangle OBA'$ and $\frac{OA}{OB} = \frac{OB'}{OA'}$.

Similarly, since $(AC') \parallel (CA')$, we get that $\frac{OA}{OC} = \frac{OC'}{OA'}$.

Therefore $\frac{OB}{OC} = \frac{OC'}{OB'}$. Applying the SAS similarity condition, we get that $\triangle OBC' \sim \sim \triangle OCB'$. Therefore, $(BC') \parallel (CB')$.

The case $(AB) \parallel (A'B')$ is similar.

15.7. Observe that the statement is equivalent to Pappus' theorem.

15.8. To do *(a)*, suppose that the parallelogram is formed by the two pairs of parallel lines $(AB) \parallel (A'B')$ and $(BC) \parallel (B'C')$ and $\ell = (AC)$ in the notation of Desargues' theorem (15.4).

To do *(b)*, suppose that the parallelogram is formed by the two pairs of parallel lines $(AB') \parallel (A'B)$ and $(BC') \parallel (B'C)$ and $\ell = (AC')$ in the notation of Pappus' theorem (15.5).

15.9. Draw $a = (KN)$, $b = (KL)$, $c = (LM)$, $d = (MN)$, mark $P = b \cap d$, and continue.

15.10. Assume there is a duality. Choose two distinct parallel lines ℓ and m. Let L and M be their dual points. Set $s = (ML)$, then its dual point S has to lie on both ℓ and m — a contradiction.

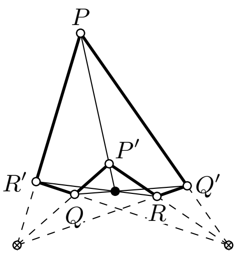
15.12. Assume $M = (a, b)$ and the line s is given by the equation $p\cdot x + q\cdot y = 1$. Then $M \in s$ is equivalent to $p\cdot a + q\cdot b = 1$.

The latter is equivalent to $m \ni S$ where m is the line given by the equation $a\cdot x + b\cdot y = 1$ and $S = (p, q)$.

To extend this bijection to the whole projective plane, assume that (1) the ideal line corresponds to the origin and (2) the ideal point given by the pencil of the lines $b\cdot x - a\cdot y = c$ for different values of c corresponds to the line given by the equation $a\cdot x + b\cdot y = 0$.

15.14. Assume one set of concurrent lines a, b, c, and another set of concurrent lines a', b', c' are given. Set

$$P = b \cap c', \qquad Q = c \cap a', \qquad R = a \cap b',$$
$$P' = b' \cap c, \qquad Q' = c' \cap a, \qquad R' = a' \cap b.$$



Then the lines (PP'), (QQ'), and (RR') are concurrent.

(Note that the obtained configuration of nine points and nine lines is the same as in the original theorem and the obtained result is its reformulation.)

15.15. Assume (AA') and (BB') are the given lines and C is the given point. Apply the dual Desargues' theorem (15.13) to construct C' so that

(AA'), (BB'), and (CC') are concurrent. Since $(AA') \parallel (BB')$, we get that $(AA') \parallel$ $\parallel (BB') \parallel (CC')$.

Now assume that P is the given point and $(R'Q)$, $(P'R)$ are the given parallel lines. Try to construct point Q' as in the dual Pappus' theorem (see the solution of 15.14).

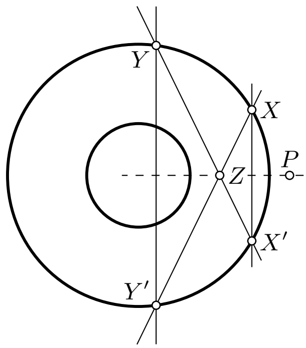
15.17. Suppose $p = (QR)$; denote by q and r the dual lines produced by the construction. Then, by Claim 15.16, P is the point of intersection of q and r.

15.18. The line v polar to V is tangent to Γ. Since $V \in p$, by Claim 15.16, we get that $P \in v$; that is, $(PV) = v$. Hence the statement follows.

15.19. Choose a point P outside of the bigger circle. Construct the lines dual to P for both circles. Note that these two lines are parallel.

Assume that the lines intersect the bigger circle at two pairs of points X, X' and Y, Y'. Set $Z =$ $= (XY) \cap (X'Y')$. Note that the line (PZ) passes thru the common center.

The center is the intersection of (PZ) and another line constructed in the same way.



15.20. Construct polar lines to two points on ℓ. Denote by L the intersection of these two lines. Note that ℓ is polar to L and therefore $(OL) \perp \ell$.

15.21. Let A, B, C, and D be the point provided by Axiom p-III. Given a line ℓ, we can assume that $A \notin \ell$; otherwise, permute the labels of the points. Then by axioms p-I and p-II, the three lines (AB), (AC), and (AD) intersect ℓ at distinct points. In particular, ℓ contains at least three points.

15.22. Let A, B, C, and D be the point provided by Axiom p-III. Show that the lines (AB), (BC), (CD), and (DA) satisfy Axiom p-III$'$. The proof of the converse is similar.

15.23. Let ℓ be a line with $n+1$ points on it.

By Axiom p-III, given any line m, there is a point P that does not lie on ℓ nor on m.

By axioms p-I and p-II, there is a bijection between the lines passing thru P and the points on ℓ. In particular, exactly $n+1$ lines passing thru P.

In the same way, there is a bijection between the lines passing thru P and the points on m. Hence *(a)* follows.

Fix a point X. By Axiom p-I, any point Y in the plane lies in a unique line passing thru X. From part *(a)*, each such line contains X and yet n point. Hence *(b)* follows.

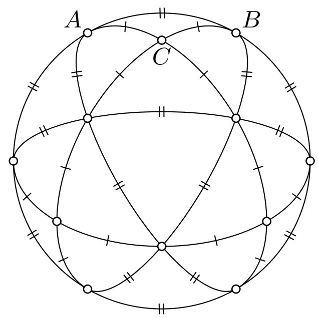
To solve *(c)*, show that the equation $n^2+n+1 = 10$ does not admit an integer solution and then apply part *(b)*.

To solve *(d)*, count the number of lines crossing a given line using the part *(a)* and apply *(b)*.

16.2. Applying the Pythagorean theorem, we get that

$$\cos AB_s = \cos AC_s \cdot \cos BC_s = \tfrac{1}{2}.$$

Therefore, $AB_s = \frac{\pi}{3}$.



Alternatively, look at the tessellation of a hemisphere on the picture; it is made from 12 copies of $\triangle_s ABC$ and yet 4 equilateral spherical triangles. From the symmetry of this tessellation, it follows that $[AB]_s$ occupies $\frac{1}{6}$ of the equator; that is, $AB_s = \frac{\pi}{3}$.

16.6. Consider the inversion of the base in a sphere with the center at the tip of the cone and apply Theorem 16.3.

16.7. Note that points on Ω do not move. Moreover, the points inside Ω are mapped outside of Ω and the other way around.

Further, note that this map sends circles to circles; moreover, the perpendicular circles are mapped to perpendicular circles. In particular, the circles perpendicular to Ω are mapped to themselves.

Consider arbitrary point $P \notin \Omega$. Suppose that P' denotes the inverse of P in Ω. Choose two distinct circles that pass thru P and P'. According to Corollary 10.17, $\Gamma_1 \perp \Omega$ and $\Gamma_2 \perp \Omega$.

Therefore, the inversion in Ω sends Γ_1 to itself, and the same holds for Γ_2.

The image of P has to lie on Γ_1 and Γ_2. Since the image of P is distinct from P, we get that it has to be P'.

16.8. Apply Theorem 16.3*b*.

16.9. Set $z = P'Q'$. Note that $\frac{y}{z} \to 1$ as $x \to 0$.

It remains to show that

$$\lim_{x\to 0} \frac{z}{x} = \frac{2}{1 + OP^2}.$$

Recall that the stereographic projection is the inversion in the sphere Υ with the center at the south pole S restricted to the plane Π. Show that there is a plane Λ passing thru S, P, Q, P', and Q'. In the plane Λ, the map $Q \mapsto Q'$ is an inversion in the circle $\Upsilon \cap \Lambda$.

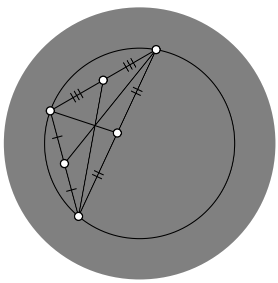
This reduces the problem to Euclidean plane geometry. The remaining calculations in Λ are similar to those in the proof of Lemma 13.10.

16.10. *(a).* Observe and use that $OA' = OB' = OC'$.

(b). Note that the medians of spherical triangle ABC map to the medians of Euclidean a triangle $A'B'C'$. It remains to apply Theorem 8.5 for $\triangle A'B'C'$.

17.1. Let N, O, S, P, P', and $\hat{P}$ be as on the diagram on page 136.

Note that $OQ = \frac{1}{x}$ and therefore we need to show that $O\hat{P} = 2/(x + \frac{1}{x})$. To do this, show and use that $\triangle SOP \sim \triangle SP'N \sim \triangle P'\hat{P}P$ and $2{\cdot}SO = NS$.



17.3. Consider the bijection $P \leftrightarrow \hat{P}$ of the h-plane with absolute Ω. Note that $\hat{P} \in [A_iB_i]$ if and only if $P \in \Gamma_i$.

17.5. The observation follows since the reflection across the perpendicular bisector of $[PQ]$ is a motion of the Euclidean plane, and a motion of the h-plane as well.

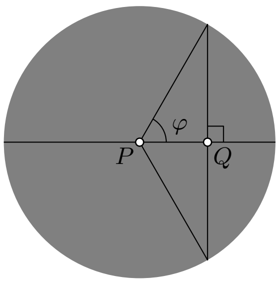
Without loss of generality, we may assume that the center of the circumcircle coincides with the center of the absolute. In this case, the h-medians of the triangle coincide with the Euclidean medians. It remains to apply Theorem 8.5.

17.6. Let $\hat{\ell}$ and $\hat{m}$ denote the h-lines in the conformal model that correspond to ℓ and m. We need to show that $\hat{\ell} \perp \hat{m}$ as arcs in the Euclidean plane.

The point Z, where s meets t, is the center of the circle Γ containing $\hat{\ell}$.

If $\hat{m}$ is passing thru Z, then the inversion in Γ exchanges the ideal points of $\hat{\ell}$. In particular, $\hat{\ell}$ maps to itself. Hence the result.

17.7. Let Q be the foot point of P on the line and φ be the angle of parallelism. We can assume that P is the center of the absolute. Therefore $PQ = \cos\varphi$ and



$$PQ_h = \tfrac{1}{2}\cdot \ln \frac{1+\cos\varphi}{1-\cos\varphi}.$$

17.8. Apply 17.7 for $\varphi = \frac{\pi}{3}$.

17.9. Note that $b = \frac{1}{2}\cdot \ln\frac{1+t}{1-t}$; therefore

❶ $$\operatorname{ch} b = \tfrac{1}{2}\cdot\left(\sqrt{\tfrac{1+t}{1-t}} + \sqrt{\tfrac{1-t}{1+t}}\right) = \frac{1}{\sqrt{1-t^2}}.$$

In the same way, we get that

❷ $$\operatorname{ch} c = \frac{1}{\sqrt{1-u^2}}.$$

Let X and Y are the ideal points of $(BC)_h$. Applying the Pythagorean theorem (6.4) again, we get that $CX = CY = \sqrt{1-t^2}$. Therefore,

$$a = \tfrac{1}{2}\cdot \ln\frac{\sqrt{1-t^2}+s}{\sqrt{1-t^2}-s},$$

and

❸ $$\operatorname{ch} a = \tfrac{1}{2}\cdot\left(\sqrt{\frac{\sqrt{1-t^2}+s}{\sqrt{1-t^2}-s}} + \sqrt{\frac{\sqrt{1-t^2}-s}{\sqrt{1-t^2}+s}}\right) = \frac{\sqrt{1-t^2}}{\sqrt{1-t^2-s^2}} = \frac{\sqrt{1-t^2}}{\sqrt{1-u^2}}.$$

Finally, note that ❶, ❷, and ❸ imply the theorem.

17.11. In the Euclidean plane, the circle Γ_2 is tangent to k; that is, the point T of intersection of Γ_2 and k is unique. It defines a unique line (PT) parallel to ℓ.

18.1. Use that $|z|^2 = z\cdot\bar{z}$ for $z = v$, w, and $v\cdot w$.

18.2. Given a quadrangle $ABCD$, we can choose the complex coordinates so that A has complex coordinate 0. Rewrite the terms in the Ptolemy's inequality in terms of the complex coordinates u, v, and w of B, C, and D; apply the identity and the triangle inequality.

18.3. Let z, v, and w denote the complex coordinates of Z, V, and W respectively. Then

$$\begin{aligned}\measuredangle ZVW + \measuredangle VWZ + \measuredangle WZV &\equiv \arg \tfrac{w-v}{z-v} + \arg \tfrac{z-w}{v-w} + \arg \tfrac{v-z}{w-z} \equiv \\ &\equiv \arg \tfrac{(w-v)\cdot(z-w)\cdot(v-z)}{(z-v)\cdot(v-w)\cdot(w-z)} \equiv \\ &\equiv \arg(-1) \equiv \pi.\end{aligned}$$

18.4. Note and use that $\measuredangle EOV = \measuredangle WOZ = \arg v$ and $\frac{OW}{OZ} = \frac{OZ}{OW} = |v|$.

18.6. Note that

$$\arg \frac{(v-u)\cdot(z-w)}{(v-w)\cdot(z-u)} \equiv \arg \frac{v-u}{z-u} + \arg \frac{z-w}{v-w} = \measuredangle ZUV + \measuredangle VWZ.$$

The statement follows since the value $\frac{(v-u)\cdot(z-w)}{(v-w)\cdot(z-u)}$ is real if and only if

$$2\cdot \arg \frac{(v-u)\cdot(z-w)}{(v-w)\cdot(z-u)} \equiv 0.$$

18.8. We can choose the complex coordinates so that the points O, E, A, B, and C have coordinates 0, 1, $1+i$, $2+i$, and $3+i$ respectively.

Set $\measuredangle EOA = \alpha$, $\measuredangle EOB = \beta$, and $\measuredangle EOC = \gamma$. Note that

$$\begin{aligned}\alpha + \beta + \gamma &\equiv \arg(1+i) + \arg(2+i) + \arg(3+i) \equiv \\ &\equiv \arg[(1+i)\cdot(2+i)\cdot(3+i)] \equiv \arg[10\cdot i] = \tfrac{\pi}{2}.\end{aligned}$$

Note that these three angles are acute and conclude that $\alpha + \beta + \gamma = \frac{\pi}{2}$.

18.9. The identity can be checked by straightforward computations.

By Theorem 18.5, five from six cross-ratios in this identity are real. Therefore so is the sixth cross-ratio; it remains to apply the theorem again.

18.10. Use 3.7 and 3.10 to show that $\angle UAB$, $\angle BVA$, and $\angle ABW$ have the same sign. Note that by SAS we have that

$$\frac{AU}{AB} = \frac{VB}{VA} = \frac{BA}{BW} \quad \text{and} \quad \measuredangle UAB = \measuredangle BVA = \measuredangle ABW.$$

The later means that $|\frac{u-a}{b-a}| = |\frac{b-v}{a-v}| = |\frac{a-b}{w-b}|$, and $\arg \frac{b-a}{u-a} = \arg \frac{a-v}{b-v} = \arg \frac{a-b}{w-b}$. It implies the first two equalities in

❶
$$\frac{b-a}{u-a} = \frac{a-v}{b-v} = \frac{w-b}{a-b} = \frac{w-v}{u-v};$$

the last equality holds since

$$\frac{(b-a)+(a-v)+(w-b)}{(u-a)+(b-v)+(a-b)} = \frac{w-v}{u-v}.$$

To prove *(b)*, rewrite ❶ using angles and distances between the points and apply SAS.

18.14. Show that the inverse of each elementary transformation is elementary and use Proposition 18.12.

18.15. The fractional linear transformation

$$f(z) = \frac{(z_1 - z_\infty)\cdot(z - z_0)}{(z_1 - z_0)\cdot(z - z_\infty)}$$

meets the conditions.

To show the uniqueness, assume there is another fractional linear transformation $g(z)$ that meets the conditions. Then the composition $h = g \circ f^{-1}$ is a fractional linear transformation; set $h(z) = \frac{a\cdot z+b}{c\cdot z+d}$.

Note that $h(\infty) = \infty$; therefore, $c = 0$. Further, $h(0) = 0$ implies $b = 0$. Finally, since $h(1) = 1$, we get that $\frac{a}{d} = 1$. Therefore, h is the *identity*; that is, $h(z) = z$ for any z. It follows that $g = f$.

18.16. Let Z' be the inverse of the point Z. Assume that the circle of the inversion has center W and radius r. Let z, z', and w denote the complex coordinate of the points Z, Z', and W respectively.

By the definition of inversion, $\arg(z - w) = \arg(z' - w)$ and $|z - w|\cdot|z' - w| = r^2$. It follows that $(\bar z' - \bar w)\cdot(z - w) = r^2$. Equivalently,

$$z' = \overline{\left(\frac{\bar w\cdot z + [r^2 - |w|^2]}{z - w}\right)}.$$

18.18. Check the statement for each elementary transformation. Then apply Proposition 18.12.

18.20. Note that $f = \frac{a\cdot z+b}{c\cdot z+d}$ preserves the unit circle $|z| = 1$. Use Corollary 10.26 and Proposition 18.12 to show that f commutes with the inversion $z \mapsto 1/\bar z$. In other words, $1/\overline{f(z)} = f(1/\bar z)$ or

$$\frac{\bar c\cdot\bar z + \bar d}{\bar a\cdot\bar z + \bar b} = \frac{a/\bar z + b}{c/\bar z + d}$$

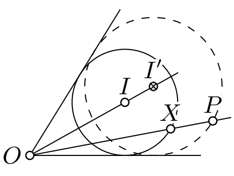
for any $z \in \hat{\mathbb{C}}$. The latter identity leads to the required statement. The condition $|w| < |v|$ follows since $f(0) \in \mathbb{D}$.

18.21. Note that the inverses of the points z and w have complex coordinates $1/\bar z$ and $1/\bar w$. Apply 12.24 and simplify.

The second part follows since the function $x \mapsto \operatorname{th}(\frac{1}{2}\cdot x)$ is increasing.

18.22. Apply Schwarz–Pick theorem for a function f such that $f(0) = 0$ and then apply Lemma 12.8.

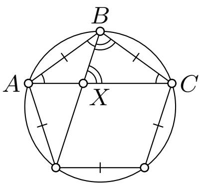
19.3. Let O be the point of intersection of the lines. Construct a circle Γ, tangent to both lines, that crosses $[OP)$; denote its center by I. Suppose that X denotes one of the points of intersections of Γ and $[OP)$.



Construct a point $I' \in [OI)$ such that $\frac{OI'}{OI} = \frac{OP}{OX}$. Observe that the circle passing thru P and centered at I' is a solution.

19.5. To construct $\sqrt{a\cdot b}$: (1) construct points A, B, and $D \in [AB]$ such that $AD = a$ and $BD = b$; (2) construct the circle Γ on the diameter $[AB]$; (3) draw the line ℓ thru D perpendicular to (AB); (4) let C be an intersection of Γ and ℓ. Then $DC = \sqrt{a\cdot b}$.

The construction of $\frac{a^2}{b}$ is analogous.



19.7, *(a).* Look at the diagram; show that the angles marked the same way have the same angle measure. Conclude that $XC = BC$ and $\triangle ABC \sim \triangle AXB$. Therefore

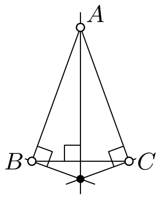
$$\frac{AB}{AC} = \frac{AX}{AB} = \frac{AC - AB}{AB} = \frac{AC}{AB} - 1.$$

It remains to solve for $\frac{AC}{AB}$.

(b). Choose two points P and Q and use the compass-and-ruler calculator to construct two points V and W such that $VW = \frac{1+\sqrt{5}}{2} \cdot PQ$. Then construct a pentagon with the sides PQ and diagonals VW.

19.8. Note that with a set-square we can construct a line parallel to a given line thru the given point. It remains to modify the construction in 14.5.

19.10. Assume that two vertices have rational coordinates, say (a_1, b_1) and (a_2, b_2). Find the coordinates of the third vertex. Use that the number $\sqrt{3}$ is irrational to show that the third vertex is an irrational point.



19.11. Observe that three perpendiculars on the diagram meet at one point if and only if the triangle is isosceles. Use this observation couple of times to verify that the given triangle is equilateral.

19.12. Consider the perspective projection $(x, y) \mapsto (\frac{1}{x}, \frac{y}{x})$ (see page 121). Let $A = (1, 1)$, $B = (3, 1)$, and $M = (2, 1)$. Note that M is the midpoint of $[AB]$.

Their images are $A' = (1, 1)$, $B' = (\frac{1}{3}, \frac{1}{3})$, and $M' = (\frac{1}{2}, \frac{1}{2})$. Clearly, M' is not the midpoint of $[A'B']$.

19.14. The set of ruler and compass is strictly stronger than a set-square which is strictly stronger than a ruler and parallel tool. To prove it, use exercises 5.22, 14.8, 19.10, and Proposition 7.1.

20.1. Assume the contrary; that is, there is a point $W \in [XY]$ such that $W \notin \blacktriangle ABC$.

Without loss of generality, we may assume that W and A lie on the opposite sides of the line (BC).

It implies that both segments $[WX]$ and $[WY]$ intersect (BC). By Axiom II, $W \in (BC)$ — a contradiction.

20.3. To prove the "only if" part, consider the line passing thru the vertex that is parallel to the opposite side.

To prove the "if" part, use Pasch's theorem (3.12).

20.4. Assume the contrary; that is, a solid square $\mathcal{Q}$ can be presented as a union of a finite collection of segments $[A_1B_1], \dots, [A_nB_n]$ and one-point sets $\{C_1\}, \dots, \{C_k\}$.

Note that $\mathcal{Q}$ contains an infinite number of mutually nonparallel segments. Therefore, we can choose a segment $[PQ]$ in $\mathcal{Q}$ that is not parallel to any of the segments $[A_1B_1], \dots, [A_nB_n]$.

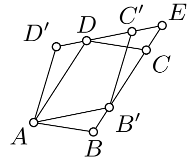
It follows that $[PQ]$ has at most one common point with each of the sets $[A_iB_i]$ and $\{C_i\}$. Since $[PQ]$ contains an infinite number of points, we arrive at a contradiction.

20.5. Show that among elementary sets only one-point sets can be subsets of a circle. It remains to note that any circle contains an infinite number of points.

20.13. Suppose that E denotes the point of intersection of the lines (BC) and $(C'D')$.

Use Proposition 20.12 to prove the following two identities:

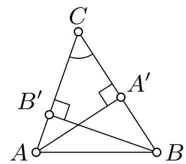
$$\text{area}(\blacksquare AB'ED) = \text{area}(\blacksquare ABCD),$$
$$\text{area}(\blacksquare AB'ED) = \text{area}(\blacksquare AB'C'D').$$



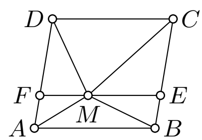
20.15. Without loss of generality, we may assume that the angles ABC and BCA are acute.

Let A' and B' denote the foot points of A and B on (BC) and (AC) respectively. Note that $h_A = AA'$ and $h_B = BB'$.

Note that $\triangle AA'C \sim \triangle BB'C$; indeed the angle at C is shared and the angles at A' and B' are right. In particular $\frac{AA'}{BB'} = \frac{AC}{BC}$ or, equivalently, $h_A \cdot BC = h_B \cdot AC$.



20.16. Draw the line ℓ thru M parallel to $[AB]$ and $[CD]$; it subdivides $\blacksquare ABCD$ into two solid parallelograms which will be denoted by $\blacksquare ABEF$ and $\blacksquare CDFE$. In particular,



$$\text{area}(\blacksquare ABCD) = \text{area}(\blacksquare ABEF) + \text{area}(\blacksquare CDFE).$$

By Proposition 20.12 and Theorem 20.14 we get that

$$\text{area}(\blacktriangle ABM) = \tfrac{1}{2} \cdot \text{area}(\blacksquare ABEF),$$
$$\text{area}(\blacktriangle CDM) = \tfrac{1}{2} \cdot \text{area}(\blacksquare CDFE)$$

and hence the result.

20.17. Let h_A and h_C denote the distances from A and C to the line (BD) respectively. According to Theorem 20.14,

$$\text{area}(\blacktriangle ABM) = \tfrac{1}{2} \cdot h_A \cdot BM; \qquad \text{area}(\blacktriangle BCM) = \tfrac{1}{2} \cdot h_C \cdot BM;$$
$$\text{area}(\blacktriangle CDM) = \tfrac{1}{2} \cdot h_C \cdot DM; \qquad \text{area}(\blacktriangle ABM) = \tfrac{1}{2} \cdot h_A \cdot DM.$$

Therefore

$$\text{area}(\blacktriangle ABM) \cdot \text{area}(\blacktriangle CDM) = \tfrac{1}{4} \cdot h_A \cdot h_C \cdot DM \cdot BM =$$
$$= \text{area}(\blacktriangle BCM) \cdot \text{area}(\blacktriangle DAM).$$

20.18. Let I be the incenter of $\triangle ABC$. Note that $\blacktriangle ABC$ can be subdivided into $\blacktriangle IAB$, $\blacktriangle IBC$, and $\blacktriangle ICA$.

It remains to apply Theorem 20.14 to each of these triangles and sum up the results.

20.19. Fix a polygonal set $\mathcal{P}$. Without loss of generality, we may assume that $\mathcal{P}$ is a union of a finite collection of solid triangles. Cut $\mathcal{P}$ along the extensions of the sides of all the triangles, it subdivides $\mathcal{P}$ into convex polygons. Cutting each polygon by diagonals from one vertex produces a subdivision into solid triangles.

20.20. Assuming $a > b$, we subdivided $\mathcal{Q}_c$ into $\mathcal{Q}_{a-b}$ and four triangles congruent to $\mathcal{T}$. Therefore

➊ $$\operatorname{area}\mathcal{Q}_c = \operatorname{area}\mathcal{Q}_{a-b} + 4\cdot \operatorname{area}\mathcal{T}.$$

According to Theorem 20.14, $\operatorname{area}\mathcal{T} = \frac{1}{2}\cdot a\cdot b$. Therefore, the identity ➊ can be written as

$$c^2 = (a-b)^2 + 2\cdot a\cdot b.$$

Simplifying, we get the Pythagorean theorem.

Case $a = b$ is simpler. Case $b > a$ can be done in the same way.

20.21. If X is a point inside of $\triangle ABC$, then $\blacktriangle ABC$ is subdivided into $\blacktriangle ABX$, $\blacktriangle BCX$, and $\blacktriangle CAX$. Therefore

$$\operatorname{area}(\blacktriangle ABX) + \operatorname{area}(\blacktriangle BCX) + \operatorname{area}(\blacktriangle CAX) = \operatorname{area}(\blacktriangle ABC).$$

Set $a = AB = BC = CA$. Let h_1, h_2, and h_3 denote the distances from X to the sides $[AB]$, $[BC]$, and $[CA]$. Then by Theorem 20.14,

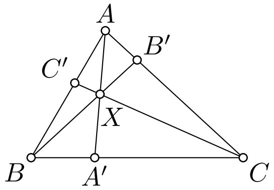
$$\operatorname{area}(\blacktriangle ABX) = \tfrac{1}{2}\cdot h_1\cdot a, \qquad \operatorname{area}(\blacktriangle BCX) = \tfrac{1}{2}\cdot h_2\cdot a, \qquad \operatorname{area}(\blacktriangle CAX) = \tfrac{1}{2}\cdot h_3\cdot a.$$

Therefore,

$$h_1 + h_2 + h_3 = \tfrac{2}{a}\cdot \operatorname{area}(\blacktriangle ABC).$$

20.23. Apply Claim 20.22 to show that

$$\frac{\operatorname{area}(\blacktriangle ABB')}{\operatorname{area}(\blacktriangle BCB')} = \frac{\operatorname{area}(\blacktriangle AXB')}{\operatorname{area}(\blacktriangle XCB')} = \frac{AB'}{B'C}.$$



And observe that

$$\operatorname{area}(\blacktriangle ABB') = \operatorname{area}(\blacktriangle ABX) + \operatorname{area}(\blacktriangle AXB'),$$
$$\operatorname{area}(\blacktriangle BCB') = \operatorname{area}(\blacktriangle BCX) + \operatorname{area}(\blacktriangle XCB').$$

It implies the first identity; the rest is analogous.

20.24. To prove *(a)*, apply Claim 20.22 twice to the triangles OL_iL_j, OL_jM_i, and OM_iM_j.

To prove part *(b)*, use Claim 20.22 to rewrite the left-hand side using the areas of triangles OL_1L_2, OL_2L_3, OL_3L_4, and OL_4L_1. Further, use part *(a)* to rewrite it using areas of OM_1M_2, OM_2M_3, OM_3M_4, and OM_4M_1 and apply Claim 20.22 again to get the right-hand side.

20.25. Let $\mathcal{P}_n$ and $\mathcal{Q}_n$ be the solid regular n-gons so that Γ is inscribed in $\mathcal{Q}_n$ and circumscribed around $\mathcal{P}_n$. Clearly, $\mathcal{P}_n \subset \mathcal{D} \subset \mathcal{Q}_n$.

Show that $\frac{\operatorname{area}\mathcal{P}_n}{\operatorname{area}\mathcal{Q}_n} = (\cos\frac{\pi}{n})^2$; in particular,

$$\frac{\operatorname{area}\mathcal{P}_n}{\operatorname{area}\mathcal{Q}_n} \to 1 \quad \text{as} \quad n \to \infty.$$

Next show that $\operatorname{area}\mathcal{Q}_n < 100$, say for all $n \geqslant 100$.

These two statements imply that $(\operatorname{area}\mathcal{Q}_n - \operatorname{area}\mathcal{P}_n) \to 0$. Hence the result.

Index

Used resources

[1] A. V. Akopyan. *Geometry in figures*. 2017.

[2] A. D. Aleksandrov. "Minimal foundations of geometry". *Siberian Math. J.* 35.6 (1994), 1057–1069.

[3] F. Bachmann. *Aufbau der Geometrie aus dem Spiegelungsbegriff*. 1959.

[4] E. Beltrami. "Teoria fondamentale degli spazii di curvatura costante". *Annali. di Mat., ser II* 2 (1868), 232–255. [Translated by J. Stillwell in *Sources of Hyperbolic Geometry*, pp. 41–62 (1996).]

[5] G. D. Birkhoff. "A set of postulates for plane geometry, based on scale and protractor". *Ann. of Math. (2)* 33.2 (1932), 329–345.

[6] J. Bolyai. *Appendix*. 1832. [Translated by Ferenc Kárteszi in *Appendix. The theory of space*, (1987).]

[7] O. Byrne. *The first six books of the Elements of Euclid: in which coloured diagrams and symbols are used instead of letters for the greater ease of learners*. 1847. URL: `https://github.com/jemmybutton/byrne-`

[8] *Euclid's Elements*.

[9] J. Hadamard. *Leçons de géométrie élémentaire: Géométrie plane*. [Translated by M. Saul in *Lessons in geometry. I. Plane geometry*, (2008).]

[10] А. П. Киселёв. *Элементарная геометрия*. [Translated by A. Givental in *Kiselev's geometry*, (2006).]

[11] J. H. Lambert. "Theorie der parallellinien". *Leipziger Magazin für reine und angewandte Mathematik* 1.2 (1786), 137–164.

[12] A.-M. Legendre. "Eléments de géométrie" (1794).

[13] Н. И. Лобачевский. "О началах геометрии". *Казанский вестник* 25—28 (1829—1830).

[14] N. I. Lobatschewsky. "Geometrische untersuchungen zur theorie der parallellinien". 1840. [Translated by G. B. Halsted in *The Theory of Parallels*, (2015).]

[15] В. В. Прасолов. *Задачи по планиметрии*. 1986. [Translated by D. Leites in *Problems in plane and solid geometry*, (2006).]

[16] G. G. Saccheri. *Euclides ab omni nævo vindicatus*. 1733. [Translated by G. B. Halsted in *Euclides vindicatus*, (1986).]

[17] И. Ф. Шарыгин. *Геометрия 7–9*. 1997.

[18] A. Tarski. "What is elementary geometry?" *The axiomatic method (edited by L. Henkin, P. Suppes and A. Tarski)*. 1959, 16–29.