Second Edition

Solutions Manual of

Introduction to

TOPOLOGY

For BS 4-Years, M.Sc. Mathematics

Z.R. Bhatti

According to New Syllabus Approved by the University of Punjab, Govt. College University, Lahore, Lahore College University for Women, Kinnaird College, F C College, University of Lahore, UCP, Comsat, FAST, GCU, Faisalabad, Global Institute, Lahore, UMT, University of Sargodha, University of Gujrat, International Islamic University, Islamabad, Peshawar University, University of AJK, Bahauddin Zakariya University, Multan, Quaid-i-Azam University, Islamabad, University of Education, Lahore, Islamia University, Bahawalpur.

ILMI KITAB KHANA
Kabir Street, Urdu Bazar, Lahore.

Contents

Chapter 1

TOPOLOGICAL SPACES

EXERCISE 1

Short Questions

Q.3 Solve / answer the following short questions:

(i) Define a topological space.

Sol: Let X be a nonempty set. A collection τ of subsets of X is said to be a topology on X if the following conditions are satisfied:

(i) The union of any number of members of τ belongs to τ.

(ii) The intersection of finite number of members of τ belongs to τ.

(iii) The empty set ϕ and the set X itself belong to τ.

The pair (X,τ) is called a topological space.

(ii) List all possible topologies on the set $X = \{0,1\}$.

Sol: $X = \{0,1\}$

All topologies on X are

$$\tau_1 = \{\phi, X\}, \qquad \tau_1 = \{\phi, \{0\}, X\},$$
$$\tau_2 = \{\phi, \{1\}, X\}, \quad \tau_2 = \{\phi, \{0\}, \{1\}, X\}$$

(iii) If $X = \{0,1\}$, find all possible Sierpinski spaces.

Sol: $X = \{0,1\}$

Consider the following topologies on X:

$$\tau_1 = \{\phi, \{0\}, X\}, \qquad \tau_2 = \{\phi, \{1\}, X\}$$

$(X,\tau_1), (X,\tau_2)$ are all possible Sierpinski spaces.

(iv) If $X = \{a,b,c,d,e,f\}$, find three Sierpinski spaces.

Sol: $X = \{a,b,c,d,e,f\}$

Consider the following three topologies on X:

$$\tau_1 = \{\phi, \{a\}, X\}, \qquad \tau_2 = \{\phi, \{a,f\}, X\}$$
$$\tau_3 = \{\phi, \{b,c,f\}, X\}$$

$(X,\tau_1), (X,\tau_2), (X,\tau_3)$ are three Sierpinski spaces.

(v) **Let $\tau = \{\phi, A_q, R\}$, where $A_q = (q,\infty), q \in Q$ (Q is the set of rational numbers). Whether $\cup\{A_q : q \in Q, q > \sqrt{2}\} \in \tau$.**

Sol: It is clear by the definition of A_q that

$$\cup\{A_q : q \in Q, q > \sqrt{2}\} = (\sqrt{2},\infty)$$

Since $\sqrt{2}$ is irrational, so $(\sqrt{2},\infty) \notin \tau$. This shows that

$$\cup\{A_q : q \in Q, q > \sqrt{2}\} \notin \tau$$

(vi) **Let $A_q = (-\infty, q), q \in Q$, where Q is the set of rational numbers. Show that $\tau = \{\phi, A_q, R\}$ is not a topology on the set of real numbers R.**

Sol: We observe that

$$A = \cup\{A_q : q \in Q, q < \sqrt{2}\} = (-\infty, \sqrt{2})$$

is the union of members of τ, but $A \notin \tau$, because $\sqrt{2}$ is irrational. This shows that union of members of τ is not in τ, so τ is not a topology on R.

(vii) **List all topologies on $X = \{a,b\}$ which consist of exactly four members.**

Sol: $X = \{a,b\}$

All the possible subsets of X are

$$\phi, \{a\}, \{b\}, X$$

Since $\{a\}, \{b\}$ form the partition of X, so

$$\tau = \{\phi, \{a\}, \{b\}, X\}$$

is a topology on X.

Since the four subsets of X are not totally ordered by inclusion, so τ is the only topology on X consisting of four members.

(viii) **Let X be an indiscrete topological space and A be a singleton subset of X. Find A^d.**

Sol: Since A is a singleton set, so let $A = \{a\}$. Since X is an indiscrete topological space, so X itself is the only nonempty open set. Obviously, a is not the limit point of A because open set X does not contain any point of A different from a. Any point $x \neq a$ of X is a limit point of A because the open set X containing x contains a point a of A different from x. Hence

$$A^d = X - A = A^c$$

(ix) **If A is any subset of a cofinite topological space X, then find $\overline{A}$.**

Sol: Finite subsets of X and X itself are the only closed sets in X. Since $\overline{A}$ is the smallest closed superset of A, so