Lecture Notes for Analog Electronics

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Class Notes 1

1 Basic Principles

In electromagnetism, voltage is a *unit* of either electrical potential or EMF. In electronics, including the text, the term "voltage" refers to the physical *quantity* of either potential or EMF. Note that we will use SI units, as does the text.

As usual, the sign convention for current $I = dq/dt$ is that I is positive in the direction which positive electrical charge moves.

We will begin by considering DC (*i.e.* constant in time) voltages and currents to introduce Ohm's Law and Kirchoff's Laws. We will soon see, however, that these generalize to AC.

1.1 Ohm's Law

For a resistor R, as in the Fig. 1 below, the voltage drop from point a to b, $V = V_{ab} = V_a - V_b$ is given by $V = IR$.

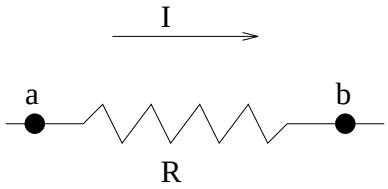

Figure 1: Voltage drop across a resistor.

A device (*e.g.* a resistor) which obeys Ohm's Law is said to be ohmic.
The power dissipated by the resistor is $P = VI = I^2R = V^2/R$.

1.2 Kirchoff's Laws

Consider an electrical circuit, that is a closed conductive path (for example a battery connected to a resistor via conductive wire), or a network of interconnected paths.

1. For any node of the circuit $\sum_{\text{in}} I = \sum_{\text{out}} I$. Note that the choice of "in" or "out" for any circuit segment is arbitrary, but it must remain consistent. So for the example of Fig. 2 we have $I_1 = I_2 + I_3$.

2. For any closed circuit, the sum of the circuit EMFs (*e.g.* batteries, generators) is equal to the sum of the circuit voltage drops: $\sum \mathcal{E} = \sum V$.

Three simple, but important, applications of these "laws" follow.

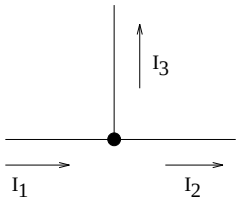

Figure 2: A current node.

1.2.1 Resistors in series

Two resistors, R_1 and R_2, connected in series have voltage drop $V = I(R_1 + R_2)$. That is, they have a combined resistance R_s given by their sum:

$$R_s = R_1 + R_2$$

This generalizes for n series resistors to $R_s = \sum_{i=1}^{n} R_i$.

1.2.2 Resistors in parallel

Two resistors, R_1 and R_2, connected in parallel have voltage drop $V = IR_p$, where

$$R_p = [(1/R_1) + (1/R_2)]^{-1}$$

This generalizes for n parallel resistors to

$$1/R_p = \sum_{i=1}^{n} 1/R_i$$

1.2.3 Voltage Divider

The circuit of Fig. 3 is called a voltage divider. It is one of the most useful and important circuit elements we will encounter. The relationship between $V_{\text{in}} = V_{ac}$ and $V_{\text{out}} = V_{bc}$ is given by

$$V_{\text{out}} = V_{\text{in}} \left[\frac{R_2}{R_1 + R_2} \right]$$

1.3 Voltage and Current Sources

A *voltage source* delivers a constant voltage regardless of the current it produces. It is an idealization. For example a battery can be thought of as a voltage source in series with a small resistor (the "internal resistance" of the battery). When we indicate a voltage V input to a circuit, this is to be considered a voltage source unless otherwise stated.

A *current source* delivers a constant current regardless of the output voltage. Again, this is an idealization, which can be a good approximation in practice over a certain range of output current, which is referred to as the *compliance* range.

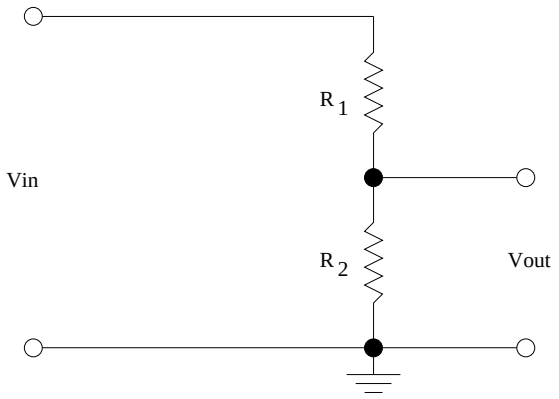

Figure 3: A voltage divider.

1.4 Thevenin's Theorem

Thevenin's theorem states that any circuit consisting of resistors and EMFs has an equivalent circuit consisting of a single voltage source $V_{\rm TH}$ in series with a single resistor $R_{\rm TH}$.

The concept of "load" is useful at this point. Consider a partial circuit with two output points held at potential difference $V_{\rm out}$ which are not connected to anything. A resistor R_L placed across the output will complete the circuit, allowing current to flow through R_L. The resistor R_L is often said to be the "load" for the circuit. A load connected to the output of our voltage divider circuit is shown in Fig. 4

The prescription for finding the Thevenin equivalent quantities $V_{\rm TH}$ and $R_{\rm TH}$ is as follows:

- For an "open circuit" ($R_L \to \infty$), then $V_{\rm TH} = V_{\rm out}$.
- For a "short circuit" ($R_L \to 0$), then $R_{\rm TH} = V_{\rm TH}/I_{\rm short}$.

An example of this using the voltage divider circuit follows. We wish to find the Thevenin equivalent circuit for the voltage divider.

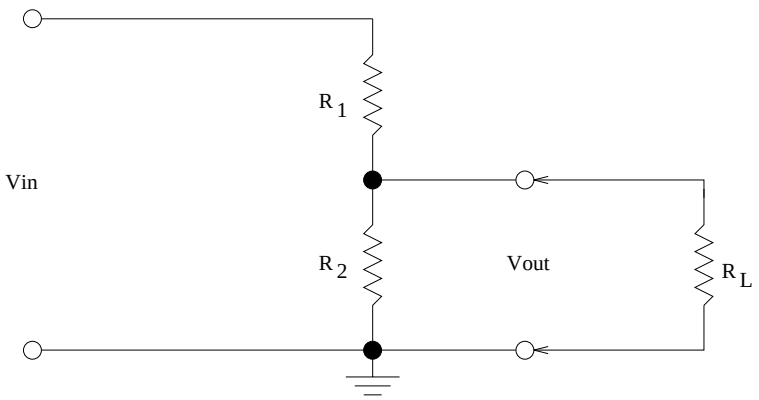

Figure 4: Adding a "load" resistor R_L.

The goal is to deduce $V_{\rm TH}$ and $R_{\rm TH}$ to yield the equivalent circuit shown in Fig. 5.

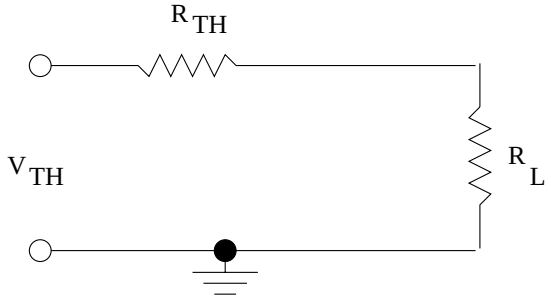

Figure 5: The Thevenin equivalent circuit.

To get $V_{\rm TH}$ we are supposed to evaluate $V_{\rm out}$ when R_L is not connected. This is just our voltage divider result:

$$V_{\rm TH} = V_{\rm in}\left[\frac{R_2}{R_1 + R_2}\right]$$

Now, the short circuit gives, by Ohm's Law, $V_{\rm in} = I_{\rm short}R_1$. Solving for $I_{\rm short}$ and combining with the $V_{\rm TH}$ result gives

$$R_{\rm TH} = V_{\rm TH}/I_{\rm short} = \frac{R_1 R_2}{R_1 + R_2}$$

Note that this is the equivalent parallel resistance of R_1 and R_2.

This concept turns out to be very useful, especially when different circuits are connected together, and is very closely related to the concepts of input and output impedance (or resistance), as we shall see.

1.5 Thevenin Theorem (contd.)

Recall that the Thevenin Theorem states that any collection of resistors and EMFs is equivalent to a circuit of the form shown within the box labelled "Circuit A" in the figure below. As before, the load resistor R_L is not part of the Thevenin circuit. The Thevenin idea, however, is most useful when one considers two circuits or circuit elements, with the first circuit's output providing the input for the second circuit. In Fig. 6, the output of the first circuit (A), consisitng of $V_{\rm TH}$ and $R_{\rm TH}$, is fed to the second circuit element (B), which consists simply of a load resistance (R_L) to ground. This simple configuration represents, in a general way, a very broad range of analog electronics.

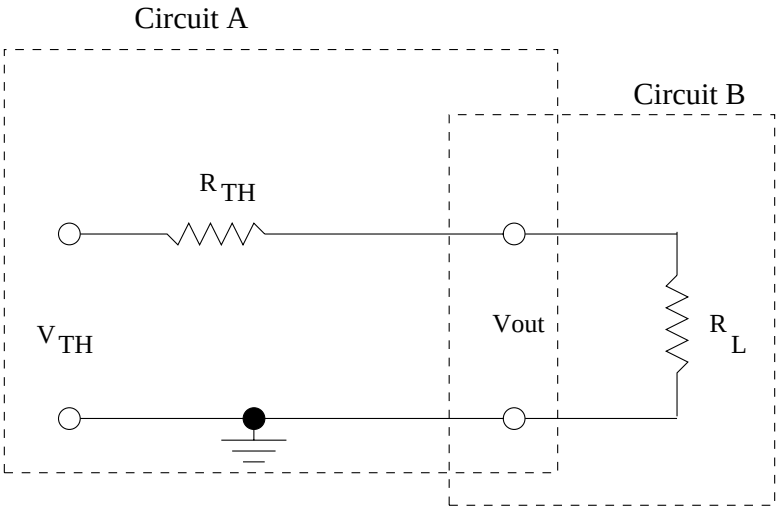

Figure 6: Two interacting circuits.

1.5.1 Avoiding Circuit Loading

$V_{\rm TH}$ is a voltage source. In the limit that $R_{\rm TH} \to 0$ the output voltage delivered to the load R_L remains at constant voltage. For finite $R_{\rm TH}$, the output voltage is reduced from $V_{\rm TH}$ by an amount $IR_{\rm TH}$, where I is the current of the complete circuit, which depends upon the value of the load resistance R_L: $I = V_{\rm TH}/(R_{\rm TH} + R_L)$.

Therefore, $R_{\rm TH}$ determines to what extent the output of the first circuit behaves as an ideal voltage source. An approximately ideal behavior turns out to be quite desirable in most cases, as $V_{\rm out}$ can be considered constant, independent of what load is connected. Since our combined equivalent circuit ($A + B$) forms a simple voltage divider, we can easily see what the requirement for $R_{\rm TH}$ can be found from the following:

$$V_{\rm out} = V_{\rm TH}\left[\frac{R_L}{R_{\rm TH} + R_L}\right] = \frac{V_{\rm TH}}{1 + (R_{\rm TH}/R_L)}$$

Thus, we should try to *keep the ratio* $R_{\rm TH}/R_L$ *small* in order to approximate ideal behavior and avoid "loading the circuit". A maximum ratio of 1/10 is often used as a design rule of thumb.

A good power supply will have a very small $R_{\rm TH}$, typically much less than an ohm. For a battery this is referred to as its internal resistance. The dimming of one's car headlights when the starter is engaged is a measure of the internal resistance of the car battery.

1.5.2 Input and Output Impedance

Our simple example can also be used to illustrate the important concepts of input and output resistance. (Shortly, we will generalize our discussion and substitute the term "impedance" for resistance. We can get a head start by using the common terms "input impedance" and "output impedance" at this point.)

- The output impedance of circuit A is simply its Thevenin equivalent resistance $R_{\rm TH}$. The output impedance is sometimes called "source impedance".
- The input impedance of circuit B is its resistance to ground from the circuit input. In this case it is simply R_L.

It is generally possible to reduce two complicated circuits, which are connected to each other as an input/output pair, to an equivalent circuit like our example. The input and output impedances can then be measured using the simple voltage divider equations.

2 RC Circuits in Time Domain

2.0.3 Capacitors

Capacitors typically consist of two electrodes separated by a non-conducting gap. The quantitiy capacitance C is related to the charge on the electrodes ($+Q$ on one and $-Q$ on the other) and the voltage difference across the capacitor by

$$C = Q/V_C$$

Capacitance is a purely geometric quantity. For example, for two planar parallel electrodes each of area A and separated by a vacuum gap d, the capacitance is (ignoring fringe fields) $C = \epsilon_0 A/d$, where ϵ_0 is the permittivity of vacuum. If a dielectric having dielectric constant κ is placed in the gap, then $\epsilon_0 \rightarrow \kappa\epsilon_0 \equiv \epsilon$. The SI unit of capacitance is the Farad. Typical laboratory capacitors range from $\sim$ 1pF to $\sim 1\mu$F.

For DC voltages, no current passes through a capacitor. It "blocks DC". When a time varying potential is applied, we can differentiate our defining expression above to get

$$I = C\frac{dV_C}{dt} \tag{1}$$

for the current passing through the capacitor.

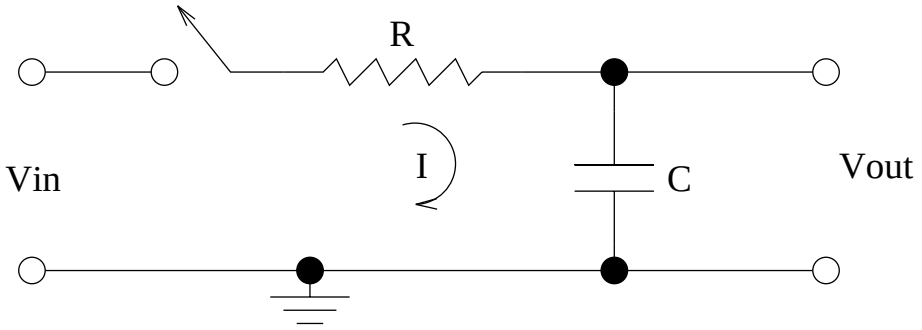

Figure 7: RC circuit — integrator.

2.0.4 A Basic RC Circuit

Consider the basic RC circuit in Fig. 7. We will start by assuming that $V_{\rm in}$ is a DC voltage source (*e.g.* a battery) and the time variation is introduced by the closing of a switch at time $t = 0$. We wish to solve for $V_{\rm out}$ as a function of time.

Applying Ohm's Law across R gives $V_{\rm in} - V_{\rm out} = IR$. The same current I passes through the capacitor according to $I = C(dV/dt)$. Substituting and rearranging gives (let $V \equiv V_C = V_{\rm out}$):

$$\frac{dV}{dt} + \frac{1}{RC}V = \frac{1}{RC}V_{\rm in} \tag{2}$$

The homogeneous solution is $V = Ae^{-t/RC}$, where A is a constant, and a particular solution is $V = V_{\rm in}$. The initial condition $V(0) = 0$ determines A, and we find the solution

$$V(t) = V_{\rm in}\left[1 - e^{-t/RC}\right] \tag{3}$$

This is the usual capacitor "charge up" solution.

Similarly, a capacitor with a voltage V_i across it which is discharged through a resistor to ground starting at $t = 0$ (for example by closing a switch) can in similar fashion be found to obey

$$V(t) = V_i e^{-t/RC}$$

2.0.5 The "RC Time"

In both cases above, the rate of charge/discharge is determined by the product RC which has the dimensions of time. This can be measured in the lab as the time during charge-up or discharge that the voltage comes to within $1/e$ of its asymptotic value. So in our charge-up example, Equation 3, this would correspond to the time required for $V_{\rm out}$ to rise from zero to 63% of $V_{\rm in}$.

2.0.6 RC Integrator

From Equation 2, we see that if $V_{\rm out} \ll V_{\rm in}$ then the solution to our RC circuit becomes

$$V_{\rm out} = \frac{1}{RC}\int V_{\rm in}(t)dt \tag{4}$$

Note that in this case V_{in} can be any function of time. Also note from our solution Eqn. 3 that the limit $V_{\text{out}} \ll V_{\text{in}}$ corresponds roughly to $t \ll RC$. Within this approximation, we see clearly from Eqn. 4 why the circuit above is sometimes called an "integrator".

2.0.7 RC Differentiator

Let's rearrange our RC circuit as shown in Fig. 8.

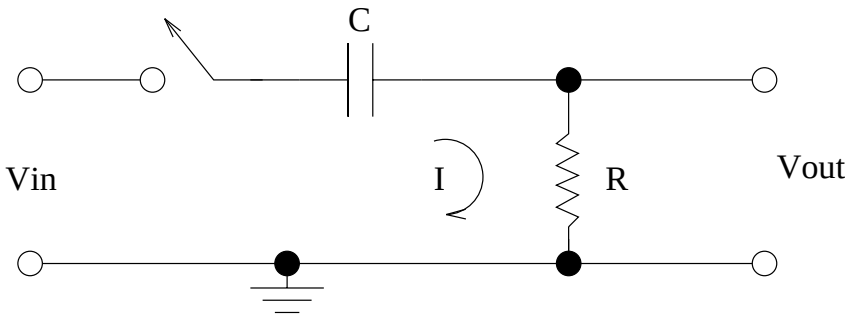

Figure 8: RC circuit — differentiator.

Applying Kirchoff's second Law, we have $V_{\text{in}} = V_C + V_R$, where we identify $V_R = V_{\text{out}}$. By Ohm's Law, $V_R = IR$, where $I = C(dV_C/dt)$ by Eqn. 1. Putting this together gives

$$V_{\text{out}} = RC\frac{d}{dt}(V_{\text{in}} - V_{\text{out}})$$

In the limit $V_{\text{in}} \gg V_{\text{out}}$, we have a differentiator:

$$V_{\text{out}} = RC\frac{dV_{\text{in}}}{dt}$$

By a similar analysis to that of Section 2.0.6, we would see the limit of validity is the opposite of the integrator, *i.e.* $t \gg RC$.

Class Notes 3

3 Circuit Analysis in Frequency Domain

We now need to turn to the analysis of passive circuits (involving EMFs, resistors, capacitors, and inductors) in frequency domain. Using the technique of the complex impedance, we will be able to analyze time-dependent circuits algebraically, rather than by solving differential equations. We will start by reviewing complex algebra and setting some notational conventions. It will probably not be particularly useful to use the text for this discussion, and it could lead to more confusion. Skimming the text and noting results might be useful.

3.1 Complex Algebra and Notation

Let $\tilde{V}$ be the complex representation of V. Then we can write

$$\tilde{V} = \Re(\tilde{V}) + \imath\Im(\tilde{V}) = Ve^{\imath\theta} = V\left[\cos\theta + \imath\sin\theta\right]$$

where $\imath = \sqrt{-1}$. V is the (real) amplitude:

$$V = \sqrt{\tilde{V}\tilde{V}^*} = \left[\Re^2(\tilde{V}) + \Im^2(\tilde{V})\right]^{1/2}$$

where $*$ denotes complex conjugation. The operation of determining the amplitude of a complex quantity is called taking the *modulus*. The phase θ is

$$\theta = \tan^{-1}\left[\Im(\tilde{V})/\Re(\tilde{V})\right]$$

So for a numerical example, let a voltage have a real part of 5 volts and an imaginary part of 3 volts. Then $\tilde{V} = 5 + 3\imath = \sqrt{34}e^{\imath\tan^{-1}(3/5)}$.

Note that we write the amplitude of $\tilde{V}$, formed by taking its modulus, simply as V. It is often written $|\tilde{V}|$. We will also use this notation if there might be confusion in some context. Since the amplitude will in general be frequency dependent, it will also be written as $V(\omega)$. We will most often be interested in results expressed as amplitudes, although we will also look at the phase.

3.2 Ohm's Law Generalized

Our technique is essentially that of the Fourier transform, although we will not need to actually invoke that formalism. Therefore, we will analyze our circuits using a single Fourier frequency component, $\omega = 2\pi f$. This is perfectly general, of course, as we can add (or integrate) over frequencies if need be to recover a result in time domain. Let our complex Fourier components of voltage and current be written as $\tilde{V} = Ve^{\imath(\omega t+\phi_1)}$ and $\tilde{I} = Ie^{\imath(\omega t+\phi_2)}$.

Now, we wish to generalize Ohm's Law by replacing $V = IR$ by $\tilde{V} = \tilde{I}\tilde{Z}$, where $\tilde{Z}$ is the (complex) impedance of a circuit element. Let's see if this can work. We already know that a resistor R takes this form. What about capacitors and inductors?

Our expression for the current through a capacitor, $I = C(dV/dt)$ becomes

$$\tilde{I} = C\frac{d}{dt}Ve^{\imath(\omega t+\phi_1)} = \imath\omega C\tilde{V}$$

Thus, we have an expression of the form $\tilde{V} = \tilde{I}\tilde{Z}_C$ for the impedance of a capacitor, $\tilde{Z}_C$, if we make the identification $\tilde{Z}_C = 1/(\imath\omega C)$.

For an inductor of self-inductance L, the voltage *drop* across the inductor is given by Lenz's Law: $V = L(dI/dt)$. (Note that the voltage drop has the opposite sign of the induced EMF, which is usually how Lenz's Law is expressed.) Our complex generalization leads to

$$\tilde{V} = L\frac{d}{dt}\tilde{I} = L\frac{d}{dt}Ie^{\imath(\omega t+\phi_2)} = \imath\omega L\tilde{I}$$

So again the form of Ohm's Law is satisfied if we make the identification $\tilde{Z}_L = \imath\omega L$.

To summarize our results, Ohm's Law in the complex form $\tilde{V} = \tilde{I}\tilde{Z}$ can be used to analyze circuits which include resistors, capacitors, and inductors if we use the following:

- resistor of resistance R: $\tilde{Z}_R = R$
- capacitor of capacitance C: $\tilde{Z}_C = 1/(\imath\omega C) = -\imath/(\omega C)$
- inductor of self-inductance L: $\tilde{Z}_L = \imath\omega L$

3.2.1 Combining Impedances

It is significant to point out that because the algebraic form of Ohm's Law is preserved, impedances follow the same rules for combination in series and parallel as we obtained for resistors previously. So, for example, two capacitors in parallel would have an equivalent impedance given by $1/\tilde{Z}_p = 1/\tilde{Z}_1 + 1/\tilde{Z}_2$. Using our definition $\tilde{Z}_C = -\imath/\omega C$, we then recover the familiar expression $C_p = C_1 + C_2$. So we have for any two impedances in series (clearly generalizing to more than two):

$$\tilde{Z}_s = \tilde{Z}_1 + \tilde{Z}_2$$

And for two impedances in parallel:

$$\tilde{Z}_p = \left[1/\tilde{Z}_1 + 1/\tilde{Z}_2\right]^{-1} = \frac{\tilde{Z}_1\tilde{Z}_2}{\tilde{Z}_1 + \tilde{Z}_2}$$

And, accordingly, our result for a voltage divider generalizes (see Fig. 9) to

$$\tilde{V}_{\text{out}} = \tilde{V}_{\text{in}}\left[\frac{\tilde{Z}_2}{\tilde{Z}_1 + \tilde{Z}_2}\right] \tag{5}$$

Now we are ready to apply this technique to some examples.

3.3 A High-Pass RC Filter

The configuration we wish to analyze is shown in Fig. 10. Note that it is the same as Fig. 7 of the notes. However, this time we apply a voltage which is sinusoidal: $\tilde{V}_{\text{in}}(t) = V_{\text{in}}e^{\imath(\omega t+\phi)}$. As an example of another common variation in notation, the figure indicates that the input is sinusoidal ("AC") by using the symbol shown for the input. Note also that the input and output voltages are represented in the figure only by their amplitudes V_{in} and V_{out}, which also is common. This is fine, since the method we are using to analyze the circuit (complex impedances) shouldn't necessarily enter into how we describe the physical circuit.

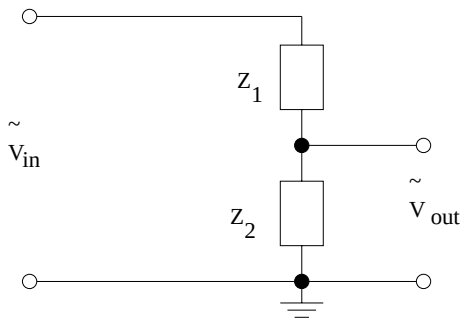

Figure 9: The voltage divider generalized.

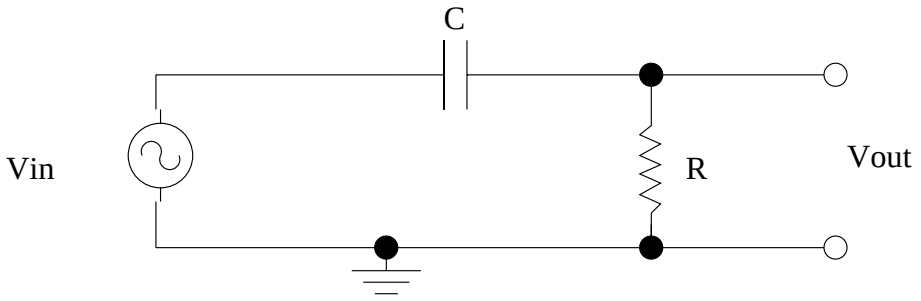

Figure 10: A high-pass filter.

We see that we have a generalized voltage divider of the form discussed in the previous section. Therefore, from Eqn. 5 we can write down the result if we substitute $\tilde{Z}_1 = \tilde{Z}_C = -\imath/(\omega C)$ and $\tilde{Z}_2 = \tilde{Z}_R = R$:

$$\tilde{V}_{\text{out}} = \tilde{V}_{\text{in}} \left[\frac{R}{R - \imath/(\omega C)} \right]$$

At this point our result is general, and includes both amplitude and phase information. Often, we are only interested in amplitudes. We can divide by $\tilde{V}_{\text{in}}$ on both sides and find the amplitude of this ratio (by multiplying by the complex conjugate then taking the square root). The result is often referred to as the *transfer function* of the circuit, which we can designate by $T(\omega)$.

$$T(\omega) \equiv \frac{|\tilde{V}_{\text{out}}|}{|\tilde{V}_{\text{in}}|} = \frac{V_{\text{out}}}{V_{\text{in}}} = \frac{\omega RC}{[1 + (\omega RC)^2]^{1/2}} \tag{6}$$

Examine the behavior of this function. Its maximum value is one and minimum is zero. You should convince yourself that this circuit attenuates low frequencies and "passes" (transmits with little attenuation) high frequencies, hence the term *high-pass filter*. The cutoff between high and low frequencies is conventionally described as the frequency at which the transfer function is $1/\sqrt{2}$. This is approximately equal to an attenuation of 3 *decibels*, which is a description often used in engineering (see below). From Eqn. 6 we see that $T = 1/\sqrt{2}$ occurs at a frequency

$$2\pi f_{3\text{db}} = \omega_{3\text{db}} = 1/(RC) \tag{7}$$

The decibel scale works as follows: db$= 20 \log_{10}(A_1/A_2)$, where A_1 and A_2 represent any real quantity, but usually are amplitudes. So a ratio of 10 corresponds to 20 db, a ratio of 2 corresponds to 6 db, $\sqrt{2}$ is approximately 3 db, *etc.*

3.4 A Low-Pass RC Filter

An analogy with the analysis above, we can analyze a low-pass filter, as shown in Fig. 11.

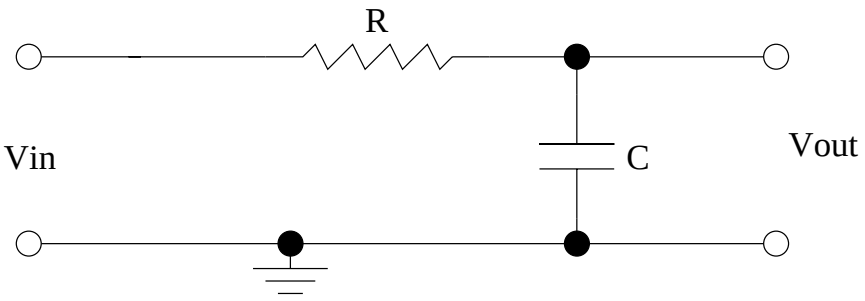

Figure 11: A low-pass filter.

You should find the following result for the transfer function:

$$T(\omega) \equiv \frac{|\tilde{V}_{\rm out}|}{|\tilde{V}_{\rm in}|} = \frac{V_{\rm out}}{V_{\rm in}} = \frac{1}{[1+(\omega RC)^2]^{1/2}} \tag{8}$$

You should verify that this indeed exhibits "low pass" behavior. And that the 3 db frequency is the same as we found for the high-pass filter:

$$2\pi f_{\rm 3db} = \omega_{\rm 3db} = 1/(RC) \tag{9}$$

We note that the two circuits above are equivalent to the circuits we called "differentiator" and "integrator" in Section 2. However, the concept of high-pass and low-pass filters is much more general, as it does not rely on an approximation.

An aside. One can compare our results for the RC circuit using the complex impedance technique with what one would obtain by starting with the differential equation (in time) for an RC circuit we obtained in Section 2, taking the Fourier transform of that equation, then solving (algebraically) for the transform of $V_{\rm out}$. It should be the same as our result for the amplitude $V_{\rm out}$ using impedances. After all, that is what the impedance technique is doing: transforming our time-domain formuation to one in frequency domain, which, because of the possibility of analysis using a single Fourier frequency component, is particularly simple. This is discussed in more detail in the next notes.

Class Notes 4

3.5 Frequency Domain Analysis (contd.)

Before we look at some more examples using our technique of complex impedance, let's look at some related general concepts.

3.5.1 Reactance

First, just a redefintion of what we already have learned. The term *reactance* is often used in place of impedance for capacitors and inductors. Reviewing our definitions of impedances from Section 3.2 we define the reactance of a capacitor $\tilde{X}_C$ to just be equal to its impedance: $\tilde{X}_C \equiv -\imath/(\omega C)$. Similarly, for an inductor $\tilde{X}_L \equiv \imath\omega L$. This is the notation used in the text.

However, an alternative but common useage is to define the reactances as real quantities. This is done simply by dropping the $\imath$ from the definitions above. The various reactances present in a circuit can by combined to form a single quantity X, which is then equal to the imaginary part of the impedance. So, for example a circuit with R, L, and C in series would have total impedance

$$\tilde{Z} = R + \imath X = R + \imath(X_L + X_C) = R + \imath(\omega L - \frac{1}{\omega C})$$

A circuit which is "reactive" is one for which X is non-negligible compared with R.

3.5.2 General Solution

As stated before, our technique involves solving for a single Fourier frequency component such as $\tilde{V} = Ve^{\imath(\omega t+\phi)}$. You may wonder how our results generalize to other frequencies and to input waveforms other than pure sine waves. The answer in words is that we Fourier decompose the input and then use these decomposition amplitudes to weight the output we found for a single frequency, V_{out}. We can formalize this within the context of the Fourier transform, whch will also allow us to see how our time-domain differential equation became transformed to an algebraic equation in frequency domain.

Consider the example of the RC low-pass filter, or integrator, circuit of Fig. 7. We obtained the differential equation given by Eq. 2. We wish to take the Fourier transform of this equation. Define the Fourier transform of $V(t)$ as

$$v(\omega) \equiv \mathcal{F}\{V(t)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dt e^{-\imath\omega t} V(t) \tag{10}$$

Recall that $\mathcal{F}\{dV/dt\} = \imath\omega\mathcal{F}\{V\}$. Therefore our differential equation becomes

$$\imath\omega v(\omega) + v(\omega)/(RC) = \mathcal{F}\{V_{\text{in}}(t)\}/(RC) \tag{11}$$

Solving for $v(\omega)$ gives

$$v(\omega) = \frac{\mathcal{F}\{V_{\text{in}}(t)\}}{1 + \imath\omega RC} \tag{12}$$

The general solution is then the real part of the inverse Fourier transform:

$$\tilde{V}(t) = \mathcal{F}^{-1}\{v(\omega)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\omega' e^{\imath\omega' t} v(\omega') \tag{13}$$

In the specific case we have considered so far of a single Fourier component of frequency ω, *i.e.* $\tilde{V}_{\rm in} = V_i e^{\imath\omega t}$, then $\mathcal{F}\{\tilde{V}_{\rm in}(t)\} = \sqrt{2\pi}\delta(\omega - \omega')$, and we recover our previous result for the transfer function:

$$\tilde{T} = \tilde{V}/\tilde{V}_{\rm in} = \frac{1}{1 + \imath\omega RC} \tag{14}$$

For an arbitrary functional form for $V_{\rm in}(t)$, one could use Eqns. 12 and 13. Note that one would go through the same steps if $V_{\rm in}(t)$ were written as a Fourier series rather than a Fourier integral. Note also that the procedure carried out to give Eqn. 11 is formally equivalent to our use of the complex impedances: In both cases the differential equation is converted to an algebraic equation.

3.6 Phase Shift

We now need to discuss finding the phase ϕ of our solution. To do this, we proceed as previously, for example like the high-pass filter, but this time we preserve the phase information by not taking the modulus of $\tilde{V}_{\rm out}$. The input to a circuit has the form $\tilde{V}_{\rm in} = V_{\rm in} e^{\imath(\omega t + \phi_1)}$, and the output $\tilde{V}_{\rm out} = V_{\rm out} e^{\imath(\omega t + \phi_2)}$. We are usually only interested in the phase difference $\phi_2 - \phi_1$ between input and output, so, for convenience, we can choose $\phi_1 = 0$ and set the *phase shift* to be $\phi_2 \equiv \phi$. Physically, we must choose the real or imaginary part of these expressions. Conventionally, the real part is used. So we have:

$$V_{\rm in}(t) = \Re(\tilde{V}_{\rm in}) = V_{\rm in}(\omega)\cos(\omega t)$$

and

$$V_{\rm out}(t) = \Re(\tilde{V}_{\rm out}) = V_{\rm out}(\omega)\cos(\omega t + \phi)$$

Let's return to our example of the high-pass filter to see how to calculate the phase shift. We rewrite the expression from Section 3.3 and then multiply numerator and denominator by the complex conjugate of the denominator:

$$\tilde{V}_{\rm out} = \tilde{V}_{\rm in}\left[\frac{R}{R - \imath/(\omega C)}\right] = V_{\rm in} e^{\imath\omega t} \frac{1 + \imath/(\omega RC)}{1 + 1/(\omega RC)^2}$$

By recalling the general form $a + \imath b = \sqrt{a^2 + b^2}\, e^{\imath\phi}$, where $\phi = \tan^{-1}(b/a)$, we can write

$$1 + \imath/(\omega RC) = \left[1 + \left(\frac{1}{\omega RC}\right)^2\right]^{1/2} e^{\imath\phi}$$

allowing us to read off the phase shift:

$$\phi = \tan^{-1}\left(1/(\omega RC)\right) \tag{15}$$

Our solution for $\tilde{V}_{\rm out}$ is then

$$\tilde{V}_{\rm out} = \frac{V_{\rm in} e^{\imath\omega t + \phi}}{\left[1 + (\frac{1}{\omega RC})^2\right]^{1/2}}$$

This, of course, yields the same $|\tilde{V}_{\rm out}|$ as we found before in Eqn. 6 of Section 3.3. But now we also have included the phase information. The "real" time-dependent solution is then just the real part of this:

$$V_{\rm out}(t) = \Re(\tilde{V}_{\rm out}) = V_{\rm out}\cos(\omega t + \phi)$$

where ϕ is given by Eqn. 15.

3.7 Power in Reactive Circuits

Recall that for DC voltages and currents the power associated with a circuit element carrying current I with voltage change V is just $P = VI$. Now, for time-varying voltages and currents we have to be more careful. We could still define an instantaneous power as the product $V(t)I(t)$. However, it is generally more useful to average the power over time.

3.7.1 General Result for AC

Since we are considering Fourier components, we will average the results over one period $T = 1/f = 2\pi/\omega$. Therefore, the time-averaged power is

$$< P >= \frac{1}{T}\int_0^T V(t)I(t)dt$$

where the brackets indicate the time average. Let the voltage and current be out of phase by an arbitrary phase angle ϕ. So we have $V(t) = V_0\cos(\omega t)$ and $I(t) = I_0\cos(\omega t + \phi)$. We can plug these into the expression for $< P >$ and simplify using the following: $\cos(\omega t + \phi) = \cos(\omega t)\cos(\phi) - \sin(\omega t)\sin(\phi)$; $\int_0^T \sin(\omega t)\cos(\omega t)dt = 0$; and $(1/T)\int_0^T \sin^2(\omega t)dt = (1/T)\int_0^T \cos^2(\omega t)dt = 1/2$. This yields

$$< P >= \frac{1}{2}V_0 I_0\cos\phi = V_{\rm RMS}I_{\rm RMS}\cos\phi \tag{16}$$

In the latter expression we have used the "root mean squared", or *RMS*, amplitudes. Using voltage as an example, the RMS and standard amplitudes are related by

$$V_{\rm RMS} \equiv \left[\frac{1}{T}\int_0^T V^2(t)dt\right]^{1/2} = \left[\frac{1}{T}\int_0^T V_0^2\cos^2(\omega t)dt\right]^{1/2} = V_0/\sqrt{2} \tag{17}$$

3.7.2 Power Using Complex Quantities

Our results above can be simply expressed in terms of $\tilde{V}$ and $\tilde{I}$. Equivalent to above, we start with $\tilde{V}(t) = V_0 e^{\imath\omega t}$ and $\tilde{I}(t) = I_0 e^{\imath(\omega t+\phi)}$. By noting that

$$\Re(\tilde{V}^*\tilde{I}) = \Re\left(V_0 I_0(\cos\phi + \imath\sin\phi)\right) = V_0 I_0\cos\phi$$

we identify an expression for average power which is equivalent to Eqn. 16 :

$$< P >= \frac{1}{2}\Re(\tilde{V}^*\tilde{I}) = \frac{1}{2}\Re(\tilde{V}\tilde{I}^*) \tag{18}$$

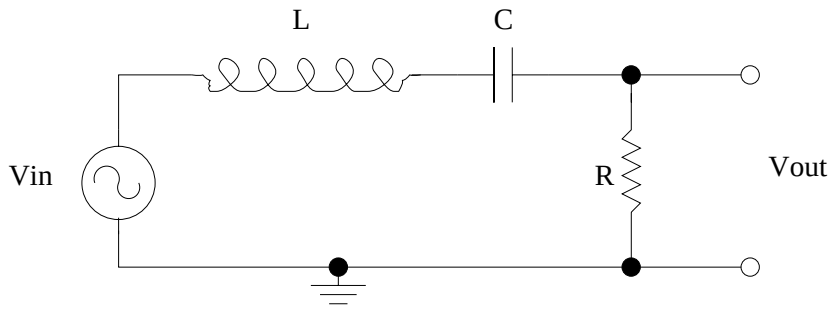

Figure 12: A RLC circuit. Several filter types are possible depending upon how $V_{\rm out}$ is chosen. In the case shown, the circuit gives a resonant output.

3.8 An RLC Circuit Example

We can apply our technique of impedance to increasingly more intricate examples, with no more effort than a commensurate increase in the amount of algebra. The RLC circuit of Fig. 12 exemplifies some new qualitative behavior.

We can again calculate the output using our generalized voltage divider result of Eqn. 5. In this case, the $\tilde{Z}_1$ consists of the inductor and capacitor in series, and $\tilde{Z}_2$ is simply R. So,

$$\tilde{Z}_1 = \imath\omega L - \imath/(\omega C) = \frac{\imath L}{\omega}\left(\omega^2 - \omega_0^2\right)$$

where we have defined the *LC resonant frequency* $\omega_0 \equiv 1/\sqrt{LC}$. We then obtain for the transfer function:

$$T(\omega) \equiv \frac{|\tilde{V}_{\rm out}|}{|\tilde{V}_{\rm in}|} = \frac{R}{|R + \tilde{Z}_1|} = \frac{\omega\gamma}{\left[\omega^2\gamma^2 + (\omega^2 - \omega_0^2)^2\right]^{1/2}}$$

where $\gamma \equiv R/L$ is the "R-L frequency".

$T(\omega)$ indeed exhibits a resonance at $\omega = \omega_0$. The quality factor Q, defined as the ratio of ω_0 to the width of the resonance is given by $Q \approx \omega_0/(2\gamma)$ for $\gamma \ll \omega_0$. Such circuits have many applications. For example, a high-Q circuit, where $V_{\rm in}(t)$ is the signal on an antenna, can be used as a receiver.

As was shown in class, we achieve different behavior if we choose to place the output across the capacitor or inductor, rather than across the resistor, as above. Rather than a resonant circuit, choosing $V_{\rm out} = V_C$ yields a low-pass filter of the form

$$T(\omega) = \frac{|-\imath/(\omega C)|}{|R + \tilde{Z}_1|} = \frac{\omega_0^2}{\left[\omega^2\gamma^2 + (\omega^2 - \omega_0^2)^2\right]^{1/2}}$$

The cutoff frequency is ω_0, and for $\omega \gg \omega_0$ then $T \sim \omega^{-2}$ ("12 db per octave"), which more closely approaches ideal step function-like behavior than the RC low pass filter, for which $T \sim \omega^{-1}$ for $\omega \gg \omega_0$ ("6 db per octave"). As you might suspect, choosing $V_{\rm out} = V_L$ provides a high-pass filter with cutoff at ω_0 and $T \sim \omega^{-2}$ for $\omega \ll \omega_0$.

3.9 More Filters

3.9.1 Combining Filter Sections

Filter circuits can be combined to produce new filters with modified functionality. An example is the homework problem (6) of page 59 of the text, where a high-pass and a low-pass filter are combined to form a "band-pass" filter. As discussed at length in Section 1.5, it is important to design a "stiff" circuit, in which the next circuit element does not load the previous one, by requiring that the output impedance of the first be much smaller than the input impedance of the second. We can standardize this inequality by using a factor of 10 for the ratio $|\tilde{Z}_{\rm in}|/|\tilde{Z}_{\rm out}|$.

3.9.2 More Powerful Filters

This technique of cascading filter elements to produce a better filter is discussed in detail in Chapter 5 of the text. In general, the transfer functions of such filters take the form (for the low-pass case):

$$T(\omega) = \left[1 + \alpha_n (f/f_c)^{2n}\right]^{-1/2}$$

where f_c is the 3 db frequency, α_n is a coefficient depending upon the type of filter, and n is the filter "order," often equal to the number of filtering capacitors.

3.9.3 Active Filters

Filters involving LC circuits are very good, better than the simple RC filters, as discussed above. Unfortunately, inductors are, in practice, not ideal lumped circuit elements and are difficult to fabricate. In addition, filters made entirely from passive elements tend to have a lot of attenuation. For these reasons active filters are most commonly used where good filtering is required. These typically use operational amplifiers (which we will discuss later), which can be configured to behave like inductors, and can have provide arbitrary voltage gain. Again, this is discussed in some detail in Chapter 5. When we discuss op amps later, we will look at some examples of very simple active filters. At high frequencies (for example RF), op amps fail, and one most fall back on inductors.

4 Diode Circuits

The figure below is from Lab 2, which gives the circuit symbol for a diode and a drawing of a diode from the lab. Diodes are quite common and useful devices. One can think of a diode as a device which allows current to flow in only one direction. This is an over-simplification, but a good approximation.

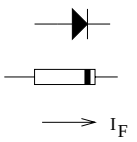

Figure 13: Symbol and drawing for diodes.

A diode is fabricated from a *pn* junction. Semi-conductors such as silicon or germanium can be "doped" with small concentrations of specific impurities to yield a material which conducts electricity via electron transport (*n*-type) or via holes (*p*-type). When these are brounght together to form a *pn* junction, electrons (holes) migrate away from the *n*-type (*p*-type) side, as shown in Fig. 14. This redistribution of charge gives rise to a potential gap ΔV across the junction, as depicted in the figure. This gap is $\Delta V \approx 0.7$ V for silicon and ≈ 0.3 V for germanium.

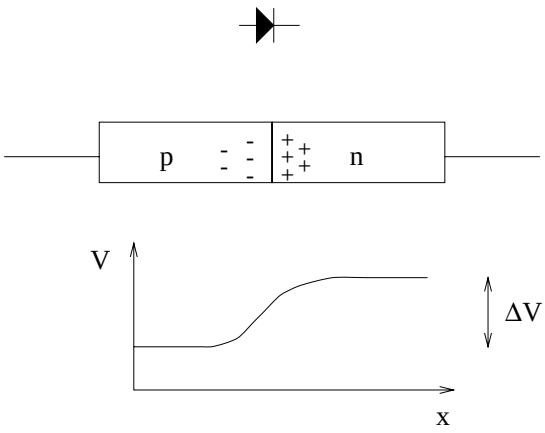

Figure 14: A *pn* junction, forming a voltage gap across the junction.

When a diode is now connected to an external voltage, this can effectively increase or decrease the potential gap. This gives rise to very different behavior, depending upon the polarity of this external voltage, as shown by the typical V-I plot of Fig. 15. When the diode is "reverse biased," as depicted in the figure, the gap increases, and very little current flows across the junction (until eventually at ~ 100 V field breakdown occurs). Conversely, a "forward biased" configuration decreases the gap, approaching zero for an external voltage equal to the gap, and current can flow easily. An analysis of the physics gives the form

$$I = I_S \left[e^{eV/kT} - 1 \right]$$

where I_S is a constant, V is the applied voltage, and $kT/e = 26$ mV at room temperature.

Thus, when reverse biased, the diode behaves much like an open switch; and when forward biased, for currents of about 10 mA or greater, the diode gives a nearly constant voltage drop of ≈ 0.6 V.

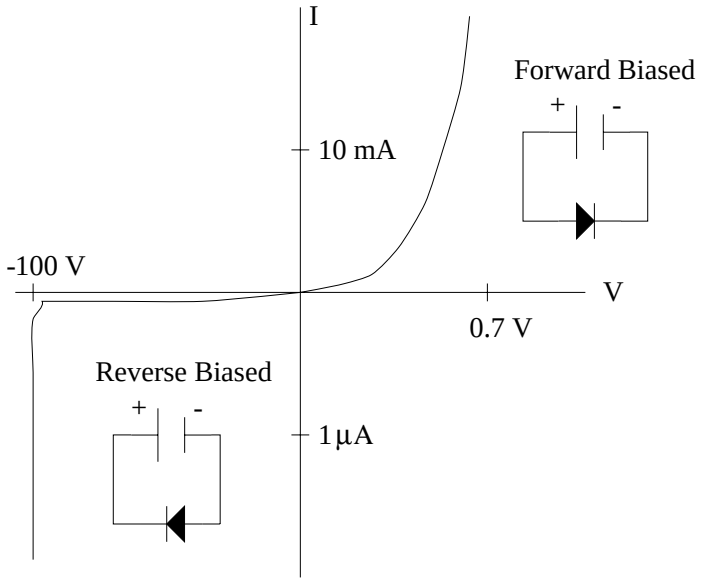

Figure 15: The V-I behavior of a diode.

5 Transistors and Transistor Circuits

Although I will not follow the text in detail for the discussion of transistors, I will follow the text's philosophy. Unless one gets into device fabrication, it is generally not important to understand the inner workings of transistors. This is difficult, and the descriptions which one gets by getting into the intrinsic properties are not particularly satisfying. Rather, it is usually enough to understand the extrinsic properties of transistors, treating them for the most part as a black box, with a little discussion about the subtleties which arise from within the black box.

In practice, one usually confronts transistors as components of pre-packaged circuits, for example in the operational amplifier circuits which we will study later. However, I have found that it is very useful to understand transistor behavior even if one rarely builds a transistor circuit in practice. The ability to analyze the circuit of an instrument or device is quite valuable.

We will start, as with Chapter 2 of the text, with bipolar transistors. There are other common technologies used, particularly FET's, which we will discuss later. However, most of what you know can be carried over directly by analogy. Also, we will assume *npn* type transistors, except where it is necessary to discuss *pnp*. For circuit calculations, one simply reverses all signs of relevant currents and voltages in order to translate *npn* to *pnp*.

5.1 Connections and Operating Mode

Below we have the basic connection definitions for bipolar transistors as taken from the text. As indicated in the figure, and as you determined in lab, the base-emitter and base-collector pairs behave somewhat like diodes. Do not take this too literally. In particular, for the base-collector pair this description is far off the mark. We will refer to the transistor connections as C, B, and E.

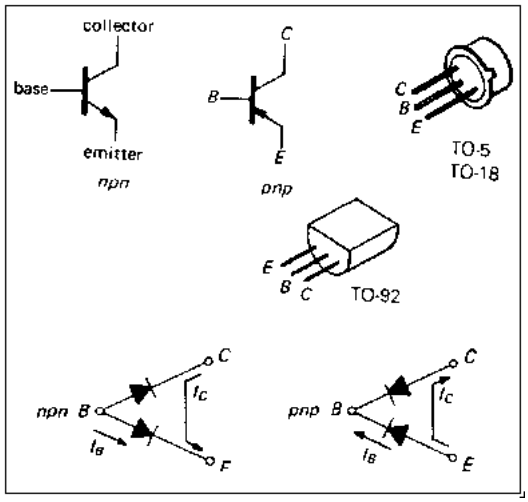

Figure 16: Bipolar transistor connections.

5.1.1 Rules for Operation

Let's start by stating what needs to be done to a transistor to make it operate as a transistor. Suppose we have the following:

1. $V_C > V_E$, by at least a *few* $\times 0.1$ V.
2. $V_B > V_E$
3. $V_C > V_B$
4. We do not exceed maximum ratings for voltage differences or currents.

When these conditions are *not* met, then (approximately) no current flows in or out of the transistor. When these conditions *are* met, then current can flow into the collector (and out the emitter) in proportion to the current flowing into the base:

$$I_C = h_{\rm FE} I_B = \beta I_B \tag{19}$$

where $h_{\rm FE} = \beta$ is the *current gain*. (We will use the β notation in these notes.) The value of the current gain varies from transistor type to type, and within each type, too. However, typically $\beta \approx 100$. Unless otherwise specified, we will assume $\beta = 100$ when we need a number. From Figure 17 below and Kirchoff's first law, we have the following relationship among the currents:

$$I_E = I_B + I_C = I_B + \beta I_B = (\beta + 1) I_B \approx I_C \tag{20}$$

As we will see below, the transistor will "try" to achieve its nominal β. This will not always be possible, in which case the transistor will still be on, but $I_C < \beta I_B$. In this case, the transistor is said to be "saturated".

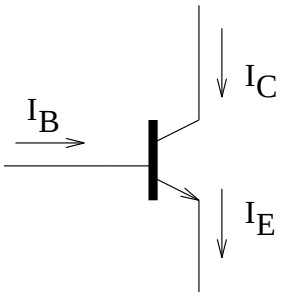

Figure 17: Transistor currents.

Because $\beta \gg 1$, the main utility of the transistor becomes evident: We are able to control a large current $I_C \approx I_E$ with a small current I_B. The simplest such control is in the form of a switch. Note that in our second condition above we require that the base-emitter "diode" be forward biased, *i.e.* that $V_{\rm BE} \equiv V_B - V_E$ be positive. In fact, the base-emitter pair does behave much like a diode. So when it is forward biased, current can easily flow, and the voltage drop quickly reaches its asymptotic value of ≈ 0.6 V. Unless otherwise noted, we will generally assume that, when the transistor is in operation, we have

$$V_{\rm BE} \equiv V_B - V_E \approx 0.6 {\rm Volts} \tag{21}$$

5.1.2 Transistor Switch and Saturation

From the preceding discussion, the most straightforward way to turn the transistor "on" or "off" is by controlling $V_{\rm BE}$. This is illustrated by the circuit below which was introduced in Lab 2. We will follow the lab steps again here.

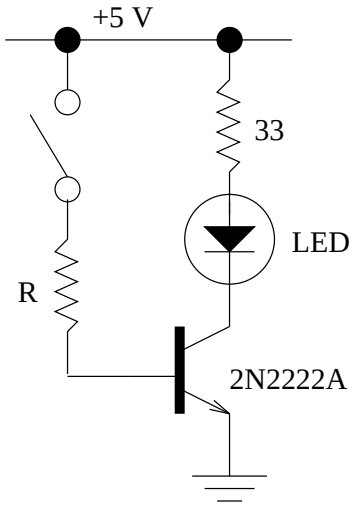

Figure 18: A transistor switch.

First, let $R = 10$ kΩ. When the switch is open, $I_C = \beta I_B = 0$, of course. When the switch is closed, then $V_{\rm BE}$ becomes positive and $V_B = V_E + 0.6 = 0.6$ V. $I_B = (5 - 0.6)/10^4 = 0.44$ mA. Hence, $I_C = \beta I_B = 44$ mA. Then, assuming negligible voltage drop across the LED, $V_C = 5 - 33 \times 0.044 = 3.5$ V. So, $V_{\rm CE} > 0$ and $V_{\rm CB} > 0$. So this should work just fine.

Substituting $R = 1$ kΩ gives $I_B = 4.4$ mA and $\beta I_B = 440$ mA. Setting this equal to I_C would give $V_C = -9.5$ V. This is not possible. In order to stay in operation $V_{\rm CE}$ must be positive, and depending upon the transistor species, usually can only go as low as ≈ 0.2 V. (Appendix K of the text, pages 1066-1067, gives data for a typical model.) Hence, I_C is limited to a maximum value of $I_C = (5 - 0.2)/33 \approx 150$ mA. So, effectively, the current gain has been reduced to $\beta = I_C/I_B = 150/4.4 = 34$. In this mode of operation, the transistor is said to be *saturated*. It turns out that for high-speed switching applications, for example in computers, the transistors are generally operated in a partially saturated mode, for reasons discussed in Section 2.02 of the text.

5.2 Notation

We will now look at some other typical transistor configurations, including the emitter follower, the current source, and the common-emitter amplifier. But first we need to set some notation. We will often be considering voltages or currents which consist of a time varying signal superposed with a constant DC value. That is,

$$V(t) = V_0 + v(t)\,; \quad I(t) = I_O + i(t)$$

where V_0 and I_0 are the DC quantities, and v or i represent time-varying signals. Hence,

$$\Delta V = v\,; \quad \Delta I = i$$

Typically, we can consider v or i to be sinusoidal functions, *e.g.* $v(t) = v_o \cos(\omega t + \phi)$, and their amplitudes v_o and i_o (sometimes also written as v or i when their is no confusion) are small compared with V_0 or I_0, respectively.

5.3 Emitter Follower

The basic emitter follower configuration is shown below in Figure 19. An input is fed to the base. The collector is held (by a voltage source) to a constant DC voltage, $V_{\rm CC}$. The emitter connects to a resistor to ground and an output. As we shall see, the most useful characteristic of this circuit is a large input impedance and a small output impedance.

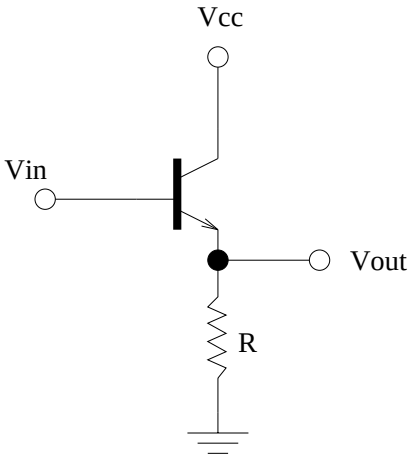

Figure 19: Basic emitter follower.

For an operating transistor we have $V_{\rm out} = V_E = V_B - 0.6$. Hence, $v_{\rm out} = v_E = v_B$. From this, we can determine the voltage gain G, equivalent to the transfer function, for the emitter follower:

$$G \equiv v_{\rm out}/v_{\rm in} = v_E/v_B = 1 \tag{22}$$

From Eqn. 20, $I_E = (\beta+1)I_B \Rightarrow i_E = (\beta+1)i_B$. Therefore, we see that the follower exhibits "current gain" of output to input equal to $\beta + 1$. Assuming the output connection draws negligible current, we have by Ohm's Law $i_E = v_E/R$. Using this in the previous expression and solving for i_B gives $i_B = i_E/(\beta + 1) = (v_B/R)/(\beta + 1)$. Now we can define the input impedance of the follower:

$$Z_{\rm in} = v_{\rm in}/i_{\rm in} = v_B/i_B = R(\beta + 1) \tag{23}$$

By applying the Thevenin definition for equivalent impedance, we can also determine the output impedance of the follower:

$$Z_{\rm out} = v_{\rm in}/i_E = \frac{v_{\rm in}}{(\beta + 1)i_B} = \frac{Z_{\rm source}}{\beta + 1} \tag{24}$$

where $Z_{\rm source}$ is the source (*i.e.* output) impedance of the circuit which gave rise to $v_{\rm in}$. Hence, the emitter follower effectively increases input impedance (compared to R) by a factor $\beta + 1 \approx 100$ and reduces output impedance, relative to that of the source impedance of the previous circuit element, by a factor $\beta + 1 \approx 100$. We will return to this point next time.

Class Notes 6

Following our discussion last time of the basic transistor switch and emitter follower, we will likewise introduce the basic relations for two other transistor circuit configurations: the current source and the common-emitter amplifier. We will then return to the issue of input and output impedance so that we can build realistic circuits using these configurations.

5.4 Transistor Current Source

Figure 20 illustrates the basic configuration for a single-transistor current source. $V_{\rm CC}$ is a constant positive voltage from a DC power supply. Hence, the base voltage V_B is also a constant, with $V_B = V_{\rm CC}R_2/(R_1 + R_2)$. R_L represents a load which we intend to power with a current which is approximately independent of the specific value of R_L.

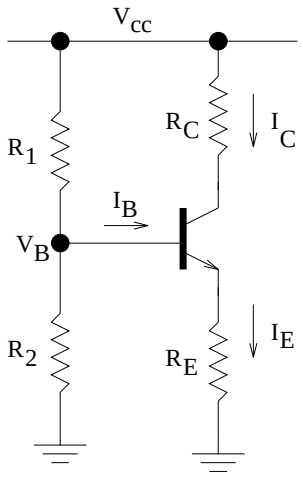

Figure 20: Basic transistor current source.

When the transistor is on, we have $I_E = (\beta+1)I_B$. In addition, we have $V_E = V_B - 0.6$; and $V_E = I_E R_E = (\beta+1)I_B R_E$. Solving for I_B in this last equation gives $I_B = V_E/((\beta+1)R_E)$. We can combine these to solve for the current which passes through R_L:

$$I_L = I_C = \beta I_B = \beta \frac{V_E}{(\beta+1)R_E} = \frac{\beta}{\beta+1}\frac{V_B - 0.6}{R_E} \approx \frac{V_B - 0.6}{R_E} \tag{25}$$

Hence, we see that indeed I_L is independent of R_L.

Of course, there are limitations to the range of R_L for which the current source behavior is reasonable. Recall that the transistor will shut down if $V_B \leq V_E$ or if $V_{\rm CE}$ is less than ≈ 0.2 V. These criteria determine the *compliance* of the current source, that is its useful operating range. So, for example, if we have $V_{\rm CC} = 15$ V and $V_B = 5$ V in our circuit above, then $V_E = 5 - 0.6 = 4.4$ V, and the range of compliance for the collector voltage V_C will be approximately 4.6 V to 15 V.

5.5 Common-emitter Amplifier

Figure 21 represents the basic configuration of the common-emitter amplifier. To determine the output for this circuit, we assume at this point that the input is a sum of a DC offset voltage V_0 and a time-varying signal v_{in}, as discussed last time. (In the next section we will discuss how to achieve these.) V_0 provides the transistor "bias", so that $V_B > V_E$, and the signal of interest is v_{in}.

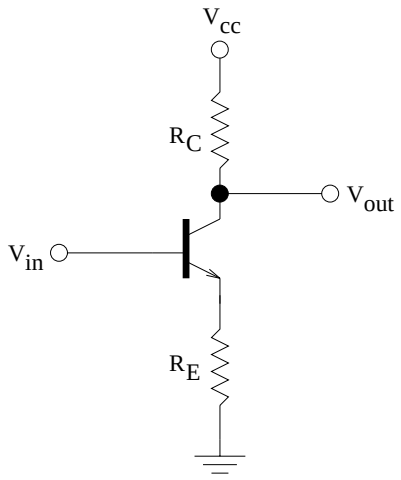

Figure 21: Basic common-emitter amplifier.

The incoming signal shows up on the emitter: $v_{\text{in}} = \Delta(V_E + 0.6) = \Delta V_E \equiv v_E$. And by Ohm's Law, $i_E = v_E/R_E = v_B/R_E$. As we found previously, $i_E = i_C + i_B \approx i_C$. Now, the voltage at the output is $V_{\text{out}} = V_C = V_{\text{CC}} - I_C R_C$. And therefore, $\Delta V_{\text{out}} \equiv v_{\text{out}} = -i_C R_C$. Putting all of this together, $v_{\text{out}} = -i_C R_C \approx -i_E R_C = -(v_B/R_E)\, R_C$, giving the voltage gain G:

$$G \equiv v_{\text{out}}/v_{\text{in}} = -R_C/R_E \tag{26}$$

5.6 Circuit Biasing and Input

Now we need to figure out how to provide inputs to our basic circuits. In Fig. 22 below we show the input network for a common-emitter amplifier. The same considerations we apply here apply equally to the input of an emitter follower. The idea is that the voltage divider R_1 and R_2 provide the DC bias voltage (V_0 in our discussion above), and the time varying signal is input through the capacitor (which blocks the DC). We need to figure out what design criteria should be applied to this design.

We need to make sure that our input circuit does not load the amplifier, C is chosen to give a reasonable RC cutoff, and that the gain of the amplifier is what we want. We will start by designing the DC component of the input network, that is choosing R_1 and R_2. It is helpful when designing the input network to consider the equivalent circuit shown in Fig. 23. The diode and resistor labelled Z_{in} represent the transistor input: the voltage drop across the base-emitter "diode" and the input impedance from Eqn. 23. R_{TH} is the Thenenin equivalent resistance for the DC input network.

So our design procedure can be as follows:

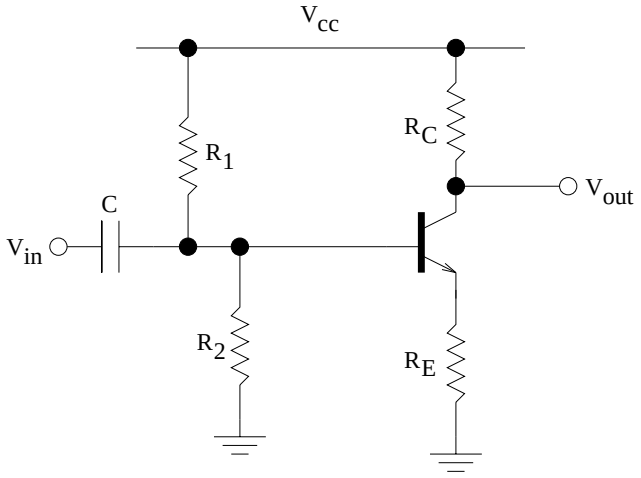

Figure 22: Common-emitter amplifier with input network.

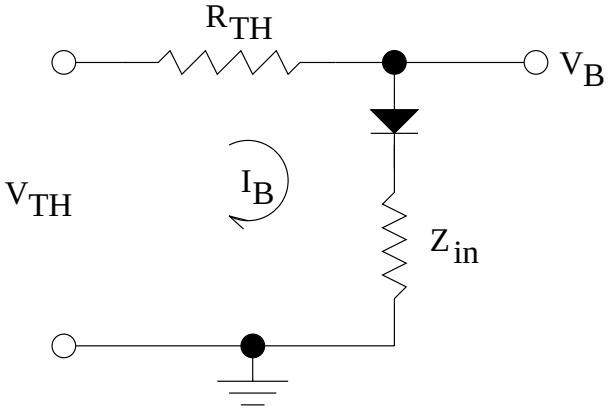

Figure 23: Equivalent circuit for design of DC input network.

1. Choose $R_{\text{TH}} \ll Z_{\text{in}} = R_E(\beta + 1)$.
2. Determine R_1 and R_2 based on the equivalent circuit.
3. Choose C to provide a proper high-pass cutoff frequency.
4. Choose the amplifier gain, if need be.

5.7 Transistor Differential Amplifier

Differential amplifiers are in general very useful. They consist of two inputs and one output, as indicated by the generic symbol in Fig. 24. The output is proportional to the *difference* between the two inputs, where the proportionality constant is the gain. One can think of this as one of the two inputs (labelled "−") being inverted and then added to the other non-inverting input (labelled "+"). Operational amplifiers ("op amps"), which we will soon study, are fancy differential amplifiers, and are represented by the same symbol as that of Fig. 24.

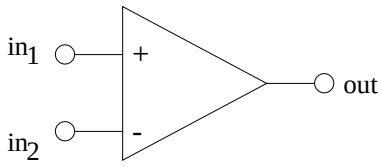

Figure 24: Symbol for a differential amplifier or op amp.

This technique is commonly used to mitigate noise pickup. For example, a signal which is to be transmitted and subject to noise pickup can first be replicated and inverted. This "differential pair" is then transmitted and then received by a differential amplifier. Any noise pickup will be approximately equal for the two inputs, and hence will not appear in the output of the differential amplifer. This "common mode" noise is rejected. This is often quantified by the common-mode rejection ratio (CMRR) which is the ratio of differential gain to common-mode gain. Clearly, a large CMRR is good.

5.7.1 A Simple Design

The circuit shown in Fig. 25 represents a differential amplifier design. It looks like two common-emitter amplifiers whose emitters are tied together at point A. In fact, the circuit does behave in this way. It is simplest to analyze its output if one writes each input as the sum of two terms, a sum and a difference. Consider two signals v_1 and v_2. In general, we can rewrite these as $v_1 = < v > + \Delta v/2$ and $v_2 = < v > - \Delta v/2$, where $< v > = (v_1 + v_2)/2$ is the average and $\Delta v = v_1 - v_2$ is the difference. Therefore, we can break down the response of the circuit to be due to the response to a common-mode input ($< v >$) and a difference (Δv) input.

Let's analyze the difference signal first. Therefore, consider two inputs $v_1 = \Delta v/2$ and $v_2 = -\Delta v/2$. The signals at the emitters then follow the inputs, as usual, so that at point A we have $v_A = v_{E1} + v_{E2} = v_1 + v_2 = 0$. Following the common-emitter amplifier derivation, we have $v_{\rm out1} = -i_C R_C$, where $i_C \approx i_E = v_E/R_E = v_{\rm in1}/R_E$. Hence, $v_{\rm out1} = -(R_C/R_E)v_1$ and $v_{\rm out2} = -(R_C/R_E)v_2$. We define the *differential gain* $G_{\rm diff}$ as the ratio of the output to the input difference. So

$$G_{\rm diff1} \equiv v_{\rm out1}/\Delta v = -(R_C/R_E)v_1/(2v_1) = -R_C/(2R_E)$$

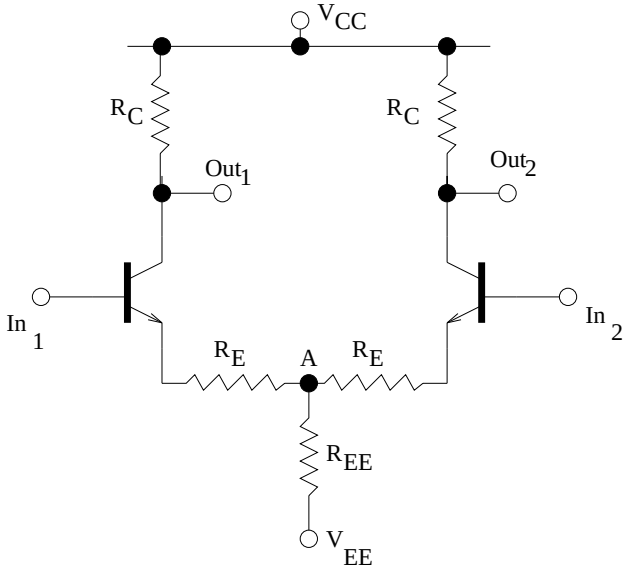

Figure 25: Differential amplifier design.

and similarly for output 2

$$G_{\rm diff2} \equiv v_{\rm out2}/\Delta v = -(R_C/R_E)v_2/(-2v_2) = R_C/(2R_E)$$

Generally, only one of the two ouputs is used. Referring back to Fig. 24, we see that if we were to choose our one output to be the one labelled "$\rm out_2$", then "$\rm in_1$" would correspond to "+" (non-inverting input) and "$\rm in_2$" would correspond to "−" (inverting input). Keeping in mind these results for the relative signs, it is usual to write the differential gain as a positive quantity:

$$G_{\rm diff} = \frac{R_C}{2R_E} \tag{27}$$

where the sign depends upon which is used.

Now consider the common mode part of the inputs: $v_1 = v_2 =< v >$. We have the following relations:

$$i_{EE} = i_{E1} + i_{E2} = 2i_E\,;$$

$$V_A = V_{EE} + I_{EE}R_{EE} \Rightarrow v_A = i_{EE}R_{EE} = 2i_E R_{EE}\,;$$

$$i_E = \frac{v_E - v_A}{R_E} = \frac{v_{\rm in} - 2i_E R_{EE}}{R_E}$$

Solving for i_E in the last equation gives:

$$i_E = v_{\rm in}\left(\frac{1}{R_E + 2R_{EE}}\right)$$

Again following the derivation for the the common-emitter amplifie, we have $v_{\rm out} = -i_C R_C \approx -i_E R_C$. So each output has the same common-mode gain:

$$G_{\rm com} \equiv \frac{v_{\rm out}}{v_{\rm in}} = -\frac{R_C}{R_E + 2R_{EE}} \tag{28}$$

The ratio of the differential to common-mode gain (ignoring the sign) then gives the CMRR:

$$\text{CMRR} = \frac{R_E + 2R_{EE}}{2R_E} \approx \frac{R_{EE}}{R_E} \tag{29}$$

where for a typical design $R_{EE} \gg R_E$.

Building on what we learned, we can easily improve our differential amplifier design by adding an emitter-follower stage to the output and a replacing the resistor R_{EE} with a current source. This is discussed briefly in the next two sections.

5.7.2 Adding a Follower

The output impedance of the common-emitter configuration, as used in the differential amplifer, $Z_{\text{out}} \approx R_C$, is not always as small as one would like. This can be easily improved by adding an emitter follower to the output. Hence, the input of the follower would be connected to the output of the differential amplifier. As discussed before, the follower then produces an output impedance which is $\approx \beta + 1$ times smaller than the preceding stage. Hence, in this case, we would have $Z_{\text{out}} \approx R_C/\beta$. The follower's emitter resistor (call it R'_E), of course, has to be consistent with our impedance non-loading criteria, in this case $\beta R'_E \gg R_C$.

5.7.3 Adding a Current Source

In our expression for CMRR above, we see that a large R_{EE} improves performance. However, this can also significantly load the voltage source of V_{EE}, producing non-ideal behavior. Hence, R_{EE} is limited in practice. A solution to this which is commonly used is to replace R_{EE} by the output of a current source. In other words, point "A" in Fig. 25 would be connected to the collector of a current source such as that of Fig. 20. This can be justified by noting that the dynamic impedance provided by R_{EE} is given by v_A/i_{EE}. By limiting variations in i_{EE}, as provided by a current source, one effectively achieves a large dynamic impedance.

To implement this one has to decide what quiescent current is required for the current source and what the quiescent voltage of the collector should be. The latter is given by the quiescent voltage at the inputs of the differential amplifier. For example, if the inputs are DC ground, then point "A" will be at approximately -0.6 V, depending upon any voltage drop across R_E.

5.7.4 No Emitter Resistor

One variation of the above is to remove the emitter resistor. In this case one replaces R_E in the expressions above with the intrinsic emitter resistance discussed in Section 5.8.1 below:

$$R_E \to r_e = 25\text{mV}/I_C$$

To be exact, one should replace R_E in our equations with the series resistance of R_E and r_e: $R_E \to R_E + r_e$. However, in most practical situations $R_E \gg r_e$.

5.8 More on Transistor Circuits

5.8.1 Intrinsic Emitter Resistance

One consequence of the Ebers-Moll equation, which we will discuss later, is that the transistor emitter has an effective resistance which is given by

$$r_e = 25\text{mV}/I_C$$

This is illustrated in Fig. 26. Essentially one can treat this as any other resistance. So in most of our examples so far in which an emitter resistor R_E is present, one can simply replace R_E by the series sum $R_E + r_e$. Numerically, typical values reveal that r_e is safely ignored. For example, $I_C = 1$ mA gives $r_e = 25\ \Omega$, whereas R_E might be typically ~ 1 kΩ. The exception is an emitter follower output, where the output voltage is divided by r_e and R_E. In some cases an external emitter resistor R_E is omitted, in which case $R_E \to r_e$ in our previous expressions.

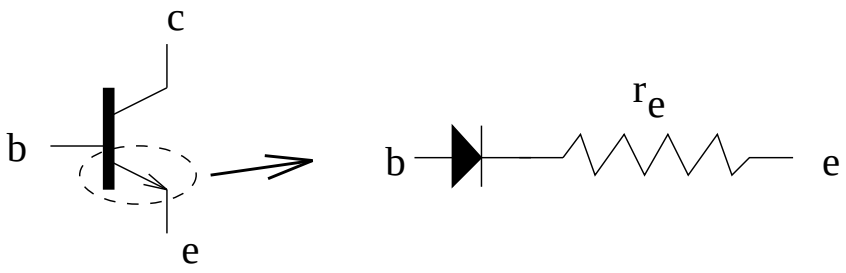

Figure 26: Intrinsic emitter resistance.

5.8.2 Input and Output Impedance of the Common-Emitter Amplifer

For convenience, the basic common-emitter amplifier is reproduced below. The calculation of the input impedance does not differ from that we used for the emitter follower in Section 5.3. That is, the input impedance is $Z_{\text{in}} = R_E(\beta + 1)$. The output impedance is quite different from that of the emitter follower, however. Consider our definition of output impedance in terms of the Thevenin equivalent circuit:

$$Z_{\text{out}} = \frac{v_{\text{out}}}{i(R_L \to 0)}$$

The numerator is just the usual v_{out} we calculated in Eqn. 26. Hence, $v_{\text{out}} = v_{\text{in}}(R_C/R_E)$. The short current is just i_C, and since $i_C = (\beta/(\beta+1))i_E \approx i_E = v_{\text{in}}/R_E$, then we have our result

$$Z_{\text{in}} = (\beta + 1)R_E \ ; \qquad Z_{\text{out}} \approx R_C \tag{30}$$

Note that these results apply equally well to the differential amplifier configuration, which is, as we said before, essentially two coupled common-emitter amplifiers.

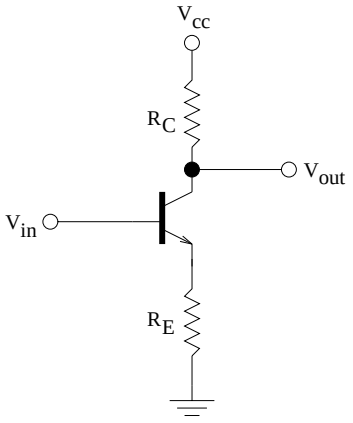

Figure 27: Basic common-emitter amplifier.

5.8.3 DC Connections and Signals

We already discussed in class the fact that for a configuration like that of the input network of Fig. 22, that the input time-varying signal $v_{\rm in}$ is not affected by the DC offsets of the resistor connections. In other words, R_1 and R_2 appear, for a time-varying signal, to be both connected to ground. Hence, when designing the cutoff frequency for the input high-pass filter, the effective resistance is just the usual parallel resistance of R_1, R_2, and the transistor input impedance $R_E(\beta + 1)$.

5.9 Ebers-Moll Equation and Transistor Realism

With the exception of saturation effects and a mention of the intrinsic emitter resistance r_e, we have so far considered transistors in a reather idealized manner. To understand many of the most important aspects of transistor circuits, this approach is reasonable. For example, we have treated the current gain β of a non-saturated transistor to be independent of currents, temperature, etc. In reality, this is not the case. One of the finer points of circuit design is to take care to eliminate a strong dependence of the circuit behavior on such complications. We start with the Ebers-Moll equation, which gives a foundation for understanding one class of complications.

5.9.1 Ebers-Moll Equation

Our simple relationship for collector current for an operating transistor, $I_C = \beta I_B$ is an idealization. We can see from the plots of Appendix K (*cf* pg. 1076-7) that β indeed does depend on various parameters. A more precise description is via the Ebers-Moll equation:

$$I_C = I_S \left[e^{V_{\rm BE}/V_T} - 1 \right] \approx I_S e^{V_{\rm BE}/V_T} \tag{31}$$

where $V_T \equiv kT/e = (25.3\ {\rm mV})(T/298\ {\rm K})$, $I_S = I_S(T)$ is the saturation current, and $V_{\rm BE} \equiv V_B - V_E$, as usual. Since typically $V_{\rm BE} \approx 600{\rm mV} \gg V_T$, then the exponential term is much larger than 1, and $I_S \ll I_C$. Since I_B is also a function of $V_{\rm BE}$, then we see that $\beta = I_C/I_B$ can be thought of as a good approximation for a rather complicated situation, and in fact β is itself a function of I_C (or $V_{\rm BE}$, as well as of temperature.

We see that I_C is not intrinsically a function of I_B, but rather is controlled by $V_{\rm BE}$. For this reason, and others, it is often stated that transistor gain is really a *transconductance* gain. This means that it takes a *voltage input* and converts it to a *current output*. So we write, in general,

$$g_m = i_{\rm out}/v_{\rm in}$$

as the transconductance gain. We then recover voltage gain by multiplying g_m by the resistor at the output which converts the output current to a voltage. For example, for the common-emitter amplifier we have $i_{\rm out} = -v_{\rm in}/R_E$ and

$$G = g_m R_C = -R_C/R_E$$

as before.

The base-emitter "diode" implies a relationship between I_B and $V_{\rm BE}$ of the form $V_{\rm BE} = V_0 \ln(I_B/I_0)$, where $V_0 \approx 0.6$ V and I_0 is a constant. If this form for $V_{\rm BE}$ is plugged into Eqn. 31, we recover our previous relationship $I_C = \beta I_B$, where the current gain β is a combination of the various factors which are slowly-varying functions of temperature and currents.

Another consequence of Ebers-Moll equation is that we see where the intrinsic emitter resistance r_e, which we introduced last time, comes from. By definition,

$$1/r_e = i_E/v_{\rm BE} \approx i_C/v_{\rm BE} = \frac{dI_C}{dV_{\rm BE}} \, .$$

From Eqn. 31, the derivative is simply I_C/V_T. So we have

$$r_e = V_T/I_C \tag{32}$$

where V_T is again as above.

5.9.2 The Current Mirror

Figure 28 shows a very commonly used current source circuit known as the *current mirror*. Understanding its principle of operation requires the Ebers-Moll equation. The "programming current" I_P defines the collector current of the left-hand transistor. (The base currents should be negligibly small.) From the Ebers-Moll equation, this collector current then uniquely determines $V_{\rm BE}$. The collector-base connection transfers this well-defined base voltage to the collector, thus maintaining the voltage drop across the programming resistor. The right-hand transistor is "matched" to the left-hand one. That is, the pair were manufactured together to have nearly identical properties. So this right-hand transistor assumes a nearly identical collector current to that which is programmed. Thus the load current becomes $I_L = I_P$. Besides transferring the program current to a load at another point of the circuit, the current mirror also has the advantage of having a larger range of compliance than the standard single-transistor current source we studied earlier.

5.9.3 Other Non-ideal Effects

The following represent some of the important departures from ideal transistor behavior:

- $V_{\rm BE} = V_{\rm BE}(T)$. As discussed above, the base-emitter "diode" includes a Boltzmann factor temperature dependence. This can be linearized, as given in the text, to yield approximately

$$\frac{\Delta V_{\rm BE}}{\Delta T} \approx -2.1 \times 10^{-3} \text{ V/°C}$$

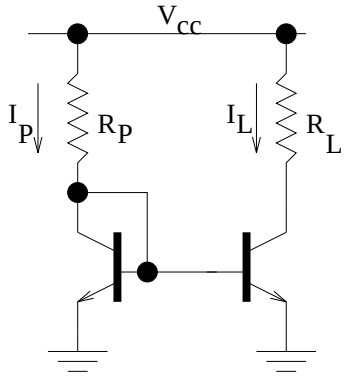

Figure 28: Current mirror.

- *Early Effect.* $V_{\rm BE}$ depends on $V_{\rm CE}$:

$$\frac{\Delta V_{\rm BE}}{\Delta V_{\rm CE}} \approx -1 \times 10^{-4}$$

- *Miller effect.* This affects high-frequency response. The reverse-biased "diode" between base and collector produces a capacitive coupling. Just as emitter resistance is effectively multiplied by $\beta + 1$ for input signals, so too this $C_{\rm CB}$, which is ususally a few pF, appears to input signals as a capacitance $(1 + G)C_{\rm CB}$ to ground, where G is the voltage gain of the transistor configuration. Hence, when combined with input source resistance, this is effectively a low-pass RC filter, and the amplifier response for frequencies above the RC cutoff will be greatly reduced. The usual solution for mitigating the Miller effect is to reduce the source impedance. This can be effectively done by coupling to a second transistor with small source resistance at base. The cascode configuration, discussed in the text, uses this. Another example is the single-input DC differential amplifier, for which there is no collector resistor at the input transistor (this eliminates ΔV_C even though the source resistance may be non-negligible), and the output transistor has grounded base (therefore with very small source resistance),

- Variation in gain. The β may be quite different from transistor to transistor, even of the same model. Therefore circuit designs should not rely on a specific gain, other than to assume that $\beta \gg 1$.

To illustrate this last point, consider our earlier one transistor current source. We determined that the load current can be written

$$I_L = \left(\frac{\beta}{\beta + 1}\right)\left[\frac{V_B - V_{\rm BE}}{R_E + r_e}\right] \tag{33}$$

where the intrinsic emitter resistance r_e has been included. Therefore, the variation in I_L induced by variation in β is

$$\frac{\Delta I_L}{I_L} = \frac{1}{I_L}\frac{dI_L}{d\beta}\Delta\beta = \left(\frac{1}{\beta + 1}\right)\frac{\Delta\beta}{\beta}$$

Hence, variations in β are attenuated by the factor $\beta + 1$. So this represents a good design.

The variation in the output of this current source resulting from the Early effect can be evaluated similarly:

$$\frac{\Delta I_L}{I_L} = \frac{1}{I_L}\frac{dI_L}{dV_{\rm BE}}\Delta V_{\rm BE} = -\frac{\Delta V_{\rm BE}}{V_B - V_{\rm BE}} = \frac{1\times 10^{-4}}{V_B - V_{\rm BE}}\Delta V_{\rm CE}$$

which can be evaluated using the compliance range for $\Delta V_{\rm CE}$.

Temperature dependence can now be estimated, as well. Using our current source, again, to exemplify this point, we see that temperature dependence can show up both in $V_{\rm BE}$ and β. The former effect can be evaluated using the chain rule and the result from the previous paragraph:

$$\frac{dI_L}{dT} = \frac{dI_L}{dV_{\rm BE}}\frac{dV_{\rm BE}}{dT} \approx \frac{2.1\ {\rm mV/^\circ C}}{R_E}$$

Therefore, we see that temperature dependence is $\propto 1/R_E$. As before, R_E is in general replaced by the sum $R_E + r_e$. In the case where the external resistor is omitted, then the typically small r_e values can induce a large temperature dependence (*cf* problem 7 at the end of Chapter 2 of the text). Similarly, using previous results, we can estimate the effect of allowing $\beta = \beta(T)$:

$$\frac{dI_L}{dT} = \frac{dI_L}{d\beta}\frac{d\beta}{dT} = \frac{I_L}{\beta + 1}\left(\frac{1}{\beta}\frac{d\beta}{dT}\right)$$

where the term in parentheses, the fractional gain temperature dependence, is often a known parameter (*cf* problem 2d at the end of Chapter 2 of the text).

Class Notes 9

6 Op-Amp Basics

The operational amplifier is one of the most useful and important components of analog electronics. They are widely used in popular electronics. Their primary limitation is that they are not especially fast: The typical performance degrades rapidly for frequencies greater than about 1 MHz, although some models are designed specifically to handle higher frequencies.

The primary use of op-amps in amplifier and related circuits is closely connected to the concept of negative feedback. Feedback represents a vast and interesting topic in itself. We will discuss it in rudimentary terms a bit later. However, it is possible to get a feeling for the two primary types of amplifier circuits, inverting and non-inverting, by simply postulating a few simple rules (the "golden rules"). We will start in this way, and then go back to understand their origin in terms of feedback.

6.1 The Golden Rules

The op-amp is in essence a differential amplifer of the type we discussed in Section 5.7 with the refinements we discussed (current source load, follower output stage), plus more, all nicely debugged, characterized, and packaged for use. Examples are the 741 and 411 models which we use in lab. These two differ most significantly in that the 411 uses JFET transistors at the inputs in order to achieve a very large input impedance ($Z_{\rm in} \sim 10^9\ \Omega$), whereas the 741 is an all-bipolar design ($Z_{\rm in} \sim 10^6\ \Omega$).

The other important fact about op-amps is that their *open-loop gain* is huge. This is the gain that would be measured from a configuration like Fig. 29, in which there is no feedback loop from output back to input. A typical open-loop voltage gain is $\sim 10^4$–10^5. By using negative feedback, we throw most of that away! We will soon discuss why, however, this might actually be a smart thing to do.

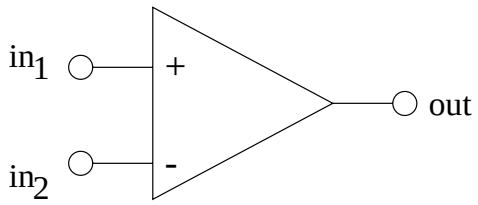

Figure 29: Operational amplifier.

The golden rules are idealizations of op-amp behavior, but are nevertheless very useful for describing overall performance. They are applicable whenever op-amps are configured with negative feedback, as in the two amplifier circuits discussed below. These rules consist of the following two statements:

1. The voltage difference between the inputs, $V_+ - V_-$, is zero.
   (Negative feedback will ensure that this is the case.)

2. The inputs draw no current.
   (This is true in the approximation that the $Z_{\rm in}$ of the op-amp is much larger than any other current path available to the inputs.)

When we assume ideal op-amp behavior, it means that we consider the golden rules to be exact. We now use these rules to analyze the two most common op-amp configurations.

6.2 Inverting Amplifier

The inverting amplifier configuration is shown in Fig. 30. It is "inverting" because our signal input comes to the "−" input, and therefore has the opposite sign to the output. The negative feedback is provided by the resistor R_2 connecting output to input.

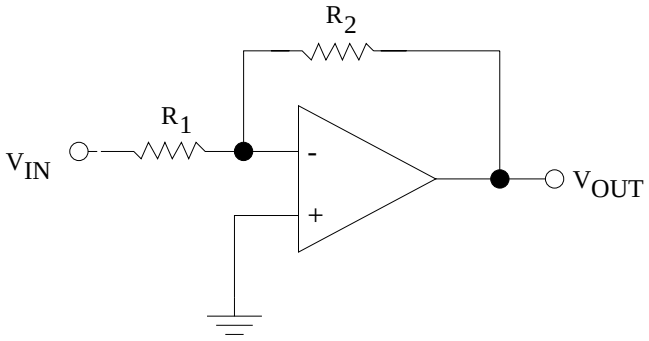

Figure 30: Inverting amplifier configuration.

We can use our rules to analyze this circuit. Since input + is connected to ground, then by rule 1, input − is also at ground. For this reason, the input − is said to be at *virtual ground*. Therefore, the voltage drop across R_1 is $v_{\rm in} - v_- = v_{\rm in}$, and the voltage drop across R_2 is $v_{\rm out} - v_- = v_{\rm out}$. So, applying Kirchoff's first law to the node at input −, we have, using golden rule 2:

$$i_- = 0 = i_{\rm in} + i_{\rm out} = v_{\rm in}/R_1 + v_{\rm out}/R_2$$

or

$$G = v_{\rm out}/v_{\rm in} = -R_2/R_1 \tag{34}$$

The input impedance, as always, is the impedance to ground for an input signal. Since the − input is at (virtual) ground, then the input impedance is simply R_1:

$$Z_{\rm in} = R_1 \tag{35}$$

The output impedance is very small ($< 1\ \Omega$), and we will discuss this again soon.

6.3 Non-inverting Amplifier

This configuration is given in Fig. 31. Again, its basic properties are easy to analyze in terms of the golden rules.

$$v_{\rm in} = v_+ = v_- = v_{\rm out}\left[\frac{R_1}{R_1 + R_2}\right]$$

where the last expression is from our voltage divider result. Therefore, rearranging gives

$$G = v_{\rm out}/v_{\rm in} = \frac{R_1 + R_2}{R_1} = 1 + \frac{R_2}{R_1} \tag{36}$$

The input impedance in this case is given by the intrinsic op-amp input impedance. As mentioned above, this is very large, and is typically in the following range:

$$Z_{\rm in} \sim 10^8 \ \text{ to } \ 10^{12}\ \Omega \tag{37}$$

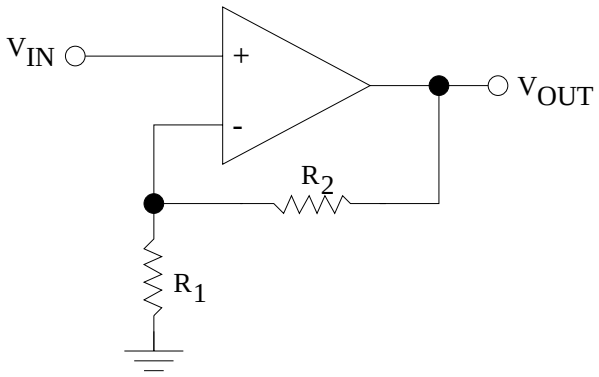

Figure 31: Non-inverting amplifier configuration.

6.4 Departures from Ideal

It is no surprise that the golden rules are not exact. On the other hand, they generally describe most, if not all, observed op-amp behavior. Here are some departures from ideal performance.

- *Offset voltage*, $V_{\rm OS}$. Recall that the input of the op-amp is a differential pair. If the two transistors are not perfectly matched, an offset will show up as a non-zero DC offset at the output. As you found in Lab 4, this can be zeroed externally. This offset adjustment amounts to changing the ratio of currents coming from the emitters of the two input transistors.

- *Bias current*, $I_{\rm bias}$. The transistor inputs actually do draw some current, regardless of golden rule 2. Those which use bipolar input transistors (*e.g.* the 741) draw more current than those which use FETs (*e.g.* the 411). The bias current is defined to be the average of the currents of the two inputs.

- *Offset current*, $I_{\rm OS}$. This is the difference between the input bias currents. Each bias current, after passing through an input resitive network, will effectively offer a voltage to the op-amp input. Therefore, an offset of the two currents will show up as a voltage offset at the output.

Perhaps the best way to beat these efects, if they are a problem for a particular application, is to choose op-amps which have good specifications. For example, $I_{\rm OS}$ can be a problem for bi-polar designs, in which case choosing a design with FET inputs will usually solve the problem. However, if one has to deal with this, it is good to know what to do. Figure 32 shows how this might be accomplished. Without the 10 kΩ resistors, this represents a non-inverting amplifier with voltage gain of $1 + (10^5/10^2) \approx 1000$. The modified design in the figure gives a DC path from ground to the op-amp inputs which are aproximately equal in resistance (10 kΩ), while maintaining the same gain.

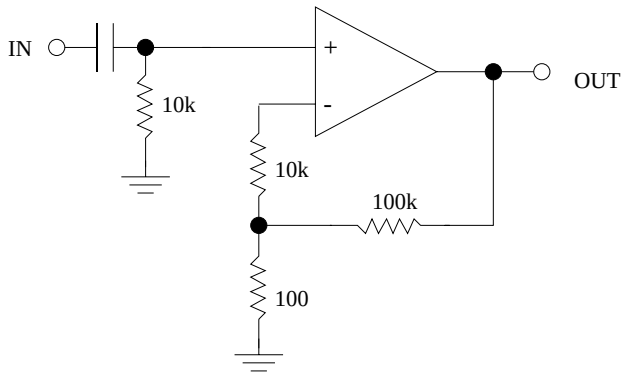

Figure 32: Non-inverting amplifier designed to minimize effect of $I_{\rm OS}$.

Similarly, the inverting amplifier configuration can be modified to mitigate offset currents. In this case one would put a resistance from the $-$ input to ground which is balanced by the R_1 and R_2 in parallel (see Fig. 30).

It is important to note that, just as we found for transistor circuits, one shpould always *provide a DC path to ground for op-amp inputs.* Otherwise, charge will build up on the effective capacitance of the inputs and the large gain will convert this voltage ($= Q/C$) into a large and uncontrolled output voltage offset.

However, our modified designs to fight $I_{\rm OS}$ have made our op-amp designs worse in a general sense. For the non-inverting design, we have turned the very large input impedance into a not very spectacular 10 kΩ. In the inverting case, we have made the virtual ground into an approximation. One way around this, if one is concerned only with AC signals, is to place a capacitor in the feedback loop. For the non-inverting amplifier, this would go in series with the resistor R_1 to ground. Therefore, as stated before, it is best, where important, to simply choose better op-amps!

6.5 Frequency-dependent Feedback

Below are examples of simple integrator and differentiator circuits which result from making the feedback path have frequency dependence, in these cases single-capacitor RC filters. It is also possible to modify non-inverting configurations in a similar way. For example, problem (3) on page 251 of the text asks about adding a "rolloff" capacitor in this way. Again, one would simply modify our derivations of the basic inverting and non-inverting gain formulae by the replacements $R \to Z$, as necessary.

6.5.1 Integrator

Using the golden rules for the circuit of Fig. 33, we have

$$\frac{v_{\rm in} - v_-}{R} = \frac{v_{\rm in}}{R} = i_{\rm in} = i_{\rm out} = -C\frac{d(v_{\rm out} - v_-)}{dt} = -C\frac{dv_{\rm out}}{dt}$$

So, solving for the output gives

$$v_{\rm out} = -\frac{1}{RC}\int v_{\rm in} dt \tag{38}$$

And for a single Fourier component ω, this gives for the gain

$$G(\omega) = -\frac{1}{\omega RC} \tag{39}$$

Therefore, to the extent that the golden rules hold, this circuit represents an *ideal integrator* and a low-pass filter. Because of the presence of the op-amp, this is an example of an *active filter*. In practice, one may need to supply a resistor in parallel with the capacitor to give a DC path for the feedback.

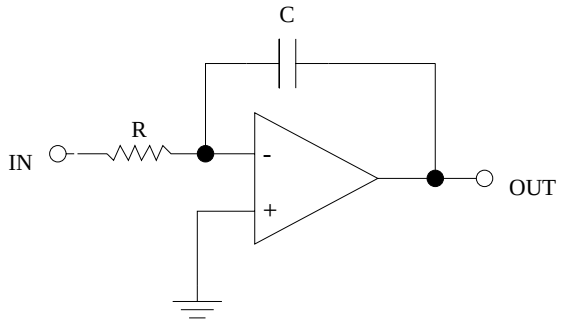

Figure 33: Op-amp integrator or low-pass filter.

6.5.2 Differentiator

The circuit of Fig. 34 can be analyzed in analogy to the integrator. We find the following:

$$v_{\rm out} = -RC\,\frac{dv_{\rm in}}{dt} \tag{40}$$

$$G(\omega) = -\omega RC \tag{41}$$

So this ideally represents a perfect differentiator and an active high-pass filter. In practice, one may need to provide a capacitor in parallel with the feedback resistor. (The gain cannot really increase with frequency indefinitely!)

6.6 Negative Feedback

As we mentioned above, the first of our Golden Rules for op-amps required the use of negative feedback. We illustrated this with the two basic negative feedback configurations: the inverting and the non-inverting configurations. In this section we will discuss negative feedback in a very general way, followed by some examples illustrating how negative feedback can be used to improve performance.

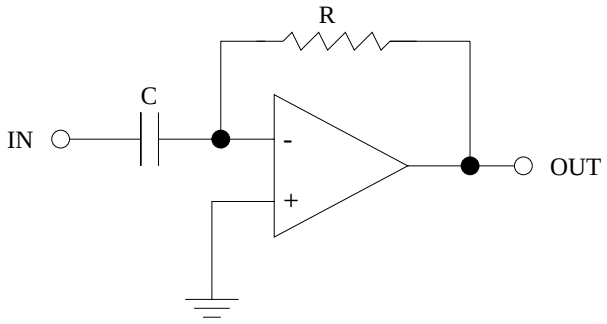

Figure 34: Op-amp differentiator or high-pass filter.

6.6.1 Gain

Consider the rather abstract schematic of a negative feedback amplifier system shown in Fig. 35. The symbol $\otimes$ is meant to indicate that negative feedback is being added to the input. The op-amp device itself has intrinsic gain A. This is called the op-amp's *open-loop gain* since this is the gain the op-amp would have in the absence of the feedback loop. The quantity B is the fraction of the output which is fed back to the input. For example, for the non-inverting amplifier this is simply given by the feedback voltage divider: $B = R_1/(R_1 + R_2)$. The gain of the device is, as usual, $G = v_{\rm out}/v_{\rm in}$. G is often called the *closed-loop gain*. To complete the terminology, the product AB is called the *loop gain*.

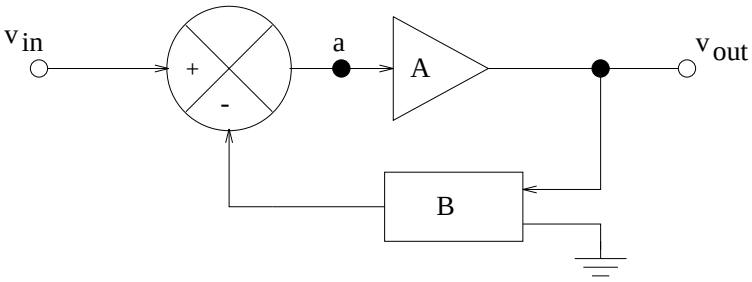

Figure 35: General negative feedback configuration.

As a result of the negative feedback, the voltage at the point labelled "a" in the figure is

$$v_a = v_{\rm in} - Bv_{\rm out}$$

The amplifier then applies its open-loop gain to this voltage to produce $v_{\rm out}$:

$$v_{\rm out} = Av_a = Av_{\rm in} - ABv_{\rm out}$$

Now we can solve for the closed-loop gain:

$$v_{\rm out}/v_{\rm in} \equiv G = \frac{A}{1 + AB} \tag{42}$$

Note that there is nothing in our derivation which precludes having B (or A) be a function of frequency.

6.6.2 Input and Output Impedance

We can now also calculate the effect that the closed-loop configuration has on the input and output impedance. The figure below is meant to clearly show the relationship between the definitions of input and output impedances and the other quantities of the circuit. The quantity R_i represents the open-loop input impedance of the op-amp, that is, the impedance the hardware had in the absence of any negative feedback loop. Similarly, R_o represents the Thevenin source (output) impedance of the open-loop device.

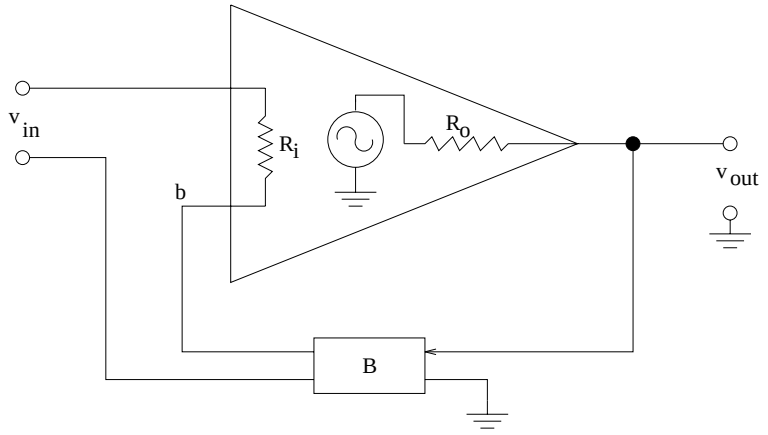

Figure 36: Schematic to illustrate the input and output impedance of a negative feedback configuration.

We start the calculation of Z_{in} with the definition $Z_{\text{in}} = v_{\text{in}}/i_{\text{in}}$. Let us calculate the current passing through R_i:

$$i_{\text{in}} = \frac{v_{\text{in}} - v_b}{R_i} = \frac{v_{\text{in}} - Bv_{\text{out}}}{R_i}$$

Substituting the result of Eqn. 42 gives

$$i_{\text{in}} = \frac{1}{R_i}\left[v_{\text{in}} - Bv_{\text{in}}\left(\frac{A}{1+AB}\right)\right]$$

Rearanging allows one to obtain

$$Z_{\text{in}} = v_{\text{in}}/i_{\text{in}} = R_i\left[1 + AB\right] \tag{43}$$

A similar procedure allows the calculation of $Z_{\text{out}} \equiv v_{\text{open}}/i_{\text{short}}$. We have $v_{\text{open}} = v_{\text{out}}$ and the shorted current is what gets when the load has zero input impedance. This means that all of the current from the amplifier goes into the load, leaving none for the feedback loop. Hence, $B = 0$ and

$$i_{\text{short}} = A\left(v_{\text{in}} - Bv_{\text{out}}\right)/R_o = Av_{\text{in}}/R_o = \frac{Av_{\text{out}}}{R_oG} = \left(\frac{Av_{\text{out}}}{R_o}\right)\left(\frac{1+AB}{A}\right) = \frac{v_{\text{out}}}{R_o}\left(1+AB\right)$$

This gives our result

$$Z_{\text{out}} = v_{\text{open}}/i_{\text{short}} = \frac{R_o}{1+AB} \tag{44}$$

Therefore, the efect of the closed loop circuit is to improve both input and output impedances by the identical loop-gain factor $1 + AB \approx AB$. So for a typical op-amp like a 741 with $A = 10^3$, $R_i = 1$ MΩ, and $R_o = 100\ \Omega$, then if we have a loop with $B = 0.1$ we get $Z_{\text{in}} = 100$ MΩ and $Z_{\text{out}} = 1\ \Omega$.

6.6.3 Examples of Negative Feedback Benefits

We just demonstrated that the input and output impedance of a device employing negative feedback are both improved by a factor $1 + AB \approx AB$, the device loop gain. Now we give a simple example of the gain equation Eqn. 42 in action.

An op-amp may typically have an open-loop gain A which varies by at least an order of magnitude over a useful range of frequency. Let $A_{\text{max}} = 10^4$ and $A_{\text{min}} = 10^3$, and let $B = 0.1$. We then calculate for the corresponding closed-loop gain extremes:

$$G_{\text{max}} = \frac{10^4}{1 + 10^3} \approx 10(1 - 10^{-3})$$

$$G_{\text{min}} = \frac{10^3}{1 + 10^2} \approx 10(1 - 10^{-2})$$

Hence, the factor of 10 open-loop gain variation has been reduced to a 1% variation. This is typical of negative feedback. It attenuates errors which appear within the feedback loop, either internal or external to the op-amp proper.

In general, the benefits of negative feedback go as the loop gain factor AB. For most op-amps, A is very large, starting at $> 10^5$ for $f < 100$ Hz. A large gain G can be achieved with large A and relatively small B, at the expense of somewhat poorer performance relative to a smaller gain, large B choice, which will tend to very good stability and error compensation properties. An extreme example of the latter choice is the "op-amp follower" circuit, consisting of a non-inverting amplifier (see Fig. 31) with $R_2 = 0$ and R_1 removed. In this case, $B = 1$, giving $G = A/(1 + A) \approx 1$.

Another interesting feature of negative feedback is one we discussed briefly in class. The qualitative statement is that any signal irregularity which is put into the feedback loop will, in the limit $B \to 1$, be taken out of the output. This reasoning is as follows. Imagine a small, steady signal v_s which is added within the feedback loop. This is returned to the output with the opposite sign after passing through the feedback loop. In the limit $B = 1$ the output and feedback are identical ($G = 1$) and the cancellation of v_s is complete. An example of this is that of placing a "push-pull" output stage to the op-amp output in order to boost output current. (See text Section 2.15.) The push-pull circuits, while boosting current, also exhibit "cross-over distortion", as we discussed in class and in the text. However, when the stage is placed within the op-amp negative feedback loop, this distortion can essentially be removed, at least when the loop gain AB is large.

6.7 Compensation in Op-amps

Recall that an RC filter introduces a phase shift between 0 and $\pi/2$. If one cascades these filters, the phase shifts can accumulate, producing at some frequency ω_π the possibility of a phase shift of $\pm\pi$. This is dangerous for op-amp circuits employing negative feedback, as a phase shift of π converts negative feedback to *positive* feedback. This in turn tends to

compound circuit instabilities and can lead to oscillating circuits (as we do on purpose for the RC relaxation oscillator).

So it is perhaps easy to simply not include such phase shifts in the feedback loop. However, at high frequencies ($f \sim 1$ MHz or more), unintended stray capacitances can become significant. In fact, within the op-amp circuits themselves, this is almost impossible to eliminate. Most manufacturers of op-amps confront this issue by intentionally reducing the open-loop gain at high frequency. This is called *compensation*. It is carried out by bypassing one of the internal amplifier stages with a high-pass filter. The effect of this is illustrated in Fig. 37. It is a so-called "Bode plot", $\log_{10}(A)$ *vs* $\log_{10}(f)$, showing how the intrinsic gain of a compensated op-amp (like the 741 or 411) decreases with frequency much sooner than one without compensation. The goal is to achieve $A < 1$ at ω_π, which is typically at frequencies of 5 to 10 MHz. (One other piece of terminology: The frequency at which the op-amp open-loop gain, A, is unity, is called f_T, and gives a good indication of how fast the op-amp is.

Compensation accounts for why op-amps are not very fast devices: The contribution of the higher frequency Fourier terms are intentionally attenuated. However, for comparators, which we turn to next, negative feedback is not used. Hence, their speed is typically much greater.

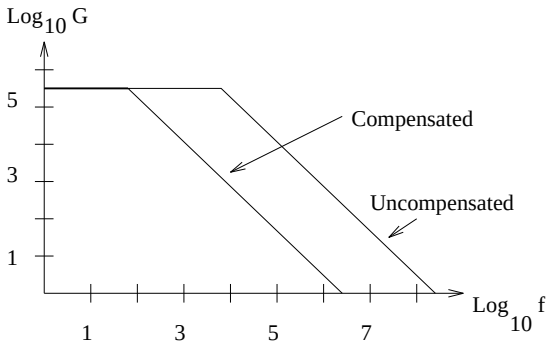

Figure 37: Bode plot showing effect of op-amp compensation.

7 Comparator Circuits

7.1 Simple Comparator

A comparator can be thought of as a fast, high-gain op-amp which is not used with negative feedback. This basic idea is shown in Fig. 38. The comparator has large open-loop gain A. The function of a comparator is to decide which of the two inputs has larger voltage. We have in the limit of very large A

$$v_{\rm out} = A(v_+ - v_-) = \begin{cases} +V_{\rm max} & v_+ > v_- \\ -|V_{\rm min}| & v_+ < v_- \end{cases}$$

where $V_{\rm max}$ and $V_{\rm min}$ are aprroximately the power supply voltages. Therefore, the comparator converts an analog input signal into an output with two possible states. Hence, this can be thought of as a 1-bit analog to digital converter (A/D or ADC). The comparator circuit does not use negative feedback, and so purposefully violates Golden Rule 1. In fact, as we shall see below, comparator circuits often employ *positive* feedback to ensure that nothing intermediate between the two extreme output states is utilized. Finally, without negative feedback, there is no need to do compensation Thus there is more gain at high frequency, meaning faster response. Also, the amplifier can be optimized for speed at the expense of linearity. Comparators, like op-amps, are readily available as integrated circuit chips, such as the model 311 (LM311 or LF311) which we have in lab. Table 9.3 (pages 584-5) of the text lists some of the possibilities on the market.

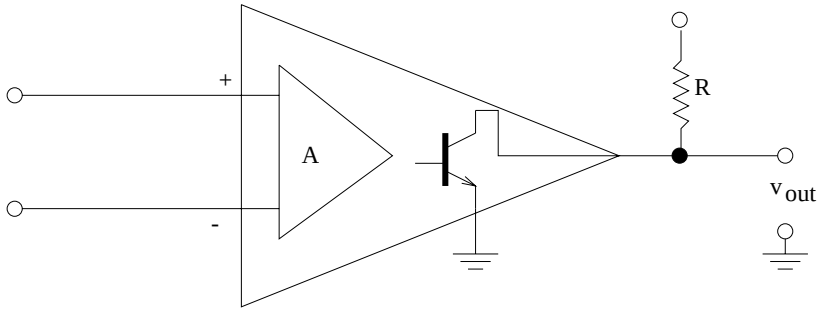

Figure 38: Comparator model.

We have shown explicitly in Fig. 38 the output stage consisting of a transistor with collector connected to the comparator output. This is the *open collector* output, and is typical. It is used in the 311 comparators we use in lab. We are obliged to complete the circuit by providing a "pull-up" resistor R. The transistor emitter is also available as an external connection. It should be connected to whatever is the lower of the two output voltage states we require. This is chosen to be ground in the figure. The high-gain differential amplifier of the comparator has output connected to the base of this transistor. When that

is *low* it will, after passing through an inverter, turn the transistor on. In this case, current sill pass through R and to the emitter connection. This current produces a voltage drop across R which pulls the output voltage (very close) to the emitter voltage (ground in our example). Typically $R \approx 1$ kΩ. When the comparator inputs are in the complementary inequality, the transistor is switched off and the output voltage goes to the voltage held by R, which is +5 V in our example. Using outputs of 0 and +5 V are typical, since these voltages correspond (roughly) to the TTL convention of digital electronics.

7.2 Schmitt Trigger

A typical circuit using a comparator is shown in Fig. 39. The output goes to one of its two possible states depending upon whether the input v_- is greater than or less than the "threshold" determined by v_+. Positive feedback is used to help reinforce the chosen output state. In this configuration, called the Schmitt trigger, two thresholds can be set, depending upon which state the output is in. The way this works is illustrated in Fig. 40. V_h and V_l refer to threshold voltages which are set up at the comparator + input by the resistor divider chain. As long as $R_3 \gg R_4$, the output states will still be determined by the pull-up resistor R_4. For the circuit in the figure, these states are 0 and +5 V. The resistor divider, then sets V_+ at different values, depending upon which state the ouput is in. Whether the connection to $+V_1$ and R_1 is required or not depends upon whether a positive threshold is required when $V_{\rm out} = 0$.

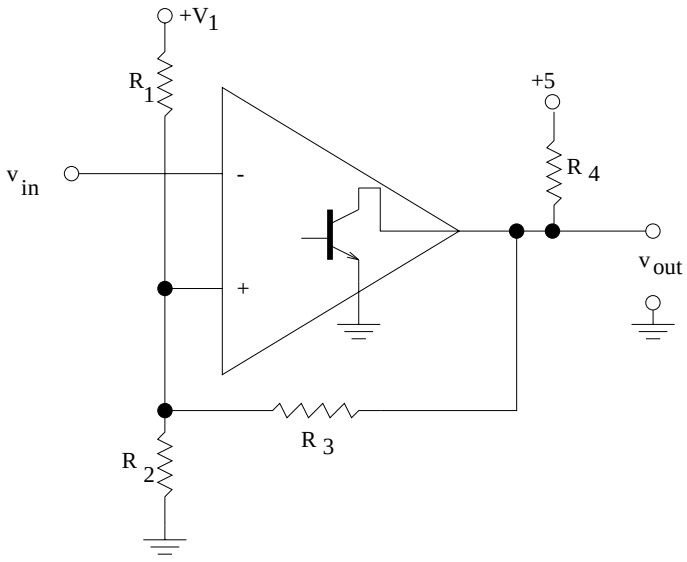

Figure 39: Schmitt trigger.

Referring to Fig. 40, we start with $V_{\rm in} = V_- < V_+$. The output is in the +5 V state. In this case the threshold produced by the voltage divider, V_h, is the larger value due to the contribution of $V_{\rm out}$. When the input crosses the threshold, the output changes to the other state, 0 V. The divider then gives a lower threshold V_l. Having two thresholds provides comparator stability and noise immunity. Any noise which is $<< (V_h - V_l)$ will not affect the operation of the comparator.

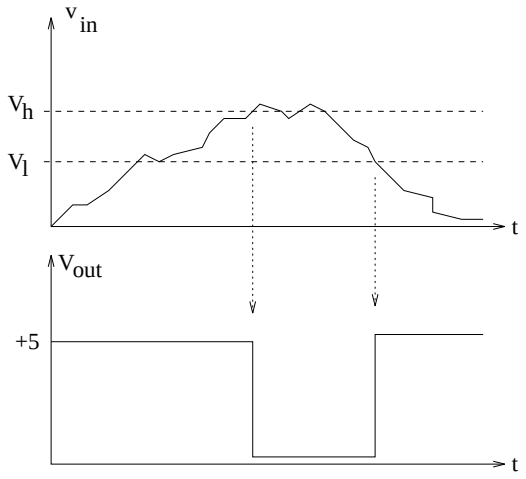

Figure 40: Examples of Schmitt trigger signals versus time. Top: $v_{\rm in}$; the dashed lines indicate the two thresholds set up at the + input of the comparator. Bottom: $v_{\rm out}$.

Note that the resistor R_1 is not necessary if $V_{\rm l} = 0$. Also, a negative threshold could be set in two ways. The resistor chain forming the threshold could be connected to negative voltage, rather than ground, or the emitter of the output transistor could be connected to negative voltage, thus producing an output with low state at this negative voltage.

7.3 RC Relaxation Oscillator

The circuit of Fig. 41 uses both positive and negative feedback. It is called an RC relaxation oscillator. Note that the positive feedback is a Schmitt configuration. So we expect to have two thresholds. The output voltages are set up to be either +5 V (pull up) or −5 V (emitter connection). Analysis of the voltage divider reveals that the corresponding two threshold at V_+ will be ±1 V. When the output is +5 V, the capacitor C is charged up through the resistor R. The RC part of the circuit is shown in Fig. 42. As we found in class, the voltage across the capacitor, and hence the − input to the comparator, is given (after applying initial conditions) by

$$V_c(t) = V_0 - \frac{3V_0}{2} e^{(t_1 - t)/RC}$$

where t_1 is the time at which the comparator output is first at $V_0 = +5$ V. Hence, the charge up curve will eventually cross the +1 V threshold, forcing the comparator to the −5 V state, and thereby starting a ramp-down of the capacitor voltage given by

$$V_c(t) = -V_0 + \frac{3V_0}{2} e^{(t_2 - t)/RC}$$

where t_2 is the time at which the output switched to −5 V. This ramp down will cross the −1 V threshold, and the whole process will therefore repeat indefinitely. The output will be a square wave, whereas V_c resembles a triangle wave. This is a common technique for building an oscillator.

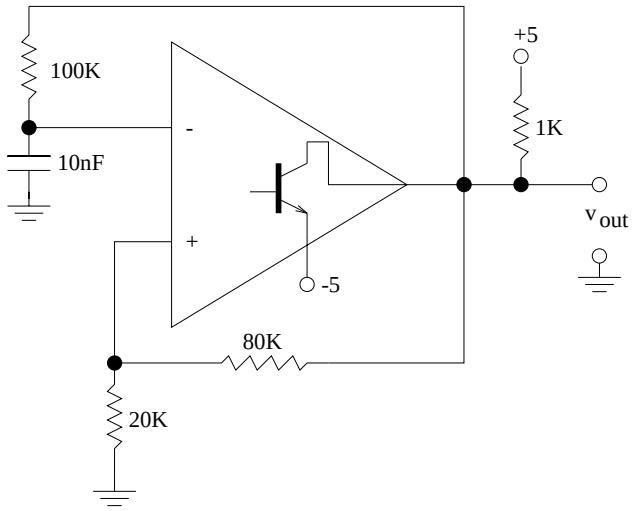

Figure 41: RC relaxation oscillator.

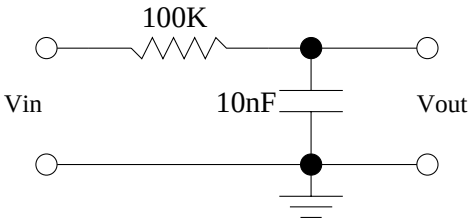

Figure 42: RC circuit with V_{in} from the comparator output and V_{out} going to the $-$ comparator input of previous figure.

8 Radio Basics

In this section we will discuss some basic concepts concerning signal modulation, generation, receiving, and demodulation. Some of these concepts are quite general and see applications in many areas. However, the most familiar perhaps is that of broadcast radio generation and receiving, hence the title of the section.

We will begin with a simplified discussion of amplitude modulation (AM). From this, we can see how to carry over many of the concepts to other forms of signal modulation and reception of signals.

8.1 The Case for Modulation

Consider the familiar example of radio signals which carry audio information. The audio itself has a typical frequency range of

$$20\text{Hz} < f_{\text{audio}} < 20\text{kHz}$$

Hence, audio has an effective *bandwidth* of about 20 kHz. Even if it were possible to broadcast signals of such low frequency in the electromagnetic spectrum, there would be a multitude of confusion resulting from the interference between competing broadcasts.

On the other hand, electromagnetic signals in the radio-frequency (RF) range, have frequencies roughly from several hundred kHz to several hundred MHz. An audio signal which modulates an RF "carrier" of, say, 20 Mhz, uses only the range 20.00 ± 0.02 MHz. Another broadcast "channel" with a carrier frequency only 100 kHz removed will have give interference with its own signal at 20.10 ± 0.02 MHz. Hence, with a carrier at much higher frequency than the signal, many channels can co-exist with little or no interference.

We will look at several techniques for signal modulation, beginning with amplitude modulation. It is important to remember that the signals do not have to be audio, that is only a familiar example. The signals could be *any* form of information which can be converted to an electromagnetic signal. Another familiar example is the modulation of computer-generated signals for transmission over telephone lines.

8.2 Amplitude Modulation

Figure 43 gives the general scheme. Each frequency, $\omega_m = 2\pi f_m$, which represents information is "mixed" with the high-frequency carrier, $\omega_c = 2\pi f_c$, to produce an output signal of the form

$$V_s(t) = A\,[1 + m\cos\omega_m t]\cos\omega_c t \tag{45}$$

where A is a constant and the constant $m \leq 1$ is known as the modulation index. We see that the carrier amplitude $A\cos(\omega_c t)$ is modulated by the factor $1+m\cos(\omega_m t)$, where $m = 0$ represents the limit of no modulation and $m = 1$ is a miximally modulated signal.

By using the identity

$$\cos x \cos y = \frac{1}{2}\left[\cos(x+y) + \cos(x-y)\right]$$

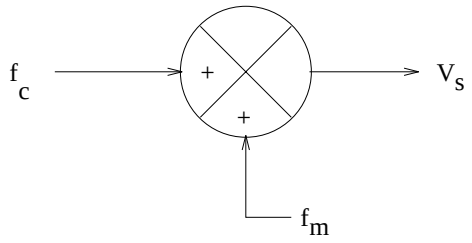

Figure 43: Schematic of modulation.

we can do a "poor man's" Fourier transform of V_s:

$$V_s(t) = A\cos\omega_c t + \frac{1}{2}Am\left[\cos\left((\omega_c+\omega_m)t\right) + \cos\left((\omega_c-\omega_m)t\right)\right] \tag{46}$$

So we have a central carrier frequency plus two side-bands at $f_c \pm f_m$.

One simple way to achieve an amplitude modulated signal is to use an amplifier for which the input is the carrier signal and the amplifier power itself is modulated by the signal, *e.g.* $V_{\rm CC} - V_{\rm EE} = V_0 + V_1\cos\omega_m t$, where V_0 is the DC offset amd we identify $m \propto V_1/V_0$.

8.3 Detection of AM

8.3.1 Heterodyne Detection

We first consider the simple, but subtle, radio receiver shown in Fig. 44. A real receiver might include at the input an antenna followed by an LC bandpass filter, with tunable capacitor. The filter is a resonant circuit with a sharp peak at the carrier frequency of the broadcast $\omega_c = 1/\sqrt{LC}$. The Q of the filter is set so that the width of the peak of the transfer function matches the bandwidth $\Delta\omega$ of the modulating signal, roughly from $\omega_c - \omega_m$ to $\omega_c + \omega_m$. With this addition, and without the amplified output, the passive "crystal" radio receiver looks like this.

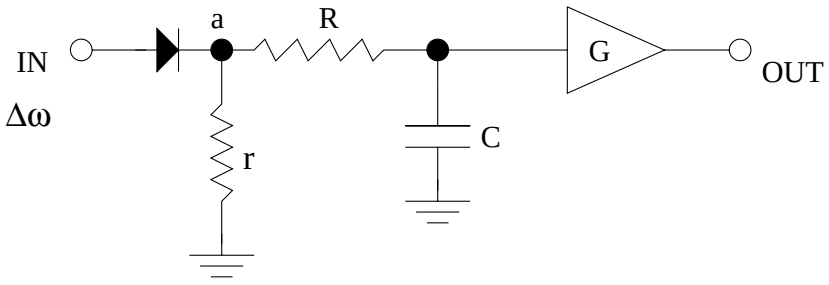

Figure 44: Simple AM receiver.

The resistor R and capacitor clearly form a low-pass filter. The cutoff frequency would be set between ω_m and ω_c in order to keep the information encoded by the low-frequency modulations, and remove the carrier. However, without the diode, the effect would be to throw away all of the information, too, since as we saw from Eqn. 46, all of the frequencies

of interest are actually in a narrow band centered about the carrier frequency. Without the diode, the system is linear, and no signal will be present at the output.

The diode is non-linear; recall its V-I curve. In order to illustrate how this works, we assume a specific form for the response of a forward-biased diode as $I = bV^2$, where b is a constant. A resistor r is inserted between point a and ground (Fig. 44) in order to convert this diode current to a voltage to be presented to the low-pass filter. Now let V be the linear combination of two signals: $V = V_1 \cos\omega_1 t + V_2 \cos\omega_2 t$. This then gives rise to an output current

$$I = bV_1^2 \cos^2\omega_1 t + bV_2^2 \cos^2\omega_2 t + 2bV_1V_2 \cos\omega_1 t \cos\omega_2 t$$

Again using trigonometric identities to form the poor man's Fourier transform, this becomes

$$2I/b = V_1^2 + V_2^2 + V_1^2 \cos 2\omega_1 t + V_2^2 \cos 2\omega_2 t + 2V_1V_2 \left[\cos((\omega_1 + \omega_2)t) + \cos((\omega_1 - \omega_2)t)\right]$$

Therefore, from the original two frequencies, the diode has produced harmonics (twice the original), as well as the sum and difference.

In the case of our simplified AM broadcast signal of Eqn. 46, where three frequencies are originally present (ω_c and $\omega_c \pm \omega_m$), the effect of the diode is easily generalized from the steps above using the substitutions $\omega_1 = \omega_c$ and $\omega_2 = \omega_c + \omega_m$ or $\omega_2 = \omega_c - \omega_m$. We find that the output of the diode will include DC, the first harmonics of all three frequecies, as well as the six possible sum and difference frequencies. Of particular interest for our receiver is the difference frequency between the carrier and the modulated carrier. For example,

$$\omega_c - (\omega_c - \omega_m) = \omega_m$$

Therefore, we do in fact recover a Fourier component corresponding to our original modulating signal. This can then be separated from the higher frequencies using the low-pass filter and amplifier. This represents a simple example of so-called *heterodyne detection*, in which different frequencies are combined in order to extract a difference frequency.

As an aside, we note that with our example $I = bV^2$, we have squared the input. When we examine this in frequency domain (Fourier transform) and low-pass filter the result (averaging), we have effectively formed the so-called *power spectrum* of the input, also called the *spectral power density*.

8.3.2 Harmonic Distortion

Note that we intentionally introduced a non-linear element (the diode) to our system. An unintentional non-linearity in a circuit, for example in an audio amplifier circuit, can introduce additional frequencies as demonstrated above. In particular, our diode with the $I = bV^2$ behavior introduced first harmonics of the original frequencies at twice the original. In general, a non-linearity may include any number of higher-order terms: $I = b_1V + b_2V^2 + b_3V^3 + \cdots$, where each additional power can generate the next higher harmonic. For example, a non-zero b_3 will generate a 2nd harmonic of the original ω at 3ω. The introduction of harmonics of the input signal is called *harmonic distortion*. Since the pattern of harmonics is what distinguishes musical instrument types to the ear, the introduction of non-linearities should be avoided in high-fidelity audio amplifiers.

8.3.3 Homodyne Detection

An example of this technique is given in the text, pages 653 and 889. It uses a phase-locked loop (PLL) circuit at the input of the receiver. Recall that the PLL circuit is designed

to produce an output which is proportional to shifts in phase of the input. Since one can consider the modulation of the carrier to be a phase shift (by amount $\omega_m t$), the output of the PLL can then produce a voltage signal proportional to these phase shifts, which in turn is used to provide active rectification of the input at the frequency of the modulation. The essential non-linear behavior of the diode discussed above is provided in this case by an active voltage multiplier. This type of PLL circuit is actually more relevant to FM detection, which is discussed below.

8.3.4 Superheterodyne Detection

This technique is illustrated in the text, pages 895-6. It is essentially a fancy version of our simple heterodyne detector above. In this case, the simple passive LC bandpass filter at the input is replaced by a local oscillator and mixer. An example is given in Figure 13.41 of the text. Consider an input carrier of frequency 10 MHz which has amplitude modulated at some much lower frequency. This signal is mixed with a local oscillator of fixed frequency greater than the carrier. In the example of the text, the local oscillator has frequency tuned to be $f_{\rm LO} = 10.455$ MHz, exactly 455 kHz greater than the carrier. As with our earlier diode example, the mixed signal includes the difference frequency, in this case 455 kHz, which in turn has nearby sideband frequencies which differ from 455 kHz by the audio modulation frequencies. From this point on, the detection is carried out as in the simple heterodyne example. One advantage here is that a relatively high-frequency carrier, which in general will be difficult to condition using conventional electronics is effectively reduced to a more manageable frequency, in the example from 10 MHz to 455 kHz. The other advantage is that the band-pass tuning which follows the mixer is always centered at a constant 455 kHz. So the tuning is accomplished by adjusting the oscillator, rather than the filter.

8.4 Other Modulation Schemes

Recall from Eqn. 45 that for AM the amplitude is modulated by varying the frequency ω_m. However, to preserve the information, the generation and receipt of the amplitude must be linear. In addition, most noise sources will naturally appear as voltages, and hence will add to the AM signal. On the other hand, phase and frequency modulation (FM) do not suffer from these complications. Hence, where fidelity is important, these schemes have intrinsic advantages. Radio broadcast by FM also has the additional advantage, by dint of historical accident, of occupying a higher frequency band, thus allowing easy accomodation of a full audio bandwidth. However, unlike the AM radio band, the FM band signals do not reflect from the ionosphere, and therefore can not be transmitted over very large distances (at night).

8.4.1 Phase Modulation

A carrier of frequency ω_c is *phase modulated* if the resulting signal has the form

$$V(t) = V_0 \cos(\omega_c t + A_p \cos \omega_m t) \tag{47}$$

where V_0 and A_p are constants and ω_m is the modulating frequency, as before. This can be expanded, and for $A_p \ll 1$ can also be simplified:

$$V(t)/V_0 \;=\; \cos\omega_c t\, \cos(A_p \cos\omega_m t) - \sin\omega_c t\, \sin(A_p \cos\omega_m t)$$

$$\begin{aligned} &\approx \cos\omega_c t - A_p \sin\omega_c t \, \cos\omega_m t \\ &= \cos\omega_c t - \frac{1}{2} A_p \left[\sin((\omega_c + \omega_m)t) + \sin((\omega_c - \omega_m)t)\right] \end{aligned} \tag{48}$$

As for AM, two new sidebands have appeared, but now they are 90° out of phase with respect to the carrier.

8.4.2 Frequency Modulation

The phase of a sinusoidal function, when frequency is a function of time, can in general be expressed as

$$\phi = \int \omega dt$$

Now suppose the frequency is modulated by a frequency ω about some central carrier frequency

$$\omega = \omega_c + A_f \cos\omega_m t$$

where A_f is a constant. Then the phase becomes

$$\phi = \omega_c t + \frac{A_f}{\omega_m} \sin\omega_m t$$

Here, A_f is called the *frequency deviation* and A_f/ω_m is the *modulation index* for FM. Carrying out steps analogous to those for Eqn. 48 gives the following expression for the FM signal:

$$V(t)/V_0 = \cos\phi = \cos\omega_c t + \frac{A_f}{2\omega_m} \left[\cos((\omega_c + \omega_m)t) - \cos((\omega_c - \omega_m)t)\right] \tag{49}$$

So again the Fourier spectrum is similar to what we found for AM, except now one of the two sidebands has amplitude of opposite sign.

8.5 FM Detection

In the AM detection schemes discussed above, the diode or other non-linear element is used to extract an output signal proportional to $\cos\omega_m t$, and hence provide a reproduction of the original modulation, for example in the form of an audio signal. For FM detection we need to replace the diode with something which can provide a voltage output proportional to the input frequency modulated signal. We explored such a technique in Lab 5 in the form of the phase-locked loop circuit. The PLL scheme is reproduced in Fig. 45. (For this application, the counter is omitted.) Recall that the signal before the VCO, labelled V_{out}, is proportional to input phase shifts. This is exactly what we need to detect the phase shift introduced by FM. All that is left is to feed V_{out} to a low-pass filter and amplifier, as before.

An apparent practical limitation of this technique for FM radio reception is that PLLs do not operate at these high frequencies ($\sim$ 100 MHz). This is overcome by using the technique discussed above at the front-end of the superheterodyne receiver. The input is mixed using a local oscillator and the resulting lower frequency (455 kHz in our example) modulated signal is then input to the PLL. Another technique, called quadrature detection is briefly discusses in the text, page 652.

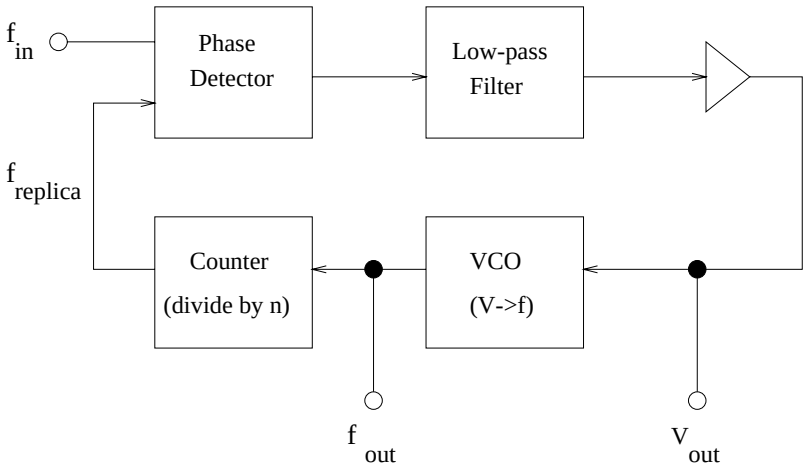

Figure 45: PLL schematic. V_{out} provides the FM signal.