A First Course in Linear Algebra

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To my wife, Pat.

Contents

Preface

This text is designed to teach the concepts and techniques of basic linear algebra as a rigorous mathematical subject. Besides computational proficiency, there is an emphasis on understanding definitions and theorems, as well as reading, understanding and creating proofs. A strictly logical organization, complete and exceedingly detailed proofs of every theorem, advice on techniques for reading and writing proofs, and a selection of challenging theoretical exercises will slowly provide the novice with the tools and confidence to be able to study other mathematical topics in a rigorous fashion.

Most students taking a course in linear algebra will have completed courses in differential and integral calculus, and maybe also multivariate calculus, and will typically be second-year students in university. This level of mathematical maturity is expected, however there is little or no requirement to know calculus itself to use this book successfully. With complete details for every proof, for nearly every example, and for solutions to a majority of the exercises, the book is ideal for self-study, for those of any age.

While there is an abundance of guidance in the use of the software system, Sage, there is no attempt to address the problems of numerical linear algebra, which are arguably continuous in nature. Similarly, there is little emphasis on a geometric approach to problems of linear algebra. While this may contradict the experience of many experienced mathematicians, the approach here is consciously algebraic. As a result, the student should be well-prepared to encounter groups, rings and fields in future courses in algebra, or other areas of discrete mathematics.

How to Use This Book

While the book is divided into chapters, the main organizational unit is the thirty-seven sections. Each contains a selection of definitions, theorems, and examples interspersed with commentary. If you are enrolled in a course, read the section *before* class and then answer the section's reading questions as preparation for class.

The version available for viewing in a web browser is the most complete, integrating all of the components of the book. Consider acquainting yourself with this version. Knowls are indicated by a dashed underlines and will allow you to seamlessly remind yourself of the content of definitions, theorems, examples, exercises, subsections and more. Use them liberally.

Historically, mathematics texts have numbered definitions and theorems. We have instead adopted a strategy more appropriate to the heavy cross-referencing, linking and knowling afforded by modern media. Mimicking an approach taken by Donald Knuth, we have given items short titles and associated acronyms. You will become comfortable with this scheme after a short time, and might even come to appreciate its inherent advantages. In the web version, each chapter has a list of ten or so important items from that chapter, and you will find yourself recognizing some of these acronyms with no extra effort beyond the normal amount of study. Bruno Mello suggests that some say an acronym should be pronouncable as a word (such as “radar”), and otherwise is an abbreviation. We will not be so strict in our use of the term.

Exercises come in three flavors, indicated by the first letter of their label. “C” indicates a problem that is essentially computational. “T” represents a problem that is more theoretical, usually requiring a solution that is as rigorous as a proof. “M” stands for problems that are “medium”, “moderate”, “midway”, “mediate” or “median”, but never “mediocre.” Their statements could feel computational, but their solutions require a more thorough understanding of the concepts or theory, while perhaps not being as rigorous as a proof. Of course, such a tripartite division will be subject to interpretation. Otherwise, larger numerical values indicate greater perceived difficulty, with gaps allowing for the contribution of new problems from readers. Many, but not all, exercises have complete solutions. These are indicated by daggers in the PDF and print versions, with solutions available in an online supplement, while in the web version a solution is indicated by a knowl right after the problem statement. Resist the urge to peek early. Working the exercises diligently is the best way to master the material.

The Archetypes are a collection of twenty-four archetypical examples. The open source lexical database, WordNet, defines an archetype as “something that serves as a model or a basis for making copies.” We employ the word in the first sense here. By carefully choosing the examples we hope to provide at least one example that is interesting and appropriate for many of the theorems and definitions, and also provide counterexamples to conjectures (and especially counterexamples to converses of theorems). Each archetype has numerous computational results which you could strive to duplicate as you encounter new definitions and theorems. There are some exercises which will help guide you in this quest.

Supplements

Print versions of the book (either a physical copy or a PDF version) have significant material available as supplements. Solutions are contained in the Exercise Manual. Advice on the use of the open source mathematical software system, Sage, is contained in another supplement. (Look for a linear algebra “Quick Reference” sheet at the Sage website.) The Archetypes are available in a PDF form which could be used as a workbook. Flashcards, with the statement of every definition and theorem, in order of appearance, are also available.

Freedom

This book is copyrighted by its author. Some would say it is his “intellectual property,” a distasteful phrase if there ever was one. Rather than exercise all the restrictions provided by the government-granted monopoly that is copyright, the author has granted you a license, the GNU Free Documentation License (GFDL). In summary it says you may receive an electronic copy at no cost via electronic networks and you may make copies forever. So your copy of the book never has to go “out-of-print.” You may redistribute copies and you may make changes to your copy for your own use. However, you have one major responsibility in accepting this license. If you make changes and distribute the changed version, then you must offer the same license for the new version, you must acknowledge the original author’s work, and you must indicate where you have made changes.

In practice, if you see a change that needs to be made (like correcting an error, or adding a particularly nice theoretical exercise), you may just wish to donate the change to the author rather than create and maintain a new version. Such donations are highly encouraged and gratefully accepted. You may notice the large number of small mistakes that have been corrected by readers that have come before you. Pay it forward.

So, in one word, the book really is “free” (as in “no cost”). But the open license employed is vastly different than “free to download, all rights reserved.” Most importantly, you know that this book, and its ideas, are not the property of anyone. Or they are the property of everyone. Either way, this book has its own inherent “freedom,” separate from those who contribute to it. Much of this philosophy is embodied in the following quote:

> If nature has made any one thing less susceptible than all others of exclusive property, it is the action of the thinking power called an idea, which an individual may exclusively possess as long as he keeps it to himself; but the moment it is divulged, it forces itself into the possession of every one, and the receiver cannot dispossess himself of it. Its peculiar character, too, is that no one possesses the less, because every other possesses the whole of it. He who receives an idea from me, receives instruction himself without lessening mine; as he who lights his taper at mine, receives light without darkening me. That ideas should freely spread from one to another over the globe, for the moral and mutual instruction of man, and improvement of his condition, seems to have been peculiarly and benevolently designed by nature, when she made them, like fire, expansible over all space, without lessening their density in any point, and like the air in which we breathe, move, and have our physical being, incapable of confinement or exclusive appropriation.
>
> Thomas Jefferson
> Letter to Isaac McPherson
> August 13, 1813

To the Instructor

The first half of this text (through Chapter M) is a course in matrix algebra, though the foundation of some more advanced ideas is also being formed in these early sections (such as Theorem NMUS, which presages invertible linear transformations). Vectors are presented exclusively as column vectors (not transposes of row vectors), and linear combinations are presented very early. Spans, null spaces, column spaces and row spaces are also presented early, simply as sets, saving most of their vector space properties for later, so they are familiar objects before being scrutinized carefully.

You cannot do everything early, so in particular matrix multiplication comes later than usual. However, with a definition built on linear combinations of column vectors, it should seem more natural than the more frequent definition using dot products of rows with columns. And this delay emphasizes that linear algebra is built upon vector addition and scalar multiplication. Of course, matrix inverses must wait for matrix multiplication, but this does not prevent nonsingular matrices from occurring sooner. Vector space properties are hinted at when vector and matrix operations are first defined, but the notion of a vector space is saved for a more axiomatic treatment later (Chapter VS). Once bases and dimension have been explored in the context of vector spaces, linear transformations and their matrix representation follow. The predominant purpose of the book is the four sections of Chapter R, which introduces the student to representations of vectors and matrices, change-of-basis, and orthonormal diagonalization (the spectral theorem). This final chapter pulls together all the important ideas of the previous chapters.

Our vector spaces use the complex numbers as the field of scalars. This avoids the fiction of complex eigenvalues being used to form scalar multiples of eigenvectors. The presence of the complex numbers in the earliest sections should not frighten students who need a review, since they will not be used heavily until much later, and Section CNO provides a quick review.

Linear algebra is an ideal subject for the novice mathematics student to learn how to develop a subject precisely, with all the rigor mathematics requires. Unfortunately, much of this rigor seems to have escaped the standard calculus curriculum, so for many university students this is their first exposure to careful definitions and theorems, and the expectation that they fully understand them, to say nothing of the expectation that they become proficient in formulating their own proofs. We have tried to make this text as helpful as possible with this transition. Every definition is stated carefully, set apart from the text. Likewise, every theorem is carefully stated, and almost every one has a complete proof. Theorems usually have just one conclusion, so they can be referenced precisely later. Definitions and theorems are cataloged in order of their appearance (Definitions and Theorems in the Reference chapter at the end of the book). Along the way, there are discussions of some more important ideas relating to formulating proofs (Proof Techniques), which is partly advice and partly a primer on logic.

Collecting responses to the Reading Questions prior to covering material in class will require students to learn how to read the material. Sections are designed to be covered in a fifty-minute lecture. Later sections are longer, but as students become more proficient at reading the text, it is possible to survey these longer sections at the same pace. With solutions to many of the exercises, students may be given the freedom to work homework at their own pace and style (individually, in groups, with an instructor's help, etc.). To compensate and keep students from falling behind, I give an examination on each chapter.

Sage is a powerful open source program for advanced mathematics. It is especially robust for linear algebra. We have included an abundance of material which will help the student (and instructor) learn how to use Sage for the study of linear algebra and how to understand linear algebra better with Sage. This material is tightly integrated with the web version of the book and will become even easier to use since the technology for interfaces to Sage continues to rapidly evolve. Sage is highly capable for mathematical research as well, and so should be a tool that students can use in subsequent courses and careers.

Conclusion

Linear algebra is a beautiful subject. I have enjoyed preparing this exposition and making it widely available. Much of my motivation for writing this book is captured by the sentiments expressed by H.M. Cundy and A.P. Rollet in their Preface to the First Edition of *Mathematical Models* (1952), especially the final sentence,

> This book was born in the classroom, and arose from the spontaneous interest of a Mathematical Sixth in the construction of simple models. A desire to show that even in mathematics one could have fun led to an exhibition of the results and attracted considerable attention throughout the school. Since then the Sherborne collection has grown, ideas have come from many sources, and widespread interest has been shown. It seems therefore desirable to give permanent form to the lessons of experience so that others can benefit by them and be encouraged to undertake similar work.

Foremost, I hope that students find their time spent with this book profitable. I hope that instructors find it flexible enough to fit the needs of their course. You can always find the latest version, and keep current with any changes, at the book's website (`http://linear.pugetsound.edu`). I appreciate receiving suggestions, corrections, and other comments, so please do contact me.

Robert A. Beezer
Tacoma, Washington
December 2012

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Many aspects of the book have been influenced by insightful questions and creative suggestions from the students who have labored through the book in our courses. For example, the flashcards with theorems and definitions are a direct result of a student suggestion. I will single out a handful of students at the University of Puget Sound who have been especially adept at finding and reporting mathematically significant typographical errors: Jake Linenthal, Christie Su, Kim Le, Sarah McQuate, Andy Zimmer, Travis Osborne, Andrew Tapay, Mark Shoemaker, Tasha Underhill, Tim Zitzer, Elizabeth Million, Steve Canfield, Jinshil Yi, Cliff Berger, Preston Van Buren, Duncan Bennett, Dan Messenger, Caden Robinson, Glenna Toomey, Tyler Ueltschi, Kyle Whitcomb, Anna Dovzhik, Chris Spalding and Jenna Fontaine. All the students of the Fall 2012 Math 290 sections were very helpful and patient through the major changes required in making Version 3.00.

I have tried to be as original as possible in the organization and presentation of this beautiful subject. However, I have been influenced by many years of teaching from another excellent textbook, *Introduction to Linear Algebra* by L.W. Johnson, R.D. Reiss and J.T. Arnold. When I have needed inspiration for the correct approach to particularly important proofs, I have learned to eventually consult two other textbooks. Sheldon Axler's *Linear Algebra Done Right* is a highly original exposition, while Ben Noble's *Applied Linear Algebra* frequently strikes just the right note between rigor and intuition. Noble's excellent book is highly recommended, even though its publication dates to 1969.

Conversion to various electronic formats have greatly depended on assistance from: Eitan Gurari, author of the powerful LaTeX translator, `tex4ht`; Davide Cervone, author of `jsMath` and `MathJax`; and Carl Witty, who advised and tested the Sony

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Incorporation of Sage code is made possible by the entire community of Sage developers and users, who create and refine the mathematical routines, the user interfaces and applications in educational settings. Technical and logistical aspects of incorporating Sage code in open textbooks was supported by a grant from the United States National Science Foundation (DUE-1022574), which has been administered by the American Institute of Mathematics, and in particular, David Farmer. The support and assistance of my fellow Principal Investigators, Jason Grout, Tom Judson, Kiran Kedlaya, Sandra Laursen, Susan Lynds, and William Stein is especially appreciated.

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Finally, in every respect, the production and distribution of this book has been accomplished with open source software. The range of individuals and projects is far too great to pretend to list them all. This project is an attempt to pay it forward.

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Chapter SLE
Systems of Linear Equations

We will motivate our study of linear algebra by studying solutions to systems of linear equations. While the focus of this chapter is on the practical matter of how to find, and describe, these solutions, we will also be setting ourselves up for more theoretical ideas that will appear later.

Section WILA
What is Linear Algebra?

We begin our study of linear algebra with an introduction and a motivational example.

Subsection LA
Linear + Algebra

The subject of linear algebra can be partially explained by the meaning of the two terms comprising the title. “Linear” is a term you will appreciate better at the end of this course, and indeed, attaining this appreciation could be taken as one of the primary goals of this course. However for now, you can understand it to mean anything that is “straight” or “flat.” For example in the xy-plane you might be accustomed to describing straight lines (is there any other kind?) as the set of solutions to an equation of the form $y = mx + b$, where the slope m and the y-intercept b are constants that together describe the line. If you have studied multivariate calculus, then you will have encountered planes. Living in three dimensions, with coordinates described by triples (x, y, z), they can be described as the set of solutions to equations of the form $ax + by + cz = d$, where a, b, c, d are constants that together determine the plane. While we might describe planes as “flat,” lines in three dimensions might be described as “straight.” From a multivariate calculus course you will recall that lines are sets of points described by equations such as $x = 3t - 4$, $y = -7t + 2$, $z = 9t$, where t is a parameter that can take on any value.

Another view of this notion of “flatness” is to recognize that the sets of points just described are solutions to equations of a relatively simple form. These equations involve addition and multiplication only. We will have a need for subtraction, and occasionally we will divide, but mostly you can describe “linear” equations as involving only addition and multiplication. Here are some examples of typical equations we will see in the next few sections:

$$2x + 3y - 4z = 13 \qquad 4x_1 + 5x_2 - x_3 + x_4 + x_5 = 0 \qquad 9a - 2b + 7c + 2d = -7$$

What we will not see are equations like:

$$xy + 5yz = 13 \qquad x_1 + x_2^3/x_4 - x_3x_4x_5^2 = 0 \qquad \tan(ab) + \log(c - d) = -7$$

The exception will be that we will on occasion need to take a square root.

You have probably heard the word "algebra" frequently in your mathematical preparation for this course. Most likely, you have spent a good ten to fifteen years learning the algebra of the real numbers, along with some introduction to the very similar algebra of complex numbers (see Section CNO). However, there are many new algebras to learn and use, and likely linear algebra will be your second algebra. Like learning a second language, the necessary adjustments can be challenging at times, but the rewards are many. And it will make learning your third and fourth algebras even easier. Perhaps you have heard of "groups" and "rings" (or maybe you have studied them already), which are excellent examples of other algebras with very interesting properties and applications. In any event, prepare yourself to learn a new algebra and realize that some of the old rules you used for the real numbers may no longer apply to this *new* algebra you will be learning!

The brief discussion above about lines and planes suggests that linear algebra has an inherently geometric nature, and this is true. Examples in two and three dimensions can be used to provide valuable insight into important concepts of this course. However, much of the power of linear algebra will be the ability to work with "flat" or "straight" objects in higher dimensions, without concerning ourselves with visualizing the situation. While much of our intuition will come from examples in two and three dimensions, we will maintain an *algebraic* approach to the subject, with the geometry being secondary. Others may wish to switch this emphasis around, and that can lead to a very fruitful and beneficial course, but here and now we are laying our bias bare.

Subsection AA
An Application

We conclude this section with a rather involved example that will highlight some of the power and techniques of linear algebra. Work through all of the details with pencil and paper, until you believe all the assertions made. However, in this introductory example, do not concern yourself with how some of the results are obtained or how you might be expected to solve a similar problem. We will come back to this example later and expose some of the techniques used and properties exploited. For now, use your background in mathematics to convince yourself that everything said here really is correct.

Example TMP Trail Mix Packaging
Suppose you are the production manager at a food-packaging plant and one of your product lines is trail mix, a healthy snack popular with hikers and backpackers, containing raisins, peanuts and hard-shelled chocolate pieces. By adjusting the mix of these three ingredients, you are able to sell three varieties of this item. The fancy version is sold in half-kilogram packages at outdoor supply stores and has more chocolate and fewer raisins, thus commanding a higher price. The standard version is sold in one kilogram packages in grocery stores and gas station mini-markets. Since the standard version has roughly equal amounts of each ingredient, it is not as expensive as the fancy version. Finally, a bulk version is sold in bins at grocery stores for consumers to load into plastic bags in amounts of their choosing. To appeal to the shoppers that like bulk items for their economy and healthfulness, this mix has many more raisins (at the expense of chocolate) and therefore sells for less.

Your production facilities have limited storage space and early each morning you are able to receive and store 380 kilograms of raisins, 500 kilograms of peanuts and 620 kilograms of chocolate pieces. As production manager, one of your most important duties is to decide how much of each version of trail mix to make every day. Clearly, you can have up to 1500 kilograms of raw ingredients available each day, so to be the most productive you will likely produce 1500 kilograms of trail mix each day. Also, you would prefer not to have any ingredients leftover each day, so that your final product is as fresh as possible and so that you can receive the maximum delivery the next morning. But how should these ingredients be allocated to the mixing of the bulk, standard and fancy versions? First, we need a little more information about the mixes. Workers mix the ingredients in 15 kilogram batches, and each row of the table below gives a recipe for a 15 kilogram batch. There is some additional information on the costs of the ingredients and the price the manufacturer can charge for the different versions of the trail mix.

	Raisins (kg/batch)	Peanuts (kg/batch)	Chocolate (kg/batch)	Cost ($/kg)	Sale Price ($/kg)
Bulk	7	6	2	3.69	4.99
Standard	6	4	5	3.86	5.50
Fancy	2	5	8	4.45	6.50
Storage (kg)	380	500	620		
Cost ($/kg)	2.55	4.65	4.80		

As production manager, it is important to realize that you only have three decisions to make — the amount of bulk mix to make, the amount of standard mix to make and the amount of fancy mix to make. Everything else is beyond your control or is handled by another department within the company. Principally, you are also limited by the amount of raw ingredients you can store each day. Let us denote the amount of each mix to produce each day, measured in kilograms, by the variable quantities b, s and f. Your production schedule can be described as values of b, s and f that do several things. First, we cannot make negative quantities of each mix, so

$$b \geq 0 \qquad\qquad s \geq 0 \qquad\qquad f \geq 0$$

Second, if we want to consume all of our ingredients each day, the storage capacities lead to three (linear) equations, one for each ingredient,

$$\begin{aligned}
\frac{7}{15}b + \frac{6}{15}s + \frac{2}{15}f &= 380 && \text{(raisins)}\\
\frac{6}{15}b + \frac{4}{15}s + \frac{5}{15}f &= 500 && \text{(peanuts)}\\
\frac{2}{15}b + \frac{5}{15}s + \frac{8}{15}f &= 620 && \text{(chocolate)}
\end{aligned}$$

It happens that this system of three equations has just one solution. In other words, as production manager, your job is easy, since there is but one way to use up all of your raw ingredients making trail mix. This single solution is

$$b = 300 \text{ kg} \qquad\qquad s = 300 \text{ kg} \qquad\qquad f = 900 \text{ kg}.$$

We do not yet have the tools to explain why this solution is the only one, but it should be simple for you to verify that this is indeed a solution. (Go ahead, we will wait.) Determining solutions such as this, and establishing that they are unique, will be the main motivation for our initial study of linear algebra.

So we have solved the problem of making sure that we make the best use of our limited storage space, and each day use up all of the raw ingredients that are shipped to us. Additionally, as production manager, you must report weekly to the CEO of the company, and you know he will be more interested in the profit derived from your decisions than in the actual production levels. So you compute,

$$300(4.99 - 3.69) + 300(5.50 - 3.86) + 900(6.50 - 4.45) = 2727.00$$

for a daily profit of $2,727 from this production schedule. The computation of the daily profit is also beyond our control, though it is definitely of interest, and it too looks like a "linear" computation.

As often happens, things do not stay the same for long, and now the marketing department has suggested that your company's trail mix products standardize on every mix being one-third peanuts. Adjusting the peanut portion of each recipe by also adjusting the chocolate portion leads to revised recipes, and slightly different costs for the bulk and standard mixes, as given in the following table.

	Raisins (kg/batch)	Peanuts (kg/batch)	Chocolate (kg/batch)	Cost ($/kg)	Sale Price ($/kg)
Bulk	7	5	3	3.70	4.99
Standard	6	5	4	3.85	5.50
Fancy	2	5	8	4.45	6.50
Storage (kg)	380	500	620		
Cost ($/kg)	2.55	4.65	4.80		

In a similar fashion as before, we desire values of b, s and f so that

$$b \geq 0 \qquad\qquad s \geq 0 \qquad\qquad f \geq 0$$

and

$$\begin{aligned}
\frac{7}{15}b + \frac{6}{15}s + \frac{2}{15}f &= 380 && \text{(raisins)}\\
\frac{5}{15}b + \frac{5}{15}s + \frac{5}{15}f &= 500 && \text{(peanuts)}\\
\frac{3}{15}b + \frac{4}{15}s + \frac{8}{15}f &= 620 && \text{(chocolate)}
\end{aligned}$$

It now happens that this system of equations has *infinitely* many solutions, as we will now demonstrate. Let f remain a variable quantity. Then if we make f kilograms of the fancy mix, we will make $4f - 3300$ kilograms of the bulk mix and $-5f + 4800$ kilograms of the standard mix. Let us now verify that, for any choice of f, the values of $b = 4f - 3300$ and $s = -5f + 4800$ will yield a production schedule that exhausts all of the day's supply of raw ingredients (right now, do not be concerned about how you might derive expressions like these for b and s). Grab your pencil and paper and play along.

$$\begin{aligned}
\frac{7}{15}(4f - 3300) + \frac{6}{15}(-5f + 4800) + \frac{2}{15}f &= 0f + \frac{5700}{15} = 380\\
\frac{5}{15}(4f - 3300) + \frac{5}{15}(-5f + 4800) + \frac{5}{15}f &= 0f + \frac{7500}{15} = 500\\
\frac{3}{15}(4f - 3300) + \frac{4}{15}(-5f + 4800) + \frac{8}{15}f &= 0f + \frac{9300}{15} = 620
\end{aligned}$$

Convince yourself that these expressions for b and s allow us to vary f and obtain an infinite number of possibilities for solutions to the three equations that describe our storage capacities. As a practical matter, there really are not an infinite number of solutions, since we are unlikely to want to end the day with a fractional number of bags of fancy mix, so our allowable values of f should probably be integers. More importantly, we need to remember that we cannot make negative amounts of each mix! Where does this lead us? Positive quantities of the bulk mix requires that

$$b \geq 0 \quad\Rightarrow\quad 4f - 3300 \geq 0 \quad\Rightarrow\quad f \geq 825$$

Similarly for the standard mix,

$$s \geq 0 \quad\Rightarrow\quad -5f + 4800 \geq 0 \quad\Rightarrow\quad f \leq 960$$

So, as production manager, you really have to choose a value of f from the finite set

$$\{825,\, 826,\, \ldots,\, 960\}$$

leaving you with 136 choices, each of which will exhaust the day's supply of raw ingredients. Pause now and think about which *you* would choose.

Recalling your weekly meeting with the CEO suggests that you might want to choose a production schedule that yields the biggest possible profit for the company. So you compute an expression for the profit based on your as yet undetermined decision for the value of f,

$$\begin{aligned}&(4f - 3300)(4.99 - 3.70) + (-5f + 4800)(5.50 - 3.85) + (f)(6.50 - 4.45)\\&\quad = -1.04f + 3663\end{aligned}$$

Since f has a negative coefficient it would appear that mixing fancy mix is detrimental to your profit and should be avoided. So you will make the decision to set daily fancy mix production at $f = 825$. This has the effect of setting $b = 4(825) - 3300 = 0$ and we stop producing bulk mix entirely. So the remainder of your daily production is standard mix at the level of $s = -5(825) + 4800 = 675$ kilograms and the resulting daily profit is $(-1.04)(825) + 3663 = 2805$. It is a pleasant surprise that daily profit has risen to \$2,805, but this is not the most important part of the story. What is important here is that there are a large number of ways to produce trail mix that use all of the day's worth of raw ingredients *and* you were able to easily choose the one that netted the largest profit. Notice too how all of the above computations look "linear."

In the food industry, things do not stay the same for long, and now the sales department says that increased competition has led to the decision to stay competitive and charge just \$5.25 for a kilogram of the standard mix, rather than the previous \$5.50 per kilogram. This decision has no effect on the possibilities for the production schedule, but will affect the decision based on profit considerations. So you revisit just the profit computation, suitably adjusted for the new selling price of standard mix,

$$\begin{aligned}&(4f - 3300)(4.99 - 3.70) + (-5f + 4800)(5.25 - 3.85) + (f)(6.50 - 4.45)\\&\quad = 0.21f + 2463\end{aligned}$$

Now it would appear that fancy mix is beneficial to the company's profit since the value of f has a positive coefficient. So you take the decision to make as much fancy mix as possible, setting $f = 960$. This leads to $s = -5(960) + 4800 = 0$ and the increased competition has driven you out of the standard mix market all together. The remainder of production is therefore bulk mix at a daily level of $b = 4(960) - 3300 = 540$ kilograms and the resulting daily profit is $0.21(960) + 2463 = 2664.60$. A daily profit of \$2,664.60 is less than it used to be, but as production manager, you have made the best of a difficult situation and shown the sales department that the best course is to pull out of the highly competitive standard mix market completely. △

This example is taken from a field of mathematics variously known by names such as operations research, systems science, or management science. More specifically, this is a prototypical example of problems that are solved by the techniques of "linear programming."

There is a lot going on under the hood in this example. The heart of the matter is the solution to systems of linear equations, which is the topic of the next few sections, and a recurrent theme throughout this course. We will return to this example on several occasions to reveal some of the reasons for its behavior.

Reading Questions

1. Is the equation $x^2 + xy + \tan(y^3) = 0$ linear or not? Why or why not?
2. Find all solutions to the system of two linear equations $2x + 3y = -8$, $x - y = 6$.
3. Describe how the production manager might explain the importance of the procedures described in the trail mix application (Subsection WILA.AA).

Exercises

C10 In Example TMP the first table lists the cost (per kilogram) to manufacture each of the three varieties of trail mix (bulk, standard, fancy). For example, it costs \$3.69 to make one kilogram of the bulk variety. Re-compute each of these three costs and notice that the computations are linear in character.

M70† In Example TMP two different prices were considered for marketing standard mix with the revised recipes (one-third peanuts in each recipe). Selling standard mix at \$5.50 resulted in selling the minimum amount of the fancy mix and no bulk mix. At \$5.25 it was best for profits to sell the maximum amount of fancy mix and then sell no standard mix. Determine a selling price for standard mix that allows for maximum profits while still selling some of each type of mix.

Section SSLE
Solving Systems of Linear Equations

We will motivate our study of linear algebra by considering the problem of solving several linear equations simultaneously. The word "solve" tends to get abused somewhat, as in "solve this problem." When talking about equations we understand a more precise meaning: find *all* of the values of some variable quantities that make an equation, or several equations, simultaneously true.

Subsection SLE
Systems of Linear Equations

Our first example is of a type we will not pursue further. While it has two equations, the first is not linear. So this is a good example to come back to later, especially after you have seen Theorem PSSLS.

Example STNE Solving two (nonlinear) equations
Suppose we desire the simultaneous solutions of the two equations,

$$\begin{aligned} x^2 + y^2 &= 1 \\ -x + \sqrt{3}y &= 0 \end{aligned}$$

You can easily check by substitution that $x = \frac{\sqrt{3}}{2}$, $y = \frac{1}{2}$ and $x = -\frac{\sqrt{3}}{2}$, $y = -\frac{1}{2}$ are both solutions. We need to also convince ourselves that these are the *only* solutions. To see this, plot each equation on the xy-plane, which means to plot $(x,\, y)$ pairs that make an individual equation true. In this case we get a circle centered at the origin with radius 1 and a straight line through the origin with slope $\frac{1}{\sqrt{3}}$. The intersections of these two curves are our desired simultaneous solutions, and so we believe from our plot that the two solutions we know already are indeed the only ones. We like to write solutions as sets, so in this case we write the set of solutions as

$$S = \left\{ \left(\tfrac{\sqrt{3}}{2},\, \tfrac{1}{2}\right),\, \left(-\tfrac{\sqrt{3}}{2},\, -\tfrac{1}{2}\right) \right\}$$

△

In order to discuss systems of linear equations carefully, we need a precise definition. And before we do that, we will introduce our periodic discussions about "Proof Techniques." Linear algebra is an excellent setting for learning how to read, understand and formulate proofs. But this is a difficult step in your development as a mathematician, so we have included a series of short essays containing advice and explanations to help you along. These will be referenced in the text as needed, and are also collected as a list you can consult when you want to return to re-read them. (Which is strongly encouraged!)

With a definition next, now is the time for the first of our proof techniques. So study Proof Technique D. We'll be right here when you get back. See you in a bit.

Definition SLE System of Linear Equations
A **system of linear equations** is a collection of m equations in the variable quantities $x_1,\, x_2,\, x_3, \ldots, x_n$ of the form,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\ &\vdots \end{aligned}$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m$$

where the values of a_{ij}, b_i and x_j, $1 \leq i \leq m$, $1 \leq j \leq n$, are from the set of complex numbers, $\mathbb{C}$. □

Do not let the mention of the complex numbers, $\mathbb{C}$, rattle you. We will stick with real numbers exclusively for many more sections, and it will sometimes seem like we only work with integers! However, we want to leave the possibility of complex numbers open, and there will be occasions in subsequent sections where they are necessary. You can review the basic properties of complex numbers in Section CNO, but these facts will not be critical until we reach Section O.

Now we make the notion of a solution to a linear system precise.

Definition SSLE Solution of a System of Linear Equations
A **solution** of a system of linear equations in n variables, x_1, x_2, x_3, ..., x_n (such as the system given in Definition SLE), is an ordered list of n complex numbers, s_1, s_2, s_3, ..., s_n such that if we substitute s_1 for x_1, s_2 for x_2, s_3 for x_3, ..., s_n for x_n, then for every equation of the system the left side will equal the right side, i.e. each equation is true simultaneously. □

More typically, we will write a solution in a form like $x_1 = 12$, $x_2 = -7$, $x_3 = 2$ to mean that $s_1 = 12$, $s_2 = -7$, $s_3 = 2$ in the notation of Definition SSLE. To discuss *all* of the possible solutions to a system of linear equations, we now define the set of all solutions. (So Section SET is now applicable, and you may want to go and familiarize yourself with what is there.)

Definition SSSLE Solution Set of a System of Linear Equations
The **solution set** of a linear system of equations is the set which contains every solution to the system, and nothing more. □

Be aware that a solution set can be infinite, or there can be no solutions, in which case we write the solution set as the empty set, $\emptyset = \{\}$ (Definition ES). Here is an example to illustrate using the notation introduced in Definition SLE and the notion of a solution (Definition SSLE).

Example NSE Notation for a system of equations
Given the system of linear equations,

$$\begin{aligned} x_1 + 2x_2 + x_4 &= 7 \\ x_1 + x_2 + x_3 - x_4 &= 3 \\ 3x_1 + x_2 + 5x_3 - 7x_4 &= 1 \end{aligned}$$

we have $n = 4$ variables and $m = 3$ equations. Also,

$$\begin{array}{lllll} a_{11} = 1 & a_{12} = 2 & a_{13} = 0 & a_{14} = 1 & b_1 = 7 \\ a_{21} = 1 & a_{22} = 1 & a_{23} = 1 & a_{24} = -1 & b_2 = 3 \\ a_{31} = 3 & a_{32} = 1 & a_{33} = 5 & a_{34} = -7 & b_3 = 1 \end{array}$$

Additionally, convince yourself that $x_1 = -2$, $x_2 = 4$, $x_3 = 2$, $x_4 = 1$ is one solution (Definition SSLE), but it is not the only one! For example, another solution is $x_1 = -12$, $x_2 = 11$, $x_3 = 1$, $x_4 = -3$, and there are more to be found. So the solution set contains at least two elements. △

We will often shorten the term "system of linear equations" to "system of equations" leaving the linear aspect implied. After all, this is a book about *linear* algebra.

Subsection PSS
Possibilities for Solution Sets

The next example illustrates the possibilities for the solution set of a system of linear equations. We will not be too formal here, and the necessary theorems to back up our claims will come in subsequent sections. So read for feeling and come back later to revisit this example.

Example TTS Three typical systems
Consider the system of two equations with two variables,

$$\begin{aligned} 2x_1 + 3x_2 &= 3 \\ x_1 - x_2 &= 4 \end{aligned}$$

If we plot the solutions to each of these equations separately on the x_1x_2-plane, we get two lines, one with negative slope, the other with positive slope. They have exactly one point in common, $(x_1, x_2) = (3, -1)$, which is the solution $x_1 = 3$, $x_2 = -1$. From the geometry, we believe that this is the only solution to the system of equations, and so we say it is unique.

Now adjust the system with a different second equation,

$$\begin{aligned} 2x_1 + 3x_2 &= 3 \\ 4x_1 + 6x_2 &= 6 \end{aligned}$$

A plot of the solutions to these equations individually results in two lines, one on top of the other! There are infinitely many pairs of points that make both equations true. We will learn shortly how to describe this infinite solution set precisely (see Example SAA, Theorem VFSLS). Notice now how the second equation is just a multiple of the first.

One more minor adjustment provides a third system of linear equations,

$$\begin{aligned} 2x_1 + 3x_2 &= 3 \\ 4x_1 + 6x_2 &= 10 \end{aligned}$$

A plot now reveals two lines with identical slopes, i.e. parallel lines. They have no points in common, and so the system has a solution set that is empty, $S = \emptyset$. △

This example exhibits all of the typical behaviors of a system of equations. A subsequent theorem will tell us that every system of linear equations has a solution set that is empty, contains a single solution or contains infinitely many solutions (Theorem PSSLS). Example STNE yielded exactly two solutions, but this does not contradict the forthcoming theorem. The equations in Example STNE are not linear because they do not match the form of Definition SLE, and so we cannot apply Theorem PSSLS in this case.

Subsection ESEO
Equivalent Systems and Equation Operations

With all this talk about finding solution sets for systems of linear equations, you might be ready to begin learning how to find these solution sets yourself. We begin with our first definition that takes a common word and gives it a very precise meaning in the context of systems of linear equations.

Definition ESYS Equivalent Systems
Two systems of linear equations are **equivalent** if their solution sets are equal. □

Notice here that the two systems of equations could *look* very different (i.e. not be equal), but still have equal solution sets, and we would then call the systems equivalent. Two linear equations in two variables might be plotted as two lines that intersect in a single point. A different system, with three equations in two variables might have a plot that is three lines, all intersecting at a common point, with this common point identical to the intersection point for the first system. By our definition, we could then say these two very different looking systems of equations are equivalent, since they have identical solution sets. It is really like a weaker form of equality, where we allow the systems to be different in some respects, but we use the term equivalent to highlight the situation when their solution sets are equal.

With this definition, we can begin to describe our strategy for solving linear systems. Given a system of linear equations that looks difficult to solve, we would like to have an *equivalent* system that is easy to solve. Since the systems will have equal solution sets, we can solve the “easy” system and get the solution set to the “difficult” system. Here come the tools for making this strategy viable.

Definition EO Equation Operations
Given a system of linear equations, the following three operations will transform the system into a different one, and each operation is known as an **equation operation**.

1. Swap the locations of two equations in the list of equations.

2. Multiply each term of an equation by a nonzero quantity.

3. Multiply each term of one equation by some quantity, and add these terms to a second equation, on both sides of the equality. Leave the first equation the same after this operation, but replace the second equation by the new one.

□

These descriptions might seem a bit vague, but the proof or the examples that follow should make it clear what is meant by each. We will shortly prove a key theorem about equation operations and solutions to linear systems of equations.

We are about to give a rather involved proof, so a discussion about just what a theorem really is would be timely. Stop and read Proof Technique T first.

In the theorem we are about to prove, the conclusion is that two systems are equivalent. By Definition ESYS this translates to requiring that solution sets be equal for the two systems. So we are being asked to show *that two sets are equal*. How do we do this? Well, there is a very standard technique, and we will use it repeatedly through the course. If you have not done so already, head to Section SET and familiarize yourself with sets, their operations, and especially the notion of set equality, Definition SE, and the nearby discussion about its use.

The following theorem has a rather long proof. This chapter contains a few very necessary theorems like this, with proofs that you can safely skip on a first reading. You might come back to them later, when you are more comfortable with reading and studying proofs.

Theorem EOPSS Equation Operations Preserve Solution Sets
If we apply one of the three equation operations of Definition EO to a system of linear equations (Definition SLE), then the original system and the transformed system are equivalent.

Proof. We take each equation operation in turn and show that the solution sets of the two systems are equal, using the definition of set equality (Definition SE).

1. It will not be our habit in proofs to resort to saying statements are “obvious,” but in this case, it should be. There is nothing about the *order* in which we write linear equations that affects their solutions, so the solution set will be equal if the systems only differ by a rearrangement of the order of the equations.

2. Suppose $\alpha \neq 0$ is a number. Let us choose to multiply the terms of equation i by α to build the new system of equations,

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1\\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2\\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3\\
&\vdots\\
\alpha a_{i1}x_1 + \alpha a_{i2}x_2 + \alpha a_{i3}x_3 + \cdots + \alpha a_{in}x_n &= \alpha b_i\\
&\vdots\\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

Let S denote the solutions to the system in the statement of the theorem, and let T denote the solutions to the transformed system.

(a) Show $S \subseteq T$. Suppose $(x_1,\, x_2,\, x_3,\, \ldots, x_n) = (\beta_1,\, \beta_2,\, \beta_3,\, \ldots, \beta_n) \in S$ is a solution to the original system. Ignoring the i-th equation for a moment, we know it makes all the other equations of the transformed system true. We also know that

$$a_{i1}\beta_1 + a_{i2}\beta_2 + a_{i3}\beta_3 + \cdots + a_{in}\beta_n = b_i$$

which we can multiply by α to get

$$\alpha a_{i1}\beta_1 + \alpha a_{i2}\beta_2 + \alpha a_{i3}\beta_3 + \cdots + \alpha a_{in}\beta_n = \alpha b_i$$

This says that the i-th equation of the transformed system is also true, so we have established that $(\beta_1,\, \beta_2,\, \beta_3,\, \ldots, \beta_n) \in T$, and therefore $S \subseteq T$.

(b) Now show $T \subseteq S$. Suppose $(x_1,\, x_2,\, x_3,\, \ldots, x_n) = (\beta_1,\, \beta_2,\, \beta_3,\, \ldots, \beta_n) \in T$ is a solution to the transformed system. Ignoring the i-th equation for a moment, we know it makes all the other equations of the original system true. We also know that

$$\alpha a_{i1}\beta_1 + \alpha a_{i2}\beta_2 + \alpha a_{i3}\beta_3 + \cdots + \alpha a_{in}\beta_n = \alpha b_i$$

which we can multiply by $\frac{1}{\alpha}$, since $\alpha \neq 0$, to get

$$a_{i1}\beta_1 + a_{i2}\beta_2 + a_{i3}\beta_3 + \cdots + a_{in}\beta_n = b_i$$

This says that the i-th equation of the original system is also true, so we have established that $(\beta_1,\, \beta_2,\, \beta_3,\, \ldots, \beta_n) \in S$, and therefore $T \subseteq S$. Locate the key point where we required that $\alpha \neq 0$, and consider what would happen if $\alpha = 0$.

3. Suppose α is a number. Let us choose to multiply the terms of equation i by α and add them to equation j in order to build the new system of equations,

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1\\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2\\
a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n &= b_3\\
&\vdots\\
(\alpha a_{i1} + a_{j1})x_1 + (\alpha a_{i2} + a_{j2})x_2 + \cdots + (\alpha a_{in} + a_{jn})x_n &= \alpha b_i + b_j\\
&\vdots\\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

Let S denote the solutions to the system in the statement of the theorem, and let T denote the solutions to the transformed system.

(a) Show $S \subseteq T$. Suppose $(x_1,\, x_2,\, x_3,\, \ldots, x_n) = (\beta_1,\, \beta_2,\, \beta_3,\, \ldots, \beta_n) \in S$ is a solution to the original system. Ignoring the j-th equation for a moment, we know this solution makes all the other equations of the transformed system true. Using the fact that the solution makes the i-th and j-th equations of the original system true, we find

$$\begin{aligned}
&(\alpha a_{i1} + a_{j1})\beta_1 + (\alpha a_{i2} + a_{j2})\beta_2 + \cdots + (\alpha a_{in} + a_{jn})\beta_n \\
&\quad= (\alpha a_{i1}\beta_1 + \alpha a_{i2}\beta_2 + \cdots + \alpha a_{in}\beta_n) + (a_{j1}\beta_1 + a_{j2}\beta_2 + \cdots + a_{jn}\beta_n) \\
&\quad= \alpha(a_{i1}\beta_1 + a_{i2}\beta_2 + \cdots + a_{in}\beta_n) + (a_{j1}\beta_1 + a_{j2}\beta_2 + \cdots + a_{jn}\beta_n) \\
&\quad= \alpha b_i + b_j.
\end{aligned}$$

This says that the j-th equation of the transformed system is also true, so we have established that $(\beta_1,\, \beta_2,\, \beta_3,\, \ldots, \beta_n) \in T$, and therefore $S \subseteq T$.

(b) Now show $T \subseteq S$. Suppose $(x_1,\, x_2,\, x_3,\, \ldots, x_n) = (\beta_1,\, \beta_2,\, \beta_3,\, \ldots, \beta_n) \in T$ is a solution to the transformed system. Ignoring the j-th equation for a moment, we know it makes all the other equations of the original system true. We then find

$$\begin{aligned}
&a_{j1}\beta_1 + \cdots + a_{jn}\beta_n \\
&\quad= a_{j1}\beta_1 + \cdots + a_{jn}\beta_n + \alpha b_i - \alpha b_i \\
&\quad= a_{j1}\beta_1 + \cdots + a_{jn}\beta_n + (\alpha a_{i1}\beta_1 + \cdots + \alpha a_{in}\beta_n) - \alpha b_i \\
&\quad= a_{j1}\beta_1 + \alpha a_{i1}\beta_1 + \cdots + a_{jn}\beta_n + \alpha a_{in}\beta_n - \alpha b_i \\
&\quad= (\alpha a_{i1} + a_{j1})\beta_1 + \cdots + (\alpha a_{in} + a_{jn})\beta_n - \alpha b_i \\
&\quad= \alpha b_i + b_j - \alpha b_i \\
&\quad= b_j
\end{aligned}$$

This says that the j-th equation of the original system is also true, so we have established that $(\beta_1,\, \beta_2,\, \beta_3,\, \ldots, \beta_n) \in S$, and therefore $T \subseteq S$.

Why did we not need to require that $\alpha \neq 0$ for this row operation? In other words, how does the third statement of the theorem read when $\alpha = 0$? Does our proof require some extra care when $\alpha = 0$? Compare your answers with the similar situation for the second row operation. (See Exercise SSLE.T20.)

■

Theorem EOPSS is the necessary tool to complete our strategy for solving systems of equations. We will use equation operations to move from one system to another, all the while keeping the solution set the same. With the right sequence of operations, we will arrive at a simpler equation to solve. The next two examples illustrate this idea, while saving some of the details for later.

Example US Three equations, one solution
We solve the following system by a sequence of equation operations.

$$\begin{aligned}
x_1 + 2x_2 + 2x_3 &= 4 \\
x_1 + 3x_2 + 3x_3 &= 5 \\
2x_1 + 6x_2 + 5x_3 &= 6
\end{aligned}$$

$\alpha = -1$ times equation 1, add to equation 2:

$$\begin{aligned}
x_1 + 2x_2 + 2x_3 &= 4 \\
0x_1 + 1x_2 + 1x_3 &= 1
\end{aligned}$$

$$2x_1 + 6x_2 + 5x_3 = 6$$

$\alpha = -2$ times equation 1, add to equation 3:

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= 4 \\ 0x_1 + 1x_2 + 1x_3 &= 1 \\ 0x_1 + 2x_2 + 1x_3 &= -2 \end{aligned}$$

$\alpha = -2$ times equation 2, add to equation 3:

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= 4 \\ 0x_1 + 1x_2 + 1x_3 &= 1 \\ 0x_1 + 0x_2 - 1x_3 &= -4 \end{aligned}$$

$\alpha = -1$ times equation 3:

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= 4 \\ 0x_1 + 1x_2 + 1x_3 &= 1 \\ 0x_1 + 0x_2 + 1x_3 &= 4 \end{aligned}$$

which can be written more clearly as

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= 4 \\ x_2 + x_3 &= 1 \\ x_3 &= 4 \end{aligned}$$

This is now a very easy system of equations to solve. The third equation requires that $x_3 = 4$ to be true. Making this substitution into equation 2 we arrive at $x_2 = -3$, and finally, substituting these values of x_2 and x_3 into the first equation, we find that $x_1 = 2$. Note too that this is the only solution to this final system of equations, since we were forced to choose these values to make the equations true. Since we performed equation operations on each system to obtain the next one in the list, all of the systems listed here are all equivalent to each other by Theorem EOPSS. Thus $(x_1,\, x_2,\, x_3) = (2, -3, 4)$ is the unique solution to the *original* system of equations (and all of the other intermediate systems of equations listed as we transformed one into another). △

Example IS Three equations, infinitely many solutions
The following system of equations made an appearance earlier in this section (Example NSE), where we listed *one* of its solutions. Now, we will try to find all of the solutions to this system. Do not concern yourself too much about why we choose this particular sequence of equation operations, just believe that the work we do is all correct.

$$\begin{aligned} x_1 + 2x_2 + 0x_3 + x_4 &= 7 \\ x_1 + x_2 + x_3 - x_4 &= 3 \\ 3x_1 + x_2 + 5x_3 - 7x_4 &= 1 \end{aligned}$$

$\alpha = -1$ times equation 1, add to equation 2:

$$\begin{aligned} x_1 + 2x_2 + 0x_3 + x_4 &= 7 \\ 0x_1 - x_2 + x_3 - 2x_4 &= -4 \\ 3x_1 + x_2 + 5x_3 - 7x_4 &= 1 \end{aligned}$$

$\alpha = -3$ times equation 1, add to equation 3:

$$x_1 + 2x_2 + 0x_3 + x_4 = 7$$

$$\begin{aligned} 0x_1 - x_2 + x_3 - 2x_4 &= -4 \\ 0x_1 - 5x_2 + 5x_3 - 10x_4 &= -20 \end{aligned}$$

$\alpha = -5$ times equation 2, add to equation 3:

$$\begin{aligned} x_1 + 2x_2 + 0x_3 + x_4 &= 7 \\ 0x_1 - x_2 + x_3 - 2x_4 &= -4 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 &= 0 \end{aligned}$$

$\alpha = -1$ times equation 2:

$$\begin{aligned} x_1 + 2x_2 + 0x_3 + x_4 &= 7 \\ 0x_1 + x_2 - x_3 + 2x_4 &= 4 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 &= 0 \end{aligned}$$

$\alpha = -2$ times equation 2, add to equation 1:

$$\begin{aligned} x_1 + 0x_2 + 2x_3 - 3x_4 &= -1 \\ 0x_1 + x_2 - x_3 + 2x_4 &= 4 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 &= 0 \end{aligned}$$

which can be written more clearly as

$$\begin{aligned} x_1 + 2x_3 - 3x_4 &= -1 \\ x_2 - x_3 + 2x_4 &= 4 \\ 0 &= 0 \end{aligned}$$

What does the equation $0 = 0$ mean? We can choose *any* values for x_1, x_2, x_3, x_4 and this equation will be true, so we only need to consider further the first two equations, since the third is true no matter what. We can analyze the second equation without consideration of the variable x_1. It would appear that there is considerable latitude in how we can choose x_2, x_3, x_4 and make this equation true. Let us choose x_3 and x_4 to be *anything* we please, say $x_3 = a$ and $x_4 = b$.

Now we can take these arbitrary values for x_3 and x_4, substitute them in equation 1, to obtain

$$\begin{aligned} x_1 + 2a - 3b &= -1 \\ x_1 &= -1 - 2a + 3b \end{aligned}$$

Similarly, equation 2 becomes

$$\begin{aligned} x_2 - a + 2b &= 4 \\ x_2 &= 4 + a - 2b \end{aligned}$$

So our arbitrary choices of values for x_3 and x_4 (a and b) translate into specific values of x_1 and x_2. The lone solution given in Example NSE was obtained by choosing $a = 2$ and $b = 1$. Now we can easily and quickly find many more (infinitely more). Suppose we choose $a = 5$ and $b = -2$, then we compute

$$\begin{aligned} x_1 &= -1 - 2(5) + 3(-2) = -17 \\ x_2 &= 4 + 5 - 2(-2) = 13 \end{aligned}$$

and you can verify that $(x_1,\, x_2,\, x_3,\, x_4) = (-17,\, 13,\, 5,\, -2)$ makes all three equations true. The entire solution set is written as

$$S = \left\{ (-1 - 2a + 3b,\, 4 + a - 2b,\, a,\, b) \middle|\, a \in \mathbb{C},\, b \in \mathbb{C} \right\}$$

It would be instructive to finish off your study of this example by taking the general form of the solutions given in this set and substituting them into each of the three equations and verify that they are true in each case (Exercise SSLE.M40). △

In the next section we will describe how to use equation operations to systematically solve any system of linear equations. But first, read one of our more important pieces of advice about speaking and writing mathematics. See Proof Technique L.

Before attacking the exercises in this section, it will be helpful to read some advice on getting started on the construction of a proof. See Proof Technique GS.

Reading Questions

1. How many solutions does the system of equations $3x+2y=4$, $6x+4y=8$ have? Explain your answer.
2. How many solutions does the system of equations $3x+2y=4$, $6x+4y=-2$ have? Explain your answer.
3. What do we mean when we say mathematics is a language?

Exercises

C10 Find a solution to the system in Example IS where $x_3 = 6$ and $x_4 = 2$. Find two other solutions to the system. Find a solution where $x_1 = -17$ and $x_2 = 14$. How many possible answers are there to each of these questions?

C20 Each archetype (Archetypes) that is a system of equations begins by listing some specific solutions. Verify the specific solutions listed in the following archetypes by evaluating the system of equations with the solutions listed.

Archetype A, Archetype B, Archetype C, Archetype D, Archetype E, Archetype F, Archetype G, Archetype H, Archetype I, Archetype J

C30† Find all solutions to the linear system:

$$\begin{aligned} x + y &= 5 \\ 2x - y &= 3 \end{aligned}$$

C31 Find all solutions to the linear system:

$$\begin{aligned} 3x + 2y &= 1 \\ x - y &= 2 \\ 4x + 2y &= 2 \end{aligned}$$

C32 Find all solutions to the linear system:

$$\begin{aligned} x + 2y &= 8 \\ x - y &= 2 \\ x + y &= 4 \end{aligned}$$

C33 Find all solutions to the linear system:

$$\begin{aligned} x + y - z &= -1 \\ x - y - z &= -1 \\ z &= 2 \end{aligned}$$

C34 Find all solutions to the linear system:

$$\begin{aligned} x + y - z &= -5 \\ x - y - z &= -3 \\ x + y - z &= 0 \end{aligned}$$

C50† A three-digit number has two properties. The tens-digit and the ones-digit add up to 5. If the number is written with the digits in the reverse order, and then subtracted from the original number, the result is 792. Use a system of equations to find all of the three-digit numbers with these properties.

C51† Find all of the six-digit numbers in which the first digit is one less than the second, the third digit is half the second, the fourth digit is three times the third and the last two digits form a number that equals the sum of the fourth and fifth. The sum of all the digits is 24. (From *The MENSA Puzzle Calendar* for January 9, 2006.)

C52† Driving along, Terry notices that the last four digits on his car's odometer are palindromic. A mile later, the last five digits are palindromic. After driving another mile, the middle four digits are palindromic. One more mile, and all six are palindromic. What was the odometer reading when Terry first looked at it? Form a linear system of equations that expresses the requirements of this puzzle. (*Car Talk* Puzzler, National Public Radio, Week of January 21, 2008) (A car odometer displays six digits and a sequence is a **palindrome** if it reads the same left-to-right as right-to-left.)

M10† Each sentence below has at least two meanings. Identify the source of the double meaning, and rewrite the sentence (at least twice) to clearly convey each meaning.

1. They are baking potatoes.
2. He bought many ripe pears and apricots.
3. She likes his sculpture.
4. I decided on the bus.

M11† Discuss the difference in meaning of each of the following three almost identical sentences, which all have the same grammatical structure. (These are due to Keith Devlin.)

1. She saw him in the park with a dog.
2. She saw him in the park with a fountain.
3. She saw him in the park with a telescope.

M12† The following sentence, due to Noam Chomsky, has a correct grammatical structure, but is meaningless. Critique its faults. "Colorless green ideas sleep furiously." (Chomsky, Noam. *Syntactic Structures*, The Hague/Paris: Mouton, 1957. p. 15.)

M13† Read the following sentence and form a mental picture of the situation.
The baby cried and the mother picked it up.
What *assumptions* did you make about the situation?

M14 Discuss the difference in meaning of the following two almost identical sentences, which have nearly identical grammatical structure. (This antanaclasis is often attributed to the comedian Groucho Marx, but has earlier roots.)

1. Time flies like an arrow.
2. Fruit flies like a banana.

M30† This problem appears in a middle-school mathematics textbook: Together Dan and Diane have \$20. Together Diane and Donna have \$15. How much do the three of them have in total? (*Transition Mathematics*, Second Edition, Scott Foresman Addison Wesley, 1998. Problem 5–1.19.)

M40 Solutions to the system in Example IS are given as

$$(x_1,\, x_2,\, x_3,\, x_4) = (-1 - 2a + 3b,\, 4 + a - 2b,\, a,\, b)$$

Evaluate the three equations of the original system with these expressions in a and b and verify that each equation is true, no matter what values are chosen for a and b.

M70† We have seen in this section that systems of linear equations have limited possibilities for solution sets, and we will shortly prove Theorem PSSLS that describes these possibilities exactly. This exercise will show that if we relax the requirement that our equations be linear, then the possibilities expand greatly. Consider a system of two equations in the two variables x and y, where the departure from linearity involves simply squaring the variables.

$$x^2 - y^2 = 1$$
$$x^2 + y^2 = 4$$

After solving this system of *nonlinear* equations, replace the second equation in turn by $x^2 + 2x + y^2 = 3$, $x^2 + y^2 = 1$, $x^2 - 4x + y^2 = -3$, $-x^2 + y^2 = 1$ and solve each resulting system of two equations in two variables. (This exercise includes suggestions from Don Kreher.)

T10† Proof Technique D asks you to formulate a definition of what it means for a whole number to be odd. What is your definition? (Do not say "the opposite of even.") Is 6 odd? Is 11 odd? Justify your answers by using your definition.

T20† Explain why the second equation operation in Definition EO requires that the scalar be nonzero, while in the third equation operation this restriction on the scalar is not present.

Section RREF
Reduced Row-Echelon Form

After solving a few systems of equations, you will recognize that it does not matter so much *what* we call our variables, as opposed to what numbers act as their coefficients. A system in the variables x_1, x_2, x_3 would behave the same if we changed the names of the variables to a, b, c and kept all the constants the same and in the same places. In this section, we will isolate the key bits of information about a system of equations into something called a matrix, and then use this matrix to systematically solve the equations. Along the way we will obtain one of our most important and useful computational tools.

Subsection MVNSE
Matrix and Vector Notation for Systems of Equations

Definition M Matrix
An $m \times n$ **matrix** is a rectangular layout of numbers from $\mathbb{C}$ having m rows and n columns. We will use upper-case Latin letters from the start of the alphabet (A, B, $C, \dots$) to denote matrices and squared-off brackets to delimit the layout. Many use large parentheses instead of brackets — the distinction is not important. Rows of a matrix will be referenced starting at the top and working down (i.e. row 1 is at the top) and columns will be referenced starting from the left (i.e. column 1 is at the left). For a matrix A, the notation $[A]_{ij}$ will refer to the complex number in row i and column j of A. □

Be careful with this notation for individual entries, since it is easy to think that $[A]_{ij}$ refers to the *whole* matrix. It does not. It is just a *number*, but is a convenient way to talk about the individual entries simultaneously. This notation will get a heavy workout once we get to Chapter M.

Example AM A matrix

$$B = \begin{bmatrix} -1 & 2 & 5 & 3 \\ 1 & 0 & -6 & 1 \\ -4 & 2 & 2 & -2 \end{bmatrix}$$

is a matrix with $m = 3$ rows and $n = 4$ columns. We can say that $[B]_{2,3} = -6$ while $[B]_{3,4} = -2$. △

When we do equation operations on system of equations, the names of the variables really are not very important. Use x_1, x_2, x_3, or a, b, c, or x, y, z, it really does not matter. In this subsection we will describe some notation that will make it easier to describe linear systems, solve the systems and describe the solution sets. Here is a list of definitions, laden with notation.

Definition CV Column Vector
A **column vector** of **size** m is an ordered list of m numbers, which is written in order vertically, starting at the top and proceeding to the bottom. At times, we will refer to a column vector as simply a **vector**. Column vectors will be written in bold, usually with lower case Latin letter from the end of the alphabet such as $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$, $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$. Some books like to write vectors with arrows, such as $\vec{u}$. Writing by hand, some like to put arrows on top of the symbol, or a tilde underneath the symbol, as in $\underset{\sim}{u}$. To refer to the **entry** or **component** of vector $\mathbf{v}$ in location i of the list, we write $[\mathbf{v}]_i$. □

Be careful with this notation. While the symbols $[\mathbf{v}]_i$ might look somewhat substantial, as an object this represents just one entry of a vector, which is just a single complex number.

Definition ZCV Zero Column Vector
The **zero vector** of size m is the column vector of size m where each entry is the number zero,

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

or defined much more compactly, $\left[\mathbf{0}\right]_i = 0$ for $1 \leq i \leq m$. □

Definition CM Coefficient Matrix
For a system of linear equations,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

the **coefficient matrix** is the $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

□

Definition VOC Vector of Constants
For a system of linear equations,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

the **vector of constants** is the column vector of size m

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{bmatrix}$$

□

Definition SOLV Solution Vector
For a system of linear equations,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \end{aligned}$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m$$

the **solution vector** is the column vector of size n

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

□

The solution vector may do double-duty on occasion. It might refer to a list of variable quantities at one point, and subsequently refer to values of those variables that actually form a particular solution to that system.

Definition MRLS Matrix Representation of a Linear System
If A is the coefficient matrix of a system of linear equations and $\mathbf{b}$ is the vector of constants, then we will write $\mathcal{LS}(A,\,\mathbf{b})$ as a shorthand expression for the system of linear equations, which we will refer to as the **matrix representation** of the linear system. □

Example NSLE Notation for systems of linear equations
The system of linear equations

$$\begin{aligned} 2x_1 + 4x_2 - 3x_3 + 5x_4 + x_5 &= 9 \\ 3x_1 + x_2 + \quad\quad x_4 - 3x_5 &= 0 \\ -2x_1 + 7x_2 - 5x_3 + 2x_4 + 2x_5 &= -3 \end{aligned}$$

has coefficient matrix

$$A = \begin{bmatrix} 2 & 4 & -3 & 5 & 1 \\ 3 & 1 & 0 & 1 & -3 \\ -2 & 7 & -5 & 2 & 2 \end{bmatrix}$$

and vector of constants

$$\mathbf{b} = \begin{bmatrix} 9 \\ 0 \\ -3 \end{bmatrix}$$

and so will be referenced as $\mathcal{LS}(A,\,\mathbf{b})$. △

Definition AM Augmented Matrix
Suppose we have a system of m equations in n variables, with coefficient matrix A and vector of constants $\mathbf{b}$. Then the **augmented matrix** of the system of equations is the $m \times (n+1)$ matrix whose first n columns are the columns of A and whose last column $(n+1)$ is the column vector $\mathbf{b}$. This matrix will be written as $[\,A \,|\, \mathbf{b}]$. □

The augmented matrix *represents* all the important information in the system of equations, since the names of the variables have been ignored, and the only connection with the variables is the location of their coefficients in the matrix. It is important to realize that the augmented matrix is just that, a matrix, and *not* a system of equations. In particular, the augmented matrix does not have any "solutions," though it will be useful for finding solutions to the system of equations that it is associated with. (Think about your objects, and review Proof Technique L.) However, notice that an augmented matrix always belongs to some system of equations, and vice versa, so it is tempting to try and blur the distinction between the two. Here is a quick example.

Example AMAA Augmented matrix for Archetype A

Archetype A is the following system of 3 equations in 3 variables.

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 1\\ 2x_1 + x_2 + x_3 &= 8\\ x_1 + x_2 &= 5 \end{aligned}$$

Here is its augmented matrix.

$$\begin{bmatrix} 1 & -1 & 2 & 1\\ 2 & 1 & 1 & 8\\ 1 & 1 & 0 & 5 \end{bmatrix}$$

△

Subsection RO
Row Operations

An augmented matrix for a system of equations will save us the tedium of continually writing down the names of the variables as we solve the system. It will also release us from any dependence on the actual names of the variables. We have seen how certain operations we can perform on equations (Definition EO) will preserve their solutions (Theorem EOPSS). The next two definitions and the following theorem carry over these ideas to augmented matrices.

Definition RO Row Operations
The following three operations will transform an $m \times n$ matrix into a different matrix of the same size, and each is known as a **row operation**.

1. Swap the locations of two rows.
2. Multiply each entry of a single row by a nonzero quantity.
3. Multiply each entry of one row by some quantity, and add these values to the entries in the same columns of a second row. Leave the first row the same after this operation, but replace the second row by the new values.

We will use a symbolic shorthand to describe these row operations:

1. $R_i \leftrightarrow R_j$: Swap the location of rows i and j.
2. αR_i: Multiply row i by the nonzero scalar α.
3. $\alpha R_i + R_j$: Multiply row i by the scalar α and add to row j.

□

Definition REM Row-Equivalent Matrices
Two matrices, A and B, are **row-equivalent** if one can be obtained from the other by a sequence of row operations. □

Example TREM Two row-equivalent matrices
The matrices

$$A = \begin{bmatrix} 2 & -1 & 3 & 4\\ 5 & 2 & -2 & 3\\ 1 & 1 & 0 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 0 & 6\\ 3 & 0 & -2 & -9\\ 2 & -1 & 3 & 4 \end{bmatrix}$$

are row-equivalent as can be seen from

$$\begin{bmatrix} 2 & -1 & 3 & 4\\ 5 & 2 & -2 & 3\\ 1 & 1 & 0 & 6 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 0 & 6\\ 5 & 2 & -2 & 3\\ 2 & -1 & 3 & 4 \end{bmatrix} \xrightarrow{-2R_1 + R_2} \begin{bmatrix} 1 & 1 & 0 & 6\\ 3 & 0 & -2 & -9\\ 2 & -1 & 3 & 4 \end{bmatrix}$$

We can also say that any pair of these three matrices are row-equivalent. △

Notice that each of the three row operations is reversible (Exercise RREF.T10), so we do not have to be careful about the distinction between "A is row-equivalent to B" and "B is row-equivalent to A." (Exercise RREF.T11)

The preceding definitions are designed to make the following theorem possible. It says that row-equivalent matrices represent systems of linear equations that have identical solution sets.

Theorem REMES Row-Equivalent Matrices represent Equivalent Systems
Suppose that A and B are row-equivalent augmented matrices. Then the systems of linear equations that they represent are equivalent systems.

Proof. If we perform a single row operation on an augmented matrix, it will have the same effect as if we did the analogous equation operation on the system of equations the matrix represents. By exactly the same methods as we used in the proof of Theorem EOPSS we can see that each of these row operations will preserve the set of solutions for the system of equations the matrix represents. ■

So at this point, our strategy is to begin with a system of equations, represent the system by an augmented matrix, perform row operations (which will preserve solutions for the system) to get a "simpler" augmented matrix, convert back to a "simpler" system of equations and then solve that system, knowing that its solutions are those of the original system. Here is a rehash of Example US as an exercise in using our new tools.

Example USR Three equations, one solution, reprised
We solve the following system using augmented matrices and row operations. This is the same system of equations solved in Example US using equation operations.

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= 4 \\ x_1 + 3x_2 + 3x_3 &= 5 \\ 2x_1 + 6x_2 + 5x_3 &= 6 \end{aligned}$$

Form the augmented matrix,

$$A = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 1 & 3 & 3 & 5 \\ 2 & 6 & 5 & 6 \end{bmatrix}$$

and apply row operations,

$$\xrightarrow{-1R_1+R_2} \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 1 & 1 \\ 2 & 6 & 5 & 6 \end{bmatrix} \xrightarrow{-2R_1+R_3} \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 1 & -2 \end{bmatrix}$$

$$\xrightarrow{-2R_2+R_3} \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -4 \end{bmatrix} \xrightarrow{-1R_3} \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

So the matrix

$$B = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

is row equivalent to A and by Theorem REMES the system of equations below has the same solution set as the original system of equations.

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &= 4 \\ x_2 + x_3 &= 1 \\ x_3 &= 4 \end{aligned}$$

Solving this "simpler" system is straightforward and is identical to the process in Example US. △

Subsection RREF
Reduced Row-Echelon Form

The preceding example amply illustrates the definitions and theorems we have seen so far. But it still leaves two questions unanswered. Exactly what is this "simpler" form for a matrix, and just how do we get it? Here is the answer to the first question, a definition of reduced row-echelon form.

Definition RREF Reduced Row-Echelon Form
A matrix is in **reduced row-echelon form** if it meets all of the following conditions:

1. If there is a row where every entry is zero, then this row lies below any other row that contains a nonzero entry.

2. The leftmost nonzero entry of a row is equal to 1.

3. The leftmost nonzero entry of a row is the only nonzero entry in its column.

4. Consider any two different leftmost nonzero entries, one located in row i, column j and the other located in row s, column t. If $s > i$, then $t > j$.

A row of only zero entries is called a **zero row** and the leftmost nonzero entry of a nonzero row is a **leading 1**. A column containing a leading 1 will be called a **pivot column**. The number of nonzero rows will be denoted by r, which is also equal to the number of leading 1's and the number of pivot columns.

The set of column indices for the pivot columns will be denoted by $D = \{d_1,\, d_2,\, d_3,\, \ldots,\, d_r\}$ where $d_1 < d_2 < d_3 < \cdots < d_r$, while the columns that are not pivot columns will be denoted as $F = \{f_1,\, f_2,\, f_3,\, \ldots,\, f_{n-r}\}$ where $f_1 < f_2 < f_3 < \cdots < f_{n-r}$.

□

The principal feature of reduced row-echelon form is the pattern of leading 1's guaranteed by conditions (2) and (4), reminiscent of a flight of geese, or steps in a staircase, or water cascading down a mountain stream.

There are a number of new terms and notation introduced in this definition, which should make you suspect that this is an important definition. Given all there is to digest here, we will mostly save the use of D and F until Section TSS. However, one important point to make here is that all of these terms and notation apply to a matrix. Sometimes we will employ these terms and sets for an augmented matrix, and other times it might be a coefficient matrix. So always give some thought to exactly which type of matrix you are analyzing.

Example RREF A matrix in reduced row-echelon form
The matrix C is in reduced row-echelon form.

$$C = \begin{bmatrix} 1 & -3 & 0 & 6 & 0 & 0 & -5 & 9 \\ 0 & 0 & 0 & 0 & 1 & 0 & 3 & -7 \\ 0 & 0 & 0 & 0 & 0 & 1 & 7 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This matrix has two zero rows and three pivot columns. So $r = 3$. Columns 1, 5, and 6 are the three pivot columns, so $D = \{1,\, 5,\, 6\}$ and then $F = \{2,\, 3,\, 4,\, 7,\, 8\}$.△

Example NRREF A matrix not in reduced row-echelon form
The matrix E is not in reduced row-echelon form, as it fails each of the four

requirements once.

$$E = \begin{bmatrix} 1 & 0 & -3 & 0 & 6 & 0 & 7 & -5 & 9 \\ 0 & 0 & 0 & 5 & 0 & 1 & 0 & 3 & -7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 7 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Our next theorem has a "constructive" proof. Learn about the meaning of this term in Proof Technique C.

Theorem REMEF Row-Equivalent Matrix in Echelon Form
Suppose A is a matrix. Then there is a matrix B so that

1. *A and B are row-equivalent.*
2. *B is in reduced row-echelon form.*

Proof. Suppose that A has m rows and n columns. We will describe a process for converting A into B via row operations. This procedure is known as **Gauss-Jordan elimination**. Tracing through this procedure will be easier if you recognize that i refers to a row that is being converted, j refers to a column that is being converted, and r keeps track of the number of nonzero rows. Here we go.

1. Set $j = 0$ and $r = 0$.
2. Increase j by 1. If j now equals $n + 1$, then stop.
3. Examine the entries of A in column j located in rows $r + 1$ through m. If all of these entries are zero, then go to Step 2.
4. Choose a row from rows $r + 1$ through m with a nonzero entry in column j. Let i denote the index for this row.
5. Increase r by 1.
6. Use the first row operation to swap rows i and r.
7. Use the second row operation to convert the entry in row r and column j to a 1.
8. Use the third row operation with row r to convert every other entry of column j to zero.
9. Go to Step 2.

The result of this procedure is that the matrix A is converted to a matrix in reduced row-echelon form, which we will refer to as B. We need to now prove this claim by showing that the converted matrix has the requisite properties of Definition RREF. First, the matrix is only converted through row operations (Steps 6, 7, 8), so A and B are row-equivalent (Definition REM).

It is a bit more work to be certain that B is in reduced row-echelon form. We claim that as we begin Step 2, the first j columns of the matrix are in reduced row-echelon form with r nonzero rows. Certainly this is true at the start when $j = 0$, since the matrix has no columns and so vacuously meets the conditions of Definition RREF with $r = 0$ nonzero rows.

In Step 2 we increase j by 1 and begin to work with the next column. There are two possible outcomes for Step 3. Suppose that every entry of column j in rows $r + 1$ through m is zero. Then with no changes we recognize that the first j columns of the matrix has its first r rows still in reduced-row echelon form, with the final $m - r$ rows still all zero.

Suppose instead that the entry in row i of column j is nonzero. Notice that since $r + 1 \leq i \leq m$, we know the first $j - 1$ entries of this row are all zero. Now, in Step 5 we increase r by 1, and then embark on building a new nonzero row. In Step 6 we swap row r and row i. In the first j columns, the first $r - 1$ rows remain in reduced row-echelon form after the swap. In Step 7 we multiply row r by a nonzero scalar, creating a 1 in the entry in column j of row i, and not changing any other rows. This new leading 1 is the first nonzero entry in its row, and is located to the right of all the leading 1's in the preceding $r - 1$ rows. With Step 8 we insure that every entry in the column with this new leading 1 is now zero, as required for reduced row-echelon form. Also, rows $r+1$ through m are now all zeros in the first j columns, so we now only have one new nonzero row, consistent with our increase of r by one. Furthermore, since the first $j - 1$ entries of row r are zero, the employment of the third row operation does not destroy any of the necessary features of rows 1 through $r - 1$ and rows $r + 1$ through m, in columns 1 through $j - 1$.

So at this stage, the first j columns of the matrix are in reduced row-echelon form. When Step 2 finally increases j to $n + 1$, then the procedure is completed and the full n columns of the matrix are in reduced row-echelon form, with the value of r correctly recording the number of nonzero rows. ■

The procedure given in the proof of Theorem REMEF can be more precisely described using a pseudo-code version of a computer program. Single-letter variables, like `m, n, i, j, r` have the same meanings as above. `:=` is assignment of the value on the right to the variable on the left, `A[i,j]` is the equivalent of the matrix entry $[A]_{ij}$, while `==` is an equality test and `<>` is a "not equals" test.

```
input m, n and A
r := 0
for j := 1 to n
   i := r+1
   while i <= m and A[i,j] == 0
       i := i+1
   if i < m+1
       r := r+1
       swap rows i and r of A (row op 1)
       scale A[r,j] to a leading 1 (row op 2)
       for k := 1 to m, k <> r
           make A[k,j] zero (row op 3, employing row r)
output r and A
```

Notice that as a practical matter the "and" used in the conditional statement of the while statement should be of the "short-circuit" variety so that the array access that follows is not out-of-bounds.

So now we can put it all together. Begin with a system of linear equations (Definition SLE), and represent the system by its augmented matrix (Definition AM). Use row operations (Definition RO) to convert this matrix into reduced row-echelon form (Definition RREF), using the procedure outlined in the proof of Theorem REMEF. Theorem REMEF also tells us we can always accomplish this, and that the result is row-equivalent (Definition REM) to the original augmented matrix. Since the matrix in reduced-row echelon form has the same solution set, we can analyze the row-reduced version instead of the original matrix, viewing it as the augmented matrix of a different system of equations. The beauty of augmented matrices in reduced row-echelon form is that the solution sets to the systems they represent can be easily determined, as we will see in the next few examples and in the next section.

We will see through the course that almost every interesting property of a matrix can be discerned by looking at a row-equivalent matrix in reduced row-echelon form. For this reason it is important to know that the matrix B is guaranteed to exist by Theorem REMEF is also unique.

Two proof techniques are applicable to the proof. First, head out and read two proof techniques: Proof Technique CD and Proof Technique U.

Theorem RREFU Reduced Row-Echelon Form is Unique
Suppose that A is an $m \times n$ matrix and that B and C are $m \times n$ matrices that are row-equivalent to A and in reduced row-echelon form. Then $B = C$.

Proof. We need to begin with no assumptions about any relationships between B and C, other than they are both in reduced row-echelon form, and they are both row-equivalent to A.

If B and C are both row-equivalent to A, then they are row-equivalent to each other. Repeated row operations on a matrix combine the rows with each other using operations that are linear, and are identical in each column. A key observation for this proof is that each individual row of B is linearly related to the rows of C. This relationship is different for each row of B, but once we fix a row, the relationship is the same across columns. More precisely, there are scalars δ_{ik}, $1 \leq i, k \leq m$ such that for any $1 \leq i \leq m$, $1 \leq j \leq n$,

$$[B]_{ij} = \sum_{k=1}^{m} \delta_{ik} [C]_{kj}$$

You should read this as saying that an entry of row i of B (in column j) is a linear function of the entries of all the rows of C that are also in column j, and the scalars (δ_{ik}) depend on which row of B we are considering (the i subscript on δ_{ik}), but are the same for every column (no dependence on j in δ_{ik}). This idea may be complicated now, but will feel more familiar once we discuss "linear combinations" (Definition LCCV) and moreso when we discuss "row spaces" (Definition RSM). For now, spend some time carefully working Exercise RREF.M40, which is designed to illustrate the origins of this expression. This completes our exploitation of the row-equivalence of B and C.

We now repeatedly exploit the fact that B and C are in reduced row-echelon form. Recall that a pivot column is all zeros, except a single one. More carefully, if R is a matrix in reduced row-echelon form, and d_ℓ is the index of a pivot column, then $[R]_{kd_\ell} = 1$ precisely when $k = \ell$ and is otherwise zero. Notice also that any entry of R that is both below the entry in row ℓ *and* to the left of column d_ℓ is also zero (with below and left understood to include equality). In other words, look at examples of matrices in reduced row-echelon form and choose a leading 1 (with a box around it). The rest of the column is also zeros, and the lower left "quadrant" of the matrix that begins here is totally zeros.

Assuming no relationship about the form of B and C, let B have r nonzero rows and denote the pivot columns as $D = \{d_1, d_2, d_3, \ldots, d_r\}$. For C let r' denote the number of nonzero rows and denote the pivot columns as
$D' = \{ d'_1, d'_2, d'_3, \ldots, d'_{r'}\}$ (Definition RREF). There are four steps in the proof, and the first three are about showing that B and C have the same number of pivot columns, in the same places. In other words, the "primed" symbols are a necessary fiction.

First Step. Suppose that $d_1 < d'_1$. Then

$$\begin{aligned} 1 &= [B]_{1d_1} && \text{Definition RREF} \\ &= \sum_{k=1}^{m} \delta_{1k} [C]_{kd_1} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{m} \delta_{1k}(0) && d_1 < d'_1\\
&= 0
\end{aligned}
$$

The entries of C are all zero since they are left and below of the leading 1 in row 1 and column d'_1 of C. This is a contradiction, so we know that $d_1 \geq d'_1$. By an entirely similar argument, reversing the roles of B and C, we could conclude that $d_1 \leq d'_1$. Together this means that $d_1 = d'_1$.

Second Step. Suppose that we have determined that $d_1 = d'_1$, $d_2 = d'_2$, $d_3 = d'_3$, $\dots d_p = d'_p$. Let us now show that $d_{p+1} = d'_{p+1}$. Working towards a contradiction, suppose that $d_{p+1} < d'_{p+1}$. For $1 \leq \ell \leq p$,

$$
\begin{aligned}
0 &= [B]_{p+1,d_\ell} && \text{Definition RREF}\\
&= \sum_{k=1}^{m} \delta_{p+1,k}\,[C]_{kd_\ell}\\
&= \sum_{k=1}^{m} \delta_{p+1,k}\,[C]_{kd'_\ell}\\
&= \delta_{p+1,\ell}\,[C]_{\ell d'_\ell} + \sum_{\substack{k=1\\k\neq\ell}}^{m} \delta_{p+1,k}\,[C]_{kd'_\ell} && \text{Property CACN}\\
&= \delta_{p+1,\ell}(1) + \sum_{\substack{k=1\\k\neq\ell}}^{m} \delta_{p+1,k}(0) && \text{Definition RREF}\\
&= \delta_{p+1,\ell}
\end{aligned}
$$

Now,

$$
\begin{aligned}
1 &= [B]_{p+1,d_{p+1}} && \text{Definition RREF}\\
&= \sum_{k=1}^{m} \delta_{p+1,k}\,[C]_{kd_{p+1}}\\
&= \sum_{k=1}^{p} \delta_{p+1,k}\,[C]_{kd_{p+1}} + \sum_{k=p+1}^{m} \delta_{p+1,k}\,[C]_{kd_{p+1}} && \text{Property AACN}\\
&= \sum_{k=1}^{p} (0)\,[C]_{kd_{p+1}} + \sum_{k=p+1}^{m} \delta_{p+1,k}\,[C]_{kd_{p+1}}\\
&= \sum_{k=p+1}^{m} \delta_{p+1,k}\,[C]_{kd_{p+1}}\\
&= \sum_{k=p+1}^{m} \delta_{p+1,k}(0) && d_{p+1} < d'_{p+1}\\
&= 0
\end{aligned}
$$

This contradiction shows that $d_{p+1} \geq d'_{p+1}$. By an entirely similar argument, we could conclude that $d_{p+1} \leq d'_{p+1}$, and therefore $d_{p+1} = d'_{p+1}$.

Third Step. Now we establish that $r = r'$. Suppose that $r' < r$. By the arguments above, we know that $d_1 = d'_1$, $d_2 = d'_2$, $d_3 = d'_3$, $\dots$, $d_{r'} = d'_{r'}$. For $1 \leq \ell \leq r' < r$,

$$
\begin{aligned}
0 &= [B]_{rd_\ell} && \text{Definition RREF}\\
&= \sum_{k=1}^{m} \delta_{rk}\,[C]_{kd_\ell}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=1}^{r'} \delta_{rk} \left[C\right]_{kd_\ell} + \sum_{k=r'+1}^{m} \delta_{rk} \left[C\right]_{kd_\ell} && \text{Property AACN} \\
&= \sum_{k=1}^{r'} \delta_{rk} \left[C\right]_{kd_\ell} + \sum_{k=r'+1}^{m} \delta_{rk}(0) && \text{Property AACN} \\
&= \sum_{k=1}^{r'} \delta_{rk} \left[C\right]_{kd_\ell} \\
&= \sum_{k=1}^{r'} \delta_{rk} \left[C\right]_{kd'_\ell} \\
&= \delta_{r\ell} \left[C\right]_{\ell d'_\ell} + \sum_{\substack{k=1\\k\neq\ell}}^{r'} \delta_{rk} \left[C\right]_{kd'_\ell} && \text{Property CACN} \\
&= \delta_{r\ell}(1) + \sum_{\substack{k=1\\k\neq\ell}}^{r'} \delta_{rk}(0) && \text{Definition RREF} \\
&= \delta_{r\ell}
\end{aligned}
$$

Now examine the entries of row r of B,

$$
\begin{aligned}
\left[B\right]_{rj} &= \sum_{k=1}^{m} \delta_{rk} \left[C\right]_{kj} \\
&= \sum_{k=1}^{r'} \delta_{rk} \left[C\right]_{kj} + \sum_{k=r'+1}^{m} \delta_{rk} \left[C\right]_{kj} && \text{Property CACN} \\
&= \sum_{k=1}^{r'} \delta_{rk} \left[C\right]_{kj} + \sum_{k=r'+1}^{m} \delta_{rk}(0) && \text{Definition RREF} \\
&= \sum_{k=1}^{r'} \delta_{rk} \left[C\right]_{kj} \\
&= \sum_{k=1}^{r'} (0) \left[C\right]_{kj} \\
&= 0
\end{aligned}
$$

So row r is a totally zero row, contradicting that this should be the bottommost nonzero row of B. So $r' \geq r$. By an entirely similar argument, reversing the roles of B and C, we would conclude that $r' \leq r$ and therefore $r = r'$. Thus, combining the first three steps we can say that $D = D'$. In other words, B and C have the same pivot columns, in the same locations.

Fourth Step. In this final step, we will not argue by contradiction. Our intent is to determine the values of the δ_{ij}. Notice that we can use the values of the d_i interchangeably for B and C. Here we go,

$$
\begin{aligned}
1 &= \left[B\right]_{id_i} && \text{Definition RREF} \\
&= \sum_{k=1}^{m} \delta_{ik} \left[C\right]_{kd_i} \\
&= \delta_{ii} \left[C\right]_{id_i} + \sum_{\substack{k=1\\k\neq i}}^{m} \delta_{ik} \left[C\right]_{kd_i} && \text{Property CACN}
\end{aligned}
$$

$$= \delta_{ii}(1) + \sum_{\substack{k=1\\k\neq i}}^{m} \delta_{ik}(0) \qquad \text{Definition RREF}$$

$$= \delta_{ii}$$

and for $\ell \neq i$

$$\begin{aligned}
0 &= [B]_{id_\ell} && \text{Definition RREF}\\
&= \sum_{k=1}^{m} \delta_{ik} [C]_{kd_\ell}\\
&= \delta_{i\ell} [C]_{\ell d_\ell} + \sum_{\substack{k=1\\k\neq \ell}}^{m} \delta_{ik} [C]_{kd_\ell} && \text{Property CACN}\\
&= \delta_{i\ell}(1) + \sum_{\substack{k=1\\k\neq \ell}}^{m} \delta_{ik}(0) && \text{Definition RREF}\\
&= \delta_{i\ell}
\end{aligned}$$

Finally, having determined the values of the δ_{ij}, we can show that $B = C$. For $1 \leq i \leq m$, $1 \leq j \leq n$,

$$\begin{aligned}
[B]_{ij} &= \sum_{k=1}^{m} \delta_{ik} [C]_{kj}\\
&= \delta_{ii} [C]_{ij} + \sum_{\substack{k=1\\k\neq i}}^{m} \delta_{ik} [C]_{kj} && \text{Property CACN}\\
&= (1) [C]_{ij} + \sum_{\substack{k=1\\k\neq i}}^{m} (0) [C]_{kj}\\
&= [C]_{ij}
\end{aligned}$$

So B and C have equal values in every entry, and so are the same matrix. ■

We will now run through some examples of using these definitions and theorems to solve some systems of equations. From now on, when we have a matrix in reduced row-echelon form, we will mark the leading 1's with a small box. This will help you count, and identify, the pivot columns. In your work, you can box 'em, circle 'em or write 'em in a different color — just identify 'em somehow. This device will prove very useful later and is a *very good habit* to start developing *right now.*

Example SAB Solutions for Archetype B
Let us find the solutions to the following system of equations,

$$\begin{aligned}
-7x_1 - 6x_2 - 12x_3 &= -33\\
5x_1 + 5x_2 + 7x_3 &= 24\\
x_1 + 4x_3 &= 5
\end{aligned}$$

First, form the augmented matrix,

$$\begin{bmatrix} -7 & -6 & -12 & -33\\ 5 & 5 & 7 & 24\\ 1 & 0 & 4 & 5 \end{bmatrix}$$

and work to reduced row-echelon form, first with $j = 1$,

$$\xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 4 & 5\\ 5 & 5 & 7 & 24\\ -7 & -6 & -12 & -33 \end{bmatrix} \xrightarrow{-5R_1 + R_2} \begin{bmatrix} 1 & 0 & 4 & 5\\ 0 & 5 & -13 & -1\\ -7 & -6 & -12 & -33 \end{bmatrix}$$

$$\xrightarrow{7R_1+R_3}\begin{bmatrix}\boxed{1} & 0 & 4 & 5\\ 0 & 5 & -13 & -1\\ 0 & -6 & 16 & 2\end{bmatrix}$$

Now, with $j = 2$,

$$\xrightarrow{\frac{1}{5}R_2}\begin{bmatrix}\boxed{1} & 0 & 4 & 5\\ 0 & 1 & \frac{-13}{5} & \frac{-1}{5}\\ 0 & -6 & 16 & 2\end{bmatrix}\xrightarrow{6R_2+R_3}\begin{bmatrix}\boxed{1} & 0 & 4 & 5\\ 0 & \boxed{1} & \frac{-13}{5} & \frac{-1}{5}\\ 0 & 0 & \frac{2}{5} & \frac{4}{5}\end{bmatrix}$$

And finally, with $j = 3$,

$$\xrightarrow{\frac{5}{2}R_3}\begin{bmatrix}\boxed{1} & 0 & 4 & 5\\ 0 & \boxed{1} & \frac{-13}{5} & \frac{-1}{5}\\ 0 & 0 & 1 & 2\end{bmatrix}\xrightarrow{\frac{13}{5}R_3+R_2}\begin{bmatrix}\boxed{1} & 0 & 4 & 5\\ 0 & \boxed{1} & 0 & 5\\ 0 & 0 & 1 & 2\end{bmatrix}$$

$$\xrightarrow{-4R_3+R_1}\begin{bmatrix}\boxed{1} & 0 & 0 & -3\\ 0 & \boxed{1} & 0 & 5\\ 0 & 0 & \boxed{1} & 2\end{bmatrix}$$

This is now the augmented matrix of a very simple system of equations, namely $x_1 = -3$, $x_2 = 5$, $x_3 = 2$, which has an obvious solution. Furthermore, we can see that this is the *only* solution to this system, so we have determined the entire solution set,

$$S = \left\{\begin{bmatrix}-3\\ 5\\ 2\end{bmatrix}\right\}$$

You might compare this example with the procedure we used in Example US. △

Archetypes A and B are meant to contrast each other in many respects. So let us solve Archetype A now.

Example SAA Solutions for Archetype A

Let us find the solutions to the following system of equations,

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 1\\ 2x_1 + x_2 + x_3 &= 8\\ x_1 + x_2 &= 5 \end{aligned}$$

First, form the augmented matrix,

$$\begin{bmatrix}1 & -1 & 2 & 1\\ 2 & 1 & 1 & 8\\ 1 & 1 & 0 & 5\end{bmatrix}$$

and work to reduced row-echelon form, first with $j = 1$,

$$\xrightarrow{-2R_1+R_2}\begin{bmatrix}1 & -1 & 2 & 1\\ 0 & 3 & -3 & 6\\ 1 & 1 & 0 & 5\end{bmatrix}\xrightarrow{-1R_1+R_3}\begin{bmatrix}\boxed{1} & -1 & 2 & 1\\ 0 & 3 & -3 & 6\\ 0 & 2 & -2 & 4\end{bmatrix}$$

Now, with $j = 2$,

$$\xrightarrow{\frac{1}{3}R_2}\begin{bmatrix}\boxed{1} & -1 & 2 & 1\\ 0 & 1 & -1 & 2\\ 0 & 2 & -2 & 4\end{bmatrix}\xrightarrow{1R_2+R_1}\begin{bmatrix}\boxed{1} & 0 & 1 & 3\\ 0 & 1 & -1 & 2\\ 0 & 2 & -2 & 4\end{bmatrix}$$

$$\xrightarrow{-2R_2+R_3}\begin{bmatrix}\boxed{1} & 0 & 1 & 3\\ 0 & \boxed{1} & -1 & 2\\ 0 & 0 & 0 & 0\end{bmatrix}$$

The system of equations represented by this augmented matrix needs to be considered a bit differently than that for Archetype B. First, the last row of the matrix is the equation $0 = 0$, which is *always* true, so it imposes no restrictions on our possible solutions and therefore we can safely ignore it as we analyze the other two equations. These equations are,

$$\begin{aligned} x_1 + x_3 &= 3 \\ x_2 - x_3 &= 2. \end{aligned}$$

While this system is fairly easy to solve, it also appears to have a multitude of solutions. For example, choose $x_3 = 1$ and see that then $x_1 = 2$ and $x_2 = 3$ will together form a solution. Or choose $x_3 = 0$, and then discover that $x_1 = 3$ and $x_2 = 2$ lead to a solution. Try it yourself: pick *any* value of x_3 you please, and figure out what x_1 and x_2 should be to make the first and second equations (respectively) true. We'll wait while you do that. Because of this behavior, we say that x_3 is a "free" or "independent" variable. But why do we vary x_3 and not some other variable? For now, notice that the third column of the augmented matrix is not a pivot column. With this idea, we can rearrange the two equations, solving each for the variable whose index is the same as the column index of a pivot column.

$$\begin{aligned} x_1 &= 3 - x_3 \\ x_2 &= 2 + x_3 \end{aligned}$$

To write the set of solution vectors in set notation, we have

$$S = \left\{ \left.\begin{bmatrix} 3 - x_3 \\ 2 + x_3 \\ x_3 \end{bmatrix} \right| x_3 \in \mathbb{C} \right\}$$

We will learn more in the next section about systems with infinitely many solutions and how to express their solution sets. Right now, you might look back at Example IS. △

Example SAE Solutions for Archetype E

Let us find the solutions to the following system of equations,

$$\begin{aligned} 2x_1 + x_2 + 7x_3 - 7x_4 &= 2 \\ -3x_1 + 4x_2 - 5x_3 - 6x_4 &= 3 \\ x_1 + x_2 + 4x_3 - 5x_4 &= 2 \end{aligned}$$

First, form the augmented matrix,

$$\begin{bmatrix} 2 & 1 & 7 & -7 & 2 \\ -3 & 4 & -5 & -6 & 3 \\ 1 & 1 & 4 & -5 & 2 \end{bmatrix}$$

and work to reduced row-echelon form, first with $j = 1$,

$$\xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 1 & 4 & -5 & 2 \\ -3 & 4 & -5 & -6 & 3 \\ 2 & 1 & 7 & -7 & 2 \end{bmatrix} \xrightarrow{3R_1 + R_2} \begin{bmatrix} 1 & 1 & 4 & -5 & 2 \\ 0 & 7 & 7 & -21 & 9 \\ 2 & 1 & 7 & -7 & 2 \end{bmatrix}$$

$$\xrightarrow{-2R_1 + R_3} \begin{bmatrix} \boxed{1} & 1 & 4 & -5 & 2 \\ 0 & 7 & 7 & -21 & 9 \\ 0 & -1 & -1 & 3 & -2 \end{bmatrix}$$

Now, with $j = 2$,

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} \boxed{1} & 1 & 4 & -5 & 2 \\ 0 & -1 & -1 & 3 & -2 \\ 0 & 7 & 7 & -21 & 9 \end{bmatrix} \xrightarrow{-1R_2} \begin{bmatrix} \boxed{1} & 1 & 4 & -5 & 2 \\ 0 & 1 & 1 & -3 & 2 \\ 0 & 7 & 7 & -21 & 9 \end{bmatrix}$$

$$\xrightarrow{-1R_2+R_1}\begin{bmatrix}\boxed{1} & 0 & 3 & -2 & 0\\ 0 & 1 & 1 & -3 & 2\\ 0 & 7 & 7 & -21 & 9\end{bmatrix}\xrightarrow{-7R_2+R_3}\begin{bmatrix}\boxed{1} & 0 & 3 & -2 & 0\\ 0 & \boxed{1} & 1 & -3 & 2\\ 0 & 0 & 0 & 0 & -5\end{bmatrix}$$

And finally, with $j = 4$,

$$\xrightarrow{-\frac{1}{5}R_3}\begin{bmatrix}\boxed{1} & 0 & 3 & -2 & 0\\ 0 & \boxed{1} & 1 & -3 & 2\\ 0 & 0 & 0 & 0 & 1\end{bmatrix}\xrightarrow{-2R_3+R_2}\begin{bmatrix}\boxed{1} & 0 & 3 & -2 & 0\\ 0 & \boxed{1} & 1 & -3 & 0\\ 0 & 0 & 0 & 0 & \boxed{1}\end{bmatrix}$$

Let us analyze the equations in the system represented by this augmented matrix. The third equation will read $0 = 1$. This is patently false, all the time. No choice of values for our variables will ever make it true. We are done. Since we cannot even make the last equation true, we have no hope of making all of the equations simultaneously true. So this system has no solutions, and its solution set is the empty set, $\emptyset = \{\ \}$ (Definition ES).

Notice that we could have reached this conclusion sooner. After performing the row operation $-7R_2 + R_3$, we can see that the third equation reads $0 = -5$, a false statement. Since the system represented by this matrix has no solutions, none of the systems represented has any solutions. However, for this example, we have chosen to bring the matrix all the way to reduced row-echelon form as practice. △

These three examples (Example SAB, Example SAA, Example SAE) illustrate the full range of possibilities for a system of linear equations — no solutions, one solution, or infinitely many solutions. In the next section we will examine these three scenarios more closely.

We (and everybody else) will often speak of "row-reducing" a matrix. This is an informal way of saying we begin with a matrix A and then analyze *the* matrix B that is row-equivalent to A and in reduced row-echelon form. So the term **row-reduce** is used as a verb, but describes something a bit more complicated, since we do not really change A. Theorem REMEF tells us that this process will always be successful and Theorem RREFU tells us that B will be unambiguous. Typically, an investigation of A will proceed by analyzing B and applying theorems whose hypotheses include the row-equivalence of A and B, and usually the hypothesis that B is in reduced row-echelon form.

Reading Questions

1. Is the matrix below in reduced row-echelon form? Why or why not?

$$\begin{bmatrix}1 & 5 & 0 & 6 & 8\\ 0 & 0 & 1 & 2 & 0\\ 0 & 0 & 0 & 0 & 1\end{bmatrix}$$

2. Use row operations to convert the matrix below to reduced row-echelon form and report the final matrix.

$$\begin{bmatrix}2 & 1 & 8\\ -1 & 1 & -1\\ -2 & 5 & 4\end{bmatrix}$$

3. Find all the solutions to the system below by using an augmented matrix and row operations. Report your final matrix in reduced row-echelon form and the set of solutions.

$$\begin{aligned}2x_1 + 3x_2 - x_3 &= 0\\ x_1 + 2x_2 + x_3 &= 3\\ x_1 + 3x_2 + 3x_3 &= 7\end{aligned}$$

Exercises

C05 Each archetype below is a system of equations. Form the augmented matrix of the system of equations, convert the matrix to reduced row-echelon form by using equation operations and then describe the solution set of the original system of equations.

Archetype A, Archetype B, Archetype C, Archetype D, Archetype E, Archetype F, Archetype G, Archetype H, Archetype I, Archetype J

For problems C10–C19, find all solutions to the system of linear equations. Use your favorite computing device to row-reduce the augmented matrices for the systems, and write the solutions as a set, using correct set notation.

C10†

$$\begin{aligned} 2x_1 - 3x_2 + x_3 + 7x_4 &= 14 \\ 2x_1 + 8x_2 - 4x_3 + 5x_4 &= -1 \\ x_1 + 3x_2 - 3x_3 &= 4 \\ -5x_1 + 2x_2 + 3x_3 + 4x_4 &= -19 \end{aligned}$$

C11†

$$\begin{aligned} 3x_1 + 4x_2 - x_3 + 2x_4 &= 6 \\ x_1 - 2x_2 + 3x_3 + x_4 &= 2 \\ 10x_2 - 10x_3 - x_4 &= 1 \end{aligned}$$

C12†

$$\begin{aligned} 2x_1 + 4x_2 + 5x_3 + 7x_4 &= -26 \\ x_1 + 2x_2 + x_3 - x_4 &= -4 \\ -2x_1 - 4x_2 + x_3 + 11x_4 &= -10 \end{aligned}$$

C13†

$$\begin{aligned} x_1 + 2x_2 + 8x_3 - 7x_4 &= -2 \\ 3x_1 + 2x_2 + 12x_3 - 5x_4 &= 6 \\ -x_1 + x_2 + x_3 - 5x_4 &= -10 \end{aligned}$$

C14†

$$\begin{aligned} 2x_1 + x_2 + 7x_3 - 2x_4 &= 4 \\ 3x_1 - 2x_2 + 11x_4 &= 13 \\ x_1 + x_2 + 5x_3 - 3x_4 &= 1 \end{aligned}$$

C15†

$$\begin{aligned} 2x_1 + 3x_2 - x_3 - 9x_4 &= -16 \\ x_1 + 2x_2 + x_3 &= 0 \\ -x_1 + 2x_2 + 3x_3 + 4x_4 &= 8 \end{aligned}$$

C16†

$$\begin{aligned} 2x_1 + 3x_2 + 19x_3 - 4x_4 &= 2 \\ x_1 + 2x_2 + 12x_3 - 3x_4 &= 1 \\ -x_1 + 2x_2 + 8x_3 - 5x_4 &= 1 \end{aligned}$$

C17†

$$\begin{aligned} -x_1 + 5x_2 &= -8 \\ -2x_1 + 5x_2 + 5x_3 + 2x_4 &= 9 \\ -3x_1 - x_2 + 3x_3 + x_4 &= 3 \end{aligned}$$

$$7x_1 + 6x_2 + 5x_3 + x_4 = 30$$

C18†

$$\begin{aligned} x_1 + 2x_2 - 4x_3 - x_4 &= 32 \\ x_1 + 3x_2 - 7x_3 - x_5 &= 33 \\ x_1 + 2x_3 - 2x_4 + 3x_5 &= 22 \end{aligned}$$

C19†

$$\begin{aligned} 2x_1 + x_2 &= 6 \\ -x_1 - x_2 &= -2 \\ 3x_1 + 4x_2 &= 4 \\ 3x_1 + 5x_2 &= 2 \end{aligned}$$

For problems C30–C33, row-reduce the matrix without the aid of a calculator, indicating the row operations you are using at each step using the notation of Definition RO.

C30†

$$\begin{bmatrix} 2 & 1 & 5 & 10 \\ 1 & -3 & -1 & -2 \\ 4 & -2 & 6 & 12 \end{bmatrix}$$

C31†

$$\begin{bmatrix} 1 & 2 & -4 \\ -3 & -1 & -3 \\ -2 & 1 & -7 \end{bmatrix}$$

C32†

$$\begin{bmatrix} 1 & 1 & 1 \\ -4 & -3 & -2 \\ 3 & 2 & 1 \end{bmatrix}$$

C33†

$$\begin{bmatrix} 1 & 2 & -1 & -1 \\ 2 & 4 & -1 & 4 \\ -1 & -2 & 3 & 5 \end{bmatrix}$$

M40† Consider the two 3×4 matrices below

$$B = \begin{bmatrix} 1 & 3 & -2 & 2 \\ -1 & -2 & -1 & -1 \\ -1 & -5 & 8 & -3 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 1 & 4 & 0 \\ -1 & -1 & -4 & 1 \end{bmatrix}$$

1. Row-reduce each matrix and determine that the reduced row-echelon forms of B and C are identical. From this argue that B and C are row-equivalent.

2. In the proof of Theorem RREFU, we begin by arguing that entries of row-equivalent matrices are related by way of certain scalars and sums. In this example, we would write that entries of B from row i that are in column j are linearly related to the entries of C in column j from all three rows

$$\left[B\right]_{ij} = \delta_{i1}\left[C\right]_{1j} + \delta_{i2}\left[C\right]_{2j} + \delta_{i3}\left[C\right]_{3j} \qquad 1 \leq j \leq 4$$

   For each $1 \leq i \leq 3$ find the corresponding three scalars in this relationship. So your answer will be nine scalars, determined three at a time.

M45† You keep a number of lizards, mice and peacocks as pets. There are a total of 108 legs and 30 tails in your menagerie. You have twice as many mice as lizards. How many of each creature do you have?

M50† A parking lot has 66 vehicles (cars, trucks, motorcycles and bicycles) in it. There are four times as many cars as trucks. The total number of tires (4 per car or truck, 2 per motorcycle or bicycle) is 252. How many cars are there? How many bicycles?

T10† Prove that each of the three row operations (Definition RO) is reversible. More precisely, if the matrix B is obtained from A by application of a single row operation, show that there is a single row operation that will transform B back into A.

T11 Suppose that A, B and C are $m \times n$ matrices. Use the definition of row-equivalence (Definition REM) to prove the following three facts.

1. A is row-equivalent to A.
2. If A is row-equivalent to B, then B is row-equivalent to A.
3. If A is row-equivalent to B, and B is row-equivalent to C, then A is row-equivalent to C.

A relationship that satisfies these three properties is known as an **equivalence relation**, an important idea in the study of various algebras. This is a formal way of saying that a relationship behaves like equality, without requiring the relationship to be as strict as equality itself. We will see it again in Theorem SER.

T12 Suppose that B is an $m \times n$ matrix in reduced row-echelon form. Build a new, likely smaller, $k \times \ell$ matrix C as follows. Keep any collection of k adjacent rows, $k \leq m$. From these rows, keep columns 1 through ℓ, $\ell \leq n$. Prove that C is in reduced row-echelon form.

T13 Generalize Exercise RREF.T12 by just keeping any k rows, and not requiring the rows to be adjacent. Prove that any such matrix C is in reduced row-echelon form.

Section TSS
Types of Solution Sets

We will now be more careful about analyzing the reduced row-echelon form derived from the augmented matrix of a system of linear equations. In particular, we will see how to systematically handle the situation when we have infinitely many solutions to a system, and we will prove that every system of linear equations has either zero, one or infinitely many solutions. With these tools, we will be able to routinely solve any linear system.

Subsection CS
Consistent Systems

The computer scientist Donald Knuth said, “Science is what we understand well enough to explain to a computer. Art is everything else.” In this section we will remove solving systems of equations from the realm of art, and into the realm of science. We begin with a definition.

Definition CS Consistent System
A system of linear equations is **consistent** if it has at least one solution. Otherwise, the system is called **inconsistent**. □

We will want to first recognize when a system is inconsistent or consistent, and in the case of consistent systems we will be able to further refine the types of solutions possible. We will do this by analyzing the reduced row-echelon form of a matrix, using the value of r, and the sets of column indices, D and F, first defined back in Definition RREF.

Use of the notation for the elements of D and F can be a bit confusing, since we have subscripted variables that are in turn equal to integers used to index the matrix. However, many questions about matrices and systems of equations can be answered once we know r, D and F. The choice of the letters D and F refer to our upcoming definition of dependent and free variables (Definition IDV). An example will help us begin to get comfortable with this aspect of reduced row-echelon form.

Example RREFN Reduced row-echelon form notation
For the 5×9 matrix

$$B = \begin{bmatrix} \boxed{1} & 5 & 0 & 0 & 2 & 8 & 0 & 5 & -1 \\ 0 & 0 & \boxed{1} & 0 & 4 & 7 & 0 & 2 & 0 \\ 0 & 0 & 0 & \boxed{1} & 3 & 9 & 0 & 3 & -6 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 4 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

in reduced row-echelon form we have

$$\begin{array}{lllll} r = 4 & & & & \\ d_1 = 1 & d_2 = 3 & d_3 = 4 & d_4 = 7 & \\ f_1 = 2 & f_2 = 5 & f_3 = 6 & f_4 = 8 & f_5 = 9 \end{array}$$

Notice that the sets

$$D = \{d_1, d_2, d_3, d_4\} = \{1, 3, 4, 7\} \qquad F = \{f_1, f_2, f_3, f_4, f_5\} = \{2, 5, 6, 8, 9\}$$

have nothing in common and together account for all of the columns of B (we say it is a **partition** of the set of column indices). △

The number r is the single most important piece of information we can get from the reduced row-echelon form of a matrix. It is defined as the number of nonzero rows, but since each nonzero row has a leading 1, it is also the number of leading

1's present. For each leading 1, we have a pivot column, so r is also the number of pivot columns. Repeating ourselves, r is the number of nonzero rows, the number of leading 1's *and* the number of pivot columns. Across different situations, each of these interpretations of the meaning of r will be useful, though it may be most helpful to think in terms of pivot columns.

Before proving some theorems about the possibilities for solution sets to systems of equations, let us analyze one particular system with an infinite solution set very carefully as an example. We will use this technique frequently, and shortly we will refine it slightly.

Archetypes I and J are both fairly large for doing computations by hand (though not impossibly large). Their properties are very similar, so we will frequently analyze the situation in Archetype I, and leave you the joy of analyzing Archetype J yourself. So work through Archetype I with the text, by hand and/or with a computer, and then tackle Archetype J yourself (and check your results with those listed). Notice too that the archetypes describing systems of equations each lists the values of r, D and F. Here we go...

Example ISSI Describing infinite solution sets, Archetype I
Archetype I is the system of $m = 4$ equations in $n = 7$ variables.

$$\begin{aligned}
x_1 + 4x_2 - x_4 + 7x_6 - 9x_7 &= 3\\
2x_1 + 8x_2 - x_3 + 3x_4 + 9x_5 - 13x_6 + 7x_7 &= 9\\
2x_3 - 3x_4 - 4x_5 + 12x_6 - 8x_7 &= 1\\
-x_1 - 4x_2 + 2x_3 + 4x_4 + 8x_5 - 31x_6 + 37x_7 &= 4
\end{aligned}$$

This system has a 4×8 augmented matrix that is row-equivalent to the following matrix (check this!), and which is in reduced row-echelon form (the existence of this matrix is guaranteed by Theorem REMEF and its uniqueness is guaranteed by Theorem RREFU),

$$\begin{bmatrix}
\boxed{1} & 4 & 0 & 0 & 2 & 1 & -3 & 4\\
0 & 0 & \boxed{1} & 0 & 1 & -3 & 5 & 2\\
0 & 0 & 0 & \boxed{1} & 2 & -6 & 6 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

So we find that $r = 3$ and

$$D = \{d_1,\, d_2,\, d_3\} = \{1,\, 3,\, 4\} \qquad F = \{f_1,\, f_2,\, f_3,\, f_4,\, f_5\} = \{2,\, 5,\, 6,\, 7,\, 8\}$$

Let i denote any one of the $r = 3$ nonzero rows. Then the index d_i is a pivot column. It will be easy in this case to use the equation represented by row i to write an expression for the variable x_{d_i}. It will be a linear function of the variables x_{f_1}, x_{f_2}, x_{f_3}, x_{f_4} (notice that $f_5 = 8$ does not reference a variable, but does tell us the final column is not a pivot column). We will now construct these three expressions. Notice that using subscripts upon subscripts takes some getting used to.

$$\begin{aligned}
&(i = 1) \qquad & x_{d_1} &= x_1 = 4 - 4x_2 - 2x_5 - x_6 + 3x_7\\
&(i = 2) \qquad & x_{d_2} &= x_3 = 2 - x_5 + 3x_6 - 5x_7\\
&(i = 3) \qquad & x_{d_3} &= x_4 = 1 - 2x_5 + 6x_6 - 6x_7
\end{aligned}$$

Each element of the set $F = \{f_1,\, f_2,\, f_3,\, f_4,\, f_5\} = \{2,\, 5,\, 6,\, 7,\, 8\}$ is the index of a variable, except for $f_5 = 8$. We refer to $x_{f_1} = x_2$, $x_{f_2} = x_5$, $x_{f_3} = x_6$ and $x_{f_4} = x_7$ as "free" (or "independent") variables since they are allowed to assume any possible combination of values that we can imagine and we can continue on to build a solution to the system by solving individual equations for the values of the other ("dependent") variables.

Each element of the set $D = \{d_1,\, d_2,\, d_3\} = \{1,\, 3,\, 4\}$ is the index of a variable. We refer to the variables $x_{d_1} = x_1$, $x_{d_2} = x_3$ and $x_{d_3} = x_4$ as "dependent" variables since they *depend* on the *independent* variables. More precisely, for each possible choice of values for the independent variables we get *exactly one* set of values for the dependent variables that combine to form a solution of the system.

To express the solutions as a set, we write

$$\left\{\left.\begin{bmatrix}4-4x_2-2x_5-x_6+3x_7\\x_2\\2-x_5+3x_6-5x_7\\1-2x_5+6x_6-6x_7\\x_5\\x_6\\x_7\end{bmatrix}\right|\, x_2,\,x_5,\,x_6,\,x_7\in\mathbb{C}\right\}$$

The condition that $x_2,\, x_5,\, x_6,\, x_7 \in \mathbb{C}$ is how we specify that the variables $x_2,\, x_5,\, x_6,\, x_7$ are "free" to assume any possible values.

This systematic approach to solving a system of equations will allow us to create a precise description of the solution set for any consistent system once we have found the reduced row-echelon form of the augmented matrix. It will work just as well when the set of free variables is empty and we get just a single solution. And we could program a computer to do it! Now have a whack at Archetype J (Exercise TSS.C10), mimicking the discussion in this example. We'll still be here when you get back. △

Using the reduced row-echelon form of the augmented matrix of a system of equations to determine the nature of the solution set of the system is a very key idea. So let us look at one more example like the last one. But first a definition, and then the example. We mix our metaphors a bit when we call variables free versus dependent. Maybe we should call dependent variables "enslaved"?

Definition IDV Independent and Dependent Variables
Suppose A is the augmented matrix of a consistent system of linear equations and B is a row-equivalent matrix in reduced row-echelon form. Suppose j is the index of a pivot column of B. Then the variable x_j is **dependent**. A variable that is not dependent is called **independent** or **free**. □

If you studied this definition carefully, you might wonder what to do if the system has n variables and column $n+1$ is a pivot column? We will see shortly, by Theorem RCLS, that this never happens for a consistent system.

Example FDV Free and dependent variables
Consider the system of five equations in five variables,

$$\begin{aligned}x_1-x_2-2x_3+x_4+11x_5&=13\\x_1-x_2+x_3+x_4+5x_5&=16\\2x_1-2x_2+x_4+10x_5&=21\\2x_1-2x_2-x_3+3x_4+20x_5&=38\\2x_1-2x_2+x_3+x_4+8x_5&=22\end{aligned}$$

whose augmented matrix row-reduces to

$$\begin{bmatrix}\boxed{1}&-1&0&0&3&6\\0&0&\boxed{1}&0&-2&1\\0&0&0&\boxed{1}&4&9\\0&0&0&0&0&0\\0&0&0&0&0&0\end{bmatrix}$$

Columns 1, 3 and 4 are pivot columns, so $D=\{1,\,3,\,4\}$. From this we know that the variables x_1, x_3 and x_4 will be dependent variables, and each of the $r=3$ nonzero rows of the row-reduced matrix will yield an expression for one of these three variables. The set F is all the remaining column indices, $F = \{2,\, 5,\, 6\}$. The column index 6 in F means that the final column is not a pivot column, and thus the system is consistent (Theorem RCLS). The remaining indices in F indicate free variables, so x_2 and x_5 (the remaining variables) are our free variables. The resulting three equations that describe our solution set are then,

$$
\begin{aligned}
(x_{d_1} = x_1) &\qquad x_1 = 6 + x_2 - 3x_5 \\
(x_{d_2} = x_3) &\qquad x_3 = 1 + 2x_5 \\
(x_{d_3} = x_4) &\qquad x_4 = 9 - 4x_5
\end{aligned}
$$

Make sure you understand where these three equations came from, and notice how the location of the pivot columns determined the variables on the left-hand side of each equation. We can compactly describe the solution set as,

$$
S = \left\{ \left.\begin{bmatrix} 6 + x_2 - 3x_5 \\ x_2 \\ 1 + 2x_5 \\ 9 - 4x_5 \\ x_5 \end{bmatrix} \right|\, x_2,\, x_5 \in \mathbb{C} \right\}
$$

Notice how we express the freedom for x_2 and x_5: $x_2,\, x_5 \in \mathbb{C}$. △

Sets are an important part of algebra, and we have seen a few already. Being comfortable with sets is important for understanding and writing proofs. If you have not already, pay a visit now to Section SET.

We can now use the values of m, n, r, and the independent and dependent variables to categorize the solution sets for linear systems through a sequence of theorems.

Through the following sequence of proofs, you will want to consult three proof techniques. See Proof Technique E, Proof Technique N, Proof Technique CP.

First we have an important theorem that explores the distinction between consistent and inconsistent linear systems.

Theorem RCLS Recognizing Consistency of a Linear System
Suppose A is the augmented matrix of a system of linear equations with n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Then the system of equations is inconsistent if and only if column $n+1$ of B is a pivot column.

Proof. ($\Leftarrow$) The first half of the proof begins with the assumption that column $n+1$ of B is a pivot column. Then the leading 1 of row r is located in column $n+1$ of B and so row r of B begins with n consecutive zeros, finishing with the leading 1. This is a representation of the equation $0 = 1$, which is false. Since this equation is false for any collection of values we might choose for the variables, there are no solutions for the system of equations, and the system is inconsistent.

($\Rightarrow$) For the second half of the proof, we wish to show that if we assume the system is inconsistent, then column $n+1$ of B is a pivot column. But instead of proving this directly, we will form the logically equivalent statement that is the contrapositive, and prove that instead (see Proof Technique CP). Turning the implication around, and negating each portion, we arrive at the logically equivalent statement: if column $n+1$ of B is not a pivot column, then the system of equations is consistent.

If column $n+1$ of B is not a pivot column, the leading 1 for row r is located somewhere in columns 1 through n. Then *every* preceding row's leading 1 is also located in columns 1 through n. In other words, since the last leading 1 is not in the last column, no leading 1 for any row is in the last column, due to the echelon layout of the leading 1's (Definition RREF). We will now construct a solution to the system by setting each dependent variable to the entry of the final column in the row with the corresponding leading 1, and setting each free variable to zero. That sentence is pretty vague, so let us be more precise. Using our notation for the sets D and F from the reduced row-echelon form (Definition RREF):

$$x_{d_i} = [B]_{i,n+1}, \quad 1 \leq i \leq r \qquad\qquad x_{f_i} = 0, \quad 1 \leq i \leq n-r$$

These values for the variables make the equations represented by the first r rows of B all true (convince yourself of this). Rows numbered greater than r (if any) are all zero rows, hence represent the equation $0 = 0$ and are also all true. We have now identified one solution to the system represented by B, and hence a solution to the system represented by A (Theorem REMES). So we can say the system is consistent (Definition CS). ■

The beauty of this theorem being an equivalence is that we can unequivocally test to see if a system is consistent or inconsistent by looking at just a single entry of the reduced row-echelon form matrix. We could program a computer to do it!

Notice that for a consistent system the row-reduced augmented matrix has $n+1 \in F$, so the largest element of F does not refer to a variable. Also, for an inconsistent system, $n+1 \in D$, and it then does not make much sense to discuss whether or not variables are free or dependent since there is no solution. Take a look back at Definition IDV and see why we did not need to consider the possibility of referencing x_{n+1} as a dependent variable.

With the characterization of Theorem RCLS, we can explore the relationships between r and n for a consistent system. We can distinguish between the case of a unique solution and infinitely many solutions, and furthermore, we recognize that these are the only two possibilities.

Theorem CSRN Consistent Systems, r and n
Suppose A is the augmented matrix of a consistent *system of linear equations with n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon form with r pivot columns. Then $r \leq n$. If $r = n$, then the system has a unique solution, and if $r < n$, then the system has infinitely many solutions.*

Proof. This theorem contains three implications that we must establish. Notice first that B has $n+1$ columns, so there can be at most $n+1$ pivot columns, i.e. $r \leq n+1$. If $r = n+1$, then every column of B is a pivot column, and in particular, the last column is a pivot column. So Theorem RCLS tells us that the system is inconsistent, contrary to our hypothesis. We are left with $r \leq n$.

When $r = n$, we find $n - r = 0$ free variables (i.e. $F = \{n+1\}$) and the only solution is given by setting the n variables to the the first n entries of column $n+1$ of B.

When $r < n$, we have $n - r > 0$ free variables. Choose one free variable and set all the other free variables to zero. Now, set the chosen free variable to any fixed value. It is possible to then determine the values of the dependent variables to create a solution to the system. By setting the chosen free variable to different values, in this manner we can create infinitely many solutions. ■

Subsection FV
Free Variables

The next theorem simply states a conclusion from the final paragraph of the previous proof, allowing us to state explicitly the number of free variables for a consistent system.

Theorem FVCS Free Variables for Consistent Systems
Suppose A is the augmented matrix of a consistent *system of linear equations with n variables. Suppose also that B is a row-equivalent matrix in reduced row-echelon*

form with r rows that are not completely zeros. Then the solution set can be described with $n - r$ free variables.

Proof. See the proof of Theorem CSRN. ■

Example CFV Counting free variables
For each archetype that is a system of equations, the values of n and r are listed. Many also contain a few sample solutions. We can use this information profitably, as illustrated by four examples.

1. Archetype A has $n = 3$ and $r = 2$. It can be seen to be consistent by the sample solutions given. Its solution set then has $n - r = 1$ free variables, and therefore will be infinite.

2. Archetype B has $n = 3$ and $r = 3$. It can be seen to be consistent by the single sample solution given. Its solution set can then be described with $n - r = 0$ free variables, and therefore will have just the single solution.

3. Archetype H has $n = 2$ and $r = 3$. In this case, column 3 must be a pivot column, so by Theorem RCLS, the system is inconsistent. We should not try to apply Theorem FVCS to count free variables, since the theorem only applies to consistent systems. (What would happen if you did try to incorrectly apply Theorem FVCS?)

4. Archetype E has $n = 4$ and $r = 3$. However, by looking at the reduced row-echelon form of the augmented matrix, we find that column 5 is a pivot column. By Theorem RCLS we recognize the system as inconsistent.

△

We have accomplished a lot so far, but our main goal has been the following theorem, which is now very simple to prove. The proof is so simple that we ought to call it a corollary, but the result is important enough that it deserves to be called a theorem. (See Proof Technique LC.) Notice that this theorem was presaged first by Example TTS and further foreshadowed by other examples.

Theorem PSSLS Possible Solution Sets for Linear Systems
A system of linear equations has no solutions, a unique solution or infinitely many solutions.

Proof. By its definition, a system is either inconsistent or consistent (Definition CS). The first case describes systems with no solutions. For consistent systems, we have the remaining two possibilities as guaranteed by, and described in, Theorem CSRN. ■

Here is a diagram that consolidates several of our theorems from this section, and which is of practical use when you analyze systems of equations. Note this presumes we have the reduced row-echelon form of the augmented matrix of the system to analyze.

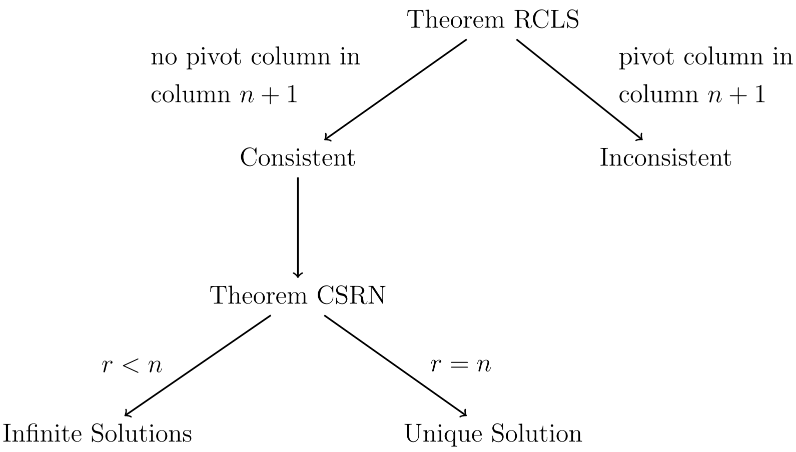

Diagram DTSLS: Decision Tree for Solving Linear Systems

We have one more theorem to round out our set of tools for determining solution sets to systems of linear equations.

Theorem CMVEI Consistent, More Variables than Equations, Infinite solutions
Suppose a consistent system of linear equations has m equations in n variables. If $n > m$, then the system has infinitely many solutions.

Proof. Suppose that the augmented matrix of the system of equations is row-equivalent to B, a matrix in reduced row-echelon form with r nonzero rows. Because B has m rows in total, the number of nonzero rows is less than or equal to m. In other words, $r \leq m$. Follow this with the hypothesis that $n > m$ and we find that the system has a solution set described by at least one free variable because

$$n - r \geq n - m > 0.$$

A consistent system with free variables will have an infinite number of solutions, as given by Theorem CSRN. ■

Notice that to use this theorem we need only know that the system is consistent, together with the values of m and n. We do not necessarily have to compute a row-equivalent reduced row-echelon form matrix, even though we discussed such a matrix in the proof. This is the substance of the following example.

Example OSGMD One solution gives many, Archetype D
Archetype D is the system of $m = 3$ equations in $n = 4$ variables,

$$\begin{aligned} 2x_1 + x_2 + 7x_3 - 7x_4 &= 8 \\ -3x_1 + 4x_2 - 5x_3 - 6x_4 &= -12 \\ x_1 + x_2 + 4x_3 - 5x_4 &= 4 \end{aligned}$$

and the solution $x_1 = 0$, $x_2 = 1$, $x_3 = 2$, $x_4 = 1$ can be checked easily by substitution. Having been *handed* this solution, we know the system is consistent. This, together with $n > m$, allows us to apply Theorem CMVEI and conclude that the system has infinitely many solutions. △

These theorems give us the procedures and implications that allow us to completely solve any system of linear equations. The main computational tool is using row operations to convert an augmented matrix into reduced row-echelon form. Here is a broad outline of how we would instruct a computer to solve a system of linear equations.

1. Represent a system of linear equations in n variables by an augmented matrix (an array is the appropriate data structure in most computer languages).

2. Convert the matrix to a row-equivalent matrix in reduced row-echelon form using the procedure from the proof of Theorem REMEF. Identify the location of the pivot columns, and their number r.

3. If column $n + 1$ is a pivot column, output the statement that the system is inconsistent and halt.

4. If column $n + 1$ is not a pivot column, there are two possibilities:

   (a) $r = n$ and the solution is unique. It can be read off directly from the entries in rows 1 through n of column $n + 1$.

   (b) $r < n$ and there are infinitely many solutions. If only a single solution is needed, set all the free variables to zero and read off the dependent variable values from column $n + 1$, as in the second half of the proof of Theorem RCLS. If the entire solution set is required, figure out some nice compact way to describe it, since your finite computer is not big enough to hold all the solutions (we will have such a way soon).

The above makes it all sound a bit simpler than it really is. In practice, row operations employ division (usually to get a leading entry of a row to convert to a leading 1) and that will introduce round-off errors. Entries that should be zero sometimes end up being very, very small nonzero entries, or small entries lead to overflow errors when used as divisors. A variety of strategies can be employed to minimize these sorts of errors, and this is one of the main topics in the important subject known as numerical linear algebra.

In this section we have gained a foolproof procedure for solving any system of linear equations, no matter how many equations or variables. We also have a handful of theorems that allow us to determine partial information about a solution set without actually constructing the whole set itself. Donald Knuth would be proud.

Reading Questions

1. How can we *easily* recognize when a system of linear equations is inconsistent or not?
2. Suppose we have converted the augmented matrix of a system of equations into reduced row-echelon form. How do we then identify the dependent and independent (free) variables?
3. What are the possible solution sets for a system of linear equations?

Exercises

C10 In the spirit of Example ISSI, describe the infinite solution set for Archetype J.

For Exercises C21–C28, find the solution set of the system of linear equations. Give the values of n and r, and interpret your answers in light of the theorems of this section.

C21†

$$\begin{aligned} x_1 + 4x_2 + 3x_3 - x_4 &= 5 \\ x_1 - x_2 + x_3 + 2x_4 &= 6 \\ 4x_1 + x_2 + 6x_3 + 5x_4 &= 9 \end{aligned}$$

C22†

$$x_1 - 2x_2 + x_3 - x_4 = 3$$

$$\begin{aligned} 2x_1 - 4x_2 + x_3 + x_4 &= 2 \\ x_1 - 2x_2 - 2x_3 + 3x_4 &= 1 \end{aligned}$$

C23†

$$\begin{aligned} x_1 - 2x_2 + x_3 - x_4 &= 3 \\ x_1 + x_2 + x_3 - x_4 &= 1 \\ x_1 \quad + x_3 - x_4 &= 2 \end{aligned}$$

C24†

$$\begin{aligned} x_1 - 2x_2 + x_3 - x_4 &= 2 \\ x_1 + x_2 + x_3 - x_4 &= 2 \\ x_1 \quad + x_3 - x_4 &= 2 \end{aligned}$$

C25†

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 - x_2 + x_3 &= 2 \\ 3x_1 + x_2 + x_3 &= 4 \\ x_2 + 2x_3 &= 6 \end{aligned}$$

C26†

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 - x_2 + x_3 &= 2 \\ 3x_1 + x_2 + x_3 &= 4 \\ 5x_2 + 2x_3 &= 1 \end{aligned}$$

C27†

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ 2x_1 - x_2 + x_3 &= 2 \\ x_1 - 8x_2 - 7x_3 &= 1 \\ x_2 + x_3 &= 0 \end{aligned}$$

C28†

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 - x_2 + x_3 &= 2 \\ x_1 - 8x_2 - 7x_3 &= 1 \\ x_2 + x_3 &= 0 \end{aligned}$$

M45† The details for Archetype J include several sample solutions. Verify that one of these solutions is correct (any one, but just one). Based *only* on this evidence, and especially without doing any row operations, explain how you know this system of linear equations has infinitely many solutions.

M46 Consider Archetype J, and specifically the row-reduced version of the augmented matrix of the system of equations, denoted as B here, and the values of r, D and F immediately following. Determine the values of the entries

$$[B]_{1,d_1} \quad [B]_{3,d_3} \quad [B]_{1,d_3} \quad [B]_{3,d_1} \quad [B]_{d_1,1} \quad [B]_{d_3,3} \quad [B]_{d_1,3} \quad [B]_{d_3,1} \quad [B]_{1,f_1} \quad [B]_{3,f_1}$$

(See Exercise TSS.M70 for a generalization.)

For Exercises M51–M57 say **as much as possible** about each system's solution set. Be sure to make it clear which theorems you are using to reach your conclusions.

M51† A consistent system of 8 equations in 6 variables.

M52† A consistent system of 6 equations in 8 variables.

M53† A system of 5 equations in 9 variables.

M54† A system with 12 equations in 35 variables.

M56† A system with 6 equations in 12 variables.

M57† A system with 8 equations and 6 variables. The reduced row-echelon form of the augmented matrix of the system has 7 pivot columns.

M60 Without doing any computations, and without examining any solutions, say as much as possible about the form of the solution set for each archetype that is a system of equations.

Archetype A, Archetype B, Archetype C, Archetype D, Archetype E, Archetype F, Archetype G, Archetype H, Archetype I, Archetype J

M70 Suppose that B is a matrix in reduced row-echelon form that is equivalent to the augmented matrix of a system of equations with m equations in n variables. Let r, D and F be as defined in Definition RREF. What can you conclude, in general, about the following entries?

$$[B]_{1,d_1} \quad [B]_{3,d_3} \quad [B]_{1,d_3} \quad [B]_{3,d_1} \quad [B]_{d_1,1} \quad [B]_{d_3,3} \quad [B]_{d_1,3} \quad [B]_{d_3,1} \quad [B]_{1,f_1} \quad [B]_{3,f_1}$$

If you cannot conclude anything about an entry, then say so. (See Exercise TSS.M46.)

T10† An inconsistent system may have $r > n$. If we try (incorrectly!) to apply Theorem FVCS to such a system, how many free variables would we discover?

T11† Suppose A is the augmented matrix of a system of linear equations in n variables. and that B is a row-equivalent matrix in reduced row-echelon form with r pivot columns. If $r = n + 1$, prove that the system of equations is inconsistent.

T20 Suppose that B is a matrix in reduced row-echelon form that is equivalent to the augmented matrix of a system of equations with m equations in n variables. Let r, D and F be as defined in Definition RREF. Prove that $d_k \geq k$ for all $1 \leq k \leq r$. Then suppose that $r \geq 2$ and $1 \leq k < \ell \leq r$ and determine what can you conclude, in general, about the following entries.

$$[B]_{k,d_k} \quad [B]_{k,d_\ell} \quad [B]_{\ell,d_k} \quad [B]_{d_k,k} \quad [B]_{d_k,\ell} \quad [B]_{d_\ell,k} \quad [B]_{d_k,f_\ell} \quad [B]_{d_\ell,f_k}$$

If you cannot conclude anything about an entry, then say so. (See Exercise TSS.M46 and Exercise TSS.M70.)

T40† Suppose that the coefficient matrix of a consistent system of linear equations has two columns that are identical. Prove that the system has infinitely many solutions.

T41† Consider the system of linear equations $\mathcal{LS}(A, \mathbf{b})$, and suppose that every element of the vector of constants $\mathbf{b}$ is a common multiple of the corresponding element of a certain column of A. More precisely, there is a complex number α, and a column index j, such that $[\mathbf{b}]_i = \alpha [A]_{ij}$ for all i. Prove that the system is consistent.

Section HSE
Homogeneous Systems of Equations

In this section we specialize to systems of linear equations where every equation has a zero as its constant term. Along the way, we will begin to express more and more ideas in the language of matrices and begin a move away from writing out whole systems of equations. The ideas initiated in this section will carry through the remainder of the course.

Subsection SHS
Solutions of Homogeneous Systems

As usual, we begin with a definition.

Definition HS Homogeneous System
A system of linear equations, $\mathcal{LS}(A,\,\mathbf{b})$ is **homogeneous** if the vector of constants is the zero vector, in other words, if $\mathbf{b}=\mathbf{0}$. □

Example AHSAC Archetype C as a homogeneous system
For each archetype that is a system of equations, we have formulated a similar, yet different, homogeneous system of equations by replacing each equation's constant term with a zero. To wit, for Archetype C, we can convert the original system of equations into the homogeneous system,

$$\begin{aligned} 2x_1-3x_2+x_3-6x_4&=0\\ 4x_1+x_2+2x_3+9x_4&=0\\ 3x_1+x_2+x_3+8x_4&=0 \end{aligned}$$

Can you quickly find a solution to this system without row-reducing the augmented matrix? △

As you might have discovered by studying Example AHSAC, setting each variable to zero will *always* be a solution of a homogeneous system. This is the substance of the following theorem.

Theorem HSC Homogeneous Systems are Consistent
Suppose that a system of linear equations is homogeneous. Then the system is consistent and one solution is found by setting each variable to zero.

Proof. Set each variable of the system to zero. When substituting these values into each equation, the left-hand side evaluates to zero, no matter what the coefficients are. Since a homogeneous system has zero on the right-hand side of each equation as the constant term, each equation is true. With one demonstrated solution, we can call the system consistent. ■

Since this solution is so obvious, we now define it as the trivial solution.

Definition TSHSE Trivial Solution to Homogeneous Systems of Equations
Suppose a homogeneous system of linear equations has n variables. The solution $x_1=0$, $x_2=0$, …, $x_n=0$ (i.e. $\mathbf{x}=\mathbf{0}$) is called the **trivial solution**. □

Here are three typical examples, which we will reference throughout this section. Work through the row operations as we bring each to reduced row-echelon form. Also notice what is similar in each example, and what differs.

Example HUSAB Homogeneous, unique solution, Archetype B
Archetype B can be converted to the homogeneous system,

$$\begin{aligned} -7x_1-6x_2-12x_3&=0\\ 5x_1+5x_2+7x_3&=0 \end{aligned}$$

$$x_1 + 4x_3 = 0$$

whose augmented matrix row-reduces to

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \end{bmatrix}$$

By Theorem HSC, the system is consistent, and so the computation $n - r = 3 - 3 = 0$ means the solution set contains just a single solution. Then, this lone solution must be the trivial solution. △

Example HISAA Homogeneous, infinite solutions, Archetype A
Archetype A can be converted to the homogeneous system,

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 0 \\ 2x_1 + x_2 + x_3 &= 0 \\ x_1 + x_2 \qquad &= 0 \end{aligned}$$

whose augmented matrix row-reduces to

$$\begin{bmatrix} \boxed{1} & 0 & 1 & 0 \\ 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By Theorem HSC, the system is consistent, and so the computation $n - r = 3 - 2 = 1$ means the solution set contains one free variable by Theorem FVCS, and hence has infinitely many solutions. We can describe this solution set using the free variable x_3,

$$S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle|\, x_1 = -x_3,\, x_2 = x_3 \right\} = \left\{ \begin{bmatrix} -x_3 \\ x_3 \\ x_3 \end{bmatrix} \middle|\, x_3 \in \mathbb{C} \right\}$$

Geometrically, these are points in three dimensions that lie on a line through the origin. △

Example HISAD Homogeneous, infinite solutions, Archetype D
Archetype D (and identically, Archetype E) can be converted to the homogeneous system,

$$\begin{aligned} 2x_1 + x_2 + 7x_3 - 7x_4 &= 0 \\ -3x_1 + 4x_2 - 5x_3 - 6x_4 &= 0 \\ x_1 + x_2 + 4x_3 - 5x_4 &= 0 \end{aligned}$$

whose augmented matrix row-reduces to

$$\begin{bmatrix} \boxed{1} & 0 & 3 & -2 & 0 \\ 0 & \boxed{1} & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

By Theorem HSC, the system is consistent, and so the computation $n - r = 4 - 2 = 2$ means the solution set contains two free variables by Theorem FVCS, and hence has infinitely many solutions. We can describe this solution set using the free variables x_3 and x_4,

$$S = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \middle|\, x_1 = -3x_3 + 2x_4,\, x_2 = -x_3 + 3x_4 \right\}$$

$$= \left\{ \begin{bmatrix} -3x_3 + 2x_4 \\ -x_3 + 3x_4 \\ x_3 \\ x_4 \end{bmatrix} \middle| \, x_3,\, x_4 \in \mathbb{C} \right\}$$

△

After working through these examples, you might perform the same computations for the slightly larger example, Archetype J.

Notice that when we do row operations on the augmented matrix of a homogeneous system of linear equations the last column of the matrix is all zeros. Any one of the three allowable row operations will convert zeros to zeros and thus, the final column of the matrix in reduced row-echelon form will also be all zeros. So in this case, we may be as likely to reference only the coefficient matrix and presume that we remember that the final column begins with zeros, and after any number of row operations is still zero.

Example HISAD suggests the following theorem.

Theorem HMVEI Homogeneous, More Variables than Equations, Infinite solutions
Suppose that a homogeneous system of linear equations has m equations and n variables with $n > m$. Then the system has infinitely many solutions.

Proof. We are assuming the system is homogeneous, so Theorem HSC says it is consistent. Then the hypothesis that $n > m$, together with Theorem CMVEI, gives infinitely many solutions. ■

Example HUSAB and Example HISAA are concerned with homogeneous systems where $n = m$ and expose a fundamental distinction between the two examples. One has a unique solution, while the other has infinitely many. These are exactly the only two possibilities for a homogeneous system and illustrate that each is possible (unlike the case when $n > m$ where Theorem HMVEI tells us that there is only one possibility for a homogeneous system).

Subsection NSM
Null Space of a Matrix

The set of solutions to a homogeneous system (which by Theorem HSC is never empty) is of enough interest to warrant its own name. However, we define it as a property of the coefficient matrix, not as a property of some system of equations.

Definition NSM Null Space of a Matrix
The **null space** of a matrix A, denoted $\mathcal{N}(A)$, is the set of all the vectors that are solutions to the homogeneous system $\mathcal{LS}(A, \mathbf{0})$. □

In the Archetypes (Archetypes) each example that is a system of equations also has a corresponding homogeneous system of equations listed, and several sample solutions are given. These solutions will be elements of the null space of the coefficient matrix. We will look at one example.

Example NSEAI Null space elements of Archetype I
The write-up for Archetype I lists several solutions of the corresponding homogeneous system. Here are two, written as solution vectors. We can say that they are in the

null space of the coefficient matrix for the system of equations in Archetype I.

$$\mathbf{x} = \begin{bmatrix} 3 \\ 0 \\ -5 \\ -6 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} -4 \\ 1 \\ -3 \\ -2 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

However, the vector

$$\mathbf{z} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

is not in the null space, since it is not a solution to the homogeneous system. For example, it fails to even make the first equation true. △

Here are two (prototypical) examples of the computation of the null space of a matrix.

Example CNS1 Computing a null space, no. 1
Let us compute the null space of

$$A = \begin{bmatrix} 2 & -1 & 7 & -3 & -8 \\ 1 & 0 & 2 & 4 & 9 \\ 2 & 2 & -2 & -1 & 8 \end{bmatrix}$$

which we write as $\mathcal{N}(A)$. Translating Definition NSM, we simply desire to solve the homogeneous system $\mathcal{LS}(A, \mathbf{0})$. So we row-reduce the augmented matrix to obtain

$$\begin{bmatrix} \boxed{1} & 0 & 2 & 0 & 1 & 0 \\ 0 & \boxed{1} & -3 & 0 & 4 & 0 \\ 0 & 0 & 0 & \boxed{1} & 2 & 0 \end{bmatrix}$$

The variables (of the homogeneous system) x_3 and x_5 are free (since columns 1, 2 and 4 are pivot columns), so we arrange the equations represented by the matrix in reduced row-echelon form to

$$\begin{aligned} x_1 &= -2x_3 - x_5 \\ x_2 &= 3x_3 - 4x_5 \\ x_4 &= -2x_5 \end{aligned}$$

So we can write the infinite solution set as sets using column vectors,

$$\mathcal{N}(A) = \left\{ \left. \begin{bmatrix} -2x_3 - x_5 \\ 3x_3 - 4x_5 \\ x_3 \\ -2x_5 \\ x_5 \end{bmatrix} \right| x_3,\, x_5 \in \mathbb{C} \right\}$$

△

Example CNS2 Computing a null space, no. 2

Let us compute the null space of

$$C = \begin{bmatrix} -4 & 6 & 1 \\ -1 & 4 & 1 \\ 5 & 6 & 7 \\ 4 & 7 & 1 \end{bmatrix}$$

which we write as $\mathcal{N}(C)$. Translating Definition NSM, we simply desire to solve the homogeneous system $\mathcal{LS}(C, \mathbf{0})$. So we row-reduce the augmented matrix to obtain

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are no free variables in the homogeneous system represented by the row-reduced matrix, so there is only the trivial solution, the zero vector, $\mathbf{0}$. So we can write the (trivial) solution set as

$$\mathcal{N}(C) = \{\mathbf{0}\} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Reading Questions

1. What is *always* true of the solution set for a homogeneous system of equations?
2. Suppose a homogeneous system of equations has 13 variables and 8 equations. How many solutions will it have? Why?
3. Describe, using only words, the null space of a matrix. (So in particular, do not use any symbols.)

Exercises

C10 Each Archetype (Archetypes) that is a system of equations has a corresponding homogeneous system with the same coefficient matrix. Compute the set of solutions for each. Notice that these solution sets are the null spaces of the coefficient matrices.

Archetype A, Archetype B, Archetype C, Archetype D/Archetype E, Archetype F, Archetype G/Archetype H, Archetype I, Archetype J

C20 Archetype K and Archetype L are simply 5×5 matrices (i.e. they are not systems of equations). Compute the null space of each matrix.

For Exercises C21-C23, solve the given homogeneous linear system. Compare your results to the results of the corresponding exercise in Section TSS.

C21†

$$\begin{aligned} x_1 + 4x_2 + 3x_3 - x_4 &= 0 \\ x_1 - x_2 + x_3 + 2x_4 &= 0 \\ 4x_1 + x_2 + 6x_3 + 5x_4 &= 0 \end{aligned}$$

C22†

$$\begin{aligned} x_1 - 2x_2 + x_3 - x_4 &= 0 \\ 2x_1 - 4x_2 + x_3 + x_4 &= 0 \\ x_1 - 2x_2 - 2x_3 + 3x_4 &= 0 \end{aligned}$$

C23†

$$\begin{aligned} x_1 - 2x_2 + x_3 - x_4 &= 0 \\ x_1 + x_2 + x_3 - x_4 &= 0 \end{aligned}$$

$$x_1 \quad + x_3 - x_4 = 0$$

For Exercises C25-C27, solve the given homogeneous linear system. Compare your results to the results of the corresponding exercise in Section TSS.

C25†

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ 2x_1 - x_2 + x_3 &= 0 \\ 3x_1 + x_2 + x_3 &= 0 \\ x_2 + 2x_3 &= 0 \end{aligned}$$

C26†

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ 2x_1 - x_2 + x_3 &= 0 \\ 3x_1 + x_2 + x_3 &= 0 \\ 5x_2 + 2x_3 &= 0 \end{aligned}$$

C27†

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ 2x_1 - x_2 + x_3 &= 0 \\ x_1 - 8x_2 - 7x_3 &= 0 \\ x_2 + x_3 &= 0 \end{aligned}$$

C30† Compute the null space of the matrix A, $\mathcal{N}(A)$.

$$A = \begin{bmatrix} 2 & 4 & 1 & 3 & 8 \\ -1 & -2 & -1 & -1 & 1 \\ 2 & 4 & 0 & -3 & 4 \\ 2 & 4 & -1 & -7 & 4 \end{bmatrix}$$

C31† Find the null space of the matrix B, $\mathcal{N}(B)$.

$$B = \begin{bmatrix} -6 & 4 & -36 & 6 \\ 2 & -1 & 10 & -1 \\ -3 & 2 & -18 & 3 \end{bmatrix}$$

M45 Without doing any computations, and without examining any solutions, say as much as possible about the form of the solution set for corresponding homogeneous system of equations of each archetype that is a system of equations.

Archetype A, Archetype B, Archetype C, Archetype D/Archetype E, Archetype F, Archetype G/Archetype H, Archetype I, Archetype J

For Exercises M50–M52 say **as much as possible** about each system's solution set. Be sure to make it clear which theorems you are using to reach your conclusions.

M50† A homogeneous system of 8 equations in 8 variables.

M51† A homogeneous system of 8 equations in 9 variables.

M52† A homogeneous system of 8 equations in 7 variables.

T10† Prove or disprove: A system of linear equations is homogeneous if and only if the system has the zero vector as a solution.

T11† Suppose that two systems of linear equations are equivalent. Prove that if the first system is homogeneous, then the second system is homogeneous. Notice that this will allow us to conclude that two equivalent systems are either both homogeneous or both not homogeneous.

T12 Give an alternate proof of Theorem HSC that uses Theorem RCLS.

T20† Consider the homogeneous system of linear equations $\mathcal{LS}(A,\,\mathbf{0})$, and suppose that $\mathbf{u}=\begin{bmatrix}u_1\\u_2\\u_3\\\vdots\\u_n\end{bmatrix}$ is one solution to the system of equations. Prove that $\mathbf{v}=\begin{bmatrix}4u_1\\4u_2\\4u_3\\\vdots\\4u_n\end{bmatrix}$ is also a solution to $\mathcal{LS}(A,\,\mathbf{0})$.

Section NM
Nonsingular Matrices

In this section we specialize further and consider matrices with equal numbers of rows and columns, which when considered as coefficient matrices lead to systems with equal numbers of equations and variables. We will see in the second half of the course (Chapter D, Chapter E, Chapter LT, Chapter R) that these matrices are especially important.

Subsection NM
Nonsingular Matrices

Our theorems will now establish connections between systems of equations (homogeneous or otherwise), augmented matrices representing those systems, coefficient matrices, constant vectors, the reduced row-echelon form of matrices (augmented and coefficient) and solution sets. Be very careful in your reading, writing and speaking about systems of equations, matrices and sets of vectors. A system of equations is not a matrix, a matrix is not a solution set, and a solution set is not a system of equations. Now would be a great time to review the discussion about speaking and writing mathematics in Proof Technique L.

Definition SQM Square Matrix
A matrix with m rows and n columns is **square** if $m = n$. In this case, we say the matrix has **size** n. To emphasize the situation when a matrix is not square, we will call it **rectangular**. □

We can now present one of the central definitions of linear algebra.

Definition NM Nonsingular Matrix
Suppose A is a square matrix. Suppose further that the solution set to the homogeneous linear system of equations $\mathcal{LS}(A, \mathbf{0})$ is $\{\mathbf{0}\}$, in other words, the system has *only* the trivial solution. Then we say that A is a **nonsingular** matrix. Otherwise we say A is a **singular** matrix. □

We can investigate whether any square matrix is nonsingular or not, no matter if the matrix is derived somehow from a system of equations or if it is simply a matrix. The definition says that to perform this investigation we must construct a very specific system of equations (homogeneous, with the matrix as the coefficient matrix) and look at its solution set. We will have theorems in this section that connect nonsingular matrices with systems of equations, creating more opportunities for confusion. Convince yourself now of two observations, (1) we can decide nonsingularity for any square matrix, and (2) the determination of nonsingularity involves the solution set for a certain homogeneous system of equations.

Notice that it makes no sense to call a system of equations nonsingular (the term does not apply to a system of equations), nor does it make any sense to call a 5×7 matrix singular (the matrix is not square).

Example S A singular matrix, Archetype A
Example HISAA shows that the coefficient matrix derived from Archetype A, specifically the 3×3 matrix,

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

is a singular matrix since there are nontrivial solutions to the homogeneous system $\mathcal{LS}(A, \mathbf{0})$. △

Example NM A nonsingular matrix, Archetype B
Example HUSAB shows that the coefficient matrix derived from Archetype B, specifically the 3×3 matrix,

$$B = \begin{bmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{bmatrix}$$

is a nonsingular matrix since the homogeneous system, $\mathcal{LS}(B, \mathbf{0})$, has only the trivial solution. △

Notice that we will not discuss Example HISAD as being a singular or nonsingular coefficient matrix since the matrix is not square.

The next theorem combines with our main computational technique (row reducing a matrix) to make it easy to recognize a nonsingular matrix. But first a definition.

Definition IM Identity Matrix
The $m \times m$ **identity matrix**, I_m, is defined by

$$[I_m]_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \qquad 1 \leq i, j \leq m$$

□

Example IM An identity matrix
The 4×4 identity matrix is

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

△

Notice that an identity matrix is square, and in reduced row-echelon form. Also, every column is a pivot column, and every possible pivot column appears once.

Theorem NMRRI Nonsingular Matrices Row Reduce to the Identity matrix
Suppose that A is a square matrix and B is a row-equivalent matrix in reduced row-echelon form. Then A is nonsingular if and only if B is the identity matrix.

Proof. ($\Leftarrow$) Suppose B is the identity matrix. When the augmented matrix $[A \mid \mathbf{0}]$ is row-reduced, the result is $[B \mid \mathbf{0}] = [I_n \mid \mathbf{0}]$. The number of nonzero rows is equal to the number of variables in the linear system of equations $\mathcal{LS}(A, \mathbf{0})$, so $n = r$ and Theorem FVCS gives $n - r = 0$ free variables. Thus, the homogeneous system $\mathcal{LS}(A, \mathbf{0})$ has just one solution, which must be the trivial solution. This is exactly the definition of a nonsingular matrix (Definition NM).

($\Rightarrow$) If A is nonsingular, then the homogeneous system $\mathcal{LS}(A, \mathbf{0})$ has a unique solution, and has no free variables in the description of the solution set. The homogeneous system is consistent (Theorem HSC) so Theorem FVCS applies and tells us there are $n - r$ free variables. Thus, $n - r = 0$, and so $n = r$. So B has n pivot columns among its total of n columns. This is enough to force B to be the $n \times n$ identity matrix I_n (see Exercise NM.T12). ■

Notice that since this theorem is an equivalence it will always allow us to determine if a matrix is either nonsingular or singular. Here are two examples of this, continuing our study of Archetype A and Archetype B.

Example SRR Singular matrix, row-reduced
We have the coefficient matrix for Archetype A and a row-equivalent matrix B in

reduced row-echelon form,

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & -1 \\ 0 & 0 & 0 \end{bmatrix} = B$$

Since B is not the 3×3 identity matrix, Theorem NMRRI tells us that A is a singular matrix. △

Example NSR Nonsingular matrix, row-reduced
We have the coefficient matrix for Archetype B and a row-equivalent matrix B in reduced row-echelon form,

$$A = \begin{bmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix} = B$$

Since B is the 3×3 identity matrix, Theorem NMRRI tells us that A is a nonsingular matrix. △

Subsection NSNM
Null Space of a Nonsingular Matrix

Nonsingular matrices and their null spaces are intimately related, as the next two examples illustrate.

Example NSS Null space of a singular matrix
Given the singular coefficient matrix from Archetype A, the null space is the set of solutions to the homogeneous system of equations $\mathcal{LS}(A, \mathbf{0})$ has a solution set and null space constructed in Example HISAA as an infinite set of vectors.

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad \mathcal{N}(A) = \left\{ \begin{bmatrix} -x_3 \\ x_3 \\ x_3 \end{bmatrix} \middle|\, x_3 \in \mathbb{C} \right\}$$

△

Example NSNM Null space of a nonsingular matrix
Given the nonsingular coefficient matrix from Archetype B, the solution set to the homogeneous system $\mathcal{LS}(A, \mathbf{0})$ is constructed in Example HUSAB and contains only the trivial solution, so the null space of A has only a single element,

$$A = \begin{bmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{bmatrix} \qquad \mathcal{N}(A) = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

△

These two examples illustrate the next theorem, which is another equivalence.

Theorem NMTNS Nonsingular Matrices have Trivial Null Spaces
Suppose that A is a square matrix. Then A is nonsingular if and only if the null space of A is the set containing only the zero vector, i.e. $\mathcal{N}(A) = \{\mathbf{0}\}$.

Proof. The null space of a square *matrix*, A, is equal to the set of solutions to the homogeneous *system*, $\mathcal{LS}(A, \mathbf{0})$. A *matrix* is nonsingular if and only if the set of solutions to the homogeneous *system*, $\mathcal{LS}(A, \mathbf{0})$, has only a trivial solution. These two observations may be chained together to construct the two proofs necessary for each half of this theorem. ■

The next theorem pulls a lot of big ideas together. Theorem NMUS tells us that we can learn much about solutions to a system of linear equations with a square coefficient matrix by just examining a similar homogeneous system.

Theorem NMUS Nonsingular Matrices and Unique Solutions
Suppose that A is a square matrix. A is a nonsingular matrix if and only if the system $\mathcal{LS}(A,\,\mathbf{b})$ has a unique solution for every choice of the constant vector $\mathbf{b}$.

Proof. ($\Leftarrow$) The hypothesis for this half of the proof is that the system $\mathcal{LS}(A,\,\mathbf{b})$ has a unique solution for *every* choice of the constant vector $\mathbf{b}$. We will make a very specific choice for $\mathbf{b}$: $\mathbf{b} = \mathbf{0}$. Then we know that the system $\mathcal{LS}(A,\,\mathbf{0})$ has a unique solution. But this is precisely the definition of what it means for A to be nonsingular (Definition NM). That almost seems too easy! Notice that we have not used the full power of our hypothesis, but there is nothing that says we must use a hypothesis to its fullest.

($\Rightarrow$) We assume that A is nonsingular of size $n \times n$, so we know there is a sequence of row operations that will convert A into the identity matrix I_n (Theorem NMRRI). Form the augmented matrix $A' = [\,A \,|\, \mathbf{b}]$ and apply this same sequence of row operations to A'. The result will be the matrix $B' = [\,I_n \,|\, \mathbf{c}]$, which is in reduced row-echelon form with $r = n$. Then the augmented matrix B' represents the (extremely simple) system of equations $x_i = [\mathbf{c}]_i$, $1 \leq i \leq n$. The vector $\mathbf{c}$ is clearly a solution, so the system is consistent (Definition CS). With a consistent system, we use Theorem FVCS to count free variables. We find that there are $n - r = n - n = 0$ free variables, and so we therefore know that the solution is unique. (This half of the proof was suggested by Asa Scherer.) ■

This theorem helps to explain part of our interest in nonsingular matrices. If a matrix is nonsingular, then no matter what vector of constants we pair it with, using the matrix as the coefficient matrix will *always* yield a linear system of equations with a solution, and the solution is unique. To determine if a matrix has this property (nonsingularity) it is enough to just solve one linear system, the homogeneous system with the matrix as coefficient matrix and the zero vector as the vector of constants (or any other vector of constants, see Exercise MM.T10).

Formulating the negation of the second part of this theorem is a good exercise. A singular matrix has the property that for *some* value of the vector $\mathbf{b}$, the system $\mathcal{LS}(A,\,\mathbf{b})$ does not have a unique solution (which means that it has no solution or infinitely many solutions). We will be able to say more about this case later (see the discussion following Theorem PSPHS).

Square matrices that are nonsingular have a long list of interesting properties, which we will start to catalog in the following, recurring, theorem. Of course, singular matrices will then have all of the opposite properties. The following theorem is a list of equivalences.

We want to understand just what is involved with understanding and proving a theorem that says several conditions are equivalent. So have a look at Proof Technique ME before studying the first in this series of theorems.

Theorem NME1 Nonsingular Matrix Equivalences, Round 1
Suppose that A is a square matrix. The following are equivalent.

1. *A is nonsingular.*
2. *A row-reduces to the identity matrix.*
3. *The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.*
4. *The linear system $\mathcal{LS}(A,\,\mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.*

Proof. The statement that A is nonsingular is equivalent to each of the subsequent statements by, in turn, Theorem NMRRI, Theorem NMTNS and Theorem NMUS. So the statement of this theorem is just a convenient way to organize all these results. ■

Finally, you may have wondered why we refer to a matrix as *nonsingular* when it creates systems of equations with *single* solutions (Theorem NMUS)! I have wondered the same thing. We will have an opportunity to address this when we get to Theorem SMZD. Can you wait that long?

Reading Questions

1. In your own words state the definition of a nonsingular matrix.
2. What is the *easiest* way to recognize if a square matrix is nonsingular or not?
3. Suppose we have a system of equations and its coefficient matrix is nonsingular. What can you say about the solution set for this system?

Exercises

In Exercises C30–C33 determine if the matrix is nonsingular or singular. Give reasons for your answer.

C30†

$$\begin{bmatrix} -3 & 1 & 2 & 8 \\ 2 & 0 & 3 & 4 \\ 1 & 2 & 7 & -4 \\ 5 & -1 & 2 & 0 \end{bmatrix}$$

C31†

$$\begin{bmatrix} 2 & 3 & 1 & 4 \\ 1 & 1 & 1 & 0 \\ -1 & 2 & 3 & 5 \\ 1 & 2 & 1 & 3 \end{bmatrix}$$

C32†

$$\begin{bmatrix} 9 & 3 & 2 & 4 \\ 5 & -6 & 1 & 3 \\ 4 & 1 & 3 & -5 \end{bmatrix}$$

C33†

$$\begin{bmatrix} -1 & 2 & 0 & 3 \\ 1 & -3 & -2 & 4 \\ -2 & 0 & 4 & 3 \\ -3 & 1 & -2 & 3 \end{bmatrix}$$

C40 Each of the archetypes below is a system of equations with a square coefficient matrix, or is itself a square matrix. Determine if these matrices are nonsingular, or singular. Comment on the null space of each matrix.

Archetype A, Archetype B, Archetype F, Archetype K, Archetype L

C50† Find the null space of the matrix E below.

$$E = \begin{bmatrix} 2 & 1 & -1 & -9 \\ 2 & 2 & -6 & -6 \\ 1 & 2 & -8 & 0 \\ -1 & 2 & -12 & 12 \end{bmatrix}$$

M30† Let A be the coefficient matrix of the system of equations below. Is A nonsingular or singular? Explain what you could infer about the solution set for the system based only on what you have learned about A being singular or nonsingular.

$$\begin{aligned} -x_1 + 5x_2 &= -8 \\ -2x_1 + 5x_2 + 5x_3 + 2x_4 &= 9 \\ -3x_1 - x_2 + 3x_3 + x_4 &= 3 \\ 7x_1 + 6x_2 + 5x_3 + x_4 &= 30 \end{aligned}$$

For Exercises M51–M52 say **as much as possible** about each system's solution set. Be sure to make it clear which theorems you are using to reach your conclusions.

M51† 6 equations in 6 variables, singular coefficient matrix.

M52† A system with a nonsingular coefficient matrix, not homogeneous.

T10† Suppose that A is a square matrix, and B is a matrix in reduced row-echelon form that is row-equivalent to A. Prove that if A is singular, then the last row of B is a zero row.

T12 Using (Definition RREF) and (Definition IM) carefully, give a proof of the following equivalence: A is a square matrix in reduced row-echelon form where every column is a pivot column if and only if A is the identity matrix.

T30† Suppose that A is a nonsingular matrix and A is row-equivalent to the matrix B. Prove that B is nonsingular.

T31† Suppose that A is a square matrix of size $n \times n$ and that we know there is a *single* vector $\mathbf{b} \in \mathbb{C}^n$ such that the system $\mathcal{LS}(A,\, \mathbf{b})$ has a unique solution. Prove that A is a nonsingular matrix. (Notice that this is very similar to Theorem NMUS, but is not exactly the same.)

T90† Provide an alternative for the second half of the proof of Theorem NMUS, without appealing to properties of the reduced row-echelon form of the coefficient matrix. In other words, prove that if A is nonsingular, then $\mathcal{LS}(A,\, \mathbf{b})$ has a unique solution for every choice of the constant vector $\mathbf{b}$. Construct this proof without using Theorem REMEF or Theorem RREFU.

Chapter V
Vectors

We have worked extensively in the last chapter with matrices, and some with vectors. In this chapter we will develop the properties of vectors, while preparing to study vector spaces (Chapter VS). Initially we will depart from our study of systems of linear equations, but in Section LC we will forge a connection between linear combinations and systems of linear equations in Theorem SLSLC. This connection will allow us to understand systems of linear equations at a higher level, while consequently discussing them less frequently.

Section VO
Vector Operations

In this section we define some new operations involving vectors, and collect some basic properties of these operations. Begin by recalling our definition of a column vector as an ordered list of complex numbers, written vertically (Definition CV). The collection of all possible vectors of a fixed size is a commonly used set, so we start with its definition.

Subsection CV
Column Vectors

Definition VSCV Vector Space of Column Vectors
The vector space $\mathbb{C}^m$ is the set of all column vectors (Definition CV) of size m with entries from the set of complex numbers, $\mathbb{C}$. □

When a set similar to this is defined using only column vectors where all the entries are from the real numbers, it is written as $\mathbb{R}^m$ and is known as **Euclidean m-space**.

The term **vector** is used in a variety of different ways. We have defined it as an ordered list written vertically. It could simply be an ordered list of numbers, and perhaps written as $\langle 2,\,3,\,-1,\,6\rangle$. Or it could be interpreted as a point in m dimensions, such as $(3,\,4,\,-2)$ representing a point in three dimensions relative to x, y and z axes. With an interpretation as a point, we can construct an arrow from the origin to the point which is consistent with the notion that a vector has direction and magnitude.

All of these ideas can be shown to be related and equivalent, so keep that in mind as you connect the ideas of this course with ideas from other disciplines. For now, we will stick with the idea that a vector is just a list of numbers, in some particular order.

Subsection VEASM Vector Equality, Addition, Scalar Multiplication

We start our study of this set by first defining what it means for two vectors to be the same.

Definition CVE Column Vector Equality
Suppose that $\mathbf{u}$, $\mathbf{v} \in \mathbb{C}^m$. Then $\mathbf{u}$ and $\mathbf{v}$ are **equal**, written $\mathbf{u} = \mathbf{v}$ if

$$[\mathbf{u}]_i = [\mathbf{v}]_i \qquad\qquad 1 \le i \le m$$

□

Now this may seem like a silly (or even stupid) thing to say so carefully. Of course two vectors are equal if they are equal for each corresponding entry! Well, this is not as silly as it appears. We will see a few occasions later where the obvious definition is *not* the right one. And besides, in doing mathematics we need to be very careful about making all the necessary definitions and making them unambiguous. And we have done that here.

Notice now that the symbol "=" is now doing triple-duty. We know from our earlier education what it means for two numbers (real or complex) to be equal, and we take this for granted. In Definition SE we defined what it meant for two sets to be equal. Now we have defined what it means for two vectors to be equal, and that definition builds on our definition for when two numbers are equal when we use the condition $u_i = v_i$ for all $1 \le i \le m$. So think carefully about your objects when you see an equal sign and think about just which notion of equality you have encountered. This will be especially important when you are asked to construct proofs whose conclusion states that two objects are equal. If you have an electronic copy of the book, such as the PDF version, searching on "Definition CVE" can be an instructive exercise. See how often, and where, the definition is employed.

OK, let us do an example of vector equality that begins to hint at the utility of this definition.

Example VESE Vector equality for a system of equations
Consider the system of linear equations in Archetype B,

$$\begin{aligned} -7x_1 - 6x_2 - 12x_3 &= -33 \\ 5x_1 + 5x_2 + 7x_3 &= 24 \\ x_1 + 4x_3 &= 5 \end{aligned}$$

Note the use of three equals signs — each indicates an equality of numbers (the linear expressions are numbers when we evaluate them with fixed values of the variable quantities). Now write the vector equality,

$$\begin{bmatrix} -7x_1 - 6x_2 - 12x_3 \\ 5x_1 + 5x_2 + 7x_3 \\ x_1 + 4x_3 \end{bmatrix} = \begin{bmatrix} -33 \\ 24 \\ 5 \end{bmatrix}.$$

By Definition CVE, this *single* equality (of two column vectors) translates into *three* simultaneous equalities of numbers that form the system of equations. So with this new notion of vector equality we can become less reliant on referring to *systems* of *simultaneous* equations. There is more to vector equality than just this, but this is a good example for starters and we will develop it further. △

We will now define two operations on the set $\mathbb{C}^m$. By this we mean well-defined procedures that somehow convert vectors into other vectors. Here are two of the most basic definitions of the entire course.

Definition CVA Column Vector Addition

Suppose that $\mathbf{u}$, $\mathbf{v} \in \mathbb{C}^m$. The **sum** of $\mathbf{u}$ and $\mathbf{v}$ is the vector $\mathbf{u}+\mathbf{v}$ defined by

$$[\mathbf{u}+\mathbf{v}]_i = [\mathbf{u}]_i + [\mathbf{v}]_i \qquad 1 \leq i \leq m$$

□

So vector addition takes two vectors of the same size and combines them (in a natural way!) to create a new vector of the same size. Notice that this definition is required, even if we agree that this is the obvious, right, natural or correct way to do it. Notice too that the symbol '+' is being recycled. We all know how to add *numbers*, but now we have the same symbol extended to double-duty and we use it to indicate how to add two new objects, vectors. And this definition of our new meaning is built on our previous meaning of addition via the expressions $u_i + v_i$. Think about your objects, especially when doing proofs. Vector addition is easy, here is an example from $\mathbb{C}^4$.

Example VA Addition of two vectors in $\mathbb{C}^4$
If

$$\mathbf{u} = \begin{bmatrix} 2 \\ -3 \\ 4 \\ 2 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} -1 \\ 5 \\ 2 \\ -7 \end{bmatrix}$$

then

$$\mathbf{u}+\mathbf{v} = \begin{bmatrix} 2 \\ -3 \\ 4 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 5 \\ 2 \\ -7 \end{bmatrix} = \begin{bmatrix} 2+(-1) \\ -3+5 \\ 4+2 \\ 2+(-7) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 6 \\ -5 \end{bmatrix}$$

△

Our second operation takes two objects of different types, specifically a number and a vector, and combines them to create another vector. In this context we call a number a **scalar** in order to emphasize that it is not a vector.

Definition CVSM Column Vector Scalar Multiplication
Suppose $\mathbf{u} \in \mathbb{C}^m$ and $\alpha \in \mathbb{C}$, then the **scalar multiple** of $\mathbf{u}$ by α is the vector $\alpha\mathbf{u}$ defined by

$$[\alpha\mathbf{u}]_i = \alpha\,[\mathbf{u}]_i \qquad 1 \leq i \leq m$$

□

Notice that we are doing a kind of multiplication here, but we are *defining* a new type, perhaps in what appears to be a natural way. We use juxtaposition (smashing two symbols together side-by-side) to denote this operation rather than using a symbol like we did with vector addition. So this can be another source of confusion. When two symbols are next to each other, are we doing regular old multiplication, the kind we have done for years, or are we doing scalar vector multiplication, the operation we just defined? Think about your objects — if the first object is a scalar, and the second is a vector, then it *must* be that we are doing our new operation, and the *result* of this operation will be another vector.

Notice how consistency in notation can be an aid here. If we write scalars as lower case Greek letters from the start of the alphabet (such as α, β, …) and write vectors in bold Latin letters from the end of the alphabet ($\mathbf{u}$, $\mathbf{v}$, …), then we have some hints about what type of objects we are working with. This can be a blessing *and* a curse, since when we go read another book about linear algebra, or read an application in another discipline (physics, economics, …) the types of notation employed may be very different and hence unfamiliar.

Again, computationally, vector scalar multiplication is very easy.

Example CVSM Scalar multiplication in $\mathbb{C}^5$
If

$$\mathbf{u} = \begin{bmatrix} 3 \\ 1 \\ -2 \\ 4 \\ -1 \end{bmatrix}$$

and $\alpha = 6$, then

$$\alpha\mathbf{u} = 6 \begin{bmatrix} 3 \\ 1 \\ -2 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 6(3) \\ 6(1) \\ 6(-2) \\ 6(4) \\ 6(-1) \end{bmatrix} = \begin{bmatrix} 18 \\ 6 \\ -12 \\ 24 \\ -6 \end{bmatrix}.$$

△

Subsection VSP
Vector Space Properties

With definitions of vector addition and scalar multiplication we can state, and prove, several properties of each operation, and some properties that involve their interplay. We now collect ten of them here for later reference.

Theorem VSPCV Vector Space Properties of Column Vectors
Suppose that $\mathbb{C}^m$ is the set of column vectors of size m (Definition VSCV) with addition and scalar multiplication as defined in Definition CVA and Definition CVSM. Then

- ACC Additive Closure, Column Vectors

  If $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$, then $\mathbf{u} + \mathbf{v} \in \mathbb{C}^m$.

- SCC Scalar Closure, Column Vectors

  If $\alpha \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $\alpha\mathbf{u} \in \mathbb{C}^m$.

- CC Commutativity, Column Vectors

  If $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$, then $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

- AAC Additive Associativity, Column Vectors

  If $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{C}^m$, then $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.

- ZC Zero Vector, Column Vectors

  There is a vector, $\mathbf{0}$, called the zero vector, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in \mathbb{C}^m$.

- AIC Additive Inverses, Column Vectors

  If $\mathbf{u} \in \mathbb{C}^m$, then there exists a vector $-\mathbf{u} \in \mathbb{C}^m$ so that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.

- SMAC Scalar Multiplication Associativity, Column Vectors

  If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $\alpha(\beta\mathbf{u}) = (\alpha\beta)\mathbf{u}$.

- DVAC Distributivity across Vector Addition, Column Vectors

  If $\alpha \in \mathbb{C}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$, then $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$.

- DSAC Distributivity across Scalar Addition, Column Vectors

  If $\alpha, \beta \in \mathbb{C}$ and $\mathbf{u} \in \mathbb{C}^m$, then $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$.

- OC One, Column Vectors

  If $\mathbf{u} \in \mathbb{C}^m$*, then* $1\mathbf{u} = \mathbf{u}$*.*

Proof. While some of these properties seem very obvious, they all require proof. However, the proofs are not very interesting, and border on tedious. We will prove one version of distributivity very carefully, and you can test your proof-building skills on some of the others. We need to establish an equality, so we will do so by beginning with one side of the equality, apply various definitions and theorems (listed to the right of each step) to massage the expression from the left into the expression on the right. Here we go with a proof of Property DSAC.

For $1 \leq i \leq m$,

$$\begin{aligned}
[(\alpha+\beta)\mathbf{u}]_i &= (\alpha+\beta)\,[\mathbf{u}]_i && \text{Definition CVSM}\\
&= \alpha\,[\mathbf{u}]_i + \beta\,[\mathbf{u}]_i && \text{Property DCN}\\
&= [\alpha\mathbf{u}]_i + [\beta\mathbf{u}]_i && \text{Definition CVSM}\\
&= [\alpha\mathbf{u} + \beta\mathbf{u}]_i && \text{Definition CVA}
\end{aligned}$$

Since the individual components of the vectors $(\alpha+\beta)\mathbf{u}$ and $\alpha\mathbf{u}+\beta\mathbf{u}$ are equal for *all* i, $1 \leq i \leq m$, Definition CVE tells us the vectors are equal. ∎

Many of the conclusions of our theorems can be characterized as "identities," especially when we are establishing basic properties of operations such as those in this section. Most of the properties listed in Theorem VSPCV are examples. So some advice about the style we use for proving identities is appropriate right now. Have a look at Proof Technique PI.

Be careful with the notion of the vector $-\mathbf{u}$. This is a vector that we add to $\mathbf{u}$ so that the result is the particular vector $\mathbf{0}$. This is basically a property of vector addition. It happens that we can compute $-\mathbf{u}$ using the *other* operation, scalar multiplication. We can prove this directly by writing that

$$[-\mathbf{u}]_i = -\,[\mathbf{u}]_i = (-1)\,[\mathbf{u}]_i = [(-1)\mathbf{u}]_i$$

We will see later how to derive this property as a *consequence* of several of the ten properties listed in Theorem VSPCV.

Similarly, we will often write something you would immediately recognize as "vector subtraction." This could be placed on a firm theoretical foundation — as you can do yourself with Exercise VO.T30.

A final note. Property AAC implies that we do not have to be careful about how we "parenthesize" the addition of vectors. In other words, there is nothing to be gained by writing $(\mathbf{u}+\mathbf{v})+(\mathbf{w}+(\mathbf{x}+\mathbf{y}))$ rather than $\mathbf{u}+\mathbf{v}+\mathbf{w}+\mathbf{x}+\mathbf{y}$, since we get the same result no matter which order we choose to perform the four additions. So we will not be careful about using parentheses this way.

Reading Questions

1. Where have you seen vectors used before in other courses? How were they different?
2. In words only, when are two vectors equal?
3. Perform the following computation with vector operations

$$2\begin{bmatrix}1\\5\\0\end{bmatrix} + (-3)\begin{bmatrix}7\\6\\5\end{bmatrix}$$

Exercises

C10† Compute

$$4\begin{bmatrix}2\\-3\\4\\1\\0\end{bmatrix}+(-2)\begin{bmatrix}1\\2\\-5\\2\\4\end{bmatrix}+\begin{bmatrix}-1\\3\\0\\1\\2\end{bmatrix}$$

C11† Solve the given vector equation for x, or explain why no solution exists:

$$3\begin{bmatrix}1\\2\\-1\end{bmatrix}+4\begin{bmatrix}2\\0\\x\end{bmatrix}=\begin{bmatrix}11\\6\\17\end{bmatrix}$$

C12† Solve the given vector equation for α, or explain why no solution exists:

$$\alpha\begin{bmatrix}1\\2\\-1\end{bmatrix}+4\begin{bmatrix}3\\4\\2\end{bmatrix}=\begin{bmatrix}-1\\0\\4\end{bmatrix}$$

C13† Solve the given vector equation for α, or explain why no solution exists:

$$\alpha\begin{bmatrix}3\\2\\-2\end{bmatrix}+\begin{bmatrix}6\\1\\2\end{bmatrix}=\begin{bmatrix}0\\-3\\6\end{bmatrix}$$

C14† Find α and β that solve the vector equation.

$$\alpha\begin{bmatrix}1\\0\end{bmatrix}+\beta\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}3\\2\end{bmatrix}$$

C15† Find α and β that solve the vector equation.

$$\alpha\begin{bmatrix}2\\1\end{bmatrix}+\beta\begin{bmatrix}1\\3\end{bmatrix}=\begin{bmatrix}5\\0\end{bmatrix}$$

T05† Provide reasons (mostly vector space properties) as justification for each of the seven steps of the following proof.

Theorem For any vectors $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}\in\mathbb{C}^m$, if $\mathbf{u}+\mathbf{v}=\mathbf{u}+\mathbf{w}$, then $\mathbf{v}=\mathbf{w}$.

Proof: Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}\in\mathbb{C}^m$, and suppose $\mathbf{u}+\mathbf{v}=\mathbf{u}+\mathbf{w}$.

$$\begin{aligned}\mathbf{v}&=\mathbf{0}+\mathbf{v}\\&=(-\mathbf{u}+\mathbf{u})+\mathbf{v}\\&=-\mathbf{u}+(\mathbf{u}+\mathbf{v})\\&=-\mathbf{u}+(\mathbf{u}+\mathbf{w})\\&=(-\mathbf{u}+\mathbf{u})+\mathbf{w}\\&=\mathbf{0}+\mathbf{w}\\&=\mathbf{w}\end{aligned}$$

T06† Provide reasons (mostly vector space properties) as justification for each of the six steps of the following proof.

Theorem For any vector $\mathbf{u}\in\mathbb{C}^m$, $0\mathbf{u}=\mathbf{0}$.

Proof: Let $\mathbf{u}\in\mathbb{C}^m$.

$$\begin{aligned}\mathbf{0}&=0\mathbf{u}+(-0\mathbf{u})\\&=(0+0)\mathbf{u}+(-0\mathbf{u})\end{aligned}$$

$$\begin{aligned}&= (0\mathbf{u} + 0\mathbf{u}) + (-0\mathbf{u})\\&= 0\mathbf{u} + (0\mathbf{u} + (-0\mathbf{u}))\\&= 0\mathbf{u} + \mathbf{0}\\&= 0\mathbf{u}\end{aligned}$$

T07† Provide reasons (mostly vector space properties) as justification for each of the six steps of the following proof.

Theorem For any scalar c, $c\,\mathbf{0} = \mathbf{0}$.

Proof: Let c be an arbitrary scalar.

$$\begin{aligned}\mathbf{0} &= c\mathbf{0} + (-c\mathbf{0})\\&= c(\mathbf{0} + \mathbf{0}) + (-c\mathbf{0})\\&= (c\mathbf{0} + c\mathbf{0}) + (-c\mathbf{0})\\&= c\mathbf{0} + (c\mathbf{0} + (-c\mathbf{0}))\\&= c\mathbf{0} + \mathbf{0}\\&= c\mathbf{0}\end{aligned}$$

T13† Prove Property CC of Theorem VSPCV. Write your proof in the style of the proof of Property DSAC given in this section.

T17 Prove Property SMAC of Theorem VSPCV. Write your proof in the style of the proof of Property DSAC given in this section.

T18 Prove Property DVAC of Theorem VSPCV. Write your proof in the style of the proof of Property DSAC given in this section.

Exercises T30, T31 and T32 are about making a careful definition of "vector subtraction".

T30 Suppose $\mathbf{u}$ and $\mathbf{v}$ are two vectors in $\mathbb{C}^m$. Define a new operation, called "subtraction," as the new vector denoted $\mathbf{u} - \mathbf{v}$ and defined by

$$[\mathbf{u} - \mathbf{v}]_i = [\mathbf{u}]_i - [\mathbf{v}]_i \qquad 1 \leq i \leq m$$

Prove that we can express the subtraction of two vectors in terms of our two basic operations. More precisely, prove that $\mathbf{u} - \mathbf{v} = \mathbf{u} + (-1)\mathbf{v}$. So in a sense, subtraction is not something new and different, but is just a convenience. Mimic the style of similar proofs in this section.

T31 Prove, by giving counterexamples, that vector subtraction is not commutative and not associative.

T32 Prove that vector subtraction obeys a distributive property. Specifically, prove that $\alpha\,(\mathbf{u} - \mathbf{v}) = \alpha\mathbf{u} - \alpha\mathbf{v}$.
Can you give two different proofs? Distinguish your two proofs by using the alternate descriptions of vector subtraction provided by Exercise VO.T30.

Section LC
Linear Combinations

In Section VO we defined vector addition and scalar multiplication. These two operations combine nicely to give us a construction known as a **linear combination**, a construct that we will work with throughout this course.

Subsection LC
Linear Combinations

Definition LCCV Linear Combination of Column Vectors
Given n vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n$ from $\mathbb{C}^m$ and n scalars $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$, their **linear combination** is the vector

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_n\mathbf{u}_n$$

□

So this definition takes an equal number of scalars and vectors, combines them using our two new operations (scalar multiplication and vector addition) and creates a single brand-new vector, of the same size as the original vectors. When a definition or theorem employs a linear combination, think about the nature of the objects that go into its creation (lists of scalars and vectors), and the type of object that results (a single vector). Computationally, a linear combination is pretty easy.

Example TLC Two linear combinations in $\mathbb{C}^6$
Suppose that

$$\alpha_1 = 1 \qquad \alpha_2 = -4 \qquad \alpha_3 = 2 \qquad \alpha_4 = -1$$

and

$$\mathbf{u}_1 = \begin{bmatrix} 2\\4\\-3\\1\\2\\9 \end{bmatrix} \qquad \mathbf{u}_2 = \begin{bmatrix} 6\\3\\0\\-2\\1\\4 \end{bmatrix} \qquad \mathbf{u}_3 = \begin{bmatrix} -5\\2\\1\\1\\-3\\0 \end{bmatrix} \qquad \mathbf{u}_4 = \begin{bmatrix} 3\\2\\-5\\7\\1\\3 \end{bmatrix}$$

then their linear combination is

$$\alpha_1\mathbf{u_1} + \alpha_2\mathbf{u_2} + \alpha_3\mathbf{u_3} + \alpha_4\mathbf{u_4} = (1)\begin{bmatrix} 2\\4\\-3\\1\\2\\9 \end{bmatrix} + (-4)\begin{bmatrix} 6\\3\\0\\-2\\1\\4 \end{bmatrix} + (2)\begin{bmatrix} -5\\2\\1\\1\\-3\\0 \end{bmatrix} + (-1)\begin{bmatrix} 3\\2\\-5\\7\\1\\3 \end{bmatrix}$$

$$= \begin{bmatrix} 2\\4\\-3\\1\\2\\9 \end{bmatrix} + \begin{bmatrix} -24\\-12\\0\\8\\-4\\-16 \end{bmatrix} + \begin{bmatrix} -10\\4\\2\\2\\-6\\0 \end{bmatrix} + \begin{bmatrix} -3\\-2\\5\\-7\\-1\\-3 \end{bmatrix} = \begin{bmatrix} -35\\-6\\4\\4\\-9\\-10 \end{bmatrix}$$

A different linear combination, of the same set of vectors, can be formed with different scalars. Take

$$\beta_1 = 3 \qquad \beta_2 = 0 \qquad \beta_3 = 5 \qquad \beta_4 = -1$$

and form the linear combination

$$\beta_1\mathbf{u_1} + \beta_2\mathbf{u_2} + \beta_3\mathbf{u_3} + \beta_4\mathbf{u_4} = (3)\begin{bmatrix}2\\4\\-3\\1\\2\\9\end{bmatrix} + (0)\begin{bmatrix}6\\3\\0\\-2\\1\\4\end{bmatrix} + (5)\begin{bmatrix}-5\\2\\1\\1\\-3\\0\end{bmatrix} + (-1)\begin{bmatrix}3\\2\\-5\\7\\1\\3\end{bmatrix}$$

$$= \begin{bmatrix}6\\12\\-9\\3\\6\\27\end{bmatrix} + \begin{bmatrix}0\\0\\0\\0\\0\\0\end{bmatrix} + \begin{bmatrix}-25\\10\\5\\5\\-15\\0\end{bmatrix} + \begin{bmatrix}-3\\-2\\5\\-7\\-1\\-3\end{bmatrix} = \begin{bmatrix}-22\\20\\1\\1\\-10\\24\end{bmatrix}$$

Notice how we could keep our set of vectors fixed, and use different sets of scalars to construct different vectors. You might build a few new linear combinations of $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, $\mathbf{u}_4$ right now. We will be right here when you get back. What vectors were you able to create? Do you think you could create the vector $\mathbf{w}$ with a "suitable" choice of four scalars?

$$\mathbf{w} = \begin{bmatrix}13\\15\\5\\-17\\2\\25\end{bmatrix}$$

Do you think you could create *any* possible vector from $\mathbb{C}^6$ by choosing the proper scalars? These last two questions are very fundamental, and time spent considering them *now* will prove beneficial later. △

Our next two examples are key ones, and a discussion about decompositions is timely. Have a look at Proof Technique DC before studying the next two examples.

Example ABLC Archetype B as a linear combination
In this example we will rewrite Archetype B in the language of vectors, vector equality and linear combinations. In Example VESE we wrote the system of $m = 3$ equations as the vector equality

$$\begin{bmatrix}-7x_1 - 6x_2 - 12x_3\\5x_1 + 5x_2 + 7x_3\\x_1 + 4x_3\end{bmatrix} = \begin{bmatrix}-33\\24\\5\end{bmatrix}$$

Now we will bust up the linear expressions on the left, first using vector addition,

$$\begin{bmatrix}-7x_1\\5x_1\\x_1\end{bmatrix} + \begin{bmatrix}-6x_2\\5x_2\\0x_2\end{bmatrix} + \begin{bmatrix}-12x_3\\7x_3\\4x_3\end{bmatrix} = \begin{bmatrix}-33\\24\\5\end{bmatrix}$$

Now we can rewrite each of these vectors as a scalar multiple of a fixed vector, where the scalar is one of the unknown variables, converting the left-hand side into a linear combination

$$x_1\begin{bmatrix}-7\\5\\1\end{bmatrix} + x_2\begin{bmatrix}-6\\5\\0\end{bmatrix} + x_3\begin{bmatrix}-12\\7\\4\end{bmatrix} = \begin{bmatrix}-33\\24\\5\end{bmatrix}$$

We can now interpret the problem of solving the system of equations as determining values for the scalar multiples that make the vector equation true. In the analysis of Archetype B, we were able to determine that it had only one solution. A quick way to see this is to row-reduce the coefficient matrix to the 3×3 identity matrix and apply Theorem NMRRI to determine that the coefficient matrix is nonsingular.

Then Theorem NMUS tells us that the system of equations has a unique solution. This solution is

$$x_1 = -3 \qquad x_2 = 5 \qquad x_3 = 2$$

So, in the context of this example, we can express the fact that these values of the variables are a solution by writing the linear combination,

$$(-3)\begin{bmatrix}-7\\5\\1\end{bmatrix} + (5)\begin{bmatrix}-6\\5\\0\end{bmatrix} + (2)\begin{bmatrix}-12\\7\\4\end{bmatrix} = \begin{bmatrix}-33\\24\\5\end{bmatrix}$$

Furthermore, these are the *only* three scalars that will accomplish this equality, since they come from a unique solution.

Notice how the three vectors in this example are the columns of the coefficient matrix of the system of equations. This is our first hint of the important interplay between the vectors that form the columns of a matrix, and the matrix itself. △

With any discussion of Archetype A or Archetype B we should be sure to contrast with the other.

Example AALC Archetype A as a linear combination
As a vector equality, Archetype A can be written as

$$\begin{bmatrix}x_1 - x_2 + 2x_3\\2x_1 + x_2 + x_3\\x_1 + x_2\end{bmatrix} = \begin{bmatrix}1\\8\\5\end{bmatrix}$$

Now bust up the linear expressions on the left, first using vector addition,

$$\begin{bmatrix}x_1\\2x_1\\x_1\end{bmatrix} + \begin{bmatrix}-x_2\\x_2\\x_2\end{bmatrix} + \begin{bmatrix}2x_3\\x_3\\0x_3\end{bmatrix} = \begin{bmatrix}1\\8\\5\end{bmatrix}$$

Rewrite each of these vectors as a scalar multiple of a fixed vector, where the scalar is one of the unknown variables, converting the left-hand side into a linear combination

$$x_1\begin{bmatrix}1\\2\\1\end{bmatrix} + x_2\begin{bmatrix}-1\\1\\1\end{bmatrix} + x_3\begin{bmatrix}2\\1\\0\end{bmatrix} = \begin{bmatrix}1\\8\\5\end{bmatrix}$$

Row-reducing the augmented matrix for Archetype A leads to the conclusion that the system is consistent and has free variables, hence infinitely many solutions. So for example, the two solutions

$$x_1 = 2 \qquad x_2 = 3 \qquad x_3 = 1$$
$$x_1 = 3 \qquad x_2 = 2 \qquad x_3 = 0$$

can be used together to say that,

$$(2)\begin{bmatrix}1\\2\\1\end{bmatrix} + (3)\begin{bmatrix}-1\\1\\1\end{bmatrix} + (1)\begin{bmatrix}2\\1\\0\end{bmatrix} = \begin{bmatrix}1\\8\\5\end{bmatrix} = (3)\begin{bmatrix}1\\2\\1\end{bmatrix} + (2)\begin{bmatrix}-1\\1\\1\end{bmatrix} + (0)\begin{bmatrix}2\\1\\0\end{bmatrix}$$

Ignore the middle of this equation, and move all the terms to the left-hand side,

$$(2)\begin{bmatrix}1\\2\\1\end{bmatrix} + (3)\begin{bmatrix}-1\\1\\1\end{bmatrix} + (1)\begin{bmatrix}2\\1\\0\end{bmatrix} + (-3)\begin{bmatrix}1\\2\\1\end{bmatrix} + (-2)\begin{bmatrix}-1\\1\\1\end{bmatrix} + (-0)\begin{bmatrix}2\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Regrouping gives

$$(-1)\begin{bmatrix}1\\2\\1\end{bmatrix} + (1)\begin{bmatrix}-1\\1\\1\end{bmatrix} + (1)\begin{bmatrix}2\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Notice that these three vectors are the columns of the coefficient matrix for the system of equations in Archetype A. This equality says there is a linear combination of those columns that equals the vector of all zeros. Give it some thought, but this says that

$$x_1 = -1 \qquad x_2 = 1 \qquad x_3 = 1$$

is a nontrivial solution to the homogeneous system of equations with the coefficient matrix for the original system in Archetype A. In particular, this demonstrates that this coefficient matrix is singular. △

There is a lot going on in the last two examples. Come back to them in a while and make some connections with the intervening material. For now, we will summarize and explain some of this behavior with a theorem.

Theorem SLSLC Solutions to Linear Systems are Linear Combinations
Denote the columns of the $m \times n$ matrix A as the vectors $\mathbf{A}_1,\,\mathbf{A}_2,\,\mathbf{A}_3,\,\ldots,\,\mathbf{A}_n$. Then $\mathbf{x} \in \mathbb{C}^n$ is a solution to the linear system of equations $\mathcal{LS}(A,\,\mathbf{b})$ if and only if $\mathbf{b}$ equals the linear combination of the columns of A formed with the entries of $\mathbf{x}$,

$$[\mathbf{x}]_1\,\mathbf{A}_1 + [\mathbf{x}]_2\,\mathbf{A}_2 + [\mathbf{x}]_3\,\mathbf{A}_3 + \cdots + [\mathbf{x}]_n\,\mathbf{A}_n = \mathbf{b}$$

Proof. The proof of this theorem is as much about a change in notation as it is about making logical deductions. Write the system of equations $\mathcal{LS}(A,\,\mathbf{b})$ as

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1\\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2\\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3\\
&\vdots\\
a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m
\end{aligned}$$

Notice then that the entry of the coefficient matrix A in row i and column j has two names: a_{ij} as the coefficient of x_j in equation i of the system and $[\mathbf{A}_j]_i$ as the i-th entry of the column vector in column j of the coefficient matrix A. Likewise, entry i of $\mathbf{b}$ has two names: b_i from the linear system and $[\mathbf{b}]_i$ as an entry of a vector. Our theorem is an equivalence (Proof Technique E) so we need to prove both "directions."

($\Leftarrow$) Suppose we have the vector equality between $\mathbf{b}$ and the linear combination of the columns of A. Then for $1 \leq i \leq m$,

$$\begin{aligned}
b_i &= [\mathbf{b}]_i && \text{Definition CV}\\
&= \left[[\mathbf{x}]_1\,\mathbf{A}_1 + [\mathbf{x}]_2\,\mathbf{A}_2 + [\mathbf{x}]_3\,\mathbf{A}_3 + \cdots + [\mathbf{x}]_n\,\mathbf{A}_n\right]_i && \text{Hypothesis}\\
&= \left[[\mathbf{x}]_1\,\mathbf{A}_1\right]_i + \left[[\mathbf{x}]_2\,\mathbf{A}_2\right]_i + \left[[\mathbf{x}]_3\,\mathbf{A}_3\right]_i + \cdots + \left[[\mathbf{x}]_n\,\mathbf{A}_n\right]_i && \text{Definition CVA}\\
&= [\mathbf{x}]_1\,[\mathbf{A}_1]_i + [\mathbf{x}]_2\,[\mathbf{A}_2]_i + [\mathbf{x}]_3\,[\mathbf{A}_3]_i + \cdots + [\mathbf{x}]_n\,[\mathbf{A}_n]_i && \text{Definition CVSM}\\
&= [\mathbf{x}]_1\,a_{i1} + [\mathbf{x}]_2\,a_{i2} + [\mathbf{x}]_3\,a_{i3} + \cdots + [\mathbf{x}]_n\,a_{in} && \text{Definition CV}\\
&= a_{i1}\,[\mathbf{x}]_1 + a_{i2}\,[\mathbf{x}]_2 + a_{i3}\,[\mathbf{x}]_3 + \cdots + a_{in}\,[\mathbf{x}]_n && \text{Property CMCN}
\end{aligned}$$

This says that the entries of $\mathbf{x}$ form a solution to equation i of $\mathcal{LS}(A,\,\mathbf{b})$ for all $1 \leq i \leq m$, in other words, $\mathbf{x}$ is a solution to $\mathcal{LS}(A,\,\mathbf{b})$.

($\Rightarrow$) Suppose now that $\mathbf{x}$ is a solution to the linear system $\mathcal{LS}(A,\,\mathbf{b})$. Then for all $1 \leq i \leq m$,

$$\begin{aligned}
[\mathbf{b}]_i &= b_i && \text{Definition CV}\\
&= a_{i1}\,[\mathbf{x}]_1 + a_{i2}\,[\mathbf{x}]_2 + a_{i3}\,[\mathbf{x}]_3 + \cdots + a_{in}\,[\mathbf{x}]_n && \text{Hypothesis}\\
&= [\mathbf{x}]_1\,a_{i1} + [\mathbf{x}]_2\,a_{i2} + [\mathbf{x}]_3\,a_{i3} + \cdots + [\mathbf{x}]_n\,a_{in} && \text{Property CMCN}\\
&= [\mathbf{x}]_1\,[\mathbf{A}_1]_i + [\mathbf{x}]_2\,[\mathbf{A}_2]_i + [\mathbf{x}]_3\,[\mathbf{A}_3]_i + \cdots + [\mathbf{x}]_n\,[\mathbf{A}_n]_i && \text{Definition CV}\\
&= \left[[\mathbf{x}]_1\,\mathbf{A}_1\right]_i + \left[[\mathbf{x}]_2\,\mathbf{A}_2\right]_i + \left[[\mathbf{x}]_3\,\mathbf{A}_3\right]_i + \cdots + \left[[\mathbf{x}]_n\,\mathbf{A}_n\right]_i && \text{Definition CVSM}
\end{aligned}$$

$$= [[\mathbf{x}]_1 \mathbf{A}_1 + [\mathbf{x}]_2 \mathbf{A}_2 + [\mathbf{x}]_3 \mathbf{A}_3 + \cdots + [\mathbf{x}]_n \mathbf{A}_n]_i \qquad \text{Definition CVA}$$

So the entries of the vector $\mathbf{b}$, and the entries of the vector that is the linear combination of the columns of A, agree for all $1 \leq i \leq m$. By Definition CVE we see that the two vectors are equal, as desired. ■

In other words, this theorem tells us that solutions to systems of equations are linear combinations of the n column vectors of the coefficient matrix ($\mathbf{A}_j$) which yield the constant vector $\mathbf{b}$. Or said another way, a solution to a system of equations $\mathcal{LS}(A, \mathbf{b})$ is an answer to the question "How can I form the vector $\mathbf{b}$ as a linear combination of the columns of A?" Look through the archetypes that are systems of equations and examine a few of the advertised solutions. In each case use the solution to form a linear combination of the columns of the coefficient matrix and verify that the result equals the constant vector (see Exercise LC.C21).

Subsection VFSS
Vector Form of Solution Sets

We have written solutions to systems of equations as column vectors. For example Archetype B has the solution $x_1 = -3$, $x_2 = 5$, $x_3 = 2$ which we write as

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix}$$

Now, we will use column vectors and linear combinations to express *all* of the solutions to a linear system of equations in a compact and understandable way. First, here are two examples that will motivate our next theorem. This is a valuable technique, almost the equal of row-reducing a matrix, so be sure you get comfortable with it over the course of this section.

Example VFSAD Vector form of solutions for Archetype D
Archetype D is a linear system of 3 equations in 4 variables. Row-reducing the augmented matrix yields

$$\begin{bmatrix} \boxed{1} & 0 & 3 & -2 & 4 \\ 0 & \boxed{1} & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and we see $r = 2$ pivot columns. Also, $D = \{1, 2\}$ so the dependent variables are then x_1 and x_2. $F = \{3, 4, 5\}$ so the two free variables are x_3 and x_4. We will express a generic solution for the system by two slightly different methods, though both arrive at the same conclusion.

First, we will decompose (Proof Technique DC) a solution vector. Rearranging each equation represented in the row-reduced form of the augmented matrix by solving for the dependent variable in each row yields the vector equality,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 - 3x_3 + 2x_4 \\ -x_3 + 3x_4 \\ x_3 \\ x_4 \end{bmatrix}$$

Now we will use the definitions of column vector addition and scalar multiplication to express this vector as a linear combination,

$$= \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3x_3 \\ -x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} 2x_4 \\ 3x_4 \\ 0 \\ x_4 \end{bmatrix} \qquad \text{Definition CVA}$$

$$= \begin{bmatrix}4\\0\\0\\0\end{bmatrix} + x_3\begin{bmatrix}-3\\-1\\1\\0\end{bmatrix} + x_4\begin{bmatrix}2\\3\\0\\1\end{bmatrix} \qquad \text{Definition CVSM}$$

We will develop the same linear combination a bit quicker, using three steps. While the method above is instructive, the method below will be our preferred approach.

Step 1. Write the vector of variables as a fixed vector, plus a linear combination of $n - r$ vectors, using the free variables as the scalars.

$$\mathbf{x} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}\\ \\ \\ \\ \end{bmatrix} + x_3\begin{bmatrix}\\ \\ \\ \\ \end{bmatrix} + x_4\begin{bmatrix}\\ \\ \\ \\ \end{bmatrix}$$

Step 2. Use 0's and 1's to ensure equality for the entries of the vectors with indices in F (corresponding to the free variables).

$$\mathbf{x} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}\\ \\0\\0\end{bmatrix} + x_3\begin{bmatrix}\\ \\1\\0\end{bmatrix} + x_4\begin{bmatrix}\\ \\0\\1\end{bmatrix}$$

Step 3. For each dependent variable, use the augmented matrix to formulate an equation expressing the dependent variable as a constant plus multiples of the free variables. Convert this equation into entries of the vectors that ensure equality for each dependent variable, one at a time.

$$x_1 = 4 - 3x_3 + 2x_4 \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}4\\ \\0\\0\end{bmatrix} + x_3\begin{bmatrix}-3\\ \\1\\0\end{bmatrix} + x_4\begin{bmatrix}2\\ \\0\\1\end{bmatrix}$$

$$x_2 = 0 - 1x_3 + 3x_4 \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}4\\0\\0\\0\end{bmatrix} + x_3\begin{bmatrix}-3\\-1\\1\\0\end{bmatrix} + x_4\begin{bmatrix}2\\3\\0\\1\end{bmatrix}$$

This final *form* of a typical solution is especially pleasing and useful. For example, we can build solutions quickly by choosing values for our free variables, and then compute a linear combination. Such as

$$x_3 = 2,\ x_4 = -5 \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}4\\0\\0\\0\end{bmatrix} + (2)\begin{bmatrix}-3\\-1\\1\\0\end{bmatrix} + (-5)\begin{bmatrix}2\\3\\0\\1\end{bmatrix} = \begin{bmatrix}-12\\-17\\2\\-5\end{bmatrix}$$

or,

$$x_3 = 1,\ x_4 = 3 \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}4\\0\\0\\0\end{bmatrix} + (1)\begin{bmatrix}-3\\-1\\1\\0\end{bmatrix} + (3)\begin{bmatrix}2\\3\\0\\1\end{bmatrix} = \begin{bmatrix}7\\8\\1\\3\end{bmatrix}$$

You will find the second solution listed in the write-up for Archetype D, and you might check the first solution by substituting it back into the original equations.

While this form is useful for quickly creating solutions, it is even better because it tells us *exactly* what every solution looks like. We know the solution set is infinite, which is pretty big, but now we can say that a solution is some multiple of $\begin{bmatrix}-3\\-1\\1\\0\end{bmatrix}$

plus a multiple of $\begin{bmatrix}2\\3\\0\\1\end{bmatrix}$ plus the fixed vector $\begin{bmatrix}4\\0\\0\\0\end{bmatrix}$. Period. So it only takes us *three* vectors to describe the entire infinite solution set, provided we also agree on how to combine the three vectors into a linear combination. △

This is such an important and fundamental technique, we will do another example.

Example VFS Vector form of solutions
Consider a linear system of $m = 5$ equations in $n = 7$ variables, having the augmented matrix A.

$$A = \begin{bmatrix} 2 & 1 & -1 & -2 & 2 & 1 & 5 & 21 \\ 1 & 1 & -3 & 1 & 1 & 1 & 2 & -5 \\ 1 & 2 & -8 & 5 & 1 & 1 & -6 & -15 \\ 3 & 3 & -9 & 3 & 6 & 5 & 2 & -24 \\ -2 & -1 & 1 & 2 & 1 & 1 & -9 & -30 \end{bmatrix}$$

Row-reducing we obtain the matrix

$$B = \begin{bmatrix} \boxed{1} & 0 & 2 & -3 & 0 & 0 & 9 & 15 \\ 0 & \boxed{1} & -5 & 4 & 0 & 0 & -8 & -10 \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 & -6 & 11 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 7 & -21 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and we see $r = 4$ pivot columns. Also, $D = \{1, 2, 5, 6\}$ so the dependent variables are then x_1, x_2, x_5, and x_6. $F = \{3, 4, 7, 8\}$ so the $n - r = 3$ free variables are x_3, x_4 and x_7. We will express a generic solution for the system by two different methods: both a decomposition and a construction.

First, we will decompose (Proof Technique DC) a solution vector. Rearranging each equation represented in the row-reduced form of the augmented matrix by solving for the dependent variable in each row yields the vector equality,

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\\x_6\\x_7\end{bmatrix} = \begin{bmatrix} 15 - 2x_3 + 3x_4 - 9x_7 \\ -10 + 5x_3 - 4x_4 + 8x_7 \\ x_3 \\ x_4 \\ 11 + 6x_7 \\ -21 - 7x_7 \\ x_7 \end{bmatrix}$$

Now we will use the definitions of column vector addition and scalar multiplication to decompose this generic solution vector as a linear combination,

$$= \begin{bmatrix}15\\-10\\0\\0\\11\\-21\\0\end{bmatrix} + \begin{bmatrix}-2x_3\\5x_3\\x_3\\0\\0\\0\\0\end{bmatrix} + \begin{bmatrix}3x_4\\-4x_4\\0\\x_4\\0\\0\\0\end{bmatrix} + \begin{bmatrix}-9x_7\\8x_7\\0\\0\\6x_7\\-7x_7\\x_7\end{bmatrix} \qquad \text{Definition CVA}$$

$$= \begin{bmatrix}15\\-10\\0\\0\\11\\-21\\0\end{bmatrix} + x_3\begin{bmatrix}-2\\5\\1\\0\\0\\0\\0\end{bmatrix} + x_4\begin{bmatrix}3\\-4\\0\\1\\0\\0\\0\end{bmatrix} + x_7\begin{bmatrix}-9\\8\\0\\0\\6\\-7\\1\end{bmatrix} \qquad \text{Definition CVSM}$$

We will now develop the same linear combination a bit quicker, using three steps.

While the method above is instructive, the method below will be our preferred approach.

Step 1. Write the vector of variables as a fixed vector, plus a linear combination of $n-r$ vectors, using the free variables as the scalars.

$$\mathbf{x} = \begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4\\ x_5\\ x_6\\ x_7 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \\ \\ \end{bmatrix} + x_3\begin{bmatrix} \\ \\ \\ \\ \\ \\ \end{bmatrix} + x_4\begin{bmatrix} \\ \\ \\ \\ \\ \\ \end{bmatrix} + x_7\begin{bmatrix} \\ \\ \\ \\ \\ \\ \end{bmatrix}$$

Step 2. Use 0's and 1's to ensure equality for the entries of the vectors with indices in F (corresponding to the free variables).

$$\mathbf{x} = \begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4\\ x_5\\ x_6\\ x_7 \end{bmatrix} = \begin{bmatrix} \\ \\ 0\\ 0\\ \\ \\ 0 \end{bmatrix} + x_3\begin{bmatrix} \\ \\ 1\\ 0\\ \\ \\ 0 \end{bmatrix} + x_4\begin{bmatrix} \\ \\ 0\\ 1\\ \\ \\ 0 \end{bmatrix} + x_7\begin{bmatrix} \\ \\ 0\\ 0\\ \\ \\ 1 \end{bmatrix}$$

Step 3. For each dependent variable, use the augmented matrix to formulate an equation expressing the dependent variable as a constant plus multiples of the free variables. Convert this equation into entries of the vectors that ensure equality for each dependent variable, one at a time.

$$x_1 = 15 - 2x_3 + 3x_4 - 9x_7 \;\Rightarrow$$

$$\mathbf{x} = \begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4\\ x_5\\ x_6\\ x_7 \end{bmatrix} = \begin{bmatrix} 15\\ \\ 0\\ 0\\ \\ \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -2\\ \\ 1\\ 0\\ \\ \\ 0 \end{bmatrix} + x_4\begin{bmatrix} 3\\ \\ 0\\ 1\\ \\ \\ 0 \end{bmatrix} + x_7\begin{bmatrix} -9\\ \\ 0\\ 0\\ \\ \\ 1 \end{bmatrix}$$

$$x_2 = -10 + 5x_3 - 4x_4 + 8x_7 \;\Rightarrow$$

$$\mathbf{x} = \begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4\\ x_5\\ x_6\\ x_7 \end{bmatrix} = \begin{bmatrix} 15\\ -10\\ 0\\ 0\\ \\ \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -2\\ 5\\ 1\\ 0\\ \\ \\ 0 \end{bmatrix} + x_4\begin{bmatrix} 3\\ -4\\ 0\\ 1\\ \\ \\ 0 \end{bmatrix} + x_7\begin{bmatrix} -9\\ 8\\ 0\\ 0\\ \\ \\ 1 \end{bmatrix}$$

$$x_5 = 11 + 6x_7 \;\Rightarrow$$

$$\mathbf{x} = \begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4\\ x_5\\ x_6\\ x_7 \end{bmatrix} = \begin{bmatrix} 15\\ -10\\ 0\\ 0\\ 11\\ \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -2\\ 5\\ 1\\ 0\\ 0\\ \\ 0 \end{bmatrix} + x_4\begin{bmatrix} 3\\ -4\\ 0\\ 1\\ 0\\ \\ 0 \end{bmatrix} + x_7\begin{bmatrix} -9\\ 8\\ 0\\ 0\\ 6\\ \\ 1 \end{bmatrix}$$

$$x_6 = -21 - 7x_7 \;\Rightarrow$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 15 \\ -10 \\ 0 \\ 0 \\ 11 \\ -21 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 5 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ -4 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_7 \begin{bmatrix} -9 \\ 8 \\ 0 \\ 0 \\ 6 \\ -7 \\ 1 \end{bmatrix}$$

This final *form* of a typical solution is especially pleasing and useful. For example, we can build solutions quickly by choosing values for our free variables, and then compute a linear combination. For example

$$x_3 = 2,\ x_4 = -4,\ x_7 = 3 \qquad \Rightarrow$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 15 \\ -10 \\ 0 \\ 0 \\ 11 \\ -21 \\ 0 \end{bmatrix} + (2) \begin{bmatrix} -2 \\ 5 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + (-4) \begin{bmatrix} 3 \\ -4 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + (3) \begin{bmatrix} -9 \\ 8 \\ 0 \\ 0 \\ 6 \\ -7 \\ 1 \end{bmatrix} = \begin{bmatrix} -28 \\ 40 \\ 2 \\ -4 \\ 29 \\ -42 \\ 3 \end{bmatrix}$$

or perhaps,

$$x_3 = 5,\ x_4 = 2,\ x_7 = 1 \qquad \Rightarrow$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 15 \\ -10 \\ 0 \\ 0 \\ 11 \\ -21 \\ 0 \end{bmatrix} + (5) \begin{bmatrix} -2 \\ 5 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + (2) \begin{bmatrix} 3 \\ -4 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + (1) \begin{bmatrix} -9 \\ 8 \\ 0 \\ 0 \\ 6 \\ -7 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 15 \\ 5 \\ 2 \\ 17 \\ -28 \\ 1 \end{bmatrix}$$

or even,

$$x_3 = 0,\ x_4 = 0,\ x_7 = 0 \qquad \Rightarrow$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 15 \\ -10 \\ 0 \\ 0 \\ 11 \\ -21 \\ 0 \end{bmatrix} + (0) \begin{bmatrix} -2 \\ 5 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + (0) \begin{bmatrix} 3 \\ -4 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + (0) \begin{bmatrix} -9 \\ 8 \\ 0 \\ 0 \\ 6 \\ -7 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 \\ -10 \\ 0 \\ 0 \\ 11 \\ -21 \\ 0 \end{bmatrix}$$

So we can compactly express *all* of the solutions to this linear system with just 4 fixed vectors, provided we agree how to combine them in a linear combinations to create solution vectors.

Suppose you were told that the vector $\mathbf{w}$ below was a solution to this system of equations. Could you turn the problem around and write $\mathbf{w}$ as a linear combination of the four vectors $\mathbf{c}$, $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$? (See Exercise LC.M11.)

$$\mathbf{w} = \begin{bmatrix} 100 \\ -75 \\ 7 \\ 9 \\ -37 \\ 35 \\ -8 \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} 15 \\ -10 \\ 0 \\ 0 \\ 11 \\ -21 \\ 0 \end{bmatrix} \qquad \mathbf{u}_1 = \begin{bmatrix} -2 \\ 5 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{u}_2 = \begin{bmatrix} 3 \\ -4 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{u}_3 = \begin{bmatrix} -9 \\ 8 \\ 0 \\ 0 \\ 6 \\ -7 \\ 1 \end{bmatrix}$$

Did you think a few weeks ago that you could so quickly and easily list *all* the solutions to a linear system of 5 equations in 7 variables?

We will now formalize the last two (important) examples as a theorem. The statement of this theorem is a bit scary, and the proof is scarier. For now, be sure to convice yourself, by working through the examples and exercises, that the statement just describes the procedure of the two immediately previous examples.

Theorem VFSLS Vector Form of Solutions to Linear Systems
Suppose that $[A \mid \mathbf{b}]$ *is the augmented matrix for a consistent linear system* $\mathcal{LS}(A,\, \mathbf{b})$ *of* m *equations in* n *variables. Let* B *be a row-equivalent* $m \times (n+1)$ *matrix in reduced row-echelon form. Suppose that* B *has* r *pivot columns, with indices* $D = \{d_1,\, d_2,\, d_3,\, \ldots,\, d_r\}$, *while the* $n-r$ *non-pivot columns have indices in* $F = \{f_1,\, f_2,\, f_3,\, \ldots,\, f_{n-r},\, n+1\}$. *Define vectors* $\mathbf{c}$, $\mathbf{u}_j$, $1 \leq j \leq n-r$ *of size* n *by*

$$[\mathbf{c}]_i = \begin{cases} 0 & \text{if } i \in F \\ [B]_{k,n+1} & \text{if } i \in D,\ i = d_k \end{cases}$$

$$[\mathbf{u}_j]_i = \begin{cases} 1 & \text{if } i \in F,\ i = f_j \\ 0 & \text{if } i \in F,\ i \neq f_j \\ -[B]_{k,f_j} & \text{if } i \in D,\ i = d_k \end{cases}.$$

Then the set of solutions to the system of equations $\mathcal{LS}(A,\, \mathbf{b})$ *is*

$$S = \{\mathbf{c} + \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_{n-r}\mathbf{u}_{n-r} \mid \alpha_1,\, \alpha_2,\, \alpha_3,\, \ldots,\, \alpha_{n-r} \in \mathbb{C}\}$$

Proof. First, $\mathcal{LS}(A,\, \mathbf{b})$ is equivalent to the linear system of equations that has the matrix B as its augmented matrix (Theorem REMES), so we need only show that S is the solution set for the system with B as its augmented matrix. The conclusion of this theorem is that the solution set is equal to the set S, so we will apply Definition SE.

We begin by showing that every element of S is indeed a solution to the system. Let $\alpha_1,\, \alpha_2,\, \alpha_3,\, \ldots,\, \alpha_{n-r}$ be one choice of the scalars used to describe elements of S. So an arbitrary element of S, which we will consider as a proposed solution is

$$\mathbf{x} = \mathbf{c} + \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_{n-r}\mathbf{u}_{n-r}$$

When $r+1 \leq \ell \leq m$, row ℓ of the matrix B is a zero row, so the equation represented by that row is always true, no matter which solution vector we propose. So concentrate on rows representing equations $1 \leq \ell \leq r$. We evaluate equation ℓ of the system represented by B with the proposed solution vector $\mathbf{x}$ and refer to the value of the left-hand side of the equation as β_ℓ,

$$\beta_\ell = [B]_{\ell 1}[\mathbf{x}]_1 + [B]_{\ell 2}[\mathbf{x}]_2 + [B]_{\ell 3}[\mathbf{x}]_3 + \cdots + [B]_{\ell n}[\mathbf{x}]_n$$

Since $[B]_{\ell d_i} = 0$ for all $1 \leq i \leq r$, except that $[B]_{\ell d_\ell} = 1$, we see that β_ℓ simplifies to

$$\beta_\ell = [\mathbf{x}]_{d_\ell} + [B]_{\ell f_1}[\mathbf{x}]_{f_1} + [B]_{\ell f_2}[\mathbf{x}]_{f_2} + [B]_{\ell f_3}[\mathbf{x}]_{f_3} + \cdots + [B]_{\ell f_{n-r}}[\mathbf{x}]_{f_{n-r}}$$

Notice that for $1 \leq i \leq n-r$

$$\begin{aligned} [\mathbf{x}]_{f_i} &= [\mathbf{c}]_{f_i} + \alpha_1[\mathbf{u}_1]_{f_i} + \alpha_2[\mathbf{u}_2]_{f_i} + \cdots + \alpha_i[\mathbf{u}_i]_{f_i} + \cdots + \alpha_{n-r}[\mathbf{u_{n-r}}]_{f_i} \\ &= 0 + \alpha_1(0) + \alpha_2(0) + \cdots + \alpha_i(1) + \cdots + \alpha_{n-r}(0) \\ &= \alpha_i \end{aligned}$$

So β_ℓ simplifies further, and we expand the first term

$$\begin{aligned} \beta_\ell &= [\mathbf{x}]_{d_\ell} + [B]_{\ell f_1}\alpha_1 + [B]_{\ell f_2}\alpha_2 + [B]_{\ell f_3}\alpha_3 + \cdots + [B]_{\ell f_{n-r}}\alpha_{n-r} \\ &= [\mathbf{c} + \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_{n-r}\mathbf{u}_{n-r}]_{d_\ell} + \\ &\qquad [B]_{\ell f_1}\alpha_1 + [B]_{\ell f_2}\alpha_2 + [B]_{\ell f_3}\alpha_3 + \cdots + [B]_{\ell f_{n-r}}\alpha_{n-r} \\ &= [\mathbf{c}]_{d_\ell} + \alpha_1[\mathbf{u}_1]_{d_\ell} + \alpha_2[\mathbf{u}_2]_{d_\ell} + \alpha_3[\mathbf{u}_3]_{d_\ell} + \cdots + \alpha_{n-r}[\mathbf{u_{n-r}}]_{d_\ell} + \\ &\qquad [B]_{\ell f_1}\alpha_1 + [B]_{\ell f_2}\alpha_2 + [B]_{\ell f_3}\alpha_3 + \cdots + [B]_{\ell f_{n-r}}\alpha_{n-r} \end{aligned}$$

$$= [B]_{\ell,n+1} + \alpha_1(-[B]_{\ell f_1}) + \alpha_2(-[B]_{\ell f_2}) + \alpha_3(-[B]_{\ell f_3}) + \cdots + \alpha_{n-r}(-[B]_{\ell f_{n-r}}) + [B]_{\ell f_1}\alpha_1 + [B]_{\ell f_2}\alpha_2 + [B]_{\ell f_3}\alpha_3 + \cdots + [B]_{\ell f_{n-r}}\alpha_{n-r}$$
$$= [B]_{\ell,n+1}$$

So β_ℓ began as the left-hand side of equation ℓ of the system represented by B and we now know it equals $[B]_{\ell,n+1}$, the constant term for equation ℓ of this system. So the arbitrarily chosen vector from S makes every equation of the system true, and therefore is a solution to the system. So all the elements of S are solutions to the system.

For the second half of the proof, assume that $\mathbf{x}$ is a solution vector for the system having B as its augmented matrix. For convenience and clarity, denote the entries of $\mathbf{x}$ by x_i, in other words, $x_i = [\mathbf{x}]_i$. We desire to show that this solution vector is also an element of the set S. Begin with the observation that a solution vector's entries makes equation ℓ of the system true for all $1 \leq \ell \leq m$,

$$[B]_{\ell,1}x_1 + [B]_{\ell,2}x_2 + [B]_{\ell,3}x_3 + \cdots + [B]_{\ell,n}x_n = [B]_{\ell,n+1}$$

When $\ell \leq r$, the pivot columns of B have zero entries in row ℓ with the exception of column d_ℓ, which will contain a 1. So for $1 \leq \ell \leq r$, equation ℓ simplifies to

$$1x_{d_\ell} + [B]_{\ell,f_1}x_{f_1} + [B]_{\ell,f_2}x_{f_2} + [B]_{\ell,f_3}x_{f_3} + \cdots + [B]_{\ell,f_{n-r}}x_{f_{n-r}} = [B]_{\ell,n+1}$$

This allows us to write,

$$\begin{aligned}
[\mathbf{x}]_{d_\ell} &= x_{d_\ell} \\
&= [B]_{\ell,n+1} - [B]_{\ell,f_1}x_{f_1} - [B]_{\ell,f_2}x_{f_2} - [B]_{\ell,f_3}x_{f_3} - \cdots - [B]_{\ell,f_{n-r}}x_{f_{n-r}} \\
&= [\mathbf{c}]_{d_\ell} + x_{f_1}[\mathbf{u}_1]_{d_\ell} + x_{f_2}[\mathbf{u}_2]_{d_\ell} + x_{f_3}[\mathbf{u}_3]_{d_\ell} + \cdots + x_{f_{n-r}}[\mathbf{u}_{n-r}]_{d_\ell} \\
&= \left[\mathbf{c} + x_{f_1}\mathbf{u}_1 + x_{f_2}\mathbf{u}_2 + x_{f_3}\mathbf{u}_3 + \cdots + x_{f_{n-r}}\mathbf{u}_{n-r}\right]_{d_\ell}
\end{aligned}$$

This tells us that the entries of the solution vector $\mathbf{x}$ corresponding to dependent variables (indices in D), are equal to those of a vector in the set S. We still need to check the other entries of the solution vector $\mathbf{x}$ corresponding to the free variables (indices in F) to see if they are equal to the entries of the same vector in the set S. To this end, suppose $i \in F$ and $i = f_j$. Then

$$\begin{aligned}
[\mathbf{x}]_i &= x_i = x_{f_j} \\
&= 0 + 0x_{f_1} + 0x_{f_2} + 0x_{f_3} + \cdots + 0x_{f_{j-1}} + 1x_{f_j} + 0x_{f_{j+1}} + \cdots + 0x_{f_{n-r}} \\
&= [\mathbf{c}]_i + x_{f_1}[\mathbf{u}_1]_i + x_{f_2}[\mathbf{u}_2]_i + x_{f_3}[\mathbf{u}_3]_i + \cdots + x_{f_j}[\mathbf{u}_j]_i + \cdots + x_{f_{n-r}}[\mathbf{u}_{n-r}]_i \\
&= \left[\mathbf{c} + x_{f_1}\mathbf{u}_1 + x_{f_2}\mathbf{u}_2 + \cdots + x_{f_{n-r}}\mathbf{u}_{n-r}\right]_i
\end{aligned}$$

So entries of $\mathbf{x}$ and $\mathbf{c} + x_{f_1}\mathbf{u}_1 + x_{f_2}\mathbf{u}_2 + \cdots + x_{f_{n-r}}\mathbf{u}_{n-r}$ are equal and therefore by Definition CVE they are equal vectors. Since $x_{f_1}, x_{f_2}, x_{f_3}, \ldots, x_{f_{n-r}}$ are scalars, this shows us that $\mathbf{x}$ qualifies for membership in S. So the set S contains all of the solutions to the system. ■

Note that both halves of the proof of Theorem VFSLS indicate that $\alpha_i = [\mathbf{x}]_{f_i}$. In other words, the arbitrary scalars, α_i, in the description of the set S actually have more meaning — they are the values of the free variables $[\mathbf{x}]_{f_i}$, $1 \leq i \leq n-r$. So we will often exploit this observation in our descriptions of solution sets.

Theorem VFSLS formalizes what happened in the three steps of Example VFSAD. The theorem will be useful in proving other theorems, and it it is useful since it tells us an exact procedure for simply describing an infinite solution set. We could program a computer to implement it, once we have the augmented matrix row-reduced and have checked that the system is consistent. By Knuth's definition, this completes our conversion of linear equation solving from art into science. Notice that it even applies (but is overkill) in the case of a unique solution. However, as a practical matter, I prefer the three-step process of Example VFSAD when I need to describe an infinite solution set. So let us practice some more, but with a bigger example.

Example VFSAI Vector form of solutions for Archetype I
Archetype I is a linear system of $m = 4$ equations in $n = 7$ variables. Row-reducing the augmented matrix yields

$$\begin{bmatrix} \boxed{1} & 4 & 0 & 0 & 2 & 1 & -3 & 4 \\ 0 & 0 & \boxed{1} & 0 & 1 & -3 & 5 & 2 \\ 0 & 0 & 0 & \boxed{1} & 2 & -6 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and we see $r = 3$ pivot columns, with indices $D = \{1, 3, 4\}$. So the $r = 3$ dependent variables are x_1, x_3, x_4. The non-pivot columns have indices in $F = \{2, 5, 6, 7, 8\}$, so the $n - r = 4$ free variables are x_2, x_5, x_6, x_7.

Step 1. Write the vector of variables ($\mathbf{x}$) as a fixed vector ($\mathbf{c}$), plus a linear combination of $n - r = 4$ vectors ($\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$), using the free variables as the scalars.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \\ \\ \\ \end{bmatrix} + x_2 \begin{bmatrix} \\ \\ \\ \\ \\ \\ \\ \end{bmatrix} + x_5 \begin{bmatrix} \\ \\ \\ \\ \\ \\ \\ \end{bmatrix} + x_6 \begin{bmatrix} \\ \\ \\ \\ \\ \\ \\ \end{bmatrix} + x_7 \begin{bmatrix} \\ \\ \\ \\ \\ \\ \\ \end{bmatrix}$$

Step 2. For each free variable, use 0's and 1's to ensure equality for the corresponding entry of the vectors. Take note of the pattern of 0's and 1's at this stage, because this is the best look you will have at it. We will state an important theorem in the next section and the proof will essentially rely on this observation.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} \\ 0 \\ \\ \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} \\ 1 \\ \\ \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} \\ 0 \\ \\ \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} \\ 0 \\ \\ \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_7 \begin{bmatrix} \\ 0 \\ \\ \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Step 3. For each dependent variable, use the augmented matrix to formulate an equation expressing the dependent variable as a constant plus multiples of the free variables. Convert this equation into entries of the vectors that ensure equality for each dependent variable, one at a time.

$$x_1 = 4 - 4x_2 - 2x_5 - 1x_6 + 3x_7 \quad \Rightarrow$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ \\ \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -4 \\ 1 \\ \\ \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ 0 \\ \\ \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} -1 \\ 0 \\ \\ \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_7 \begin{bmatrix} 3 \\ 0 \\ \\ \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$x_3 = 2 + 0x_2 - x_5 + 3x_6 - 5x_7 \quad \Rightarrow$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \\ \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -4 \\ 1 \\ 0 \\ \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ 0 \\ -1 \\ \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} -1 \\ 0 \\ 3 \\ \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_7 \begin{bmatrix} 3 \\ 0 \\ -5 \\ \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$x_4 = 1 + 0x_2 - 2x_5 + 6x_6 - 6x_7 \quad \Rightarrow$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ 0 \\ -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_6 \begin{bmatrix} -1 \\ 0 \\ 3 \\ 6 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_7 \begin{bmatrix} 3 \\ 0 \\ -5 \\ -6 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

We can now use this final expression to quickly build solutions to the system. You might try to recreate each of the solutions listed in the write-up for Archetype I. (Hint: look at the values of the free variables in each solution, and notice that the vector $\mathbf{c}$ has 0's in these locations.)

Even better, we have a description of the infinite solution set, based on just 5 vectors, which we combine in linear combinations to produce solutions.

Whenever we discuss Archetype I you know that is your cue to go work through Archetype J by yourself. Remember to take note of the 0/1 pattern at the conclusion of Step 2. Have fun — we won't go anywhere while you're away. △

This technique is so important, that we will do one more example. However, an important distinction will be that this system is homogeneous.

Example VFSAL Vector form of solutions for Archetype L
Archetype L is presented simply as the 5×5 matrix

$$L = \begin{bmatrix} -2 & -1 & -2 & -4 & 4 \\ -6 & -5 & -4 & -4 & 6 \\ 10 & 7 & 7 & 10 & -13 \\ -7 & -5 & -6 & -9 & 10 \\ -4 & -3 & -4 & -6 & 6 \end{bmatrix}$$

We will employ this matrix here as the coefficient matrix of a homogeneous system and reference this matrix as L. So we are solving the homogeneous system $\mathcal{LS}(L, \mathbf{0})$ having $m = 5$ equations in $n = 5$ variables. If we built the augmented matrix, we would add a sixth column to L containing all zeros. As we did row operations, this sixth column would remain all zeros. So instead we will row-reduce the coefficient matrix, and mentally remember the missing sixth column of zeros. This row-reduced matrix is

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 1 & -2 \\ 0 & \boxed{1} & 0 & -2 & 2 \\ 0 & 0 & \boxed{1} & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and we see $r = 3$ pivot columns, with indices $D = \{1,\, 2,\, 3\}$. So the $r = 3$ dependent variables are x_1, x_2, x_3. The non-pivot columns have indices $F = \{4,\, 5\}$, so the $n - r = 2$ free variables are x_4, x_5. Notice that if we had included the all-zero vector of constants to form the augmented matrix for the system, then the index 6 would have appeared in the set F, and subsequently would have been ignored when listing the free variables. So nothing is lost by not creating an augmented matrix (in the case of a homogenous system). And maybe it is an improvement, since now *every* index in F can be used to reference a variable of the linear system.

Step 1. Write the vector of variables ($\mathbf{x}$) as a fixed vector ($\mathbf{c}$), plus a linear combination of $n - r = 2$ vectors ($\mathbf{u}_1$, $\mathbf{u}_2$), using the free variables as the scalars.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix} + x_4 \begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix} + x_5 \begin{bmatrix} \\ \\ \\ \\ \\ \end{bmatrix}$$

Step 2. For each free variable, use 0's and 1's to ensure equality for the corresponding entry of the vectors. Take note of the pattern of 0's and 1's at this stage, even if it is not as illuminating as in other examples.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} \\ \\ \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} \\ \\ \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} \\ \\ \\ 0 \\ 1 \end{bmatrix}$$

Step 3. For each dependent variable, use the augmented matrix to formulate an equation expressing the dependent variable as a constant plus multiples of the free variables. Do not forget about the "missing" sixth column being full of zeros. Convert this equation into entries of the vectors that ensure equality for each dependent variable, one at a time.

$$x_1 = 0 - 1x_4 + 2x_5 \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ \\ \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ \\ \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 2 \\ \\ \\ 0 \\ 1 \end{bmatrix}$$

$$x_2 = 0 + 2x_4 - 2x_5 \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 2 \\ \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 2 \\ -2 \\ \\ 0 \\ 1 \end{bmatrix}$$

$$x_3 = 0 - 2x_4 + 1x_5 \quad \Rightarrow \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 2 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

The vector $\mathbf{c}$ will always have 0's in the entries corresponding to free variables. However, since we are solving a homogeneous system, the row-reduced augmented matrix has zeros in column $n + 1 = 6$, and hence *all* the entries of $\mathbf{c}$ are zero. So we can write

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \mathbf{0} + x_4 \begin{bmatrix} -1 \\ 2 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \\ 1 \end{bmatrix} = x_4 \begin{bmatrix} -1 \\ 2 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

It will always happen that the solutions to a homogeneous system has $\mathbf{c} = \mathbf{0}$ (even in the case of a unique solution?). So our expression for the solutions is a bit more pleasing. In this example it says that the solutions are *all possible* linear combinations of the two vectors $\mathbf{u}_1 = \begin{bmatrix}-1\\2\\-2\\1\\0\end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix}2\\-2\\1\\0\\1\end{bmatrix}$, with no mention of any fixed vector entering into the linear combination.

This observation will motivate our next section and the main definition of that section, and after that we will conclude the section by formalizing this situation. △

Subsection PSHS
Particular Solutions, Homogeneous Solutions

The next theorem tells us that in order to find all of the solutions to a linear system of equations, it is sufficient to find just one solution, and then find all of the solutions to the corresponding homogeneous system. This explains part of our interest in the null space, the set of all solutions to a homogeneous system.

Theorem PSPHS Particular Solution Plus Homogeneous Solutions
Suppose that $\mathbf{w}$ *is one solution to the linear system of equations* $\mathcal{LS}(A,\,\mathbf{b})$*. Then* $\mathbf{y}$ *is a solution to* $\mathcal{LS}(A,\,\mathbf{b})$ *if and only if* $\mathbf{y} = \mathbf{w} + \mathbf{z}$ *for some vector* $\mathbf{z} \in \mathcal{N}(A)$*.*

Proof. Let $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n$ be the columns of the coefficient matrix A.

(⇐) Suppose $\mathbf{y} = \mathbf{w} + \mathbf{z}$ and $\mathbf{z} \in \mathcal{N}(A)$. Then

$$\begin{aligned}
\mathbf{b} &= [\mathbf{w}]_1 \mathbf{A}_1 + [\mathbf{w}]_2 \mathbf{A}_2 + [\mathbf{w}]_3 \mathbf{A}_3 + \cdots + [\mathbf{w}]_n \mathbf{A}_n && \text{Theorem SLSLC}\\
&= [\mathbf{w}]_1 \mathbf{A}_1 + [\mathbf{w}]_2 \mathbf{A}_2 + [\mathbf{w}]_3 \mathbf{A}_3 + \cdots + [\mathbf{w}]_n \mathbf{A}_n + \mathbf{0} && \text{Property ZC}\\
&= [\mathbf{w}]_1 \mathbf{A}_1 + [\mathbf{w}]_2 \mathbf{A}_2 + [\mathbf{w}]_3 \mathbf{A}_3 + \cdots + [\mathbf{w}]_n \mathbf{A}_n && \text{Theorem SLSLC}\\
&\quad + [\mathbf{z}]_1 \mathbf{A}_1 + [\mathbf{z}]_2 \mathbf{A}_2 + [\mathbf{z}]_3 \mathbf{A}_3 + \cdots + [\mathbf{z}]_n \mathbf{A}_n\\
&= ([\mathbf{w}]_1 + [\mathbf{z}]_1) \mathbf{A}_1 + ([\mathbf{w}]_2 + [\mathbf{z}]_2) \mathbf{A}_2 && \text{Theorem VSPCV}\\
&\quad ([\mathbf{w}]_3 + [\mathbf{z}]_3) \mathbf{A}_3 + \cdots + ([\mathbf{w}]_n + [\mathbf{z}]_n) \mathbf{A}_n\\
&= + [\mathbf{w}+\mathbf{z}]_1 \mathbf{A}_1 + [\mathbf{w}+\mathbf{z}]_2 \mathbf{A}_2 + \cdots + [\mathbf{w}+\mathbf{z}]_n \mathbf{A}_n && \text{Definition CVA}\\
&= [\mathbf{y}]_1 \mathbf{A}_1 + [\mathbf{y}]_2 \mathbf{A}_2 + [\mathbf{y}]_3 \mathbf{A}_3 + \cdots + [\mathbf{y}]_n \mathbf{A}_n && \text{Definition of } \mathbf{y}
\end{aligned}$$

Applying Theorem SLSLC we see that the vector $\mathbf{y}$ is a solution to $\mathcal{LS}(A,\,\mathbf{b})$.

(⇒) Suppose $\mathbf{y}$ is a solution to $\mathcal{LS}(A,\,\mathbf{b})$. Then

$$\begin{aligned}
\mathbf{0} &= \mathbf{b} - \mathbf{b}\\
&= [\mathbf{y}]_1 \mathbf{A}_1 + [\mathbf{y}]_2 \mathbf{A}_2 + [\mathbf{y}]_3 \mathbf{A}_3 + \cdots + [\mathbf{y}]_n \mathbf{A}_n && \text{Theorem SLSLC}\\
&\quad - ([\mathbf{w}]_1 \mathbf{A}_1 + [\mathbf{w}]_2 \mathbf{A}_2 + [\mathbf{w}]_3 \mathbf{A}_3 + \cdots + [\mathbf{w}]_n \mathbf{A}_n)\\
&= ([\mathbf{y}]_1 - [\mathbf{w}]_1) \mathbf{A}_1 + ([\mathbf{y}]_2 - [\mathbf{w}]_2) \mathbf{A}_2 && \text{Theorem VSPCV}\\
&\quad + ([\mathbf{y}]_3 - [\mathbf{w}]_3) \mathbf{A}_3 + \cdots + ([\mathbf{y}]_n - [\mathbf{w}]_n) \mathbf{A}_n\\
&= [\mathbf{y}-\mathbf{w}]_1 \mathbf{A}_1 + [\mathbf{y}-\mathbf{w}]_2 \mathbf{A}_2 && \text{Definition CVA}\\
&\quad + [\mathbf{y}-\mathbf{w}]_3 \mathbf{A}_3 + \cdots + [\mathbf{y}-\mathbf{w}]_n \mathbf{A}_n
\end{aligned}$$

By Theorem SLSLC we see that the vector $\mathbf{y} - \mathbf{w}$ is a solution to the homogeneous system $\mathcal{LS}(A,\,\mathbf{0})$ and by Definition NSM, $\mathbf{y} - \mathbf{w} \in \mathcal{N}(A)$. In other words, $\mathbf{y} - \mathbf{w} = \mathbf{z}$ for some vector $\mathbf{z} \in \mathcal{N}(A)$. Rewritten, this is $\mathbf{y} = \mathbf{w} + \mathbf{z}$, as desired. ■

After proving Theorem NMUS we commented (insufficiently) on the negation of one half of the theorem. Nonsingular coefficient matrices lead to unique solutions for every choice of the vector of constants. What does this say about singular matrices? A singular matrix A has a nontrivial null space (Theorem NMTNS). For a given vector of constants, $\mathbf{b}$, the system $\mathcal{LS}(A,\,\mathbf{b})$ could be inconsistent, meaning there are no solutions. But if there is at least one solution ($\mathbf{w}$), then Theorem PSPHS tells us there will be infinitely many solutions because of the role of the infinite null space for a singular matrix. So a system of equations with a singular coefficient matrix *never* has a unique solution. Notice that this is the contrapositive of the statement in Exercise NM.T31. With a singular coefficient matrix, either there are no solutions, or infinitely many solutions, depending on the choice of the vector of constants ($\mathbf{b}$).

Example PSHS Particular solutions, homogeneous solutions, Archetype D
Archetype D is a consistent system of equations with a nontrivial null space. Let A denote the coefficient matrix of this system. The write-up for this system begins with three solutions,

$$\mathbf{y}_1 = \begin{bmatrix}0\\1\\2\\1\end{bmatrix} \qquad \mathbf{y}_2 = \begin{bmatrix}4\\0\\0\\0\end{bmatrix} \qquad \mathbf{y}_3 = \begin{bmatrix}7\\8\\1\\3\end{bmatrix}$$

We will choose to have $\mathbf{y}_1$ play the role of $\mathbf{w}$ in the statement of Theorem PSPHS, any one of the three vectors listed here (or others) could have been chosen. To illustrate the theorem, we should be able to write each of these three solutions as the vector $\mathbf{w}$ plus a solution to the corresponding homogeneous system of equations. Since $\mathbf{0}$ is always a solution to a homogeneous system we can easily write

$$\mathbf{y}_1 = \mathbf{w} = \mathbf{w} + \mathbf{0}.$$

The vectors $\mathbf{y}_2$ and $\mathbf{y}_3$ will require a bit more effort. Solutions to the homogeneous system $\mathcal{LS}(A,\,\mathbf{0})$ are exactly the elements of the null space of the coefficient matrix, which by an application of Theorem VFSLS is

$$\mathcal{N}(A) = \left\{ x_3 \begin{bmatrix}-3\\-1\\1\\0\end{bmatrix} + x_4 \begin{bmatrix}2\\3\\0\\1\end{bmatrix} \middle|\, x_3,\, x_4 \in \mathbb{C} \right\}$$

Then

$$\mathbf{y}_2 = \begin{bmatrix}4\\0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\1\\2\\1\end{bmatrix} + \begin{bmatrix}4\\-1\\-2\\-1\end{bmatrix} = \begin{bmatrix}0\\1\\2\\1\end{bmatrix} + \left((-2)\begin{bmatrix}-3\\-1\\1\\0\end{bmatrix} + (-1)\begin{bmatrix}2\\3\\0\\1\end{bmatrix}\right) = \mathbf{w} + \mathbf{z}_2$$

where

$$\mathbf{z}_2 = \begin{bmatrix}4\\-1\\-2\\-1\end{bmatrix} = (-2)\begin{bmatrix}-3\\-1\\1\\0\end{bmatrix} + (-1)\begin{bmatrix}2\\3\\0\\1\end{bmatrix}$$

is obviously a solution of the homogeneous system since it is written as a linear combination of the vectors describing the null space of the coefficient matrix (or as a check, you could just evaluate the equations in the homogeneous system with $\mathbf{z}_2$).

Again

$$\mathbf{y}_3 = \begin{bmatrix}7\\8\\1\\3\end{bmatrix} = \begin{bmatrix}0\\1\\2\\1\end{bmatrix} + \begin{bmatrix}7\\7\\-1\\2\end{bmatrix} = \begin{bmatrix}0\\1\\2\\1\end{bmatrix} + \left((-1)\begin{bmatrix}-3\\-1\\1\\0\end{bmatrix} + 2\begin{bmatrix}2\\3\\0\\1\end{bmatrix}\right) = \mathbf{w} + \mathbf{z}_3$$

where

$$\mathbf{z}_3 = \begin{bmatrix}7\\7\\-1\\2\end{bmatrix} = (-1)\begin{bmatrix}-3\\-1\\1\\0\end{bmatrix} + 2\begin{bmatrix}2\\3\\0\\1\end{bmatrix}$$

is obviously a solution of the homogeneous system since it is written as a linear combination of the vectors describing the null space of the coefficient matrix (or as a check, you could just evaluate the equations in the homogeneous system with $\mathbf{z}_2$).

Here is another view of this theorem, in the context of this example. Grab two new solutions of the original system of equations, say

$$\mathbf{y}_4 = \begin{bmatrix} 11 \\ 0 \\ -3 \\ -1 \end{bmatrix} \qquad \mathbf{y}_5 = \begin{bmatrix} -4 \\ 2 \\ 4 \\ 2 \end{bmatrix}$$

and form their difference,

$$\mathbf{u} = \begin{bmatrix} 11 \\ 0 \\ -3 \\ -1 \end{bmatrix} - \begin{bmatrix} -4 \\ 2 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 15 \\ -2 \\ -7 \\ -3 \end{bmatrix}.$$

It is no accident that **u** is a solution to the homogeneous system (check this!). In other words, the difference between any two solutions to a linear system of equations is an element of the null space of the coefficient matrix. This is an equivalent way to state Theorem PSPHS. (See Exercise MM.T50). △

The ideas of this subsection will appear again in Chapter LT when we discuss pre-images of linear transformations (Definition PI).

Reading Questions

1. Earlier, a reading question asked you to solve the system of equations

$$\begin{aligned} 2x_1 + 3x_2 - x_3 &= 0 \\ x_1 + 2x_2 + x_3 &= 3 \\ x_1 + 3x_2 + 3x_3 &= 7 \end{aligned}$$

   Use a linear combination to rewrite this system of equations as a vector equality.

2. Find a linear combination of the vectors

$$S = \left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ -5 \end{bmatrix} \right\}$$

   that equals the vector $\begin{bmatrix} 1 \\ -9 \\ 11 \end{bmatrix}$.

3. The matrix below is the augmented matrix of a system of equations, row-reduced to reduced row-echelon form. Write the vector form of the solutions to the system.

$$\begin{bmatrix} \boxed{1} & 3 & 0 & 6 & 0 & 9 \\ 0 & 0 & \boxed{1} & -2 & 0 & -8 \\ 0 & 0 & 0 & 0 & \boxed{1} & 3 \end{bmatrix}$$

Exercises

C21† Consider each archetype that is a system of equations. For individual solutions listed (both for the original system and the corresponding homogeneous system) express the vector of constants as a linear combination of the columns of the coefficient matrix, as guaranteed by Theorem SLSLC. Verify this equality by computing the linear combination. For systems with no solutions, recognize that it is then impossible to write the vector of constants as a linear combination of the columns of the coefficient matrix. Note too, for homogeneous systems, that the solutions give rise to linear combinations that equal the zero vector.

Archetype A, Archetype B, Archetype C, Archetype D, Archetype E, Archetype F, Archetype G, Archetype H, Archetype I, Archetype J

C22† Consider each archetype that is a system of equations. Write elements of the solution set in vector form, as guaranteed by Theorem VFSLS.

Archetype A, Archetype B, Archetype C, Archetype D, Archetype E, Archetype F, Archetype G, Archetype H, Archetype I, Archetype J

C40† Find the vector form of the solutions to the system of equations below.

$$\begin{aligned} 2x_1 - 4x_2 + 3x_3 + x_5 &= 6 \\ x_1 - 2x_2 - 2x_3 + 14x_4 - 4x_5 &= 15 \\ x_1 - 2x_2 + x_3 + 2x_4 + x_5 &= -1 \\ -2x_1 + 4x_2 - 12x_4 + x_5 &= -7 \end{aligned}$$

C41† Find the vector form of the solutions to the system of equations below.

$$\begin{aligned} -2x_1 - 1x_2 - 8x_3 + 8x_4 + 4x_5 - 9x_6 - 1x_7 - 1x_8 - 18x_9 &= 3 \\ 3x_1 - 2x_2 + 5x_3 + 2x_4 - 2x_5 - 5x_6 + 1x_7 + 2x_8 + 15x_9 &= 10 \\ 4x_1 - 2x_2 + 8x_3 + 2x_5 - 14x_6 - 2x_8 + 2x_9 &= 36 \\ -1x_1 + 2x_2 + 1x_3 - 6x_4 + 7x_6 - 1x_7 - 3x_9 &= -8 \\ 3x_1 + 2x_2 + 13x_3 - 14x_4 - 1x_5 + 5x_6 - 1x_8 + 12x_9 &= 15 \\ -2x_1 + 2x_2 - 2x_3 - 4x_4 + 1x_5 + 6x_6 - 2x_7 - 2x_8 - 15x_9 &= -7 \end{aligned}$$

M10† Example TLC asks if the vector

$$\mathbf{w} = \begin{bmatrix} 13 \\ 15 \\ 5 \\ -17 \\ 2 \\ 25 \end{bmatrix}$$

can be written as a linear combination of the four vectors

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 4 \\ -3 \\ 1 \\ 2 \\ 9 \end{bmatrix} \qquad \mathbf{u}_2 = \begin{bmatrix} 6 \\ 3 \\ 0 \\ -2 \\ 1 \\ 4 \end{bmatrix} \qquad \mathbf{u}_3 = \begin{bmatrix} -5 \\ 2 \\ 1 \\ 1 \\ -3 \\ 0 \end{bmatrix} \qquad \mathbf{u}_4 = \begin{bmatrix} 3 \\ 2 \\ -5 \\ 7 \\ 1 \\ 3 \end{bmatrix}$$

Can it? Can any vector in $\mathbb{C}^6$ be written as a linear combination of the four vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, $\mathbf{u}_4$?

M11† At the end of Example VFS, the vector $\mathbf{w}$ is claimed to be a solution to the linear system under discussion. Verify that $\mathbf{w}$ really is a solution. Then determine the four scalars that express $\mathbf{w}$ as a linear combination of $\mathbf{c}$, $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$.

Section SS
Spanning Sets

In this section we will provide an extremely compact way to describe an infinite set of vectors, making use of linear combinations. This will give us a convenient way to describe the solution set of a linear system, the null space of a matrix, and many other sets of vectors.

Subsection SSV
Span of a Set of Vectors

In Example VFSAL we saw the solution set of a homogeneous system described as all possible linear combinations of two particular vectors. This is a useful way to construct or describe infinite sets of vectors, so we encapsulate the idea in a definition.

Definition SSCV Span of a Set of Column Vectors
Given a set of vectors $S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_p\}$, their **span**, $\langle S\rangle$, is the set of all possible linear combinations of $\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_p$. Symbolically,

$$\begin{aligned}\langle S\rangle &= \{\,\alpha_1\mathbf{u}_1+\alpha_2\mathbf{u}_2+\alpha_3\mathbf{u}_3+\cdots+\alpha_p\mathbf{u}_p \mid \alpha_i\in\mathbb{C},\, 1\leq i\leq p\}\\ &= \left\{\sum_{i=1}^{p}\alpha_i\mathbf{u}_i \,\middle|\, \alpha_i\in\mathbb{C},\, 1\leq i\leq p\right\}\end{aligned}$$

□

The span is just a set of vectors, though in all but one situation it is an infinite set. (Just when is it not infinite?) So we start with a finite collection of vectors S (p of them to be precise), and use this finite set to describe an infinite set of vectors, $\langle S\rangle$. Confusing the *finite* set S with the *infinite* set $\langle S\rangle$ is one of the most persistent problems in understanding introductory linear algebra. We will see this construction repeatedly, so let us work through some examples to get comfortable with it. The most obvious question about a set is if a particular item of the correct type is in the set, or not in the set.

Example ABS A basic span
Consider the set of 5 vectors, S, from $\mathbb{C}^4$

$$S=\left\{\begin{bmatrix}1\\1\\3\\1\end{bmatrix},\,\begin{bmatrix}2\\1\\2\\-1\end{bmatrix},\,\begin{bmatrix}7\\3\\5\\-5\end{bmatrix},\,\begin{bmatrix}1\\1\\-1\\2\end{bmatrix},\,\begin{bmatrix}-1\\0\\9\\0\end{bmatrix}\right\}$$

and consider the infinite set of vectors $\langle S\rangle$ formed from all possible linear combinations of the elements of S. Here are four vectors we definitely know are elements of $\langle S\rangle$, since we will construct them in accordance with Definition SSCV,

$$\mathbf{w}=(2)\begin{bmatrix}1\\1\\3\\1\end{bmatrix}+(1)\begin{bmatrix}2\\1\\2\\-1\end{bmatrix}+(-1)\begin{bmatrix}7\\3\\5\\-5\end{bmatrix}+(2)\begin{bmatrix}1\\1\\-1\\2\end{bmatrix}+(3)\begin{bmatrix}-1\\0\\9\\0\end{bmatrix}=\begin{bmatrix}-4\\2\\28\\10\end{bmatrix}$$

$$\mathbf{x}=(5)\begin{bmatrix}1\\1\\3\\1\end{bmatrix}+(-6)\begin{bmatrix}2\\1\\2\\-1\end{bmatrix}+(-3)\begin{bmatrix}7\\3\\5\\-5\end{bmatrix}+(4)\begin{bmatrix}1\\1\\-1\\2\end{bmatrix}+(2)\begin{bmatrix}-1\\0\\9\\0\end{bmatrix}=\begin{bmatrix}-26\\-6\\2\\34\end{bmatrix}$$

$$\mathbf{y} = (1)\begin{bmatrix}1\\1\\3\\1\end{bmatrix} + (0)\begin{bmatrix}2\\1\\2\\-1\end{bmatrix} + (1)\begin{bmatrix}7\\3\\5\\-5\end{bmatrix} + (0)\begin{bmatrix}1\\1\\-1\\2\end{bmatrix} + (1)\begin{bmatrix}-1\\0\\9\\0\end{bmatrix} = \begin{bmatrix}7\\4\\17\\-4\end{bmatrix}$$

$$\mathbf{z} = (0)\begin{bmatrix}1\\1\\3\\1\end{bmatrix} + (0)\begin{bmatrix}2\\1\\2\\-1\end{bmatrix} + (0)\begin{bmatrix}7\\3\\5\\-5\end{bmatrix} + (0)\begin{bmatrix}1\\1\\-1\\2\end{bmatrix} + (0)\begin{bmatrix}-1\\0\\9\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

The purpose of a set is to collect objects with some common property, and to exclude objects without that property. So the most fundamental question about a set is if a given object is an element of the set or not. Let us learn more about $\langle S\rangle$ by investigating which vectors are elements of the set, and which are not.

First, is $\mathbf{u} = \begin{bmatrix}-15\\-6\\19\\5\end{bmatrix}$ an element of $\langle S\rangle$? We are asking if there are scalars α_1, α_2, α_3, α_4, α_5 such that

$$\alpha_1\begin{bmatrix}1\\1\\3\\1\end{bmatrix} + \alpha_2\begin{bmatrix}2\\1\\2\\-1\end{bmatrix} + \alpha_3\begin{bmatrix}7\\3\\5\\-5\end{bmatrix} + \alpha_4\begin{bmatrix}1\\1\\-1\\2\end{bmatrix} + \alpha_5\begin{bmatrix}-1\\0\\9\\0\end{bmatrix} = \mathbf{u} = \begin{bmatrix}-15\\-6\\19\\5\end{bmatrix}$$

Applying Theorem SLSLC we recognize the search for these scalars as a solution to a linear system of equations with augmented matrix

$$\begin{bmatrix}1 & 2 & 7 & 1 & -1 & -15\\ 1 & 1 & 3 & 1 & 0 & -6\\ 3 & 2 & 5 & -1 & 9 & 19\\ 1 & -1 & -5 & 2 & 0 & 5\end{bmatrix}$$

which row-reduces to

$$\begin{bmatrix}\boxed{1} & 0 & -1 & 0 & 3 & 10\\ 0 & \boxed{1} & 4 & 0 & -1 & -9\\ 0 & 0 & 0 & \boxed{1} & -2 & -7\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

At this point, we see that the system is consistent (Theorem RCLS), so we know there *is* a solution for the five scalars α_1, α_2, α_3, α_4, α_5. This is enough evidence for us to say that $\mathbf{u} \in \langle S\rangle$. If we wished further evidence, we could compute an actual solution, say

$$\alpha_1 = 2 \qquad \alpha_2 = 1 \qquad \alpha_3 = -2 \qquad \alpha_4 = -3 \qquad \alpha_5 = 2$$

This particular solution allows us to write

$$(2)\begin{bmatrix}1\\1\\3\\1\end{bmatrix} + (1)\begin{bmatrix}2\\1\\2\\-1\end{bmatrix} + (-2)\begin{bmatrix}7\\3\\5\\-5\end{bmatrix} + (-3)\begin{bmatrix}1\\1\\-1\\2\end{bmatrix} + (2)\begin{bmatrix}-1\\0\\9\\0\end{bmatrix} = \mathbf{u} = \begin{bmatrix}-15\\-6\\19\\5\end{bmatrix}$$

making it even more obvious that $\mathbf{u} \in \langle S\rangle$.

Let us do it again. Is $\mathbf{v} = \begin{bmatrix}3\\1\\2\\-1\end{bmatrix}$ an element of $\langle S\rangle$? We are asking if there are scalars α_1, α_2, α_3, α_4, α_5 such that

$$\alpha_1\begin{bmatrix}1\\1\\3\\1\end{bmatrix} + \alpha_2\begin{bmatrix}2\\1\\2\\-1\end{bmatrix} + \alpha_3\begin{bmatrix}7\\3\\5\\-5\end{bmatrix} + \alpha_4\begin{bmatrix}1\\1\\-1\\2\end{bmatrix} + \alpha_5\begin{bmatrix}-1\\0\\9\\0\end{bmatrix} = \mathbf{v} = \begin{bmatrix}3\\1\\2\\-1\end{bmatrix}$$

Applying Theorem SLSLC we recognize the search for these scalars as a solution to a linear system of equations with augmented matrix

$$\begin{bmatrix} 1 & 2 & 7 & 1 & -1 & 3 \\ 1 & 1 & 3 & 1 & 0 & 1 \\ 3 & 2 & 5 & -1 & 9 & 2 \\ 1 & -1 & -5 & 2 & 0 & -1 \end{bmatrix}$$

which row-reduces to

$$\begin{bmatrix} \boxed{1} & 0 & -1 & 0 & 3 & 0 \\ 0 & \boxed{1} & 4 & 0 & -1 & 0 \\ 0 & 0 & 0 & \boxed{1} & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

At this point, we see that the system is inconsistent by Theorem RCLS, so we know there *is not* a solution for the five scalars α_1, α_2, α_3, α_4, α_5. This is enough evidence for us to say that $\mathbf{v} \notin \langle S \rangle$. End of story. △

Example SCAA Span of the columns of Archetype A
Begin with the finite set of three vectors of size 3

$$S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3\} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix},\, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix},\, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

and consider the infinite set $\langle S \rangle$. The vectors of S could have been chosen to be anything, but for reasons that will become clear later, we have chosen the three columns of the coefficient matrix in Archetype A.

First, as an example, note that

$$\mathbf{v} = (5) \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + (-3) \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + (7) \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 22 \\ 14 \\ 2 \end{bmatrix}$$

is in $\langle S \rangle$, since it is a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$. We write this succinctly as $\mathbf{v} \in \langle S \rangle$. There is nothing magical about the scalars $\alpha_1 = 5$, $\alpha_2 = -3$, $\alpha_3 = 7$, they could have been chosen to be anything. So repeat this part of the example yourself, using different values of α_1, α_2, α_3. What happens if you choose all three scalars to be zero?

So we know how to quickly construct sample elements of the set $\langle S \rangle$. A slightly different question arises when you are handed a vector of the correct size and asked if it is an element of $\langle S \rangle$. For example, is $\mathbf{w} = \begin{bmatrix} 1 \\ 8 \\ 5 \end{bmatrix}$ in $\langle S \rangle$? More succinctly, $\mathbf{w} \in \langle S \rangle$?

To answer this question, we will look for scalars α_1, α_2, α_3 so that

$$\alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \alpha_3 \mathbf{u}_3 = \mathbf{w}$$

By Theorem SLSLC solutions to this vector equation are solutions to the system of equations

$$\begin{aligned} \alpha_1 - \alpha_2 + 2\alpha_3 &= 1 \\ 2\alpha_1 + \alpha_2 + \alpha_3 &= 8 \\ \alpha_1 + \alpha_2 &= 5 \end{aligned}$$

Building the augmented matrix for this linear system, and row-reducing, gives

$$\begin{bmatrix} \boxed{1} & 0 & 1 & 3 \\ 0 & \boxed{1} & -1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This system has infinitely many solutions (there is a free variable in x_3), but all we need is one solution vector. The solution,

$$\alpha_1 = 2 \qquad \alpha_2 = 3 \qquad \alpha_3 = 1$$

tells us that

$$(2)\mathbf{u}_1 + (3)\mathbf{u}_2 + (1)\mathbf{u}_3 = \mathbf{w}$$

so we are convinced that $\mathbf{w}$ really is in $\langle S\rangle$. Notice that there are an infinite number of ways to answer this question affirmatively. We could choose a different solution, this time choosing the free variable to be zero,

$$\alpha_1 = 3 \qquad \alpha_2 = 2 \qquad \alpha_3 = 0$$

shows us that

$$(3)\mathbf{u}_1 + (2)\mathbf{u}_2 + (0)\mathbf{u}_3 = \mathbf{w}$$

Verifying the arithmetic in this second solution will make it obvious that $\mathbf{w}$ is in this span. And of course, we now realize that there are an infinite number of ways to realize $\mathbf{w}$ as element of $\langle S\rangle$.

Let us ask the same type of question again, but this time with $\mathbf{y} = \begin{bmatrix}2\\4\\3\end{bmatrix}$, i.e. is $\mathbf{y} \in \langle S\rangle$?

So we will look for scalars α_1, α_2, α_3 so that

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 = \mathbf{y}$$

By Theorem SLSLC solutions to this vector equation are the solutions to the system of equations

$$\begin{aligned}\alpha_1 - \alpha_2 + 2\alpha_3 &= 2\\ 2\alpha_1 + \alpha_2 + \alpha_3 &= 4\\ \alpha_1 + \alpha_2 &= 3\end{aligned}$$

Building the augmented matrix for this linear system, and row-reducing, gives

$$\begin{bmatrix}\boxed{1} & 0 & 1 & 0\\ 0 & \boxed{1} & -1 & 0\\ 0 & 0 & 0 & \boxed{1}\end{bmatrix}$$

This system is inconsistent (there is a pivot column in the last column, Theorem RCLS), so there are no scalars α_1, α_2, α_3 that will create a linear combination of $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ that equals $\mathbf{y}$. More precisely, $\mathbf{y} \notin \langle S\rangle$.

There are three things to observe in this example. (1) It is easy to construct vectors in $\langle S\rangle$. (2) It is possible that some vectors are in $\langle S\rangle$ (e.g. $\mathbf{w}$), while others are not (e.g. $\mathbf{y}$). (3) Deciding if a given vector is in $\langle S\rangle$ leads to solving a linear system of equations and asking if the system is consistent.

With a computer program in hand to solve systems of linear equations, could you create a program to decide if a vector was, or was not, in the span of a given set of vectors? Is this art or science?

This example was built on vectors from the columns of the coefficient matrix of Archetype A. Study the determination that $\mathbf{v} \in \langle S\rangle$ and see if you can connect it with some of the other properties of Archetype A. △

Having analyzed Archetype A in Example SCAA, we will of course subject Archetype B to a similar investigation.

Example SCAB Span of the columns of Archetype B
Begin with the finite set of three vectors of size 3 that are the columns of the

coefficient matrix in Archetype B,

$$R = \{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3\} = \left\{\begin{bmatrix}-7\\5\\1\end{bmatrix},\,\begin{bmatrix}-6\\5\\0\end{bmatrix},\,\begin{bmatrix}-12\\7\\4\end{bmatrix}\right\}$$

and consider the infinite set $\langle R\rangle$.

First, as an example, note that

$$\mathbf{x} = (2)\begin{bmatrix}-7\\5\\1\end{bmatrix} + (4)\begin{bmatrix}-6\\5\\0\end{bmatrix} + (-3)\begin{bmatrix}-12\\7\\4\end{bmatrix} = \begin{bmatrix}-2\\9\\-10\end{bmatrix}$$

is in $\langle R\rangle$, since it is a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$. In other words, $\mathbf{x} \in \langle R\rangle$. Try some different values of α_1, α_2, α_3 yourself, and see what vectors you can create as elements of $\langle R\rangle$.

Now ask if a given vector is an element of $\langle R\rangle$. For example, is $\mathbf{z} = \begin{bmatrix}-33\\24\\5\end{bmatrix}$ in $\langle R\rangle$? Is $\mathbf{z} \in \langle R\rangle$?

To answer this question, we will look for scalars α_1, α_2, α_3 so that

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 = \mathbf{z}$$

By Theorem SLSLC solutions to this vector equation are the solutions to the system of equations

$$\begin{aligned}-7\alpha_1 - 6\alpha_2 - 12\alpha_3 &= -33\\ 5\alpha_1 + 5\alpha_2 + 7\alpha_3 &= 24\\ \alpha_1 + 4\alpha_3 &= 5\end{aligned}$$

Building the augmented matrix for this linear system, and row-reducing, gives

$$\begin{bmatrix}\boxed{1} & 0 & 0 & -3\\ 0 & \boxed{1} & 0 & 5\\ 0 & 0 & \boxed{1} & 2\end{bmatrix}$$

This system has a unique solution,

$$\alpha_1 = -3 \qquad \alpha_2 = 5 \qquad \alpha_3 = 2$$

telling us that

$$(-3)\mathbf{v}_1 + (5)\mathbf{v}_2 + (2)\mathbf{v}_3 = \mathbf{z}$$

so we are convinced that $\mathbf{z}$ really is in $\langle R\rangle$. Notice that in this case we have only one way to answer the question affirmatively since the solution is unique.

Let us ask about another vector, say is $\mathbf{x} = \begin{bmatrix}-7\\8\\-3\end{bmatrix}$ in $\langle R\rangle$? Is $\mathbf{x} \in \langle R\rangle$?

We desire scalars α_1, α_2, α_3 so that

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 = \mathbf{x}$$

By Theorem SLSLC solutions to this vector equation are the solutions to the system of equations

$$\begin{aligned}-7\alpha_1 - 6\alpha_2 - 12\alpha_3 &= -7\\ 5\alpha_1 + 5\alpha_2 + 7\alpha_3 &= 8\\ \alpha_1 + 4\alpha_3 &= -3\end{aligned}$$

Building the augmented matrix for this linear system, and row-reducing, gives

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 1 \\ 0 & \boxed{1} & 0 & 2 \\ 0 & 0 & \boxed{1} & -1 \end{bmatrix}$$

This system has a unique solution,

$$\alpha_1 = 1 \qquad \alpha_2 = 2 \qquad \alpha_3 = -1$$

telling us that

$$(1)\mathbf{v}_1 + (2)\mathbf{v}_2 + (-1)\mathbf{v}_3 = \mathbf{x}$$

so we are convinced that $\mathbf{x}$ really is in $\langle R\rangle$. Notice that in this case we again have only one way to answer the question affirmatively since the solution is again unique.

We could continue to test other vectors for membership in $\langle R\rangle$, but there is no point. A question about membership in $\langle R\rangle$ inevitably leads to a system of three equations in the three variables α_1, α_2, α_3 with a coefficient matrix whose columns are the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$. This particular coefficient matrix is nonsingular, so by Theorem NMUS, the system is guaranteed to have a solution. (This solution is unique, but that is not critical here.) So *no matter* which vector we might have chosen for $\mathbf{z}$, we would have been *certain* to discover that it was an element of $\langle R\rangle$. Stated differently, every vector of size 3 is in $\langle R\rangle$, or $\langle R\rangle = \mathbb{C}^3$.

Compare this example with Example SCAA, and see if you can connect $\mathbf{z}$ with some aspects of the write-up for Archetype B. △

Subsection SSNS Spanning Sets of Null Spaces

We saw in Example VFSAL that when a system of equations is homogeneous the solution set can be expressed in the form described by Theorem VFSLS where the vector $\mathbf{c}$ is the zero vector. We can essentially ignore this vector, so that the remainder of the typical expression for a solution looks like an arbitrary linear combination, where the scalars are the free variables and the vectors are $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_{n-r}$. Which sounds a lot like a span. This is the substance of the next theorem.

Theorem SSNS Spanning Sets for Null Spaces
Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form. Suppose that B has r pivot columns, with indices given by $D = \{d_1, d_2, d_3, \ldots, d_r\}$, while the $n-r$ non-pivot columns have indices $F = \{f_1, f_2, f_3, \ldots, f_{n-r}, n+1\}$. Construct the $n-r$ vectors $\mathbf{z}_j$, $1 \le j \le n-r$ of size n,

$$[\mathbf{z}_j]_i = \begin{cases} 1 & \text{if } i \in F,\ i = f_j \\ 0 & \text{if } i \in F,\ i \neq f_j \\ -[B]_{k,f_j} & \text{if } i \in D,\ i = d_k \end{cases}$$

Then the null space of A is given by

$$\mathcal{N}(A) = \langle \{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \ldots, \mathbf{z}_{n-r}\} \rangle$$

Proof. Consider the homogeneous system with A as a coefficient matrix, $\mathcal{LS}(A, \mathbf{0})$. Its set of solutions, S, is by Definition NSM, the null space of A, $\mathcal{N}(A)$. Let B' denote the result of row-reducing the augmented matrix of this homogeneous system. Since the system is homogeneous, the final column of the augmented matrix will be all zeros, and after any number of row operations (Definition RO), the column will still be all zeros. So B' has a final column that is totally zeros.

Now apply Theorem VFSLS to B', after noting that our homogeneous system must be consistent (Theorem HSC). The vector $\mathbf{c}$ has zeros for each entry that has an index in F. For entries with their index in D, the value is $-\left[B'\right]_{k,n+1}$, but for B' any entry in the final column (index $n+1$) is zero. So $\mathbf{c}=\mathbf{0}$. The vectors $\mathbf{z}_j$, $1\leq j\leq n-r$ are identical to the vectors $\mathbf{u}_j$, $1\leq j\leq n-r$ described in Theorem VFSLS. Putting it all together and applying Definition SSCV in the final step,

$$\begin{aligned}\mathcal{N}(A) &= S\\ &= \left\{\mathbf{c}+\alpha_1\mathbf{u}_1+\alpha_2\mathbf{u}_2+\alpha_3\mathbf{u}_3+\cdots+\alpha_{n-r}\mathbf{u}_{n-r}\middle|\,\alpha_1,\,\alpha_2,\,\alpha_3,\,\ldots,\,\alpha_{n-r}\in\mathbb{C}\right\}\\ &= \left\{\alpha_1\mathbf{u}_1+\alpha_2\mathbf{u}_2+\alpha_3\mathbf{u}_3+\cdots+\alpha_{n-r}\mathbf{u}_{n-r}\middle|\,\alpha_1,\,\alpha_2,\,\alpha_3,\,\ldots,\,\alpha_{n-r}\in\mathbb{C}\right\}\\ &= \left\langle\left\{\mathbf{z}_1,\,\mathbf{z}_2,\,\mathbf{z}_3,\,\ldots,\,\mathbf{z}_{n-r}\right\}\right\rangle\end{aligned}$$

■

Notice that the hypotheses of Theorem VFSLS and Theorem SSNS are slightly different. In the former, B is the row-reduced version of an augmented matrix of a linear system, while in the latter, B is the row-reduced version of an arbitrary matrix. Understanding this subtlety now will avoid confusion later.

Example SSNS Spanning set of a null space
Find a set of vectors, S, so that the null space of the matrix A below is the span of S, that is, $\langle S\rangle=\mathcal{N}(A)$.

$$A=\begin{bmatrix}1 & 3 & 3 & -1 & -5\\ 2 & 5 & 7 & 1 & 1\\ 1 & 1 & 5 & 1 & 5\\ -1 & -4 & -2 & 0 & 4\end{bmatrix}$$

The null space of A is the set of all solutions to the homogeneous system $\mathcal{LS}(A,\,\mathbf{0})$. If we find the vector form of the solutions to this homogeneous system (Theorem VFSLS) then the vectors $\mathbf{u}_j$, $1\leq j\leq n-r$ in the linear combination are exactly the vectors $\mathbf{z}_j$, $1\leq j\leq n-r$ described in Theorem SSNS. So we can mimic Example VFSAL to arrive at these vectors (rather than being a slave to the formulas in the statement of the theorem).

Begin by row-reducing A. The result is

$$\begin{bmatrix}\boxed{1} & 0 & 6 & 0 & 4\\ 0 & \boxed{1} & -1 & 0 & -2\\ 0 & 0 & 0 & \boxed{1} & 3\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

With $D=\{1,\,2,\,4\}$ and $F=\{3,\,5\}$ we recognize that x_3 and x_5 are free variables and we can interpret each nonzero row as an expression for the dependent variables x_1, x_2, x_4 (respectively) in the free variables x_3 and x_5. With this we can write the vector form of a solution vector as

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=\begin{bmatrix}-6x_3-4x_5\\x_3+2x_5\\x_3\\-3x_5\\x_5\end{bmatrix}=x_3\begin{bmatrix}-6\\1\\1\\0\\0\end{bmatrix}+x_5\begin{bmatrix}-4\\2\\0\\-3\\1\end{bmatrix}$$

Then in the notation of Theorem SSNS,

$$\mathbf{z}_1=\begin{bmatrix}-6\\1\\1\\0\\0\end{bmatrix}\qquad\qquad\mathbf{z}_2=\begin{bmatrix}-4\\2\\0\\-3\\1\end{bmatrix}$$

and

$$\mathcal{N}(A) = \langle\{\mathbf{z}_1,\, \mathbf{z}_2\}\rangle = \left\langle\left\{\begin{bmatrix}-6\\1\\1\\0\\0\end{bmatrix},\, \begin{bmatrix}-4\\2\\0\\-3\\1\end{bmatrix}\right\}\right\rangle$$

△

Example NSDS Null space directly as a span
Let us express the null space of A as the span of a set of vectors, applying Theorem SSNS as economically as possible, without reference to the underlying homogeneous system of equations (in contrast to Example SSNS).

$$A = \begin{bmatrix} 2 & 1 & 5 & 1 & 5 & 1\\ 1 & 1 & 3 & 1 & 6 & -1\\ -1 & 1 & -1 & 0 & 4 & -3\\ -3 & 2 & -4 & -4 & -7 & 0\\ 3 & -1 & 5 & 2 & 2 & 3 \end{bmatrix}$$

Theorem SSNS creates vectors for the span by first row-reducing the matrix in question. The row-reduced version of A is

$$B = \begin{bmatrix} \boxed{1} & 0 & 2 & 0 & -1 & 2\\ 0 & \boxed{1} & 1 & 0 & 3 & -1\\ 0 & 0 & 0 & \boxed{1} & 4 & -2\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We will mechanically follow the prescription of Theorem SSNS. Here we go, in two big steps.

First, the non-pivot columns have indices $F = \{3,\, 5,\, 6\}$, so we will construct the $n - r = 6 - 3 = 3$ vectors with a pattern of zeros and ones dictated by the indices in F. This is the realization of the first two lines of the three-case definition of the vectors $\mathbf{z}_j$, $1 \leq j \leq n - r$.

$$\mathbf{z}_1 = \begin{bmatrix} \\ \\ 1\\ \\ 0\\ 0 \end{bmatrix} \qquad \mathbf{z}_2 = \begin{bmatrix} \\ \\ 0\\ \\ 1\\ 0 \end{bmatrix} \qquad \mathbf{z}_3 = \begin{bmatrix} \\ \\ 0\\ \\ 0\\ 1 \end{bmatrix}$$

Each of these vectors arises due to the presence of a column that is not a pivot column. The remaining entries of each vector are the entries of the non-pivot column, negated, and distributed into the empty slots in order (these slots have indices in the set D, so also refer to pivot columns). This is the realization of the third line of the three-case definition of the vectors $\mathbf{z}_j$, $1 \leq j \leq n - r$.

$$\mathbf{z}_1 = \begin{bmatrix} -2\\ -1\\ 1\\ 0\\ 0\\ 0 \end{bmatrix} \qquad \mathbf{z}_2 = \begin{bmatrix} 1\\ -3\\ 0\\ -4\\ 1\\ 0 \end{bmatrix} \qquad \mathbf{z}_3 = \begin{bmatrix} -2\\ 1\\ 0\\ 2\\ 0\\ 1 \end{bmatrix}$$

So, by Theorem SSNS, we have

$$\mathcal{N}(A) = \langle\{\mathbf{z}_1,\,\mathbf{z}_2,\,\mathbf{z}_3\}\rangle = \left\langle\left\{\begin{bmatrix}-2\\-1\\1\\0\\0\\0\end{bmatrix},\,\begin{bmatrix}1\\-3\\0\\-4\\1\\0\end{bmatrix},\,\begin{bmatrix}-2\\1\\0\\2\\0\\1\end{bmatrix}\right\}\right\rangle$$

We know that the null space of A is the solution set of the homogeneous system $\mathcal{LS}(A,\,\mathbf{0})$, but nowhere in this application of Theorem SSNS have we found occasion to reference the variables or equations of this system. These details are all buried in the proof of Theorem SSNS. △

Here is an example that will simultaneously exercise the span construction and Theorem SSNS, while also pointing the way to the next section.

Example SCAD Span of the columns of Archetype D
Begin with the set of four vectors of size 3

$$T = \{\mathbf{w}_1,\,\mathbf{w}_2,\,\mathbf{w}_3,\,\mathbf{w}_4\} = \left\{\begin{bmatrix}2\\-3\\1\end{bmatrix},\,\begin{bmatrix}1\\4\\1\end{bmatrix},\,\begin{bmatrix}7\\-5\\4\end{bmatrix},\,\begin{bmatrix}-7\\-6\\-5\end{bmatrix}\right\}$$

and consider the infinite set $W = \langle T\rangle$. The vectors of T have been chosen as the four columns of the coefficient matrix in Archetype D. Check that the vector

$$\mathbf{z}_2 = \begin{bmatrix}2\\3\\0\\1\end{bmatrix}$$

is a solution to the homogeneous system $\mathcal{LS}(D,\,\mathbf{0})$ (it is the vector $\mathbf{z}_2$ provided by the description of the null space of the coefficient matrix D from Theorem SSNS).

Applying Theorem SLSLC, we can write the linear combination,

$$2\mathbf{w}_1 + 3\mathbf{w}_2 + 0\mathbf{w}_3 + 1\mathbf{w}_4 = \mathbf{0}$$

which we can solve for $\mathbf{w}_4$,

$$\mathbf{w}_4 = (-2)\mathbf{w}_1 + (-3)\mathbf{w}_2.$$

This equation says that whenever we encounter the vector $\mathbf{w}_4$, we can replace it with a specific linear combination of the vectors $\mathbf{w}_1$ and $\mathbf{w}_2$. So using $\mathbf{w}_4$ in the set T, along with $\mathbf{w}_1$ and $\mathbf{w}_2$, is excessive. An example of what we mean here can be illustrated by the computation,

$$\begin{aligned}
&5\mathbf{w}_1 + (-4)\mathbf{w}_2 + 6\mathbf{w}_3 + (-3)\mathbf{w}_4\\
&\quad= 5\mathbf{w}_1 + (-4)\mathbf{w}_2 + 6\mathbf{w}_3 + (-3)\left((-2)\mathbf{w}_1 + (-3)\mathbf{w}_2\right)\\
&\quad= 5\mathbf{w}_1 + (-4)\mathbf{w}_2 + 6\mathbf{w}_3 + (6\mathbf{w}_1 + 9\mathbf{w}_2)\\
&\quad= 11\mathbf{w}_1 + 5\mathbf{w}_2 + 6\mathbf{w}_3
\end{aligned}$$

So what began as a linear combination of the vectors $\mathbf{w}_1,\,\mathbf{w}_2,\,\mathbf{w}_3,\,\mathbf{w}_4$ has been reduced to a linear combination of the vectors $\mathbf{w}_1,\,\mathbf{w}_2,\,\mathbf{w}_3$. A careful proof using our definition of set equality (Definition SE) would now allow us to conclude that this reduction is possible for any vector in W, so

$$W = \langle\{\mathbf{w}_1,\,\mathbf{w}_2,\,\mathbf{w}_3\}\rangle$$

So the span of our set of vectors, W, has not changed, but we have *described* it by the span of a set of *three* vectors, rather than *four*. Furthermore, we can achieve yet another, similar, reduction.

Check that the vector

$$\mathbf{z}_1 = \begin{bmatrix} -3 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

is a solution to the homogeneous system $\mathcal{LS}(D,\ \mathbf{0})$ (it is the vector $\mathbf{z}_1$ provided by the description of the null space of the coefficient matrix D from Theorem SSNS). Applying Theorem SLSLC, we can write the linear combination,

$$(-3)\mathbf{w}_1 + (-1)\mathbf{w}_2 + 1\mathbf{w}_3 = \mathbf{0}$$

which we can solve for $\mathbf{w}_3$,

$$\mathbf{w}_3 = 3\mathbf{w}_1 + 1\mathbf{w}_2$$

This equation says that whenever we encounter the vector $\mathbf{w}_3$, we can replace it with a specific linear combination of the vectors $\mathbf{w}_1$ and $\mathbf{w}_2$. So, as before, the vector $\mathbf{w}_3$ is not needed in the description of W, provided we have $\mathbf{w}_1$ and $\mathbf{w}_2$ available. In particular, a careful proof (such as is done in Example RSC5) would show that

$$W = \langle\{\mathbf{w}_1,\ \mathbf{w}_2\}\rangle$$

So W began life as the span of a set of four vectors, and we have now shown (utilizing solutions to a homogeneous system) that W can also be described as the span of a set of just two vectors. Convince yourself that we cannot go any further. In other words, it is not possible to dismiss either $\mathbf{w}_1$ or $\mathbf{w}_2$ in a similar fashion and winnow the set down to just one vector.

What was it about the original set of four vectors that allowed us to declare certain vectors as surplus? And just which vectors were we able to dismiss? And why did we have to stop once we had two vectors remaining? The answers to these questions motivate "linear independence," our next section and next definition, and so are worth considering carefully *now*. △

Reading Questions

1. Let S be the set of three vectors below.

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix},\ \begin{bmatrix} 3 \\ -4 \\ 2 \end{bmatrix},\ \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} \right\}$$

   Let $W = \langle S\rangle$ be the span of S. Is the vector $\begin{bmatrix} -1 \\ 8 \\ -4 \end{bmatrix}$ in W? Give an explanation of the reason for your answer.

2. Use S and W from the previous question. Is the vector $\begin{bmatrix} 6 \\ 5 \\ -1 \end{bmatrix}$ in W? Give an explanation of the reason for your answer.

3. For the matrix A below, find a set S so that $\langle S\rangle = \mathcal{N}(A)$, where $\mathcal{N}(A)$ is the null space of A. (See Theorem SSNS.)

$$A = \begin{bmatrix} 1 & 3 & 1 & 9 \\ 2 & 1 & -3 & 8 \\ 1 & 1 & -1 & 5 \end{bmatrix}$$

Exercises

C22† For each archetype that is a system of equations, consider the corresponding homogeneous system of equations. Write elements of the solution set to these homogeneous systems in vector form, as guaranteed by Theorem VFSLS. Then write the null space of the coefficient matrix of each system as the span of a set of vectors, as described in Theorem SSNS.

Archetype A, Archetype B, Archetype C, Archetype D/Archetype E, Archetype F, Archetype G/Archetype H, Archetype I, Archetype J

C23† Archetype K and Archetype L are defined as matrices. Use Theorem SSNS directly to find a set S so that $\langle S\rangle$ is the null space of the matrix. Do not make any reference to the associated homogeneous system of equations in your solution.

C40† Suppose that $S=\left\{\begin{bmatrix}2\\-1\\3\\4\end{bmatrix},\,\begin{bmatrix}3\\2\\-2\\1\end{bmatrix}\right\}$. Let $W=\langle S\rangle$ and let $\mathbf{x}=\begin{bmatrix}5\\8\\-12\\-5\end{bmatrix}$. Is $\mathbf{x}\in W$? If so, provide an explicit linear combination that demonstrates this.

C41† Suppose that $S=\left\{\begin{bmatrix}2\\-1\\3\\4\end{bmatrix},\,\begin{bmatrix}3\\2\\-2\\1\end{bmatrix}\right\}$. Let $W=\langle S\rangle$ and let $\mathbf{y}=\begin{bmatrix}5\\1\\3\\5\end{bmatrix}$. Is $\mathbf{y}\in W$? If so, provide an explicit linear combination that demonstrates this.

C42† Suppose $R=\left\{\begin{bmatrix}2\\-1\\3\\4\\0\end{bmatrix},\,\begin{bmatrix}1\\1\\2\\2\\-1\end{bmatrix},\,\begin{bmatrix}3\\-1\\0\\3\\-2\end{bmatrix}\right\}$. Is $\mathbf{y}=\begin{bmatrix}1\\-1\\-8\\-4\\-3\end{bmatrix}$ in $\langle R\rangle$?

C43† Suppose $R=\left\{\begin{bmatrix}2\\-1\\3\\4\\0\end{bmatrix},\,\begin{bmatrix}1\\1\\2\\2\\-1\end{bmatrix},\,\begin{bmatrix}3\\-1\\0\\3\\-2\end{bmatrix}\right\}$. Is $\mathbf{z}=\begin{bmatrix}1\\1\\5\\3\\1\end{bmatrix}$ in $\langle R\rangle$?

C44† Suppose that $S=\left\{\begin{bmatrix}-1\\2\\1\end{bmatrix},\,\begin{bmatrix}3\\1\\2\end{bmatrix},\,\begin{bmatrix}1\\5\\4\end{bmatrix},\,\begin{bmatrix}-6\\5\\1\end{bmatrix}\right\}$. Let $W=\langle S\rangle$ and let $\mathbf{y}=\begin{bmatrix}-5\\3\\0\end{bmatrix}$. Is $\mathbf{y}\in W$? If so, provide an explicit linear combination that demonstrates this.

C45† Suppose that $S=\left\{\begin{bmatrix}-1\\2\\1\end{bmatrix},\,\begin{bmatrix}3\\1\\2\end{bmatrix},\,\begin{bmatrix}1\\5\\4\end{bmatrix},\,\begin{bmatrix}-6\\5\\1\end{bmatrix}\right\}$. Let $W=\langle S\rangle$ and let $\mathbf{w}=\begin{bmatrix}2\\1\\3\end{bmatrix}$. Is $\mathbf{w}\in W$? If so, provide an explicit linear combination that demonstrates this.

C50† Let A be the matrix below.

1. Find a set S so that $\mathcal{N}(A)=\langle S\rangle$.
2. If $\mathbf{z}=\begin{bmatrix}3\\-5\\1\\2\end{bmatrix}$, then show directly that $\mathbf{z}\in\mathcal{N}(A)$.
3. Write $\mathbf{z}$ as a linear combination of the vectors in S.

$$A=\begin{bmatrix}2&3&1&4\\1&2&1&3\\-1&0&1&1\end{bmatrix}$$

C60† For the matrix A below, find a set of vectors S so that the span of S equals the null space of A, $\langle S\rangle=\mathcal{N}(A)$.

$$A=\begin{bmatrix}1&1&6&-8\\1&-2&0&1\\-2&1&-6&7\end{bmatrix}$$

M10† Consider the set of all size 2 vectors in the Cartesian plane $\mathbb{R}^2$.

1. Give a geometric description of the span of a single vector.
2. How can you tell if two vectors span the entire plane, without doing any row reduction or calculation?

M11† Consider the set of all size 3 vectors in Cartesian 3-space $\mathbb{R}^3$.

1. Give a geometric description of the span of a single vector.
2. Describe the possibilities for the span of two vectors.
3. Describe the possibilities for the span of three vectors.

M12† Let $\mathbf{u} = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$.

1. Find a vector $\mathbf{w}_1$, different from $\mathbf{u}$ and $\mathbf{v}$, so that $\langle\{\mathbf{u}, \mathbf{v}, \mathbf{w}_1\}\rangle = \langle\{\mathbf{u}, \mathbf{v}\}\rangle$.
2. Find a vector $\mathbf{w}_2$ so that $\langle\{\mathbf{u}, \mathbf{v}, \mathbf{w}_2\}\rangle \neq \langle\{\mathbf{u}, \mathbf{v}\}\rangle$.

M20 In Example SCAD we began with the four columns of the coefficient matrix of Archetype D, and used these columns in a span construction. Then we methodically argued that we could remove the last column, then the third column, and create the same set by just doing a span construction with the first two columns. We claimed we could not go any further, and had removed as many vectors as possible. Provide a convincing argument for why a third vector cannot be removed.

M21† In the spirit of Example SCAD, begin with the four columns of the coefficient matrix of Archetype C, and use these columns in a span construction to build the set S. Argue that S can be expressed as the span of just three of the columns of the coefficient matrix (saying exactly which three) and in the spirit of Exercise SS.M20 argue that no one of these three vectors can be removed and still have a span construction create S.

T10† Suppose that $\mathbf{v}_1,\ \mathbf{v}_2 \in \mathbb{C}^m$. Prove that

$$\langle\{\mathbf{v}_1,\ \mathbf{v}_2\}\rangle = \langle\{\mathbf{v}_1,\ \mathbf{v}_2,\ 5\mathbf{v}_1 + 3\mathbf{v}_2\}\rangle$$

T20† Suppose that S is a set of vectors from $\mathbb{C}^m$. Prove that the zero vector, $\mathbf{0}$, is an element of $\langle S\rangle$.

T21 Suppose that S is a set of vectors from $\mathbb{C}^m$ and $\mathbf{x},\ \mathbf{y} \in \langle S\rangle$. Prove that $\mathbf{x} + \mathbf{y} \in \langle S\rangle$.

T22 Suppose that S is a set of vectors from $\mathbb{C}^m$, $\alpha \in \mathbb{C}$, and $\mathbf{x} \in \langle S\rangle$. Prove that $\alpha\mathbf{x} \in \langle S\rangle$.

Section LI
Linear Independence

"Linear independence" is one of the most fundamental conceptual ideas in linear algebra, along with the notion of a span. So this section, and the subsequent Section LDS, will explore this new idea.

Subsection LISV
Linearly Independent Sets of Vectors

Theorem SLSLC tells us that a solution to a homogeneous system of equations is a linear combination of the columns of the coefficient matrix that equals the zero vector. We used just this situation to our advantage (twice!) in Example SCAD where we reduced the set of vectors used in a span construction from four down to two, by declaring certain vectors as surplus. The next two definitions will allow us to formalize this situation.

Definition RLDCV Relation of Linear Dependence for Column Vectors
Given a set of vectors $S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_n\}$, a true statement of the form

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_n\mathbf{u}_n = \mathbf{0}$$

is a **relation of linear dependence** on S. If this statement is formed in a trivial fashion, i.e. $\alpha_i = 0$, $1 \leq i \leq n$, then we say it is the **trivial relation of linear dependence** on S. □

Definition LICV Linear Independence of Column Vectors
The set of vectors $S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_n\}$ is **linearly dependent** if there is a relation of linear dependence on S that is not trivial. In the case where the *only* relation of linear dependence on S is the trivial one, then S is a **linearly independent** set of vectors. □

Notice that a relation of linear dependence is an *equation*. Though most of it is a linear combination, it is not a linear combination (that would be a vector). Linear independence is a property of a *set* of vectors. It is easy to take a set of vectors, and an equal number of scalars, *all zero*, and form a linear combination that equals the zero vector. When the easy way is the *only* way, then we say the set is linearly independent. Here are a couple of examples.

Example LDS Linearly dependent set in $\mathbb{C}^5$
Consider the set of $n = 4$ vectors from $\mathbb{C}^5$,

$$S = \left\{ \begin{bmatrix}2\\-1\\3\\1\\2\end{bmatrix},\, \begin{bmatrix}1\\2\\-1\\5\\2\end{bmatrix},\, \begin{bmatrix}2\\1\\-3\\6\\1\end{bmatrix},\, \begin{bmatrix}-6\\7\\-1\\0\\1\end{bmatrix} \right\}$$

To determine linear independence we first form a relation of linear dependence,

$$\alpha_1 \begin{bmatrix}2\\-1\\3\\1\\2\end{bmatrix} + \alpha_2 \begin{bmatrix}1\\2\\-1\\5\\2\end{bmatrix} + \alpha_3 \begin{bmatrix}2\\1\\-3\\6\\1\end{bmatrix} + \alpha_4 \begin{bmatrix}-6\\7\\-1\\0\\1\end{bmatrix} = \mathbf{0}$$

We know that $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ is a solution to this equation, but that is of no interest whatsoever. That is *always* the case, no matter what four vectors we might have chosen. We are curious to know if there are other, nontrivial, solutions. Theorem SLSLC tells us that we can find such solutions as solutions to the homogeneous system $\mathcal{LS}(A,\, \mathbf{0})$ where the coefficient matrix has these four vectors as columns, which we then row-reduce

$$A = \begin{bmatrix} 2 & 1 & 2 & -6 \\ -1 & 2 & 1 & 7 \\ 3 & -1 & -3 & -1 \\ 1 & 5 & 6 & 0 \\ 2 & 2 & 1 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & -2 \\ 0 & \boxed{1} & 0 & 4 \\ 0 & 0 & \boxed{1} & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We could solve this homogeneous system completely, but for this example all we need is one nontrivial solution. Setting the lone free variable to any nonzero value, such as $x_4 = 1$, yields the nontrivial solution

$$\mathbf{x} = \begin{bmatrix} 2 \\ -4 \\ 3 \\ 1 \end{bmatrix}$$

completing our application of Theorem SLSLC, we have

$$2\begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \\ 2 \end{bmatrix} + (-4)\begin{bmatrix} 1 \\ 2 \\ -1 \\ 5 \\ 2 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 1 \\ -3 \\ 6 \\ 1 \end{bmatrix} + 1\begin{bmatrix} -6 \\ 7 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \mathbf{0}$$

This is a relation of linear dependence on S that is not trivial, so we conclude that S is linearly dependent. △

Example LIS Linearly independent set in $\mathbb{C}^5$
Consider the set of $n = 4$ vectors from $\mathbb{C}^5$,

$$T = \left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -3 \\ 6 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 7 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

To determine linear independence we first form a relation of linear dependence,

$$\alpha_1\begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \\ 2 \end{bmatrix} + \alpha_2\begin{bmatrix} 1 \\ 2 \\ -1 \\ 5 \\ 2 \end{bmatrix} + \alpha_3\begin{bmatrix} 2 \\ 1 \\ -3 \\ 6 \\ 1 \end{bmatrix} + \alpha_4\begin{bmatrix} -6 \\ 7 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \mathbf{0}$$

We know that $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$ is a solution to this equation, but that is of no interest whatsoever. That is *always* the case, no matter what four vectors we might have chosen. We are curious to know if there are other, nontrivial, solutions. Theorem SLSLC tells us that we can find such solutions as solution to the homogeneous system $\mathcal{LS}(B, \mathbf{0})$ where the coefficient matrix has these four vectors as columns. Row-reducing this coefficient matrix yields,

$$B = \begin{bmatrix} 2 & 1 & 2 & -6 \\ -1 & 2 & 1 & 7 \\ 3 & -1 & -3 & -1 \\ 1 & 5 & 6 & 1 \\ 2 & 2 & 1 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From the form of this matrix, we see that there are no free variables, so the solution is unique, and because the system is homogeneous, this unique solution is the trivial solution. So we now know that there is but one way to combine the four vectors of T into a relation of linear dependence, and that one way is the easy and obvious way. In this situation we say that the set, T, is linearly independent. △

Example LDS and Example LIS relied on solving a homogeneous system of equations to determine linear independence. We can codify this process in a time-saving theorem.

Theorem LIVHS Linearly Independent Vectors and Homogeneous Systems
Suppose that $S = \{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3,\,\ldots,\,\mathbf{v}_n\} \subseteq \mathbb{C}^m$ *is a set of vectors and* A *is the* $m \times n$ *matrix whose columns are the vectors in* S*. Then* S *is a linearly independent set if and only if the homogeneous system* $\mathcal{LS}(A,\,\mathbf{0})$ *has a unique solution.*

Proof. ($\Leftarrow$) Suppose that $\mathcal{LS}(A,\,\mathbf{0})$ has a unique solution. Since it is a homogeneous system, this solution must be the trivial solution $\mathbf{x} = \mathbf{0}$. By Theorem SLSLC, this means that the only relation of linear dependence on S is the trivial one. So S is linearly independent.

($\Rightarrow$) We will prove the contrapositive. Suppose that $\mathcal{LS}(A,\,\mathbf{0})$ does not have a unique solution. Since it is a homogeneous system, it is consistent (Theorem HSC), and so must have infinitely many solutions (Theorem PSSLS). One of these infinitely many solutions must be nontrivial (in fact, almost all of them are), so choose one. By Theorem SLSLC this nontrivial solution will give a nontrivial relation of linear dependence on S, so we can conclude that S is a linearly dependent set. ■

Since Theorem LIVHS is an equivalence, we can use it to determine the linear independence or dependence of any set of column vectors, just by creating a matrix and analyzing the row-reduced form. Let us illustrate this with two more examples.

Example LIHS Linearly independent, homogeneous system
Is the set of vectors

$$S = \left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \\ 2 \end{bmatrix},\, \begin{bmatrix} 6 \\ 2 \\ -1 \\ 3 \\ 4 \end{bmatrix},\, \begin{bmatrix} 4 \\ 3 \\ -4 \\ 5 \\ 1 \end{bmatrix} \right\}$$

linearly independent or linearly dependent?

Theorem LIVHS suggests we study the matrix, A, whose columns are the vectors in S. Specifically, we are interested in the size of the solution set for the homogeneous system $\mathcal{LS}(A,\,\mathbf{0})$, so we row-reduce A.

$$A = \begin{bmatrix} 2 & 6 & 4 \\ -1 & 2 & 3 \\ 3 & -1 & -4 \\ 4 & 3 & 5 \\ 2 & 4 & 1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now, $r = 3$, so there are $n - r = 3 - 3 = 0$ free variables and we see that $\mathcal{LS}(A,\,\mathbf{0})$ has a unique solution (Theorem HSC, Theorem FVCS). By Theorem LIVHS, the set S is linearly independent. △

Example LDHS Linearly dependent, homogeneous system
Is the set of vectors

$$S = \left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \\ 4 \\ 2 \end{bmatrix},\, \begin{bmatrix} 6 \\ 2 \\ -1 \\ 3 \\ 4 \end{bmatrix},\, \begin{bmatrix} 4 \\ 3 \\ -4 \\ -1 \\ 2 \end{bmatrix} \right\}$$

linearly independent or linearly dependent?

Theorem LIVHS suggests we study the matrix, A, whose columns are the vectors in S. Specifically, we are interested in the size of the solution set for the homogeneous system $\mathcal{LS}(A,\,\mathbf{0})$, so we row-reduce A.

$$A = \begin{bmatrix} 2 & 6 & 4 \\ -1 & 2 & 3 \\ 3 & -1 & -4 \\ 4 & 3 & -1 \\ 2 & 4 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now, $r = 2$, so there are $n - r = 3 - 2 = 1$ free variables and we see that $\mathcal{LS}(A, \mathbf{0})$ has infinitely many solutions (Theorem HSC, Theorem FVCS). By Theorem LIVHS, the set S is linearly dependent. △

As an equivalence, Theorem LIVHS gives us a straightforward way to determine if a set of vectors is linearly independent or dependent.

Review Example LIHS and Example LDHS. They are very similar, differing only in the last two slots of the third vector. This resulted in slightly different matrices when row-reduced, and slightly different values of r, the number of nonzero rows. Notice, too, that we are less interested in the actual solution set, and more interested in its form or size. These observations allow us to make a slight improvement in Theorem LIVHS.

Theorem LIVRN Linearly Independent Vectors, r and n
Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\} \subseteq \mathbb{C}^m$ *is a set of vectors and* A *is the* $m \times n$ *matrix whose columns are the vectors in* S. *Let* B *be a matrix in reduced row-echelon form that is row-equivalent to* A *and let* r *denote the number of pivot columns in* B. *Then* S *is linearly independent if and only if* $n = r$.

Proof. Theorem LIVHS says the linear independence of S is equivalent to the homogeneous linear system $\mathcal{LS}(A, \mathbf{0})$ having a unique solution. Since $\mathcal{LS}(A, \mathbf{0})$ is consistent (Theorem HSC) we can apply Theorem CSRN to see that the solution is unique exactly when $n = r$. ■

So now here is an example of the most straightforward way to determine if a set of column vectors is linearly independent or linearly dependent. While this method can be quick and easy, do not forget the logical progression from the definition of linear independence through homogeneous system of equations which makes it possible.

Example LDRN Linearly dependent, r and n
Is the set of vectors

$$S = \left\{ \begin{bmatrix} 2 \\ -1 \\ 3 \\ 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 9 \\ -6 \\ -2 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 4 \\ 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \\ 1 \\ 4 \\ 3 \\ 2 \end{bmatrix} \right\}$$

linearly independent or linearly dependent?

Theorem LIVRN suggests we place these vectors into a matrix as columns and analyze the row-reduced version of the matrix,

$$\begin{bmatrix} 2 & 9 & 1 & -3 & 6 \\ -1 & -6 & 1 & 1 & -2 \\ 3 & -2 & 1 & 4 & 1 \\ 1 & 3 & 0 & 2 & 4 \\ 0 & 2 & 0 & 1 & 3 \\ 3 & 1 & 1 & 2 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 0 & -1 \\ 0 & \boxed{1} & 0 & 0 & 1 \\ 0 & 0 & \boxed{1} & 0 & 2 \\ 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now we need only compute that $r = 4 < 5 = n$ to recognize, via Theorem LIVRN that S is a linearly dependent set. Boom! △

Example LLDS Large linearly dependent set in $\mathbb{C}^4$
Consider the set of $n = 9$ vectors from $\mathbb{C}^4$,

$$R = \left\{ \begin{bmatrix} -1 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 7 \\ 1 \\ -3 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 2 \\ 9 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -6 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 5 \\ 3 \end{bmatrix}, \begin{bmatrix} -6 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

To employ Theorem LIVHS, we form a 4×9 matrix, C, whose columns are the vectors in B

$$C = \begin{bmatrix} -1 & 7 & 1 & 0 & 5 & 2 & 3 & 1 & -6 \\ 3 & 1 & 2 & 4 & -2 & 1 & 0 & 1 & -1 \\ 1 & -3 & -1 & 2 & 4 & -6 & -3 & 5 & 1 \\ 2 & 6 & -2 & 9 & 3 & 4 & 1 & 3 & 1 \end{bmatrix}.$$

To determine if the homogeneous system $\mathcal{LS}(C, \mathbf{0})$ has a unique solution or not, we would normally row-reduce this matrix. But in this particular example, we can do better. Theorem HMVEI tells us that since the system is homogeneous with $n = 9$ variables in $m = 4$ equations, and $n > m$, there must be infinitely many solutions. Since there is not a unique solution, Theorem LIVHS says the set is linearly dependent. △

The situation in Example LLDS is slick enough to warrant formulating as a theorem.

Theorem MVSLD More Vectors than Size implies Linear Dependence
Suppose that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\} \subseteq \mathbb{C}^m$ *and* $n > m$*. Then* S *is a linearly dependent set.*

Proof. Form the $m \times n$ matrix A whose columns are $\mathbf{u}_i$, $1 \leq i \leq n$. Consider the homogeneous system $\mathcal{LS}(A, \mathbf{0})$. By Theorem HMVEI this system has infinitely many solutions. Since the system does not have a unique solution, Theorem LIVHS says the columns of A form a linearly dependent set, as desired. ■

Subsection LINM
Linear Independence and Nonsingular Matrices

We will now specialize to sets of n vectors from $\mathbb{C}^n$. This will put Theorem MVSLD off-limits, while Theorem LIVHS will involve square matrices. Let us begin by contrasting Archetype A and Archetype B.

Example LDCAA Linearly dependent columns in Archetype A
Archetype A is a system of linear equations with coefficient matrix,

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Do the columns of this matrix form a linearly independent or dependent set? By Example S we know that A is singular. According to the definition of nonsingular matrices, Definition NM, the homogeneous system $\mathcal{LS}(A, \mathbf{0})$ has infinitely many solutions. So by Theorem LIVHS, the columns of A form a linearly dependent set. △

Example LICAB Linearly independent columns in Archetype B
Archetype B is a system of linear equations with coefficient matrix,

$$B = \begin{bmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{bmatrix}$$

Do the columns of this matrix form a linearly independent or dependent set? By Example NM we know that B is nonsingular. According to the definition of nonsingular matrices, Definition NM, the homogeneous system $\mathcal{LS}(A, \mathbf{0})$ has a unique solution. So by Theorem LIVHS, the columns of B form a linearly independent set. △

That Archetype A and Archetype B have opposite properties for the columns of their coefficient matrices is no accident. Here is the theorem, and then we will update our equivalences for nonsingular matrices, Theorem NME1.

Theorem NMLIC Nonsingular Matrices have Linearly Independent Columns
Suppose that A is a square matrix. Then A is nonsingular if and only if the columns of A form a linearly independent set.

Proof. This is a proof where we can chain together equivalences, rather than proving the two halves separately.

$$\begin{aligned} A \text{ nonsingular} &\iff \mathcal{LS}(A, \mathbf{0}) \text{ has a unique solution} && \text{Definition NM} \\ &\iff \text{columns of } A \text{ are linearly independent} && \text{Theorem LIVHS} \end{aligned}$$

■

Here is the update to Theorem NME1.

Theorem NME2 Nonsingular Matrix Equivalences, Round 2
Suppose that A is a square matrix. The following are equivalent.

1. *A is nonsingular.*

2. *A row-reduces to the identity matrix.*

3. *The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.*

4. *The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.*

5. *The columns of A form a linearly independent set.*

Proof. Theorem NMLIC is yet another equivalence for a nonsingular matrix, so we can add it to the list in Theorem NME1. ■

Subsection NSSLI
Null Spaces, Spans, Linear Independence

In Subsection SS.SSNS we proved Theorem SSNS which provided $n - r$ vectors that could be used with the span construction to build the entire null space of a matrix. As we have hinted in Example SCAD, and as we will see again going forward, linearly dependent sets carry redundant vectors with them when used in building a set as a span. Our aim now is to show that the vectors provided by Theorem SSNS form a linearly independent set, so in one sense they are as efficient as possible a way to describe the null space. Notice that the vectors $\mathbf{z}_j$, $1 \leq j \leq n - r$ first appear in the vector form of solutions to arbitrary linear systems (Theorem VFSLS). The exact same vectors appear again in the span construction in the conclusion of Theorem SSNS. Since this second theorem specializes to homogeneous systems the only real difference is that the vector $\mathbf{c}$ in Theorem VFSLS is the zero vector for a homogeneous system. Finally, Theorem BNS will now show that these same vectors are a linearly independent set. We will set the stage for the proof of this theorem with a moderately large example. Study the example carefully, as it will make it easier to understand the proof.

Example LINSB Linear independence of null space basis
Suppose that we are interested in the null space of a 3×7 matrix, A, which row-reduces to

$$B = \begin{bmatrix} \boxed{1} & 0 & -2 & 4 & 0 & 3 & 9 \\ 0 & \boxed{1} & 5 & 6 & 0 & 7 & 1 \\ 0 & 0 & 0 & 0 & \boxed{1} & 8 & -5 \end{bmatrix}$$

The set $F = \{3,\,4,\,6,\,7\}$ is the set of indices for our four free variables that would be used in a description of the solution set for the homogeneous system $\mathcal{LS}(A,\,\mathbf{0})$. Applying Theorem SSNS we can begin to construct a set of four vectors whose span is the null space of A, a set of vectors we will reference as T.

$$\mathcal{N}(A) = \langle T\rangle = \langle\{\mathbf{z}_1,\,\mathbf{z}_2,\,\mathbf{z}_3,\,\mathbf{z}_4\}\rangle = \left\langle\left\{\begin{bmatrix} \\ \\ 1 \\ 0 \\ \\ 0 \\ 0 \end{bmatrix},\,\begin{bmatrix} \\ \\ 0 \\ 1 \\ \\ 0 \\ 0 \end{bmatrix},\,\begin{bmatrix} \\ \\ 0 \\ 0 \\ \\ 1 \\ 0 \end{bmatrix},\,\begin{bmatrix} \\ \\ 0 \\ 0 \\ \\ 0 \\ 1 \end{bmatrix}\right\}\right\rangle$$

So far, we have constructed as much of these individual vectors as we can, based just on the knowledge of the contents of the set F. This has allowed us to determine the entries in slots 3, 4, 6 and 7, while we have left slots 1, 2 and 5 blank. Without doing any more, let us ask if T is linearly independent? Begin with a relation of linear dependence on T, and see what we can learn about the scalars,

$$\mathbf{0} = \alpha_1\mathbf{z}_1 + \alpha_2\mathbf{z}_2 + \alpha_3\mathbf{z}_3 + \alpha_4\mathbf{z}_4$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \alpha_1\begin{bmatrix} \\ \\ 1 \\ 0 \\ \\ 0 \\ 0 \end{bmatrix} + \alpha_2\begin{bmatrix} \\ \\ 0 \\ 1 \\ \\ 0 \\ 0 \end{bmatrix} + \alpha_3\begin{bmatrix} \\ \\ 0 \\ 0 \\ \\ 1 \\ 0 \end{bmatrix} + \alpha_4\begin{bmatrix} \\ \\ 0 \\ 0 \\ \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \\ \\ \alpha_1 \\ 0 \\ \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \\ \\ 0 \\ \alpha_2 \\ \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \\ \\ 0 \\ 0 \\ \\ \alpha_3 \\ 0 \end{bmatrix} + \begin{bmatrix} \\ \\ 0 \\ 0 \\ \\ 0 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} \\ \\ \alpha_1 \\ \alpha_2 \\ \\ \alpha_3 \\ \alpha_4 \end{bmatrix}$$

Applying Definition CVE to the two ends of this chain of equalities, we see that $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$. So the only relation of linear dependence on the set T is a trivial one. By Definition LICV the set T is linearly independent. The important feature of this example is how the "pattern of zeros and ones" in the four vectors led to the conclusion of linear independence. △

The proof of Theorem BNS is really quite straightforward, and relies on the "pattern of zeros and ones" that arise in the vectors $\mathbf{z}_i$, $1 \leq i \leq n-r$ in the entries that arise with the locations of the non-pivot columns. Play along with Example LINSB as you study the proof. Also, take a look at Example VFSAD, Example VFSAI and Example VFSAL, especially at the conclusion of Step 2 (temporarily ignore the construction of the constant vector, $\mathbf{c}$). This proof is also a good first example of how to prove a conclusion that states a set is linearly independent.

Theorem BNS Basis for Null Spaces
Suppose that A is an $m\times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r pivot columns. Let $D = \{d_1,\,d_2,\,d_3,\,\ldots,\,d_r\}$ and $F =$

$\{f_1,\,f_2,\,f_3,\,\ldots,\,f_{n-r}\}$ be the sets of column indices where B does and does not (respectively) have pivot columns. Construct the $n-r$ vectors $\mathbf{z}_j$, $1\leq j\leq n-r$ of size n as

$$\left[\mathbf{z}_j\right]_i=\begin{cases}1 & \text{if } i\in F,\ i=f_j\\ 0 & \text{if } i\in F,\ i\neq f_j\\ -\left[B\right]_{k,f_j} & \text{if } i\in D,\ i=d_k\end{cases}$$

Define the set $S=\{\mathbf{z}_1,\,\mathbf{z}_2,\,\mathbf{z}_3,\,\ldots,\,\mathbf{z}_{n-r}\}$. Then

1. *$\mathcal{N}(A)=\langle S\rangle$.*
2. *S is a linearly independent set.*

Proof. Notice first that the vectors $\mathbf{z}_j$, $1\leq j\leq n-r$ are exactly the same as the $n-r$ vectors defined in Theorem SSNS. Also, the hypotheses of Theorem SSNS are the same as the hypotheses of the theorem we are currently proving. So it is then simply the conclusion of Theorem SSNS that tells us that $\mathcal{N}(A)=\langle S\rangle$. That was the easy half, but the second part is not much harder. What is new here is the claim that S is a linearly independent set.

To prove the linear independence of a set, we need to start with a relation of linear dependence and somehow conclude that the scalars involved *must all be zero*, i.e. that the relation of linear dependence only happens in the trivial fashion. So to establish the linear independence of S, we start with

$$\alpha_1\mathbf{z}_1+\alpha_2\mathbf{z}_2+\alpha_3\mathbf{z}_3+\cdots+\alpha_{n-r}\mathbf{z}_{n-r}=\mathbf{0}.$$

For each j, $1\leq j\leq n-r$, consider the equality of the individual entries of the vectors on both sides of this equality in position f_j,

$$\begin{aligned}
0&=\left[\mathbf{0}\right]_{f_j}\\
&=\left[\alpha_1\mathbf{z}_1+\alpha_2\mathbf{z}_2+\alpha_3\mathbf{z}_3+\cdots+\alpha_{n-r}\mathbf{z}_{n-r}\right]_{f_j} && \text{Definition CVE}\\
&=\left[\alpha_1\mathbf{z}_1\right]_{f_j}+\left[\alpha_2\mathbf{z}_2\right]_{f_j}+\left[\alpha_3\mathbf{z}_3\right]_{f_j}+\cdots+\left[\alpha_{n-r}\mathbf{z}_{n-r}\right]_{f_j} && \text{Definition CVA}\\
&=\alpha_1\left[\mathbf{z}_1\right]_{f_j}+\alpha_2\left[\mathbf{z}_2\right]_{f_j}+\alpha_3\left[\mathbf{z}_3\right]_{f_j}+\cdots+\\
&\qquad\alpha_{j-1}\left[\mathbf{z}_{j-1}\right]_{f_j}+\alpha_j\left[\mathbf{z}_j\right]_{f_j}+\alpha_{j+1}\left[\mathbf{z}_{j+1}\right]_{f_j}+\cdots+\\
&\qquad\alpha_{n-r}\left[\mathbf{z}_{n-r}\right]_{f_j} && \text{Definition CVSM}\\
&=\alpha_1(0)+\alpha_2(0)+\alpha_3(0)+\cdots+\\
&\qquad\alpha_{j-1}(0)+\alpha_j(1)+\alpha_{j+1}(0)+\cdots+\alpha_{n-r}(0) && \text{Definition of } \mathbf{z}_j\\
&=\alpha_j
\end{aligned}$$

So for all j, $1\leq j\leq n-r$, we have $\alpha_j=0$, which is the conclusion that tells us that the *only* relation of linear dependence on $S=\{\mathbf{z}_1,\,\mathbf{z}_2,\,\mathbf{z}_3,\,\ldots,\,\mathbf{z}_{n-r}\}$ is the trivial one. Hence, by Definition LICV the set is linearly independent, as desired. ■

Example NSLIL Null space spanned by linearly independent set, Archetype L
In Example VFSAL we previewed Theorem SSNS by finding a set of two vectors such that their span was the null space for the matrix in Archetype L. Writing the matrix as L, we have

$$\mathcal{N}(L)=\left\langle\left\{\begin{bmatrix}-1\\2\\-2\\1\\0\end{bmatrix},\,\begin{bmatrix}2\\-2\\1\\0\\1\end{bmatrix}\right\}\right\rangle$$

Solving the homogeneous system $\mathcal{LS}(L,\,\mathbf{0})$ resulted in recognizing x_4 and x_5 as the free variables. So look in entries 4 and 5 of the two vectors above and notice the pattern of zeros and ones that provides the linear independence of the set. △

Reading Questions

1. Let S be the set of three vectors below.
$$S = \left\{ \begin{bmatrix}1\\2\\-1\end{bmatrix}, \begin{bmatrix}3\\-4\\2\end{bmatrix}, \begin{bmatrix}4\\-2\\1\end{bmatrix} \right\}$$
Is S linearly independent or linearly dependent? Explain why.
2. Let S be the set of three vectors below.
$$S = \left\{ \begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}3\\2\\2\end{bmatrix}, \begin{bmatrix}4\\3\\-4\end{bmatrix} \right\}$$
Is S linearly independent or linearly dependent? Explain why.
3. Is the matrix below singular or nonsingular? Explain your answer using only the final conclusion you reached in the previous question, along with one new theorem.
$$\begin{bmatrix}1 & 3 & 4\\-1 & 2 & 3\\0 & 2 & -4\end{bmatrix}$$

Exercises

Determine if the sets of vectors in Exercises C20–C25 are linearly independent or linearly dependent. When the set is linearly dependent, exhibit a nontrivial relation of linear dependence.

C20† $\left\{ \begin{bmatrix}1\\-2\\1\end{bmatrix}, \begin{bmatrix}2\\-1\\3\end{bmatrix}, \begin{bmatrix}1\\5\\0\end{bmatrix} \right\}$

C21† $\left\{ \begin{bmatrix}-1\\2\\4\\2\end{bmatrix}, \begin{bmatrix}3\\3\\-1\\3\end{bmatrix}, \begin{bmatrix}7\\3\\-6\\4\end{bmatrix} \right\}$

C22† $\left\{ \begin{bmatrix}-2\\-1\\-1\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}, \begin{bmatrix}3\\3\\6\end{bmatrix}, \begin{bmatrix}-5\\-4\\-6\end{bmatrix}, \begin{bmatrix}4\\4\\7\end{bmatrix} \right\}$

C23† $\left\{ \begin{bmatrix}1\\-2\\2\\5\\3\end{bmatrix}, \begin{bmatrix}3\\3\\1\\2\\-4\end{bmatrix}, \begin{bmatrix}2\\1\\2\\-1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\\2\\2\end{bmatrix} \right\}$

C24† $\left\{ \begin{bmatrix}1\\2\\-1\\0\\1\end{bmatrix}, \begin{bmatrix}3\\2\\-1\\2\\2\end{bmatrix}, \begin{bmatrix}4\\4\\-2\\2\\3\end{bmatrix}, \begin{bmatrix}-1\\2\\-1\\-2\\0\end{bmatrix} \right\}$

C25† $\left\{ \begin{bmatrix}2\\1\\3\\-1\\2\end{bmatrix}, \begin{bmatrix}4\\-2\\1\\3\\2\end{bmatrix}, \begin{bmatrix}10\\-7\\0\\10\\4\end{bmatrix} \right\}$

C30† For the matrix B below, find a set S that is linearly independent and spans the null space of B, that is, $\mathcal{N}(B) = \langle S \rangle$.
$$B = \begin{bmatrix}-3 & 1 & -2 & 7\\-1 & 2 & 1 & 4\\1 & 1 & 2 & -1\end{bmatrix}$$

C31† For the matrix A below, find a linearly independent set S so that the null space of A is spanned by S, that is, $\mathcal{N}(A) = \langle S\rangle$.

$$A = \begin{bmatrix} -1 & -2 & 2 & 1 & 5 \\ 1 & 2 & 1 & 1 & 5 \\ 3 & 6 & 1 & 2 & 7 \\ 2 & 4 & 0 & 1 & 2 \end{bmatrix}$$

C32† Find a set of column vectors, T, such that (1) the span of T is the null space of B, $\langle T\rangle = \mathcal{N}(B)$ and (2) T is a linearly independent set.

$$B = \begin{bmatrix} 2 & 1 & 1 & 1 \\ -4 & -3 & 1 & -7 \\ 1 & 1 & -1 & 3 \end{bmatrix}$$

C33† Find a set S so that S is linearly independent and $\mathcal{N}(A) = \langle S\rangle$, where $\mathcal{N}(A)$ is the null space of the matrix A below.

$$A = \begin{bmatrix} 2 & 3 & 3 & 1 & 4 \\ 1 & 1 & -1 & -1 & -3 \\ 3 & 2 & -8 & -1 & 1 \end{bmatrix}$$

C50 Consider each archetype that is a system of equations and consider the solutions listed for the homogeneous version of the archetype. (If only the trivial solution is listed, then assume this is the only solution to the system.) From the solution set, determine if the columns of the coefficient matrix form a linearly independent or linearly dependent set. In the case of a linearly dependent set, use one of the sample solutions to provide a nontrivial relation of linear dependence on the set of columns of the coefficient matrix (Definition RLD). Indicate when Theorem MVSLD applies and connect this with the number of variables and equations in the system of equations.

Archetype A, Archetype B, Archetype C, Archetype D/Archetype E, Archetype F, Archetype G/Archetype H, Archetype I, Archetype J

C51 For each archetype that is a system of equations consider the homogeneous version. Write elements of the solution set in vector form (Theorem VFSLS) and from this extract the vectors $\mathbf{z}_j$ described in Theorem BNS. These vectors are used in a span construction to describe the null space of the coefficient matrix for each archetype. What does it mean when we write a null space as $\langle\{\ \}\rangle$?

Archetype A, Archetype B, Archetype C, Archetype D/Archetype E, Archetype F, Archetype G/Archetype H, Archetype I, Archetype J

C52 For each archetype that is a system of equations consider the homogeneous version. Sample solutions are given and a linearly independent spanning set is given for the null space of the coefficient matrix. Write each of the sample solutions individually as a linear combination of the vectors in the spanning set for the null space of the coefficient matrix.

Archetype A, Archetype B, Archetype C, Archetype D/Archetype E, Archetype F, Archetype G/Archetype H, Archetype I, Archetype J

C60† For the matrix A below, find a set of vectors S so that (1) S is linearly independent, and (2) the span of S equals the null space of A, $\langle S\rangle = \mathcal{N}(A)$. (See Exercise SS.C60.)

$$A = \begin{bmatrix} 1 & 1 & 6 & -8 \\ 1 & -2 & 0 & 1 \\ -2 & 1 & -6 & 7 \end{bmatrix}$$

M20† Suppose that $S = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\}$ is a set of three vectors from $\mathbb{C}^{873}$. Prove that the set

$$T = \{2\mathbf{v}_1 + 3\mathbf{v}_2 + \mathbf{v}_3,\ \mathbf{v}_1 - \mathbf{v}_2 - 2\mathbf{v}_3,\ 2\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3\}$$

is linearly dependent.

M21† Suppose that $S = \{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3\}$ is a linearly independent set of three vectors from $\mathbb{C}^{873}$. Prove that the set

$$T = \{2\mathbf{v}_1 + 3\mathbf{v}_2 + \mathbf{v}_3,\, \mathbf{v}_1 - \mathbf{v}_2 + 2\mathbf{v}_3,\, 2\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3\}$$

is linearly independent.

M50† Consider the set of vectors from $\mathbb{C}^3$, W, given below. Find a set T that contains three vectors from W and such that $W = \langle T\rangle$.

$$W = \langle\{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3,\, \mathbf{v}_4,\, \mathbf{v}_5\}\rangle = \left\langle\left\{\begin{bmatrix}2\\1\\1\end{bmatrix},\, \begin{bmatrix}-1\\-1\\1\end{bmatrix},\, \begin{bmatrix}1\\2\\3\end{bmatrix},\, \begin{bmatrix}3\\1\\3\end{bmatrix},\, \begin{bmatrix}0\\1\\-3\end{bmatrix}\right\}\right\rangle$$

M51† Consider the subspace $W = \langle\{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3,\, \mathbf{v}_4\}\rangle$. Find a set S so that (1) S is a subset of W, (2) S is linearly independent, and (3) $W = \langle S\rangle$. Write each vector not included in S as a linear combination of the vectors that are in S.

$$\mathbf{v}_1 = \begin{bmatrix}1\\-1\\2\end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix}4\\-4\\8\end{bmatrix} \qquad \mathbf{v}_3 = \begin{bmatrix}-3\\2\\-7\end{bmatrix} \qquad \mathbf{v}_4 = \begin{bmatrix}2\\1\\7\end{bmatrix}$$

T10 Prove that if a set of vectors contains the zero vector, then the set is linearly dependent. (Ed. "The zero vector is death to linearly independent sets.")

T12 Suppose that S is a linearly independent set of vectors, and T is a subset of S, $T \subseteq S$ (Definition SSET). Prove that T is linearly independent.

T13 Suppose that T is a linearly dependent set of vectors, and T is a subset of S, $T \subseteq S$ (Definition SSET). Prove that S is linearly dependent.

T15† Suppose that $\{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3,\, \ldots,\, \mathbf{v}_n\}$ is a set of vectors. Prove that

$$\{\mathbf{v}_1 - \mathbf{v}_2,\, \mathbf{v}_2 - \mathbf{v}_3,\, \mathbf{v}_3 - \mathbf{v}_4,\, \ldots,\, \mathbf{v}_n - \mathbf{v}_1\}$$

is a linearly dependent set.

T20† Suppose that $\{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3,\, \mathbf{v}_4\}$ is a linearly independent set in $\mathbb{C}^{35}$. Prove that

$$\{\mathbf{v}_1,\, \mathbf{v}_1 + \mathbf{v}_2,\, \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3,\, \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4\}$$

is a linearly independent set.

T50† Suppose that A is an $m \times n$ matrix with linearly independent columns and the linear system $\mathcal{LS}(A,\, \mathbf{b})$ is consistent. Show that this system has a unique solution. (Notice that we are not requiring A to be square.)

Section LDS
Linear Dependence and Spans

In any linearly dependent set there is always one vector that can be written as a linear combination of the others. This is the substance of the upcoming Theorem DLDS. Perhaps this will explain the use of the word "dependent." In a linearly dependent set, at least one vector "depends" on the others (via a linear combination).

Indeed, because Theorem DLDS is an equivalence (Proof Technique E) some authors use this condition as a definition (Proof Technique D) of linear dependence. Then linear independence is defined as the logical opposite of linear dependence. Of course, we have *chosen* to take Definition LICV as our definition, and then follow with Theorem DLDS as a theorem.

Subsection LDSS
Linearly Dependent Sets and Spans

If we use a linearly dependent set to construct a span, then we can *always* create the same infinite set with a starting set that is one vector smaller in size. We will illustrate this behavior in Example RSC5. However, this will not be possible if we build a span from a linearly independent set. So in a certain sense, using a linearly independent set to formulate a span is the best possible way — there are not any extra vectors being used to build up all the necessary linear combinations. OK, here is the theorem, and then the example.

Theorem DLDS Dependency in Linearly Dependent Sets
Suppose that $S = \{\mathbf{u}_1,\,\mathbf{u}_2,\,\mathbf{u}_3,\,\ldots,\,\mathbf{u}_n\}$ *is a set of vectors. Then S is a linearly dependent set if and only if there is an index t,* $1 \leq t \leq n$ *such that* $\mathbf{u_t}$ *is a linear combination of the vectors* $\mathbf{u}_1,\,\mathbf{u}_2,\,\mathbf{u}_3,\,\ldots,\,\mathbf{u}_{t-1},\,\mathbf{u}_{t+1},\,\ldots,\,\mathbf{u}_n$.

Proof. (⇒) Suppose that S is linearly dependent, so there exists a nontrivial relation of linear dependence by Definition LICV. That is, there are scalars, α_i, $1 \leq i \leq n$, which are not all zero, such that

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_n\mathbf{u}_n = \mathbf{0}.$$

Since the α_i cannot all be zero, choose one, say α_t, that is nonzero. Then,

$$\begin{aligned}
\mathbf{u}_t &= \frac{-1}{\alpha_t}\left(-\alpha_t\mathbf{u}_t\right) && \text{Property MICN}\\
&= \frac{-1}{\alpha_t}\left(\alpha_1\mathbf{u}_1 + \cdots + \alpha_{t-1}\mathbf{u}_{t-1} + \alpha_{t+1}\mathbf{u}_{t+1} + \cdots + \alpha_n\mathbf{u}_n\right) && \text{Theorem VSPCV}\\
&= \frac{-\alpha_1}{\alpha_t}\mathbf{u}_1 + \cdots + \frac{-\alpha_{t-1}}{\alpha_t}\mathbf{u}_{t-1} + \frac{-\alpha_{t+1}}{\alpha_t}\mathbf{u}_{t+1} + \cdots + \frac{-\alpha_n}{\alpha_t}\mathbf{u}_n && \text{Theorem VSPCV}
\end{aligned}$$

Since the values of $\frac{\alpha_i}{\alpha_t}$ are again scalars, we have expressed $\mathbf{u}_t$ as a linear combination of the other elements of S.

(⇐) Assume that the vector $\mathbf{u}_t$ is a linear combination of the other vectors in S. Write this linear combination, denoting the relevant scalars as $\beta_1,\,\beta_2,\,\ldots,\,\beta_{t-1}$, $\beta_{t+1},\,\ldots,\,\beta_n$, as

$$\mathbf{u_t} = \beta_1\mathbf{u}_1 + \beta_2\mathbf{u}_2 + \cdots + \beta_{t-1}\mathbf{u}_{t-1} + \beta_{t+1}\mathbf{u}_{t+1} + \cdots + \beta_n\mathbf{u}_n$$

Then we have

$$\begin{aligned}
&\beta_1\mathbf{u}_1 + \cdots + \beta_{t-1}\mathbf{u}_{t-1} + (-1)\mathbf{u}_t + \beta_{t+1}\mathbf{u}_{t+1} + \cdots + \beta_n\mathbf{u}_n\\
&\qquad = \mathbf{u}_t + (-1)\mathbf{u}_t && \text{Theorem VSPCV}\\
&\qquad = \left(1 + (-1)\right)\mathbf{u}_t && \text{Property DSAC}\\
&\qquad = 0\mathbf{u}_t && \text{Property AICN}
\end{aligned}$$

$$= \mathbf{0} \qquad \text{Definition CVSM}$$

So the scalars $\beta_1,\,\beta_2,\,\beta_3,\,\ldots,\,\beta_{t-1},\,\beta_t = -1, \beta_{t+1},\,\ldots,\,\beta_n$ provide a *nontrivial* linear combination of the vectors in S, thus establishing that S is a linearly dependent set (Definition LICV). ■

This theorem can be used, sometimes repeatedly, to whittle down the size of a set of vectors used in a span construction. We have seen some of this already in Example SCAD, but in the next example we will detail some of the subtleties.

Example RSC5 Reducing a span in $\mathbb{C}^5$
Consider the set of $n = 4$ vectors from $\mathbb{C}^5$,

$$R = \{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3,\,\mathbf{v}_4\} = \left\{ \begin{bmatrix}1\\2\\-1\\3\\2\end{bmatrix}, \begin{bmatrix}2\\1\\3\\1\\2\end{bmatrix}, \begin{bmatrix}0\\-7\\6\\-11\\-2\end{bmatrix}, \begin{bmatrix}4\\1\\2\\1\\6\end{bmatrix} \right\}$$

and define $V = \langle R\rangle$.

To employ Theorem LIVHS, we form a 5×4 matrix, D, and row-reduce to understand solutions to the homogeneous system $\mathcal{LS}(D,\,\mathbf{0})$,

$$D = \begin{bmatrix}1&2&0&4\\2&1&-7&1\\-1&3&6&2\\3&1&-11&1\\2&2&-2&6\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}\boxed{1}&0&0&4\\0&\boxed{1}&0&0\\0&0&\boxed{1}&1\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

We can find infinitely many solutions to this system, most of them nontrivial, and we choose any one we like to build a relation of linear dependence on R. Let us begin with $x_4 = 1$, to find the solution

$$\begin{bmatrix}-4\\0\\-1\\1\end{bmatrix}$$

So we can write the relation of linear dependence,

$$(-4)\mathbf{v}_1 + 0\mathbf{v}_2 + (-1)\mathbf{v}_3 + 1\mathbf{v}_4 = \mathbf{0}$$

Theorem DLDS guarantees that we can solve this relation of linear dependence for *some* vector in R, but the choice of which one is up to us. Notice however that $\mathbf{v}_2$ has a zero coefficient. In this case, we cannot choose to solve for $\mathbf{v}_2$. Maybe some other relation of linear dependence would produce a nonzero coefficient for $\mathbf{v}_2$ if we just had to solve for this vector. Unfortunately, this example has been engineered to *always* produce a zero coefficient here, as you can see from solving the homogeneous system. Every solution has $x_2 = 0$!

OK, if we are convinced that we cannot solve for $\mathbf{v}_2$, let us instead solve for $\mathbf{v}_3$,

$$\mathbf{v}_3 = (-4)\mathbf{v}_1 + 0\mathbf{v}_2 + 1\mathbf{v}_4 = (-4)\mathbf{v}_1 + 1\mathbf{v}_4$$

We now claim that this particular equation will allow us to write

$$V = \langle R\rangle = \langle\{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3,\,\mathbf{v}_4\}\rangle = \langle\{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_4\}\rangle$$

in essence declaring $\mathbf{v}_3$ as surplus for the task of building V as a span. This claim is an equality of two sets, so we will use Definition SE to establish it carefully. Let $R' = \{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_4\}$ and $V' = \langle R'\rangle$. We want to show that $V = V'$.

First show that $V' \subseteq V$. Since every vector of R' is in R, any vector we can construct in V' as a linear combination of vectors from R' can also be constructed as a vector in V by the same linear combination of the same vectors in R. That was easy, now turn it around.

Next show that $V \subseteq V'$. Choose any $\mathbf{v}$ from V. So there are scalars α_1, α_2, α_3, α_4 such that

$$\begin{aligned}\mathbf{v} &= \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\mathbf{v}_3 + \alpha_4\mathbf{v}_4\\ &= \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \alpha_3\left((-4)\mathbf{v}_1 + 1\mathbf{v}_4\right) + \alpha_4\mathbf{v}_4\\ &= \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \left((-4\alpha_3)\mathbf{v}_1 + \alpha_3\mathbf{v}_4\right) + \alpha_4\mathbf{v}_4\\ &= \left(\alpha_1 - 4\alpha_3\right)\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \left(\alpha_3 + \alpha_4\right)\mathbf{v}_4.\end{aligned}$$

This equation says that $\mathbf{v}$ can then be written as a linear combination of the vectors in R' and hence qualifies for membership in V'. So $V \subseteq V'$ and we have established that $V = V'$.

If R' was also linearly dependent (it is not), we could reduce the set even further. Notice that we could have chosen to eliminate any one of $\mathbf{v}_1$, $\mathbf{v}_3$ or $\mathbf{v}_4$, but somehow $\mathbf{v}_2$ is essential to the creation of V since it cannot be replaced by any linear combination of $\mathbf{v}_1$, $\mathbf{v}_3$ or $\mathbf{v}_4$. △

Subsection COV
Casting Out Vectors

In Example RSC5 we used four vectors to create a span. With a relation of linear dependence in hand, we were able to "toss out" one of these four vectors and create the same span from a subset of just three vectors from the original set of four. We did have to take some care as to just which vector we tossed out. In the next example, we will be more methodical about just how we choose to eliminate vectors from a linearly dependent set while preserving a span.

Example COV Casting out vectors
We begin with a set S containing seven vectors from $\mathbb{C}^4$,

$$S = \left\{ \begin{bmatrix}1\\2\\0\\-1\end{bmatrix},\ \begin{bmatrix}4\\8\\0\\-4\end{bmatrix},\ \begin{bmatrix}0\\-1\\2\\2\end{bmatrix},\ \begin{bmatrix}-1\\3\\-3\\4\end{bmatrix},\ \begin{bmatrix}0\\9\\-4\\8\end{bmatrix},\ \begin{bmatrix}7\\-13\\12\\-31\end{bmatrix},\ \begin{bmatrix}-9\\7\\-8\\37\end{bmatrix} \right\}$$

and define $W = \langle S\rangle$.

The set S is obviously linearly dependent by Theorem MVSLD, since we have $n = 7$ vectors from $\mathbb{C}^4$. So we can slim down S some, and still create W as the span of a smaller set of vectors.

As a device for identifying relations of linear dependence among the vectors of S, we place the seven column vectors of S into a matrix as columns,

$$A = [\mathbf{A}_1|\mathbf{A}_2|\mathbf{A}_3|\dots|\mathbf{A}_7] = \begin{bmatrix}1 & 4 & 0 & -1 & 0 & 7 & -9\\ 2 & 8 & -1 & 3 & 9 & -13 & 7\\ 0 & 0 & 2 & -3 & -4 & 12 & -8\\ -1 & -4 & 2 & 4 & 8 & -31 & 37\end{bmatrix}$$

By Theorem SLSLC a nontrivial solution to $\mathcal{LS}(A,\,\mathbf{0})$ will give us a nontrivial relation of linear dependence (Definition RLDCV) on the columns of A (which are the elements of the set S). The row-reduced form for A is the matrix

$$B = \begin{bmatrix}\boxed{1} & 4 & 0 & 0 & 2 & 1 & -3\\ 0 & 0 & \boxed{1} & 0 & 1 & -3 & 5\\ 0 & 0 & 0 & \boxed{1} & 2 & -6 & 6\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

so we can easily create solutions to the homogeneous system $\mathcal{LS}(A,\,\mathbf{0})$ using the free variables x_2, x_5, x_6, x_7. Any such solution will provide a relation of linear dependence on the columns of B. These solutions will allow us to solve for one column vector as a linear combination of some others, in the spirit of Theorem

DLDS, and remove that vector from the set. We will set about forming these linear combinations methodically.

Set the free variable $x_2 = 1$, and set the other free variables to zero. Then a solution to $\mathcal{LS}(A,\ \mathbf{0})$ is

$$\mathbf{x} = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

which can be used to create the linear combination

$$(-4)\mathbf{A}_1 + 1\mathbf{A}_2 + 0\mathbf{A}_3 + 0\mathbf{A}_4 + 0\mathbf{A}_5 + 0\mathbf{A}_6 + 0\mathbf{A}_7 = \mathbf{0}$$

This can then be arranged and solved for $\mathbf{A}_2$, resulting in $\mathbf{A}_2$ expressed as a linear combination of $\{\mathbf{A}_1,\ \mathbf{A}_3,\ \mathbf{A}_4\}$,

$$\mathbf{A}_2 = 4\mathbf{A}_1 + 0\mathbf{A}_3 + 0\mathbf{A}_4$$

This means that $\mathbf{A}_2$ is surplus, and we can create W just as well with a smaller set with this vector removed,

$$W = \langle\{\mathbf{A}_1,\ \mathbf{A}_3,\ \mathbf{A}_4,\ \mathbf{A}_5,\ \mathbf{A}_6,\ \mathbf{A}_7\}\rangle$$

Technically, this set equality for W requires a proof, in the spirit of Example RSC5, but we will bypass this requirement here, and in the next few paragraphs.

Now, set the free variable $x_5 = 1$, and set the other free variables to zero. Then a solution to $\mathcal{LS}(B,\ \mathbf{0})$ is

$$\mathbf{x} = \begin{bmatrix} -2 \\ 0 \\ -1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

which can be used to create the linear combination

$$(-2)\mathbf{A}_1 + 0\mathbf{A}_2 + (-1)\mathbf{A}_3 + (-2)\mathbf{A}_4 + 1\mathbf{A}_5 + 0\mathbf{A}_6 + 0\mathbf{A}_7 = \mathbf{0}$$

This can then be arranged and solved for $\mathbf{A}_5$, resulting in $\mathbf{A}_5$ expressed as a linear combination of $\{\mathbf{A}_1,\ \mathbf{A}_3,\ \mathbf{A}_4\}$,

$$\mathbf{A}_5 = 2\mathbf{A}_1 + 1\mathbf{A}_3 + 2\mathbf{A}_4$$

This means that $\mathbf{A}_5$ is surplus, and we can create W just as well with a smaller set with this vector removed,

$$W = \langle\{\mathbf{A}_1,\ \mathbf{A}_3,\ \mathbf{A}_4,\ \mathbf{A}_6,\ \mathbf{A}_7\}\rangle$$

Do it again, set the free variable $x_6 = 1$, and set the other free variables to zero. Then a solution to $\mathcal{LS}(B,\ \mathbf{0})$ is

$$\mathbf{x} = \begin{bmatrix} -1 \\ 0 \\ 3 \\ 6 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

which can be used to create the linear combination

$$(-1)\mathbf{A}_1 + 0\mathbf{A}_2 + 3\mathbf{A}_3 + 6\mathbf{A}_4 + 0\mathbf{A}_5 + 1\mathbf{A}_6 + 0\mathbf{A}_7 = \mathbf{0}$$

This can then be arranged and solved for $\mathbf{A}_6$, resulting in $\mathbf{A}_6$ expressed as a linear combination of $\{\mathbf{A}_1,\,\mathbf{A}_3,\,\mathbf{A}_4\}$,

$$\mathbf{A}_6 = 1\mathbf{A}_1 + (-3)\mathbf{A}_3 + (-6)\mathbf{A}_4$$

This means that $\mathbf{A}_6$ is surplus, and we can create W just as well with a smaller set with this vector removed,

$$W = \langle\{\mathbf{A}_1,\,\mathbf{A}_3,\,\mathbf{A}_4,\,\mathbf{A}_7\}\rangle$$

Set the free variable $x_7 = 1$, and set the other free variables to zero. Then a solution to $\mathcal{LS}(B,\,\mathbf{0})$ is

$$\mathbf{x} = \begin{bmatrix} 3 \\ 0 \\ -5 \\ -6 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

which can be used to create the linear combination

$$3\mathbf{A}_1 + 0\mathbf{A}_2 + (-5)\mathbf{A}_3 + (-6)\mathbf{A}_4 + 0\mathbf{A}_5 + 0\mathbf{A}_6 + 1\mathbf{A}_7 = \mathbf{0}$$

This can then be arranged and solved for $\mathbf{A}_7$, resulting in $\mathbf{A}_7$ expressed as a linear combination of $\{\mathbf{A}_1,\,\mathbf{A}_3,\,\mathbf{A}_4\}$,

$$\mathbf{A}_7 = (-3)\mathbf{A}_1 + 5\mathbf{A}_3 + 6\mathbf{A}_4$$

This means that $\mathbf{A}_7$ is surplus, and we can create W just as well with a smaller set with this vector removed,

$$W = \langle\{\mathbf{A}_1,\,\mathbf{A}_3,\,\mathbf{A}_4\}\rangle$$

You might think we could keep this up, but we have run out of free variables. And not coincidentally, the set $\{\mathbf{A}_1,\,\mathbf{A}_3,\,\mathbf{A}_4\}$ is linearly independent (check this!). It should be clear how each free variable was used to eliminate the a column from the set used to span the column space, as this will be the essence of the proof of the next theorem. The column vectors in S were not chosen entirely at random, they are the columns of Archetype I. See if you can mimic this example using the columns of Archetype J. Go ahead, we'll go grab a cup of coffee and be back before you finish up.

For extra credit, notice that the vector

$$\mathbf{b} = \begin{bmatrix} 3 \\ 9 \\ 1 \\ 4 \end{bmatrix}$$

is the vector of constants in the definition of Archetype I. Since the system $\mathcal{LS}(A,\,\mathbf{b})$ is consistent, we know by Theorem SLSLC that $\mathbf{b}$ is a linear combination of the columns of A, or stated equivalently, $\mathbf{b} \in W$. This means that $\mathbf{b}$ must also be a linear combination of just the three columns $\mathbf{A}_1,\,\mathbf{A}_3,\,\mathbf{A}_4$. Can you find such a linear combination? Did you notice that there is just a single (unique) answer? Hmmmm.
△

Example COV deserves your careful attention, since this important example motivates the following very fundamental theorem.

Theorem BS Basis of a Span
Suppose that $S = \{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3,\,\ldots,\,\mathbf{v}_n\}$ *is a set of column vectors. Define* $W = \langle S\rangle$

and let A be the matrix whose columns are the vectors from S. Let B be the reduced row-echelon form of A, with $D = \{d_1,\, d_2,\, d_3,\, \ldots,\, d_r\}$ the set of indices for the pivot columns of B. Then

1. $T = \{\mathbf{v}_{d_1},\, \mathbf{v}_{d_2},\, \mathbf{v}_{d_3},\, \ldots\, \mathbf{v}_{d_r}\}$ *is a linearly independent set.*
2. $W = \langle T\rangle$.

Proof. To prove that T is linearly independent, begin with a relation of linear dependence on T,

$$\mathbf{0} = \alpha_1\mathbf{v}_{d_1} + \alpha_2\mathbf{v}_{d_2} + \alpha_3\mathbf{v}_{d_3} + \ldots + \alpha_r\mathbf{v}_{d_r}$$

and we will try to conclude that the only possibility for the scalars α_i is that they are all zero. Denote the non-pivot columns of B by $F = \{f_1,\, f_2,\, f_3,\, \ldots,\, f_{n-r}\}$. Then we can preserve the equality by adding a big fat zero to the linear combination,

$$\mathbf{0} = \alpha_1\mathbf{v}_{d_1} + \alpha_2\mathbf{v}_{d_2} + \alpha_3\mathbf{v}_{d_3} + \ldots + \alpha_r\mathbf{v}_{d_r} + 0\mathbf{v}_{f_1} + 0\mathbf{v}_{f_2} + 0\mathbf{v}_{f_3} + \ldots + 0\mathbf{v}_{f_{n-r}}$$

By Theorem SLSLC, the scalars in this linear combination (suitably reordered) are a solution to the homogeneous system $\mathcal{LS}(A,\, \mathbf{0})$. But notice that this is the solution obtained by setting each free variable to zero. If we consider the description of a solution vector in the conclusion of Theorem VFSLS, in the case of a homogeneous system, then we see that if all the free variables are set to zero the resulting solution vector is trivial (all zeros). So it must be that $\alpha_i = 0$, $1 \leq i \leq r$. This implies by Definition LICV that T is a linearly independent set.

The second conclusion of this theorem is an equality of sets (Definition SE). Since T is a subset of S, any linear combination of elements of the set T can also be viewed as a linear combination of elements of the set S. So $\langle T\rangle \subseteq \langle S\rangle = W$. It remains to prove that $W = \langle S\rangle \subseteq \langle T\rangle$.

For each k, $1 \leq k \leq n-r$, form a solution $\mathbf{x}$ to $\mathcal{LS}(A,\, \mathbf{0})$ by setting the free variables as follows:

$$x_{f_1} = 0 \qquad x_{f_2} = 0 \qquad x_{f_3} = 0 \qquad \ldots \qquad x_{f_k} = 1 \qquad \ldots \qquad x_{f_{n-r}} = 0$$

By Theorem VFSLS, the remainder of this solution vector is given by,

$$x_{d_1} = -\left[B\right]_{1,f_k} \qquad x_{d_2} = -\left[B\right]_{2,f_k} \qquad x_{d_3} = -\left[B\right]_{3,f_k} \qquad \ldots \qquad x_{d_r} = -\left[B\right]_{r,f_k}$$

From this solution, we obtain a relation of linear dependence on the columns of A,

$$-\left[B\right]_{1,f_k}\mathbf{v}_{d_1} - \left[B\right]_{2,f_k}\mathbf{v}_{d_2} - \left[B\right]_{3,f_k}\mathbf{v}_{d_3} - \ldots - \left[B\right]_{r,f_k}\mathbf{v}_{d_r} + 1\mathbf{v}_{f_k} = \mathbf{0}$$

which can be arranged as the equality

$$\mathbf{v}_{f_k} = \left[B\right]_{1,f_k}\mathbf{v}_{d_1} + \left[B\right]_{2,f_k}\mathbf{v}_{d_2} + \left[B\right]_{3,f_k}\mathbf{v}_{d_3} + \ldots + \left[B\right]_{r,f_k}\mathbf{v}_{d_r}$$

Now, suppose we take an arbitrary element, $\mathbf{w}$, of $W = \langle S\rangle$ and write it as a linear combination of the elements of S, but with the terms organized according to the indices in D and F,

$$\mathbf{w} = \alpha_1\mathbf{v}_{d_1} + \alpha_2\mathbf{v}_{d_2} + \ldots + \alpha_r\mathbf{v}_{d_r} + \beta_1\mathbf{v}_{f_1} + \beta_2\mathbf{v}_{f_2} + \ldots + \beta_{n-r}\mathbf{v}_{f_{n-r}}$$

From the above, we can replace each $\mathbf{v}_{f_j}$ by a linear combination of the $\mathbf{v}_{d_i}$,

$$\begin{aligned}
\mathbf{w} = {} & \alpha_1\mathbf{v}_{d_1} + \alpha_2\mathbf{v}_{d_2} + \ldots + \alpha_r\mathbf{v}_{d_r} + \\
& \beta_1\left(\left[B\right]_{1,f_1}\mathbf{v}_{d_1} + \left[B\right]_{2,f_1}\mathbf{v}_{d_2} + \left[B\right]_{3,f_1}\mathbf{v}_{d_3} + \ldots + \left[B\right]_{r,f_1}\mathbf{v}_{d_r}\right) + \\
& \beta_2\left(\left[B\right]_{1,f_2}\mathbf{v}_{d_1} + \left[B\right]_{2,f_2}\mathbf{v}_{d_2} + \left[B\right]_{3,f_2}\mathbf{v}_{d_3} + \ldots + \left[B\right]_{r,f_2}\mathbf{v}_{d_r}\right) + \\
& \qquad \vdots \\
& \beta_{n-r}\left(\left[B\right]_{1,f_{n-r}}\mathbf{v}_{d_1} + \left[B\right]_{2,f_{n-r}}\mathbf{v}_{d_2} + \left[B\right]_{3,f_{n-r}}\mathbf{v}_{d_3} + \ldots + \left[B\right]_{r,f_{n-r}}\mathbf{v}_{d_r}\right)
\end{aligned}$$

With repeated applications of several of the properties of Theorem VSPCV we can rearrange this expression as,

$$\begin{aligned}
&= \left(\alpha_1 + \beta_1 [B]_{1,f_1} + \beta_2 [B]_{1,f_2} + \beta_3 [B]_{1,f_3} + \dots + \beta_{n-r} [B]_{1,f_{n-r}}\right) \mathbf{v}_{d_1} + \\
&\quad \left(\alpha_2 + \beta_1 [B]_{2,f_1} + \beta_2 [B]_{2,f_2} + \beta_3 [B]_{2,f_3} + \dots + \beta_{n-r} [B]_{2,f_{n-r}}\right) \mathbf{v}_{d_2} + \\
&\quad \vdots \\
&\quad \left(\alpha_r + \beta_1 [B]_{r,f_1} + \beta_2 [B]_{r,f_2} + \beta_3 [B]_{r,f_3} + \dots + \beta_{n-r} [B]_{r,f_{n-r}}\right) \mathbf{v}_{d_r}
\end{aligned}$$

This mess expresses the vector $\mathbf{w}$ as a linear combination of the vectors in

$$T = \{\mathbf{v}_{d_1}, \mathbf{v}_{d_2}, \mathbf{v}_{d_3}, \dots \mathbf{v}_{d_r}\}$$

thus saying that $\mathbf{w} \in \langle T \rangle$. Therefore, $W = \langle S \rangle \subseteq \langle T \rangle$. ■

In Example COV, we tossed-out vectors one at a time. But in each instance, we rewrote the offending vector as a linear combination of those vectors with the column indices of the pivot columns of the reduced row-echelon form of the matrix of columns. In the proof of Theorem BS, we accomplish this reduction in one big step. In Example COV we arrived at a linearly independent set at exactly the same moment that we ran out of free variables to exploit. This was not a coincidence, it is the substance of our conclusion of linear independence in Theorem BS.

Here is a straightforward application of Theorem BS.

Example RSC4 Reducing a span in $\mathbb{C}^4$
Begin with a set of five vectors from $\mathbb{C}^4$,

$$S = \left\{ \begin{bmatrix}1\\1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\2\\4\\2\end{bmatrix}, \begin{bmatrix}2\\0\\-1\\1\end{bmatrix}, \begin{bmatrix}7\\1\\-1\\4\end{bmatrix}, \begin{bmatrix}0\\2\\5\\1\end{bmatrix} \right\}$$

and let $W = \langle S \rangle$. To arrive at a (smaller) linearly independent set, follow the procedure described in Theorem BS. Place the vectors from S into a matrix as columns, and row-reduce,

$$\begin{bmatrix}1 & 2 & 2 & 7 & 0\\1 & 2 & 0 & 1 & 2\\2 & 4 & -1 & -1 & 5\\1 & 2 & 1 & 4 & 1\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}\boxed{1} & 2 & 0 & 1 & 2\\0 & 0 & \boxed{1} & 3 & -1\\0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0\end{bmatrix}$$

Columns 1 and 3 are the pivot columns ($D = \{1, 3\}$) so the set

$$T = \left\{ \begin{bmatrix}1\\1\\2\\1\end{bmatrix}, \begin{bmatrix}2\\0\\-1\\1\end{bmatrix} \right\}$$

is linearly independent and $\langle T \rangle = \langle S \rangle = W$. Boom!

Since the reduced row-echelon form of a matrix is unique (Theorem RREFU), the procedure of Theorem BS leads us to a unique set T. However, there is a wide variety of possibilities for sets T that are linearly independent and which can be employed in a span to create W. Without proof, we list two other possibilities:

$$T' = \left\{ \begin{bmatrix}2\\2\\4\\2\end{bmatrix}, \begin{bmatrix}2\\0\\-1\\1\end{bmatrix} \right\}$$

$$T^* = \left\{ \begin{bmatrix} 3\\1\\1\\2 \end{bmatrix}, \begin{bmatrix} -1\\1\\3\\0 \end{bmatrix} \right\}$$

Can you prove that T' and T^* are linearly independent sets and $W = \langle S\rangle = \langle T'\rangle = \langle T^*\rangle$? △

Example RES Reworking elements of a span
Begin with a set of five vectors from $\mathbb{C}^4$,

$$R = \left\{ \begin{bmatrix} 2\\1\\3\\2 \end{bmatrix}, \begin{bmatrix} -1\\1\\0\\1 \end{bmatrix}, \begin{bmatrix} -8\\-1\\-9\\-4 \end{bmatrix}, \begin{bmatrix} 3\\1\\-1\\-2 \end{bmatrix}, \begin{bmatrix} -10\\-1\\-1\\4 \end{bmatrix} \right\}$$

It is easy to create elements of $X = \langle R\rangle$ — we will create one at random,

$$\mathbf{y} = 6\begin{bmatrix} 2\\1\\3\\2 \end{bmatrix} + (-7)\begin{bmatrix} -1\\1\\0\\1 \end{bmatrix} + 1\begin{bmatrix} -8\\-1\\-9\\-4 \end{bmatrix} + 6\begin{bmatrix} 3\\1\\-1\\-2 \end{bmatrix} + 2\begin{bmatrix} -10\\-1\\-1\\4 \end{bmatrix} = \begin{bmatrix} 9\\2\\1\\-3 \end{bmatrix}$$

We know we can replace R by a smaller set (since it is obviously linearly dependent by Theorem MVSLD) that will create the same span. Here goes,

$$\begin{bmatrix} 2 & -1 & -8 & 3 & -10\\ 1 & 1 & -1 & 1 & -1\\ 3 & 0 & -9 & -1 & -1\\ 2 & 1 & -4 & -2 & 4 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -3 & 0 & -1\\ 0 & \boxed{1} & 2 & 0 & 2\\ 0 & 0 & 0 & \boxed{1} & -2\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So, if we collect the first, second and fourth vectors from R,

$$P = \left\{ \begin{bmatrix} 2\\1\\3\\2 \end{bmatrix}, \begin{bmatrix} -1\\1\\0\\1 \end{bmatrix}, \begin{bmatrix} 3\\1\\-1\\-2 \end{bmatrix} \right\}$$

then P is linearly independent and $\langle P\rangle = \langle R\rangle = X$ by Theorem BS. Since we built $\mathbf{y}$ as an element of $\langle R\rangle$ it must also be an element of $\langle P\rangle$. Can we write $\mathbf{y}$ as a linear combination of just the three vectors in P? The answer is, of course, yes. But let us compute an explicit linear combination just for fun. By Theorem SLSLC we can get such a linear combination by solving a system of equations with the column vectors of R as the columns of a coefficient matrix, and $\mathbf{y}$ as the vector of constants.

Employing an augmented matrix to solve this system,

$$\begin{bmatrix} 2 & -1 & 3 & 9\\ 1 & 1 & 1 & 2\\ 3 & 0 & -1 & 1\\ 2 & 1 & -2 & -3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 1\\ 0 & \boxed{1} & 0 & -1\\ 0 & 0 & \boxed{1} & 2\\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So we see, as expected, that

$$1\begin{bmatrix} 2\\1\\3\\2 \end{bmatrix} + (-1)\begin{bmatrix} -1\\1\\0\\1 \end{bmatrix} + 2\begin{bmatrix} 3\\1\\-1\\-2 \end{bmatrix} = \begin{bmatrix} 9\\2\\1\\-3 \end{bmatrix} = \mathbf{y}$$

A key feature of this example is that the linear combination that expresses $\mathbf{y}$ as a linear combination of the vectors in P is unique. This is a consequence of the linear independence of P. The linearly independent set P is smaller than R, but still just (barely) big enough to create elements of the set $X = \langle R\rangle$. There are many, many ways to write $\mathbf{y}$ as a linear combination of the five vectors in R (the appropriate system of equations to verify this claim yields two free variables in the description of the solution set), yet there is precisely one way to write $\mathbf{y}$ as a linear combination of the three vectors in P. △

Reading Questions

1. Let S be the linearly dependent set of three vectors below.

$$S = \left\{ \begin{bmatrix} 1 \\ 10 \\ 100 \\ 1000 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 23 \\ 203 \\ 2003 \end{bmatrix} \right\}$$

   Write one vector from S as a linear combination of the other two and include this vector equality in your response. (You should be able to do this on sight, rather than doing some computations.) Convert this expression into a nontrivial relation of linear dependence on S.

2. Explain why the word "dependent" is used in the definition of linear dependence.

3. Suppose that $Y = \langle P \rangle = \langle Q \rangle$, where P is a linearly dependent set and Q is linearly independent. Would you rather use P or Q to describe Y? Why?

Exercises

C20† Let T be the set of columns of the matrix B below. Define $W = \langle T \rangle$. Find a set R so that (1) R has 3 vectors, (2) R is a subset of T, and (3) $W = \langle R \rangle$.

$$B = \begin{bmatrix} -3 & 1 & -2 & 7 \\ -1 & 2 & 1 & 4 \\ 1 & 1 & 2 & -1 \end{bmatrix}$$

C40 Verify that the set $R' = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$ at the end of Example RSC5 is linearly independent.

C50† Consider the set of vectors from $\mathbb{C}^3$, W, given below. Find a linearly independent set T that contains three vectors from W and such that $\langle W \rangle = \langle T \rangle$.

$$W = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5\} = \left\{ \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix} \right\}$$

C51† Given the set S below, find a linearly independent set T so that $\langle T \rangle = \langle S \rangle$.

$$S = \left\{ \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \\ 3 \end{bmatrix} \right\}$$

C52† Let W be the span of the set of vectors S below, $W = \langle S \rangle$. Find a set T so that 1) the span of T is W, $\langle T \rangle = W$, (2) T is a linearly independent set, and (3) T is a subset of S.

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} \right\}$$

C55† Let T be the set of vectors $T = \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 6 \end{bmatrix} \right\}$. Find two different subsets of T, named R and S, so that R and S each contain three vectors, and so that $\langle R \rangle = \langle T \rangle$ and $\langle S \rangle = \langle T \rangle$. Prove that both R and S are linearly independent.

C70 Reprise Example RES by creating a new version of the vector $\mathbf{y}$. In other words, form a new, different linear combination of the vectors in R to create a new vector $\mathbf{y}$ (but do not simplify the problem too much by choosing any of the five new scalars to be zero). Then express this new $\mathbf{y}$ as a combination of the vectors in P.

M10 At the conclusion of Example RSC4 two alternative solutions, sets T' and T^*, are proposed. Verify these claims by proving that $\langle T\rangle = \langle T'\rangle$ and $\langle T\rangle = \langle T^*\rangle$.

T40† Suppose that $\mathbf{v}_1$ and $\mathbf{v}_2$ are any two vectors from $\mathbb{C}^m$. Prove the following set equality.

$$\langle\{\mathbf{v}_1,\,\mathbf{v}_2\}\rangle = \langle\{\mathbf{v}_1+\mathbf{v}_2,\,\mathbf{v}_1-\mathbf{v}_2\}\rangle$$

Section O
Orthogonality

In this section we define a couple more operations with vectors, and prove a few theorems. At first blush these definitions and results will not appear central to what follows, but we will make use of them at key points in the remainder of the course (such as Section MINM, Section OD). Because we have chosen to use $\mathbb{C}$ as our set of scalars, this subsection is a bit more, uh, ...complex than it would be for the real numbers. We will explain as we go along how things get easier for the real numbers $\mathbb{R}$. If you have not already, now would be a good time to review some of the basic properties of arithmetic with complex numbers described in Section CNO. With that done, we can extend the basics of complex number arithmetic to our study of vectors in $\mathbb{C}^m$.

Subsection CAV
Complex Arithmetic and Vectors

We know how the addition and multiplication of complex numbers is employed in defining the operations for vectors in $\mathbb{C}^m$ (Definition CVA and Definition CVSM). We can also extend the idea of the conjugate to vectors.

Definition CCCV Complex Conjugate of a Column Vector
Suppose that $\mathbf{u}$ is a vector from $\mathbb{C}^m$. Then the conjugate of the vector, $\overline{\mathbf{u}}$, is defined by

$$\left[\overline{\mathbf{u}}\right]_i = \overline{\left[\mathbf{u}\right]_i} \qquad\qquad 1 \leq i \leq m$$

□

With this definition we can show that the conjugate of a column vector behaves as we would expect with regard to vector addition and scalar multiplication.

Theorem CRVA Conjugation Respects Vector Addition
Suppose $\mathbf{x}$ and $\mathbf{y}$ are two vectors from $\mathbb{C}^m$. Then

$$\overline{\mathbf{x}+\mathbf{y}} = \overline{\mathbf{x}} + \overline{\mathbf{y}}$$

Proof. For each $1 \leq i \leq m$,

$$\begin{aligned}
\left[\overline{\mathbf{x}+\mathbf{y}}\right]_i &= \overline{\left[\mathbf{x}+\mathbf{y}\right]_i} && \text{Definition CCCV}\\
&= \overline{\left[\mathbf{x}\right]_i + \left[\mathbf{y}\right]_i} && \text{Definition CVA}\\
&= \overline{\left[\mathbf{x}\right]_i} + \overline{\left[\mathbf{y}\right]_i} && \text{Theorem CCRA}\\
&= \left[\overline{\mathbf{x}}\right]_i + \left[\overline{\mathbf{y}}\right]_i && \text{Definition CCCV}\\
&= \left[\overline{\mathbf{x}} + \overline{\mathbf{y}}\right]_i && \text{Definition CVA}
\end{aligned}$$

Then by Definition CVE we have $\overline{\mathbf{x}+\mathbf{y}} = \overline{\mathbf{x}} + \overline{\mathbf{y}}$. ■

Theorem CRSM Conjugation Respects Vector Scalar Multiplication
Suppose $\mathbf{x}$ is a vector from $\mathbb{C}^m$, and $\alpha \in \mathbb{C}$ is a scalar. Then

$$\overline{\alpha\mathbf{x}} = \overline{\alpha}\,\overline{\mathbf{x}}$$

Proof. For $1 \leq i \leq m$,

$$\begin{aligned}
\left[\overline{\alpha\mathbf{x}}\right]_i &= \overline{\left[\alpha\mathbf{x}\right]_i} && \text{Definition CCCV}\\
&= \overline{\alpha\left[\mathbf{x}\right]_i} && \text{Definition CVSM}\\
&= \overline{\alpha}\,\overline{\left[\mathbf{x}\right]_i} && \text{Theorem CCRM}\\
&= \overline{\alpha}\,\left[\overline{\mathbf{x}}\right]_i && \text{Definition CCCV}
\end{aligned}$$

$$= [\overline{\alpha}\,\overline{\mathbf{x}}]_i \qquad \text{Definition CVSM}$$

Then by Definition CVE we have $\overline{\alpha\mathbf{x}} = \overline{\alpha}\,\overline{\mathbf{x}}$. ■

These two theorems together tell us how we can "push" complex conjugation through linear combinations.

Subsection IP
Inner products

Definition IP Inner Product
Given the vectors $\mathbf{u},\ \mathbf{v} \in \mathbb{C}^m$ the **inner product** of $\mathbf{u}$ and $\mathbf{v}$ is the scalar quantity in $\mathbb{C}$,

$$\langle \mathbf{u},\ \mathbf{v}\rangle = \overline{[\mathbf{u}]_1}\,[\mathbf{v}]_1 + \overline{[\mathbf{u}]_2}\,[\mathbf{v}]_2 + \overline{[\mathbf{u}]_3}\,[\mathbf{v}]_3 + \cdots + \overline{[\mathbf{u}]_m}\,[\mathbf{v}]_m = \sum_{i=1}^{m} \overline{[\mathbf{u}]_i}\,[\mathbf{v}]_i$$

□

This operation is a bit different in that we begin with two vectors but produce a scalar. Computing one is straightforward.

Example CSIP Computing some inner products
The inner product of

$$\mathbf{u} = \begin{bmatrix} 2+3i \\ 5+2i \\ -3+i \end{bmatrix} \qquad \text{and} \qquad \mathbf{v} = \begin{bmatrix} 1+2i \\ -4+5i \\ 0+5i \end{bmatrix}$$

is

$$\begin{aligned}
\langle \mathbf{u},\ \mathbf{v}\rangle &= (\overline{2+3i})(1+2i) + (\overline{5+2i})(-4+5i) + (\overline{-3+i})(0+5i) \\
&= (2-3i)(1+2i) + (5-2i)(-4+5i) + (-3-i)(0+5i) \\
&= (8+i) + (-10+33i) + (5-15i) \\
&= 3+19i
\end{aligned}$$

The inner product of

$$\mathbf{w} = \begin{bmatrix} 2 \\ 4 \\ -3 \\ 2 \\ 8 \end{bmatrix} \qquad \text{and} \qquad \mathbf{x} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ -1 \\ -2 \end{bmatrix}$$

is

$$\begin{aligned}
\langle \mathbf{w},\ \mathbf{x}\rangle &= (\overline{2})3 + (\overline{4})1 + (\overline{-3})0 + (\overline{2})(-1) + (\overline{8})(-2) \\
&= 2(3) + 4(1) + (-3)0 + 2(-1) + 8(-2) = -8.
\end{aligned}$$

△

In the case where the entries of our vectors are all real numbers (as in the second part of Example CSIP), the computation of the inner product may look familiar and be known to you as a **dot product** or **scalar product**. So you can view the inner product as a generalization of the scalar product to vectors from $\mathbb{C}^m$ (rather than $\mathbb{R}^m$).

Note that we have chosen to conjugate the entries of the *first* vector listed in the inner product, while it is almost equally feasible to conjugate entries from the *second* vector instead. In particular, prior to Version 2.90, we did use the latter definition, and this has now changed to the former, with resulting adjustments propogated up through Section CB (only). However, conjugating the first vector leads to much nicer formulas for certain matrix decompositions and also shortens some proofs.

There are several quick theorems we can now prove, and they will each be useful later.

Theorem IPVA Inner Product and Vector Addition
Suppose $\mathbf{u},\ \mathbf{v},\ \mathbf{w} \in \mathbb{C}^m$. *Then*

1. $\langle \mathbf{u}+\mathbf{v},\, \mathbf{w}\rangle = \langle \mathbf{u},\, \mathbf{w}\rangle + \langle \mathbf{v},\, \mathbf{w}\rangle$
2. $\langle \mathbf{u},\, \mathbf{v}+\mathbf{w}\rangle = \langle \mathbf{u},\, \mathbf{v}\rangle + \langle \mathbf{u},\, \mathbf{w}\rangle$

Proof. The proofs of the two parts are very similar, with the second one requiring just a bit more effort due to the conjugation that occurs. We will prove part 1 and you can prove part 2 (Exercise O.T10).

$$\begin{aligned}
\langle \mathbf{u}+\mathbf{v},\, \mathbf{w}\rangle &= \sum_{i=1}^{m} \overline{[\mathbf{u}+\mathbf{v}]_i}\, [\mathbf{w}]_i && \text{Definition IP}\\
&= \sum_{i=1}^{m} \left(\overline{[\mathbf{u}]_i + [\mathbf{v}]_i}\right) [\mathbf{w}]_i && \text{Definition CVA}\\
&= \sum_{i=1}^{m} \left(\overline{[\mathbf{u}]_i} + \overline{[\mathbf{v}]_i}\right) [\mathbf{w}]_i && \text{Theorem CCRA}\\
&= \sum_{i=1}^{m} \overline{[\mathbf{u}]_i}\, [\mathbf{w}]_i + \overline{[\mathbf{v}]_i}\, [\mathbf{w}]_i && \text{Property DCN}\\
&= \sum_{i=1}^{m} \overline{[\mathbf{u}]_i}\, [\mathbf{w}]_i + \sum_{i=1}^{m} \overline{[\mathbf{v}]_i}\, [\mathbf{w}]_i && \text{Property CACN}\\
&= \langle \mathbf{u},\, \mathbf{w}\rangle + \langle \mathbf{v},\, \mathbf{w}\rangle && \text{Definition IP}
\end{aligned}$$

■

Theorem IPSM Inner Product and Scalar Multiplication
Suppose $\mathbf{u},\ \mathbf{v} \in \mathbb{C}^m$ *and* $\alpha \in \mathbb{C}$. *Then*

1. $\langle \alpha\mathbf{u},\, \mathbf{v}\rangle = \overline{\alpha}\, \langle \mathbf{u},\, \mathbf{v}\rangle$
2. $\langle \mathbf{u},\, \alpha\mathbf{v}\rangle = \alpha\, \langle \mathbf{u},\, \mathbf{v}\rangle$

Proof. The proofs of the two parts are very similar, with the second one requiring just a bit more effort due to the conjugation that occurs. We will prove part 1 and you can prove part 2 (Exercise O.T11).

$$\begin{aligned}
\langle \alpha\mathbf{u},\, \mathbf{v}\rangle &= \sum_{i=1}^{m} \overline{[\alpha\mathbf{u}]_i}\, [\mathbf{v}]_i && \text{Definition IP}\\
&= \sum_{i=1}^{m} \overline{\alpha\, [\mathbf{u}]_i}\, [\mathbf{v}]_i && \text{Definition CVSM}\\
&= \sum_{i=1}^{m} \overline{\alpha}\, \overline{[\mathbf{u}]_i}\, [\mathbf{v}]_i && \text{Theorem CCRM}\\
&= \overline{\alpha} \sum_{i=1}^{m} \overline{[\mathbf{u}]_i}\, [\mathbf{v}]_i && \text{Property DCN}\\
&= \overline{\alpha}\, \langle \mathbf{u},\, \mathbf{v}\rangle && \text{Definition IP}
\end{aligned}$$

■

Theorem IPAC Inner Product is Anti-Commutative
Suppose that $\mathbf{u}$ *and* $\mathbf{v}$ *are vectors in* $\mathbb{C}^m$. *Then* $\langle\mathbf{u},\,\mathbf{v}\rangle = \overline{\langle\mathbf{v},\,\mathbf{u}\rangle}$.

Proof.

$$
\begin{aligned}
\langle\mathbf{u},\,\mathbf{v}\rangle &= \sum_{i=1}^{m} \overline{[\mathbf{u}]_i}\,[\mathbf{v}]_i && \text{Definition IP}\\
&= \sum_{i=1}^{m} \overline{[\mathbf{u}]_i}\,\overline{\overline{[\mathbf{v}]_i}} && \text{Theorem CCT}\\
&= \sum_{i=1}^{m} \overline{[\mathbf{u}]_i\,\overline{[\mathbf{v}]_i}} && \text{Theorem CCRM}\\
&= \overline{\left(\sum_{i=1}^{m} [\mathbf{u}]_i\,\overline{[\mathbf{v}]_i}\right)} && \text{Theorem CCRA}\\
&= \overline{\left(\sum_{i=1}^{m} \overline{[\mathbf{v}]_i}\,[\mathbf{u}]_i\right)} && \text{Property CMCN}\\
&= \overline{\langle\mathbf{v},\,\mathbf{u}\rangle} && \text{Definition IP}
\end{aligned}
$$

■

Subsection N
Norm

If treating linear algebra in a more geometric fashion, the length of a vector occurs naturally, and is what you would expect from its name. With complex numbers, we will define a similar function. Recall that if c is a complex number, then $|c|$ denotes its modulus (Definition MCN).

Definition NV Norm of a Vector
The **norm** of the vector $\mathbf{u}$ is the scalar quantity in $\mathbb{C}$

$$\|\mathbf{u}\| = \sqrt{\left|[\mathbf{u}]_1\right|^2 + \left|[\mathbf{u}]_2\right|^2 + \left|[\mathbf{u}]_3\right|^2 + \cdots + \left|[\mathbf{u}]_m\right|^2} = \sqrt{\sum_{i=1}^{m} \left|[\mathbf{u}]_i\right|^2}$$

□

Computing a norm is also easy to do.

Example CNSV Computing the norm of some vectors
The norm of

$$\mathbf{u} = \begin{bmatrix} 3+2i \\ 1-6i \\ 2+4i \\ 2+i \end{bmatrix}$$

is

$$
\begin{aligned}
\|\mathbf{u}\| &= \sqrt{|3+2i|^2 + |1-6i|^2 + |2+4i|^2 + |2+i|^2}\\
&= \sqrt{13+37+20+5} = \sqrt{75} = 5\sqrt{3}
\end{aligned}
$$

The norm of

$$\mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 4 \\ -3 \end{bmatrix}$$

is

$$\left\|\mathbf{v}\right\| = \sqrt{\left|3\right|^2 + \left|-1\right|^2 + \left|2\right|^2 + \left|4\right|^2 + \left|-3\right|^2} = \sqrt{3^2 + 1^2 + 2^2 + 4^2 + 3^2} = \sqrt{39}.$$

△

Notice how the norm of a vector with real number entries is just the length of the vector. Inner products and norms are related by the following theorem.

Theorem IPN Inner Products and Norms
Suppose that $\mathbf{u}$ *is a vector in* $\mathbb{C}^m$*. Then* $\left\|\mathbf{u}\right\|^2 = \langle\mathbf{u},\,\mathbf{u}\rangle$*.*

Proof.

$$\begin{aligned}
\left\|\mathbf{u}\right\|^2 &= \left(\sqrt{\sum_{i=1}^{m}\left|\left[\mathbf{u}\right]_i\right|^2}\right)^2 && \text{Definition NV} \\
&= \sum_{i=1}^{m}\left|\left[\mathbf{u}\right]_i\right|^2 && \text{Inverse functions} \\
&= \sum_{i=1}^{m}\overline{\left[\mathbf{u}\right]_i}\left[\mathbf{u}\right]_i && \text{Definition MCN} \\
&= \langle\mathbf{u},\,\mathbf{u}\rangle && \text{Definition IP}
\end{aligned}$$

■

When our vectors have entries only from the real numbers Theorem IPN says that the dot product of a vector with itself is equal to the length of the vector squared.

Theorem PIP Positive Inner Products
Suppose that $\mathbf{u}$ *is a vector in* $\mathbb{C}^m$*. Then* $\langle\mathbf{u},\,\mathbf{u}\rangle \geq 0$ *with equality if and only if* $\mathbf{u} = \mathbf{0}$*.*

Proof. From the proof of Theorem IPN we see that

$$\langle\mathbf{u},\,\mathbf{u}\rangle = \left|\left[\mathbf{u}\right]_1\right|^2 + \left|\left[\mathbf{u}\right]_2\right|^2 + \left|\left[\mathbf{u}\right]_3\right|^2 + \cdots + \left|\left[\mathbf{u}\right]_m\right|^2$$

Since each modulus is squared, every term is positive, and the sum must also be positive. (Notice that in general the inner product is a complex number and cannot be compared with zero, but in the special case of $\langle\mathbf{u},\,\mathbf{u}\rangle$ the result is a real number.)

The phrase, "with equality if and only if" means that we want to show that the statement $\langle\mathbf{u},\,\mathbf{u}\rangle = 0$ (i.e. with equality) is equivalent ("if and only if") to the statement $\mathbf{u} = \mathbf{0}$.

If $\mathbf{u} = \mathbf{0}$, then it is a straightforward computation to see that $\langle\mathbf{u},\,\mathbf{u}\rangle = 0$. In the other direction, assume that $\langle\mathbf{u},\,\mathbf{u}\rangle = 0$. As before, $\langle\mathbf{u},\,\mathbf{u}\rangle$ is a sum of moduli. So we have

$$0 = \langle\mathbf{u},\,\mathbf{u}\rangle = \left|\left[\mathbf{u}\right]_1\right|^2 + \left|\left[\mathbf{u}\right]_2\right|^2 + \left|\left[\mathbf{u}\right]_3\right|^2 + \cdots + \left|\left[\mathbf{u}\right]_m\right|^2$$

Now we have a sum of squares equaling zero, so each term must be zero. Then by similar logic, $\left|\left[\mathbf{u}\right]_i\right| = 0$ will imply that $\left[\mathbf{u}\right]_i = 0$, since $0 + 0i$ is the only complex number with zero modulus. Thus every entry of $\mathbf{u}$ is zero and so $\mathbf{u} = \mathbf{0}$, as desired. ■

Notice that Theorem PIP contains *three* implications:

$$\mathbf{u} \in \mathbb{C}^m \Rightarrow \langle \mathbf{u},\, \mathbf{u}\rangle \geq 0$$
$$\mathbf{u} = \mathbf{0} \Rightarrow \langle \mathbf{u},\, \mathbf{u}\rangle = 0$$
$$\langle \mathbf{u},\, \mathbf{u}\rangle = 0 \Rightarrow \mathbf{u} = \mathbf{0}$$

The results contained in Theorem PIP are summarized by saying "the inner product is **positive definite**."

Subsection OV
Orthogonal Vectors

"Orthogonal" is a generalization of "perpendicular." You may have used mutually perpendicular vectors in a physics class, or you may recall from a calculus class that perpendicular vectors have a zero dot product. We will now extend these ideas into the realm of higher dimensions and complex scalars.

Definition OV Orthogonal Vectors
A pair of vectors, $\mathbf{u}$ and $\mathbf{v}$, from $\mathbb{C}^m$ are **orthogonal** if their inner product is zero, that is, $\langle \mathbf{u},\, \mathbf{v}\rangle = 0$. □

Example TOV Two orthogonal vectors
The vectors

$$\mathbf{u} = \begin{bmatrix} 2+3i \\ 4-2i \\ 1+i \\ 1+i \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 1-i \\ 2+3i \\ 4-6i \\ 1 \end{bmatrix}$$

are orthogonal since

$$\begin{aligned}\langle \mathbf{u},\, \mathbf{v}\rangle &= (2-3i)(1-i) + (4+2i)(2+3i) + (1-i)(4-6i) + (1-i)(1)\\ &= (-1-5i) + (2+16i) + (-2-10i) + (1-i)\\ &= 0+0i.\end{aligned}$$

We extend this definition to whole sets by requiring vectors to be pairwise orthogonal. Despite using the same word, careful thought about what objects you are using will eliminate any source of confusion.

Definition OSV Orthogonal Set of Vectors
Suppose that $S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_n\}$ is a set of vectors from $\mathbb{C}^m$. Then S is an **orthogonal set** if every pair of different vectors from S is orthogonal, that is $\langle \mathbf{u}_i,\, \mathbf{u}_j\rangle = 0$ whenever $i \neq j$. □

We now define the prototypical orthogonal set, which we will reference repeatedly.

Definition SUV Standard Unit Vectors
Let $\mathbf{e}_j \in \mathbb{C}^m$, $1 \leq j \leq m$ denote the column vectors defined by

$$[\mathbf{e}_j]_i = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

Then the set

$$\{\mathbf{e}_1,\, \mathbf{e}_2,\, \mathbf{e}_3,\, \ldots,\, \mathbf{e}_m\} = \{\mathbf{e}_j \,|\, 1 \leq j \leq m\}$$

is the set of **standard unit vectors** in $\mathbb{C}^m$. □

Notice that $\mathbf{e}_j$ is identical to column j of the $m \times m$ identity matrix I_m (Definition IM) and is a pivot column for I_m, since the identity matrix is in reduced row-echelon form. These observations will often be useful. We will reserve the notation $\mathbf{e}_i$ for these vectors. It is not hard to see that the set of standard unit vectors is an orthogonal set.

Example SUVOS Standard Unit Vectors are an Orthogonal Set
Compute the inner product of two distinct vectors from the set of standard unit vectors (Definition SUV), say $\mathbf{e}_i$, $\mathbf{e}_j$, where $i \neq j$,

$$\begin{aligned}\langle \mathbf{e}_i,\, \mathbf{e}_j \rangle &= \overline{0}0 + \overline{0}0 + \cdots + \overline{1}0 + \cdots + \overline{0}0 + \cdots + \overline{0}1 + \cdots + \overline{0}0 + \overline{0}0\\ &= 0(0) + 0(0) + \cdots + 1(0) + \cdots + 0(1) + \cdots + 0(0) + 0(0)\\ &= 0\end{aligned}$$

So the set $\{\mathbf{e}_1,\, \mathbf{e}_2,\, \mathbf{e}_3,\, \ldots,\, \mathbf{e}_m\}$ is an orthogonal set. △

Example AOS An orthogonal set
The set

$$\{\mathbf{x}_1,\, \mathbf{x}_2,\, \mathbf{x}_3,\, \mathbf{x}_4\} = \left\{ \begin{bmatrix}1+i\\1\\1-i\\i\end{bmatrix},\, \begin{bmatrix}1+5i\\6+5i\\-7-i\\1-6i\end{bmatrix},\, \begin{bmatrix}-7+34i\\-8-23i\\-10+22i\\30+13i\end{bmatrix},\, \begin{bmatrix}-2-4i\\6+i\\4+3i\\6-i\end{bmatrix} \right\}$$

is an orthogonal set.

Since the inner product is anti-commutative (Theorem IPAC) we can test pairs of different vectors in any order. If the result is zero, then it will also be zero if the inner product is computed in the opposite order. This means there are six different pairs of vectors to use in an inner product computation. We will do two and you can practice your inner products on the other four.

$$\begin{aligned}\langle \mathbf{x}_1,\, \mathbf{x}_3 \rangle &= (1-i)(-7+34i) + (1)(-8-23i) + (1+i)(-10+22i) + (-i)(30+13i)\\ &= (27+41i) + (-8-23i) + (-32+12i) + (13-30i)\\ &= 0+0i\end{aligned}$$

and

$$\begin{aligned}\langle \mathbf{x}_2,\, \mathbf{x}_4 \rangle &= (1-5i)(-2-4i) + (6-5i)(6+i) + (-7+i)(4+3i) + (1+6i)(6-i)\\ &= (-22+6i) + (41-24i) + (-31-17i) + (12+35i)\\ &= 0+0i\end{aligned}$$

So far, this section has seen lots of definitions, and lots of theorems establishing un-surprising consequences of those definitions. But here is our first theorem that suggests that inner products and orthogonal vectors have some utility. It is also one of our first illustrations of how to arrive at linear independence as the conclusion of a theorem.

Theorem OSLI Orthogonal Sets are Linearly Independent
Suppose that S is an orthogonal set of nonzero vectors. Then S is linearly independent.

Proof. Let $S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_n\}$ be an orthogonal set of nonzero vectors. To prove the linear independence of S, we can appeal to the definition (Definition LICV) and begin with an arbitrary relation of linear dependence (Definition RLDCV),

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_n\mathbf{u}_n = \mathbf{0}.$$

Then, for every $1 \leq i \leq n$, we have

$$\begin{aligned}&\alpha_i \langle \mathbf{u}_i,\, \mathbf{u}_i \rangle\\ &\quad= \alpha_1(0) + \alpha_2(0) + \cdots + \alpha_i \langle \mathbf{u}_i,\, \mathbf{u}_i \rangle + \cdots + \alpha_n(0) && \text{Property ZCN}\\ &\quad= \alpha_1 \langle \mathbf{u}_i,\, \mathbf{u}_1 \rangle + \cdots + \alpha_i \langle \mathbf{u}_i,\, \mathbf{u}_i \rangle + \cdots + \alpha_n \langle \mathbf{u}_i,\, \mathbf{u}_n \rangle && \text{Definition OSV}\\ &\quad= \langle \mathbf{u}_i,\, \alpha_1\mathbf{u}_1 \rangle + \langle \mathbf{u}_i,\, \alpha_2\mathbf{u}_2 \rangle + \cdots + \langle \mathbf{u}_i,\, \alpha_n\mathbf{u}_n \rangle && \text{Theorem IPSM}\\ &\quad= \langle \mathbf{u}_i,\, \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_n\mathbf{u}_n \rangle && \text{Theorem IPVA}\\ &\quad= \langle \mathbf{u}_i,\, \mathbf{0} \rangle && \text{Definition RLDCV}\\ &\quad= 0 && \text{Definition IP}\end{aligned}$$

Because $\mathbf{u}_i$ was assumed to be nonzero, Theorem PIP says $\langle\mathbf{u}_i,\,\mathbf{u}_i\rangle$ is nonzero and thus α_i must be zero. So we conclude that $\alpha_i = 0$ for all $1 \leq i \leq n$ in any relation of linear dependence on S. But this says that S is a linearly independent set since the only way to form a relation of linear dependence is the trivial way (Definition LICV). Boom! ■

Subsection GSP Gram-Schmidt Procedure

The Gram-Schmidt Procedure is really a theorem. It says that if we begin with a linearly independent set of p vectors, S, then we can do a number of calculations with these vectors and produce an orthogonal set of p vectors, T, so that $\langle S\rangle = \langle T\rangle$. Given the large number of computations involved, it is indeed a procedure to do all the necessary computations, and it is best employed on a computer. However, it also has value in proofs where we may on occasion wish to replace a linearly independent set by an orthogonal set.

This is our first occasion to use the technique of "mathematical induction" for a proof, a technique we will see again several times, especially in Chapter D. So study the simple example described in Proof Technique I first.

Theorem GSP Gram-Schmidt Procedure
Suppose that $S = \{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3,\,\ldots,\,\mathbf{v}_p\}$ *is a linearly independent set of vectors in* $\mathbb{C}^m$. *Define the vectors* $\mathbf{u}_i$, $1 \leq i \leq p$ *by*

$$\mathbf{u}_i = \mathbf{v}_i - \frac{\langle\mathbf{u}_1,\,\mathbf{v}_i\rangle}{\langle\mathbf{u}_1,\,\mathbf{u}_1\rangle}\mathbf{u}_1 - \frac{\langle\mathbf{u}_2,\,\mathbf{v}_i\rangle}{\langle\mathbf{u}_2,\,\mathbf{u}_2\rangle}\mathbf{u}_2 - \frac{\langle\mathbf{u}_3,\,\mathbf{v}_i\rangle}{\langle\mathbf{u}_3,\,\mathbf{u}_3\rangle}\mathbf{u}_3 - \cdots - \frac{\langle\mathbf{u}_{i-1},\,\mathbf{v}_i\rangle}{\langle\mathbf{u}_{i-1},\,\mathbf{u}_{i-1}\rangle}\mathbf{u}_{i-1}$$

Let $T = \{\mathbf{u}_1,\,\mathbf{u}_2,\,\mathbf{u}_3,\,\ldots,\,\mathbf{u}_p\}$. *Then* T *is an orthogonal set of nonzero vectors, and* $\langle T\rangle = \langle S\rangle$.

Proof. We will prove the result by using induction on p (Proof Technique I). To begin, we prove that T has the desired properties when $p = 1$. In this case $\mathbf{u}_1 = \mathbf{v}_1$ and $T = \{\mathbf{u}_1\} = \{\mathbf{v}_1\} = S$. Because S and T are equal, $\langle S\rangle = \langle T\rangle$. Equally trivial, T is an orthogonal set. If $\mathbf{u}_1 = \mathbf{0}$, then S would be a linearly dependent set, a contradiction.

Suppose that the theorem is true for any set of $p-1$ linearly independent vectors. Let $S = \{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3,\,\ldots,\,\mathbf{v}_p\}$ be a linearly independent set of p vectors. Then $S' = \{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3,\,\ldots,\,\mathbf{v}_{p-1}\}$ is also linearly independent. So we can apply the theorem to S' and construct the vectors $T' = \{\mathbf{u}_1,\,\mathbf{u}_2,\,\mathbf{u}_3,\,\ldots,\,\mathbf{u}_{p-1}\}$. T' is therefore an orthogonal set of nonzero vectors and $\langle S'\rangle = \langle T'\rangle$. Define

$$\mathbf{u}_p = \mathbf{v}_p - \frac{\langle\mathbf{u}_1,\,\mathbf{v}_p\rangle}{\langle\mathbf{u}_1,\,\mathbf{u}_1\rangle}\mathbf{u}_1 - \frac{\langle\mathbf{u}_2,\,\mathbf{v}_p\rangle}{\langle\mathbf{u}_2,\,\mathbf{u}_2\rangle}\mathbf{u}_2 - \frac{\langle\mathbf{u}_3,\,\mathbf{v}_p\rangle}{\langle\mathbf{u}_3,\,\mathbf{u}_3\rangle}\mathbf{u}_3 - \cdots - \frac{\langle\mathbf{u}_{p-1},\,\mathbf{v}_p\rangle}{\langle\mathbf{u}_{p-1},\,\mathbf{u}_{p-1}\rangle}\mathbf{u}_{p-1}$$

and let $T = T' \cup \{\mathbf{u}_p\}$. We need to now show that T has several properties by building on what we know about T'. But first notice that the above equation has no problems with the denominators ($\langle\mathbf{u}_i,\,\mathbf{u}_i\rangle$) being zero, since the $\mathbf{u}_i$ are from T', which is composed of nonzero vectors.

We show that $\langle T\rangle = \langle S\rangle$, by first establishing that $\langle T\rangle \subseteq \langle S\rangle$. Suppose $\mathbf{x} \in \langle T\rangle$, so

$$\mathbf{x} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \cdots + a_p\mathbf{u}_p$$

The term $a_p\mathbf{u}_p$ is a linear combination of vectors from T' and the vector $\mathbf{v}_p$, while the remaining terms are a linear combination of vectors from T'. Since $\langle T'\rangle = \langle S'\rangle$, any term that is a multiple of a vector from T' can be rewritten as a linear combination of vectors from S'. The remaining term $a_p\mathbf{v}_p$ is a multiple of a vector in S. So we see that $\mathbf{x}$ can be rewritten as a linear combination of vectors from S, i.e. $\mathbf{x} \in \langle S\rangle$.

To show that $\langle S\rangle \subseteq \langle T\rangle$, begin with $\mathbf{y} \in \langle S\rangle$, so

$$\mathbf{y} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \cdots + a_p\mathbf{v}_p$$

Rearrange our defining equation for $\mathbf{u}_p$ by solving for $\mathbf{v}_p$. Then the term $a_p\mathbf{v}_p$ is a multiple of a linear combination of elements of T. The remaining terms are a linear combination of $\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3,\,\ldots,\,\mathbf{v}_{p-1}$, hence an element of $\langle S'\rangle = \langle T'\rangle$. Thus these remaining terms can be written as a linear combination of the vectors in T'. So $\mathbf{y}$ is a linear combination of vectors from T, i.e. $\mathbf{y} \in \langle T\rangle$.

The elements of T' are nonzero, but what about $\mathbf{u}_p$? Suppose to the contrary that $\mathbf{u}_p = \mathbf{0}$,

$$\mathbf{0} = \mathbf{u}_p = \mathbf{v}_p - \frac{\langle \mathbf{u}_1,\,\mathbf{v}_p\rangle}{\langle \mathbf{u}_1,\,\mathbf{u}_1\rangle}\mathbf{u}_1 - \frac{\langle \mathbf{u}_2,\,\mathbf{v}_p\rangle}{\langle \mathbf{u}_2,\,\mathbf{u}_2\rangle}\mathbf{u}_2 - \frac{\langle \mathbf{u}_3,\,\mathbf{v}_p\rangle}{\langle \mathbf{u}_3,\,\mathbf{u}_3\rangle}\mathbf{u}_3 - \cdots - \frac{\langle \mathbf{u}_{p-1},\,\mathbf{v}_p\rangle}{\langle \mathbf{u}_{p-1},\,\mathbf{u}_{p-1}\rangle}\mathbf{u}_{p-1}$$

$$\mathbf{v}_p = \frac{\langle \mathbf{u}_1,\,\mathbf{v}_p\rangle}{\langle \mathbf{u}_1,\,\mathbf{u}_1\rangle}\mathbf{u}_1 + \frac{\langle \mathbf{u}_2,\,\mathbf{v}_p\rangle}{\langle \mathbf{u}_2,\,\mathbf{u}_2\rangle}\mathbf{u}_2 + \frac{\langle \mathbf{u}_3,\,\mathbf{v}_p\rangle}{\langle \mathbf{u}_3,\,\mathbf{u}_3\rangle}\mathbf{u}_3 + \cdots + \frac{\langle \mathbf{u}_{p-1},\,\mathbf{v}_p\rangle}{\langle \mathbf{u}_{p-1},\,\mathbf{u}_{p-1}\rangle}\mathbf{u}_{p-1}$$

Since $\langle S'\rangle = \langle T'\rangle$ we can write the vectors $\mathbf{u}_1,\,\mathbf{u}_2,\,\mathbf{u}_3,\,\ldots,\,\mathbf{u}_{p-1}$ on the right side of this equation in terms of the vectors $\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3,\,\ldots,\,\mathbf{v}_{p-1}$ and we then have the vector $\mathbf{v}_p$ expressed as a linear combination of the other $p-1$ vectors in S, implying that S is a linearly dependent set (Theorem DLDS), contrary to our lone hypothesis about S.

Finally, it is a simple matter to establish that T is an orthogonal set, though it will not appear so simple looking. Think about your objects as you work through the following — what is a vector and what is a scalar. Since T' is an orthogonal set by induction, most pairs of elements in T are already known to be orthogonal. We just need to test "new" inner products, between $\mathbf{u}_p$ and $\mathbf{u}_i$, for $1 \leq i \leq p-1$. Here we go, using summation notation,

$$\begin{aligned}
\langle \mathbf{u}_i,\,\mathbf{u}_p\rangle &= \left\langle \mathbf{u}_i,\,\mathbf{v}_p - \sum_{k=1}^{p-1}\frac{\langle \mathbf{u}_k,\,\mathbf{v}_p\rangle}{\langle \mathbf{u}_k,\,\mathbf{u}_k\rangle}\mathbf{u}_k\right\rangle \\
&= \langle \mathbf{u}_i,\,\mathbf{v}_p\rangle - \left\langle \mathbf{u}_i,\,\sum_{k=1}^{p-1}\frac{\langle \mathbf{u}_k,\,\mathbf{v}_p\rangle}{\langle \mathbf{u}_k,\,\mathbf{u}_k\rangle}\mathbf{u}_k\right\rangle && \text{Theorem IPVA} \\
&= \langle \mathbf{u}_i,\,\mathbf{v}_p\rangle - \sum_{k=1}^{p-1}\left\langle \mathbf{u}_i,\,\frac{\langle \mathbf{u}_k,\,\mathbf{v}_p\rangle}{\langle \mathbf{u}_k,\,\mathbf{u}_k\rangle}\mathbf{u}_k\right\rangle && \text{Theorem IPVA} \\
&= \langle \mathbf{u}_i,\,\mathbf{v}_p\rangle - \sum_{k=1}^{p-1}\frac{\langle \mathbf{u}_k,\,\mathbf{v}_p\rangle}{\langle \mathbf{u}_k,\,\mathbf{u}_k\rangle}\langle \mathbf{u}_i,\,\mathbf{u}_k\rangle && \text{Theorem IPSM} \\
&= \langle \mathbf{u}_i,\,\mathbf{v}_p\rangle - \frac{\langle \mathbf{u}_i,\,\mathbf{v}_p\rangle}{\langle \mathbf{u}_i,\,\mathbf{u}_i\rangle}\langle \mathbf{u}_i,\,\mathbf{u}_i\rangle - \sum_{k\neq i}\frac{\langle \mathbf{u}_k,\,\mathbf{v}_p\rangle}{\langle \mathbf{u}_k,\,\mathbf{u}_k\rangle}(0) && \text{Induction Hypothesis} \\
&= \langle \mathbf{u}_i,\,\mathbf{v}_p\rangle - \langle \mathbf{u}_i,\,\mathbf{v}_p\rangle - \sum_{k\neq i} 0 \\
&= 0
\end{aligned}$$

■

Example GSTV Gram-Schmidt of three vectors
We will illustrate the Gram-Schmidt process with three vectors. Begin with the linearly independent (check this!) set

$$S = \{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3\} = \left\{\begin{bmatrix}1\\1+i\\1\end{bmatrix},\,\begin{bmatrix}-i\\1\\1+i\end{bmatrix},\,\begin{bmatrix}0\\i\\i\end{bmatrix}\right\}$$

Then

$$\mathbf{u}_1 = \mathbf{v_1} = \begin{bmatrix} 1 \\ 1+i \\ 1 \end{bmatrix}$$

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\langle \mathbf{u}_1,\, \mathbf{v}_2 \rangle}{\langle \mathbf{u}_1,\, \mathbf{u}_1 \rangle}\mathbf{u}_1 = \frac{1}{4}\begin{bmatrix} -2-3i \\ 1-i \\ 2+5i \end{bmatrix}$$

$$\mathbf{u}_3 = \mathbf{v}_3 - \frac{\langle \mathbf{u}_1,\, \mathbf{v}_3 \rangle}{\langle \mathbf{u}_1,\, \mathbf{u}_1 \rangle}\mathbf{u}_1 - \frac{\langle \mathbf{u}_2,\, \mathbf{v}_3 \rangle}{\langle \mathbf{u}_2,\, \mathbf{u}_2 \rangle}\mathbf{u}_2 = \frac{1}{11}\begin{bmatrix} -3-i \\ 1+3i \\ -1-i \end{bmatrix}$$

and

$$T = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3\} = \left\{ \begin{bmatrix} 1 \\ 1+i \\ 1 \end{bmatrix},\, \frac{1}{4}\begin{bmatrix} -2-3i \\ 1-i \\ 2+5i \end{bmatrix},\, \frac{1}{11}\begin{bmatrix} -3-i \\ 1+3i \\ -1-i \end{bmatrix} \right\}$$

is an orthogonal set (which you can check) of nonzero vectors and $\langle T \rangle = \langle S \rangle$ (all by Theorem GSP). Of course, as a by-product of orthogonality, the set T is also linearly independent (Theorem OSLI). △

One final definition related to orthogonal vectors.

Definition ONS OrthoNormal Set
Suppose $S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_n\}$ is an orthogonal set of vectors such that $\|\mathbf{u}_i\| = 1$ for all $1 \leq i \leq n$. Then S is an **orthonormal** set of vectors. □

Once you have an orthogonal set, it is easy to convert it to an orthonormal set — multiply each vector by the reciprocal of its norm, and the resulting vector will have norm 1. This scaling of each vector will not affect the orthogonality properties (apply Theorem IPSM).

Example ONTV Orthonormal set, three vectors
The set

$$T = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3\} = \left\{ \begin{bmatrix} 1 \\ 1+i \\ 1 \end{bmatrix},\, \frac{1}{4}\begin{bmatrix} -2-3i \\ 1-i \\ 2+5i \end{bmatrix},\, \frac{1}{11}\begin{bmatrix} -3-i \\ 1+3i \\ -1-i \end{bmatrix} \right\}$$

from Example GSTV is an orthogonal set.

We compute the norm of each vector,

$$\|\mathbf{u}_1\| = 2 \qquad \|\mathbf{u}_2\| = \frac{1}{2}\sqrt{11} \qquad \|\mathbf{u}_3\| = \frac{\sqrt{2}}{\sqrt{11}}$$

Converting each vector to a norm of 1, yields an orthonormal set,

$$\mathbf{w}_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1+i \\ 1 \end{bmatrix}$$

$$\mathbf{w}_2 = \frac{1}{\frac{1}{2}\sqrt{11}}\frac{1}{4}\begin{bmatrix} -2-3i \\ 1-i \\ 2+5i \end{bmatrix} = \frac{1}{2\sqrt{11}}\begin{bmatrix} -2-3i \\ 1-i \\ 2+5i \end{bmatrix}$$

$$\mathbf{w}_3 = \frac{1}{\frac{\sqrt{2}}{\sqrt{11}}}\frac{1}{11}\begin{bmatrix} -3-i \\ 1+3i \\ -1-i \end{bmatrix} = \frac{1}{\sqrt{22}}\begin{bmatrix} -3-i \\ 1+3i \\ -1-i \end{bmatrix}$$

△

Example ONFV Orthonormal set, four vectors

As an exercise convert the linearly independent set

$$S = \left\{ \begin{bmatrix} 1+i \\ 1 \\ 1-i \\ i \end{bmatrix},\, \begin{bmatrix} i \\ 1+i \\ -1 \\ -i \end{bmatrix},\, \begin{bmatrix} i \\ -i \\ -1+i \\ 1 \end{bmatrix},\, \begin{bmatrix} -1-i \\ i \\ 1 \\ -1 \end{bmatrix} \right\}$$

to an orthogonal set via the Gram-Schmidt Process (Theorem GSP) and then scale the vectors to norm 1 to create an orthonormal set. You should get the same set you would if you scaled the orthogonal set of Example AOS to become an orthonormal set. △

We will see orthonormal sets again in Subsection MINM.UM. They are intimately related to unitary matrices (Definition UM) through Theorem CUMOS. Some of the utility of orthonormal sets is captured by Theorem COB in Subsection B.OBC. Orthonormal sets appear once again in Section OD where they are key in orthonormal diagonalization.

Reading Questions

1. Is the set

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix},\, \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix},\, \begin{bmatrix} 8 \\ 4 \\ -2 \end{bmatrix} \right\}$$

   an orthogonal set? Why?

2. What is the distinction between an orthogonal set and an orthonormal set?

3. What is nice about the output of the Gram-Schmidt process?

Exercises

C20 Complete Example AOS by verifying that the four remaining inner products are zero.

C21 Verify that the set T created in Example GSTV by the Gram-Schmidt Procedure is an orthogonal set.

M60 Suppose that $\{\mathbf{u},\,\mathbf{v},\,\mathbf{w}\} \subseteq \mathbb{C}^n$ is an orthonormal set. Prove that $\mathbf{u}+\mathbf{v}$ is not orthogonal to $\mathbf{v}+\mathbf{w}$.

T10 Prove part 2 of the conclusion of Theorem IPVA.

T11 Prove part 2 of the conclusion of Theorem IPSM.

T20† Suppose that $\mathbf{u},\,\mathbf{v},\,\mathbf{w} \in \mathbb{C}^n$, $\alpha,\,\beta \in \mathbb{C}$ and $\mathbf{u}$ is orthogonal to both $\mathbf{v}$ and $\mathbf{w}$. Prove that $\mathbf{u}$ is orthogonal to $\alpha\mathbf{v}+\beta\mathbf{w}$.

T30 Suppose that the set S in the hypothesis of Theorem GSP is not just linearly independent, but is also orthogonal. Prove that the set T created by the Gram-Schmidt procedure is equal to S. (Note that we are getting a stronger conclusion than $\langle T\rangle = \langle S\rangle$ — the conclusion is that $T = S$.) In other words, it is pointless to apply the Gram-Schmidt procedure to a set that is already orthogonal.

Chapter M
Matrices

We have made frequent use of matrices for solving systems of equations, and we have begun to investigate a few of their properties, such as the null space and nonsingularity. In this chapter, we will take a more systematic approach to the study of matrices.

Section MO
Matrix Operations

In this section we will back up and start simple. We begin with a definition of a totally general set of matrices, and see where that takes us.

Subsection MEASM
Matrix Equality, Addition, Scalar Multiplication

Definition VSM Vector Space of $m \times n$ Matrices
The vector space M_{mn} is the set of all $m \times n$ matrices with entries from the set of complex numbers. □

Just as we made, and used, a careful definition of equality for column vectors, so too, we have precise definitions for matrices.

Definition ME Matrix Equality
The $m \times n$ matrices A and B are **equal**, written $A = B$ provided $[A]_{ij} = [B]_{ij}$ for all $1 \leq i \leq m$, $1 \leq j \leq n$. □

So equality of matrices translates to the equality of complex numbers, on an entry-by-entry basis. Notice that we now have yet another definition that uses the symbol "=" for shorthand. Whenever a theorem has a conclusion saying two matrices are equal (think about your objects), we will consider appealing to this definition as a way of formulating the top-level structure of the proof.

We will now define two operations on the set M_{mn}. Again, we will overload a symbol ('+') and a convention (juxtaposition for scalar multiplication).

Definition MA Matrix Addition
Given the $m \times n$ matrices A and B, define the **sum** of A and B as an $m \times n$ matrix, written $A + B$, according to

$$[A + B]_{ij} = [A]_{ij} + [B]_{ij} \qquad\qquad 1 \leq i \leq m,\ 1 \leq j \leq n$$

□

So matrix addition takes two matrices of the same size and combines them (in a natural way!) to create a new matrix of the same size. Perhaps this is the "obvious" thing to do, but it does not relieve us from the obligation to state it carefully.

Example MA Addition of two matrices in M_{23}
If

$$A = \begin{bmatrix} 2 & -3 & 4 \\ 1 & 0 & -7 \end{bmatrix} \qquad B = \begin{bmatrix} 6 & 2 & -4 \\ 3 & 5 & 2 \end{bmatrix}$$

then

$$\begin{aligned} A + B &= \begin{bmatrix} 2 & -3 & 4 \\ 1 & 0 & -7 \end{bmatrix} + \begin{bmatrix} 6 & 2 & -4 \\ 3 & 5 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 2+6 & -3+2 & 4+(-4) \\ 1+3 & 0+5 & -7+2 \end{bmatrix} = \begin{bmatrix} 8 & -1 & 0 \\ 4 & 5 & -5 \end{bmatrix} \end{aligned}$$

△

Our second operation takes two objects of different types, specifically a number and a matrix, and combines them to create another matrix. As with vectors, in this context we call a number a **scalar** in order to emphasize that it is not a matrix.

Definition MSM Matrix Scalar Multiplication
Given the $m \times n$ matrix A and the scalar $\alpha \in \mathbb{C}$, the **scalar multiple** of A is an $m \times n$ matrix, written αA and defined according to

$$[\alpha A]_{ij} = \alpha [A]_{ij} \qquad 1 \leq i \leq m,\, 1 \leq j \leq n$$

□

Notice again that we have yet another kind of multiplication, and it is again written putting two symbols side-by-side. Computationally, scalar matrix multiplication is very easy.

Example MSM Scalar multiplication in M_{32}
If

$$A = \begin{bmatrix} 2 & 8 \\ -3 & 5 \\ 0 & 1 \end{bmatrix}$$

and $\alpha = 7$, then

$$\alpha A = 7 \begin{bmatrix} 2 & 8 \\ -3 & 5 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7(2) & 7(8) \\ 7(-3) & 7(5) \\ 7(0) & 7(1) \end{bmatrix} = \begin{bmatrix} 14 & 56 \\ -21 & 35 \\ 0 & 7 \end{bmatrix}$$

△

Subsection VSP
Vector Space Properties

With definitions of matrix addition and scalar multiplication we can now state, and prove, several properties of each operation, and some properties that involve their interplay. We now collect ten of them here for later reference.

Theorem VSPM Vector Space Properties of Matrices
Suppose that M_{mn} is the set of all $m \times n$ matrices (Definition VSM) with addition and scalar multiplication as defined in Definition MA and Definition MSM. Then

- ACM Additive Closure, Matrices
  If $A,\, B \in M_{mn}$, then $A + B \in M_{mn}$.
- SCM Scalar Closure, Matrices
  If $\alpha \in \mathbb{C}$ and $A \in M_{mn}$, then $\alpha A \in M_{mn}$.
- CM Commutativity, Matrices
  If $A,\, B \in M_{mn}$, then $A + B = B + A$.

- AAM Additive Associativity, Matrices

  If $A, B, C \in M_{mn}$, *then* $A + (B + C) = (A + B) + C$.

- ZM Zero Matrix, Matrices

  There is a matrix, $\mathcal{O}$, *called the zero matrix, such that* $A + \mathcal{O} = A$ *for all* $A \in M_{mn}$.

- AIM Additive Inverses, Matrices

  If $A \in M_{mn}$, *then there exists a matrix* $-A \in M_{mn}$ *so that* $A + (-A) = \mathcal{O}$.

- SMAM Scalar Multiplication Associativity, Matrices

  If $\alpha, \beta \in \mathbb{C}$ *and* $A \in M_{mn}$, *then* $\alpha(\beta A) = (\alpha\beta)A$.

- DMAM Distributivity across Matrix Addition, Matrices

  If $\alpha \in \mathbb{C}$ *and* $A, B \in M_{mn}$, *then* $\alpha(A + B) = \alpha A + \alpha B$.

- DSAM Distributivity across Scalar Addition, Matrices

  If $\alpha, \beta \in \mathbb{C}$ *and* $A \in M_{mn}$, *then* $(\alpha + \beta)A = \alpha A + \beta A$.

- OM One, Matrices

  If $A \in M_{mn}$, *then* $1A = A$.

Proof. While some of these properties seem very obvious, they all require proof. However, the proofs are not very interesting, and border on tedious. We will prove one version of distributivity very carefully, and you can test your proof-building skills on some of the others. We will give our new notation for matrix entries a workout here. Compare the style of the proofs here with those given for vectors in Theorem VSPCV — while the objects here are more complicated, our notation makes the proofs cleaner.

To prove Property DSAM, $(\alpha+\beta)A = \alpha A+\beta A$, we need to establish the equality of two matrices (see Proof Technique GS). Definition ME says we need to establish the equality of their entries, one-by-one. How do we do this, when we do not even know how many entries the two matrices might have? This is where the notation for matrix entries, given in Definition M, comes into play. Ready? Here we go.

For *any* i and j, $1 \le i \le m$, $1 \le j \le n$,

$$\begin{aligned}
[(\alpha + \beta)A]_{ij} &= (\alpha + \beta) [A]_{ij} && \text{Definition MSM}\\
&= \alpha [A]_{ij} + \beta [A]_{ij} && \text{Distributivity in } \mathbb{C}\\
&= [\alpha A]_{ij} + [\beta A]_{ij} && \text{Definition MSM}\\
&= [\alpha A + \beta A]_{ij} && \text{Definition MA}
\end{aligned}$$

There are several things to notice here. (1) Each equals sign is an equality of scalars (numbers). (2) The two ends of the equation, being true for any i and j, allow us to conclude the equality of the matrices by Definition ME. (3) There are several plus signs, and several instances of juxtaposition. Identify each one, and state exactly what operation is being represented by each. ■

For now, note the similarities between Theorem VSPM about matrices and Theorem VSPCV about vectors.

The zero matrix described in this theorem, $\mathcal{O}$, is what you would expect — a matrix full of zeros.

Definition ZM Zero Matrix
The $m \times n$ **zero matrix** is written as $\mathcal{O} = \mathcal{O}_{m\times n}$ and defined by $[\mathcal{O}]_{ij} = 0$, for all $1 \le i \le m$, $1 \le j \le n$. □

Subsection TSM
Transposes and Symmetric Matrices

We describe one more common operation we can perform on matrices. Informally, to transpose a matrix is to build a new matrix by swapping its rows and columns.

Definition TM Transpose of a Matrix
Given an $m \times n$ matrix A, its **transpose** is the $n \times m$ matrix A^t given by

$$\left[A^t\right]_{ij} = [A]_{ji}\,, \quad 1 \leq i \leq n,\, 1 \leq j \leq m.$$

□

Example TM Transpose of a 3×4 matrix
Suppose

$$D = \begin{bmatrix} 3 & 7 & 2 & -3 \\ -1 & 4 & 2 & 8 \\ 0 & 3 & -2 & 5 \end{bmatrix}.$$

We could formulate the transpose, entry-by-entry, using the definition. But it is easier to just systematically rewrite rows as columns (or vice-versa). The form of the definition given will be more useful in proofs. So we have

$$D^t = \begin{bmatrix} 3 & -1 & 0 \\ 7 & 4 & 3 \\ 2 & 2 & -2 \\ -3 & 8 & 5 \end{bmatrix}$$

△

It will sometimes happen that a matrix is equal to its transpose. In this case, we will call a matrix **symmetric**. These matrices occur naturally in certain situations, and also have some nice properties, so it is worth stating the definition carefully. Informally a matrix is symmetric if we can "flip" it about the main diagonal (upper-left corner, running down to the lower-right corner) and have it look unchanged.

Definition SYM Symmetric Matrix
The matrix A is **symmetric** if $A = A^t$. □

Example SYM A symmetric 5×5 matrix
The matrix

$$E = \begin{bmatrix} 2 & 3 & -9 & 5 & 7 \\ 3 & 1 & 6 & -2 & -3 \\ -9 & 6 & 0 & -1 & 9 \\ 5 & -2 & -1 & 4 & -8 \\ 7 & -3 & 9 & -8 & -3 \end{bmatrix}$$

is symmetric. △

You might have noticed that Definition SYM did not specify the size of the matrix A, as has been our custom. That is because it was not necessary. An alternative would have been to state the definition just for square matrices, but this is the substance of the next proof.

Before reading the next proof, we want to offer you some advice about how to become more proficient at constructing proofs. Perhaps you can apply this advice to the next theorem. Have a peek at Proof Technique P now.

Theorem SMS Symmetric Matrices are Square
Suppose that A is a symmetric matrix. Then A is square.

Proof. We start by specifying A's size, without assuming it is square, since we are trying to *prove* that, so we cannot also assume it. Suppose A is an $m \times n$ matrix. Because A is symmetric, we know by Definition SYM that $A = A^t$. So, in particular, Definition ME requires that A and A^t must have the same size. The size of A^t is $n \times m$. Because A has m rows and A^t has n rows, we conclude that $m = n$, and hence A must be square by Definition SQM. ■

We finish this section with three easy theorems, but they illustrate the interplay of our three new operations, our new notation, and the techniques used to prove matrix equalities.

Theorem TMA Transpose and Matrix Addition
Suppose that A and B are $m \times n$ matrices. Then $(A + B)^t = A^t + B^t$.

Proof. The statement to be proved is an equality of matrices, so we work entry-by-entry and use Definition ME. Think carefully about the objects involved here, and the many uses of the plus sign. For $1 \leq i \leq m$, $1 \leq j \leq n$,

$$\begin{aligned}
\left[(A+B)^t\right]_{ij} &= [A+B]_{ji} && \text{Definition TM}\\
&= [A]_{ji} + [B]_{ji} && \text{Definition MA}\\
&= \left[A^t\right]_{ij} + \left[B^t\right]_{ij} && \text{Definition TM}\\
&= \left[A^t + B^t\right]_{ij} && \text{Definition MA}
\end{aligned}$$

Since the matrices $(A+B)^t$ and $A^t + B^t$ agree at each entry, Definition ME tells us the two matrices are equal. ■

Theorem TMSM Transpose and Matrix Scalar Multiplication
Suppose that $\alpha \in \mathbb{C}$ and A is an $m \times n$ matrix. Then $(\alpha A)^t = \alpha A^t$.

Proof. The statement to be proved is an equality of matrices, so we work entry-by-entry and use Definition ME. Notice that the desired equality is of $n \times m$ matrices, and think carefully about the objects involved here, plus the many uses of juxtaposition. For $1 \leq i \leq m$, $1 \leq j \leq n$,

$$\begin{aligned}
\left[(\alpha A)^t\right]_{ji} &= [\alpha A]_{ij} && \text{Definition TM}\\
&= \alpha\,[A]_{ij} && \text{Definition MSM}\\
&= \alpha \left[A^t\right]_{ji} && \text{Definition TM}\\
&= \left[\alpha A^t\right]_{ji} && \text{Definition MSM}
\end{aligned}$$

Since the matrices $(\alpha A)^t$ and αA^t agree at each entry, Definition ME tells us the two matrices are equal. ■

Theorem TT Transpose of a Transpose
Suppose that A is an $m \times n$ matrix. Then $\left(A^t\right)^t = A$.

Proof. We again want to prove an equality of matrices, so we work entry-by-entry and use Definition ME. For $1 \leq i \leq m$, $1 \leq j \leq n$,

$$\begin{aligned}
\left[\left(A^t\right)^t\right]_{ij} &= \left[A^t\right]_{ji} && \text{Definition TM}\\
&= [A]_{ij} && \text{Definition TM}
\end{aligned}$$

Since the matrices $\left(A^t\right)^t$ and A agree at each entry, Definition ME tells us the two matrices are equal. ■

Subsection MCC
Matrices and Complex Conjugation

As we did with vectors (Definition CCCV), we can define what it means to take the conjugate of a matrix.

Definition CCM Complex Conjugate of a Matrix
Suppose A is an $m \times n$ matrix. Then the **conjugate** of A, written $\overline{A}$ is an $m \times n$ matrix defined by

$$\left[\overline{A}\right]_{ij} = \overline{[A]_{ij}}$$

□

Example CCM Complex conjugate of a matrix
If

$$A = \begin{bmatrix} 2-i & 3 & 5+4i \\ -3+6i & 2-3i & 0 \end{bmatrix}$$

then

$$\overline{A} = \begin{bmatrix} 2+i & 3 & 5-4i \\ -3-6i & 2+3i & 0 \end{bmatrix}$$

The interplay between the conjugate of a matrix and the two operations on matrices is what you might expect.

Theorem CRMA Conjugation Respects Matrix Addition
Suppose that A and B are $m \times n$ matrices. Then $\overline{A+B} = \overline{A} + \overline{B}$.

Proof. For $1 \leq i \leq m$, $1 \leq j \leq n$,

$$\begin{aligned}
\left[\overline{A+B}\right]_{ij} &= \overline{[A+B]_{ij}} && \text{Definition CCM} \\
&= \overline{[A]_{ij} + [B]_{ij}} && \text{Definition MA} \\
&= \overline{[A]_{ij}} + \overline{[B]_{ij}} && \text{Theorem CCRA} \\
&= \left[\overline{A}\right]_{ij} + \left[\overline{B}\right]_{ij} && \text{Definition CCM} \\
&= \left[\overline{A} + \overline{B}\right]_{ij} && \text{Definition MA}
\end{aligned}$$

Since the matrices $\overline{A+B}$ and $\overline{A} + \overline{B}$ are equal in each entry, Definition ME says that $\overline{A+B} = \overline{A} + \overline{B}$. ■

Theorem CRMSM Conjugation Respects Matrix Scalar Multiplication
Suppose that $\alpha \in \mathbb{C}$ and A is an $m \times n$ matrix. Then $\overline{\alpha A} = \overline{\alpha}\,\overline{A}$.

Proof. For $1 \leq i \leq m$, $1 \leq j \leq n$,

$$\begin{aligned}
\left[\overline{\alpha A}\right]_{ij} &= \overline{[\alpha A]_{ij}} && \text{Definition CCM} \\
&= \overline{\alpha\,[A]_{ij}} && \text{Definition MSM} \\
&= \overline{\alpha}\,\overline{[A]_{ij}} && \text{Theorem CCRM} \\
&= \overline{\alpha}\,\left[\overline{A}\right]_{ij} && \text{Definition CCM} \\
&= \left[\overline{\alpha}\,\overline{A}\right]_{ij} && \text{Definition MSM}
\end{aligned}$$

Since the matrices $\overline{\alpha A}$ and $\overline{\alpha}\,\overline{A}$ are equal in each entry, Definition ME says that $\overline{\alpha A} = \overline{\alpha}\,\overline{A}$. ■

Theorem CCM Conjugate of the Conjugate of a Matrix
Suppose that A is an $m \times n$ matrix. Then $\overline{\left(\overline{A}\right)} = A$.

Proof. For $1 \leq i \leq m$, $1 \leq j \leq n$,

$$\begin{aligned}
\left[\overline{\left(\overline{A}\right)}\right]_{ij} &= \overline{\left[\overline{A}\right]_{ij}} && \text{Definition CCM} \\
&= \overline{\overline{[A]_{ij}}} && \text{Definition CCM} \\
&= [A]_{ij} && \text{Theorem CCT}
\end{aligned}$$

Since the matrices $\overline{\left(\overline{A}\right)}$ and A are equal in each entry, Definition ME says that $\overline{\left(\overline{A}\right)} = A$. ■

Finally, we will need the following result about matrix conjugation and transposes later.

Theorem MCT Matrix Conjugation and Transposes
Suppose that A is an $m \times n$ matrix. Then $\overline{(A^t)} = \left(\overline{A}\right)^t$.

Proof. For $1 \leq i \leq m$, $1 \leq j \leq n$,

$$\begin{aligned}
\left[\overline{(A^t)}\right]_{ji} &= \overline{\left[A^t\right]_{ji}} && \text{Definition CCM}\\
&= \overline{[A]_{ij}} && \text{Definition TM}\\
&= \left[\overline{A}\right]_{ij} && \text{Definition CCM}\\
&= \left[\left(\overline{A}\right)^t\right]_{ji} && \text{Definition TM}
\end{aligned}$$

Since the matrices $\overline{(A^t)}$ and $\left(\overline{A}\right)^t$ are equal in each entry, Definition ME says that $\overline{(A^t)} = \left(\overline{A}\right)^t$. ■

Subsection AM
Adjoint of a Matrix

The combination of transposing and conjugating a matrix will be important in subsequent sections, such as Subsection MINM.UM and Section OD. We make a key definition here and prove some basic results in the same spirit as those above.

Definition A Adjoint
If A is a matrix, then its **adjoint** is $A^* = \left(\overline{A}\right)^t$. □

You will see the adjoint written elsewhere variously as A^H, A^* or $A^\dagger$. Notice that Theorem MCT says it does not really matter if we conjugate and then transpose, or transpose and then conjugate.

Theorem AMA Adjoint and Matrix Addition
Suppose A and B are matrices of the same size. Then $(A+B)^* = A^* + B^*$.

Proof.

$$\begin{aligned}
(A+B)^* &= \left(\overline{A+B}\right)^t && \text{Definition A}\\
&= \left(\overline{A}+\overline{B}\right)^t && \text{Theorem CRMA}\\
&= \left(\overline{A}\right)^t + \left(\overline{B}\right)^t && \text{Theorem TMA}\\
&= A^* + B^* && \text{Definition A}
\end{aligned}$$

■

Theorem AMSM Adjoint and Matrix Scalar Multiplication
Suppose $\alpha \in \mathbb{C}$ is a scalar and A is a matrix. Then $(\alpha A)^* = \overline{\alpha} A^*$.

Proof.

$$\begin{aligned}
(\alpha A)^* &= \left(\overline{\alpha A}\right)^t && \text{Definition A}\\
&= \left(\overline{\alpha}\,\overline{A}\right)^t && \text{Theorem CRMSM}\\
&= \overline{\alpha}\left(\overline{A}\right)^t && \text{Theorem TMSM}
\end{aligned}$$

$$= \overline{\alpha} A^* \qquad \text{Definition A}$$

■

Theorem AA Adjoint of an Adjoint
Suppose that A is a matrix. Then $(A^*)^* = A$.

Proof.

$$\begin{aligned}
(A^*)^* &= \left(\overline{(A^*)}\right)^t && \text{Definition A}\\
&= \overline{\left((A^*)^t\right)} && \text{Theorem MCT}\\
&= \overline{\left(\left(\left(\overline{A}\right)^t\right)^t\right)} && \text{Definition A}\\
&= \overline{\left(\overline{A}\right)} && \text{Theorem TT}\\
&= A && \text{Theorem CCM}
\end{aligned}$$

■

Take note of how the theorems in this section, while simple, build on earlier theorems and definitions and never descend to the level of entry-by-entry proofs based on Definition ME. In other words, the equal signs that appear in the previous proofs are equalities of matrices, not scalars (which is the opposite of a proof like that of Theorem TMA).

Reading Questions

1. Perform the following matrix computation.

$$(6)\begin{bmatrix} 2 & -2 & 8 & 1 \\ 4 & 5 & -1 & 3 \\ 7 & -3 & 0 & 2 \end{bmatrix} + (-2)\begin{bmatrix} 2 & 7 & 1 & 2 \\ 3 & -1 & 0 & 5 \\ 1 & 7 & 3 & 3 \end{bmatrix}$$

2. Theorem VSPM reminds you of what previous theorem? How strong is the similarity?
3. Compute the transpose of the matrix below.

$$\begin{bmatrix} 6 & 8 & 4 \\ -2 & 1 & 0 \\ 9 & -5 & 6 \end{bmatrix}$$

Exercises

C10† Let $A = \begin{bmatrix} 1 & 4 & -3 \\ 6 & 3 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 2 & 1 \\ -2 & -6 & 5 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 4 \\ 4 & 0 \\ -2 & 2 \end{bmatrix}$. Let $\alpha = 4$ and $\beta = 1/2$. Perform the following calculations: (1) $A + B$, (2) $A + C$, (3) $B^t + C$, (4) $A + B^t$, (5) βC, (6) $4A - 3B$, (7) $A^t + \alpha C$, (8) $A + B - C^t$, (9) $4A + 2B - 5C^t$.

C11† Solve the given vector equation for x, or explain why no solution exists:

$$2\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & x \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 5 & -2 \end{bmatrix}$$

C12† Solve the given matrix equation for α, or explain why no solution exists:

$$\alpha\begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & -1 \end{bmatrix} + \begin{bmatrix} 4 & 3 & -6 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 12 & 6 \\ 6 & 4 & -2 \end{bmatrix}$$

C13† Solve the given matrix equation for α, or explain why no solution exists:

$$\alpha\begin{bmatrix} 3 & 1 \\ 2 & 0 \\ 1 & 4 \end{bmatrix} - \begin{bmatrix} 4 & 1 \\ 3 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -2 \\ 2 & 6 \end{bmatrix}$$

C14† Find α and β that solve the following equation:

$$\alpha\begin{bmatrix}1 & 2\\ 4 & 1\end{bmatrix} + \beta\begin{bmatrix}2 & 1\\ 3 & 1\end{bmatrix} = \begin{bmatrix}-1 & 4\\ 6 & 1\end{bmatrix}$$

In Chapter V we defined the operations of vector addition and vector scalar multiplication in Definition CVA and Definition CVSM. These two operations formed the underpinnings of the remainder of the chapter. We have now defined similar operations for matrices in Definition MA and Definition MSM. You will have noticed the resulting similarities between Theorem VSPCV and Theorem VSPM.
In Exercises M20–M25, you will be asked to extend these similarities to other fundamental definitions and concepts we first saw in Chapter V. This sequence of problems was suggested by Martin Jackson.

M20 Suppose $S = \{B_1, B_2, B_3, \ldots, B_p\}$ is a set of matrices from M_{mn}. Formulate appropriate definitions for the following terms and give an example of the use of each.

1. A linear combination of elements of S.
2. A relation of linear dependence on S, both trivial and nontrivial.
3. S is a linearly independent set.
4. $\langle S\rangle$.

M21† Show that the set S is linearly independent in $M_{2,2}$.

$$S = \left\{\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right\}$$

M22† Determine if the set S below is linearly independent in $M_{2,3}$.

$$\left\{\begin{bmatrix}-2 & 3 & 4\\ -1 & 3 & -2\end{bmatrix}, \begin{bmatrix}4 & -2 & 2\\ 0 & -1 & 1\end{bmatrix}, \begin{bmatrix}-1 & -2 & -2\\ 2 & 2 & 2\end{bmatrix}, \begin{bmatrix}-1 & 1 & 0\\ -1 & 0 & -2\end{bmatrix}, \begin{bmatrix}-1 & 2 & -2\\ 0 & -1 & -2\end{bmatrix}\right\}$$

M23† Determine if the matrix A is in the span of S. In other words, is $A \in \langle S\rangle$? If so write A as a linear combination of the elements of S.

$$A = \begin{bmatrix}-13 & 24 & 2\\ -8 & -2 & -20\end{bmatrix}$$

$$S = \left\{\begin{bmatrix}-2 & 3 & 4\\ -1 & 3 & -2\end{bmatrix}, \begin{bmatrix}4 & -2 & 2\\ 0 & -1 & 1\end{bmatrix}, \begin{bmatrix}-1 & -2 & -2\\ 2 & 2 & 2\end{bmatrix}, \begin{bmatrix}-1 & 1 & 0\\ -1 & 0 & -2\end{bmatrix}, \begin{bmatrix}-1 & 2 & -2\\ 0 & -1 & -2\end{bmatrix}\right\}$$

M24† Suppose Y is the set of all 3×3 symmetric matrices (Definition SYM). Find a set T so that T is linearly independent and $\langle T\rangle = Y$.

M25 Define a subset of $M_{3,3}$ by

$$U_{33} = \left\{A \in M_{3,3} \mid [A]_{ij} = 0 \text{ whenever } i > j\right\}$$

Find a set R so that R is linearly independent and $\langle R\rangle = U_{33}$.

T13† Prove Property CM of Theorem VSPM. Write your proof in the style of the proof of Property DSAM given in this section.

T14 Prove Property AAM of Theorem VSPM. Write your proof in the style of the proof of Property DSAM given in this section.

T17 Prove Property SMAM of Theorem VSPM. Write your proof in the style of the proof of Property DSAM given in this section.

T18 Prove Property DMAM of Theorem VSPM. Write your proof in the style of the proof of Property DSAM given in this section.

A matrix A is **skew-symmetric** if $A^t = -A$ Exercises T30–T37 employ this definition.

T30 Prove that a skew-symmetric matrix is square. (Hint: study the proof of Theorem SMS.)

T31 Prove that a skew-symmetric matrix must have zeros for its diagonal elements. In other words, if A is skew-symmetric of size n, then $[A]_{ii} = 0$ for $1 \leq i \leq n$. (Hint: carefully construct an example of a 3×3 skew-symmetric matrix before attempting a proof.)

T32 Prove that a matrix A is both skew-symmetric and symmetric if and only if A is the zero matrix. (Hint: one half of this proof is very easy, the other half takes a little more work.)

T33 Suppose A and B are both skew-symmetric matrices of the same size and $\alpha,\ \beta \in \mathbb{C}$. Prove that $\alpha A + \beta B$ is a skew-symmetric matrix.

T34 Suppose A is a square matrix. Prove that $A + A^t$ is a symmetric matrix.

T35 Suppose A is a square matrix. Prove that $A - A^t$ is a skew-symmetric matrix.

T36 Suppose A is a square matrix. Prove that there is a symmetric matrix B and a skew-symmetric matrix C such that $A = B + C$. In other words, any square matrix can be decomposed into a symmetric matrix and a skew-symmetric matrix (Proof Technique DC). (Hint: consider building a proof on Exercise MO.T34 and Exercise MO.T35.)

T37 Prove that the decomposition in Exercise MO.T36 is unique (see Proof Technique U). (Hint: a proof can turn on Exercise MO.T31.)

Section MM
Matrix Multiplication

We know how to add vectors and how to multiply them by scalars. Together, these operations give us the possibility of making linear combinations. Similarly, we know how to add matrices and how to multiply matrices by scalars. In this section we mix all these ideas together and produce an operation known as "matrix multiplication." This will lead to some results that are both surprising and central. We begin with a definition of how to multiply a vector by a matrix.

Subsection MVP
Matrix-Vector Product

We have repeatedly seen the importance of forming linear combinations of the columns of a matrix. As one example of this, the oft-used Theorem SLSLC, said that every solution to a system of linear equations gives rise to a linear combination of the column vectors of the coefficient matrix that equals the vector of constants. This theorem, and others, motivate the following central definition.

Definition MVP Matrix-Vector Product
Suppose A is an $m \times n$ matrix with columns $\mathbf{A}_1,\,\mathbf{A}_2,\,\mathbf{A}_3,\,\ldots,\,\mathbf{A}_n$ and $\mathbf{u}$ is a vector of size n. Then the **matrix-vector product** of A with $\mathbf{u}$ is the linear combination

$$A\mathbf{u} = [\mathbf{u}]_1\,\mathbf{A}_1 + [\mathbf{u}]_2\,\mathbf{A}_2 + [\mathbf{u}]_3\,\mathbf{A}_3 + \cdots + [\mathbf{u}]_n\,\mathbf{A}_n$$

□

So, the matrix-vector product is yet another version of "multiplication," at least in the sense that we have yet again overloaded juxtaposition of two symbols as our notation. Remember your objects, an $m \times n$ matrix times a vector of size n will create a vector of size m. So if A is rectangular, then the size of the vector changes. With all the linear combinations we have performed so far, this computation should now seem second nature.

Example MTV A matrix times a vector
Consider

$$A = \begin{bmatrix} 1 & 4 & 2 & 3 & 4 \\ -3 & 2 & 0 & 1 & -2 \\ 1 & 6 & -3 & -1 & 5 \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ -2 \\ 3 \\ -1 \end{bmatrix}$$

Then

$$A\mathbf{u} = 2\begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix} + (-2)\begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} + 3\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} + (-1)\begin{bmatrix} 4 \\ -2 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \\ 6 \end{bmatrix}.$$

△

We can now represent systems of linear equations compactly with a matrix-vector product (Definition MVP) and column vector equality (Definition CVE). This finally yields a very popular alternative to our unconventional $\mathcal{LS}(A,\,\mathbf{b})$ notation.

Theorem SLEMM Systems of Linear Equations as Matrix Multiplication
The set of solutions to the linear system $\mathcal{LS}(A,\,\mathbf{b})$ equals the set of solutions for $\mathbf{x}$ in the vector equation $A\mathbf{x} = \mathbf{b}$.

Proof. This theorem says that two sets (of solutions) are equal. So we need to show that one set of solutions is a subset of the other, and vice versa (Definition SE). Let

$\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{A}_3$, …, $\mathbf{A}_n$ be the columns of A. Both of these set inclusions then follow from the following chain of equivalences (Proof Technique E),

$$\begin{aligned} &\mathbf{x} \text{ is a solution to } \mathcal{LS}(A, \mathbf{b}) && \\ \iff & [\mathbf{x}]_1 \mathbf{A}_1 + [\mathbf{x}]_2 \mathbf{A}_2 + [\mathbf{x}]_3 \mathbf{A}_3 + \cdots + [\mathbf{x}]_n \mathbf{A}_n = \mathbf{b} && \text{Theorem SLSLC} \\ \iff & \mathbf{x} \text{ is a solution to } A\mathbf{x} = \mathbf{b} && \text{Definition MVP} \end{aligned}$$

Example MNSLE Matrix notation for systems of linear equations
Consider the system of linear equations from Example NSLE.

$$\begin{aligned} 2x_1 + 4x_2 - 3x_3 + 5x_4 + x_5 &= 9 \\ 3x_1 + x_2 + x_4 - 3x_5 &= 0 \\ -2x_1 + 7x_2 - 5x_3 + 2x_4 + 2x_5 &= -3 \end{aligned}$$

has coefficient matrix and vector of constants

$$A = \begin{bmatrix} 2 & 4 & -3 & 5 & 1 \\ 3 & 1 & 0 & 1 & -3 \\ -2 & 7 & -5 & 2 & 2 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 9 \\ 0 \\ -3 \end{bmatrix}$$

and so will be described compactly by the vector equation $A\mathbf{x} = \mathbf{b}$. △

The matrix-vector product is a very natural computation. We have motivated it by its connections with systems of equations, but here is another example.

Example MBC Money's best cities
Every year *Money* magazine selects several cities in the United States as the "best" cities to live in, based on a wide array of statistics about each city. This is an example of how the editors of *Money* might arrive at a single number that consolidates the statistics about a city. We will analyze Los Angeles, Chicago and New York City, based on four criteria: average high temperature in July (Farenheit), number of colleges and universities in a 30-mile radius, number of toxic waste sites in the Superfund environmental clean-up program and a personal crime index based on FBI statistics (average = 100, smaller is safer). It should be apparent how to generalize the example to a greater number of cities and a greater number of statistics.

We begin by building a table of statistics. The rows will be labeled with the cities, and the columns with statistical categories. These values are from *Money*'s website in early 2005.

City	Temp	Colleges	Superfund	Crime
Los Angeles	77	28	93	254
Chicago	84	38	85	363
New York	84	99	1	193

Conceivably these data might reside in a spreadsheet. Now we must combine the statistics for each city. We could accomplish this by weighting each category, scaling the values and summing them. The sizes of the weights would depend upon the numerical size of each statistic generally, but more importantly, they would reflect the editors opinions or beliefs about which statistics were most important to their readers. Is the crime index more important than the number of colleges and universities? Of course, there is no right answer to this question.

Suppose the editors finally decide on the following weights to employ: temperature, 0.23; colleges, 0.46; Superfund, −0.05; crime, −0.20. Notice how negative weights are used for undesirable statistics. Then, for example, the editors would compute for Los Angeles,

$$(0.23)(77) + (0.46)(28) + (-0.05)(93) + (-0.20)(254) = -24.86$$

This computation might remind you of an inner product, but we will produce the computations for all of the cities as a matrix-vector product. Write the table of raw statistics as a matrix

$$T = \begin{bmatrix} 77 & 28 & 93 & 254 \\ 84 & 38 & 85 & 363 \\ 84 & 99 & 1 & 193 \end{bmatrix}$$

and the weights as a vector

$$\mathbf{w} = \begin{bmatrix} 0.23 \\ 0.46 \\ -0.05 \\ -0.20 \end{bmatrix}$$

then the matrix-vector product (Definition MVP) yields

$$T\mathbf{w} = (0.23)\begin{bmatrix} 77 \\ 84 \\ 84 \end{bmatrix} + (0.46)\begin{bmatrix} 28 \\ 38 \\ 99 \end{bmatrix} + (-0.05)\begin{bmatrix} 93 \\ 85 \\ 1 \end{bmatrix} + (-0.20)\begin{bmatrix} 254 \\ 363 \\ 193 \end{bmatrix} = \begin{bmatrix} -24.86 \\ -40.05 \\ 26.21 \end{bmatrix}$$

This vector contains a single number for each of the cities being studied, so the editors would rank New York best (26.21), Los Angeles next (−24.86), and Chicago third (−40.05). Of course, the mayor's offices in Chicago and Los Angeles are free to counter with a different set of weights that cause their city to be ranked best. These alternative weights would be chosen to play to each cities' strengths, and minimize their problem areas.

If a speadsheet were used to make these computations, a row of weights would be entered somewhere near the table of data and the formulas in the spreadsheet would effect a matrix-vector product. This example is meant to illustrate how "linear" computations (addition, multiplication) can be organized as a matrix-vector product.

Another example would be the matrix of numerical scores on examinations and exercises for students in a class. The rows would be indexed by students and the columns would be indexed by exams and assignments. The instructor could then assign weights to the different exams and assignments, and via a matrix-vector product, compute a single score for each student. △

Later (much later) we will need the following theorem, which is really a technical lemma (see Proof Technique LC). Since we are in a position to prove it now, we will. But you can safely skip it for the moment, if you promise to come back later to study the proof when the theorem is employed. At that point you will also be able to understand the comments in the paragraph following the proof.

Theorem EMMVP Equal Matrices and Matrix-Vector Products
Suppose that A and B are $m \times n$ matrices such that $A\mathbf{x} = B\mathbf{x}$ for every $\mathbf{x} \in \mathbb{C}^n$. Then $A = B$.

Proof. We are assuming $A\mathbf{x} = B\mathbf{x}$ for all $\mathbf{x} \in \mathbb{C}^n$, so we can employ this equality for *any* choice of the vector $\mathbf{x}$. However, we will limit our use of this equality to the standard unit vectors, $\mathbf{e}_j$, $1 \leq j \leq n$ (Definition SUV). For all $1 \leq j \leq n$, $1 \leq i \leq m$,

$$\begin{aligned}
&[A]_{ij} \\
&= 0\,[A]_{i1} + \cdots + 0\,[A]_{i,j-1} + 1\,[A]_{ij} + 0\,[A]_{i,j+1} + \cdots + 0\,[A]_{in} && \text{Theorem PCNA} \\
&= [A]_{i1}\,[\mathbf{e}_j]_1 + [A]_{i2}\,[\mathbf{e}_j]_2 + [A]_{i3}\,[\mathbf{e}_j]_3 + \cdots + [A]_{in}\,[\mathbf{e}_j]_n && \text{Definition SUV} \\
&= [A\mathbf{e}_j]_i && \text{Definition MVP} \\
&= [B\mathbf{e}_j]_i && \text{Definition CVE} \\
&= [B]_{i1}\,[\mathbf{e}_j]_1 + [B]_{i2}\,[\mathbf{e}_j]_2 + [B]_{i3}\,[\mathbf{e}_j]_3 + \cdots + [B]_{in}\,[\mathbf{e}_j]_n && \text{Definition MVP} \\
&= 0\,[B]_{i1} + \cdots + 0\,[B]_{i,j-1} + 1\,[B]_{ij} + 0\,[B]_{i,j+1} + \cdots + 0\,[B]_{in} && \text{Definition SUV} \\
&= [B]_{ij} && \text{Theorem PCNA}
\end{aligned}$$

So by Definition ME the matrices A and B are equal, as desired. ■

You might notice from studying the proof that the hypotheses of this theorem could be "weakened" (i.e. made less restrictive). We need only suppose the equality of the matrix-vector products for just the standard unit vectors (Definition SUV) or any other spanning set (Definition SSVS) of $\mathbb{C}^n$ (Exercise LISS.T40). However, in practice, when we apply this theorem the stronger hypothesis will be in effect so this version of the theorem will suffice for our purposes. (If we changed the statement of the theorem to have the less restrictive hypothesis, then we would call the theorem "stronger.")

Subsection MM Matrix Multiplication

We now define how to multiply two matrices together. Stop for a minute and think about how you might define this new operation.

Many books would present this definition much earlier in the course. However, we have taken great care to delay it as long as possible and to present as many ideas as practical based mostly on the notion of linear combinations. Towards the conclusion of the course, or when you perhaps take a second course in linear algebra, you may be in a position to appreciate the reasons for this. For now, understand that matrix multiplication is a central definition and perhaps you will appreciate its importance more by having saved it for later.

Definition MM Matrix Multiplication
Suppose A is an $m \times n$ matrix and $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3, \ldots, \mathbf{B}_p$ are the columns of an $n \times p$ matrix B. Then the **matrix product** of A with B is the $m \times p$ matrix where column i is the matrix-vector product $A\mathbf{B}_i$. Symbolically,

$$AB = A\left[\mathbf{B}_1|\mathbf{B}_2|\mathbf{B}_3|\ldots|\mathbf{B}_p\right] = \left[A\mathbf{B}_1|A\mathbf{B}_2|A\mathbf{B}_3|\ldots|A\mathbf{B}_p\right].$$

□

Example PTM Product of two matrices
Set

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 & 6 \\ 0 & -4 & 1 & 2 & 3 \\ -5 & 1 & 2 & -3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 6 & 2 & 1 \\ -1 & 4 & 3 & 2 \\ 1 & 1 & 2 & 3 \\ 6 & 4 & -1 & 2 \\ 1 & -2 & 3 & 0 \end{bmatrix}$$

Then

$$AB = \left[A\begin{bmatrix} 1 \\ -1 \\ 1 \\ 6 \\ 1 \end{bmatrix} \middle| A\begin{bmatrix} 6 \\ 4 \\ 1 \\ 4 \\ -2 \end{bmatrix} \middle| A\begin{bmatrix} 2 \\ 3 \\ 2 \\ -1 \\ 3 \end{bmatrix} \middle| A\begin{bmatrix} 1 \\ 2 \\ 3 \\ 2 \\ 0 \end{bmatrix} \right] = \begin{bmatrix} 28 & 17 & 20 & 10 \\ 20 & -13 & -3 & -1 \\ -18 & -44 & 12 & -3 \end{bmatrix}.$$

△

Is this the definition of matrix multiplication you expected? Perhaps our previous operations for matrices caused you to think that we might multiply two matrices of the *same* size, *entry-by-entry*? Notice that our current definition uses matrices of different sizes (though the number of columns in the first must equal the number of rows in the second), and the result is of a third size. Notice too in the previous example that we cannot even consider the product BA, since the sizes of the two matrices in this order are not right.

But it gets weirder than that. Many of your old ideas about "multiplication" will not apply to matrix multiplication, but some still will. So make no assumptions, and do not do anything until you have a theorem that says you can. Even if the sizes are right, matrix multiplication is not commutative — order matters.

Example MMNC Matrix multiplication is not commutative
Set

$$A = \begin{bmatrix} 1 & 3 \\ -1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 0 \\ 5 & 1 \end{bmatrix}.$$

Then we have two square, 2×2 matrices, so Definition MM allows us to multiply them in either order. We find

$$AB = \begin{bmatrix} 19 & 3 \\ 6 & 2 \end{bmatrix} \qquad BA = \begin{bmatrix} 4 & 12 \\ 4 & 17 \end{bmatrix}$$

and $AB \neq BA$. Not even close. It should not be hard for you to construct other pairs of matrices that do not commute (try a couple of 3×3's). Can you find a pair of non-identical matrices that *do* commute? △

Subsection MMEE
Matrix Multiplication, Entry-by-Entry

While certain "natural" properties of multiplication do not hold, many more do. In the next subsection, we will state and prove the relevant theorems. But first, we need a theorem that provides an alternate means of multiplying two matrices. In many texts, this would be given as the *definition* of matrix multiplication. We prefer to turn it around and have the following formula as a consequence of our definition. It will prove useful for proofs of matrix equality, where we need to examine products of matrices, entry-by-entry.

Theorem EMP Entries of Matrix Products
Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Then for $1 \leq i \leq m$, $1 \leq j \leq p$, the individual entries of AB are given by

$$\begin{aligned} [AB]_{ij} &= [A]_{i1} [B]_{1j} + [A]_{i2} [B]_{2j} + [A]_{i3} [B]_{3j} + \cdots + [A]_{in} [B]_{nj} \\ &= \sum_{k=1}^{n} [A]_{ik} [B]_{kj} \end{aligned}$$

Proof. Let the vectors $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \ldots, \mathbf{A}_n$ denote the columns of A and let the vectors $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3, \ldots, \mathbf{B}_p$ denote the columns of B. Then for $1 \leq i \leq m, 1 \leq j \leq p$,

$$\begin{aligned} [AB]_{ij} &= [A\mathbf{B}_j]_i && \text{Definition MM} \\ &= \left[[\mathbf{B}_j]_1 \mathbf{A}_1 + [\mathbf{B}_j]_2 \mathbf{A}_2 + \cdots + [\mathbf{B}_j]_n \mathbf{A}_n\right]_i && \text{Definition MVP} \\ &= \left[[\mathbf{B}_j]_1 \mathbf{A}_1\right]_i + \left[[\mathbf{B}_j]_2 \mathbf{A}_2\right]_i + \cdots + \left[[\mathbf{B}_j]_n \mathbf{A}_n\right]_i && \text{Definition CVA} \\ &= [\mathbf{B}_j]_1 [\mathbf{A}_1]_i + [\mathbf{B}_j]_2 [\mathbf{A}_2]_i + \cdots + [\mathbf{B}_j]_n [\mathbf{A}_n]_i && \text{Definition CVSM} \\ &= [B]_{1j} [A]_{i1} + [B]_{2j} [A]_{i2} + \cdots + [B]_{nj} [A]_{in} && \text{Definition M} \\ &= [A]_{i1} [B]_{1j} + [A]_{i2} [B]_{2j} + \cdots + [A]_{in} [B]_{nj} && \text{Property CMCN} \\ &= \sum_{k=1}^{n} [A]_{ik} [B]_{kj} \end{aligned}$$

■

Example PTMEE Product of two matrices, entry-by-entry

Consider again the two matrices from Example PTM

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 & 6 \\ 0 & -4 & 1 & 2 & 3 \\ -5 & 1 & 2 & -3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 6 & 2 & 1 \\ -1 & 4 & 3 & 2 \\ 1 & 1 & 2 & 3 \\ 6 & 4 & -1 & 2 \\ 1 & -2 & 3 & 0 \end{bmatrix}$$

Then suppose we just wanted the entry of AB in the second row, third column:

$$\begin{aligned}[AB]_{23} =& [A]_{21}[B]_{13} + [A]_{22}[B]_{23} + [A]_{23}[B]_{33} + [A]_{24}[B]_{43} + [A]_{25}[B]_{53} \\ =&(0)(2) + (-4)(3) + (1)(2) + (2)(-1) + (3)(3) = -3\end{aligned}$$

Notice how there are 5 terms in the sum, since 5 is the common dimension of the two matrices (column count for A, row count for B). In the conclusion of Theorem EMP, it would be the index k that would run from 1 to 5 in this computation. Here is a bit more practice.

The entry of third row, first column:

$$\begin{aligned}[AB]_{31} =& [A]_{31}[B]_{11} + [A]_{32}[B]_{21} + [A]_{33}[B]_{31} + [A]_{34}[B]_{41} + [A]_{35}[B]_{51} \\ =&(-5)(1) + (1)(-1) + (2)(1) + (-3)(6) + (4)(1) = -18\end{aligned}$$

To get some more practice on your own, complete the computation of the other 10 entries of this product. Construct some other pairs of matrices (of compatible sizes) and compute their product two ways. First use Definition MM. Since linear combinations are straightforward for you now, this should be easy to do and to do correctly. Then do it again, using Theorem EMP. Since this process may take some practice, use your first computation to check your work. △

Theorem EMP is the way many people compute matrix products by hand. It will also be very useful for the theorems we are going to prove shortly. However, the definition (Definition MM) is frequently the most useful for its connections with deeper ideas like the null space and the upcoming column space.

Subsection PMM
Properties of Matrix Multiplication

In this subsection, we collect properties of matrix multiplication and its interaction with the zero matrix (Definition ZM), the identity matrix (Definition IM), matrix addition (Definition MA), scalar matrix multiplication (Definition MSM), the inner product (Definition IP), conjugation (Theorem MMCC), and the transpose (Definition TM). Whew! Here we go. These are great proofs to practice with, so try to concoct the proofs before reading them, they will get progressively more complicated as we go.

Theorem MMZM Matrix Multiplication and the Zero Matrix
Suppose A is an $m \times n$ matrix. Then

1. $A\mathcal{O}_{n\times p} = \mathcal{O}_{m\times p}$
2. $\mathcal{O}_{p\times m}A = \mathcal{O}_{p\times n}$

Proof. We will prove (1) and leave (2) to you. Entry-by-entry, for $1 \le i \le m$, $1 \le j \le p$,

$$\begin{aligned}[A\mathcal{O}_{n\times p}]_{ij} &= \sum_{k=1}^{n} [A]_{ik}[\mathcal{O}_{n\times p}]_{kj} && \text{Theorem EMP} \\ &= \sum_{k=1}^{n} [A]_{ik}\, 0 && \text{Definition ZM}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=1}^{n} 0 \\
&= 0 && \text{Property ZCN} \\
&= [\mathcal{O}_{m\times p}]_{ij} && \text{Definition ZM}
\end{aligned}
$$

So by the definition of matrix equality (Definition ME), the matrices $A\mathcal{O}_{n\times p}$ and $\mathcal{O}_{m\times p}$ are equal. ■

Theorem MMIM Matrix Multiplication and Identity Matrix
Suppose A is an $m \times n$ matrix. Then

1. $AI_n = A$
2. $I_m A = A$

Proof. Again, we will prove (1) and leave (2) to you. Entry-by-entry, For $1 \leq i \leq m$, $1 \leq j \leq n$,

$$
\begin{aligned}
[AI_n]_{ij} &= \sum_{k=1}^{n} [A]_{ik} [I_n]_{kj} && \text{Theorem EMP} \\
&= [A]_{ij} [I_n]_{jj} + \sum_{\substack{k=1 \\ k\neq j}}^{n} [A]_{ik} [I_n]_{kj} && \text{Property CACN} \\
&= [A]_{ij} (1) + \sum_{k=1, k\neq j}^{n} [A]_{ik} (0) && \text{Definition IM} \\
&= [A]_{ij} + \sum_{k=1, k\neq j}^{n} 0 \\
&= [A]_{ij}
\end{aligned}
$$

So the matrices A and AI_n are equal, entry-by-entry, and by the definition of matrix equality (Definition ME) we can say they are equal matrices. ■

It is this theorem that gives the identity matrix its name. It is a matrix that behaves with matrix multiplication like the scalar 1 does with scalar multiplication. To multiply by the identity matrix is to have no effect on the other matrix.

Theorem MMDAA Matrix Multiplication Distributes Across Addition
Suppose A is an $m \times n$ matrix and B and C are $n \times p$ matrices and D is a $p \times s$ matrix. Then

1. $A(B + C) = AB + AC$
2. $(B + C)D = BD + CD$

Proof. We will do (1), you do (2). Entry-by-entry, for $1 \leq i \leq m$, $1 \leq j \leq p$,

$$
\begin{aligned}
[A(B + C)]_{ij} &= \sum_{k=1}^{n} [A]_{ik} [B + C]_{kj} && \text{Theorem EMP} \\
&= \sum_{k=1}^{n} [A]_{ik} ([B]_{kj} + [C]_{kj}) && \text{Definition MA} \\
&= \sum_{k=1}^{n} [A]_{ik} [B]_{kj} + [A]_{ik} [C]_{kj} && \text{Property DCN}
\end{aligned}
$$

$$= \sum_{k=1}^{n} [A]_{ik} [B]_{kj} + \sum_{k=1}^{n} [A]_{ik} [C]_{kj} \qquad \text{Property CACN}$$
$$= [AB]_{ij} + [AC]_{ij} \qquad \text{Theorem EMP}$$
$$= [AB + AC]_{ij} \qquad \text{Definition MA}$$

So the matrices $A(B + C)$ and $AB + AC$ are equal, entry-by-entry, and by the definition of matrix equality (Definition ME) we can say they are equal matrices. ■

Theorem MMSMM Matrix Multiplication and Scalar Matrix Multiplication
Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Let α be a scalar. Then $\alpha(AB) = (\alpha A)B = A(\alpha B)$.

Proof. These are equalities of matrices. We will do the first one, the second is similar and will be good practice for you. For $1 \leq i \leq m$, $1 \leq j \leq p$,

$$[\alpha(AB)]_{ij} = \alpha [AB]_{ij} \qquad \text{Definition MSM}$$
$$= \alpha \sum_{k=1}^{n} [A]_{ik} [B]_{kj} \qquad \text{Theorem EMP}$$
$$= \sum_{k=1}^{n} \alpha [A]_{ik} [B]_{kj} \qquad \text{Property DCN}$$
$$= \sum_{k=1}^{n} [\alpha A]_{ik} [B]_{kj} \qquad \text{Definition MSM}$$
$$= [(\alpha A)B]_{ij} \qquad \text{Theorem EMP}$$

So the matrices $\alpha(AB)$ and $(\alpha A)B$ are equal, entry-by-entry, and by the definition of matrix equality (Definition ME) we can say they are equal matrices. ■

Theorem MMA Matrix Multiplication is Associative
Suppose A is an $m \times n$ matrix, B is an $n \times p$ matrix and D is a $p \times s$ matrix. Then $A(BD) = (AB)D$.

Proof. A matrix equality, so we will go entry-by-entry, no surprise there. For $1 \leq i \leq m$, $1 \leq j \leq s$,

$$[A(BD)]_{ij} = \sum_{k=1}^{n} [A]_{ik} [BD]_{kj} \qquad \text{Theorem EMP}$$
$$= \sum_{k=1}^{n} [A]_{ik} \left(\sum_{\ell=1}^{p} [B]_{k\ell} [D]_{\ell j} \right) \qquad \text{Theorem EMP}$$
$$= \sum_{k=1}^{n} \sum_{\ell=1}^{p} [A]_{ik} [B]_{k\ell} [D]_{\ell j} \qquad \text{Property DCN}$$

We can switch the order of the summation since these are finite sums,

$$= \sum_{\ell=1}^{p} \sum_{k=1}^{n} [A]_{ik} [B]_{k\ell} [D]_{\ell j} \qquad \text{Property CACN}$$

As $[D]_{\ell j}$ does not depend on the index k, we can use distributivity to move it outside of the inner sum,

$$= \sum_{\ell=1}^{p} [D]_{\ell j} \left(\sum_{k=1}^{n} [A]_{ik} [B]_{k\ell} \right) \qquad \text{Property DCN}$$

$$
\begin{aligned}
&= \sum_{\ell=1}^{p} [D]_{\ell j} [AB]_{i\ell} && \text{Theorem EMP} \\
&= \sum_{\ell=1}^{p} [AB]_{i\ell} [D]_{\ell j} && \text{Property CMCN} \\
&= [(AB)D]_{ij} && \text{Theorem EMP}
\end{aligned}
$$

So the matrices $(AB)D$ and $A(BD)$ are equal, entry-by-entry, and by the definition of matrix equality (Definition ME) we can say they are equal matrices. ■

The statement of our next theorem is technically inaccurate. If we upgrade the vectors $\mathbf{u}$, $\mathbf{v}$ to matrices with a single column, then the expression $\overline{\mathbf{u}}^t\mathbf{v}$ is a 1×1 matrix, though we will treat this small matrix as if it was simply the scalar quantity in its lone entry. When we apply Theorem MMIP there should not be any confusion. Notice that if we treat a column vector as a matrix with a single column, then we can also construct the adjoint of a vector, though we will not make this a common practice.

Theorem MMIP Matrix Multiplication and Inner Products
If we consider the vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^m$ *as* $m \times 1$ *matrices then*

$$\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\mathbf{u}}^t\mathbf{v} = \mathbf{u}^*\mathbf{v}$$

Proof.

$$
\begin{aligned}
\langle \mathbf{u}, \mathbf{v} \rangle &= \sum_{k=1}^{m} \overline{[\mathbf{u}]_k} [\mathbf{v}]_k && \text{Definition IP} \\
&= \sum_{k=1}^{m} \overline{[\mathbf{u}]_{k1}} [\mathbf{v}]_{k1} && \text{Column vectors as matrices} \\
&= \sum_{k=1}^{m} [\overline{\mathbf{u}}]_{k1} [\mathbf{v}]_{k1} && \text{Definition CCM} \\
&= \sum_{k=1}^{m} \left[\overline{\mathbf{u}}^t\right]_{1k} [\mathbf{v}]_{k1} && \text{Definition TM} \\
&= \left[\overline{\mathbf{u}}^t\mathbf{v}\right]_{11} && \text{Theorem EMP}
\end{aligned}
$$

To finish we just blur the distinction between a 1×1 matrix ($\overline{\mathbf{u}}^t\mathbf{v}$) and its lone entry. ■

Theorem MMCC Matrix Multiplication and Complex Conjugation
Suppose A *is an* $m \times n$ *matrix and* B *is an* $n \times p$ *matrix. Then* $\overline{AB} = \overline{A}\,\overline{B}$.

Proof. To obtain this matrix equality, we will work entry-by-entry. For $1 \le i \le m$, $1 \le j \le p$,

$$
\begin{aligned}
\left[\overline{AB}\right]_{ij} &= \overline{[AB]_{ij}} && \text{Definition CCM} \\
&= \overline{\sum_{k=1}^{n} [A]_{ik} [B]_{kj}} && \text{Theorem EMP} \\
&= \sum_{k=1}^{n} \overline{[A]_{ik} [B]_{kj}} && \text{Theorem CCRA} \\
&= \sum_{k=1}^{n} \overline{[A]_{ik}}\; \overline{[B]_{kj}} && \text{Theorem CCRM}
\end{aligned}
$$

$$= \sum_{k=1}^{n} \left[\overline{A}\right]_{ik} \left[\overline{B}\right]_{kj} \qquad \text{Definition CCM}$$

$$= \left[\overline{A}\,\overline{B}\right]_{ij} \qquad \text{Theorem EMP}$$

So the matrices $\overline{AB}$ and $\overline{A}\,\overline{B}$ are equal, entry-by-entry, and by the definition of matrix equality (Definition ME) we can say they are equal matrices. ■

Another theorem in this style, and it is a good one. If you have been practicing with the previous proofs you should be able to do this one yourself.

Theorem MMT Matrix Multiplication and Transposes
Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Then $(AB)^t = B^t A^t$.

Proof. This theorem may be surprising but if we check the sizes of the matrices involved, then maybe it will not seem so far-fetched. First, AB has size $m \times p$, so its transpose has size $p \times m$. The product of B^t with A^t is a $p \times n$ matrix times an $n \times m$ matrix, also resulting in a $p \times m$ matrix. So at least our objects are compatible for equality (and would not be, in general, if we did not reverse the order of the matrix multiplication).

Here we go again, entry-by-entry. For $1 \leq i \leq m$, $1 \leq j \leq p$,

$$\left[(AB)^t\right]_{ji} = [AB]_{ij} \qquad \text{Definition TM}$$

$$= \sum_{k=1}^{n} [A]_{ik} [B]_{kj} \qquad \text{Theorem EMP}$$

$$= \sum_{k=1}^{n} [B]_{kj} [A]_{ik} \qquad \text{Property CMCN}$$

$$= \sum_{k=1}^{n} \left[B^t\right]_{jk} \left[A^t\right]_{ki} \qquad \text{Definition TM}$$

$$= \left[B^t A^t\right]_{ji} \qquad \text{Theorem EMP}$$

So the matrices $(AB)^t$ and $B^t A^t$ are equal, entry-by-entry, and by the definition of matrix equality (Definition ME) we can say they are equal matrices. ■

This theorem seems odd at first glance, since we have to switch the order of A and B. But if we simply consider the sizes of the matrices involved, we can see that the switch is necessary for this reason alone. That the individual entries of the products then come along to be equal is a bonus.

As the adjoint of a matrix is a composition of a conjugate and a transpose, its interaction with matrix multiplication is similar to that of a transpose. Here is the last of our long list of basic properties of matrix multiplication.

Theorem MMAD Matrix Multiplication and Adjoints
Suppose A is an $m \times n$ matrix and B is an $n \times p$ matrix. Then $(AB)^* = B^* A^*$.

Proof.

$$(AB)^* = \left(\overline{AB}\right)^t \qquad \text{Definition A}$$

$$= \left(\overline{A}\,\overline{B}\right)^t \qquad \text{Theorem MMCC}$$

$$= \left(\overline{B}\right)^t \left(\overline{A}\right)^t \qquad \text{Theorem MMT}$$

$$= B^* A^* \qquad \text{Definition A}$$

■

Notice how none of these proofs above relied on writing out huge general matrices with lots of ellipses ("…") and trying to formulate the equalities a whole matrix at a time. This messy business is a "proof technique" to be avoided at all costs. Notice too how the proof of Theorem MMAD does not use an entry-by-entry approach, but simply builds on previous results about matrix multiplication's interaction with conjugation and transposes.

These theorems, along with Theorem VSPM and the other results in Section MO, give you the "rules" for how matrices interact with the various operations we have defined on matrices (addition, scalar multiplication, matrix multiplication, conjugation, transposes and adjoints). Use them and use them often. But do not try to do anything with a matrix that you do not have a rule for. Together, we would informally call all these operations, and the attendant theorems, "the algebra of matrices." Notice, too, that every column vector is just a $n \times 1$ matrix, so these theorems apply to column vectors also. Finally, these results, taken as a whole, may make us feel that the definition of matrix multiplication is not so unnatural.

Subsection HM
Hermitian Matrices

The adjoint of a matrix has a basic property when employed in a matrix-vector product as part of an inner product. At this point, you could even use the following result as a motivation for the definition of an adjoint.

Theorem AIP Adjoint and Inner Product
Suppose that A is an $m \times n$ matrix and $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{y} \in \mathbb{C}^m$. Then $\langle A\mathbf{x},\, \mathbf{y}\rangle = \langle \mathbf{x},\, A^*\mathbf{y}\rangle$.

Proof.

$$\begin{aligned}
\langle A\mathbf{x},\, \mathbf{y}\rangle &= \left(\overline{A\mathbf{x}}\right)^t \mathbf{y} && \text{Theorem MMIP}\\
&= \left(\overline{A}\,\overline{\mathbf{x}}\right)^t \mathbf{y} && \text{Theorem MMCC}\\
&= \overline{\mathbf{x}}^t \overline{A}^t \mathbf{y} && \text{Theorem MMT}\\
&= \overline{\mathbf{x}}^t \left(A^*\mathbf{y}\right) && \text{Definition A}\\
&= \langle \mathbf{x},\, A^*\mathbf{y}\rangle && \text{Theorem MMIP}
\end{aligned}$$

■

Sometimes a matrix is equal to its adjoint (Definition A), and these matrices have interesting properties. One of the most common situations where this occurs is when a matrix has only real number entries. Then we are simply talking about symmetric matrices (Definition SYM), so you can view this as a generalization of a symmetric matrix.

Definition HM Hermitian Matrix
The square matrix A is **Hermitian** (or **self-adjoint**) if $A = A^*$. □

Again, the set of real matrices that are Hermitian is exactly the set of symmetric matrices. In Section PEE we will uncover some amazing properties of Hermitian matrices, so when you get there, run back here to remind yourself of this definition. Further properties will also appear in Section OD. Right now we prove a fundamental result about Hermitian matrices, matrix vector products and inner products. As a characterization, this could be employed as a definition of a Hermitian matrix and some authors take this approach.

Theorem HMIP Hermitian Matrices and Inner Products
Suppose that A is a square matrix of size n. Then A is Hermitian if and only if $\langle A\mathbf{x},\, \mathbf{y}\rangle = \langle \mathbf{x},\, A\mathbf{y}\rangle$ for all $\mathbf{x},\, \mathbf{y} \in \mathbb{C}^n$.

Proof. (⇒) This is the "easy half" of the proof, and makes the rationale for a definition of Hermitian matrices most obvious. Assume A is Hermitian,

$$\begin{aligned} \langle A\mathbf{x},\,\mathbf{y}\rangle &= \langle \mathbf{x},\,A^*\mathbf{y}\rangle && \text{Theorem AIP} \\ &= \langle \mathbf{x},\,A\mathbf{y}\rangle && \text{Definition HM} \end{aligned}$$

(⇐) This "half" will take a bit more work. Assume that $\langle A\mathbf{x},\,\mathbf{y}\rangle = \langle \mathbf{x},\,A\mathbf{y}\rangle$ for all $\mathbf{x},\,\mathbf{y} \in \mathbb{C}^n$. We show that $A = A^*$ by establishing that $A\mathbf{x} = A^*\mathbf{x}$ for all $\mathbf{x}$, so we can then apply Theorem EMMVP. With only this much motivation, consider the inner product for any $\mathbf{x} \in \mathbb{C}^n$.

$$\begin{aligned} \langle A\mathbf{x} - A^*\mathbf{x},\,A\mathbf{x} - A^*\mathbf{x}\rangle &= \langle A\mathbf{x} - A^*\mathbf{x},\,A\mathbf{x}\rangle - \langle A\mathbf{x} - A^*\mathbf{x},\,A^*\mathbf{x}\rangle && \text{Theorem IPVA} \\ &= \langle A\mathbf{x} - A^*\mathbf{x},\,A\mathbf{x}\rangle - \langle A\left(A\mathbf{x} - A^*\mathbf{x}\right),\,\mathbf{x}\rangle && \text{Theorem AIP} \\ &= \langle A\mathbf{x} - A^*\mathbf{x},\,A\mathbf{x}\rangle - \langle A\mathbf{x} - A^*\mathbf{x},\,A\mathbf{x}\rangle && \text{Hypothesis} \\ &= 0 && \text{Property AICN} \end{aligned}$$

Because this first inner product equals zero, and has the same vector in each argument ($A\mathbf{x} - A^*\mathbf{x}$), Theorem PIP gives the conclusion that $A\mathbf{x} - A^*\mathbf{x} = \mathbf{0}$. With $A\mathbf{x} = A^*\mathbf{x}$ for all $\mathbf{x} \in \mathbb{C}^n$, Theorem EMMVP says $A = A^*$, which is the defining property of a Hermitian matrix (Definition HM). ■

So, informally, Hermitian matrices are those that can be tossed around from one side of an inner product to the other with reckless abandon. We will see later what this buys us.

Reading Questions

1. Form the matrix vector product of

$$\begin{bmatrix} 2 & 3 & -1 & 0 \\ 1 & -2 & 7 & 3 \\ 1 & 5 & 3 & 2 \end{bmatrix} \qquad \text{with} \qquad \begin{bmatrix} 2 \\ -3 \\ 0 \\ 5 \end{bmatrix}$$

2. Multiply together the two matrices below (in the order given).

$$\begin{bmatrix} 2 & 3 & -1 & 0 \\ 1 & -2 & 7 & 3 \\ 1 & 5 & 3 & 2 \end{bmatrix} \qquad \begin{bmatrix} 2 & 6 \\ -3 & -4 \\ 0 & 2 \\ 3 & -1 \end{bmatrix}$$

3. Rewrite the system of linear equations below as a vector equality and using a matrix-vector product. (This question does not ask for a solution to the system. But it does ask you to express the system of equations in a new form using tools from this section.)

$$\begin{aligned} 2x_1 + 3x_2 - x_3 &= 0 \\ x_1 + 2x_2 + x_3 &= 3 \\ x_1 + 3x_2 + 3x_3 &= 7 \end{aligned}$$

Exercises

C20† Compute the product of the two matrices below, AB. Do this using the definitions of the matrix-vector product (Definition MVP) and the definition of matrix multiplication (Definition MM).

$$A = \begin{bmatrix} 2 & 5 \\ -1 & 3 \\ 2 & -2 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 5 & -3 & 4 \\ 2 & 0 & 2 & -3 \end{bmatrix}$$

C21† Compute the product AB of the two matrices below using both the definition of the matrix-vector product (Definition MVP) and the definition of matrix multiplication (Definition MM).

$$A = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 0 & 1 \\ 3 & 1 & 5 \end{bmatrix}$$

C22† Compute the product AB of the two matrices below using both the definition of the matrix-vector product (Definition MVP) and the definition of matrix multiplication (Definition MM).

$$A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}$$

C23† Compute the product AB of the two matrices below using both the definition of the matrix-vector product (Definition MVP) and the definition of matrix multiplication (Definition MM).

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \\ 6 & 5 \\ 1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} -3 & 1 \\ 4 & 2 \end{bmatrix}$$

C24† Compute the product AB of the two matrices below.

$$A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 0 & 1 & -2 & -1 \\ 1 & 1 & 3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 \\ 4 \\ 0 \\ 2 \end{bmatrix}$$

C25† Compute the product AB of the two matrices below.

$$A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 0 & 1 & -2 & -1 \\ 1 & 1 & 3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} -7 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

C26† Compute the product AB of the two matrices below using both the definition of the matrix-vector product (Definition MVP) and the definition of matrix multiplication (Definition MM).

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -5 & -1 \\ 0 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}$$

C30† For the matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, find A^2, A^3, A^4. Find a general formula for A^n for any positive integer n.

C31† For the matrix $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$, find A^2, A^3, A^4. Find a general formula for A^n for any positive integer n.

C32† For the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, find A^2, A^3, A^4. Find a general formula for A^n for any positive integer n.

C33† For the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, find A^2, A^3, A^4. Find a general formula for A^n for any positive integer n.

T10† Suppose that A is a square matrix and there is a vector, $\mathbf{b}$, such that $\mathcal{LS}(A, \mathbf{b})$ has a unique solution. Prove that A is nonsingular. Give a direct proof (perhaps appealing to Theorem PSPHS) rather than just negating a sentence from the text discussing a similar situation.

T12 The conclusion of Theorem HMIP is $\langle A\mathbf{x},\, \mathbf{y}\rangle = \langle \mathbf{x},\, A^*\mathbf{y}\rangle$. Use the same hypotheses, and prove the similar conclusion: $\langle \mathbf{x},\, A\mathbf{y}\rangle = \langle A^*\mathbf{x},\, \mathbf{y}\rangle$. Two different approaches can be based on an application of Theorem HMIP. The first uses Theorem AA, while a second uses Theorem IPAC. Can you provide two proofs?

T20 Prove the second part of Theorem MMZM.

T21 Prove the second part of Theorem MMIM.

T22 Prove the second part of Theorem MMDAA.

T23† Prove the second part of Theorem MMSMM.

T31 Suppose that A is an $m \times n$ matrix and $\mathbf{x},\, \mathbf{y} \in \mathcal{N}(A)$. Prove that $\mathbf{x} + \mathbf{y} \in \mathcal{N}(A)$.

T32 Suppose that A is an $m \times n$ matrix, $\alpha \in \mathbb{C}$, and $\mathbf{x} \in \mathcal{N}(A)$. Prove that $\alpha\mathbf{x} \in \mathcal{N}(A)$.

T35 Suppose that A is an $n \times n$ matrix. Prove that A^*A and AA^* are Hermitian matrices.

T40† Suppose that A is an $m \times n$ matrix and B is an $n \times p$ matrix. Prove that the null space of B is a subset of the null space of AB, that is $\mathcal{N}(B) \subseteq \mathcal{N}(AB)$. Provide an example where the opposite is false, in other words give an example where $\mathcal{N}(AB) \not\subseteq \mathcal{N}(B)$.

T41† Suppose that A is an $n \times n$ nonsingular matrix and B is an $n \times p$ matrix. Prove that the null space of B is equal to the null space of AB, that is $\mathcal{N}(B) = \mathcal{N}(AB)$. (Compare with Exercise MM.T40.)

T50 Suppose $\mathbf{u}$ and $\mathbf{v}$ are any two solutions of the linear system $\mathcal{LS}(A,\, \mathbf{b})$. Prove that $\mathbf{u} - \mathbf{v}$ is an element of the null space of A, that is, $\mathbf{u} - \mathbf{v} \in \mathcal{N}(A)$.

T51† Give a new proof of Theorem PSPHS replacing applications of Theorem SLSLC with matrix-vector products (Theorem SLEMM).

T52† Suppose that $\mathbf{x},\, \mathbf{y} \in \mathbb{C}^n$, $\mathbf{b} \in \mathbb{C}^m$ and A is an $m \times n$ matrix. If $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{x} + \mathbf{y}$ are each a solution to the linear system $\mathcal{LS}(A,\, \mathbf{b})$, what can you say that is interesting about $\mathbf{b}$? Form an implication with the existence of the three solutions as the hypothesis and an interesting statement about $\mathcal{LS}(A,\, \mathbf{b})$ as the conclusion, and then give a proof.

Section MISLE
Matrix Inverses and Systems of Linear Equations

The inverse of a square matrix, and solutions to linear systems with square coefficient matrices, are intimately connected.

Subsection SI
Solutions and Inverses

We begin with a familiar example, performed in a novel way.

Example SABMI Solutions to Archetype B with a matrix inverse
Archetype B is the system of $m = 3$ linear equations in $n = 3$ variables,

$$\begin{aligned} -7x_1 - 6x_2 - 12x_3 &= -33 \\ 5x_1 + 5x_2 + 7x_3 &= 24 \\ x_1 + 4x_3 &= 5 \end{aligned}$$

By Theorem SLEMM we can represent this system of equations as

$$A\mathbf{x} = \mathbf{b}$$

where

$$A = \begin{bmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -33 \\ 24 \\ 5 \end{bmatrix}$$

Now, entirely unmotivated, we define the 3×3 matrix B,

$$B = \begin{bmatrix} -10 & -12 & -9 \\ \frac{13}{2} & 8 & \frac{11}{2} \\ \frac{5}{2} & 3 & \frac{5}{2} \end{bmatrix}$$

and note the remarkable fact that

$$BA = \begin{bmatrix} -10 & -12 & -9 \\ \frac{13}{2} & 8 & \frac{11}{2} \\ \frac{5}{2} & 3 & \frac{5}{2} \end{bmatrix} \begin{bmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now apply this computation to the problem of solving the system of equations,

$$\begin{aligned} \mathbf{x} &= I_3\mathbf{x} && \text{Theorem MMIM} \\ &= (BA)\mathbf{x} && \text{Substitution} \\ &= B(A\mathbf{x}) && \text{Theorem MMA} \\ &= B\mathbf{b} && \text{Substitution} \end{aligned}$$

So we have

$$\mathbf{x} = B\mathbf{b} = \begin{bmatrix} -10 & -12 & -9 \\ \frac{13}{2} & 8 & \frac{11}{2} \\ \frac{5}{2} & 3 & \frac{5}{2} \end{bmatrix} \begin{bmatrix} -33 \\ 24 \\ 5 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix}$$

So with the help and assistance of B we have been able to determine a solution to the system represented by $A\mathbf{x} = \mathbf{b}$ through judicious use of matrix multiplication. We know by Theorem NMUS that since the coefficient matrix in this example is nonsingular, there would be a unique solution, no matter what the choice of $\mathbf{b}$. The derivation above amplifies this result, since we were *forced* to conclude that $\mathbf{x} = B\mathbf{b}$ and the solution could not be anything else. You should notice that this argument would hold for any particular choice of $\mathbf{b}$. △

The matrix B of the previous example is called the inverse of A. When A and B

are combined via matrix multiplication, the result is the identity matrix, which can be inserted "in front" of $\mathbf{x}$ as the first step in finding the solution. This is entirely analogous to how we might solve a single linear equation like $3x = 12$.

$$x = 1x = \left(\frac{1}{3}(3)\right)x = \frac{1}{3}(3x) = \frac{1}{3}(12) = 4$$

Here we have obtained a solution by employing the "multiplicative inverse" of 3, $3^{-1} = \frac{1}{3}$. This works fine for any scalar multiple of x, except for zero, since zero does not have a multiplicative inverse. Consider separately the two linear equations,

$$0x = 12 \qquad\qquad 0x = 0$$

The first has no solutions, while the second has infinitely many solutions. For matrices, it is all just a little more complicated. Some matrices have inverses, some do not. And when a matrix does have an inverse, just how would we compute it? In other words, just where did that matrix B in the last example come from? Are there other matrices that might have worked just as well?

Subsection IM Inverse of a Matrix

Definition MI Matrix Inverse
Suppose A and B are square matrices of size n such that $AB = I_n$ and $BA = I_n$. Then A is **invertible** and B is the **inverse** of A. In this situation, we write $B = A^{-1}$.
□

Notice that if B is the inverse of A, then we can just as easily say A is the inverse of B, or A and B are inverses of each other.

Not every square matrix has an inverse. In Example SABMI the matrix B is the inverse of the coefficient matrix of Archetype B. To see this it only remains to check that $AB = I_3$. What about Archetype A? It is an example of a square matrix without an inverse.

Example MWIAA A matrix without an inverse, Archetype A
Consider the coefficient matrix from Archetype A,

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Suppose that A is invertible and does have an inverse, say B. Choose the vector of constants

$$\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

and consider the system of equations $\mathcal{LS}(A, \mathbf{b})$. Just as in Example SABMI, this vector equation would have the unique solution $\mathbf{x} = B\mathbf{b}$.

However, the system $\mathcal{LS}(A, \mathbf{b})$ is inconsistent. Form the augmented matrix $[\,A \,|\, \mathbf{b}]$ and row-reduce to

$$\begin{bmatrix} \boxed{1} & 0 & 1 & 0 \\ 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

which allows us to recognize the inconsistency by Theorem RCLS.

So the assumption of A's inverse leads to a logical inconsistency (the system cannot be both consistent and inconsistent), so our assumption is false. A is not invertible.

It is possible this example is less than satisfying. Just where did that particular choice of the vector **b** come from anyway? Stay tuned for an application of the future Theorem CSCS in Example CSAA. △

Let us look at one more matrix inverse before we embark on a more systematic study.

Example MI Matrix inverse
Consider the matrices,

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ -2 & -3 & 0 & -5 & -1 \\ 1 & 1 & 0 & 2 & 1 \\ -2 & -3 & -1 & -3 & -2 \\ -1 & -3 & -1 & -3 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} -3 & 3 & 6 & -1 & -2 \\ 0 & -2 & -5 & -1 & 1 \\ 1 & 2 & 4 & 1 & -1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & -1 & -2 & 0 & 1 \end{bmatrix}$$

Then

$$AB = \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ -2 & -3 & 0 & -5 & -1 \\ 1 & 1 & 0 & 2 & 1 \\ -2 & -3 & -1 & -3 & -2 \\ -1 & -3 & -1 & -3 & 1 \end{bmatrix} \begin{bmatrix} -3 & 3 & 6 & -1 & -2 \\ 0 & -2 & -5 & -1 & 1 \\ 1 & 2 & 4 & 1 & -1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & -1 & -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$BA = \begin{bmatrix} -3 & 3 & 6 & -1 & -2 \\ 0 & -2 & -5 & -1 & 1 \\ 1 & 2 & 4 & 1 & -1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & -1 & -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ -2 & -3 & 0 & -5 & -1 \\ 1 & 1 & 0 & 2 & 1 \\ -2 & -3 & -1 & -3 & -2 \\ -1 & -3 & -1 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

so by Definition MI, we can say that A is invertible and write $B = A^{-1}$. △

We will now concern ourselves less with whether or not an inverse of a matrix exists, but instead with how you can find one when it does exist. In Section MINM we will have some theorems that allow us to more quickly and easily determine just when a matrix is invertible.

Subsection CIM
Computing the Inverse of a Matrix

We have seen that the matrices from Archetype B and Archetype K both have inverses, but these inverse matrices have just dropped from the sky. How would we compute an inverse? And just when is a matrix invertible, and when is it not? Writing a putative inverse with n^2 unknowns and solving the resultant n^2 equations is one approach. Applying this approach to 2×2 matrices can get us somewhere, so just for fun, let us do it.

Theorem TTMI Two-by-Two Matrix Inverse
Suppose

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then A is invertible if and only if $ad - bc \neq 0$. When A is invertible, then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Proof. ($\Leftarrow$) Assume that $ad - bc \neq 0$. We will use the definition of the inverse of a matrix to establish that A has an inverse (Definition MI). Note that if $ad - bc \neq 0$ then the displayed formula for A^{-1} is legitimate since we are not dividing by zero).

Using this proposed formula for the inverse of A, we compute

$$AA^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \left(\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right) = \frac{1}{ad-bc} \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$A^{-1}A = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

By Definition MI this is sufficient to establish that A is invertible, and that the expression for A^{-1} is correct.

($\Rightarrow$) Assume that A is invertible, and proceed with a proof by contradiction (Proof Technique CD), by assuming also that $ad - bc = 0$. This translates to $ad = bc$. Let

$$B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$$

be a putative inverse of A.

This means that

$$I_2 = AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

Working on the matrices on two ends of this equation, we will multiply the top row by c and the bottom row by a.

$$\begin{bmatrix} c & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} ace+bcg & acf+bch \\ ace+adg & acf+adh \end{bmatrix}$$

We are assuming that $ad = bc$, so we can replace two occurrences of ad by bc in the bottom row of the right matrix.

$$\begin{bmatrix} c & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} ace+bcg & acf+bch \\ ace+bcg & acf+bch \end{bmatrix}$$

The matrix on the right now has two rows that are identical, and therefore the same must be true of the matrix on the left. Identical rows for the matrix on the left implies that $a = 0$ and $c = 0$.

With this information, the product AB becomes

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2 = AB = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix} = \begin{bmatrix} bg & bh \\ dg & dh \end{bmatrix}$$

So $bg = dh = 1$ and thus b, g, d, h are all nonzero. But then bh and dg (the "other corners") must also be nonzero, so this is (finally) a contradiction. So our assumption was false and we see that $ad - bc \neq 0$ whenever A has an inverse. ∎

There are several ways one could try to prove this theorem, but there is a continual temptation to divide by one of the eight entries involved (a through f), but we can never be sure if these numbers are zero or not. This could lead to an analysis by cases, which is messy, messy, messy. Note how the above proof never divides, but always multiplies, and how zero/nonzero considerations are handled. Pay attention to the expression $ad - bc$, as we will see it again in a while (Chapter D).

This theorem is cute, and it is nice to have a formula for the inverse, and a condition that tells us when we can use it. However, this approach becomes impractical for larger matrices, even though it is possible to demonstrate that, in theory, there is a general formula. (Think for a minute about extending this result to just 3×3 matrices. For starters, we need 18 letters!) Instead, we will work column-by-column. Let us first work an example that will motivate the main theorem and remove some of the previous mystery.

Example CMI Computing a matrix inverse
Consider the matrix defined in Example MI as,

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 & 1 \\ -2 & -3 & 0 & -5 & -1 \\ 1 & 1 & 0 & 2 & 1 \\ -2 & -3 & -1 & -3 & -2 \\ -1 & -3 & -1 & -3 & 1 \end{bmatrix}$$

For its inverse, we desire a matrix B so that $AB = I_5$. Emphasizing the structure of the columns and employing the definition of matrix multiplication Definition MM,

$$\begin{aligned} AB &= I_5 \\ A[\mathbf{B}_1|\mathbf{B}_2|\mathbf{B}_3|\mathbf{B}_4|\mathbf{B}_5] &= [\mathbf{e}_1|\mathbf{e}_2|\mathbf{e}_3|\mathbf{e}_4|\mathbf{e}_5] \\ [A\mathbf{B}_1|A\mathbf{B}_2|A\mathbf{B}_3|A\mathbf{B}_4|A\mathbf{B}_5] &= [\mathbf{e}_1|\mathbf{e}_2|\mathbf{e}_3|\mathbf{e}_4|\mathbf{e}_5] \end{aligned}$$

Equating the matrices column-by-column we have

$$A\mathbf{B}_1 = \mathbf{e}_1 \qquad A\mathbf{B}_2 = \mathbf{e}_2 \qquad A\mathbf{B}_3 = \mathbf{e}_3 \qquad A\mathbf{B}_4 = \mathbf{e}_4 \qquad A\mathbf{B}_5 = \mathbf{e}_5.$$

Since the matrix B is what we are trying to compute, we can view each column, $\mathbf{B}_i$, as a column vector of unknowns. Then we have five systems of equations to solve, each with 5 equations in 5 variables. Notice that all 5 of these systems have the same coefficient matrix. We will now solve each system in turn,

Row-reduce the augmented matrix of the linear system $\mathcal{LS}(A,\ \mathbf{e}_1)$,

$$\begin{bmatrix} 1 & 2 & 1 & 2 & 1 & 1 \\ -2 & -3 & 0 & -5 & -1 & 0 \\ 1 & 1 & 0 & 2 & 1 & 0 \\ -2 & -3 & -1 & -3 & -2 & 0 \\ -1 & -3 & -1 & -3 & 1 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 0 & -3 \\ 0 & \boxed{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 & 1 \\ 0 & 0 & 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & 0 & 0 & \boxed{1} & 1 \end{bmatrix}; \mathbf{B}_1 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Row-reduce the augmented matrix of the linear system $\mathcal{LS}(A,\ \mathbf{e}_2)$,

$$\begin{bmatrix} 1 & 2 & 1 & 2 & 1 & 0 \\ -2 & -3 & 0 & -5 & -1 & 1 \\ 1 & 1 & 0 & 2 & 1 & 0 \\ -2 & -3 & -1 & -3 & -2 & 0 \\ -1 & -3 & -1 & -3 & 1 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 0 & 3 \\ 0 & \boxed{1} & 0 & 0 & 0 & -2 \\ 0 & 0 & \boxed{1} & 0 & 0 & 2 \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & -1 \end{bmatrix}; \mathbf{B}_2 = \begin{bmatrix} 3 \\ -2 \\ 2 \\ 0 \\ -1 \end{bmatrix}$$

Row-reduce the augmented matrix of the linear system $\mathcal{LS}(A,\ \mathbf{e}_3)$,

$$\begin{bmatrix} 1 & 2 & 1 & 2 & 1 & 0 \\ -2 & -3 & 0 & -5 & -1 & 0 \\ 1 & 1 & 0 & 2 & 1 & 1 \\ -2 & -3 & -1 & -3 & -2 & 0 \\ -1 & -3 & -1 & -3 & 1 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 0 & 6 \\ 0 & \boxed{1} & 0 & 0 & 0 & -5 \\ 0 & 0 & \boxed{1} & 0 & 0 & 4 \\ 0 & 0 & 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & 0 & 0 & \boxed{1} & -2 \end{bmatrix}; \mathbf{B}_3 = \begin{bmatrix} 6 \\ -5 \\ 4 \\ 1 \\ -2 \end{bmatrix}$$

Row-reduce the augmented matrix of the linear system $\mathcal{LS}(A,\,\mathbf{e}_4)$,

$$\begin{bmatrix}1&2&1&2&1&0\\-2&-3&0&-5&-1&0\\1&1&0&2&1&0\\-2&-3&-1&-3&-2&1\\-1&-3&-1&-3&1&0\end{bmatrix}\xrightarrow{\text{RREF}}\begin{bmatrix}\boxed{1}&0&0&0&0&-1\\0&\boxed{1}&0&0&0&-1\\0&0&\boxed{1}&0&0&1\\0&0&0&\boxed{1}&0&1\\0&0&0&0&\boxed{1}&0\end{bmatrix};\mathbf{B}_4=\begin{bmatrix}-1\\-1\\1\\1\\0\end{bmatrix}$$

Row-reduce the augmented matrix of the linear system $\mathcal{LS}(A,\,\mathbf{e}_5)$,

$$\begin{bmatrix}1&2&1&2&1&0\\-2&-3&0&-5&-1&0\\1&1&0&2&1&0\\-2&-3&-1&-3&-2&0\\-1&-3&-1&-3&1&1\end{bmatrix}\xrightarrow{\text{RREF}}\begin{bmatrix}\boxed{1}&0&0&0&0&-2\\0&\boxed{1}&0&0&0&1\\0&0&\boxed{1}&0&0&-1\\0&0&0&\boxed{1}&0&0\\0&0&0&0&\boxed{1}&1\end{bmatrix};\mathbf{B}_5=\begin{bmatrix}-2\\1\\-1\\0\\1\end{bmatrix}$$

We can now collect our 5 solution vectors into the matrix B,

$$\begin{aligned}B&=[\mathbf{B}_1|\mathbf{B}_2|\mathbf{B}_3|\mathbf{B}_4|\mathbf{B}_5]\\&=\left[\begin{bmatrix}-3\\0\\1\\1\\1\end{bmatrix}\middle|\begin{bmatrix}3\\-2\\2\\0\\-1\end{bmatrix}\middle|\begin{bmatrix}6\\-5\\4\\1\\-2\end{bmatrix}\middle|\begin{bmatrix}-1\\-1\\1\\1\\0\end{bmatrix}\middle|\begin{bmatrix}-2\\1\\-1\\0\\1\end{bmatrix}\right]\\&=\begin{bmatrix}-3&3&6&-1&-2\\0&-2&-5&-1&1\\1&2&4&1&-1\\1&0&1&1&0\\1&-1&-2&0&1\end{bmatrix}\end{aligned}$$

By this method, we know that $AB=I_5$. Check that $BA=I_5$, and then we will know that we have the inverse of A. △

Notice how the five systems of equations in the preceding example were all solved by *exactly* the same sequence of row operations. Would it not be nice to avoid this obvious duplication of effort? Our main theorem for this section follows, and it mimics this previous example, while also avoiding all the overhead.

Theorem CINM Computing the Inverse of a Nonsingular Matrix
Suppose A is a nonsingular square matrix of size n. Create the $n\times 2n$ matrix M by placing the $n\times n$ identity matrix I_n to the right of the matrix A. Let N be a matrix that is row-equivalent to M and in reduced row-echelon form. Finally, let J be the matrix formed from the final n columns of N. Then $AJ=I_n$.

Proof. A is nonsingular, so by Theorem NMRRI there is a sequence of row operations that will convert A into I_n. It is this same sequence of row operations that will convert M into N, since having the identity matrix in the first n columns of N is sufficient to guarantee that N is in reduced row-echelon form.

If we consider the systems of linear equations, $\mathcal{LS}(A,\,\mathbf{e}_i)$, $1\leq i\leq n$, we see that the aforementioned sequence of row operations will also bring the augmented matrix of each of these systems into reduced row-echelon form. Furthermore, the unique solution to $\mathcal{LS}(A,\,\mathbf{e}_i)$ appears in column $n+1$ of the row-reduced augmented matrix of the system and is identical to column $n+i$ of N. Let $\mathbf{N}_1,\,\mathbf{N}_2,\,\mathbf{N}_3,\,\ldots,\,\mathbf{N}_{2n}$ denote the columns of N. So we find,

$$\begin{aligned}AJ&=A[\mathbf{N}_{n+1}|\mathbf{N}_{n+2}|\mathbf{N}_{n+3}|\ldots|\mathbf{N}_{n+n}]\\&=[A\mathbf{N}_{n+1}|A\mathbf{N}_{n+2}|A\mathbf{N}_{n+3}|\ldots|A\mathbf{N}_{n+n}] && \text{Definition MM}\end{aligned}$$

$$=[\mathbf{e}_1|\mathbf{e}_2|\mathbf{e}_3|\dots|\mathbf{e}_n]$$
$$=I_n \qquad \text{Definition IM}$$

as desired. ■

We have to be just a bit careful here about both what this theorem says and what it does not say. If A is a nonsingular matrix, then we are guaranteed a matrix B such that $AB = I_n$, and the proof gives us a process for constructing B. However, the definition of the inverse of a matrix (Definition MI) requires that $BA = I_n$ also. So at this juncture we must compute the matrix product in the "opposite" order before we claim B as the inverse of A. However, we will soon see that this is *always* the case, in Theorem OSIS, so the title of this theorem is not inaccurate.

What if A is singular? At this point we only know that Theorem CINM cannot be applied. The question of A's inverse is still open. (But see Theorem NI in the next section.)

We will finish by computing the inverse for the coefficient matrix of Archetype B, the one we just pulled from a hat in Example SABMI. There are more examples in the Archetypes (Archetypes) to practice with, though notice that it is silly to ask for the inverse of a rectangular matrix (the sizes are not right) and not every square matrix has an inverse (remember Example MWIAA?).

Example CMIAB Computing a matrix inverse, Archetype B
Archetype B has a coefficient matrix given as

$$B = \begin{bmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{bmatrix}$$

Exercising Theorem CINM we set

$$M = \begin{bmatrix} -7 & -6 & -12 & 1 & 0 & 0 \\ 5 & 5 & 7 & 0 & 1 & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 \end{bmatrix}.$$

which row reduces to

$$N = \begin{bmatrix} 1 & 0 & 0 & -10 & -12 & -9 \\ 0 & 1 & 0 & \frac{13}{2} & 8 & \frac{11}{2} \\ 0 & 0 & 1 & \frac{5}{2} & 3 & \frac{5}{2} \end{bmatrix}.$$

So

$$B^{-1} = \begin{bmatrix} -10 & -12 & -9 \\ \frac{13}{2} & 8 & \frac{11}{2} \\ \frac{5}{2} & 3 & \frac{5}{2} \end{bmatrix}$$

once we check that $B^{-1}B = I_3$ (the product in the opposite order is a consequence of the theorem). △

Subsection PMI
Properties of Matrix Inverses

The inverse of a matrix enjoys some nice properties. We collect a few here. First, a matrix can have but one inverse.

Theorem MIU Matrix Inverse is Unique
Suppose the square matrix A has an inverse. Then A^{-1} is unique.

Proof. As described in Proof Technique U, we will assume that A has two inverses. The hypothesis tells there is at least one. Suppose then that B and C are both inverses for A, so we know by Definition MI that $AB = BA = I_n$ and $AC = CA = I_n$. Then we have,

$$\begin{aligned} B &= BI_n && \text{Theorem MMIM} \\ &= B(AC) && \text{Definition MI} \\ &= (BA)C && \text{Theorem MMA} \\ &= I_nC && \text{Definition MI} \\ &= C && \text{Theorem MMIM} \end{aligned}$$

So we conclude that B and C are the same, and cannot be different. So any matrix that acts like *an* inverse, must be *the* inverse. ■

When most of us dress in the morning, we put on our socks first, followed by our shoes. In the evening we must then first remove our shoes, followed by our socks. Try to connect the conclusion of the following theorem with this everyday example.

Theorem SS Socks and Shoes
Suppose A and B are invertible matrices of size n. Then AB is an invertible matrix and $(AB)^{-1} = B^{-1}A^{-1}$.

Proof. At the risk of carrying our everyday analogies too far, the proof of this theorem is quite easy when we compare it to the workings of a dating service. We have a statement about the inverse of the matrix AB, which for all we know right now might not even exist. Suppose AB was to sign up for a dating service with two requirements for a compatible date. Upon multiplication on the left, and on the right, the result should be the identity matrix. In other words, AB's ideal date would be its inverse.

Now along comes the matrix $B^{-1}A^{-1}$ (which we know exists because our hypothesis says both A and B are invertible and we can form the product of these two matrices), also looking for a date. Let us see if $B^{-1}A^{-1}$ is a good match for AB. First they meet at a noncommittal neutral location, say a coffee shop, for quiet conversation:

$$\begin{aligned} (B^{-1}A^{-1})(AB) &= B^{-1}(A^{-1}A)B && \text{Theorem MMA} \\ &= B^{-1}I_nB && \text{Definition MI} \\ &= B^{-1}B && \text{Theorem MMIM} \\ &= I_n && \text{Definition MI} \end{aligned}$$

The first date having gone smoothly, a second, more serious, date is arranged, say dinner and a show:

$$\begin{aligned} (AB)(B^{-1}A^{-1}) &= A(BB^{-1})A^{-1} && \text{Theorem MMA} \\ &= AI_nA^{-1} && \text{Definition MI} \\ &= AA^{-1} && \text{Theorem MMIM} \\ &= I_n && \text{Definition MI} \end{aligned}$$

So the matrix $B^{-1}A^{-1}$ has met all of the requirements to be AB's inverse (date) and with the ensuing marriage proposal we can announce that $(AB)^{-1} = B^{-1}A^{-1}$. ■

Theorem MIMI Matrix Inverse of a Matrix Inverse
Suppose A is an invertible matrix. Then A^{-1} is invertible and $(A^{-1})^{-1} = A$.

Proof. As with the proof of Theorem SS, we examine if A is a suitable inverse for A^{-1} (by definition, the opposite is true).

$$AA^{-1} = I_n \qquad \text{Definition MI}$$

and

$$A^{-1}A = I_n \qquad \text{Definition MI}$$

The matrix A has met all the requirements to be the inverse of A^{-1}, and so is invertible and we can write $A = (A^{-1})^{-1}$. ■

Theorem MIT Matrix Inverse of a Transpose
Suppose A is an invertible matrix. Then A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$.

Proof. As with the proof of Theorem SS, we see if $(A^{-1})^t$ is a suitable inverse for A^t. Apply Theorem MMT to see that

$$\begin{aligned}
(A^{-1})^t A^t &= (AA^{-1})^t && \text{Theorem MMT}\\
&= I_n^t && \text{Definition MI}\\
&= I_n && \text{Definition SYM}
\end{aligned}$$

and

$$\begin{aligned}
A^t (A^{-1})^t &= (A^{-1}A)^t && \text{Theorem MMT}\\
&= I_n^t && \text{Definition MI}\\
&= I_n && \text{Definition SYM}
\end{aligned}$$

The matrix $(A^{-1})^t$ has met all the requirements to be the inverse of A^t, and so is invertible and we can write $(A^t)^{-1} = (A^{-1})^t$. ■

Theorem MISM Matrix Inverse of a Scalar Multiple
Suppose A is an invertible matrix and α is a nonzero scalar. Then $(\alpha A)^{-1} = \frac{1}{\alpha}A^{-1}$ and αA is invertible.

Proof. As with the proof of Theorem SS, we see if $\frac{1}{\alpha}A^{-1}$ is a suitable inverse for αA.

$$\begin{aligned}
\left(\frac{1}{\alpha}A^{-1}\right)(\alpha A) &= \left(\frac{1}{\alpha}\alpha\right)\left(AA^{-1}\right) && \text{Theorem MMSMM}\\
&= 1I_n && \text{Scalar multiplicative inverses}\\
&= I_n && \text{Property OM}
\end{aligned}$$

and

$$\begin{aligned}
(\alpha A)\left(\frac{1}{\alpha}A^{-1}\right) &= \left(\alpha\frac{1}{\alpha}\right)\left(A^{-1}A\right) && \text{Theorem MMSMM}\\
&= 1I_n && \text{Scalar multiplicative inverses}\\
&= I_n && \text{Property OM}
\end{aligned}$$

The matrix $\frac{1}{\alpha}A^{-1}$ has met all the requirements to be the inverse of αA, so we can write $(\alpha A)^{-1} = \frac{1}{\alpha}A^{-1}$. ■

Notice that there are some likely theorems that are missing here. For example, it would be tempting to think that $(A+B)^{-1} = A^{-1} + B^{-1}$, but this is false. Can you find a counterexample? (See Exercise MISLE.T10.)

Reading Questions

1. Compute the inverse of the matrix below.

$$\begin{bmatrix} -2 & 3 \\ -3 & 4 \end{bmatrix}$$

2. Compute the inverse of the matrix below.

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & -2 & -3 \\ -2 & 4 & 6 \end{bmatrix}$$

3. Explain why Theorem SS has the title it does. (Do not just state the theorem, explain the choice of the title making reference to the theorem itself.)

Exercises

C16† If it exists, find the inverse of $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & -1 & 1 \end{bmatrix}$, and check your answer.

C17† If it exists, find the inverse of $A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}$, and check your answer.

C18† If it exists, find the inverse of $A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 2 & 1 \\ 2 & 2 & 1 \end{bmatrix}$, and check your answer.

C19† If it exists, find the inverse of $A = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 1 \\ 2 & 2 & 1 \end{bmatrix}$, and check your answer.

C21† Verify that B is the inverse of A.

$$A = \begin{bmatrix} 1 & 1 & -1 & 2 \\ -2 & -1 & 2 & -3 \\ 1 & 1 & 0 & 2 \\ -1 & 2 & 0 & 2 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 2 & 0 & -1 \\ 8 & 4 & -1 & -1 \\ -1 & 0 & 1 & 0 \\ -6 & -3 & 1 & 1 \end{bmatrix}$$

C22† Recycle the matrices A and B from Exercise MISLE.C21 and set

$$\mathbf{c} = \begin{bmatrix} 2 \\ 1 \\ -3 \\ 2 \end{bmatrix} \qquad \mathbf{d} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Employ the matrix B to solve the two linear systems $\mathcal{LS}(A,\, \mathbf{c})$ and $\mathcal{LS}(A,\, \mathbf{d})$.

C23 If it exists, find the inverse of the 2×2 matrix

$$A = \begin{bmatrix} 7 & 3 \\ 5 & 2 \end{bmatrix}$$

and check your answer. (See Theorem TTMI.)

C24 If it exists, find the inverse of the 2×2 matrix

$$A = \begin{bmatrix} 6 & 3 \\ 4 & 2 \end{bmatrix}$$

and check your answer. (See Theorem TTMI.)

C25 At the conclusion of Example CMI, verify that $BA = I_5$ by computing the matrix product.

C26† Let

$$D = \begin{bmatrix} 1 & -1 & 3 & -2 & 1 \\ -2 & 3 & -5 & 3 & 0 \\ 1 & -1 & 4 & -2 & 2 \\ -1 & 4 & -1 & 0 & 4 \\ 1 & 0 & 5 & -2 & 5 \end{bmatrix}$$

Compute the inverse of D, D^{-1}, by forming the 5×10 matrix $[\,D \mid I_5]$ and row-reducing (Theorem CINM). Then use a calculator to compute D^{-1} directly.

C27† Let

$$E = \begin{bmatrix} 1 & -1 & 3 & -2 & 1 \\ -2 & 3 & -5 & 3 & -1 \\ 1 & -1 & 4 & -2 & 2 \\ -1 & 4 & -1 & 0 & 2 \\ 1 & 0 & 5 & -2 & 4 \end{bmatrix}$$

Compute the inverse of E, E^{-1}, by forming the 5×10 matrix $[\,E \mid I_5]$ and row-reducing (Theorem CINM). Then use a calculator to compute E^{-1} directly.

C28† Let

$$C = \begin{bmatrix} 1 & 1 & 3 & 1 \\ -2 & -1 & -4 & -1 \\ 1 & 4 & 10 & 2 \\ -2 & 0 & -4 & 5 \end{bmatrix}$$

Compute the inverse of C, C^{-1}, by forming the 4×8 matrix $[\,C \mid I_4]$ and row-reducing (Theorem CINM). Then use a calculator to compute C^{-1} directly.

C40† Find all solutions to the system of equations below, making use of the matrix inverse found in Exercise MISLE.C28.

$$\begin{aligned} x_1 + x_2 + 3x_3 + x_4 &= -4 \\ -2x_1 - x_2 - 4x_3 - x_4 &= 4 \\ x_1 + 4x_2 + 10x_3 + 2x_4 &= -20 \\ -2x_1 - 4x_3 + 5x_4 &= 9 \end{aligned}$$

C41† Use the inverse of a matrix to find all the solutions to the following system of equations.

$$\begin{aligned} x_1 + 2x_2 - x_3 &= -3 \\ 2x_1 + 5x_2 - x_3 &= -4 \\ -x_1 - 4x_2 &= 2 \end{aligned}$$

C42† Use a matrix inverse to solve the linear system of equations.

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 5 \\ x_1 - 2x_3 &= -8 \\ 2x_1 - x_2 - x_3 &= -6 \end{aligned}$$

T10† Construct an example to demonstrate that $(A + B)^{-1} = A^{-1} + B^{-1}$ is not true for all square matrices A and B of the same size.

Section MINM
Matrix Inverses and Nonsingular Matrices

We saw in Theorem CINM that if a square matrix A is nonsingular, then there is a matrix B so that $AB = I_n$. In other words, B is halfway to being an inverse of A. We will see in this section that B automatically fulfills the second condition ($BA = I_n$). Example MWIAA showed us that the coefficient matrix from Archetype A had no inverse. Not coincidentally, this coefficient matrix is singular. We will make all these connections precise now. Not many examples or definitions in this section, just theorems.

Subsection NMI
Nonsingular Matrices are Invertible

We need a couple of technical results for starters. Some books would call these minor, but essential, results "lemmas." We'll just call 'em theorems. See Proof Technique LC for more on the distinction.

The first of these technical results is interesting in that the hypothesis says something about a product of two square matrices and the conclusion then says the same thing about each individual matrix in the product. This result has an analogy in the algebra of complex numbers: suppose $\alpha,\ \beta \in \mathbb{C}$, then $\alpha\beta \neq 0$ if and only if $\alpha \neq 0$ and $\beta \neq 0$. We can view this result as suggesting that the term "nonsingular" for matrices is like the term "nonzero" for scalars.

Theorem NPNT Nonsingular Product has Nonsingular Terms
Suppose that A and B are square matrices of size n. The product AB is nonsingular if and only if A and B are both nonsingular.

Proof. ($\Rightarrow$) We will do this portion of the proof in two parts, each as a proof by contradiction (Proof Technique CD). Assume that AB is nonsingular. Establishing that B is nonsingular is the easier part, so we will do it first, but in reality, we will *need* to know that B is nonsingular when we prove that A is nonsingular.

You can also think of this proof as being a study of four possible conclusions in the table below. One of the four rows *must* happen (the list is exhaustive). In the proof we learn that the first three rows lead to contradictions, and so are impossible. That leaves the fourth row as a certainty, which is our desired conclusion.

A	B	Case
Singular	Singular	1
Nonsingular	Singular	1
Singular	Nonsingular	2
Nonsingular	Nonsingular	

Part 1. Suppose B is singular. Then there is a nonzero vector $\mathbf{z}$ that is a solution to $\mathcal{LS}(B,\ \mathbf{0})$. So

$$\begin{aligned}(AB)\mathbf{z} &= A(B\mathbf{z}) && \text{Theorem MMA}\\ &= A\mathbf{0} && \text{Theorem SLEMM}\\ &= \mathbf{0} && \text{Theorem MMZM}\end{aligned}$$

Because $\mathbf{z}$ is a nonzero solution to $\mathcal{LS}(AB,\ \mathbf{0})$, we conclude that AB is singular (Definition NM). This is a contradiction, so B is nonsingular, as desired.

Part 2. Suppose A is singular. Then there is a nonzero vector $\mathbf{y}$ that is a solution to $\mathcal{LS}(A,\ \mathbf{0})$. Now consider the linear system $\mathcal{LS}(B,\ \mathbf{y})$. Since we know B is nonsingular from Case 1, the system has a unique solution (Theorem NMUS), which we will denote as $\mathbf{w}$. We first claim $\mathbf{w}$ is not the zero vector either. Assuming the opposite, suppose that $\mathbf{w} = \mathbf{0}$ (Proof Technique CD). Then

$$\begin{aligned} \mathbf{y} &= B\mathbf{w} && \text{Theorem SLEMM} \\ &= B\mathbf{0} && \text{Hypothesis} \\ &= \mathbf{0} && \text{Theorem MMZM} \end{aligned}$$

contrary to $\mathbf{y}$ being nonzero. So $\mathbf{w} \neq \mathbf{0}$. The pieces are in place, so here we go,

$$\begin{aligned} (AB)\mathbf{w} &= A(B\mathbf{w}) && \text{Theorem MMA} \\ &= A\mathbf{y} && \text{Theorem SLEMM} \\ &= \mathbf{0} && \text{Theorem SLEMM} \end{aligned}$$

So $\mathbf{w}$ is a nonzero solution to $\mathcal{LS}(AB,\,\mathbf{0})$, and thus we can say that AB is singular (Definition NM). This is a contradiction, so A is nonsingular, as desired.

($\Leftarrow$) Now assume that both A and B are nonsingular. Suppose that $\mathbf{x} \in \mathbb{C}^n$ is a solution to $\mathcal{LS}(AB,\,\mathbf{0})$. Then

$$\begin{aligned} \mathbf{0} &= (AB)\,\mathbf{x} && \text{Theorem SLEMM} \\ &= A\,(B\mathbf{x}) && \text{Theorem MMA} \end{aligned}$$

By Theorem SLEMM, $B\mathbf{x}$ is a solution to $\mathcal{LS}(A,\,\mathbf{0})$, and by the definition of a nonsingular matrix (Definition NM), we conclude that $B\mathbf{x} = \mathbf{0}$. Now, by an entirely similar argument, the nonsingularity of B forces us to conclude that $\mathbf{x} = \mathbf{0}$. So the only solution to $\mathcal{LS}(AB,\,\mathbf{0})$ is the zero vector and we conclude that AB is nonsingular by Definition NM. ■

This is a powerful result in the "forward" direction, because it allows us to begin with a hypothesis that something complicated (the matrix product AB) has the property of being nonsingular, and we can then conclude that the simpler constituents (A and B individually) then also have the property of being nonsingular. If we had thought that the matrix product was an artificial construction, results like this would make us begin to think twice.

The contrapositive of this result is equally interesting. It says that A or B (or both) is a singular matrix if and only if the product AB is singular. Notice how the negation of the theorem's conclusion (A and B both nonsingular) becomes the statement "at least one of A and B is singular." (See Proof Technique CP.)

Theorem OSIS One-Sided Inverse is Sufficient
Suppose A and B are square matrices of size n such that $AB = I_n$. Then $BA = I_n$.

Proof. The matrix I_n is nonsingular (since it row-reduces easily to I_n, Theorem NMRRI). So A and B are nonsingular by Theorem NPNT, so in particular B is nonsingular. We can therefore apply Theorem CINM to assert the existence of a matrix C so that $BC = I_n$. This application of Theorem CINM could be a bit confusing, mostly because of the names of the matrices involved. B is nonsingular, so there must be a "right-inverse" for B, and we are calling it C.

Now

$$\begin{aligned} BA &= (BA)I_n && \text{Theorem MMIM} \\ &= (BA)(BC) && \text{Theorem CINM} \\ &= B(AB)C && \text{Theorem MMA} \\ &= BI_nC && \text{Hypothesis} \\ &= BC && \text{Theorem MMIM} \\ &= I_n && \text{Theorem CINM} \end{aligned}$$

which is the desired conclusion. ■

So Theorem OSIS tells us that if A is nonsingular, then the matrix B guaranteed by Theorem CINM will be both a "right-inverse" and a "left-inverse" for A, so A is invertible and $A^{-1} = B$.

So if you have a nonsingular matrix, A, you can use the procedure described in Theorem CINM to find an inverse for A. If A is singular, then the procedure in Theorem CINM will fail as the first n columns of M will not row-reduce to the identity matrix. However, we can say a bit more. When A is singular, then A does not have an inverse (which is very different from saying that the procedure in Theorem CINM fails to find an inverse). This may feel like we are splitting hairs, but it is important that we do not make unfounded assumptions. These observations motivate the next theorem.

Theorem NI Nonsingularity is Invertibility
Suppose that A is a square matrix. Then A is nonsingular if and only if A is invertible.

Proof. ($\Leftarrow$) Since A is invertible, we can write $I_n = AA^{-1}$ (Definition MI). Notice that I_n is nonsingular (Theorem NMRRI) so Theorem NPNT implies that A (and A^{-1}) is nonsingular.

($\Rightarrow$) Suppose now that A is nonsingular. By Theorem CINM we find B so that $AB = I_n$. Then Theorem OSIS tells us that $BA = I_n$. So B is A's inverse, and by construction, A is invertible. ■

So for a square matrix, the properties of having an inverse and of having a trivial null space are one and the same. Cannot have one without the other.

Theorem NME3 Nonsingular Matrix Equivalences, Round 3
Suppose that A is a square matrix of size n. The following are equivalent.

1. *A is nonsingular.*
2. *A row-reduces to the identity matrix.*
3. *The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.*
4. *The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.*
5. *The columns of A are a linearly independent set.*
6. *A is invertible.*

Proof. We can update our list of equivalences for nonsingular matrices (Theorem NME2) with the equivalent condition from Theorem NI. ■

In the case that A is a nonsingular coefficient matrix of a system of equations, the inverse allows us to very quickly compute the unique solution, for any vector of constants.

Theorem SNCM Solution with Nonsingular Coefficient Matrix
Suppose that A is nonsingular. Then the unique solution to $\mathcal{LS}(A, \mathbf{b})$ is $A^{-1}\mathbf{b}$.

Proof. By Theorem NMUS we know already that $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every choice of $\mathbf{b}$. We need to show that the expression stated is indeed a solution (*the* solution). That is easy, just "plug it in" to the vector equation representation of the system (Theorem SLEMM),

$$\begin{aligned} A\left(A^{-1}\mathbf{b}\right) &= \left(AA^{-1}\right)\mathbf{b} && \text{Theorem MMA} \\ &= I_n\mathbf{b} && \text{Definition MI} \end{aligned}$$

$$= \mathbf{b} \qquad \text{Theorem MMIM}$$

Since $A\mathbf{x} = \mathbf{b}$ is true when we substitute $A^{-1}\mathbf{b}$ for $\mathbf{x}$, $A^{-1}\mathbf{b}$ is a (the!) solution to $\mathcal{LS}(A,\,\mathbf{b})$. ■

Subsection UM
Unitary Matrices

Recall that the adjoint of a matrix is $A^* = \left(\overline{A}\right)^t$ (Definition A).

Definition UM Unitary Matrices
Suppose that U is a square matrix of size n such that $U^*U = I_n$. Then we say U is **unitary**. □

This condition may seem rather far-fetched at first glance. Would there be *any* matrix that behaved this way? Well, yes, here is one.

Example UM3 Unitary matrix of size 3

$$U = \begin{bmatrix} \frac{1+i}{\sqrt{5}} & \frac{3+2\,i}{\sqrt{55}} & \frac{2+2i}{\sqrt{22}} \\ \frac{1-i}{\sqrt{5}} & \frac{2+2\,i}{\sqrt{55}} & \frac{-3+i}{\sqrt{22}} \\ \frac{i}{\sqrt{5}} & \frac{3-5\,i}{\sqrt{55}} & -\frac{2}{\sqrt{22}} \end{bmatrix}$$

The computations get a bit tiresome, but if you work your way through the computation of U^*U, you *will* arrive at the 3×3 identity matrix I_3. △

Unitary matrices do not have to look quite so gruesome. Here is a larger one that is a bit more pleasing.

Example UPM Unitary permutation matrix
The matrix

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

is unitary as can be easily checked. Notice that it is just a rearrangement of the columns of the 5×5 identity matrix, I_5 (Definition IM).

An interesting exercise is to build another 5×5 unitary matrix, R, using a different rearrangement of the columns of I_5. Then form the product PR. This will be another unitary matrix (Exercise MINM.T10). If you were to build all $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$ matrices of this type you would have a set that remains closed under matrix multiplication. It is an example of another algebraic structure known as a **group** since together the set and the one operation (matrix multiplication here) is closed, associative, has an identity (I_5), and inverses (Theorem UMI). Notice though that the operation in this group is not commutative! △

If a matrix A has only real number entries (we say it is a **real matrix**) then the defining property of being unitary simplifies to $A^tA = I_n$. In this case we, and everybody else, call the matrix **orthogonal**, so you may often encounter this term in your other reading when the complex numbers are not under consideration.

Unitary matrices have easily computed inverses. They also have columns that form orthonormal sets. Here are the theorems that show us that unitary matrices are not as strange as they might initially appear.

Theorem UMI Unitary Matrices are Invertible
Suppose that U is a unitary matrix of size n. Then U is nonsingular, and $U^{-1} = U^*$.

Proof. By Definition UM, we know that $U^*U = I_n$. The matrix I_n is nonsingular (since it row-reduces easily to I_n, Theorem NMRRI). So by Theorem NPNT, U and U^* are both nonsingular matrices.

The equation $U^*U = I_n$ gets us halfway to an inverse of U, and Theorem OSIS tells us that then $UU^* = I_n$ also. So U and U^* are inverses of each other (Definition MI). ■

Theorem CUMOS Columns of Unitary Matrices are Orthonormal Sets
Suppose that $S = \{\mathbf{A}_1,\,\mathbf{A}_2,\,\mathbf{A}_3,\,\dots,\,\mathbf{A}_n\}$ is the set of columns of a square matrix A of size n. Then A is a unitary matrix if and only if S is an orthonormal set.

Proof. The proof revolves around recognizing that a typical entry of the product A^*A is an inner product of columns of A. Here are the details to support this claim.

$$\begin{aligned}
\left[A^*A\right]_{ij} &= \sum_{k=1}^{n} \left[A^*\right]_{ik}\left[A\right]_{kj} && \text{Theorem EMP}\\
&= \sum_{k=1}^{n} \left[\overline{A}^t\right]_{ik}\left[A\right]_{kj} && \text{Theorem EMP}\\
&= \sum_{k=1}^{n} \left[\overline{A}\right]_{ki}\left[A\right]_{kj} && \text{Definition TM}\\
&= \sum_{k=1}^{n} \overline{\left[A\right]_{ki}}\left[A\right]_{kj} && \text{Definition CCM}\\
&= \sum_{k=1}^{n} \overline{\left[\mathbf{A}_i\right]_{k}}\left[\mathbf{A}_j\right]_{k} && \\
&= \left\langle \mathbf{A}_i,\,\mathbf{A}_j\right\rangle && \text{Definition IP}
\end{aligned}$$

We now employ this equality in a chain of equivalences,

$$\begin{aligned}
&S = \{\mathbf{A}_1,\,\mathbf{A}_2,\,\mathbf{A}_3,\,\dots,\,\mathbf{A}_n\} \text{ is an orthonormal set}\\
\iff& \left\langle \mathbf{A}_i,\,\mathbf{A}_j\right\rangle = \begin{cases} 0 & \text{if } i \neq j\\ 1 & \text{if } i = j\end{cases} && \text{Definition ONS}\\
\iff& \left[A^*A\right]_{ij} = \begin{cases} 0 & \text{if } i \neq j\\ 1 & \text{if } i = j\end{cases} && \\
\iff& \left[A^*A\right]_{ij} = \left[I_n\right]_{ij},\ 1 \leq i \leq n,\ 1 \leq j \leq n && \text{Definition IM}\\
\iff& A^*A = I_n && \text{Definition ME}\\
\iff& A \text{ is a unitary matrix} && \text{Definition UM}
\end{aligned}$$

■

Example OSMC Orthonormal set from matrix columns
The matrix

$$U = \begin{bmatrix} \frac{1+i}{\sqrt{5}} & \frac{3+2\,i}{\sqrt{55}} & \frac{2+2i}{\sqrt{22}}\\ \frac{1-i}{\sqrt{5}} & \frac{2+2\,i}{\sqrt{55}} & \frac{-3+i}{\sqrt{22}}\\ \frac{i}{\sqrt{5}} & \frac{3-5\,i}{\sqrt{55}} & -\frac{2}{\sqrt{22}} \end{bmatrix}$$

from Example UM3 is a unitary matrix. By Theorem CUMOS, its columns

$$\left\{\begin{bmatrix}\frac{1+i}{\sqrt{5}}\\ \frac{1-i}{\sqrt{5}}\\ \frac{i}{\sqrt{5}}\end{bmatrix},\ \begin{bmatrix}\frac{3+2\,i}{\sqrt{55}}\\ \frac{2+2\,i}{\sqrt{55}}\\ \frac{3-5\,i}{\sqrt{55}}\end{bmatrix},\ \begin{bmatrix}\frac{2+2i}{\sqrt{22}}\\ \frac{-3+i}{\sqrt{22}}\\ -\frac{2}{\sqrt{22}}\end{bmatrix}\right\}$$

form an orthonormal set. You might find checking the six inner products of pairs of these vectors easier than doing the matrix product U^*U. Or, because the inner product is anti-commutative (Theorem IPAC) you only need check three inner products (see Exercise MINM.T12). △

When using vectors and matrices that only have real number entries, orthogonal matrices are those matrices with inverses that equal their transpose. Similarly, the inner product is the familiar dot product. Keep this special case in mind as you read the next theorem.

Theorem UMPIP Unitary Matrices Preserve Inner Products
Suppose that U is a unitary matrix of size n and $\mathbf{u}$ and $\mathbf{v}$ are two vectors from $\mathbb{C}^n$. Then

$$\langle U\mathbf{u},\, U\mathbf{v}\rangle = \langle \mathbf{u},\, \mathbf{v}\rangle \qquad \text{and} \qquad \left\|U\mathbf{v}\right\| = \left\|\mathbf{v}\right\|$$

Proof.

$$\begin{aligned}
\langle U\mathbf{u},\, U\mathbf{v}\rangle &= \left(\overline{U\mathbf{u}}\right)^t U\mathbf{v} && \text{Theorem MMIP}\\
&= \left(\overline{U}\overline{\mathbf{u}}\right)^t U\mathbf{v} && \text{Theorem MMCC}\\
&= \overline{\mathbf{u}}^t\overline{U}^t U\mathbf{v} && \text{Theorem MMT}\\
&= \overline{\mathbf{u}}^t U^* U\mathbf{v} && \text{Definition A}\\
&= \overline{\mathbf{u}}^t I_n\mathbf{v} && \text{Definition UM}\\
&= \overline{\mathbf{u}}^t\mathbf{v} && \text{Theorem MMIM}\\
&= \langle \mathbf{u},\, \mathbf{v}\rangle && \text{Theorem MMIP}
\end{aligned}$$

The second conclusion is just a specialization of the first conclusion.

$$\begin{aligned}
\left\|U\mathbf{v}\right\| &= \sqrt{\left\|U\mathbf{v}\right\|^2}\\
&= \sqrt{\langle U\mathbf{v},\, U\mathbf{v}\rangle} && \text{Theorem IPN}\\
&= \sqrt{\langle \mathbf{v},\, \mathbf{v}\rangle}\\
&= \sqrt{\left\|\mathbf{v}\right\|^2} && \text{Theorem IPN}\\
&= \left\|\mathbf{v}\right\|
\end{aligned}$$

■

Aside from the inherent interest in this theorem, it makes a bigger statement about unitary matrices. When we view vectors geometrically as directions or forces, then the norm equates to a notion of length. If we transform a vector by multiplication with a unitary matrix, then the length (norm) of that vector stays the same. If we consider column vectors with two or three slots containing only real numbers, then the inner product of two such vectors is just the dot product, and this quantity can be used to compute the angle between two vectors. When two vectors are multiplied (transformed) by the same unitary matrix, their dot product is unchanged and their individual lengths are unchanged. This results in the angle between the two vectors remaining unchanged.

A "unitary transformation" (matrix-vector products with unitary matrices) thus preserve geometrical relationships among vectors representing directions, forces, or other physical quantities. In the case of a two-slot vector with real entries, this is simply a rotation. These sorts of computations are exceedingly important in computer graphics such as games and real-time simulations, especially when increased realism is achieved by performing many such computations quickly. We will see unitary matrices again in subsequent sections (especially Theorem OD) and in each instance, consider the interpretation of the unitary matrix as a sort of geometry-preserving transformation. Some authors use the term **isometry** to highlight this behavior. We will speak loosely of a unitary matrix as being a sort of generalized rotation.

A final reminder: the terms "dot product," "symmetric matrix" and "orthogonal matrix" used in reference to vectors or matrices with real number entries are special cases of the terms "inner product," "Hermitian matrix" and "unitary matrix" that we use for vectors or matrices with complex number entries, so keep that in mind as you read elsewhere.

Reading Questions

1. Compute the inverse of the coefficient matrix of the system of equations below and use the inverse to solve the system.

$$\begin{aligned} 4x_1 + 10x_2 &= 12 \\ 2x_1 + 6x_2 &= 4 \end{aligned}$$

2. In the reading questions for Section MISLE you were asked to find the inverse of the 3×3 matrix below.

$$\begin{bmatrix} 2 & 3 & 1 \\ 1 & -2 & -3 \\ -2 & 4 & 6 \end{bmatrix}$$

Because the matrix was not nonsingular, you had no theorems at that point that would allow you to compute the inverse. Explain why you now know that the inverse does not exist (which is different than not being able to compute it) by quoting the relevant theorem's acronym.

3. Is the matrix A unitary? Why?

$$A = \begin{bmatrix} \frac{1}{\sqrt{22}}(4+2i) & \frac{1}{\sqrt{374}}(5+3i) \\ \frac{1}{\sqrt{22}}(-1-i) & \frac{1}{\sqrt{374}}(12+14i) \end{bmatrix}$$

Exercises

C20 Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. Verify that AB is nonsingular.

C40† Solve the system of equations below using the inverse of a matrix.

$$\begin{aligned} x_1 + x_2 + 3x_3 + x_4 &= 5 \\ -2x_1 - x_2 - 4x_3 - x_4 &= -7 \\ x_1 + 4x_2 + 10x_3 + 2x_4 &= 9 \\ -2x_1 - 4x_3 + 5x_4 &= 9 \end{aligned}$$

M10† Find values of x, y, z so that matrix $A = \begin{bmatrix} 1 & 2 & x \\ 3 & 0 & y \\ 1 & 1 & z \end{bmatrix}$ is invertible.

M11† Find values of x, y z so that matrix $A = \begin{bmatrix} 1 & x & 1 \\ 1 & y & 4 \\ 0 & z & 5 \end{bmatrix}$ is singular.

M15† If A and B are $n \times n$ matrices, A is nonsingular, and B is singular, show directly that AB is singular, without using Theorem NPNT.

M20† Construct an example of a 4×4 unitary matrix.

M80† Matrix multiplication interacts nicely with many operations. But not always with transforming a matrix to reduced row-echelon form. Suppose that A is an $m \times n$ matrix and B is an $n \times p$ matrix. Let P be a matrix that is row-equivalent to A and in reduced row-echelon form, Q be a matrix that is row-equivalent to B and in reduced row-echelon form, and let R be a matrix that is row-equivalent to AB and in reduced row-echelon form.

Is $PQ = R$? (In other words, with nonstandard notation, is $\text{rref}(A)\text{rref}(B) = \text{rref}(AB)$?)

Construct a counterexample to show that, in general, this statement is false. Then find a large class of matrices where if A and B are in the class, then the statement is true.

T10 Suppose that Q and P are unitary matrices of size n. Prove that QP is a unitary matrix.

T11 Prove that Hermitian matrices (Definition HM) have real entries on the diagonal. More precisely, suppose that A is a Hermitian matrix of size n. Then $[A]_{ii} \in \mathbb{R}$, $1 \leq i \leq n$.

T12 Suppose that we are checking if a square matrix of size n is unitary. Show that a straightforward application of Theorem CUMOS requires the computation of n^2 inner products when the matrix is unitary, and fewer when the matrix is not orthogonal. Then show that this maximum number of inner products can be reduced to $\frac{1}{2}n(n+1)$ in light of Theorem IPAC.

T25 The notation A^k means a repeated matrix product between k copies of the square matrix A.

1. Assume A is an $n \times n$ matrix where $A^2 = \mathcal{O}$ (which does not imply that $A = \mathcal{O}$.) Prove that $I_n - A$ is invertible by showing that $I_n + A$ is an inverse of $I_n - A$.

2. Assume that A is an $n \times n$ matrix where $A^3 = \mathcal{O}$. Prove that $I_n - A$ is invertible.

3. Form a general theorem based on your observations from parts (1) and (2) and provide a proof.

Section CRS
Column and Row Spaces

A matrix-vector product (Definition MVP) is a linear combination of the columns of the matrix and this allows us to connect matrix multiplication with systems of equations via Theorem SLSLC. Row operations are linear combinations of the rows of a matrix, and of course, reduced row-echelon form (Definition RREF) is also intimately related to solving systems of equations. In this section we will formalize these ideas with two key definitions of sets of vectors derived from a matrix.

Subsection CSSE
Column Spaces and Systems of Equations

Theorem SLSLC showed us that there is a natural correspondence between solutions to linear systems and linear combinations of the columns of the coefficient matrix. This idea motivates the following important definition.

Definition CSM Column Space of a Matrix
Suppose that A is an $m \times n$ matrix with columns $\mathbf{A}_1,\,\mathbf{A}_2,\,\mathbf{A}_3,\,\dots,\,\mathbf{A}_n$. Then the **column space** of A, written $\mathcal{C}(A)$, is the subset of $\mathbb{C}^m$ containing all linear combinations of the columns of A,

$$\mathcal{C}(A) = \langle\{\mathbf{A}_1,\,\mathbf{A}_2,\,\mathbf{A}_3,\,\dots,\,\mathbf{A}_n\}\rangle$$

□

Some authors refer to the column space of a matrix as the **range**, but we will reserve this term for use with linear transformations (Definition RLT).

Upon encountering any new set, the first question we ask is what objects are in the set, and which objects are not? Here is an example of one way to answer this question, and it will motivate a theorem that will then answer the question precisely.

Example CSMCS Column space of a matrix and consistent systems
Archetype D and Archetype E are linear systems of equations, with an identical 3×4 coefficient matrix, which we call A here. However, Archetype D is consistent, while Archetype E is not. We can explain this difference by employing the column space of the matrix A.

The column vector of constants, $\mathbf{b}$, in Archetype D is given below, and one solution listed for $\mathcal{LS}(A,\,\mathbf{b})$ is $\mathbf{x}$,

$$\mathbf{b} = \begin{bmatrix} 8 \\ -12 \\ 4 \end{bmatrix} \qquad\qquad \mathbf{x} = \begin{bmatrix} 7 \\ 8 \\ 1 \\ 3 \end{bmatrix}$$

By Theorem SLSLC, we can summarize this solution as a linear combination of the columns of A that equals $\mathbf{b}$,

$$7\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + 8\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 7 \\ -5 \\ 4 \end{bmatrix} + 3\begin{bmatrix} -7 \\ -6 \\ -5 \end{bmatrix} = \begin{bmatrix} 8 \\ -12 \\ 4 \end{bmatrix} = \mathbf{b}.$$

This equation says that $\mathbf{b}$ is a linear combination of the columns of A, and then by Definition CSM, we can say that $\mathbf{b} \in \mathcal{C}(A)$.

On the other hand, Archetype E is the linear system $\mathcal{LS}(A,\,\mathbf{c})$, where the vector of constants is

$$\mathbf{c} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$$

and this system of equations is inconsistent. This means $\mathbf{c} \not\in \mathcal{C}(A)$, for if it were, then it would equal a linear combination of the columns of A and Theorem SLSLC would lead us to a solution of the system $\mathcal{LS}(A,\,\mathbf{c})$. △

So if we fix the coefficient matrix, and vary the vector of constants, we can sometimes find consistent systems, and sometimes inconsistent systems. The vectors of constants that lead to consistent systems are exactly the elements of the column space. This is the content of the next theorem, and since it is an equivalence, it provides an alternate view of the column space.

Theorem CSCS Column Spaces and Consistent Systems
Suppose A is an $m \times n$ matrix and $\mathbf{b}$ is a vector of size m. Then $\mathbf{b} \in \mathcal{C}(A)$ if and only if $\mathcal{LS}(A,\,\mathbf{b})$ is consistent.

Proof. ($\Rightarrow$) Suppose $\mathbf{b} \in \mathcal{C}(A)$. Then we can write $\mathbf{b}$ as some linear combination of the columns of A. By Theorem SLSLC we can use the scalars from this linear combination to form a solution to $\mathcal{LS}(A,\,\mathbf{b})$, so this system is consistent.

($\Leftarrow$) If $\mathcal{LS}(A,\,\mathbf{b})$ is consistent, there is a solution that may be used with Theorem SLSLC to write $\mathbf{b}$ as a linear combination of the columns of A. This qualifies $\mathbf{b}$ for membership in $\mathcal{C}(A)$. ■

This theorem tells us that asking if the system $\mathcal{LS}(A,\,\mathbf{b})$ is consistent is exactly the same question as asking if $\mathbf{b}$ is in the column space of A. Or equivalently, it tells us that the column space of the matrix A is precisely those vectors of constants, $\mathbf{b}$, that can be paired with A to create a system of linear equations $\mathcal{LS}(A,\,\mathbf{b})$ that is consistent.

Employing Theorem SLEMM we can form the chain of equivalences

$$\mathbf{b} \in \mathcal{C}(A) \iff \mathcal{LS}(A,\,\mathbf{b}) \text{ is consistent} \iff A\mathbf{x} = \mathbf{b} \text{ for some } \mathbf{x}$$

Thus, an alternative (and popular) definition of the column space of an $m \times n$ matrix A is

$$\mathcal{C}(A) = \left\{\mathbf{y} \in \mathbb{C}^m \,\middle|\, \mathbf{y} = A\mathbf{x} \text{ for some } \mathbf{x} \in \mathbb{C}^n\right\} = \left\{A\mathbf{x} \,\middle|\, \mathbf{x} \in \mathbb{C}^n\right\} \subseteq \mathbb{C}^m$$

We recognize this as saying create *all* the matrix vector products possible with the matrix A by letting $\mathbf{x}$ range over all of the possibilities. By Definition MVP we see that this means take all possible linear combinations of the columns of A — precisely the definition of the column space (Definition CSM) we have chosen.

Notice how this formulation of the column space looks very much like the definition of the null space of a matrix (Definition NSM), but for a rectangular matrix the column vectors of $\mathcal{C}(A)$ and $\mathcal{N}(A)$ have different sizes, so the sets are very different.

Given a vector $\mathbf{b}$ and a matrix A it is now very mechanical to test if $\mathbf{b} \in \mathcal{C}(A)$. Form the linear system $\mathcal{LS}(A,\,\mathbf{b})$, row-reduce the augmented matrix, $[A \,|\, \mathbf{b}]$, and test for consistency with Theorem RCLS. Here is an example of this procedure.

Example MCSM Membership in the column space of a matrix
Consider the column space of the 3×4 matrix A,

$$A = \begin{bmatrix} 3 & 2 & 1 & -4 \\ -1 & 1 & -2 & 3 \\ 2 & -4 & 6 & -8 \end{bmatrix}$$

We first show that $\mathbf{v} = \begin{bmatrix} 18 \\ -6 \\ 12 \end{bmatrix}$ is in the column space of A, $\mathbf{v} \in \mathcal{C}(A)$. Theorem CSCS says we need only check the consistency of $\mathcal{LS}(A,\,\mathbf{v})$. Form the augmented matrix and row-reduce,

$$\begin{bmatrix} 3 & 2 & 1 & -4 & 18 \\ -1 & 1 & -2 & 3 & -6 \\ 2 & -4 & 6 & -8 & 12 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 1 & -2 & 6 \\ 0 & \boxed{1} & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the final column is not a pivot column, Theorem RCLS tells us the system is consistent and therefore by Theorem CSCS, $\mathbf{v} \in \mathcal{C}(A)$.

If we wished to demonstrate explicitly that $\mathbf{v}$ is a linear combination of the columns of A, we can find a solution (any solution) of $\mathcal{LS}(A, \mathbf{v})$ and use Theorem SLSLC to construct the desired linear combination. For example, set the free variables to $x_3 = 2$ and $x_4 = 1$. Then a solution has $x_2 = 1$ and $x_1 = 6$. Then by Theorem SLSLC,

$$\mathbf{v} = \begin{bmatrix} 18 \\ -6 \\ 12 \end{bmatrix} = 6 \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ -2 \\ 6 \end{bmatrix} + 1 \begin{bmatrix} -4 \\ 3 \\ -8 \end{bmatrix}$$

Now we show that $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$ is not in the column space of A, $\mathbf{w} \notin \mathcal{C}(A)$. Theorem CSCS says we need only check the consistency of $\mathcal{LS}(A, \mathbf{w})$. Form the augmented matrix and row-reduce,

$$\begin{bmatrix} 3 & 2 & 1 & -4 & 2 \\ -1 & 1 & -2 & 3 & 1 \\ 2 & -4 & 6 & -8 & -3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 1 & -2 & 0 \\ 0 & \boxed{1} & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

since the final column is a pivot column, Theorem RCLS tells us the system is inconsistent and therefore by Theorem CSCS, $\mathbf{w} \notin \mathcal{C}(A)$. △

Theorem CSCS completes a collection of three theorems, and one definition, that deserve comment. Many questions about spans, linear independence, null space, column spaces and similar objects can be converted to questions about systems of equations (homogeneous or not), which we understand well from our previous results, especially those in Chapter SLE. These previous results include theorems like Theorem RCLS which allows us to quickly decide consistency of a system, and Theorem BNS which allows us to describe solution sets for homogeneous systems compactly as the span of a linearly independent set of column vectors.

The table below lists these four definitions and theorems along with a brief reminder of the statement and an example of how the statement is used.

Definition NSM	
Synopsis	Null space is solution set of homogeneous system
Example	General solution sets described by Theorem PSPHS
Theorem SLSLC	
Synopsis	Solutions for linear combinations with unknown scalars
Example	Deciding membership in spans
Theorem SLEMM	
Synopsis	System of equations represented by matrix-vector product
Example	Solution to $\mathcal{LS}(A, \mathbf{b})$ is $A^{-1}\mathbf{b}$ when A is nonsingular
Theorem CSCS	
Synopsis	Column space vectors create consistent systems
Example	Deciding membership in column spaces

Subsection CSSOC
Column Space Spanned by Original Columns

So we have a foolproof, automated procedure for determining membership in $\mathcal{C}(A)$. While this works just fine a vector at a time, we would like to have a more useful description of the set $\mathcal{C}(A)$ as a whole. The next example will preview the first of two fundamental results about the column space of a matrix.

Example CSTW Column space, two ways
Consider the 5×7 matrix A,

$$\begin{bmatrix} 2 & 4 & 1 & -1 & 1 & 4 & 4 \\ 1 & 2 & 1 & 0 & 2 & 4 & 7 \\ 0 & 0 & 1 & 4 & 1 & 8 & 7 \\ 1 & 2 & -1 & 2 & 1 & 9 & 6 \\ -2 & -4 & 1 & 3 & -1 & -2 & -2 \end{bmatrix}$$

According to the definition (Definition CSM), the column space of A is

$$\mathcal{C}(A) = \left\langle \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 0 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 4 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 8 \\ 9 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ 7 \\ 7 \\ 6 \\ -2 \end{bmatrix} \right\} \right\rangle$$

While this is a concise description of an infinite set, we might be able to describe the span with fewer than seven vectors. This is the substance of Theorem BS. So we take these seven vectors and make them the columns of a matrix, which is simply the original matrix A again. Now we row-reduce,

$$\begin{bmatrix} 2 & 4 & 1 & -1 & 1 & 4 & 4 \\ 1 & 2 & 1 & 0 & 2 & 4 & 7 \\ 0 & 0 & 1 & 4 & 1 & 8 & 7 \\ 1 & 2 & -1 & 2 & 1 & 9 & 6 \\ -2 & -4 & 1 & 3 & -1 & -2 & -2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 2 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & \boxed{1} & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & \boxed{1} & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivot columns are $D = \{1,\,3,\,4,\,5\}$, so we can create the set

$$T = \left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 4 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}$$

and know that $\mathcal{C}(A) = \langle T \rangle$ and T is a linearly independent set of columns from the set of columns of A. △

We will now formalize the previous example, which will make it trivial to determine a linearly independent set of vectors that will span the column space of a matrix, and is constituted of just columns of A.

Theorem BCS Basis of the Column Space
Suppose that A is an $m \times n$ matrix with columns $\mathbf{A}_1,\,\mathbf{A}_2,\,\mathbf{A}_3,\,\ldots,\,\mathbf{A}_n$, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Let $D = \{d_1,\,d_2,\,d_3,\,\ldots,\,d_r\}$ be the set of indices for the pivot columns of B Let $T = \{\mathbf{A}_{d_1},\,\mathbf{A}_{d_2},\,\mathbf{A}_{d_3},\,\ldots,\,\mathbf{A}_{d_r}\}$. Then

1. *T is a linearly independent set.*
2. *$\mathcal{C}(A) = \langle T \rangle$.*

Proof. Definition CSM describes the column space as the span of the set of columns of A. Theorem BS tells us that we can reduce the set of vectors used in a span. If we apply Theorem BS to $\mathcal{C}(A)$, we would collect the columns of A into a matrix (which would just be A again) and bring the matrix to reduced row-echelon form, which is the matrix B in the statement of the theorem. In this case, the conclusions of Theorem BS applied to A, B and $\mathcal{C}(A)$ are exactly the conclusions we desire. ∎

This is a nice result since it gives us a handful of vectors that describe the entire column space (through the span), and we believe this set is as small as possible because we cannot create any more relations of linear dependence to trim it down further. Furthermore, we defined the column space (Definition CSM) as all linear combinations of the columns of the matrix, and the elements of the set T are still columns of the matrix (we will not be so lucky in the next two constructions of the column space).

Procedurally this theorem is extremely easy to apply. Row-reduce the original matrix, identify r pivot columns the reduced matrix, and grab the columns of the original matrix with the same indices as the pivot columns. But it is still important to study the proof of Theorem BS and its motivation in Example COV which lie at the root of this theorem. We will trot through an example all the same.

Example CSOCD Column space, original columns, Archetype D
Let us determine a compact expression for the entire column space of the coefficient matrix of the system of equations that is Archetype D. Notice that in Example CSMCS we were only determining if individual vectors were in the column space or not, now we are describing the entire column space.

To start with the application of Theorem BCS, call the coefficient matrix A and row-reduce it to reduced row-echelon form B,

$$A = \begin{bmatrix} 2 & 1 & 7 & -7 \\ -3 & 4 & -5 & -6 \\ 1 & 1 & 4 & -5 \end{bmatrix} \qquad B = \begin{bmatrix} \boxed{1} & 0 & 3 & -2 \\ 0 & \boxed{1} & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since columns 1 and 2 are pivot columns, $D = \{1, 2\}$. To construct a set that spans $\mathcal{C}(A)$, just grab the columns of A with indices in D, so

$$\mathcal{C}(A) = \left\langle \left\{ \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} \right\} \right\rangle.$$

That's it.

In Example CSMCS we determined that the vector

$$\mathbf{c} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$$

was not in the column space of A. Try to write $\mathbf{c}$ as a linear combination of the first two columns of A. What happens?

Also in Example CSMCS we determined that the vector

$$\mathbf{b} = \begin{bmatrix} 8 \\ -12 \\ 4 \end{bmatrix}$$

was in the column space of A. Try to write $\mathbf{b}$ as a linear combination of the first two columns of A. What happens? Did you find a unique solution to this question? Hmmmm. △

Subsection CSNM
Column Space of a Nonsingular Matrix

Let us specialize to square matrices and contrast the column spaces of the coefficient matrices in Archetype A and Archetype B.

Example CSAA Column space of Archetype A
The coefficient matrix in Archetype A is A, which row-reduces to B,

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Columns 1 and 2 are pivot columns, so by Theorem BCS we can write

$$\mathcal{C}(A) = \langle\{\mathbf{A}_1,\,\mathbf{A}_2\}\rangle = \left\langle\left\{\begin{bmatrix}1\\2\\1\end{bmatrix},\,\begin{bmatrix}-1\\1\\1\end{bmatrix}\right\}\right\rangle.$$

We want to show in this example that $\mathcal{C}(A) \neq \mathbb{C}^3$. So take, for example, the vector $\mathbf{b} = \begin{bmatrix}1\\3\\2\end{bmatrix}$. Then there is no solution to the system $\mathcal{LS}(A,\,\mathbf{b})$, or equivalently, it is not possible to write $\mathbf{b}$ as a linear combination of $\mathbf{A}_1$ and $\mathbf{A}_2$. Try one of these two computations yourself. (Or try both!). Since $\mathbf{b} \notin \mathcal{C}(A)$, the column space of A cannot be all of $\mathbb{C}^3$. So by varying the vector of constants, it is possible to create inconsistent systems of equations with this coefficient matrix (the vector $\mathbf{b}$ being one such example).

In Example MWIAA we wished to show that the coefficient matrix from Archetype A was not invertible as a first example of a matrix without an inverse. Our device there was to find an inconsistent linear system with A as the coefficient matrix. The vector of constants in that example was $\mathbf{b}$, deliberately chosen outside the column space of A. △

Example CSAB Column space of Archetype B
The coefficient matrix in Archetype B, call it B here, is known to be nonsingular (see Example NM). By Theorem NMUS, the linear system $\mathcal{LS}(B,\,\mathbf{b})$ has a (unique) solution for every choice of $\mathbf{b}$. Theorem CSCS then says that $\mathbf{b} \in \mathcal{C}(B)$ for all $\mathbf{b} \in \mathbb{C}^3$. Stated differently, there is no way to build an inconsistent system with the coefficient matrix B, but then we knew that already from Theorem NMUS. △

Example CSAA and Example CSAB together motivate the following equivalence, which says that nonsingular matrices have column spaces that are as big as possible.

Theorem CSNM Column Space of a Nonsingular Matrix
Suppose A is a square matrix of size n. Then A is nonsingular if and only if $\mathcal{C}(A) = \mathbb{C}^n$.

Proof. ($\Rightarrow$) Suppose A is nonsingular. We wish to establish the set equality $\mathcal{C}(A) = \mathbb{C}^n$. By Definition CSM, $\mathcal{C}(A) \subseteq \mathbb{C}^n$. To show that $\mathbb{C}^n \subseteq \mathcal{C}(A)$ choose $\mathbf{b} \in \mathbb{C}^n$. By Theorem NMUS, we know the linear system $\mathcal{LS}(A,\,\mathbf{b})$ has a (unique) solution and therefore is consistent. Theorem CSCS then says that $\mathbf{b} \in \mathcal{C}(A)$. So by Definition SE, $\mathcal{C}(A) = \mathbb{C}^n$.

($\Leftarrow$) If $\mathbf{e}_i$ is column i of the $n \times n$ identity matrix (Definition SUV) and by hypothesis $\mathcal{C}(A) = \mathbb{C}^n$, then $\mathbf{e}_i \in \mathcal{C}(A)$ for $1 \leq i \leq n$. By Theorem CSCS, the system $\mathcal{LS}(A,\,\mathbf{e}_i)$ is consistent for $1 \leq i \leq n$. Let $\mathbf{b}_i$ denote any one particular solution to $\mathcal{LS}(A,\,\mathbf{e}_i)$, $1 \leq i \leq n$.

Define the $n \times n$ matrix $B = [\mathbf{b}_1|\mathbf{b}_2|\mathbf{b}_3|\ldots|\mathbf{b}_n]$. Then

$$\begin{aligned} AB &= A\,[\mathbf{b}_1|\mathbf{b}_2|\mathbf{b}_3|\ldots|\mathbf{b}_n] \\ &= [A\mathbf{b}_1|A\mathbf{b}_2|A\mathbf{b}_3|\ldots|A\mathbf{b}_n] && \text{Definition MM} \\ &= [\mathbf{e}_1|\mathbf{e}_2|\mathbf{e}_3|\ldots|\mathbf{e}_n] \\ &= I_n && \text{Definition SUV} \end{aligned}$$

So the matrix B is a "right-inverse" for A. By Theorem NMRRI, I_n is a nonsingular matrix, so by Theorem NPNT both A and B are nonsingular. Thus, in particular, A is nonsingular. (Travis Osborne contributed to this proof.) ■

With this equivalence for nonsingular matrices we can update our list, Theorem NME3.

Theorem NME4 Nonsingular Matrix Equivalences, Round 4
Suppose that A is a square matrix of size n. The following are equivalent.

1. *A is nonsingular.*
2. *A row-reduces to the identity matrix.*
3. *The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.*
4. *The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.*
5. *The columns of A are a linearly independent set.*
6. *A is invertible.*
7. *The column space of A is $\mathbb{C}^n$, $\mathcal{C}(A) = \mathbb{C}^n$.*

Proof. Since Theorem CSNM is an equivalence, we can add it to the list in Theorem NME3. ■

Subsection RSM
Row Space of a Matrix

The rows of a matrix can be viewed as vectors, since they are just lists of numbers, arranged horizontally. So we will transpose a matrix, turning rows into columns, so we can then manipulate rows as column vectors. As a result we will be able to make some new connections between row operations and solutions to systems of equations. OK, here is the second primary definition of this section.

Definition RSM Row Space of a Matrix
Suppose A is an $m \times n$ matrix. Then the **row space** of A, $\mathcal{R}(A)$, is the column space of A^t, i.e. $\mathcal{R}(A) = \mathcal{C}(A^t)$. □

Informally, the row space is the set of all linear combinations of the rows of A. However, we write the rows as column vectors, thus the necessity of using the transpose to make the rows into columns. Additionally, with the row space defined in terms of the column space, all of the previous results of this section can be applied to row spaces.

Notice that if A is a rectangular $m \times n$ matrix, then $\mathcal{C}(A) \subseteq \mathbb{C}^m$, while $\mathcal{R}(A) \subseteq \mathbb{C}^n$ and the two sets are not comparable since they do not even hold objects of the same type. However, when A is square of size n, both $\mathcal{C}(A)$ and $\mathcal{R}(A)$ are subsets of $\mathbb{C}^n$, though usually the sets will not be equal (but see Exercise CRS.M20).

Example RSAI Row space of Archetype I
The coefficient matrix in Archetype I is

$$I = \begin{bmatrix} 1 & 4 & 0 & -1 & 0 & 7 & -9 \\ 2 & 8 & -1 & 3 & 9 & -13 & 7 \\ 0 & 0 & 2 & -3 & -4 & 12 & -8 \\ -1 & -4 & 2 & 4 & 8 & -31 & 37 \end{bmatrix}$$

To build the row space, we transpose the matrix,

$$I^t = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 4 & 8 & 0 & -4 \\ 0 & -1 & 2 & 2 \\ -1 & 3 & -3 & 4 \\ 0 & 9 & -4 & 8 \\ 7 & -13 & 12 & -31 \\ -9 & 7 & -8 & 37 \end{bmatrix}$$

Then the columns of this matrix are used in a span to build the row space,

$$\mathcal{R}(I) = \mathcal{C}\left(I^t\right) = \left\langle \left\{ \begin{bmatrix} 1 \\ 4 \\ 0 \\ -1 \\ 0 \\ 7 \\ -9 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \\ -1 \\ 3 \\ 9 \\ -13 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ -3 \\ -4 \\ 12 \\ -8 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ 2 \\ 4 \\ 8 \\ -31 \\ 37 \end{bmatrix} \right\} \right\rangle.$$

However, we can use Theorem BCS to get a slightly better description. First, row-reduce I^t,

$$\begin{bmatrix} \boxed{1} & 0 & 0 & -\frac{31}{7} \\ 0 & \boxed{1} & 0 & \frac{12}{7} \\ 0 & 0 & \boxed{1} & \frac{13}{7} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the pivot columns have indices $D = \{1,\,2,\,3\}$, the column space of I^t can be spanned by just the first three columns of I^t,

$$\mathcal{R}(I) = \mathcal{C}\left(I^t\right) = \left\langle \left\{ \begin{bmatrix} 1 \\ 4 \\ 0 \\ -1 \\ 0 \\ 7 \\ -9 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \\ -1 \\ 3 \\ 9 \\ -13 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ -3 \\ -4 \\ 12 \\ -8 \end{bmatrix} \right\} \right\rangle.$$

△

The row space would not be too interesting if it was simply the column space of the transpose. However, when we do row operations on a matrix we have no effect on the many linear combinations that can be formed with the rows of the matrix. This is stated more carefully in the following theorem.

Theorem REMRS Row-Equivalent Matrices have equal Row Spaces
Suppose A and B are row-equivalent matrices. Then $\mathcal{R}(A) = \mathcal{R}(B)$.

Proof. Two matrices are row-equivalent (Definition REM) if one can be obtained from another by a sequence of (possibly many) row operations. We will prove the theorem for two matrices that differ by a single row operation, and then this result can be applied repeatedly to get the full statement of the theorem. The row spaces of A and B are spans of the columns of their transposes. For each row operation we perform on a matrix, we can define an analogous operation on the columns. Perhaps we should call these **column operations**. Instead, we will still call them row operations, but we will apply them to the columns of the transposes.

Refer to the columns of A^t and B^t as $\mathbf{A}_i$ and $\mathbf{B}_i$, $1 \leq i \leq m$. The row operation that switches rows will just switch columns of the transposed matrices. This will have no effect on the possible linear combinations formed by the columns.

Suppose that B^t is formed from A^t by multiplying column $\mathbf{A}_t$ by $\alpha \neq 0$. In other words, $\mathbf{B}_t = \alpha\mathbf{A}_t$, and $\mathbf{B}_i = \mathbf{A}_i$ for all $i \neq t$. We need to establish that two sets are equal, $\mathcal{C}(A^t) = \mathcal{C}(B^t)$. We will take a generic element of one and show that it is contained in the other.

$$\begin{aligned}\beta_1\mathbf{B}_1+&\beta_2\mathbf{B}_2 + \beta_3\mathbf{B}_3 + \cdots + \beta_t\mathbf{B}_t + \cdots + \beta_m\mathbf{B}_m\\ &= \beta_1\mathbf{A}_1 + \beta_2\mathbf{A}_2 + \beta_3\mathbf{A}_3 + \cdots + \beta_t\left(\alpha\mathbf{A}_t\right) + \cdots + \beta_m\mathbf{A}_m\\ &= \beta_1\mathbf{A}_1 + \beta_2\mathbf{A}_2 + \beta_3\mathbf{A}_3 + \cdots + \left(\alpha\beta_t\right)\mathbf{A}_t + \cdots + \beta_m\mathbf{A}_m\end{aligned}$$

says that $\mathcal{C}(B^t) \subseteq \mathcal{C}(A^t)$. Similarly,

$$\begin{aligned}\gamma_1\mathbf{A}_1+&\gamma_2\mathbf{A}_2 + \gamma_3\mathbf{A}_3 + \cdots + \gamma_t\mathbf{A}_t + \cdots + \gamma_m\mathbf{A}_m\\ &= \gamma_1\mathbf{A}_1 + \gamma_2\mathbf{A}_2 + \gamma_3\mathbf{A}_3 + \cdots + \left(\frac{\gamma_t}{\alpha}\alpha\right)\mathbf{A}_t + \cdots + \gamma_m\mathbf{A}_m\\ &= \gamma_1\mathbf{A}_1 + \gamma_2\mathbf{A}_2 + \gamma_3\mathbf{A}_3 + \cdots + \frac{\gamma_t}{\alpha}\left(\alpha\mathbf{A}_t\right) + \cdots + \gamma_m\mathbf{A}_m\\ &= \gamma_1\mathbf{B}_1 + \gamma_2\mathbf{B}_2 + \gamma_3\mathbf{B}_3 + \cdots + \frac{\gamma_t}{\alpha}\mathbf{B}_t + \cdots + \gamma_m\mathbf{B}_m\end{aligned}$$

says that $\mathcal{C}(A^t) \subseteq \mathcal{C}(B^t)$. So $\mathcal{R}(A) = \mathcal{C}(A^t) = \mathcal{C}(B^t) = \mathcal{R}(B)$ when a single row operation of the second type is performed.

Suppose now that B^t is formed from A^t by replacing $\mathbf{A}_t$ with $\alpha\mathbf{A}_s + \mathbf{A}_t$ for some $\alpha \in \mathbb{C}$ and $s \neq t$. In other words, $\mathbf{B}_t = \alpha\mathbf{A}_s + \mathbf{A}_t$, and $\mathbf{B}_i = \mathbf{A}_i$ for $i \neq t$.

$$\begin{aligned}\beta_1\mathbf{B}_1+&\beta_2\mathbf{B}_2 + \cdots + \beta_s\mathbf{B}_s + \cdots + \beta_t\mathbf{B}_t + \cdots + \beta_m\mathbf{B}_m\\ &= \beta_1\mathbf{A}_1 + \beta_2\mathbf{A}_2 + \cdots + \beta_s\mathbf{A}_s + \cdots + \beta_t\left(\alpha\mathbf{A}_s + \mathbf{A}_t\right) + \cdots + \beta_m\mathbf{A}_m\\ &= \beta_1\mathbf{A}_1 + \beta_2\mathbf{A}_2 + \cdots + \beta_s\mathbf{A}_s + \cdots + \left(\beta_t\alpha\right)\mathbf{A}_s + \beta_t\mathbf{A}_t + \cdots + \beta_m\mathbf{A}_m\\ &= \beta_1\mathbf{A}_1 + \beta_2\mathbf{A}_2 + \cdots + \beta_s\mathbf{A}_s + \left(\beta_t\alpha\right)\mathbf{A}_s + \cdots + \beta_t\mathbf{A}_t + \cdots + \beta_m\mathbf{A}_m\\ &= \beta_1\mathbf{A}_1 + \beta_2\mathbf{A}_2 + \cdots + \left(\beta_s + \beta_t\alpha\right)\mathbf{A}_s + \cdots + \beta_t\mathbf{A}_t + \cdots + \beta_m\mathbf{A}_m\end{aligned}$$

says that $\mathcal{C}(B^t) \subseteq \mathcal{C}(A^t)$. Similarly,

$$\begin{aligned}\gamma_1\mathbf{A}_1 &+ \gamma_2\mathbf{A}_2 + \cdots + \gamma_s\mathbf{A}_s + \cdots + \gamma_t\mathbf{A}_t + \cdots + \gamma_m\mathbf{A}_m\\ &= \gamma_1\mathbf{A}_1 + \gamma_2\mathbf{A}_2 + \cdots + \gamma_s\mathbf{A}_s + \cdots + \left(-\alpha\gamma_t\mathbf{A}_s + \alpha\gamma_t\mathbf{A}_s\right) + \gamma_t\mathbf{A}_t + \cdots + \gamma_m\mathbf{A}_m\\ &= \gamma_1\mathbf{A}_1 + \gamma_2\mathbf{A}_2 + \cdots + \left(-\alpha\gamma_t + \gamma_s\right)\mathbf{A}_s + \cdots + \gamma_t\left(\alpha\mathbf{A}_s + \mathbf{A}_t\right) + \cdots + \gamma_m\mathbf{A}_m\\ &= \gamma_1\mathbf{B}_1 + \gamma_2\mathbf{B}_2 + \cdots + \left(-\alpha\gamma_t + \gamma_s\right)\mathbf{B}_s + \cdots + \gamma_t\mathbf{B}_t + \cdots + \gamma_m\mathbf{B}_m\end{aligned}$$

says that $\mathcal{C}(A^t) \subseteq \mathcal{C}(B^t)$. So $\mathcal{R}(A) = \mathcal{C}(A^t) = \mathcal{C}(B^t) = \mathcal{R}(B)$ when a single row operation of the third type is performed.

So the row space of a matrix is preserved by each row operation, and hence row spaces of row-equivalent matrices are equal sets. ■

Example RSREM Row spaces of two row-equivalent matrices
In Example TREM we saw that the matrices

$$A = \begin{bmatrix} 2 & -1 & 3 & 4\\ 5 & 2 & -2 & 3\\ 1 & 1 & 0 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 0 & 6\\ 3 & 0 & -2 & -9\\ 2 & -1 & 3 & 4 \end{bmatrix}$$

are row-equivalent by demonstrating a sequence of two row operations that converted A into B. Applying Theorem REMRS we can say

$$\mathcal{R}(A) = \left\langle\left\{\begin{bmatrix}2\\-1\\3\\4\end{bmatrix}, \begin{bmatrix}5\\2\\-2\\3\end{bmatrix}, \begin{bmatrix}1\\1\\0\\6\end{bmatrix}\right\}\right\rangle = \left\langle\left\{\begin{bmatrix}1\\1\\0\\6\end{bmatrix}, \begin{bmatrix}3\\0\\-2\\-9\end{bmatrix}, \begin{bmatrix}2\\-1\\3\\4\end{bmatrix}\right\}\right\rangle = \mathcal{R}(B)$$

Theorem REMRS is at its best when one of the row-equivalent matrices is in reduced row-echelon form. The vectors that are zero rows can be ignored. (Who needs the zero vector when building a span? See Exercise LI.T10.) The echelon pattern insures that the nonzero rows yield vectors that are linearly independent. Here is the theorem.

Theorem BRS Basis for the Row Space
Suppose that A is a matrix and B is a row-equivalent matrix in reduced row-echelon form. Let S be the set of nonzero columns of B^t. Then

1. $\mathcal{R}(A) = \langle S\rangle$.
2. *S is a linearly independent set.*

Proof. From Theorem REMRS we know that $\mathcal{R}(A) = \mathcal{R}(B)$. If B has any zero rows, these are columns of B^t that are the zero vector. We can safely toss out the zero vector in the span construction, since it can be recreated from the nonzero vectors by a linear combination where all the scalars are zero. So $\mathcal{R}(A) = \langle S\rangle$.

Suppose B has r nonzero rows and let $D = \{d_1, d_2, d_3, \ldots, d_r\}$ denote the indices of the pivot columns of B. Denote the r column vectors of B^t, the vectors in S, as $\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3, \ldots, \mathbf{B}_r$. To show that S is linearly independent, start with a relation of linear dependence

$$\alpha_1\mathbf{B}_1 + \alpha_2\mathbf{B}_2 + \alpha_3\mathbf{B}_3 + \cdots + \alpha_r\mathbf{B}_r = \mathbf{0}$$

Now consider this vector equality in location d_i. Since B is in reduced row-echelon form, the entries of column d_i of B are all zero, except for a leading 1 in row i. Thus, in B^t, row d_i is all zeros, excepting a 1 in column i. So, for $1 \leq i \leq r$,

$$\begin{aligned}
0 &= [\mathbf{0}]_{d_i} && \text{Definition ZCV}\\
&= [\alpha_1\mathbf{B}_1 + \alpha_2\mathbf{B}_2 + \alpha_3\mathbf{B}_3 + \cdots + \alpha_r\mathbf{B}_r]_{d_i} && \text{Definition RLDCV}\\
&= [\alpha_1\mathbf{B}_1]_{d_i} + [\alpha_2\mathbf{B}_2]_{d_i} + [\alpha_3\mathbf{B}_3]_{d_i} + \cdots + [\alpha_r\mathbf{B}_r]_{d_i} && \text{Definition MA}\\
&= \alpha_1[\mathbf{B}_1]_{d_i} + \alpha_2[\mathbf{B}_2]_{d_i} + \alpha_3[\mathbf{B}_3]_{d_i} + \cdots + \alpha_r[\mathbf{B}_r]_{d_i} && \text{Definition MSM}\\
&= \alpha_1(0) + \alpha_2(0) + \alpha_3(0) + \cdots + \alpha_i(1) + \cdots + \alpha_r(0) && \text{Definition RREF}\\
&= \alpha_i
\end{aligned}$$

So we conclude that $\alpha_i = 0$ for all $1 \leq i \leq r$, establishing the linear independence of S (Definition LICV). ∎

Example IAS Improving a span
Suppose in the course of analyzing a matrix (its column space, its null space, its ...) we encounter the following set of vectors, described by a span

$$X = \left\langle\left\{\begin{bmatrix}1\\2\\1\\6\\6\end{bmatrix}, \begin{bmatrix}3\\-1\\2\\-1\\6\end{bmatrix}, \begin{bmatrix}1\\-1\\0\\-1\\-2\end{bmatrix}, \begin{bmatrix}-3\\2\\-3\\6\\-10\end{bmatrix}\right\}\right\rangle$$

Let A be the matrix whose rows are the vectors in X, so by design $X = \mathcal{R}(A)$,

$$A = \begin{bmatrix}1 & 2 & 1 & 6 & 6\\3 & -1 & 2 & -1 & 6\\1 & -1 & 0 & -1 & -2\\-3 & 2 & -3 & 6 & -10\end{bmatrix}$$

Row-reduce A to form a row-equivalent matrix in reduced row-echelon form,

$$B = \begin{bmatrix}\boxed{1} & 0 & 0 & 2 & -1\\0 & \boxed{1} & 0 & 3 & 1\\0 & 0 & \boxed{1} & -2 & 5\\0 & 0 & 0 & 0 & 0\end{bmatrix}$$

Then Theorem BRS says we can grab the nonzero columns of B^t and write

$$X = \mathcal{R}(A) = \mathcal{R}(B) = \left\langle\left\{\begin{bmatrix}1\\0\\0\\2\\-1\end{bmatrix},\,\begin{bmatrix}0\\1\\0\\3\\1\end{bmatrix},\,\begin{bmatrix}0\\0\\1\\-2\\5\end{bmatrix}\right\}\right\rangle$$

These three vectors provide a much-improved description of X. There are fewer vectors, and the pattern of zeros and ones in the first three entries makes it easier to determine membership in X. △

Notice that in Example IAS all we had to do was row-reduce the right matrix and toss out a zero row. Next to row operations themselves, Theorem BRS *is probably the most powerful computational technique at your disposal* as it quickly provides a much improved description of a span, *any span* (row space, column space, ...).

Theorem BRS and the techniques of Example IAS will provide yet another description of the column space of a matrix. First we state a triviality as a theorem, so we can reference it later.

Theorem CSRST Column Space, Row Space, Transpose
Suppose A is a matrix. Then $\mathcal{C}(A) = \mathcal{R}(A^t)$.

Proof.

$$\begin{aligned}\mathcal{C}(A) &= \mathcal{C}\left(\left(A^t\right)^t\right) && \text{Theorem TT}\\ &= \mathcal{R}\left(A^t\right) && \text{Definition RSM}\end{aligned}$$

■

So to find another expression for the column space of a matrix, build its transpose, row-reduce it, toss out the zero rows, and convert the nonzero rows to column vectors to yield an improved set for the span construction. We will do Archetype I, then you do Archetype J.

Example CSROI Column space from row operations, Archetype I
To find the column space of the coefficient matrix of Archetype I, we proceed as follows. The matrix is

$$I = \begin{bmatrix}1 & 4 & 0 & -1 & 0 & 7 & -9\\ 2 & 8 & -1 & 3 & 9 & -13 & 7\\ 0 & 0 & 2 & -3 & -4 & 12 & -8\\ -1 & -4 & 2 & 4 & 8 & -31 & 37\end{bmatrix}$$

The transpose is

$$\begin{bmatrix}1 & 2 & 0 & -1\\ 4 & 8 & 0 & -4\\ 0 & -1 & 2 & 2\\ -1 & 3 & -3 & 4\\ 0 & 9 & -4 & 8\\ 7 & -13 & 12 & -31\\ -9 & 7 & -8 & 37\end{bmatrix}.$$

Row-reduced this becomes,

$$\begin{bmatrix}\boxed{1} & 0 & 0 & -\frac{31}{7}\\ 0 & \boxed{1} & 0 & \frac{12}{7}\\ 0 & 0 & \boxed{1} & \frac{13}{7}\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}.$$

Now, using Theorem CSRST and Theorem BRS

$$\mathcal{C}(I) = \mathcal{R}(I^t) = \left\langle \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -\frac{31}{7} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \frac{12}{7} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{13}{7} \end{bmatrix} \right\} \right\rangle.$$

This is a very nice description of the column space. Fewer vectors than the 7 involved in the definition, and the pattern of the zeros and ones in the first 3 slots can be used to advantage. For example, Archetype I is presented as a consistent system of equations with a vector of constants

$$\mathbf{b} = \begin{bmatrix} 3 \\ 9 \\ 1 \\ 4 \end{bmatrix}.$$

Since $\mathcal{LS}(I,\,\mathbf{b})$ is consistent, Theorem CSCS tells us that $\mathbf{b} \in \mathcal{C}(I)$. But we could see this quickly with the following computation, which really only involves any work in the 4th entry of the vectors as the scalars in the linear combination are *dictated* by the first three entries of $\mathbf{b}$.

$$\mathbf{b} = \begin{bmatrix} 3 \\ 9 \\ 1 \\ 4 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \\ 0 \\ -\frac{31}{7} \end{bmatrix} + 9 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \frac{12}{7} \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{13}{7} \end{bmatrix}$$

Can you now rapidly construct several vectors, $\mathbf{b}$, so that $\mathcal{LS}(I,\,\mathbf{b})$ is consistent, and several more so that the system is inconsistent? △

Reading Questions

1. Write the column space of the matrix below as the span of a set of three vectors and explain your choice of method.

$$\begin{bmatrix} 1 & 3 & 1 & 3 \\ 2 & 0 & 1 & 1 \\ -1 & 2 & 1 & 0 \end{bmatrix}$$

2. Suppose that A is an $n \times n$ nonsingular matrix. What can you say about its column space?

3. Is the vector $\begin{bmatrix} 0 \\ 5 \\ 2 \\ 3 \end{bmatrix}$ in the row space of the following matrix? Why or why not?

$$\begin{bmatrix} 1 & 3 & 1 & 3 \\ 2 & 0 & 1 & 1 \\ -1 & 2 & 1 & 0 \end{bmatrix}$$

Exercises

C20 For each matrix below, find a set of linearly independent vectors X so that $\langle X \rangle$ equals the clolumn space of the matrix, and a set of linearly independent vectors Y so that $\langle Y \rangle$ equals the row space of the matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & -1 & 2 & 3 \\ 1 & 1 & 2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 3 & 2 & -1 & 4 & 5 \\ 0 & 1 & 1 & 1 & 2 \end{bmatrix} \qquad C = \begin{bmatrix} 2 & 1 & 0 \\ 3 & 0 & 3 \\ 1 & 2 & -3 \\ 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}$$

From your results for these three matrices, can you formulate a conjecture about the sets X and Y?

C30† Example CSOCD expresses the column space of the coefficient matrix from Archetype D (call the matrix A here) as the span of the first two columns of A. In Example CSMCS we determined that the vector

$$\mathbf{c} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$$

was not in the column space of A and that the vector

$$\mathbf{b} = \begin{bmatrix} 8 \\ -12 \\ 4 \end{bmatrix}$$

was in the column space of A. Attempt to write $\mathbf{c}$ and $\mathbf{b}$ as linear combinations of the two vectors in the span construction for the column space in Example CSOCD and record your observations.

C31† For the matrix A below find a set of vectors T meeting the following requirements: (1) the span of T is the column space of A, that is, $\langle T\rangle = \mathcal{C}(A)$, (2) T is linearly independent, and (3) the elements of T are columns of A.

$$A = \begin{bmatrix} 2 & 1 & 4 & -1 & 2 \\ 1 & -1 & 5 & 1 & 1 \\ -1 & 2 & -7 & 0 & 1 \\ 2 & -1 & 8 & -1 & 2 \end{bmatrix}$$

C32 In Example CSAA, verify that the vector $\mathbf{b}$ is not in the column space of the coefficient matrix.

C33† Find a linearly independent set S so that the span of S, $\langle S\rangle$, is row space of the matrix B, and S is linearly independent.

$$B = \begin{bmatrix} 2 & 3 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ -1 & 2 & 3 & -4 \end{bmatrix}$$

C34† For the 3×4 matrix A and the column vector $\mathbf{y} \in \mathbb{C}^4$ given below, determine if $\mathbf{y}$ is in the row space of A. In other words, answer the question: $\mathbf{y} \in \mathcal{R}(A)$?

$$A = \begin{bmatrix} -2 & 6 & 7 & -1 \\ 7 & -3 & 0 & -3 \\ 8 & 0 & 7 & 6 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ -2 \end{bmatrix}$$

C35† For the matrix A below, find two different linearly independent sets whose spans equal the column space of A, $\mathcal{C}(A)$, such that

1. the elements are each columns of A.
2. the set is obtained by a procedure that is substantially different from the procedure you use in part (1).

$$A = \begin{bmatrix} 3 & 5 & 1 & -2 \\ 1 & 2 & 3 & 3 \\ -3 & -4 & 7 & 13 \end{bmatrix}$$

C40 The following archetypes are systems of equations. For each system, write the vector of constants as a linear combination of the vectors in the span construction for the column space provided by Theorem BCS (these vectors are listed for each of these archetypes).

Archetype A, Archetype B, Archetype C, Archetype D, Archetype E, Archetype F, Archetype G, Archetype H, Archetype I, Archetype J

C42 The following archetypes are either matrices or systems of equations with coefficient matrices. For each matrix, compute a set of column vectors such that (1) the vectors are columns of the matrix, (2) the set is linearly independent, and (3) the span of the set is the column space of the matrix. See Theorem BCS.

Archetype A, Archetype B, Archetype C, Archetype D/Archetype E, Archetype F, Archetype G/Archetype H, Archetype I, Archetype J, Archetype K, Archetype L

C50 The following archetypes are either matrices or systems of equations with coefficient matrices. For each matrix, compute a set of column vectors such that (1) the set is linearly independent, and (2) the span of the set is the row space of the matrix. See Theorem BRS.

Archetype A, Archetype B, Archetype C, Archetype D/Archetype E, Archetype F, Archetype G/Archetype H, Archetype I, Archetype J, Archetype K, Archetype L

C51 The following archetypes are either matrices or systems of equations with coefficient matrices. For each matrix, compute the column space as the span of a linearly independent set as follows: transpose the matrix, row-reduce, toss out zero rows, convert rows into column vectors. See Example CSROI.

Archetype A, Archetype B, Archetype C, Archetype D/Archetype E, Archetype F, Archetype G/Archetype H, Archetype I, Archetype J, Archetype K, Archetype L

C52 The following archetypes are systems of equations. For each different coefficient matrix build two new vectors of constants. The first should lead to a consistent system and the second should lead to an inconsistent system. Descriptions of the column space as spans of linearly independent sets of vectors with "nice patterns" of zeros and ones might be most useful and instructive in connection with this exercise. (See the end of Example CSROI.)

Archetype A, Archetype B, Archetype C, Archetype D/Archetype E, Archetype F, Archetype G/Archetype H, Archetype I, Archetype J

M10† For the matrix E below, find vectors $\mathbf{b}$ and $\mathbf{c}$ so that the system $\mathcal{LS}(E,\,\mathbf{b})$ is consistent and $\mathcal{LS}(E,\,\mathbf{c})$ is inconsistent.

$$E = \begin{bmatrix} -2 & 1 & 1 & 0 \\ 3 & -1 & 0 & 2 \\ 4 & 1 & 1 & 6 \end{bmatrix}$$

M20† Usually the column space and null space of a matrix contain vectors of different sizes. For a square matrix, though, the vectors in these two sets are the same size. Usually the two sets will be different. Construct an example of a square matrix where the column space and null space are equal.

M21† We have a variety of theorems about how to create column spaces and row spaces and they frequently involve row-reducing a matrix. Here is a procedure that some try to use to get a column space. Begin with an $m \times n$ matrix A and row-reduce to a matrix B with columns $\mathbf{B}_1,\,\mathbf{B}_2,\,\mathbf{B}_3,\,\ldots,\,\mathbf{B}_n$. Then form the column space of A as

$$\mathcal{C}(A) = \langle\{\mathbf{B}_1,\,\mathbf{B}_2,\,\mathbf{B}_3,\,\ldots,\,\mathbf{B}_n\}\rangle = \mathcal{C}(B)$$

This is *not* not a legitimate procedure, and therefore is *not* a theorem. Construct an example to show that the procedure will not in general create the column space of A.

T40† Suppose that A is an $m \times n$ matrix and B is an $n \times p$ matrix. Prove that the column space of AB is a subset of the column space of A, that is $\mathcal{C}(AB) \subseteq \mathcal{C}(A)$. Provide an example where the opposite is false, in other words give an example where $\mathcal{C}(A) \not\subseteq \mathcal{C}(AB)$. (Compare with Exercise MM.T40.)

T41† Suppose that A is an $m \times n$ matrix and B is an $n \times n$ nonsingular matrix. Prove that the column space of A is equal to the column space of AB, that is $\mathcal{C}(A) = \mathcal{C}(AB)$. (Compare with Exercise MM.T41 and Exercise CRS.T40.)

T45† Suppose that A is an $m \times n$ matrix and B is an $n \times m$ matrix where AB is a nonsingular matrix. Prove that

1. $\mathcal{N}(B) = \{\mathbf{0}\}$
2. $\mathcal{C}(B) \cap \mathcal{N}(A) = \{\mathbf{0}\}$

Discuss the case when $m = n$ in connection with Theorem NPNT.

Section FS
Four Subsets

There are four natural subsets associated with a matrix. We have met three already: the null space, the column space and the row space. In this section we will introduce a fourth, the left null space. The objective of this section is to describe one procedure that will allow us to find linearly independent sets that span each of these four sets of column vectors. Along the way, we will make a connection with the inverse of a matrix, so Theorem FS will tie together most all of this chapter (and the entire course so far).

Subsection LNS
Left Null Space

Definition LNS Left Null Space
Suppose A is an $m \times n$ matrix. Then the **left null space** is defined as $\mathcal{L}(A) = \mathcal{N}(A^t) \subseteq \mathbb{C}^m$. □

The left null space will not feature prominently in the sequel, but we can explain its name and connect it to row operations. Suppose $\mathbf{y} \in \mathcal{L}(A)$. Then by Definition LNS, $A^t\mathbf{y} = \mathbf{0}$. We can then write

$$\begin{aligned} \mathbf{0}^t &= \left(A^t\mathbf{y}\right)^t && \text{Definition LNS} \\ &= \mathbf{y}^t\left(A^t\right)^t && \text{Theorem MMT} \\ &= \mathbf{y}^t A && \text{Theorem TT} \end{aligned}$$

The product $\mathbf{y}^t A$ can be viewed as the components of $\mathbf{y}$ acting as the scalars in a linear combination of the *rows* of A. And the result is a "row vector", $\mathbf{0}^t$ that is totally zeros. When we apply a sequence of row operations to a matrix, each row of the resulting matrix is some linear combination of the rows. These observations tell us that the vectors in the left null space are scalars that record a sequence of row operations that result in a row of zeros in the row-reduced version of the matrix. We will see this idea more explicitly in the course of proving Theorem FS.

Example LNS Left null space
We will find the left null space of

$$A = \begin{bmatrix} 1 & -3 & 1 \\ -2 & 1 & 1 \\ 1 & 5 & 1 \\ 9 & -4 & 0 \end{bmatrix}$$

We transpose A and row-reduce,

$$A^t = \begin{bmatrix} 1 & -2 & 1 & 9 \\ -3 & 1 & 5 & -4 \\ 1 & 1 & 1 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 2 \\ 0 & \boxed{1} & 0 & -3 \\ 0 & 0 & \boxed{1} & 1 \end{bmatrix}$$

Applying Definition LNS and Theorem BNS we have

$$\mathcal{L}(A) = \mathcal{N}\left(A^t\right) = \left\langle\left\{\begin{bmatrix} -2 \\ 3 \\ -1 \\ 1 \end{bmatrix}\right\}\right\rangle$$

If you row-reduce A you will discover one zero row in the reduced row-echelon form. This zero row is created by a sequence of row operations, which in total amounts to a linear combination, with scalars $a_1 = -2$, $a_2 = 3$, $a_3 = -1$ and $a_4 = 1$, on the rows of A and which results in the zero vector (check this!). So the components of the vector describing the left null space of A provide a relation of linear dependence on the rows of A. △

Subsection CCS
Computing Column Spaces

We have three ways to build the column space of a matrix. First, we can use just the definition, Definition CSM, and express the column space as a span of the columns of the matrix. A second approach gives us the column space as the span of *some* of the columns of the matrix, and additionally, this set is linearly independent (Theorem BCS). Finally, we can transpose the matrix, row-reduce the transpose, kick out zero rows, and write the remaining rows as column vectors. Theorem CSRST and Theorem BRS tell us that the resulting vectors are linearly independent and their span is the column space of the original matrix.

We will now demonstrate a fourth method by way of a rather complicated example. Study this example carefully, but realize that its main purpose is to motivate a theorem that simplifies much of the apparent complexity. So other than an instructive exercise or two, the procedure we are about to describe will not be a usual approach to computing a column space.

Example CSANS Column space as null space
Let us find the column space of the matrix A below with a new approach.

$$A = \begin{bmatrix} 10 & 0 & 3 & 8 & 7 \\ -16 & -1 & -4 & -10 & -13 \\ -6 & 1 & -3 & -6 & -6 \\ 0 & 2 & -2 & -3 & -2 \\ 3 & 0 & 1 & 2 & 3 \\ -1 & -1 & 1 & 1 & 0 \end{bmatrix}$$

By Theorem CSCS we know that the column vector $\mathbf{b}$ is in the column space of A if and only if the linear system $\mathcal{LS}(A, \mathbf{b})$ is consistent. So let us try to solve this system in full generality, using a vector of variables for the vector of constants. In other words, which vectors $\mathbf{b}$ lead to consistent systems? Begin by forming the augmented matrix $[A \,|\, \mathbf{b}]$ with a general version of $\mathbf{b}$,

$$[A \,|\, \mathbf{b}] = \begin{bmatrix} 10 & 0 & 3 & 8 & 7 & b_1 \\ -16 & -1 & -4 & -10 & -13 & b_2 \\ -6 & 1 & -3 & -6 & -6 & b_3 \\ 0 & 2 & -2 & -3 & -2 & b_4 \\ 3 & 0 & 1 & 2 & 3 & b_5 \\ -1 & -1 & 1 & 1 & 0 & b_6 \end{bmatrix}$$

To identify solutions we will bring this matrix to reduced row-echelon form. Despite the presence of variables in the last column, there is nothing to stop us from doing this, except our numerical routines on calculators cannot be used, and even some of the symbolic algebra routines do some unexpected maneuvers with this computation. So do it by hand. Yes, it is a bit of work. But worth it. We'll still be here when you get back. Notice along the way that the row operations are *exactly* the same ones you would do if you were just row-reducing the coefficient matrix alone, say in connection with a homogeneous system of equations. The column with the b_i acts as a sort of bookkeeping device. There are many different possibilities for the result, depending on what order you choose to perform the row operations, but shortly we will all be on the same page. If you want to match our work right now, use row 5 to remove any occurrence of b_1 from the other entries of the last column, and use row 6 to remove any occurrence of b_2 from the last columns. We have:

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 2 & b_3 - b_4 + 2b_5 - b_6 \\ 0 & \boxed{1} & 0 & 0 & -3 & -2b_3 + 3b_4 - 3b_5 + 3b_6 \\ 0 & 0 & \boxed{1} & 0 & 1 & b_3 + b_4 + 3b_5 + 3b_6 \\ 0 & 0 & 0 & \boxed{1} & -2 & -2b_3 + b_4 - 4b_5 \\ 0 & 0 & 0 & 0 & 0 & b_1 + 3b_3 - b_4 + 3b_5 + b_6 \\ 0 & 0 & 0 & 0 & 0 & b_2 - 2b_3 + b_4 + b_5 - b_6 \end{bmatrix}$$

Our goal is to identify those vectors $\mathbf{b}$ which make $\mathcal{LS}(A, \mathbf{b})$ consistent. By Theorem RCLS we know that the consistent systems are precisely those without a pivot column in the last column. Are the expressions in the last column of rows 5 and 6 equal to zero, or are they leading 1's? The answer is: maybe. It depends on $\mathbf{b}$. With a nonzero value for either of these expressions, we would scale the row and produce a leading 1. So we get a consistent system, and $\mathbf{b}$ is in the column space, if and only if these two expressions are both simultaneously zero. In other words, members of the column space of A are exactly those vectors $\mathbf{b}$ that satisfy

$$b_1 + 3b_3 - b_4 + 3b_5 + b_6 = 0$$
$$b_2 - 2b_3 + b_4 + b_5 - b_6 = 0$$

Hmmm. Looks suspiciously like a homogeneous system of two equations with six variables. If you have been playing along (and we hope you have) then you may have a slightly different system, but you should have just two equations. Form the coefficient matrix and row-reduce (notice that the system above has a coefficient matrix that is already in reduced row-echelon form). We should all be together now with the same matrix,

$$L = \begin{bmatrix} \boxed{1} & 0 & 3 & -1 & 3 & 1 \\ 0 & \boxed{1} & -2 & 1 & 1 & -1 \end{bmatrix}$$

So, $\mathcal{C}(A) = \mathcal{N}(L)$ and we can apply Theorem BNS to obtain a linearly independent set to use in a span construction,

$$\mathcal{C}(A) = \mathcal{N}(L) = \left\langle \left\{ \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} \right\rangle$$

Whew! As a postscript to this central example, you may wish to convince yourself that the four vectors above really are elements of the column space. Do they create consistent systems with A as coefficient matrix? Can you recognize the constant vector in your description of these solution sets?

OK, that was so much fun, let us do it again. But simpler this time. And we will all get the same results all the way through. Doing row operations by hand with variables can be a bit error prone, so let us see if we can improve the process some. Rather than row-reduce a column vector $\mathbf{b}$ full of variables, let us write $\mathbf{b} = I_6\mathbf{b}$ and we will row-reduce the matrix I_6 and when we finish row-reducing, *then* we will compute the matrix-vector product. You should first convince yourself that we can operate like this (this is the subject of a future homework exercise).

Rather than augmenting A with $\mathbf{b}$, we will instead augment it with I_6 (does this feel familiar?),

$$M = \begin{bmatrix} 10 & 0 & 3 & 8 & 7 & 1 & 0 & 0 & 0 & 0 & 0 \\ -16 & -1 & -4 & -10 & -13 & 0 & 1 & 0 & 0 & 0 & 0 \\ -6 & 1 & -3 & -6 & -6 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 2 & -2 & -3 & -2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We want to row-reduce the left-hand side of this matrix, but we will apply the same row operations to the right-hand side as well. And once we get the left-hand side in reduced row-echelon form, we will continue on to put leading 1's in the final two rows, as well as making pivot columns that contain these two additional leading 1's. It is these additional row operations that will ensure that we all get to the same place, since the reduced row-echelon form is unique (Theorem RREFU),

$$N = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & -1 & 2 & -1 \\ 0 & 1 & 0 & 0 & -3 & 0 & 0 & -2 & 3 & -3 & 3 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 3 & 3 \\ 0 & 0 & 0 & 1 & -2 & 0 & 0 & -2 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 3 & -1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & 1 & -1 \end{bmatrix}$$

We are after the final six columns of this matrix, which we will multiply by $\mathbf{b}$

$$J = \begin{bmatrix} 0 & 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & -2 & 3 & -3 & 3 \\ 0 & 0 & 1 & 1 & 3 & 3 \\ 0 & 0 & -2 & 1 & -4 & 0 \\ 1 & 0 & 3 & -1 & 3 & 1 \\ 0 & 1 & -2 & 1 & 1 & -1 \end{bmatrix}$$

so

$$J\mathbf{b} = \begin{bmatrix} 0 & 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & -2 & 3 & -3 & 3 \\ 0 & 0 & 1 & 1 & 3 & 3 \\ 0 & 0 & -2 & 1 & -4 & 0 \\ 1 & 0 & 3 & -1 & 3 & 1 \\ 0 & 1 & -2 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \\ b_6 \end{bmatrix} = \begin{bmatrix} b_3 - b_4 + 2b_5 - b_6 \\ -2b_3 + 3b_4 - 3b_5 + 3b_6 \\ b_3 + b_4 + 3b_5 + 3b_6 \\ -2b_3 + b_4 - 4b_5 \\ b_1 + 3b_3 - b_4 + 3b_5 + b_6 \\ b_2 - 2b_3 + b_4 + b_5 - b_6 \end{bmatrix}$$

So by applying the same row operations that row-reduce A to the identity matrix (which we could do with a calculator once I_6 is placed alongside of A), we can then arrive at the result of row-reducing a column of symbols where the vector of constants usually resides. Since the row-reduced version of A has two zero rows, for a consistent system we require that

$$\begin{aligned} b_1 + 3b_3 - b_4 + 3b_5 + b_6 &= 0 \\ b_2 - 2b_3 + b_4 + b_5 - b_6 &= 0 \end{aligned}$$

Now we are exactly back where we were on the first go-round. Notice that we obtain the matrix L as simply the last two rows and last six columns of N. △

This example motivates the remainder of this section, so it is worth careful study. You might attempt to mimic the second approach with the coefficient matrices of Archetype I and Archetype J. We will see shortly that the matrix L contains more information about A than just the column space.

Subsection EEF
Extended Echelon Form

The final matrix that we row-reduced in Example CSANS should look familiar in most respects to the procedure we used to compute the inverse of a nonsingular matrix, Theorem CINM. We will now generalize that procedure to matrices that are not necessarily nonsingular, or even square. First a definition.

Definition EEF Extended Echelon Form
Suppose A is an $m \times n$ matrix. Extend A on its right side with the addition of an $m \times m$ identity matrix to form an $m \times (n+m)$ matrix M. Use row operations to bring M to reduced row-echelon form and call the result N. N is the **extended reduced row-echelon form** of A, and we will standardize on names for five submatrices (B, C, J, K, L) of N.

Let B denote the $m \times n$ matrix formed from the first n columns of N and let J denote the $m \times m$ matrix formed from the last m columns of N. Suppose that B has r nonzero rows. Further partition N by letting C denote the $r \times n$ matrix formed from all of the nonzero rows of B. Let K be the $r \times m$ matrix formed from the first r rows of J, while L will be the $(m-r) \times m$ matrix formed from the bottom $m-r$ rows of J. Pictorially,

$$M = [A|I_m] \xrightarrow{\text{RREF}} N = [B|J] = \left[\begin{array}{c|c} C & K \\ \hline 0 & L \end{array}\right]$$

□

Example SEEF Submatrices of extended echelon form
We illustrate Definition EEF with the matrix A,

$$A = \begin{bmatrix} 1 & -1 & -2 & 7 & 1 & 6 \\ -6 & 2 & -4 & -18 & -3 & -26 \\ 4 & -1 & 4 & 10 & 2 & 17 \\ 3 & -1 & 2 & 9 & 1 & 12 \end{bmatrix}$$

Augmenting with the 4×4 identity matrix,

$$M = \begin{bmatrix} 1 & -1 & -2 & 7 & 1 & 6 & 1 & 0 & 0 & 0 \\ -6 & 2 & -4 & -18 & -3 & -26 & 0 & 1 & 0 & 0 \\ 4 & -1 & 4 & 10 & 2 & 17 & 0 & 0 & 1 & 0 \\ 3 & -1 & 2 & 9 & 1 & 12 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and row-reducing, we obtain

$$N = \begin{bmatrix} \boxed{1} & 0 & 2 & 1 & 0 & 3 & 0 & 1 & 1 & 1 \\ 0 & \boxed{1} & 4 & -6 & 0 & -1 & 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 2 & 0 & -1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & 2 & 2 & 1 \end{bmatrix}$$

So we then obtain

$$B = \begin{bmatrix} \boxed{1} & 0 & 2 & 1 & 0 & 3 \\ 0 & \boxed{1} & 4 & -6 & 0 & -1 \\ 0 & 0 & 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} \boxed{1} & 0 & 2 & 1 & 0 & 3 \\ 0 & \boxed{1} & 4 & -6 & 0 & -1 \\ 0 & 0 & 0 & 0 & \boxed{1} & 2 \end{bmatrix}$$

$$J = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 2 & 3 & 0 \\ 0 & -1 & 0 & -2 \\ \boxed{1} & 2 & 2 & 1 \end{bmatrix}$$

$$K = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 2 & 3 & 0 \\ 0 & -1 & 0 & -2 \end{bmatrix}$$

$$L = \begin{bmatrix} \boxed{1} & 2 & 2 & 1 \end{bmatrix}$$

You can observe (or verify) the properties of the following theorem with this example. △

Theorem PEEF Properties of Extended Echelon Form
Suppose that A is an $m \times n$ matrix and that N is its extended echelon form. Then

1. *J is nonsingular.*
2. *$B = JA$.*
3. *If $\mathbf{x} \in \mathbb{C}^n$ and $\mathbf{y} \in \mathbb{C}^m$, then $A\mathbf{x} = \mathbf{y}$ if and only if $B\mathbf{x} = J\mathbf{y}$.*
4. *C is in reduced row-echelon form, has no zero rows and has r pivot columns.*
5. *L is in reduced row-echelon form, has no zero rows and has $m-r$ pivot columns.*

Proof. J is the result of applying a sequence of row operations to I_m, and therefore J and I_m are row-equivalent. $\mathcal{LS}(I_m,\,\mathbf{0})$ has only the zero solution, since I_m is nonsingular (Theorem NMRRI). Thus, $\mathcal{LS}(J,\,\mathbf{0})$ also has only the zero solution (Theorem REMES, Definition ESYS) and J is therefore nonsingular (Definition NSM).

To prove the second part of this conclusion, first convince yourself that row operations and the matrix-vector product are associative operations. By this we mean the following. Suppose that F is an $m \times n$ matrix that is row-equivalent to the matrix G. Apply to the column vector $F\mathbf{w}$ the same sequence of row operations that converts F to G. Then the result is $G\mathbf{w}$. So we can do row operations on the matrix, then do a matrix-vector product, *or* do a matrix-vector product and then do row operations on a column vector, and the result will be the same either way. Since matrix multiplication is defined by a collection of matrix-vector products (Definition MM), the matrix product FH will become GH if we apply the same sequence of row operations to FH that convert F to G. (This argument can be made more rigorous using elementary matrices from the upcoming Subsection DM.EM and the associative property of matrix multiplication established in Theorem MMA.) Now apply these observations to A.

Write $AI_n = I_mA$ and apply the row operations that convert M to N. A is converted to B, while I_m is converted to J, so we have $BI_n = JA$. Simplifying the left side gives the desired conclusion.

For the third conclusion, we now establish the two equivalences

$$A\mathbf{x} = \mathbf{y} \qquad\iff\qquad JA\mathbf{x} = J\mathbf{y} \qquad\iff\qquad B\mathbf{x} = J\mathbf{y}$$

The forward direction of the first equivalence is accomplished by multiplying both sides of the matrix equality by J, while the backward direction is accomplished by multiplying by the inverse of J (which we know exists by Theorem NI since J is nonsingular). The second equivalence is obtained simply by the substitutions given by $JA = B$.

The first r rows of N are in reduced row-echelon form, since any contiguous collection of rows taken from a matrix in reduced row-echelon form will form a matrix that is again in reduced row-echelon form (Exercise RREF.T12). Since the matrix C is formed by removing the last n entries of each these rows, the remainder is still in reduced row-echelon form. By its construction, C has no zero rows. C has r rows and each contains a leading 1, so there are r pivot columns in C.

The final $m-r$ rows of N are in reduced row-echelon form, since any contiguous collection of rows taken from a matrix in reduced row-echelon form will form a matrix that is again in reduced row-echelon form. Since the matrix L is formed by removing the first n entries of each these rows, and these entries are all zero (they form the zero rows of B), the remainder is still in reduced row-echelon form. L is the final $m-r$ rows of the nonsingular matrix J, so none of these rows can be totally zero, or J would not row-reduce to the identity matrix. L has $m-r$ rows and each contains a leading 1, so there are $m-r$ pivot columns in L. ■

Notice that in the case where A is a nonsingular matrix we know that the reduced row-echelon form of A is the identity matrix (Theorem NMRRI), so $B = I_n$. Then the second conclusion above says $JA = B = I_n$, so J is the inverse of A. Thus this theorem generalizes Theorem CINM, though the result is a "left-inverse" of A rather than a "right-inverse."

The third conclusion of Theorem PEEF is the most telling. It says that $\mathbf{x}$ is a solution to the linear system $\mathcal{LS}(A,\,\mathbf{y})$ if and only if $\mathbf{x}$ is a solution to the linear system $\mathcal{LS}(B,\,J\mathbf{y})$. Or said differently, if we row-reduce the augmented matrix $[A\,|\,\mathbf{y}]$ we will get the augmented matrix $[B\,|\,J\mathbf{y}]$. The matrix J tracks the cumulative effect of the row operations that converts A to reduced row-echelon form, here effectively applying them to the vector of constants in a system of equations having A as a coefficient matrix. When A row-reduces to a matrix with zero rows, then $J\mathbf{y}$ should also have zero entries in the same rows if the system is to be consistent.

Subsection FS Four Subsets

With all the preliminaries in place we can state our main result for this section. In essence this result will allow us to say that we can find linearly independent sets to use in span constructions for all four subsets (null space, column space, row space, left null space) by analyzing only the extended echelon form of the matrix, and specifically, just the two submatrices C and L, which will be ripe for analysis since they are already in reduced row-echelon form (Theorem PEEF).

Theorem FS Four Subsets
Suppose A is an $m \times n$ matrix with extended echelon form N. Suppose the reduced row-echelon form of A has r nonzero rows. Then C is the submatrix of N formed from the first r rows and the first n columns and L is the submatrix of N formed from the last m columns and the last $m-r$ rows. Then

1. *The null space of A is the null space of C, $\mathcal{N}(A) = \mathcal{N}(C)$.*
2. *The row space of A is the row space of C, $\mathcal{R}(A) = \mathcal{R}(C)$.*
3. *The column space of A is the null space of L, $\mathcal{C}(A) = \mathcal{N}(L)$.*
4. *The left null space of A is the row space of L, $\mathcal{L}(A) = \mathcal{R}(L)$.*

Proof. First, $\mathcal{N}(A) = \mathcal{N}(B)$ since B is row-equivalent to A (Theorem REMES). The zero rows of B represent equations that are always true in the homogeneous system $\mathcal{LS}(B,\,\mathbf{0})$, so the removal of these equations will not change the solution set. Thus, in turn, $\mathcal{N}(B) = \mathcal{N}(C)$.

Second, $\mathcal{R}(A) = \mathcal{R}(B)$ since B is row-equivalent to A (Theorem REMRS). The zero rows of B contribute nothing to the span that is the row space of B, so the removal of these rows will not change the row space. Thus, in turn, $\mathcal{R}(B) = \mathcal{R}(C)$.

Third, we prove the set equality $\mathcal{C}(A) = \mathcal{N}(L)$ with Definition SE. Begin by showing that $\mathcal{C}(A) \subseteq \mathcal{N}(L)$. Choose $\mathbf{y} \in \mathcal{C}(A) \subseteq \mathbb{C}^m$. Then there exists a vector $\mathbf{x} \in \mathbb{C}^n$ such that $A\mathbf{x} = \mathbf{y}$ (Theorem CSCS). Then for $1 \leq k \leq m-r$,

$$
\begin{aligned}
[L\mathbf{y}]_k &= [J\mathbf{y}]_{r+k} && L \text{ a submatrix of } J \\
&= [B\mathbf{x}]_{r+k} && \text{Theorem PEEF} \\
&= [\mathcal{O}\mathbf{x}]_k && \text{Zero matrix a submatrix of } B \\
&= [\mathbf{0}]_k && \text{Theorem MMZM}
\end{aligned}
$$

So, for all $1 \leq k \leq m-r$, $[L\mathbf{y}]_k = [\mathbf{0}]_k$. So by Definition CVE we have $L\mathbf{y} = \mathbf{0}$ and thus $\mathbf{y} \in \mathcal{N}(L)$.

Now, show that $\mathcal{N}(L) \subseteq \mathcal{C}(A)$. Choose $\mathbf{y} \in \mathcal{N}(L) \subseteq \mathbb{C}^m$. Form the vector $K\mathbf{y} \in \mathbb{C}^r$. The linear system $\mathcal{LS}(C,\,K\mathbf{y})$ is consistent since C is in reduced row-echelon form and has no zero rows (Theorem PEEF). Let $\mathbf{x} \in \mathbb{C}^n$ denote a solution to $\mathcal{LS}(C,\,K\mathbf{y})$.

Then for $1 \leq j \leq r$,

$$\begin{aligned}
[B\mathbf{x}]_j &= [C\mathbf{x}]_j && C \text{ a submatrix of } B\\
&= [K\mathbf{y}]_j && \mathbf{x} \text{ a solution to } \mathcal{LS}(C,\,K\mathbf{y})\\
&= [J\mathbf{y}]_j && K \text{ a submatrix of } J
\end{aligned}$$

And for $r+1 \leq k \leq m$,

$$\begin{aligned}
[B\mathbf{x}]_k &= [\mathcal{O}\mathbf{x}]_{k-r} && \text{Zero matrix a submatrix of } B\\
&= [\mathbf{0}]_{k-r} && \text{Theorem MMZM}\\
&= [L\mathbf{y}]_{k-r} && \mathbf{y} \text{ in } \mathcal{N}(L)\\
&= [J\mathbf{y}]_k && L \text{ a submatrix of } J
\end{aligned}$$

So for all $1 \leq i \leq m$, $[B\mathbf{x}]_i = [J\mathbf{y}]_i$ and by Definition CVE we have $B\mathbf{x} = J\mathbf{y}$. From Theorem PEEF we know then that $A\mathbf{x} = \mathbf{y}$, and therefore $\mathbf{y} \in \mathcal{C}(A)$ (Theorem CSCS). By Definition SE we now have $\mathcal{C}(A) = \mathcal{N}(L)$.

Fourth, we prove the set equality $\mathcal{L}(A) = \mathcal{R}(L)$ with Definition SE. Begin by showing that $\mathcal{R}(L) \subseteq \mathcal{L}(A)$. Choose $\mathbf{y} \in \mathcal{R}(L) \subseteq \mathbb{C}^m$. Then there exists a vector $\mathbf{w} \in \mathbb{C}^{m-r}$ such that $\mathbf{y} = L^t\mathbf{w}$ (Definition RSM, Theorem CSCS). Then for $1 \leq i \leq n$,

$$\begin{aligned}
\left[A^t\mathbf{y}\right]_i &= \sum_{k=1}^{m} \left[A^t\right]_{ik} [\mathbf{y}]_k && \text{Theorem EMP}\\
&= \sum_{k=1}^{m} \left[A^t\right]_{ik} \left[L^t\mathbf{w}\right]_k && \text{Definition of } \mathbf{w}\\
&= \sum_{k=1}^{m} \left[A^t\right]_{ik} \sum_{\ell=1}^{m-r} \left[L^t\right]_{k\ell} [\mathbf{w}]_\ell && \text{Theorem EMP}\\
&= \sum_{k=1}^{m} \sum_{\ell=1}^{m-r} \left[A^t\right]_{ik} \left[L^t\right]_{k\ell} [\mathbf{w}]_\ell && \text{Property DCN}\\
&= \sum_{\ell=1}^{m-r} \sum_{k=1}^{m} \left[A^t\right]_{ik} \left[L^t\right]_{k\ell} [\mathbf{w}]_\ell && \text{Property CACN}\\
&= \sum_{\ell=1}^{m-r} \left(\sum_{k=1}^{m} \left[A^t\right]_{ik} \left[L^t\right]_{k\ell}\right) [\mathbf{w}]_\ell && \text{Property DCN}\\
&= \sum_{\ell=1}^{m-r} \left(\sum_{k=1}^{m} \left[A^t\right]_{ik} \left[J^t\right]_{k,r+\ell}\right) [\mathbf{w}]_\ell && L \text{ a submatrix of } J\\
&= \sum_{\ell=1}^{m-r} \left[A^tJ^t\right]_{i,r+\ell} [\mathbf{w}]_\ell && \text{Theorem EMP}\\
&= \sum_{\ell=1}^{m-r} \left[(JA)^t\right]_{i,r+\ell} [\mathbf{w}]_\ell && \text{Theorem MMT}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\ell=1}^{m-r} \left[B^t\right]_{i,r+\ell} [\mathbf{w}]_\ell && \text{Theorem PEEF}\\
&= \sum_{\ell=1}^{m-r} 0\,[\mathbf{w}]_\ell && \text{Zero rows in } B\\
&= 0 && \text{Property ZCN}\\
&= [\mathbf{0}]_i && \text{Definition ZCV}
\end{aligned}
$$

Since $[A^t\mathbf{y}]_i = [\mathbf{0}]_i$ for $1 \leq i \leq n$, Definition CVE implies that $A^t\mathbf{y} = \mathbf{0}$. This means that $\mathbf{y} \in \mathcal{N}(A^t)$.

Now, show that $\mathcal{L}(A) \subseteq \mathcal{R}(L)$. Choose $\mathbf{y} \in \mathcal{L}(A) \subseteq \mathbb{C}^m$. The matrix J is nonsingular (Theorem PEEF), so J^t is also nonsingular (Theorem MIT) and therefore the linear system $\mathcal{LS}(J^t, \mathbf{y})$ has a unique solution. Denote this solution as $\mathbf{x} \in \mathbb{C}^m$. We will need to work with two "halves" of $\mathbf{x}$, which we will denote as $\mathbf{z}$ and $\mathbf{w}$ with formal definitions given by

$$
[z]_j = [x]_i \qquad 1 \leq j \leq r, \qquad\qquad [w]_k = [x]_{r+k} \qquad 1 \leq k \leq m-r
$$

Now, for $1 \leq j \leq r$,

$$
\begin{aligned}
\left[C^t\mathbf{z}\right]_j &= \sum_{k=1}^{r} \left[C^t\right]_{jk} [\mathbf{z}]_k && \text{Theorem EMP}\\
&= \sum_{k=1}^{r} \left[C^t\right]_{jk} [\mathbf{z}]_k + \sum_{\ell=1}^{m-r} [\mathcal{O}]_{j\ell} [\mathbf{w}]_\ell && \text{Definition ZM}\\
&= \sum_{k=1}^{r} \left[B^t\right]_{jk} [\mathbf{z}]_k + \sum_{\ell=1}^{m-r} \left[B^t\right]_{j,r+\ell} [\mathbf{w}]_\ell && C,\ \mathcal{O} \text{ submatrices of } B\\
&= \sum_{k=1}^{r} \left[B^t\right]_{jk} [\mathbf{x}]_k + \sum_{\ell=1}^{m-r} \left[B^t\right]_{j,r+\ell} [\mathbf{x}]_{r+\ell} && \text{Definitions of } \mathbf{z} \text{ and } \mathbf{w}\\
&= \sum_{k=1}^{r} \left[B^t\right]_{jk} [\mathbf{x}]_k + \sum_{k=r+1}^{m} \left[B^t\right]_{jk} [\mathbf{x}]_k && \text{Re-index second sum}\\
&= \sum_{k=1}^{m} \left[B^t\right]_{jk} [\mathbf{x}]_k && \text{Combine sums}\\
&= \sum_{k=1}^{m} \left[(JA)^t\right]_{jk} [\mathbf{x}]_k && \text{Theorem PEEF}\\
&= \sum_{k=1}^{m} \left[A^tJ^t\right]_{jk} [\mathbf{x}]_k && \text{Theorem MMT}\\
&= \sum_{k=1}^{m}\sum_{\ell=1}^{m} \left[A^t\right]_{j\ell} \left[J^t\right]_{\ell k} [\mathbf{x}]_k && \text{Theorem EMP}\\
&= \sum_{\ell=1}^{m}\sum_{k=1}^{m} \left[A^t\right]_{j\ell} \left[J^t\right]_{\ell k} [\mathbf{x}]_k && \text{Property CACN}\\
&= \sum_{\ell=1}^{m} \left[A^t\right]_{j\ell} \left(\sum_{k=1}^{m} \left[J^t\right]_{\ell k} [\mathbf{x}]_k\right) && \text{Property DCN}\\
&= \sum_{\ell=1}^{m} \left[A^t\right]_{j\ell} \left[J^t\mathbf{x}\right]_\ell && \text{Theorem EMP}
\end{aligned}
$$

$$\begin{aligned}
&= \sum_{\ell=1}^{m} \left[A^t\right]_{j\ell} [\mathbf{y}]_\ell && \text{Definition of } \mathbf{x}\\
&= \left[A^t\mathbf{y}\right]_j && \text{Theorem EMP}\\
&= [\mathbf{0}]_j && \mathbf{y} \in \mathcal{L}(A)
\end{aligned}$$

So, by Definition CVE, $C^t\mathbf{z} = \mathbf{0}$ and the vector $\mathbf{z}$ gives us a linear combination of the columns of C^t that equals the zero vector. In other words, $\mathbf{z}$ gives a relation of linear dependence on the the rows of C. However, the rows of C are a linearly independent set by Theorem BRS. According to Definition LICV we must conclude that the entries of $\mathbf{z}$ are all zero, i.e. $\mathbf{z} = \mathbf{0}$.

Now, for $1 \leq i \leq m$, we have

$$\begin{aligned}
[\mathbf{y}]_i &= \left[J^t\mathbf{x}\right]_i && \text{Definition of } \mathbf{x}\\
&= \sum_{k=1}^{m} \left[J^t\right]_{ik} [\mathbf{x}]_k && \text{Theorem EMP}\\
&= \sum_{k=1}^{r} \left[J^t\right]_{ik} [\mathbf{x}]_k + \sum_{k=r+1}^{m} \left[J^t\right]_{ik} [\mathbf{x}]_k && \text{Break apart sum}\\
&= \sum_{k=1}^{r} \left[J^t\right]_{ik} [\mathbf{z}]_k + \sum_{k=r+1}^{m} \left[J^t\right]_{ik} [\mathbf{w}]_{k-r} && \text{Definition of } \mathbf{z} \text{ and } \mathbf{w}\\
&= \sum_{k=1}^{r} \left[J^t\right]_{ik} 0 + \sum_{\ell=1}^{m-r} \left[J^t\right]_{i,r+\ell} [\mathbf{w}]_\ell && \mathbf{z} = \mathbf{0}, \text{ re-index}\\
&= 0 + \sum_{\ell=1}^{m-r} \left[L^t\right]_{i,\ell} [\mathbf{w}]_\ell && L \text{ a submatrix of } J\\
&= \left[L^t\mathbf{w}\right]_i && \text{Theorem EMP}
\end{aligned}$$

So by Definition CVE, $\mathbf{y} = L^t\mathbf{w}$. The existence of $\mathbf{w}$ implies that $\mathbf{y} \in \mathcal{R}(L)$, and therefore $\mathcal{L}(A) \subseteq \mathcal{R}(L)$. So by Definition SE we have $\mathcal{L}(A) = \mathcal{R}(L)$. ■

The first two conclusions of this theorem are nearly trivial. But they set up a pattern of results for C that is reflected in the latter two conclusions about L. In total, they tell us that we can compute all four subsets just by finding null spaces and row spaces. This theorem does not tell us exactly how to compute these subsets, but instead simply expresses them as null spaces and row spaces of matrices in reduced row-echelon form without any zero rows (C and L). A linearly independent set that spans the null space of a matrix in reduced row-echelon form can be found easily with Theorem BNS. It is an even easier matter to find a linearly independent set that spans the row space of a matrix in reduced row-echelon form with Theorem BRS, especially when there are no zero rows present. So an application of Theorem FS is typically followed by two applications each of Theorem BNS and Theorem BRS.

The situation when $r = m$ deserves comment, since now the matrix L has no rows. What is $\mathcal{C}(A)$ when we try to apply Theorem FS and encounter $\mathcal{N}(L)$? One interpretation of this situation is that L is the coefficient matrix of a homogeneous system that has no equations. How hard is it to find a solution vector to this system? Some thought will convince you that *any* proposed vector will qualify as a solution, since it makes *all* of the equations true. So every possible vector is in the null space of L and therefore $\mathcal{C}(A) = \mathcal{N}(L) = \mathbb{C}^m$. OK, perhaps this sounds like some twisted argument from *Alice in Wonderland*. Let us try another argument that might solidly convince you of this logic.

If $r=m$, when we row-reduce the augmented matrix of $\mathcal{LS}(A,\,\mathbf{b})$ the result will have no zero rows, and the first n columns will all be pivot columns, leaving none for the final column, so by Theorem RCLS the system will be consistent. By Theorem CSCS, $\mathbf{b}\in\mathcal{C}(A)$. Since $\mathbf{b}$ was arbitrary, every possible vector is in the column space of A, so we again have $\mathcal{C}(A)=\mathbb{C}^m$. The situation when a matrix has $r=m$ is known by the term **full rank**, and in the case of a square matrix coincides with nonsingularity (see Exercise FS.M50).

The properties of the matrix L described by this theorem can be explained informally as follows. A column vector $\mathbf{y}\in\mathbb{C}^m$ is in the column space of A if the linear system $\mathcal{LS}(A,\,\mathbf{y})$ is consistent (Theorem CSCS). By Theorem RCLS, the reduced row-echelon form of the augmented matrix $[\,A\,|\,\mathbf{y}]$ of a consistent system will have zeros in the bottom $m-r$ locations of the last column. By Theorem PEEF this final column is the vector $J\mathbf{y}$ and so should then have zeros in the final $m-r$ locations. But since L comprises the final $m-r$ rows of J, this condition is expressed by saying $\mathbf{y}\in\mathcal{N}(L)$.

Additionally, the rows of J are the scalars in linear combinations of the rows of A that create the rows of B. That is, the rows of J record the net effect of the sequence of row operations that takes A to its reduced row-echelon form, B. This can be seen in the equation $JA=B$ (Theorem PEEF). As such, the rows of L are scalars for linear combinations of the rows of A that yield zero rows. But such linear combinations are precisely the elements of the left null space. So any element of the row space of L is also an element of the left null space of A.

We will now illustrate Theorem FS with a few examples.

Example FS1 Four subsets, no. 1

In Example SEEF we found the five relevant submatrices of the matrix

$$A=\begin{bmatrix}1&-1&-2&7&1&6\\-6&2&-4&-18&-3&-26\\4&-1&4&10&2&17\\3&-1&2&9&1&12\end{bmatrix}$$

To apply Theorem FS we only need C and L,

$$C=\begin{bmatrix}\boxed{1}&0&2&1&0&3\\0&\boxed{1}&4&-6&0&-1\\0&0&0&0&\boxed{1}&2\end{bmatrix}\qquad\qquad L=\begin{bmatrix}\boxed{1}&2&2&1\end{bmatrix}$$

Then we use Theorem FS to obtain

$$\mathcal{N}(A)=\mathcal{N}(C)=\left\langle\left\{\begin{bmatrix}-2\\-4\\1\\0\\0\\0\end{bmatrix},\,\begin{bmatrix}-1\\6\\0\\1\\0\\0\end{bmatrix},\,\begin{bmatrix}-3\\1\\0\\0\\-2\\1\end{bmatrix}\right\}\right\rangle\qquad\text{Theorem BNS}$$

$$\mathcal{R}(A)=\mathcal{R}(C)=\left\langle\left\{\begin{bmatrix}1\\0\\2\\1\\0\\3\end{bmatrix},\,\begin{bmatrix}0\\1\\4\\-6\\0\\-1\end{bmatrix},\,\begin{bmatrix}0\\0\\0\\0\\1\\2\end{bmatrix}\right\}\right\rangle\qquad\text{Theorem BRS}$$

$$\mathcal{C}(A)=\mathcal{N}(L)=\left\langle\left\{\begin{bmatrix}-2\\1\\0\\0\end{bmatrix},\,\begin{bmatrix}-2\\0\\1\\0\end{bmatrix},\,\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}\right\}\right\rangle\qquad\text{Theorem BNS}$$

$$\mathcal{L}(A)=\mathcal{R}(L)=\left\langle\left\{\begin{bmatrix}1\\2\\2\\1\end{bmatrix}\right\}\right\rangle \qquad \text{Theorem BRS}$$

Boom! △

Example FS2 Four subsets, no. 2
Now let us return to the matrix A that we used to motivate this section in Example CSANS,

$$A=\begin{bmatrix}10 & 0 & 3 & 8 & 7\\ -16 & -1 & -4 & -10 & -13\\ -6 & 1 & -3 & -6 & -6\\ 0 & 2 & -2 & -3 & -2\\ 3 & 0 & 1 & 2 & 3\\ -1 & -1 & 1 & 1 & 0\end{bmatrix}$$

We form the matrix M by adjoining the 6×6 identity matrix I_6,

$$M=\begin{bmatrix}10 & 0 & 3 & 8 & 7 & 1 & 0 & 0 & 0 & 0 & 0\\ -16 & -1 & -4 & -10 & -13 & 0 & 1 & 0 & 0 & 0 & 0\\ -6 & 1 & -3 & -6 & -6 & 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 2 & -2 & -3 & -2 & 0 & 0 & 0 & 1 & 0 & 0\\ 3 & 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 & 1 & 0\\ -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1\end{bmatrix}$$

and row-reduce to obtain N

$$N=\begin{bmatrix}\boxed{1} & 0 & 0 & 0 & 2 & 0 & 0 & 1 & -1 & 2 & -1\\ 0 & \boxed{1} & 0 & 0 & -3 & 0 & 0 & -2 & 3 & -3 & 3\\ 0 & 0 & \boxed{1} & 0 & 1 & 0 & 0 & 1 & 1 & 3 & 3\\ 0 & 0 & 0 & \boxed{1} & -2 & 0 & 0 & -2 & 1 & -4 & 0\\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 & 3 & -1 & 3 & 1\\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} & -2 & 1 & 1 & -1\end{bmatrix}$$

To find the four subsets for A, we only need identify the 4×5 matrix C and the 2×6 matrix L,

$$C=\begin{bmatrix}\boxed{1} & 0 & 0 & 0 & 2\\ 0 & \boxed{1} & 0 & 0 & -3\\ 0 & 0 & \boxed{1} & 0 & 1\\ 0 & 0 & 0 & \boxed{1} & -2\end{bmatrix} \qquad L=\begin{bmatrix}\boxed{1} & 0 & 3 & -1 & 3 & 1\\ 0 & \boxed{1} & -2 & 1 & 1 & -1\end{bmatrix}$$

Then we apply Theorem FS,

$$\mathcal{N}(A)=\mathcal{N}(C)=\left\langle\left\{\begin{bmatrix}-2\\3\\-1\\2\\1\end{bmatrix}\right\}\right\rangle \qquad \text{Theorem BNS}$$

$$\mathcal{R}(A)=\mathcal{R}(C)=\left\langle\left\{\begin{bmatrix}1\\0\\0\\0\\2\end{bmatrix},\,\begin{bmatrix}0\\1\\0\\0\\-3\end{bmatrix},\,\begin{bmatrix}0\\0\\1\\0\\1\end{bmatrix},\,\begin{bmatrix}0\\0\\0\\1\\-2\end{bmatrix}\right\}\right\rangle \qquad \text{Theorem BRS}$$

$$\mathcal{C}(A)=\mathcal{N}(L)=\left\langle\left\{\begin{bmatrix}-3\\2\\1\\0\\0\\0\end{bmatrix},\,\begin{bmatrix}1\\-1\\0\\1\\0\\0\end{bmatrix},\,\begin{bmatrix}-3\\-1\\0\\0\\1\\0\end{bmatrix},\,\begin{bmatrix}-1\\1\\0\\0\\0\\1\end{bmatrix}\right\}\right\rangle \qquad \text{Theorem BNS}$$

$$\mathcal{L}(A) = \mathcal{R}(L) = \left\langle\left\{\begin{bmatrix}1\\0\\3\\-1\\3\\1\end{bmatrix},\,\begin{bmatrix}0\\1\\-2\\1\\1\\-1\end{bmatrix}\right\}\right\rangle \qquad \text{Theorem BRS}$$

△

The next example is just a bit different since the matrix has more rows than columns, and a trivial null space.

Example FSAG Four subsets, Archetype G
Archetype G and Archetype H are both systems of $m = 5$ equations in $n = 2$ variables. They have identical coefficient matrices, which we will denote here as the matrix G,

$$G = \begin{bmatrix}2 & 3\\ -1 & 4\\ 3 & 10\\ 3 & -1\\ 6 & 9\end{bmatrix}$$

Adjoin the 5×5 identity matrix, I_5, to form

$$M = \begin{bmatrix}2 & 3 & 1 & 0 & 0 & 0 & 0\\ -1 & 4 & 0 & 1 & 0 & 0 & 0\\ 3 & 10 & 0 & 0 & 1 & 0 & 0\\ 3 & -1 & 0 & 0 & 0 & 1 & 0\\ 6 & 9 & 0 & 0 & 0 & 0 & 1\end{bmatrix}$$

This row-reduces to

$$N = \begin{bmatrix}\boxed{1} & 0 & 0 & 0 & 0 & \frac{3}{11} & \frac{1}{33}\\ 0 & \boxed{1} & 0 & 0 & 0 & -\frac{2}{11} & \frac{1}{11}\\ 0 & 0 & \boxed{1} & 0 & 0 & 0 & -\frac{1}{3}\\ 0 & 0 & 0 & \boxed{1} & 0 & 1 & -\frac{1}{3}\\ 0 & 0 & 0 & 0 & \boxed{1} & 1 & -1\end{bmatrix}$$

The first $n = 2$ columns contain $r = 2$ leading 1's, so we obtain C as the 2×2 identity matrix and extract L from the final $m - r = 3$ rows in the final $m = 5$ columns.

$$C = \begin{bmatrix}\boxed{1} & 0\\ 0 & \boxed{1}\end{bmatrix} \qquad L = \begin{bmatrix}\boxed{1} & 0 & 0 & 0 & -\frac{1}{3}\\ 0 & \boxed{1} & 0 & 1 & -\frac{1}{3}\\ 0 & 0 & \boxed{1} & 1 & -1\end{bmatrix}$$

Then we apply Theorem FS,

$$\mathcal{N}(G) = \mathcal{N}(C) = \langle\emptyset\rangle = \{\mathbf{0}\} \qquad \text{Theorem BNS}$$

$$\mathcal{R}(G) = \mathcal{R}(C) = \left\langle\left\{\begin{bmatrix}1\\0\end{bmatrix},\,\begin{bmatrix}0\\1\end{bmatrix}\right\}\right\rangle = \mathbb{C}^2 \qquad \text{Theorem BRS}$$

$$\mathcal{C}(G) = \mathcal{N}(L) = \left\langle\left\{\begin{bmatrix}0\\-1\\-1\\1\\0\end{bmatrix},\,\begin{bmatrix}\frac{1}{3}\\\frac{1}{3}\\1\\0\\1\end{bmatrix}\right\}\right\rangle \qquad \text{Theorem BNS}$$

$$= \left\langle\left\{\begin{bmatrix}0\\-1\\-1\\1\\0\end{bmatrix},\,\begin{bmatrix}1\\1\\3\\0\\3\end{bmatrix}\right\}\right\rangle$$

$$\mathcal{L}(G) = \mathcal{R}(L) = \left\langle\left\{\begin{bmatrix}1\\0\\0\\0\\-\frac{1}{3}\end{bmatrix},\,\begin{bmatrix}0\\1\\0\\1\\-\frac{1}{3}\end{bmatrix},\,\begin{bmatrix}0\\0\\1\\1\\-1\end{bmatrix}\right\}\right\rangle \qquad \text{Theorem BRS}$$

$$= \left\langle\left\{\begin{bmatrix}3\\0\\0\\0\\-1\end{bmatrix},\,\begin{bmatrix}0\\3\\0\\3\\-1\end{bmatrix},\,\begin{bmatrix}0\\0\\1\\1\\-1\end{bmatrix}\right\}\right\rangle$$

As mentioned earlier, Archetype G is consistent, while Archetype H is inconsistent. See if you can write the two different vectors of constants from these two archetypes as linear combinations of the two vectors in $\mathcal{C}(G)$. How about the two columns of G, can you write each individually as a linear combination of the two vectors in $\mathcal{C}(G)$? They must be in the column space of G also. Are your answers unique? Do you notice anything about the scalars that appear in the linear combinations you are forming? △

Example COV and Example CSROI each describes the column space of the coefficient matrix from Archetype I as the span of a set of $r = 3$ linearly independent vectors. It is no accident that these two different sets both have the same size. If we (you?) were to calculate the column space of this matrix using the null space of the matrix L from Theorem FS then we would again find a set of 3 linearly independent vectors that span the range. More on this later.

So we have three different methods to obtain a description of the column space of a matrix as the span of a linearly independent set. Theorem BCS is sometimes useful since the vectors it specifies are equal to actual columns of the matrix. Theorem BRS and Theorem CSRST combine to create vectors with lots of zeros, and strategically placed 1's near the top of the vector. Theorem FS and the matrix L from the extended echelon form gives us a third method, which tends to create vectors with lots of zeros, and strategically placed 1's near the bottom of the vector. If we do not care about linear independence we can also appeal to Definition CSM and simply express the column space as the span of all the columns of the matrix, giving us a fourth description.

With Theorem CSRST and Definition RSM, we can compute column spaces with theorems about row spaces, and we can compute row spaces with theorems about column spaces, but in each case we must transpose the matrix first. At this point you may be overwhelmed by all the possibilities for computing column and row spaces. Diagram CSRST is meant to help. For both the column space and row space, it suggests four techniques. One is to appeal to the definition, another yields a span of a linearly independent set, and a third uses Theorem FS. A fourth suggests transposing the matrix and the dashed line implies that then the companion set of techniques can be applied. This can lead to a bit of silliness, since if you were to follow the dashed lines *twice* you would transpose the matrix twice, and by Theorem TT would accomplish nothing productive.

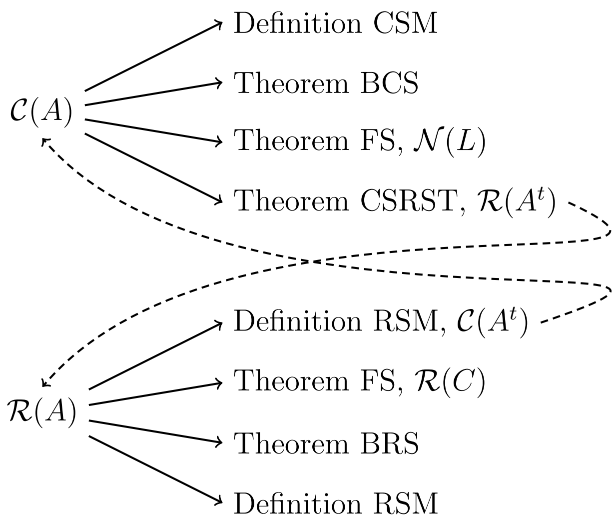

Diagram CSRST: Column Space and Row Space Techniques

Although we have many ways to describe a column space, notice that one tempting strategy will usually fail. It is not possible to simply row-reduce a matrix directly and then use the columns of the row-reduced matrix as a set whose span equals the column space. In other words, row operations *do not* preserve column spaces (however row operations do preserve row spaces, Theorem REMRS). See Exercise CRS.M21.

Reading Questions

1. Find a nontrivial element of the left null space of A.

$$A = \begin{bmatrix} 2 & 1 & -3 & 4 \\ -1 & -1 & 2 & -1 \\ 0 & -1 & 1 & 2 \end{bmatrix}$$

2. Find the matrices C and L in the extended echelon form of A.

$$A = \begin{bmatrix} -9 & 5 & -3 \\ 2 & -1 & 1 \\ -5 & 3 & -1 \end{bmatrix}$$

3. Why is Theorem FS a great conclusion to Chapter M?

Exercises

C20 Example FSAG concludes with several questions. Perform the analysis suggested by these questions.

C25† Given the matrix A below, use the extended echelon form of A to answer each part of this problem. In each part, find a linearly independent set of vectors, S, so that the span of S, $\langle S\rangle$, equals the specified set of vectors.

$$A = \begin{bmatrix} -5 & 3 & -1 \\ -1 & 1 & 1 \\ -8 & 5 & -1 \\ 3 & -2 & 0 \end{bmatrix}$$

1. The row space of A, $\mathcal{R}(A)$.
2. The column space of A, $\mathcal{C}(A)$.
3. The null space of A, $\mathcal{N}(A)$.

4. The left null space of A, $\mathcal{L}(A)$.

C26† For the matrix D below use the extended echelon form to find:

1. A linearly independent set whose span is the column space of D.
2. A linearly independent set whose span is the left null space of D.

$$D = \begin{bmatrix} -7 & -11 & -19 & -15 \\ 6 & 10 & 18 & 14 \\ 3 & 5 & 9 & 7 \\ -1 & -2 & -4 & -3 \end{bmatrix}$$

C41 The following archetypes are systems of equations. For each system, write the vector of constants as a linear combination of the vectors in the span construction for the column space provided by Theorem FS and Theorem BNS (these vectors are listed for each of these archetypes).

Archetype A, Archetype B, Archetype C, Archetype D, Archetype E, Archetype F, Archetype G, Archetype H, Archetype I, Archetype J

C43 The following archetypes are either matrices or systems of equations with coefficient matrices. For each matrix, compute the extended echelon form N and identify the matrices C and L. Using Theorem FS, Theorem BNS and Theorem BRS express the null space, the row space, the column space and left null space of each coefficient matrix as a span of a linearly independent set.

Archetype A, Archetype B, Archetype C, Archetype D/Archetype E, Archetype F, Archetype G/Archetype H, Archetype I, Archetype J, Archetype K, Archetype L

C60† For the matrix B below, find sets of vectors whose span equals the column space of B ($\mathcal{C}(B)$) and which individually meet the following extra requirements.

1. The set illustrates the definition of the column space.
2. The set is linearly independent and the members of the set are columns of B.
3. The set is linearly independent with a "nice pattern of zeros and ones" at the *top* of each vector.
4. The set is linearly independent with a "nice pattern of zeros and ones" at the *bottom* of each vector.

$$B = \begin{bmatrix} 2 & 3 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ -1 & 2 & 3 & -4 \end{bmatrix}$$

C61† Let A be the matrix below, and find the indicated sets with the requested properties.

$$A = \begin{bmatrix} 2 & -1 & 5 & -3 \\ -5 & 3 & -12 & 7 \\ 1 & 1 & 4 & -3 \end{bmatrix}$$

1. A linearly independent set S so that $\mathcal{C}(A) = \langle S \rangle$ and S is composed of columns of A.
2. A linearly independent set S so that $\mathcal{C}(A) = \langle S \rangle$ and the vectors in S have a nice pattern of zeros and ones at the top of the vectors.
3. A linearly independent set S so that $\mathcal{C}(A) = \langle S \rangle$ and the vectors in S have a nice pattern of zeros and ones at the bottom of the vectors.
4. A linearly independent set S so that $\mathcal{R}(A) = \langle S \rangle$.

M50 Suppose that A is a nonsingular matrix. Extend the four conclusions of Theorem FS in this special case and discuss connections with previous results (such as Theorem NME4).

M51 Suppose that A is a singular matrix. Extend the four conclusions of Theorem FS in this special case and discuss connections with previous results (such as Theorem NME4).

Chapter VS
Vector Spaces

We now have a computational toolkit in place and so we can begin our study of linear algebra at a more theoretical level.

Linear algebra is the study of two fundamental objects, vector spaces and linear transformations (see Chapter LT). This chapter will focus on the former. The power of mathematics is often derived from generalizing many different situations into one abstract formulation, and that is exactly what we will be doing throughout this chapter.

Section VS
Vector Spaces

In this section we present a formal definition of a vector space, which will lead to an extra increment of abstraction. Once defined, we study its most basic properties.

Subsection VS
Vector Spaces

Here is one of the two most important definitions in the entire course.

Definition VS Vector Space
Suppose that V is a set upon which we have defined two operations: (1) **vector addition**, which combines two elements of V and is denoted by "+", and (2) **scalar multiplication**, which combines a complex number with an element of V and is denoted by juxtaposition. Then V, along with the two operations, is a **vector space** over $\mathbb{C}$ if the following ten properties hold.

- AC Additive Closure

  If $\mathbf{u}$, $\mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} \in V$.

- SC Scalar Closure

  If $\alpha \in \mathbb{C}$ and $\mathbf{u} \in V$, then $\alpha\mathbf{u} \in V$.

- C Commutativity

  If $\mathbf{u}$, $\mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

- AA Additive Associativity

  If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in V$, then $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.

- Z Zero Vector

  There is a vector, $\mathbf{0}$, called the zero vector, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u} \in V$.

- AI Additive Inverses

  If $\mathbf{u}\in V$, then there exists a vector $-\mathbf{u}\in V$ so that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.

- SMA Scalar Multiplication Associativity

  If $\alpha,\ \beta\in\mathbb{C}$ and $\mathbf{u}\in V$, then $\alpha(\beta\mathbf{u})=(\alpha\beta)\mathbf{u}$.

- DVA Distributivity across Vector Addition

  If $\alpha\in\mathbb{C}$ and $\mathbf{u},\ \mathbf{v}\in V$, then $\alpha(\mathbf{u}+\mathbf{v})=\alpha\mathbf{u}+\alpha\mathbf{v}$.

- DSA Distributivity across Scalar Addition

  If $\alpha,\ \beta\in\mathbb{C}$ and $\mathbf{u}\in V$, then $(\alpha+\beta)\mathbf{u}=\alpha\mathbf{u}+\beta\mathbf{u}$.

- O One

  If $\mathbf{u}\in V$, then $1\mathbf{u}=\mathbf{u}$.

The objects in V are called **vectors**, no matter what else they might really be, simply by virtue of being elements of a vector space. □

Now, there are several important observations to make. Many of these will be easier to understand on a second or third reading, and especially after carefully studying the examples in Subsection VS.EVS.

An **axiom** is often a "self-evident" truth. Something so fundamental that we all agree it is true and accept it without proof. Typically, it would be the logical underpinning that we would begin to build theorems upon. Some might refer to the ten properties of Definition VS as axioms, implying that a vector space is a very natural object and the ten properties are the essence of a vector space. We will instead emphasize that we will begin with a definition of a vector space. After studying the remainder of this chapter, you might return here and remind yourself how all our forthcoming theorems and definitions rest on this foundation.

As we will see shortly, the objects in V can be *anything*, even though we will call them vectors. We have been working with vectors frequently, but we should stress here that these have so far just been *column* vectors — scalars arranged in a columnar list of fixed length. In a similar vein, you have used the symbol "+" for many years to represent the addition of numbers (scalars). We have extended its use to the addition of column vectors and to the addition of matrices, and now we are going to recycle it even further and let it denote vector addition in *any* possible vector space. So when describing a new vector space, we will have to *define* exactly what "+" is. Similar comments apply to scalar multiplication. Conversely, we can *define* our operations any way we like, so long as the ten properties are fulfilled (see Example CVS).

In Definition VS, the scalars do not have to be complex numbers. They can come from what are called in more advanced mathematics, "fields". Examples of fields are the set of complex numbers, the set of real numbers, the set of rational numbers, and even the finite set of "binary numbers", $\{0,\ 1\}$. There are many, many others. In this case we would call V a **vector space** over (the field) F.

A vector space is composed of three objects, a set and two operations. Some would explicitly state in the definition that V must be a nonempty set, but we can infer this from Property Z, since the set cannot be empty and contain a vector that behaves as the zero vector. Also, we usually use the same symbol for both the set and the vector space itself. Do not let this convenience fool you into thinking the operations are secondary!

This discussion has either convinced you that we are really embarking on a new level of abstraction, or it has seemed cryptic, mysterious or nonsensical. You might want to return to this section in a few days and give it another read then. In any case, let us look at some concrete examples now.

Subsection EVS
Examples of Vector Spaces

Our aim in this subsection is to give you a storehouse of examples to work with, to become comfortable with the ten vector space properties and to convince you that the multitude of examples justifies (at least initially) making such a broad definition as Definition VS. Some of our claims will be justified by reference to previous theorems, we will prove some facts from scratch, and we will do one nontrivial example completely. In other places, our usual thoroughness will be neglected, so grab paper and pencil and play along.

Example VSCV The vector space $\mathbb{C}^m$
Set: $\mathbb{C}^m$, all column vectors of size m, Definition VSCV.

Equality: Entry-wise, Definition CVE.

Vector Addition: The "usual" addition, given in Definition CVA.

Scalar Multiplication: The "usual" scalar multiplication, given in Definition CVSM.

Does this set with these operations fulfill the ten properties? Yes. And by design all we need to do is quote Theorem VSPCV. That was easy. △

Example VSM The vector space of matrices, M_{mn}
Set: M_{mn}, the set of all matrices of size $m \times n$ and entries from $\mathbb{C}$, Definition VSM.

Equality: Entry-wise, Definition ME.

Vector Addition: The "usual" addition, given in Definition MA.

Scalar Multiplication: The "usual" scalar multiplication, given in Definition MSM.

Does this set with these operations fulfill the ten properties? Yes. And all we need to do is quote Theorem VSPM. Another easy one (by design). △

So, the set of all matrices of a fixed size forms a vector space. That entitles us to call a matrix a vector, since a matrix is an element of a vector space. For example, if $A,\ B \in M_{3,4}$ then we call A and B "vectors," and we even use our previous notation for column vectors to refer to A and B. So we could legitimately write expressions like

$$\mathbf{u}+\mathbf{v} = A + B = B + A = \mathbf{v}+\mathbf{u}$$

This could lead to some confusion, but it is not too great a danger. But it is worth comment.

The previous two examples may be less than satisfying. We made all the relevant definitions long ago. And the required verifications were all handled by quoting old theorems. However, it is important to consider these two examples first. We have been studying vectors and matrices carefully (Chapter V, Chapter M), and both objects, along with their operations, have certain properties in common, as you may have noticed in comparing Theorem VSPCV with Theorem VSPM. Indeed, it is these two theorems that *motivate* us to formulate the abstract definition of a vector space, Definition VS. Now, if we prove some general theorems about vector spaces (as we will shortly in Subsection VS.VSP), we can then instantly apply the conclusions to *both* $\mathbb{C}^m$ and M_{mn}. Notice too, how we have taken six definitions and two theorems and reduced them down to two *examples*. With greater generalization and abstraction our old ideas get downgraded in stature.

Let us look at some more examples, now considering some new vector spaces.

Example VSP The vector space of polynomials, P_n
Set: P_n, the set of all polynomials of degree n or less in the variable x with coefficients from $\mathbb{C}$.

Equality:

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n = b_0 + b_1x + b_2x^2 + \cdots + b_nx^n$$

$$\text{if and only if } a_i = b_i \text{ for } 0 \leq i \leq n$$

Vector Addition:

$$(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) + (b_0 + b_1x + b_2x^2 + \cdots + b_nx^n) = \\ (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots + (a_n + b_n)x^n$$

Scalar Multiplication:

$$\alpha(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n) = (\alpha a_0) + (\alpha a_1)x + (\alpha a_2)x^2 + \cdots + (\alpha a_n)x^n$$

This set, with these operations, will fulfill the ten properties, though we will not work all the details here. However, we will make a few comments and prove one of the properties. First, the zero vector (Property Z) is what you might expect, and you can check that it has the required property.

$$\mathbf{0} = 0 + 0x + 0x^2 + \cdots + 0x^n$$

The additive inverse (Property AI) is also no surprise, though consider how we have chosen to write it.

$$-\left(a_0 + a_1x + a_2x^2 + \cdots + a_nx^n\right) = (-a_0) + (-a_1)x + (-a_2)x^2 + \cdots + (-a_n)x^n$$

Now let us prove the associativity of vector addition (Property AA). This is a bit tedious, though necessary. Throughout, the plus sign ("+") does triple-duty. You might ask yourself what each plus sign represents as you work through this proof.

$$\begin{aligned}
\mathbf{u}+&(\mathbf{v}+\mathbf{w})\\
&= (a_0 + a_1x + \cdots + a_nx^n) + ((b_0 + b_1x + \cdots + b_nx^n) + (c_0 + c_1x + \cdots + c_nx^n))\\
&= (a_0 + a_1x + \cdots + a_nx^n) + ((b_0 + c_0) + (b_1 + c_1)x + \cdots + (b_n + c_n)x^n)\\
&= (a_0 + (b_0 + c_0)) + (a_1 + (b_1 + c_1))x + \cdots + (a_n + (b_n + c_n))x^n\\
&= ((a_0 + b_0) + c_0) + ((a_1 + b_1) + c_1)x + \cdots + ((a_n + b_n) + c_n)x^n\\
&= ((a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_n + b_n)x^n) + (c_0 + c_1x + \cdots + c_nx^n)\\
&= ((a_0 + a_1x + \cdots + a_nx^n) + (b_0 + b_1x + \cdots + b_nx^n)) + (c_0 + c_1x + \cdots + c_nx^n)\\
&= (\mathbf{u}+\mathbf{v})+\mathbf{w}
\end{aligned}$$

Notice how it is the application of the associativity of the (old) addition of complex numbers in the middle of this chain of equalities that makes the whole proof happen. The remainder is successive applications of our (new) definition of vector (polynomial) addition. Proving the remainder of the ten properties is similar in style and tedium. You might try proving the commutativity of vector addition (Property C), or one of the distributivity properties (Property DVA, Property DSA). △

Example VSIS The vector space of infinite sequences
Set: $\mathbb{C}^\infty = \{(c_0,\, c_1,\, c_2,\, c_3,\, \dots) |\, c_i \in \mathbb{C},\ i \in \mathbb{N}\}$.

Equality:

$$(c_0,\, c_1,\, c_2,\, \dots) = (d_0,\, d_1,\, d_2,\, \dots) \text{ if and only if } c_i = d_i \text{ for all } i \geq 0$$

Vector Addition:

$$(c_0,\, c_1,\, c_2,\, \dots) + (d_0,\, d_1,\, d_2,\, \dots) = (c_0 + d_0,\, c_1 + d_1,\, c_2 + d_2,\, \dots)$$

Scalar Multiplication:

$$\alpha(c_0,\, c_1,\, c_2,\, c_3,\, \dots) = (\alpha c_0,\, \alpha c_1,\, \alpha c_2,\, \alpha c_3,\, \dots)$$

This should remind you of the vector space $\mathbb{C}^m$, though now our lists of scalars are written horizontally with commas as delimiters and they are allowed to be infinite in length. What does the zero vector look like (Property Z)? Additive inverses (Property AI)? Can you prove the associativity of vector addition (Property AA)? △

Example VSF The vector space of functions

Let X be any set.

Set: $F = \{f \mid f : X \to \mathbb{C}\}$.

Equality: $f = g$ if and only if $f(x) = g(x)$ for all $x \in X$.

Vector Addition: $f + g$ is the function with outputs defined by $(f + g)(x) = f(x) + g(x)$.

Scalar Multiplication: αf is the function with outputs defined by $(\alpha f)(x) = \alpha f(x)$.

So this is the set of all functions of one variable that take elements of the set X to a complex number. You might have studied functions of one variable that take a real number to a real number, and that might be a more natural set to use as X. But since we are allowing our scalars to be complex numbers, we need to specify that the range of our functions is the complex numbers. Study carefully how the definitions of the operation are made, and think about the different uses of "+" and juxtaposition. As an example of what is required when verifying that this is a vector space, consider that the zero vector (Property Z) is the function z whose definition is $z(x) = 0$ for every input $x \in X$.

Vector spaces of functions are very important in mathematics and physics, where the field of scalars may be the real numbers, so the ranges of the functions can in turn also be the set of real numbers. △

Here is a unique example.

Example VSS The singleton vector space

Set: $Z = \{\mathbf{z}\}$.

Equality: Huh?

Vector Addition: $\mathbf{z} + \mathbf{z} = \mathbf{z}$.

Scalar Multiplication: $\alpha\mathbf{z} = \mathbf{z}$.

This should look pretty wild. First, just what is $\mathbf{z}$? Column vector, matrix, polynomial, sequence, function? Mineral, plant, or animal? We aren't saying! $\mathbf{z}$ just *is*. And we have definitions of vector addition and scalar multiplication that are sufficient for an occurrence of either that may come along.

Our only concern is if this set, along with the definitions of two operations, fulfills the ten properties of Definition VS. Let us check associativity of vector addition (Property AA). For all $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in Z$,

$$\begin{aligned}\mathbf{u} + (\mathbf{v} + \mathbf{w}) &= \mathbf{z} + (\mathbf{z} + \mathbf{z})\\ &= \mathbf{z} + \mathbf{z}\\ &= (\mathbf{z} + \mathbf{z}) + \mathbf{z}\\ &= (\mathbf{u} + \mathbf{v}) + \mathbf{w}\end{aligned}$$

What is the zero vector in this vector space (Property Z)? With only one element in the set, we do not have much choice. Is $\mathbf{z} = \mathbf{0}$? It appears that $\mathbf{z}$ behaves like the zero vector should, so it gets the title. Maybe now the definition of this vector space does not seem so bizarre. It is a set whose only element is the element that behaves like the zero vector, so that lone element *is* the zero vector. △

Perhaps some of the above definitions and verifications seem obvious or like splitting hairs, but the next example should convince you that they *are* necessary. We will study this one carefully. Ready? Check your preconceptions at the door.

Example CVS The crazy vector space

Set: $C = \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{C}\}$.

Vector Addition: $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1 + 1, x_2 + y_2 + 1)$.

Scalar Multiplication: $\alpha(x_1, x_2) = (\alpha x_1 + \alpha - 1, \alpha x_2 + \alpha - 1)$.

Now, the first thing I hear you say is "You can't do that!" And my response is, "Oh yes, I can!" I am free to define my set and my operations any way I please. They may not look natural, or even useful, but we will now verify that they provide us with another example of a vector space. And that is enough. If you are adventurous, you might try first checking some of the properties yourself. What is the zero vector? Additive inverses? Can you prove associativity? Ready, here we go.

Property AC, Property SC: The result of each operation is a pair of complex numbers, so these two closure properties are fulfilled.

Property C:

$$\begin{aligned}\mathbf{u}+\mathbf{v} &= (x_1,\,x_2)+(y_1,\,y_2)=(x_1+y_1+1,\,x_2+y_2+1)\\ &= (y_1+x_1+1,\,y_2+x_2+1)=(y_1,\,y_2)+(x_1,\,x_2)\\ &= \mathbf{v}+\mathbf{u}\end{aligned}$$

Property AA:

$$\begin{aligned}\mathbf{u}+(\mathbf{v}+\mathbf{w}) &= (x_1,\,x_2)+((y_1,\,y_2)+(z_1,\,z_2))\\ &= (x_1,\,x_2)+(y_1+z_1+1,\,y_2+z_2+1)\\ &= (x_1+(y_1+z_1+1)+1,\,x_2+(y_2+z_2+1)+1)\\ &= (x_1+y_1+z_1+2,\,x_2+y_2+z_2+2)\\ &= ((x_1+y_1+1)+z_1+1,\,(x_2+y_2+1)+z_2+1)\\ &= (x_1+y_1+1,\,x_2+y_2+1)+(z_1,\,z_2)\\ &= ((x_1,\,x_2)+(y_1,\,y_2))+(z_1,\,z_2)\\ &= (\mathbf{u}+\mathbf{v})+\mathbf{w}\end{aligned}$$

Property Z: The zero vector is ... $\mathbf{0}=(-1,\,-1)$. Now I hear you say, "No, no, that can't be, it must be $(0,\,0)$!". Indulge me for a moment and let us check my proposal.

$$\mathbf{u}+\mathbf{0}=(x_1,\,x_2)+(-1,\,-1)=(x_1+(-1)+1,\,x_2+(-1)+1)=(x_1,\,x_2)=\mathbf{u}$$

Feeling better? Or worse?

Property AI: For each vector, $\mathbf{u}$, we must locate an additive inverse, $-\mathbf{u}$. Here it is, $-(x_1,\,x_2)=(-x_1-2,\,-x_2-2)$. As odd as it may look, I hope you are withholding judgment. Check:

$$\begin{aligned}\mathbf{u}+(-\mathbf{u}) &= (x_1,\,x_2)+(-x_1-2,\,-x_2-2)\\ &= (x_1+(-x_1-2)+1,\,-x_2+(x_2-2)+1)=(-1,\,-1)=\mathbf{0}\end{aligned}$$

Property SMA:

$$\begin{aligned}\alpha(\beta\mathbf{u}) &= \alpha(\beta(x_1,\,x_2))\\ &= \alpha(\beta x_1+\beta-1,\,\beta x_2+\beta-1)\\ &= (\alpha(\beta x_1+\beta-1)+\alpha-1,\,\alpha(\beta x_2+\beta-1)+\alpha-1)\\ &= ((\alpha\beta x_1+\alpha\beta-\alpha)+\alpha-1,\,(\alpha\beta x_2+\alpha\beta-\alpha)+\alpha-1)\\ &= (\alpha\beta x_1+\alpha\beta-1,\,\alpha\beta x_2+\alpha\beta-1)\\ &= (\alpha\beta)(x_1,\,x_2)\\ &= (\alpha\beta)\mathbf{u}\end{aligned}$$

Property DVA: If you have hung on so far, here is where it gets even wilder. In the next two properties we mix and mash the two operations.

$$\begin{aligned}&\alpha(\mathbf{u}+\mathbf{v})\\ &\quad= \alpha\left((x_1,\,x_2)+(y_1,\,y_2)\right)\\ &\quad= \alpha(x_1+y_1+1,\,x_2+y_2+1)\\ &\quad= (\alpha(x_1+y_1+1)+\alpha-1,\,\alpha(x_2+y_2+1)+\alpha-1)\\ &\quad= (\alpha x_1+\alpha y_1+\alpha+\alpha-1,\,\alpha x_2+\alpha y_2+\alpha+\alpha-1)\\ &\quad= (\alpha x_1+\alpha-1+\alpha y_1+\alpha-1+1,\,\alpha x_2+\alpha-1+\alpha y_2+\alpha-1+1)\end{aligned}$$

$$\begin{aligned}
&= ((\alpha x_1 + \alpha - 1) + (\alpha y_1 + \alpha - 1) + 1,\ (\alpha x_2 + \alpha - 1) + (\alpha y_2 + \alpha - 1) + 1) \\
&= (\alpha x_1 + \alpha - 1,\ \alpha x_2 + \alpha - 1) + (\alpha y_1 + \alpha - 1,\ \alpha y_2 + \alpha - 1) \\
&= \alpha(x_1,\ x_2) + \alpha(y_1,\ y_2) \\
&= \alpha\mathbf{u} + \alpha\mathbf{v}
\end{aligned}$$

Property DSA:

$$\begin{aligned}
&(\alpha + \beta)\mathbf{u} \\
&\quad = (\alpha + \beta)(x_1,\ x_2) \\
&\quad = ((\alpha + \beta)x_1 + (\alpha + \beta) - 1,\ (\alpha + \beta)x_2 + (\alpha + \beta) - 1) \\
&\quad = (\alpha x_1 + \beta x_1 + \alpha + \beta - 1,\ \alpha x_2 + \beta x_2 + \alpha + \beta - 1) \\
&\quad = (\alpha x_1 + \alpha - 1 + \beta x_1 + \beta - 1 + 1,\ \alpha x_2 + \alpha - 1 + \beta x_2 + \beta - 1 + 1) \\
&\quad = ((\alpha x_1 + \alpha - 1) + (\beta x_1 + \beta - 1) + 1,\ (\alpha x_2 + \alpha - 1) + (\beta x_2 + \beta - 1) + 1) \\
&\quad = (\alpha x_1 + \alpha - 1,\ \alpha x_2 + \alpha - 1) + (\beta x_1 + \beta - 1,\ \beta x_2 + \beta - 1) \\
&\quad = \alpha(x_1,\ x_2) + \beta(x_1,\ x_2) \\
&\quad = \alpha\mathbf{u} + \beta\mathbf{u}
\end{aligned}$$

Property O: After all that, this one is easy, but no less pleasing.

$$1\mathbf{u} = 1(x_1,\ x_2) = (x_1 + 1 - 1,\ x_2 + 1 - 1) = (x_1,\ x_2) = \mathbf{u}$$

That is it, C is a vector space, as crazy as that may seem.

Notice that in the case of the zero vector and additive inverses, we only had to propose possibilities and then verify that they were the correct choices. You might try to discover how you would arrive at these choices, though you should understand why the process of discovering them is not a necessary component of the proof itself. △

Subsection VSP
Vector Space Properties

Subsection VS.EVS has provided us with an abundance of examples of vector spaces, most of them containing useful and interesting mathematical objects along with natural operations. In this subsection we will prove some general properties of vector spaces. Some of these results will again seem obvious, but it is important to understand why it is necessary to state and prove them. A typical hypothesis will be "Let V be a vector space." From this we may assume the ten properties of Definition VS, *and nothing more*. It is like starting over, as we learn about what can happen in this new algebra we are learning. But the power of this careful approach is that we can apply these theorems to any vector space we encounter — those in the previous examples, or new ones we have not yet contemplated. Or perhaps new ones that nobody has ever contemplated. We will illustrate some of these results with examples from the crazy vector space (Example CVS), but mostly we are stating theorems and doing proofs. These proofs do not get too involved, but are not trivial either, so these are good theorems to try proving yourself before you study the proof given here. (See Proof Technique P.)

First we show that there is just one zero vector. Notice that the properties only require there to be *at least* one, and say nothing about there possibly being more. That is because we can use the ten properties of a vector space (Definition VS) to learn that there can *never* be more than one. To require that this extra condition be stated as an eleventh property would make the definition of a vector space more complicated than it needs to be.

Theorem ZVU Zero Vector is Unique
Suppose that V is a vector space. The zero vector, $\mathbf{0}$, is unique.

Proof. To prove uniqueness, a standard technique is to suppose the existence of two objects (Proof Technique U). So let $\mathbf{0}_1$ and $\mathbf{0}_2$ be two zero vectors in V. Then

$$\begin{aligned}
\mathbf{0}_1 &= \mathbf{0}_1 + \mathbf{0}_2 && \text{Property Z for } \mathbf{0}_2\\
&= \mathbf{0}_2 + \mathbf{0}_1 && \text{Property C}\\
&= \mathbf{0}_2 && \text{Property Z for } \mathbf{0}_1
\end{aligned}$$

This proves the uniqueness since the two zero vectors are really the same. ■

Theorem AIU Additive Inverses are Unique
Suppose that V is a vector space. For each $\mathbf{u} \in V$, the additive inverse, $-\mathbf{u}$, is unique.

Proof. To prove uniqueness, a standard technique is to suppose the existence of two objects (Proof Technique U). So let $-\mathbf{u}_1$ and $-\mathbf{u}_2$ be two additive inverses for $\mathbf{u}$. Then

$$\begin{aligned}
-\mathbf{u}_1 &= -\mathbf{u}_1 + \mathbf{0} && \text{Property Z}\\
&= -\mathbf{u}_1 + (\mathbf{u} + -\mathbf{u}_2) && \text{Property AI}\\
&= (-\mathbf{u}_1 + \mathbf{u}) + -\mathbf{u}_2 && \text{Property AA}\\
&= \mathbf{0} + -\mathbf{u}_2 && \text{Property AI}\\
&= -\mathbf{u}_2 && \text{Property Z}
\end{aligned}$$

So the two additive inverses are really the same. ■

As obvious as the next three theorems appear, nowhere have we guaranteed that the zero scalar, scalar multiplication and the zero vector all interact this way. Until we have proved it, anyway.

Theorem ZSSM Zero Scalar in Scalar Multiplication
Suppose that V is a vector space and $\mathbf{u} \in V$. Then $0\mathbf{u} = \mathbf{0}$.

Proof. Notice that 0 is a scalar, $\mathbf{u}$ is a vector, so Property SC says $0\mathbf{u}$ is again a vector. As such, $0\mathbf{u}$ has an additive inverse, $-(0\mathbf{u})$ by Property AI.

$$\begin{aligned}
0\mathbf{u} &= \mathbf{0} + 0\mathbf{u} && \text{Property Z}\\
&= (-(0\mathbf{u}) + 0\mathbf{u}) + 0\mathbf{u} && \text{Property AI}\\
&= -(0\mathbf{u}) + (0\mathbf{u} + 0\mathbf{u}) && \text{Property AA}\\
&= -(0\mathbf{u}) + (0 + 0)\mathbf{u} && \text{Property DSA}\\
&= -(0\mathbf{u}) + 0\mathbf{u} && \text{Property ZCN}\\
&= \mathbf{0} && \text{Property AI}
\end{aligned}$$

■

Here is another theorem that *looks* like it should be obvious, but is still in need of a proof.

Theorem ZVSM Zero Vector in Scalar Multiplication
Suppose that V is a vector space and $\alpha \in \mathbb{C}$. Then $\alpha\mathbf{0} = \mathbf{0}$.

Proof. Notice that α is a scalar, $\mathbf{0}$ is a vector, so Property SC means $\alpha\mathbf{0}$ is again a vector. As such, $\alpha\mathbf{0}$ has an additive inverse, $-(\alpha\mathbf{0})$ by Property AI.

$$\begin{aligned}
\alpha\mathbf{0} &= \mathbf{0} + \alpha\mathbf{0} && \text{Property Z}\\
&= (-(\alpha\mathbf{0}) + \alpha\mathbf{0}) + \alpha\mathbf{0} && \text{Property AI}\\
&= -(\alpha\mathbf{0}) + (\alpha\mathbf{0} + \alpha\mathbf{0}) && \text{Property AA}\\
&= -(\alpha\mathbf{0}) + \alpha\left(\mathbf{0} + \mathbf{0}\right) && \text{Property DVA}\\
&= -(\alpha\mathbf{0}) + \alpha\mathbf{0} && \text{Property Z}\\
&= \mathbf{0} && \text{Property AI}
\end{aligned}$$

Here is another one that sure looks obvious. But understand that we have chosen to use certain notation because it makes the theorem's conclusion look so nice. The theorem is not true because the notation looks so good; it still needs a proof. If we had really wanted to make this point, we might have used notation like $\mathbf{u}^\sharp$ for the additive inverse of $\mathbf{u}$. Then we would have written the defining property, Property AI, as $\mathbf{u}+\mathbf{u}^\sharp=\mathbf{0}$. This theorem would become $\mathbf{u}^\sharp=(-1)\mathbf{u}$. Not really quite as pretty, is it?

Theorem AISM Additive Inverses from Scalar Multiplication
Suppose that V is a vector space and $\mathbf{u}\in V$. Then $-\mathbf{u}=(-1)\mathbf{u}$.

Proof.

$$\begin{aligned}
-\mathbf{u} &= -\mathbf{u}+\mathbf{0} && \text{Property Z}\\
&= -\mathbf{u}+0\mathbf{u} && \text{Theorem ZSSM}\\
&= -\mathbf{u}+(1+(-1))\,\mathbf{u} && \\
&= -\mathbf{u}+(1\mathbf{u}+(-1)\mathbf{u}) && \text{Property DSA}\\
&= -\mathbf{u}+(\mathbf{u}+(-1)\mathbf{u}) && \text{Property O}\\
&= (-\mathbf{u}+\mathbf{u})+(-1)\mathbf{u} && \text{Property AA}\\
&= \mathbf{0}+(-1)\mathbf{u} && \text{Property AI}\\
&= (-1)\mathbf{u} && \text{Property Z}
\end{aligned}$$

■

Because of this theorem, we can now write linear combinations like $6\mathbf{u}_1+(-4)\mathbf{u}_2$ as $6\mathbf{u}_1-4\mathbf{u}_2$, even though we have not formally defined an operation called **vector subtraction**.

Our next theorem is a bit different from several of the others in the list. Rather than making a declaration ("the zero vector is unique") it is an implication ("if…, then…") and so can be used in proofs to convert a vector equality into two possibilities, one a scalar equality and the other a vector equality. It should remind you of the situation for complex numbers. If $\alpha,\ \beta\in\mathbb{C}$ and $\alpha\beta=0$, then $\alpha=0$ or $\beta=0$. This critical property is the driving force behind using a factorization to solve a polynomial equation.

Theorem SMEZV Scalar Multiplication Equals the Zero Vector
Suppose that V is a vector space and $\alpha\in\mathbb{C}$. If $\alpha\mathbf{u}=\mathbf{0}$, then either $\alpha=0$ or $\mathbf{u}=\mathbf{0}$.

Proof. We prove this theorem by breaking up the analysis into two cases. The first seems too trivial, and it is, but the logic of the argument is still legitimate.

Case 1. Suppose $\alpha=0$. In this case our conclusion is true (the first part of the either/or is true) and we are done. That was easy.

Case 2. Suppose $\alpha\neq 0$.

$$\begin{aligned}
\mathbf{u} &= 1\mathbf{u} && \text{Property O}\\
&= \left(\frac{1}{\alpha}\alpha\right)\mathbf{u} && \alpha\neq 0\\
&= \frac{1}{\alpha}(\alpha\mathbf{u}) && \text{Property SMA}\\
&= \frac{1}{\alpha}(\mathbf{0}) && \text{Hypothesis}\\
&= \mathbf{0} && \text{Theorem ZVSM}
\end{aligned}$$

So in this case, the conclusion is true (the second part of the either/or is true) and we are done since the conclusion was true in each of the two cases. ■

Example PCVS Properties for the Crazy Vector Space
Several of the above theorems have interesting demonstrations when applied to the crazy vector space, C (Example CVS). We are not proving anything new here, or learning anything we did not know already about C. It is just plain fun to see how these general theorems apply in a specific instance. For most of our examples, the applications are obvious or trivial, but not with C.

Suppose $\mathbf{u} \in C$. Then, as given by Theorem ZSSM,

$$0\mathbf{u} = 0(x_1,\, x_2) = (0x_1 + 0 - 1,\, 0x_2 + 0 - 1) = (-1, -1) = \mathbf{0}$$

And as given by Theorem ZVSM,

$$\begin{aligned}\alpha\mathbf{0} &= \alpha(-1,\, -1) = (\alpha(-1) + \alpha - 1,\, \alpha(-1) + \alpha - 1)\\ &= (-\alpha + \alpha - 1,\, -\alpha + \alpha - 1) = (-1,\, -1) = \mathbf{0}\end{aligned}$$

Finally, as given by Theorem AISM,

$$\begin{aligned}(-1)\mathbf{u} &= (-1)(x_1,\, x_2) = ((-1)x_1 + (-1) - 1,\, (-1)x_2 + (-1) - 1)\\ &= (-x_1 - 2,\, -x_2 - 2) = -\mathbf{u}\end{aligned}$$

△

Subsection RD
Recycling Definitions

When we say that V is a vector space, we then know we have a set of objects (the "vectors"), but we also know we have been provided with two operations ("vector addition" and "scalar multiplication") and these operations behave with these objects according to the ten properties of Definition VS. One combines two vectors and produces a vector, the other takes a scalar and a vector, producing a vector as the result. So if $\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3 \in V$ then an expression like

$$5\mathbf{u}_1 + 7\mathbf{u}_2 - 13\mathbf{u}_3$$

would be unambiguous in *any* of the vector spaces we have discussed in this section. And the resulting object would be another vector in the vector space. If you were tempted to call the above expression a linear combination, you would be right. Four of the definitions that were central to our discussions in Chapter V were stated in the context of vectors being *column vectors*, but were purposely kept broad enough that they could be applied in the context of any vector space. They only rely on the presence of scalars, vectors, vector addition and scalar multiplication to make sense. We will restate them shortly, unchanged, except that their titles and acronyms no longer refer to column vectors, and the hypothesis of being in a vector space has been added. Take the time now to look forward and review each one, and begin to form some connections to what we have done earlier and what we will be doing in subsequent sections and chapters. Specifically, compare the following pairs of definitions:

- Definition LCCV and Definition LC
- Definition SSCV and Definition SS
- Definition RLDCV and Definition RLD
- Definition LICV and Definition LI

Reading Questions

1. Comment on how the vector space $\mathbb{C}^m$ went from a theorem (Theorem VSPCV) to an example (Example VSCV).

2. In the crazy vector space, C, (Example CVS) compute the linear combination

$$2(3,\,4)+(-6)(1,\,2).$$

3. Suppose that α is a scalar and $\mathbf{0}$ is the zero vector. Why should we prove anything as obvious as $\alpha\mathbf{0}=\mathbf{0}$ such as we did in Theorem ZVSM?

Exercises

M10 Define a possibly new vector space by beginning with the set and vector addition from $\mathbb{C}^2$ (Example VSCV) but change the definition of scalar multiplication to

$$\alpha\mathbf{x}=\mathbf{0}=\begin{bmatrix}0\\0\end{bmatrix} \qquad\qquad \alpha\in\mathbb{C},\ \mathbf{x}\in\mathbb{C}^2$$

Prove that the first nine properties required for a vector space hold, but Property O does not hold.

This example shows us that we cannot expect to be able to derive Property O as a consequence of assuming the first nine properties. In other words, we cannot slim down our list of properties by jettisoning the last one, and still have the same collection of objects qualify as vector spaces.

M11† Let V be the set $\mathbb{C}^2$ with the usual vector addition, but with scalar multiplication defined by

$$\alpha\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}\alpha y\\\alpha x\end{bmatrix}$$

Determine whether or not V is a vector space with these operations.

M12† Let V be the set $\mathbb{C}^2$ with the usual scalar multiplication, but with vector addition defined by

$$\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}z\\w\end{bmatrix}=\begin{bmatrix}y+w\\x+z\end{bmatrix}$$

Determine whether or not V is a vector space with these operations.

M13† Let V be the set $M_{2,2}$ with the usual scalar multiplication, but with addition defined by $A+B=\mathcal{O}_{2,2}$ for all 2×2 matrices A and B. Determine whether or not V is a vector space with these operations.

M14† Let V be the set $M_{2,2}$ with the usual addition, but with scalar multiplication defined by $\alpha A=\mathcal{O}_{2,2}$ for all 2×2 matrices A and scalars α. Determine whether or not V is a vector space with these operations.

M15† Consider the following sets of 3×3 matrices, where the symbol $*$ indicates the position of an arbitrary complex number. Determine whether or not these sets form vector spaces with the usual operations of addition and scalar multiplication for matrices.

1. All matrices of the form $\begin{bmatrix}* & * & 1\\ * & 1 & *\\ 1 & * & *\end{bmatrix}$

2. All matrices of the form $\begin{bmatrix}* & 0 & *\\ 0 & * & 0\\ * & 0 & *\end{bmatrix}$

3. All matrices of the form $\begin{bmatrix}* & 0 & 0\\ 0 & * & 0\\ 0 & 0 & *\end{bmatrix}$ (These are the **diagonal** matrices.)

4. All matrices of the form $\begin{bmatrix}* & * & *\\ 0 & * & *\\ 0 & 0 & *\end{bmatrix}$ (These are the **upper triangular** matrices.)

M20† Explain why we need to define the vector space P_n as the set of all polynomials with degree *up to and including* n instead of the more obvious set of all polynomials of degree *exactly* n.

M21† The set of integers is denoted $\mathbb{Z}$. Does the set $\mathbb{Z}^2 = \left\{ \begin{bmatrix} m \\ n \end{bmatrix} \middle| m, n \in \mathbb{Z} \right\}$ with the operations of standard addition and scalar multiplication of vectors form a vector space?

T10 Prove each of the ten properties of Definition VS for each of the following examples of a vector space: Example VSP, Example VSIS, Example VSF, Example VSS.

The next three problems suggest that under the right situations we can "cancel." In practice, these techniques should be avoided in other proofs. Prove each of the following statements.

T21† Suppose that V is a vector space, and $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w} \in V$. If $\mathbf{w} + \mathbf{u} = \mathbf{w} + \mathbf{v}$, then $\mathbf{u} = \mathbf{v}$.

T22† Suppose V is a vector space, $\mathbf{u}$, $\mathbf{v} \in V$ and α is a nonzero scalar from $\mathbb{C}$. If $\alpha\mathbf{u} = \alpha\mathbf{v}$, then $\mathbf{u} = \mathbf{v}$.

T23† Suppose V is a vector space, $\mathbf{u} \neq \mathbf{0}$ is a vector in V and α, $\beta \in \mathbb{C}$. If $\alpha\mathbf{u} = \beta\mathbf{u}$, then $\alpha = \beta$.

T30† Suppose that V is a vector space and $\alpha \in \mathbb{C}$ is a scalar such that $\alpha\mathbf{x} = \mathbf{x}$ for every $\mathbf{x} \in V$. Prove that $\alpha = 1$. In other words, Property O is not duplicated for any other scalar but the "special" scalar, 1. (This question was suggested by James Gallagher.)

Section S
Subspaces

A subspace is a vector space that is contained within another vector space. So every subspace is a vector space in its own right, but it is also defined relative to some other (larger) vector space. We will discover shortly that we are already familiar with a wide variety of subspaces from previous sections.

Subsection S
Subspaces

Here is the principal definition for this section.

Definition S Subspace
Suppose that V and W are two vector spaces that have identical definitions of vector addition and scalar multiplication, and that W is a subset of V, $W \subseteq V$. Then W is a **subspace** of V. □

Let us look at an example of a vector space inside another vector space.

Example SC3 A subspace of $\mathbb{C}^3$
We know that $\mathbb{C}^3$ is a vector space (Example VSCV). Consider the subset,

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| \, 2x_1 - 5x_2 + 7x_3 = 0 \right\}$$

It is clear that $W \subseteq \mathbb{C}^3$, since the objects in W are column vectors of size 3. But is W a vector space? Does it satisfy the ten properties of Definition VS when we use the same operations? That is the main question.

Suppose $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ are vectors from W. Then we know that these vectors cannot be totally arbitrary, they must have gained membership in W by virtue of meeting the membership test. For example, we know that $\mathbf{x}$ must satisfy $2x_1 - 5x_2 + 7x_3 = 0$ while $\mathbf{y}$ must satisfy $2y_1 - 5y_2 + 7y_3 = 0$. Our first property (Property AC) asks the question, is $\mathbf{x} + \mathbf{y} \in W$? When our set of vectors was $\mathbb{C}^3$, this was an easy question to answer. Now it is not so obvious. Notice first that

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}$$

and we can test this vector for membership in W as follows. Because $\mathbf{x} \in W$ we know $2x_1 - 5x_2 + 7x_3 = 0$ and because $\mathbf{y} \in W$ we know $2y_1 - 5y_2 + 7y_3 = 0$. Therefore,

$$\begin{aligned} 2(x_1 + y_1) - 5(x_2 + y_2) + 7(x_3 + y_3) &= 2x_1 + 2y_1 - 5x_2 - 5y_2 + 7x_3 + 7y_3 \\ &= (2x_1 - 5x_2 + 7x_3) + (2y_1 - 5y_2 + 7y_3) \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

and by this computation we see that $\mathbf{x} + \mathbf{y} \in W$. One property down, nine to go.

If α is a scalar and $\mathbf{x} \in W$, is it always true that $\alpha\mathbf{x} \in W$? This is what we need to establish Property SC. Again, the answer is not as obvious as it was when our set of vectors was all of $\mathbb{C}^3$. Let us see. First, note that because $\mathbf{x} \in W$ we know $2x_1 - 5x_2 + 7x_3 = 0$. Therefore,

$$\alpha\mathbf{x} = \alpha \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \end{bmatrix}$$

and we can test this vector for membership in W. First, note that because $\mathbf{x} \in W$ we know $2x_1 - 5x_2 + 7x_3 = 0$. Therefore,

$$\begin{aligned} 2(\alpha x_1) - 5(\alpha x_2) + 7(\alpha x_3) &= \alpha(2x_1 - 5x_2 + 7x_3) \\ &= \alpha 0 \\ &= 0 \end{aligned}$$

and we see that indeed $\alpha\mathbf{x} \in W$. Always.

If W has a zero vector, it will be unique (Theorem ZVU). The zero vector for $\mathbb{C}^3$ should also perform the required duties when added to elements of W. So the likely candidate for a zero vector in W is the same zero vector that we know $\mathbb{C}^3$ has. You can check that $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is a zero vector in W too (Property Z).

With a zero vector, we can now ask about additive inverses (Property AI). As you might suspect, the natural candidate for an additive inverse in W is the same as the additive inverse from $\mathbb{C}^3$. However, we must insure that these additive inverses actually are elements of W. Given $\mathbf{x} \in W$, is $-\mathbf{x} \in W$?

$$-\mathbf{x} = \begin{bmatrix} -x_1 \\ -x_2 \\ -x_3 \end{bmatrix}$$

and we can test this vector for membership in W. As before, because $\mathbf{x} \in W$ we know $2x_1 - 5x_2 + 7x_3 = 0$.

$$\begin{aligned} 2(-x_1) - 5(-x_2) + 7(-x_3) &= -(2x_1 - 5x_2 + 7x_3) \\ &= -0 \\ &= 0 \end{aligned}$$

and we now believe that $-\mathbf{x} \in W$.

Is the vector addition in W commutative (Property C)? Is $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$? Of course! Nothing about restricting the scope of our set of vectors will prevent the operation from still being commutative. Indeed, the remaining five properties are unaffected by the transition to a smaller set of vectors, and so remain true. That was convenient.

So W satisfies all ten properties, is therefore a vector space, and thus earns the title of being a subspace of $\mathbb{C}^3$. △

Subsection TS Testing Subspaces

In Example SC3 we proceeded through all ten of the vector space properties before believing that a subset was a subspace. But six of the properties were easy to prove, and we can lean on some of the properties of the vector space (the superset) to make the other four easier. Here is a theorem that will make it easier to test if a subset is a vector space. A shortcut if there ever was one.

Theorem TSS Testing Subsets for Subspaces
Suppose that V is a vector space and W is a subset of V, $W \subseteq V$. Endow W with the same operations as V. Then W is a subspace if and only if three conditions are met

1. *W is nonempty, $W \neq \emptyset$.*
2. *If $\mathbf{x} \in W$ and $\mathbf{y} \in W$, then $\mathbf{x} + \mathbf{y} \in W$.*
3. *If $\alpha \in \mathbb{C}$ and $\mathbf{x} \in W$, then $\alpha\mathbf{x} \in W$.*

Proof. (⇒) We have the hypothesis that W is a subspace, so by Property Z we know that W contains a zero vector. This is enough to show that $W \neq \emptyset$. Also, since W is a vector space it satisfies the additive and scalar multiplication closure properties (Property AC, Property SC), and so exactly meets the second and third conditions. If that was easy, the other direction might require a bit more work.

(⇐) We have three properties for our hypothesis, and from this we should conclude that W has the ten defining properties of a vector space. The second and third conditions of our hypothesis are exactly Property AC and Property SC. Our hypothesis that V is a vector space implies that Property C, Property AA, Property SMA, Property DVA, Property DSA and Property O all hold. They continue to be true for vectors from W since passing to a subset, and keeping the operation the same, leaves their statements unchanged. Eight down, two to go.

Suppose $\mathbf{x} \in W$. Then by the third part of our hypothesis (scalar closure), we know that $(-1)\mathbf{x} \in W$. By Theorem AISM $(-1)\mathbf{x} = -\mathbf{x}$, so together these statements show us that $-\mathbf{x} \in W$. $-\mathbf{x}$ is the additive inverse of $\mathbf{x}$ in V, but will continue in this role when viewed as element of the subset W. So every element of W has an additive inverse that is an element of W and Property AI is completed. Just one property left.

While we have implicitly discussed the zero vector in the previous paragraph, we need to be certain that the zero vector (of V) really lives in W. Since W is nonempty, we can choose some vector $\mathbf{z} \in W$. Then by the argument in the previous paragraph, we know $-\mathbf{z} \in W$. Now by Property AI for V and then by the second part of our hypothesis (additive closure) we see that

$$\mathbf{0} = \mathbf{z} + (-\mathbf{z}) \in W$$

So W contains the zero vector from V. Since this vector performs the required duties of a zero vector in V, it will continue in that role as an element of W. This gives us, Property Z, the final property of the ten required. (Sarah Fellez contributed to this proof.) ∎

So just three conditions, plus being a subset of a known vector space, gets us all ten properties. Fabulous! This theorem can be paraphrased by saying that a subspace is "a nonempty subset (of a vector space) that is closed under vector addition and scalar multiplication."

You might want to go back and rework Example SC3 in light of this result, perhaps seeing where we can now economize or where the work done in the example mirrored the proof and where it did not. We will press on and apply this theorem in a slightly more abstract setting.

Example SP4 A subspace of P_4

P_4 is the vector space of polynomials with degree at most 4 (Example VSP). Define a subset W as

$$W = \{\, p(x) \mid p \in P_4,\ p(2) = 0 \}$$

so W is the collection of those polynomials (with degree 4 or less) whose graphs cross the x-axis at $x = 2$. Whenever we encounter a new set it is a good idea to gain a better understanding of the set by finding a few elements in the set, and a few outside it. For example $x^2 - x - 2 \in W$, while $x^4 + x^3 - 7 \notin W$.

Is W nonempty? Yes, $x - 2 \in W$.

Additive closure? Suppose $p \in W$ and $q \in W$. Is $p + q \in W$? p and q are not totally arbitrary, we know that $p(2) = 0$ and $q(2) = 0$. Then we can check $p + q$ for membership in W,

$$\begin{aligned}
(p+q)(2) &= p(2) + q(2) && \text{Addition in } P_4\\
&= 0 + 0 && p \in W,\ q \in W\\
&= 0
\end{aligned}$$

so we see that $p+q$ qualifies for membership in W.

Scalar multiplication closure? Suppose that $\alpha\in\mathbb{C}$ and $p\in W$. Then we know that $p(2)=0$. Testing αp for membership,

$$\begin{aligned}(\alpha p)(2) &= \alpha p(2) && \text{Scalar multiplication in } P_4\\ &= \alpha 0 && p\in W\\ &= 0\end{aligned}$$

so $\alpha p\in W$.

We have shown that W meets the three conditions of Theorem TSS and so qualifies as a subspace of P_4. Notice that by Definition S we now know that W is also a vector space. So all the properties of a vector space (Definition VS) and the theorems of Section VS apply in full. △

Much of the power of Theorem TSS is that we can easily establish new vector spaces if we can locate them as subsets of other vector spaces, such as the vector spaces presented in Subsection VS.EVS.

It can be as instructive to consider some subsets that are *not* subspaces. Since Theorem TSS is an equivalence (see Proof Technique E) we can be assured that a subset is not a subspace if it violates one of the three conditions, and in any example of interest this will not be the "nonempty" condition. However, since a subspace has to be a vector space in its own right, we can also search for a violation of any one of the ten defining properties in Definition VS or any inherent property of a vector space, such as those given by the basic theorems of Subsection VS.VSP. Notice also that a violation need only be for a specific vector or pair of vectors.

Example NSC2Z A non-subspace in $\mathbb{C}^2$, zero vector
Consider the subset W below as a candidate for being a subspace of $\mathbb{C}^2$

$$W=\left\{\left.\begin{bmatrix}x_1\\x_2\end{bmatrix}\right|3x_1-5x_2=12\right\}$$

The zero vector of $\mathbb{C}^2$, $\mathbf{0}=\begin{bmatrix}0\\0\end{bmatrix}$ will need to be the zero vector in W also. However, $\mathbf{0}\not\in W$ since $3(0)-5(0)=0\neq 12$. So W has no zero vector and fails Property Z of Definition VS. This subspace also fails to be closed under addition and scalar multiplication. Can you find examples of this? △

Example NSC2A A non-subspace in $\mathbb{C}^2$, additive closure
Consider the subset X below as a candidate for being a subspace of $\mathbb{C}^2$

$$X=\left\{\left.\begin{bmatrix}x_1\\x_2\end{bmatrix}\right|x_1x_2=0\right\}$$

You can check that $\mathbf{0}\in X$, so the approach of the last example will not get us anywhere. However, notice that $\mathbf{x}=\begin{bmatrix}1\\0\end{bmatrix}\in X$ and $\mathbf{y}=\begin{bmatrix}0\\1\end{bmatrix}\in X$. Yet

$$\mathbf{x}+\mathbf{y}=\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}\not\in X$$

So X fails the additive closure requirement of either Property AC or Theorem TSS, and is therefore not a subspace. △

Example NSC2S A non-subspace in $\mathbb{C}^2$, scalar multiplication closure
Consider the subset Y below as a candidate for being a subspace of $\mathbb{C}^2$

$$Y=\left\{\left.\begin{bmatrix}x_1\\x_2\end{bmatrix}\right|x_1\in\mathbb{Z},\ x_2\in\mathbb{Z}\right\}$$

$\mathbb{Z}$ is the set of integers, so we are only allowing "whole numbers" as the constituents of our vectors. Now, $\mathbf{0}\in Y$, and additive closure also holds (can you prove these claims?). So we will have to try something different. Note that $\alpha = \frac{1}{2} \in \mathbb{C}$ and $\begin{bmatrix}2\\3\end{bmatrix} \in Y$, but

$$\alpha\mathbf{x} = \frac{1}{2}\begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}1\\\frac{3}{2}\end{bmatrix} \notin Y$$

So Y fails the scalar multiplication closure requirement of either Property SC or Theorem TSS, and is therefore not a subspace. △

There are two examples of subspaces that are trivial. Suppose that V is any vector space. Then V is a subset of itself and is a vector space. By Definition S, V qualifies as a subspace of itself. The set containing just the zero vector $Z = \{\mathbf{0}\}$ is also a subspace as can be seen by applying Theorem TSS or by simple modifications of the techniques hinted at in Example VSS. Since these subspaces are so obvious (and therefore not too interesting) we will refer to them as being trivial.

Definition TS Trivial Subspaces
Given the vector space V, the subspaces V and $\{\mathbf{0}\}$ are each called a **trivial subspace**. □

We can also use Theorem TSS to prove more general statements about subspaces, as illustrated in the next theorem.

Theorem NSMS Null Space of a Matrix is a Subspace
Suppose that A is an $m \times n$ matrix. Then the null space of A, $\mathcal{N}(A)$, is a subspace of $\mathbb{C}^n$.

Proof. We will examine the three requirements of Theorem TSS. Recall that Definition NSM can be formulated as $\mathcal{N}(A) = \{\mathbf{x} \in \mathbb{C}^n |\, A\mathbf{x} = \mathbf{0}\}$.

First, $\mathbf{0} \in \mathcal{N}(A)$, which can be inferred as a consequence of Theorem HSC. So $\mathcal{N}(A) \neq \emptyset$.

Second, check additive closure by supposing that $\mathbf{x} \in \mathcal{N}(A)$ and $\mathbf{y} \in \mathcal{N}(A)$. So we know a little something about $\mathbf{x}$ and $\mathbf{y}$: $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{y} = \mathbf{0}$, and that is all we know. Question: Is $\mathbf{x} + \mathbf{y} \in \mathcal{N}(A)$? Let us check.

$$\begin{aligned} A(\mathbf{x}+\mathbf{y}) &= A\mathbf{x} + A\mathbf{y} && \text{Theorem MMDAA}\\ &= \mathbf{0} + \mathbf{0} && \mathbf{x} \in \mathcal{N}(A),\ \mathbf{y} \in \mathcal{N}(A)\\ &= \mathbf{0} && \text{Theorem VSPCV} \end{aligned}$$

So, yes, $\mathbf{x} + \mathbf{y}$ qualifies for membership in $\mathcal{N}(A)$.

Third, check scalar multiplication closure by supposing that $\alpha \in \mathbb{C}$ and $\mathbf{x} \in \mathcal{N}(A)$. So we know a little something about $\mathbf{x}$: $A\mathbf{x} = \mathbf{0}$, and that is all we know. Question: Is $\alpha\mathbf{x} \in \mathcal{N}(A)$? Let us check.

$$\begin{aligned} A(\alpha\mathbf{x}) &= \alpha(A\mathbf{x}) && \text{Theorem MMSMM}\\ &= \alpha\mathbf{0} && \mathbf{x} \in \mathcal{N}(A)\\ &= \mathbf{0} && \text{Theorem ZVSM} \end{aligned}$$

So, yes, $\alpha\mathbf{x}$ qualifies for membership in $\mathcal{N}(A)$.

Having met the three conditions in Theorem TSS we can now say that the null space of a matrix is a subspace (and hence a vector space in its own right!). ■

Here is an example where we can exercise Theorem NSMS.

Example RSNS Recasting a subspace as a null space

Consider the subset of $\mathbb{C}^5$ defined as

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \middle| \begin{array}{l} 3x_1 + x_2 - 5x_3 + 7x_4 + x_5 = 0, \\ 4x_1 + 6x_2 + 3x_3 - 6x_4 - 5x_5 = 0, \\ -2x_1 + 4x_2 + 7x_4 + x_5 = 0 \end{array} \right\}$$

It is possible to show that W is a subspace of $\mathbb{C}^5$ by checking the three conditions of Theorem TSS directly, but it will get tedious rather quickly. Instead, give W a fresh look and notice that it is a set of solutions to a homogeneous system of equations. Define the matrix

$$A = \begin{bmatrix} 3 & 1 & -5 & 7 & 1 \\ 4 & 6 & 3 & -6 & -5 \\ -2 & 4 & 0 & 7 & 1 \end{bmatrix}$$

and then recognize that $W = \mathcal{N}(A)$. By Theorem NSMS we can immediately see that W is a subspace. Boom! △

Subsection TSS
The Span of a Set

The span of a set of column vectors got a heavy workout in Chapter V and Chapter M. The definition of the span depended only on being able to formulate linear combinations. In any of our more general vector spaces we always have a definition of vector addition and of scalar multiplication. So we can build linear combinations and manufacture spans. This subsection contains two definitions that are just mild variants of definitions we have seen earlier for column vectors. If you have not already, compare them with Definition LCCV and Definition SSCV.

Definition LC Linear Combination
Suppose that V is a vector space. Given n vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n$ and n scalars $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$, their **linear combination** is the vector

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_n\mathbf{u}_n.$$

□

Example LCM A linear combination of matrices
In the vector space M_{23} of 2×3 matrices, we have the vectors

$$\mathbf{x} = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 0 & 7 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} 3 & -1 & 2 \\ 5 & 5 & 1 \end{bmatrix} \qquad \mathbf{z} = \begin{bmatrix} 4 & 2 & -4 \\ 1 & 1 & 1 \end{bmatrix}$$

and we can form linear combinations such as

$$\begin{aligned} 2\mathbf{x} + 4\mathbf{y} + (-1)\mathbf{z} &= 2\begin{bmatrix} 1 & 3 & -2 \\ 2 & 0 & 7 \end{bmatrix} + 4\begin{bmatrix} 3 & -1 & 2 \\ 5 & 5 & 1 \end{bmatrix} + (-1)\begin{bmatrix} 4 & 2 & -4 \\ 1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 6 & -4 \\ 4 & 0 & 14 \end{bmatrix} + \begin{bmatrix} 12 & -4 & 8 \\ 20 & 20 & 4 \end{bmatrix} + \begin{bmatrix} -4 & -2 & 4 \\ -1 & -1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 10 & 0 & 8 \\ 23 & 19 & 17 \end{bmatrix} \end{aligned}$$

or,

$$\begin{aligned} 4\mathbf{x} - 2\mathbf{y} + 3\mathbf{z} &= 4\begin{bmatrix} 1 & 3 & -2 \\ 2 & 0 & 7 \end{bmatrix} - 2\begin{bmatrix} 3 & -1 & 2 \\ 5 & 5 & 1 \end{bmatrix} + 3\begin{bmatrix} 4 & 2 & -4 \\ 1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 12 & -8 \\ 8 & 0 & 28 \end{bmatrix} + \begin{bmatrix} -6 & 2 & -4 \\ -10 & -10 & -2 \end{bmatrix} + \begin{bmatrix} 12 & 6 & -12 \\ 3 & 3 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 10 & 20 & -24 \\ 1 & -7 & 29 \end{bmatrix} \end{aligned}$$

△

When we realize that we can form linear combinations in any vector space, then it is natural to revisit our definition of the span of a set, since it is the set of *all* possible linear combinations of a set of vectors.

Definition SS Span of a Set
Suppose that V is a vector space. Given a set of vectors $S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_t\}$, their **span**, $\langle S\rangle$, is the set of all possible linear combinations of $\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_t$. Symbolically,

$$\begin{aligned}\langle S\rangle &= \{\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_t\mathbf{u}_t \mid \alpha_i \in \mathbb{C},\, 1 \leq i \leq t\} \\ &= \left\{\sum_{i=1}^{t} \alpha_i\mathbf{u}_i \,\middle|\, \alpha_i \in \mathbb{C},\, 1 \leq i \leq t\right\}\end{aligned}$$

□

Theorem SSS Span of a Set is a Subspace
Suppose V is a vector space. Given a set of vectors $S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_t\} \subseteq V$, their span, $\langle S\rangle$, is a subspace.

Proof. By Definition SS, the span contains linear combinations of vectors from the vector space V, so by repeated use of the closure properties, Property AC and Property SC, $\langle S\rangle$ can be seen to be a subset of V.

We will then verify the three conditions of Theorem TSS. First,

$$\begin{aligned}\mathbf{0} &= \mathbf{0} + \mathbf{0} + \mathbf{0} + \ldots + \mathbf{0} && \text{Property Z}\\ &= 0\mathbf{u}_1 + 0\mathbf{u}_2 + 0\mathbf{u}_3 + \cdots + 0\mathbf{u}_t && \text{Theorem ZSSM}\end{aligned}$$

So we have written $\mathbf{0}$ as a linear combination of the vectors in S and by Definition SS, $\mathbf{0} \in \langle S\rangle$ and therefore $\langle S\rangle \neq \emptyset$.

Second, suppose $\mathbf{x} \in \langle S\rangle$ and $\mathbf{y} \in \langle S\rangle$. Can we conclude that $\mathbf{x} + \mathbf{y} \in \langle S\rangle$? What do we know about $\mathbf{x}$ and $\mathbf{y}$ by virtue of their membership in $\langle S\rangle$? There must be scalars from $\mathbb{C}$, $\alpha_1,\, \alpha_2,\, \alpha_3,\, \ldots,\, \alpha_t$ and $\beta_1,\, \beta_2,\, \beta_3,\, \ldots,\, \beta_t$ so that

$$\begin{aligned}\mathbf{x} &= \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_t\mathbf{u}_t\\ \mathbf{y} &= \beta_1\mathbf{u}_1 + \beta_2\mathbf{u}_2 + \beta_3\mathbf{u}_3 + \cdots + \beta_t\mathbf{u}_t\end{aligned}$$

Then

$$\begin{aligned}\mathbf{x} + \mathbf{y} &= \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_t\mathbf{u}_t\\ &\qquad + \beta_1\mathbf{u}_1 + \beta_2\mathbf{u}_2 + \beta_3\mathbf{u}_3 + \cdots + \beta_t\mathbf{u}_t\\ &= \alpha_1\mathbf{u}_1 + \beta_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \beta_2\mathbf{u}_2\\ &\qquad + \alpha_3\mathbf{u}_3 + \beta_3\mathbf{u}_3 + \cdots + \alpha_t\mathbf{u}_t + \beta_t\mathbf{u}_t && \text{Property AA, Property C}\\ &= (\alpha_1 + \beta_1)\mathbf{u}_1 + (\alpha_2 + \beta_2)\mathbf{u}_2\\ &\qquad + (\alpha_3 + \beta_3)\mathbf{u}_3 + \cdots + (\alpha_t + \beta_t)\mathbf{u}_t && \text{Property DSA}\end{aligned}$$

Since each $\alpha_i + \beta_i$ is again a scalar from $\mathbb{C}$ we have expressed the vector sum $\mathbf{x} + \mathbf{y}$ as a linear combination of the vectors from S, and therefore by Definition SS we can say that $\mathbf{x} + \mathbf{y} \in \langle S\rangle$.

Third, suppose $\alpha \in \mathbb{C}$ and $\mathbf{x} \in \langle S\rangle$. Can we conclude that $\alpha\mathbf{x} \in \langle S\rangle$? What do we know about $\mathbf{x}$ by virtue of its membership in $\langle S\rangle$? There must be scalars from $\mathbb{C}$, $\alpha_1,\, \alpha_2,\, \alpha_3,\, \ldots,\, \alpha_t$ so that

$$\mathbf{x} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_t\mathbf{u}_t$$

Then

$$\alpha\mathbf{x} = \alpha\left(\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_t\mathbf{u}_t\right)$$

$$\begin{aligned} &= \alpha(\alpha_1\mathbf{u}_1) + \alpha(\alpha_2\mathbf{u}_2) + \alpha(\alpha_3\mathbf{u}_3) + \cdots + \alpha(\alpha_t\mathbf{u}_t) && \text{Property DVA} \\ &= (\alpha\alpha_1)\mathbf{u}_1 + (\alpha\alpha_2)\mathbf{u}_2 + (\alpha\alpha_3)\mathbf{u}_3 + \cdots + (\alpha\alpha_t)\mathbf{u}_t && \text{Property SMA} \end{aligned}$$

Since each $\alpha\alpha_i$ is again a scalar from $\mathbb{C}$ we have expressed the scalar multiple $\alpha\mathbf{x}$ as a linear combination of the vectors from S, and therefore by Definition SS we can say that $\alpha\mathbf{x} \in \langle S\rangle$.

With the three conditions of Theorem TSS met, we can say that $\langle S\rangle$ is a subspace (and so is also a vector space, Definition VS). (See Exercise SS.T20, Exercise SS.T21, Exercise SS.T22.) ■

Example SSP Span of a set of polynomials
In Example SP4 we proved that

$$W = \{\,p(x)|\,p \in P_4,\ p(2) = 0\}$$

is a subspace of P_4, the vector space of polynomials of degree at most 4. Since W is a vector space itself, let us construct a span within W. First let

$$S = \left\{x^4 - 4x^3 + 5x^2 - x - 2,\ 2x^4 - 3x^3 - 6x^2 + 6x + 4\right\}$$

and verify that S is a subset of W by checking that each of these two polynomials has $x = 2$ as a root. Now, if we define $U = \langle S\rangle$, then Theorem SSS tells us that U is a subspace of W. So quite quickly we have built a chain of subspaces, U inside W, and W inside P_4.

Rather than dwell on how quickly we can build subspaces, let us try to gain a better understanding of just how the span construction creates subspaces, in the context of this example. We can quickly build representative elements of U,

$$3(x^4-4x^3+5x^2-x-2)+5(2x^4-3x^3-6x^2+6x+4) = 13x^4-27x^3-15x^2+27x+14$$

and

$$(-2)(x^4-4x^3+5x^2-x-2)+8(2x^4-3x^3-6x^2+6x+4) = 14x^4-16x^3-58x^2+50x+36$$

and each of these polynomials must be in W since it is closed under addition and scalar multiplication. But you might check for yourself that both of these polynomials have $x = 2$ as a root.

I can tell you that $\mathbf{y} = 3x^4 - 7x^3 - x^2 + 7x - 2$ is not in U, but would you believe me? A first check shows that $\mathbf{y}$ does have $x = 2$ as a root, but that only shows that $\mathbf{y} \in W$. What does $\mathbf{y}$ have to do to gain membership in $U = \langle S\rangle$? It must be a linear combination of the vectors in S, $x^4 - 4x^3 + 5x^2 - x - 2$ and $2x^4 - 3x^3 - 6x^2 + 6x + 4$. So let us suppose that $\mathbf{y}$ is such a linear combination,

$$\begin{aligned} \mathbf{y} &= 3x^4 - 7x^3 - x^2 + 7x - 2 \\ &= \alpha_1(x^4 - 4x^3 + 5x^2 - x - 2) + \alpha_2(2x^4 - 3x^3 - 6x^2 + 6x + 4) \\ &= (\alpha_1 + 2\alpha_2)x^4 + (-4\alpha_1 - 3\alpha_2)x^3 + (5\alpha_1 - 6\alpha_2)x^2 \\ &\qquad + (-\alpha_1 + 6\alpha_2)x + (-2\alpha_1 + 4\alpha_2) \end{aligned}$$

Notice that operations above are done in accordance with the definition of the vector space of polynomials (Example VSP). Now, if we equate coefficients, which is the definition of equality for polynomials, then we obtain the system of five linear equations in two variables

$$\begin{aligned} \alpha_1 + 2\alpha_2 &= 3 \\ -4\alpha_1 - 3\alpha_2 &= -7 \\ 5\alpha_1 - 6\alpha_2 &= -1 \\ -\alpha_1 + 6\alpha_2 &= 7 \\ -2\alpha_1 + 4\alpha_2 &= -2 \end{aligned}$$

Build an augmented matrix from the system and row-reduce,

$$\begin{bmatrix} 1 & 2 & 3 \\ -4 & -3 & -7 \\ 5 & -6 & -1 \\ -1 & 6 & 7 \\ -2 & 4 & -2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the final column of the row-reduced augmented matrix is a pivot column, Theorem RCLS tells us the system of equations is inconsistent. Therefore, there are no scalars, α_1 and α_2, to establish $\mathbf{y}$ as a linear combination of the elements in U. So $\mathbf{y} \notin U$. △

Let us again examine membership in a span.

Example SM32 A subspace of M_{32}

The set of all 3×2 matrices forms a vector space when we use the operations of matrix addition (Definition MA) and scalar matrix multiplication (Definition MSM), as was shown in Example VSM. Consider the subset

$$S = \left\{ \begin{bmatrix} 3 & 1 \\ 4 & 2 \\ 5 & -5 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 14 & -1 \end{bmatrix}, \begin{bmatrix} 3 & -1 \\ -1 & 2 \\ -19 & -11 \end{bmatrix}, \begin{bmatrix} 4 & 2 \\ 1 & -2 \\ 14 & -2 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ -4 & 0 \\ -17 & 7 \end{bmatrix} \right\}$$

and define a new subset of vectors W in M_{32} using the span (Definition SS), $W = \langle S \rangle$. So by Theorem SSS we know that W is a subspace of M_{32}. While W is an infinite set, and this is a precise description, it would still be worthwhile to investigate whether or not W contains certain elements.

First, is

$$\mathbf{y} = \begin{bmatrix} 9 & 3 \\ 7 & 3 \\ 10 & -11 \end{bmatrix}$$

in W? To answer this, we want to determine if $\mathbf{y}$ can be written as a linear combination of the five matrices in S. Can we find scalars, α_1, α_2, α_3, α_4, α_5 so that

$$\begin{aligned} &\begin{bmatrix} 9 & 3 \\ 7 & 3 \\ 10 & -11 \end{bmatrix} \\ &= \alpha_1 \begin{bmatrix} 3 & 1 \\ 4 & 2 \\ 5 & -5 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 14 & -1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 3 & -1 \\ -1 & 2 \\ -19 & -11 \end{bmatrix} + \alpha_4 \begin{bmatrix} 4 & 2 \\ 1 & -2 \\ 14 & -2 \end{bmatrix} + \alpha_5 \begin{bmatrix} 3 & 1 \\ -4 & 0 \\ -17 & 7 \end{bmatrix} \\ &= \begin{bmatrix} 3\alpha_1 + \alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 & \alpha_1 + \alpha_2 - \alpha_3 + 2\alpha_4 + \alpha_5 \\ 4\alpha_1 + 2\alpha_2 - \alpha_3 + \alpha_4 - 4\alpha_5 & 2\alpha_1 - \alpha_2 + 2\alpha_3 - 2\alpha_4 \\ 5\alpha_1 + 14\alpha_2 - 19\alpha_3 + 14\alpha_4 - 17\alpha_5 & -5\alpha_1 - \alpha_2 - 11\alpha_3 - 2\alpha_4 + 7\alpha_5 \end{bmatrix} \end{aligned}$$

Using our definition of matrix equality (Definition ME) we can translate this statement into six equations in the five unknowns,

$$\begin{aligned} 3\alpha_1 + \alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 &= 9 \\ \alpha_1 + \alpha_2 - \alpha_3 + 2\alpha_4 + \alpha_5 &= 3 \\ 4\alpha_1 + 2\alpha_2 - \alpha_3 + \alpha_4 - 4\alpha_5 &= 7 \\ 2\alpha_1 - \alpha_2 + 2\alpha_3 - 2\alpha_4 &= 3 \\ 5\alpha_1 + 14\alpha_2 - 19\alpha_3 + 14\alpha_4 - 17\alpha_5 &= 10 \\ -5\alpha_1 - \alpha_2 - 11\alpha_3 - 2\alpha_4 + 7\alpha_5 &= -11 \end{aligned}$$

This is a linear system of equations, which we can represent with an augmented matrix and row-reduce in search of solutions. The matrix that is row-equivalent to the augmented matrix is

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 0 & \frac{5}{8} & 2 \\ 0 & \boxed{1} & 0 & 0 & \frac{-19}{4} & -1 \\ 0 & 0 & \boxed{1} & 0 & \frac{-7}{8} & 0 \\ 0 & 0 & 0 & \boxed{1} & \frac{17}{8} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So we recognize that the system is consistent since the final column is not a pivot column (Theorem RCLS), and compute $n - r = 5 - 4 = 1$ free variables (Theorem FVCS). While there are infinitely many solutions, we are only in pursuit of a single solution, so let us choose the free variable $\alpha_5 = 0$ for simplicity's sake. Then we easily see that $\alpha_1 = 2$, $\alpha_2 = -1$, $\alpha_3 = 0$, $\alpha_4 = 1$. So the scalars $\alpha_1 = 2$, $\alpha_2 = -1$, $\alpha_3 = 0$, $\alpha_4 = 1$, $\alpha_5 = 0$ will provide a linear combination of the elements of S that equals $\mathbf{y}$, as we can verify by checking,

$$\begin{bmatrix} 9 & 3 \\ 7 & 3 \\ 10 & -11 \end{bmatrix} = 2 \begin{bmatrix} 3 & 1 \\ 4 & 2 \\ 5 & -5 \end{bmatrix} + (-1) \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 14 & -1 \end{bmatrix} + (1) \begin{bmatrix} 4 & 2 \\ 1 & -2 \\ 14 & -2 \end{bmatrix}$$

So with one particular linear combination in hand, we are convinced that $\mathbf{y}$ deserves to be a member of $W = \langle S \rangle$.

Second, is

$$\mathbf{x} = \begin{bmatrix} 2 & 1 \\ 3 & 1 \\ 4 & -2 \end{bmatrix}$$

in W? To answer this, we want to determine if $\mathbf{x}$ can be written as a linear combination of the five matrices in S. Can we find scalars, α_1, α_2, α_3, α_4, α_5 so that

$$\begin{aligned} &\begin{bmatrix} 2 & 1 \\ 3 & 1 \\ 4 & -2 \end{bmatrix} \\ &= \alpha_1 \begin{bmatrix} 3 & 1 \\ 4 & 2 \\ 5 & -5 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 14 & -1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 3 & -1 \\ -1 & 2 \\ -19 & -11 \end{bmatrix} + \alpha_4 \begin{bmatrix} 4 & 2 \\ 1 & -2 \\ 14 & -2 \end{bmatrix} + \alpha_5 \begin{bmatrix} 3 & 1 \\ -4 & 0 \\ -17 & 7 \end{bmatrix} \\ &= \begin{bmatrix} 3\alpha_1 + \alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 & \alpha_1 + \alpha_2 - \alpha_3 + 2\alpha_4 + \alpha_5 \\ 4\alpha_1 + 2\alpha_2 - \alpha_3 + \alpha_4 - 4\alpha_5 & 2\alpha_1 - \alpha_2 + 2\alpha_3 - 2\alpha_4 \\ 5\alpha_1 + 14\alpha_2 - 19\alpha_3 + 14\alpha_4 - 17\alpha_5 & -5\alpha_1 - \alpha_2 - 11\alpha_3 - 2\alpha_4 + 7\alpha_5 \end{bmatrix} \end{aligned}$$

Using our definition of matrix equality (Definition ME) we can translate this statement into six equations in the five unknowns,

$$\begin{aligned} 3\alpha_1 + \alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 &= 2 \\ \alpha_1 + \alpha_2 - \alpha_3 + 2\alpha_4 + \alpha_5 &= 1 \\ 4\alpha_1 + 2\alpha_2 - \alpha_3 + \alpha_4 - 4\alpha_5 &= 3 \\ 2\alpha_1 - \alpha_2 + 2\alpha_3 - 2\alpha_4 &= 1 \\ 5\alpha_1 + 14\alpha_2 - 19\alpha_3 + 14\alpha_4 - 17\alpha_5 &= 4 \\ -5\alpha_1 - \alpha_2 - 11\alpha_3 - 2\alpha_4 + 7\alpha_5 &= -2 \end{aligned}$$

This is a linear system of equations, which we can represent with an augmented matrix and row-reduce in search of solutions. The matrix that is row-equivalent to the augmented matrix is

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 0 & \frac{5}{8} & 0 \\ 0 & \boxed{1} & 0 & 0 & -\frac{38}{8} & 0 \\ 0 & 0 & \boxed{1} & 0 & -\frac{7}{8} & 0 \\ 0 & 0 & 0 & \boxed{1} & -\frac{17}{8} & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the last column is a pivot column, Theorem RCLS tells us that the system is inconsistent. Therefore, there are no values for the scalars that will place $\mathbf{x}$ in W, and so we conclude that $\mathbf{x} \notin W$. △

Notice how Example SSP and Example SM32 contained questions about membership in a span, but these questions quickly became questions about solutions to a system of linear equations. This will be a common theme going forward.

Subsection SC
Subspace Constructions

Several of the subsets of vectors spaces that we worked with in Chapter M are also subspaces — they are closed under vector addition and scalar multiplication in $\mathbb{C}^m$.

Theorem CSMS Column Space of a Matrix is a Subspace
Suppose that A is an $m \times n$ matrix. Then $\mathcal{C}(A)$ is a subspace of $\mathbb{C}^m$.

Proof. Definition CSM shows us that $\mathcal{C}(A)$ is a subset of $\mathbb{C}^m$, and that it is defined as the span of a set of vectors from $\mathbb{C}^m$ (the columns of the matrix). Since $\mathcal{C}(A)$ is a span, Theorem SSS says it is a subspace. ■

That was easy! Notice that we could have used this same approach to prove that the null space is a subspace, since Theorem SSNS provided a description of the null space of a matrix as the span of a set of vectors. However, I much prefer the current proof of Theorem NSMS. Speaking of easy, here is a very easy theorem that exposes another of our constructions as creating subspaces.

Theorem RSMS Row Space of a Matrix is a Subspace
Suppose that A is an $m \times n$ matrix. Then $\mathcal{R}(A)$ is a subspace of $\mathbb{C}^n$.

Proof. Definition RSM says $\mathcal{R}(A) = \mathcal{C}(A^t)$, so the row space of a matrix is a column space, and every column space is a subspace by Theorem CSMS. That's enough. ■

One more.

Theorem LNSMS Left Null Space of a Matrix is a Subspace
Suppose that A is an $m \times n$ matrix. Then $\mathcal{L}(A)$ is a subspace of $\mathbb{C}^m$.

Proof. Definition LNS says $\mathcal{L}(A) = \mathcal{N}(A^t)$, so the left null space is a null space, and every null space is a subspace by Theorem NSMS. Done. ■

So the span of a set of vectors, and the null space, column space, row space and left null space of a matrix are all subspaces, and hence are all vector spaces, meaning they have all the properties detailed in Definition VS and in the basic theorems presented in Section VS. We have worked with these objects as just sets in Chapter V and Chapter M, but now we understand that they have much more structure. In particular, being closed under vector addition and scalar multiplication means a subspace is also closed under linear combinations.

Reading Questions

1. Summarize the three conditions that allow us to quickly test if a set is a subspace.
2. Consider the set of vectors

$$W = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \middle| \, 3a - 2b + c = 5 \right\}$$

   Is the set W a subspace of $\mathbb{C}^3$? Explain your answer.
3. Name five general constructions of sets of column vectors (subsets of $\mathbb{C}^m$) that we now know as subspaces.

Exercises

C15† Working within the vector space $\mathbb{C}^3$, determine if $\mathbf{b} = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$ is in the subspace W,

$$W = \left\langle \left\{ \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \right\} \right\rangle$$

C16† Working within the vector space $\mathbb{C}^4$, determine if $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ is in the subspace W,

$$W = \left\langle \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \\ 2 \end{bmatrix} \right\} \right\rangle$$

C17† Working within the vector space $\mathbb{C}^4$, determine if $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$ is in the subspace W,

$$W = \left\langle \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} \right\} \right\rangle$$

C20† Working within the vector space P_3 of polynomials of degree 3 or less, determine if $p(x) = x^3 + 6x + 4$ is in the subspace W below.

$$W = \left\langle \left\{ x^3 + x^2 + x, \, x^3 + 2x - 6, \, x^2 - 5 \right\} \right\rangle$$

C21† Consider the subspace

$$W = \left\langle \left\{ \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} -3 & 1 \\ 2 & 1 \end{bmatrix} \right\} \right\rangle$$

of the vector space of 2×2 matrices, M_{22}. Is $C = \begin{bmatrix} -3 & 3 \\ 6 & -4 \end{bmatrix}$ an element of W?

C25 Show that the set $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| \, 3x_1 - 5x_2 = 12 \right\}$ from Example NSC2Z fails Property AC and Property SC.

C26 Show that the set $Y = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| \, x_1 \in \mathbb{Z}, \, x_2 \in \mathbb{Z} \right\}$ from Example NSC2S has Property AC.

M20† In $\mathbb{C}^3$, the vector space of column vectors of size 3, prove that the set Z is a subspace.

$$Z = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| \, 4x_1 - x_2 + 5x_3 = 0 \right\}$$

T20† A square matrix A of size n is upper triangular if $[A]_{ij} = 0$ whenever $i > j$. Let UT_n be the set of all upper triangular matrices of size n. Prove that UT_n is a subspace of the vector space of all square matrices of size n, M_{nn}.

T30† Let P be the set of all polynomials, of any degree. The set P is a vector space. Let E be the subset of P consisting of all polynomials with only terms of even degree. Prove or disprove: the set E is a subspace of P.

T31† Let P be the set of all polynomials, of any degree. The set P is a vector space. Let F be the subset of P consisting of all polynomials with only terms of odd degree. Prove or disprove: the set F is a subspace of P.

Section LISS
Linear Independence and Spanning Sets

A vector space is defined as a set with two operations, meeting ten properties (Definition VS). Just as the definition of span of a set of vectors only required knowing how to add vectors and how to multiply vectors by scalars, so it is with linear independence. A definition of a linearly independent set of vectors in an arbitrary vector space only requires knowing how to form linear combinations and equating these with the zero vector. Since every vector space must have a zero vector (Property Z), we always have a zero vector at our disposal.

In this section we will also put a twist on the notion of the span of a set of vectors. Rather than beginning with a set of vectors and creating a subspace that is the span, we will instead begin with a subspace and look for a set of vectors whose span equals the subspace.

The combination of linear independence and spanning will be very important going forward.

Subsection LI
Linear Independence

Our previous definition of linear independence (Definition LICV) employed a relation of linear dependence that was a linear combination on one side of an equality and a zero vector on the other side. As a linear combination in a vector space (Definition LC) depends only on vector addition and scalar multiplication, and every vector space must have a zero vector (Property Z), we can extend our definition of linear independence from the setting of $\mathbb{C}^m$ to the setting of a general vector space V with almost no changes. Compare these next two definitions with Definition RLDCV and Definition LICV.

Definition RLD Relation of Linear Dependence
Suppose that V is a vector space. Given a set of vectors $S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_n\}$, an equation of the form

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \alpha_3\mathbf{u}_3 + \cdots + \alpha_n\mathbf{u}_n = \mathbf{0}$$

is a **relation of linear dependence** on S. If this equation is formed in a trivial fashion, i.e. $\alpha_i = 0$, $1 \leq i \leq n$, then we say it is a **trivial relation of linear dependence** on S.

□

Definition LI Linear Independence
Suppose that V is a vector space. The set of vectors $S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_n\}$ from V is **linearly dependent** if there is a relation of linear dependence on S that is not trivial. In the case where the *only* relation of linear dependence on S is the trivial one, then S is a **linearly independent** set of vectors. □

Notice the emphasis on the word "only." This might remind you of the definition of a nonsingular matrix, where if the matrix is employed as the coefficient matrix of a homogeneous system then the *only* solution is the *trivial* one.

Example LIP4 Linear independence in P_4
In the vector space of polynomials with degree 4 or less, P_4 (Example VSP) consider the set S below

$$\left\{2x^4 + 3x^3 + 2x^2 - x + 10,\, -x^4 - 2x^3 + x^2 + 5x - 8,\, 2x^4 + x^3 + 10x^2 + 17x - 2\right\}$$

Is this set of vectors linearly independent or dependent? Consider that

$$3\left(2x^4 + 3x^3 + 2x^2 - x + 10\right) + 4\left(-x^4 - 2x^3 + x^2 + 5x - 8\right)$$

$$+(-1)\left(2x^4 + x^3 + 10x^2 + 17x - 2\right) = 0x^4 + 0x^3 + 0x^2 + 0x + 0 = \mathbf{0}$$

This is a nontrivial relation of linear dependence (Definition RLD) on the set S and so convinces us that S is linearly dependent (Definition LI).

Now, I hear you say, "Where did *those* scalars come from?" Do not worry about that right now, just be sure you understand why the above explanation is sufficient to prove that S is linearly dependent. The remainder of the example will demonstrate how we might find these scalars if they had not been provided so readily.

Let us look at another set of vectors (polynomials) from P_4. Let

$$T = \left\{3x^4 - 2x^3 + 4x^2 + 6x - 1,\ -3x^4 + 1x^3 + 0x^2 + 4x + 2,\right.$$
$$\left. 4x^4 + 5x^3 - 2x^2 + 3x + 1,\ 2x^4 - 7x^3 + 4x^2 + 2x + 1\right\}$$

Suppose we have a relation of linear dependence on this set,

$$\begin{aligned}\mathbf{0} &= 0x^4 + 0x^3 + 0x^2 + 0x + 0\\ &= \alpha_1\left(3x^4 - 2x^3 + 4x^2 + 6x - 1\right) + \alpha_2\left(-3x^4 + 1x^3 + 0x^2 + 4x + 2\right)\\ &\quad + \alpha_3\left(4x^4 + 5x^3 - 2x^2 + 3x + 1\right) + \alpha_4\left(2x^4 - 7x^3 + 4x^2 + 2x + 1\right)\end{aligned}$$

Using our definitions of vector addition and scalar multiplication in P_4 (Example VSP), we arrive at,

$$\begin{aligned}&0x^4 + 0x^3 + 0x^2 + 0x + 0 =\\ &\quad\left(3\alpha_1 - 3\alpha_2 + 4\alpha_3 + 2\alpha_4\right)x^4 + \left(-2\alpha_1 + \alpha_2 + 5\alpha_3 - 7\alpha_4\right)x^3 +\\ &\quad\left(4\alpha_1 - 2\alpha_3 + 4\alpha_4\right)x^2 + \left(6\alpha_1 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4\right)x + \left(-\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4\right)\end{aligned}$$

Equating coefficients, we arrive at the homogeneous system of equations,

$$\begin{aligned}3\alpha_1 - 3\alpha_2 + 4\alpha_3 + 2\alpha_4 &= 0\\ -2\alpha_1 + \alpha_2 + 5\alpha_3 - 7\alpha_4 &= 0\\ 4\alpha_1 - 2\alpha_3 + 4\alpha_4 &= 0\\ 6\alpha_1 + 4\alpha_2 + 3\alpha_3 + 2\alpha_4 &= 0\\ -\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 &= 0\end{aligned}$$

We form the coefficient matrix of this homogeneous system of equations and row-reduce to find

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 0\\ 0 & \boxed{1} & 0 & 0\\ 0 & 0 & \boxed{1} & 0\\ 0 & 0 & 0 & \boxed{1}\\ 0 & 0 & 0 & 0\end{bmatrix}$$

We expected the system to be consistent (Theorem HSC) and so can compute $n - r = 4 - 4 = 0$ and Theorem CSRN tells us that the solution is unique. Since this is a homogeneous system, this unique solution is the trivial solution (Definition TSHSE), $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$, $\alpha_4 = 0$. So by Definition LI the set T is linearly independent.

A few observations. If we had discovered infinitely many solutions, then we could have used one of the nontrivial solutions to provide a linear combination in the manner we used to show that S was linearly dependent. It is important to realize that it is not interesting that we can create a relation of linear dependence with zero scalars — we can *always* do that — but for T, this is the *only* way to create a relation of linear dependence. It was no accident that we arrived at a homogeneous system of equations in this example, it is related to our use of the zero vector in defining a relation of linear dependence. It is easy to present a convincing statement that a set is linearly dependent (just exhibit a nontrivial relation of linear dependence) but a convincing statement of linear independence requires demonstrating that there is no relation of linear dependence other than the trivial one. Notice how we relied on theorems from Chapter SLE to provide this demonstration. Whew! There is a lot going on in this example. Spend some time with it, we will be waiting patiently right here when you get back. △

Example LIM32 Linear independence in M_{32}
Consider the two sets of vectors R and S from the vector space of all 3×2 matrices, M_{32} (Example VSM)

$$R = \left\{ \begin{bmatrix} 3 & -1 \\ 1 & 4 \\ 6 & -6 \end{bmatrix}, \begin{bmatrix} -2 & 3 \\ 1 & -3 \\ -2 & -6 \end{bmatrix}, \begin{bmatrix} 6 & -6 \\ -1 & 0 \\ 7 & -9 \end{bmatrix}, \begin{bmatrix} 7 & 9 \\ -4 & -5 \\ 2 & 5 \end{bmatrix} \right\}$$

$$S = \left\{ \begin{bmatrix} 2 & 0 \\ 1 & -1 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} -4 & 0 \\ -2 & 2 \\ -2 & -6 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -2 & 1 \\ 2 & 4 \end{bmatrix}, \begin{bmatrix} -5 & 3 \\ -10 & 7 \\ 2 & 0 \end{bmatrix} \right\}$$

One set is linearly independent, the other is not. Which is which? Let us examine R first. Build a generic relation of linear dependence (Definition RLD),

$$\alpha_1 \begin{bmatrix} 3 & -1 \\ 1 & 4 \\ 6 & -6 \end{bmatrix} + \alpha_2 \begin{bmatrix} -2 & 3 \\ 1 & -3 \\ -2 & -6 \end{bmatrix} + \alpha_3 \begin{bmatrix} 6 & -6 \\ -1 & 0 \\ 7 & -9 \end{bmatrix} + \alpha_4 \begin{bmatrix} 7 & 9 \\ -4 & -5 \\ 2 & 5 \end{bmatrix} = \mathbf{0}$$

Massaging the left-hand side with our definitions of vector addition and scalar multiplication in M_{32} (Example VSM) we obtain,

$$\begin{bmatrix} 3\alpha_1 - 2\alpha_2 + 6\alpha_3 + 7\alpha_4 & -\alpha_1 + 3\alpha_2 - 6\alpha_3 + 9\alpha_4 \\ \alpha_1 + \alpha_2 - \alpha_3 - 4\alpha_4 & 4\alpha_1 - 3\alpha_2 + -5\alpha_4 \\ 6\alpha_1 - 2\alpha_2 + 7\alpha_3 + 2\alpha_4 & -6\alpha_1 - 6\alpha_2 - 9\alpha_3 + 5\alpha_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Using our definition of matrix equality (Definition ME) to equate entries we get the homogeneous system of six equations in four variables,

$$\begin{aligned} 3\alpha_1 - 2\alpha_2 + 6\alpha_3 + 7\alpha_4 &= 0 \\ -\alpha_1 + 3\alpha_2 - 6\alpha_3 + 9\alpha_4 &= 0 \\ \alpha_1 + \alpha_2 - \alpha_3 - 4\alpha_4 &= 0 \\ 4\alpha_1 - 3\alpha_2 + -5\alpha_4 &= 0 \\ 6\alpha_1 - 2\alpha_2 + 7\alpha_3 + 2\alpha_4 &= 0 \\ -6\alpha_1 - 6\alpha_2 - 9\alpha_3 + 5\alpha_4 &= 0 \end{aligned}$$

Form the coefficient matrix of this homogeneous system and row-reduce to obtain

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Analyzing this matrix we are led to conclude that $\alpha_1 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$, $\alpha_4 = 0$. This means there is *only* a trivial relation of linear dependence on the vectors of R and so we call R a linearly independent set (Definition LI).

So it must be that S is linearly dependent. Let us see if we can find a nontrivial relation of linear dependence on S. We will begin as with R, by constructing a relation of linear dependence (Definition RLD) with unknown scalars,

$$\alpha_1 \begin{bmatrix} 2 & 0 \\ 1 & -1 \\ 1 & 3 \end{bmatrix} + \alpha_2 \begin{bmatrix} -4 & 0 \\ -2 & 2 \\ -2 & -6 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 & 1 \\ -2 & 1 \\ 2 & 4 \end{bmatrix} + \alpha_4 \begin{bmatrix} -5 & 3 \\ -10 & 7 \\ 2 & 0 \end{bmatrix} = \mathbf{0}$$

Massaging the left-hand side with our definitions of vector addition and scalar multiplication in M_{32} (Example VSM) we obtain,

$$\begin{bmatrix} 2\alpha_1 - 4\alpha_2 + \alpha_3 - 5\alpha_4 & \alpha_3 + 3\alpha_4 \\ \alpha_1 - 2\alpha_2 - 2\alpha_3 - 10\alpha_4 & -\alpha_1 + 2\alpha_2 + \alpha_3 + 7\alpha_4 \\ \alpha_1 - 2\alpha_2 + 2\alpha_3 + 2\alpha_4 & 3\alpha_1 - 6\alpha_2 + 4\alpha_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Using our definition of matrix equality (Definition ME) to equate entries we get the homogeneous system of six equations in four variables,

$$\begin{aligned} 2\alpha_1 - 4\alpha_2 + \alpha_3 - 5\alpha_4 &= 0 \\ \alpha_3 + 3\alpha_4 &= 0 \\ \alpha_1 - 2\alpha_2 - 2\alpha_3 - 10\alpha_4 &= 0 \\ -\alpha_1 + 2\alpha_2 + \alpha_3 + 7\alpha_4 &= 0 \\ \alpha_1 - 2\alpha_2 + 2\alpha_3 + 2\alpha_4 &= 0 \\ 3\alpha_1 - 6\alpha_2 + 4\alpha_3 &= 0 \end{aligned}$$

Form the coefficient matrix of this homogeneous system and row-reduce to obtain

$$\begin{bmatrix} \boxed{1} & -2 & 0 & -4 \\ 0 & 0 & \boxed{1} & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Analyzing this we see that the system is consistent (we expected this since the system is homogeneous, Theorem HSC) and has $n - r = 4 - 2 = 2$ free variables, namely α_2 and α_4. This means there are infinitely many solutions, and in particular, we can find a nontrivial solution, so long as we do not pick all of our free variables to be zero. The mere presence of a nontrivial solution for these scalars is enough to conclude that S is a linearly dependent set (Definition LI). But let us go ahead and explicitly construct a nontrivial relation of linear dependence.

Choose $\alpha_2 = 1$ and $\alpha_4 = -1$. There is nothing special about this choice, there are infinitely many possibilities, some "easier" than this one, just avoid picking both variables to be zero. (Why not?) Then we find the dependent variables to have values $\alpha_1 = -2$ and $\alpha_3 = 3$. So the relation of linear dependence,

$$(-2)\begin{bmatrix} 2 & 0 \\ 1 & -1 \\ 1 & 3 \end{bmatrix} + (1)\begin{bmatrix} -4 & 0 \\ -2 & 2 \\ -2 & -6 \end{bmatrix} + (3)\begin{bmatrix} 1 & 1 \\ -2 & 1 \\ 2 & 4 \end{bmatrix} + (-1)\begin{bmatrix} -5 & 3 \\ -10 & 7 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

is an iron-clad demonstration that S is linearly dependent. Can you construct another such demonstration? △

Example LIC Linearly independent set in the crazy vector space

Is the set $R = \{(1,\ 0),\ (6,\ 3)\}$ linearly independent in the crazy vector space C (Example CVS)?

We begin with an arbitrary relation of linear dependence on R

$$\mathbf{0} = a_1(1,\ 0) + a_2(6,\ 3) \qquad \text{Definition RLD}$$

and then massage it to a point where we can apply the definition of equality in C. Recall the definitions of vector addition and scalar multiplication in C are not what you would expect.

$$\begin{aligned} &(-1,\ -1) \\ &= \mathbf{0} && \text{Example CVS} \\ &= a_1(1,\ 0) + a_2(6,\ 3) && \text{Definition RLD} \\ &= (1a_1 + a_1 - 1,\ 0a_1 + a_1 - 1) + (6a_2 + a_2 - 1,\ 3a_2 + a_2 - 1) && \text{Example CVS} \end{aligned}$$

$$\begin{aligned} &= (2a_1 - 1,\, a_1 - 1) + (7a_2 - 1,\, 4a_2 - 1) \\ &= (2a_1 - 1 + 7a_2 - 1 + 1,\, a_1 - 1 + 4a_2 - 1 + 1) && \text{Example CVS} \\ &= (2a_1 + 7a_2 - 1,\, a_1 + 4a_2 - 1) \end{aligned}$$

Equality in C (Example CVS) then yields the two equations,

$$\begin{aligned} 2a_1 + 7a_2 - 1 &= -1 \\ a_1 + 4a_2 - 1 &= -1 \end{aligned}$$

which becomes the homogeneous system

$$\begin{aligned} 2a_1 + 7a_2 &= 0 \\ a_1 + 4a_2 &= 0 \end{aligned}$$

Since the coefficient matrix of this system is nonsingular (check this!) the system has only the trivial solution $a_1 = a_2 = 0$. By Definition LI the set R is linearly independent. Notice that even though the zero vector of C is not what we might have first suspected, a question about linear independence still concludes with a question about a homogeneous system of equations. Hmmm. △

Subsection SS
Spanning Sets

In a vector space V, suppose we are given a set of vectors $S \subseteq V$. Then we can immediately construct a subspace, $\langle S\rangle$, using Definition SS and then be assured by Theorem SSS that the construction does provide a subspace. We now turn the situation upside-down. Suppose we are first given a subspace $W \subseteq V$. Can we find a set S so that $\langle S\rangle = W$? Typically W is infinite and we are searching for a finite set of vectors S that we can combine in linear combinations and "build" all of W.

I like to think of S as the raw materials that are sufficient for the construction of W. If you have nails, lumber, wire, copper pipe, drywall, plywood, carpet, shingles, paint (and a few other things), then you can combine them in many different ways to create a house (or infinitely many different houses for that matter). A fast-food restaurant may have beef, chicken, beans, cheese, tortillas, taco shells and hot sauce and from this small list of ingredients build a wide variety of items for sale. Or maybe a better analogy comes from Ben Cordes — the additive primary colors (red, green and blue) can be combined to create many different colors by varying the intensity of each. The intensity is like a scalar multiple, and the combination of the three intensities is like vector addition. The three individual colors, red, green and blue, are the elements of the spanning set.

Because we will use terms like "spanned by" and "spanning set," there is the potential for confusion with "the span." Come back and reread the first paragraph of this subsection whenever you are uncertain about the difference. Here is the working definition.

Definition SSVS Spanning Set of a Vector Space
Suppose V is a vector space. A subset S of V is a **spanning set** of V if $\langle S\rangle = V$. In this case, we also frequently say S **spans** V. □

The definition of a spanning set requires that two sets (subspaces actually) be equal. If S is a subset of V, then $\langle S\rangle \subseteq V$, always. Thus it is usually only necessary to prove that $V \subseteq \langle S\rangle$. Now would be a good time to review Definition SE.

Example SSP4 Spanning set in P_4
In Example SP4 we showed that

$$W = \{\, p(x) |\, p \in P_4,\ p(2) = 0\}$$

is a subspace of P_4, the vector space of polynomials with degree at most 4 (Example

VSP). In this example, we will show that the set

$$S = \left\{x - 2,\, x^2 - 4x + 4,\, x^3 - 6x^2 + 12x - 8,\, x^4 - 8x^3 + 24x^2 - 32x + 16\right\}$$

is a spanning set for W. To do this, we require that $W = \langle S\rangle$. This is an equality of sets. We can check that every polynomial in S has $x = 2$ as a root and therefore $S \subseteq W$. Since W is closed under addition and scalar multiplication, $\langle S\rangle \subseteq W$ also.

So it remains to show that $W \subseteq \langle S\rangle$ (Definition SE). To do this, begin by choosing an arbitrary polynomial in W, say $r(x) = ax^4 + bx^3 + cx^2 + dx + e \in W$. This polynomial is not as arbitrary as it would appear, since we also know it must have $x = 2$ as a root. This translates to

$$0 = a(2)^4 + b(2)^3 + c(2)^2 + d(2) + e = 16a + 8b + 4c + 2d + e$$

as a condition on r.

We wish to show that r is a polynomial in $\langle S\rangle$, that is, we want to show that r can be written as a linear combination of the vectors (polynomials) in S. So let us try.

$$\begin{aligned}
r(x) &= ax^4 + bx^3 + cx^2 + dx + e\\
&= \alpha_1 (x - 2) + \alpha_2 \left(x^2 - 4x + 4\right) + \alpha_3 \left(x^3 - 6x^2 + 12x - 8\right)\\
&\quad + \alpha_4 \left(x^4 - 8x^3 + 24x^2 - 32x + 16\right)\\
&= \alpha_4 x^4 + (\alpha_3 - 8\alpha_4)\, x^3 + (\alpha_2 - 6\alpha_3 + 24\alpha_4)\, x^2\\
&\quad + (\alpha_1 - 4\alpha_2 + 12\alpha_3 - 32\alpha_4)\, x + (-2\alpha_1 + 4\alpha_2 - 8\alpha_3 + 16\alpha_4)
\end{aligned}$$

Equating coefficients (vector equality in P_4) gives the system of five equations in four variables,

$$\begin{aligned}
\alpha_4 &= a\\
\alpha_3 - 8\alpha_4 &= b\\
\alpha_2 - 6\alpha_3 + 24\alpha_4 &= c\\
\alpha_1 - 4\alpha_2 + 12\alpha_3 - 32\alpha_4 &= d\\
-2\alpha_1 + 4\alpha_2 - 8\alpha_3 + 16\alpha_4 &= e
\end{aligned}$$

Any solution to this system of equations will provide the linear combination we need to determine if $r \in \langle S\rangle$, but we need to be convinced there is a solution for any values of a, b, c, d, e that qualify r to be a member of W. So the question is: is this system of equations consistent? We will form the augmented matrix, and row-reduce. (We probably need to do this by hand, since the matrix is symbolic — reversing the order of the first four rows is the best way to start). We obtain a matrix in reduced row-echelon form

$$\begin{aligned}
&\begin{bmatrix}
\boxed{1} & 0 & 0 & 0 & 32a + 12b + 4c + d\\
0 & \boxed{1} & 0 & 0 & 24a + 6b + c\\
0 & 0 & \boxed{1} & 0 & 8a + b\\
0 & 0 & 0 & \boxed{1} & a\\
0 & 0 & 0 & 0 & 16a + 8b + 4c + 2d + e
\end{bmatrix}\\
=&\begin{bmatrix}
\boxed{1} & 0 & 0 & 0 & 32a + 12b + 4c + d\\
0 & \boxed{1} & 0 & 0 & 24a + 6b + c\\
0 & 0 & \boxed{1} & 0 & 8a + b\\
0 & 0 & 0 & \boxed{1} & a\\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\end{aligned}$$

For your results to match our first matrix, you may find it necessary to multiply the final row of your row-reduced matrix by the appropriate scalar, and/or add multiples of this row to some of the other rows. To obtain the second version of the matrix, the last entry of the last column has been simplified to zero according to the one condition we were able to impose on an arbitrary polynomial from W. Since the last column is not a pivot column, Theorem RCLS tells us this system is consistent. Therefore, *any* polynomial from W can be written as a linear combination of the polynomials in S, so $W \subseteq \langle S\rangle$. Therefore, $W = \langle S\rangle$ and S is a spanning set for W by Definition SSVS.

Notice that an alternative to row-reducing the augmented matrix by hand would be to appeal to Theorem FS by expressing the column space of the coefficient matrix as a null space, and then verifying that the condition on r guarantees that r is in the column space, thus implying that the system is always consistent. Give it a try, we will wait. This has been a complicated example, but worth studying carefully. △

Given a subspace and a set of vectors, as in Example SSP4 it can take some work to determine that the set actually is a spanning set. An even harder problem is to be confronted with a subspace and required to construct a spanning set with no guidance. We will now work an example of this flavor, but some of the steps will be unmotivated. Fortunately, we will have some better tools for this type of problem later on.

Example SSM22 Spanning set in M_{22}
In the space of all 2×2 matrices, M_{22} consider the subspace

$$Z = \left\{\begin{bmatrix} a & b \\ c & d\end{bmatrix}\middle|\, a + 3b - c - 5d = 0,\ -2a - 6b + 3c + 14d = 0\right\}$$

and find a spanning set for Z.

We need to construct a limited number of matrices in Z so that every matrix in Z can be expressed as a linear combination of this limited number of matrices. Suppose that $B = \begin{bmatrix} a & b \\ c & d\end{bmatrix}$ is a matrix in Z. Then we can form a column vector with the entries of B and write

$$\begin{bmatrix} a \\ b \\ c \\ d\end{bmatrix} \in \mathcal{N}\left(\begin{bmatrix} 1 & 3 & -1 & -5 \\ -2 & -6 & 3 & 14\end{bmatrix}\right)$$

Row-reducing this matrix and applying Theorem REMES we obtain the equivalent statement,

$$\begin{bmatrix} a \\ b \\ c \\ d\end{bmatrix} \in \mathcal{N}\left(\begin{bmatrix} \boxed{1} & 3 & 0 & -1 \\ 0 & 0 & \boxed{1} & 4\end{bmatrix}\right)$$

We can then express the subspace Z in the following equal forms,

$$\begin{aligned}
Z &= \left\{\begin{bmatrix} a & b \\ c & d\end{bmatrix}\middle|\, a + 3b - c - 5d = 0,\ -2a - 6b + 3c + 14d = 0\right\}\\
&= \left\{\begin{bmatrix} a & b \\ c & d\end{bmatrix}\middle|\, a + 3b - d = 0,\ c + 4d = 0\right\}\\
&= \left\{\begin{bmatrix} a & b \\ c & d\end{bmatrix}\middle|\, a = -3b + d,\ c = -4d\right\}\\
&= \left\{\begin{bmatrix} -3b + d & b \\ -4d & d\end{bmatrix}\middle|\, b,\ d \in \mathbb{C}\right\}\\
&= \left\{\begin{bmatrix} -3b & b \\ 0 & 0\end{bmatrix} + \begin{bmatrix} d & 0 \\ -4d & d\end{bmatrix}\middle|\, b,\ d \in \mathbb{C}\right\}\\
&= \left\{b\begin{bmatrix} -3 & 1 \\ 0 & 0\end{bmatrix} + d\begin{bmatrix} 1 & 0 \\ -4 & 1\end{bmatrix}\middle|\, b,\ d \in \mathbb{C}\right\}
\end{aligned}$$

$$= \left\langle \left\{ \begin{bmatrix} -3 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \right\} \right\rangle$$

So the set

$$Q = \left\{ \begin{bmatrix} -3 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \right\}$$

spans Z by Definition SSVS. △

Example SSC Spanning set in the crazy vector space
In Example LIC we determined that the set $R = \{(1, 0), (6, 3)\}$ is linearly independent in the crazy vector space C (Example CVS). We now show that R is a spanning set for C.

Given an arbitrary vector $(x, y) \in C$ we desire to show that it can be written as a linear combination of the elements of R. In other words, are there scalars a_1 and a_2 so that

$$(x, y) = a_1(1, 0) + a_2(6, 3)$$

We will act as if this equation is true and try to determine just what a_1 and a_2 would be (as functions of x and y). Recall that our vector space operations are unconventional and are defined in Example CVS.

$$\begin{aligned}
(x, y) &= a_1(1, 0) + a_2(6, 3) \\
&= (1a_1 + a_1 - 1,\ 0a_1 + a_1 - 1) + (6a_2 + a_2 - 1,\ 3a_2 + a_2 - 1) \\
&= (2a_1 - 1,\ a_1 - 1) + (7a_2 - 1,\ 4a_2 - 1) \\
&= (2a_1 - 1 + 7a_2 - 1 + 1,\ a_1 - 1 + 4a_2 - 1 + 1) \\
&= (2a_1 + 7a_2 - 1,\ a_1 + 4a_2 - 1)
\end{aligned}$$

Equality in C then yields the two equations,

$$\begin{aligned}
2a_1 + 7a_2 - 1 &= x \\
a_1 + 4a_2 - 1 &= y
\end{aligned}$$

which becomes the linear system with a matrix representation

$$\begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} x+1 \\ y+1 \end{bmatrix}$$

The coefficient matrix of this system is nonsingular, hence invertible (Theorem NI), and we can employ its inverse to find a solution (Theorem TTMI, Theorem SNCM),

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 2 & 7 \\ 1 & 4 \end{bmatrix}^{-1} \begin{bmatrix} x+1 \\ y+1 \end{bmatrix} = \begin{bmatrix} 4 & -7 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x+1 \\ y+1 \end{bmatrix} = \begin{bmatrix} 4x - 7y - 3 \\ -x + 2y + 1 \end{bmatrix}$$

We could chase through the above implications backwards and take the existence of these solutions as sufficient evidence for R being a spanning set for C. Instead, let us view the above as simply scratchwork and now get serious with a simple direct proof that R is a spanning set. Ready? Suppose (x, y) is any vector from C, then compute the following linear combination using the definitions of the operations in C,

$$\begin{aligned}
&(4x - 7y - 3)(1, 0) + (-x + 2y + 1)(6, 3) \\
&\quad = (1(4x - 7y - 3) + (4x - 7y - 3) - 1,\ 0(4x - 7y - 3) + (4x - 7y - 3) - 1) + \\
&\qquad (6(-x + 2y + 1) + (-x + 2y + 1) - 1,\ 3(-x + 2y + 1) + (-x + 2y + 1) - 1) \\
&\quad = (8x - 14y - 7,\ 4x - 7y - 4) + (-7x + 14y + 6,\ -4x + 8y + 3) \\
&\quad = ((8x - 14y - 7) + (-7x + 14y + 6) + 1,\ (4x - 7y - 4) + (-4x + 8y + 3) + 1) \\
&\quad = (x, y)
\end{aligned}$$

This final sequence of computations in C is sufficient to demonstrate that any element of C *can* be written (or expressed) as a linear combination of the two vectors in R, so $C \subseteq \langle R\rangle$. Since the reverse inclusion $\langle R\rangle \subseteq C$ is trivially true, $C = \langle R\rangle$ and we say R spans C (Definition SSVS). Notice that this demonstration is no more or less valid if we hide from the reader our scratchwork that suggested $a_1 = 4x - 7y - 3$ and $a_2 = -x + 2y + 1$. △

Subsection VR
Vector Representation

In Chapter R we will take up the matter of representations fully, where Theorem VRRB will be critical for Definition VR. We will now motivate and prove a critical theorem that tells us how to "represent" a vector. This theorem could wait, but working with it now will provide some extra insight into the nature of linearly independent spanning sets. First an example, then the theorem.

Example AVR A vector representation
Consider the set

$$S = \left\{ \begin{bmatrix} -7 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} -12 \\ 7 \\ 4 \end{bmatrix} \right\}$$

from the vector space $\mathbb{C}^3$. Let A be the matrix whose columns are the set S, and verify that A is nonsingular. By Theorem NMLIC the elements of S form a linearly independent set. Suppose that $\mathbf{b} \in \mathbb{C}^3$. Then $\mathcal{LS}(A, \mathbf{b})$ has a (unique) solution (Theorem NMUS) and hence is consistent. By Theorem SLSLC, $\mathbf{b} \in \langle S\rangle$. Since $\mathbf{b}$ is arbitrary, this is enough to show that $\langle S\rangle = \mathbb{C}^3$, and therefore S is a spanning set for $\mathbb{C}^3$ (Definition SSVS). (This set comes from the columns of the coefficient matrix of Archetype B.)

Now examine the situation for a particular choice of $\mathbf{b}$, say $\mathbf{b} = \begin{bmatrix} -33 \\ 24 \\ 5 \end{bmatrix}$. Because S is a spanning set for $\mathbb{C}^3$, we know we can write $\mathbf{b}$ as a linear combination of the vectors in S,

$$\begin{bmatrix} -33 \\ 24 \\ 5 \end{bmatrix} = (-3) \begin{bmatrix} -7 \\ 5 \\ 1 \end{bmatrix} + (5) \begin{bmatrix} -6 \\ 5 \\ 0 \end{bmatrix} + (2) \begin{bmatrix} -12 \\ 7 \\ 4 \end{bmatrix}.$$

The nonsingularity of the matrix A tells that the scalars in this linear combination are unique. More precisely, it is the linear independence of S that provides the uniqueness. We will refer to the scalars $a_1 = -3$, $a_2 = 5$, $a_3 = 2$ as a "representation of $\mathbf{b}$ relative to S." In other words, once we settle on S as a linearly independent set that spans $\mathbb{C}^3$, the vector $\mathbf{b}$ is recoverable just by knowing the scalars $a_1 = -3$, $a_2 = 5$, $a_3 = 2$ (use these scalars in a linear combination of the vectors in S). This is all an illustration of the following important theorem, which we prove in the setting of a general vector space. △

Theorem VRRB Vector Representation Relative to a Basis
Suppose that V is a vector space and $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_m\}$ is a linearly independent set that spans V. Let $\mathbf{w}$ be any vector in V. Then there exist unique *scalars $a_1, a_2, a_3, \ldots, a_m$ such that*

$$\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \cdots + a_m\mathbf{v}_m.$$

Proof. That $\mathbf{w}$ can be written as a linear combination of the vectors in B follows from the spanning property of the set (Definition SSVS). This is good, but not the meat of this theorem. We now know that for any choice of the vector $\mathbf{w}$ there exist *some* scalars that will create $\mathbf{w}$ as a linear combination of the basis vectors. The real question is: Is there *more* than one way to write $\mathbf{w}$ as a linear combination of $\{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3,\,\ldots,\,\mathbf{v}_m\}$? Are the scalars $a_1,\,a_2,\,a_3,\,\ldots,\,a_m$ unique? (Proof Technique U)

Assume there are two different linear combinations of $\{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3,\,\ldots,\,\mathbf{v}_m\}$ that equal the vector $\mathbf{w}$. In other words there exist scalars $a_1,\,a_2,\,a_3,\,\ldots,\,a_m$ and $b_1,\,b_2,\,b_3,\,\ldots,\,b_m$ so that

$$\begin{aligned}\mathbf{w} &= a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \cdots + a_m\mathbf{v}_m\\ \mathbf{w} &= b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + b_3\mathbf{v}_3 + \cdots + b_m\mathbf{v}_m.\end{aligned}$$

Then notice that

$$\begin{aligned}\mathbf{0} &= \mathbf{w} + (-\mathbf{w}) && \text{Property AI}\\ &= \mathbf{w} + (-1)\mathbf{w} && \text{Theorem AISM}\\ &= (a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \cdots + a_m\mathbf{v}_m)+\\ &\qquad (-1)(b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + b_3\mathbf{v}_3 + \cdots + b_m\mathbf{v}_m)\\ &= (a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \cdots + a_m\mathbf{v}_m)+\\ &\qquad (-b_1\mathbf{v}_1 - b_2\mathbf{v}_2 - b_3\mathbf{v}_3 - \ldots - b_m\mathbf{v}_m) && \text{Property DVA}\\ &= (a_1 - b_1)\mathbf{v_1} + (a_2 - b_2)\mathbf{v_2} + (a_3 - b_3)\mathbf{v_3}+\\ &\qquad \cdots + (a_m - b_m)\mathbf{v_m} && \text{Property C, Property DSA}\end{aligned}$$

But this is a relation of linear dependence on a linearly independent set of vectors (Definition RLD)! Now we are using the other assumption about B, that $\{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3,\,\ldots,\,\mathbf{v}_m\}$ is a linearly independent set. So by Definition LI it *must* happen that the scalars are all zero. That is,

$$\begin{aligned}(a_1 - b_1) &= 0 & (a_2 - b_2) &= 0 & (a_3 - b_3) &= 0 & \ldots && (a_m - b_m) &= 0\\ a_1 &= b_1 & a_2 &= b_2 & a_3 &= b_3 & \ldots && a_m &= b_m.\end{aligned}$$

And so we find that the scalars are unique. ■

The converse of Theorem VRRB is true as well, but is not important enough to rise beyond an exercise (see Exercise LISS.T51).

This is a very typical use of the hypothesis that a set is linearly independent — obtain a relation of linear dependence and then conclude that the scalars *must* all be zero. The result of this theorem tells us that we can write any vector in a vector space as a linear combination of the vectors in a linearly independent spanning set, but only just. There is only enough raw material in the spanning set to write each vector one way as a linear combination. So in this sense, we could call a linearly independent spanning set a "minimal spanning set." These sets are so important that we will give them a simpler name ("basis") and explore their properties further in the next section.

Reading Questions

1. Is the set of matrices below linearly independent or linearly dependent in the vector space M_{22}? Why or why not?

$$\left\{\begin{bmatrix}1 & 3\\ -2 & 4\end{bmatrix},\,\begin{bmatrix}-2 & 3\\ 3 & -5\end{bmatrix},\,\begin{bmatrix}0 & 9\\ -1 & 3\end{bmatrix}\right\}$$

2. Explain the difference between the following two uses of the term "span":

   1. S is a subset of the vector space V and the span of S is a subspace of V.
   2. W is a subspace of the vector space Y and T spans W.

3. The set

$$S = \left\{ \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix},\ \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix},\ \begin{bmatrix} 5 \\ 8 \\ 2 \end{bmatrix} \right\}$$

is linearly independent and spans $\mathbb{C}^3$. Write the vector $\mathbf{x} = \begin{bmatrix} -6 \\ 2 \\ 2 \end{bmatrix}$ as a linear combination of the elements of S. How many ways are there to answer this question, and which theorem allows you to say so?

Exercises

C20† In the vector space of 2×2 matrices, M_{22}, determine if the set S below is linearly independent.

$$S = \left\{ \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix},\ \begin{bmatrix} 0 & 4 \\ -1 & 2 \end{bmatrix},\ \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} \right\}$$

C21† In the crazy vector space C (Example CVS), is the set $S = \{(0,\ 2),\ (2,\ 8)\}$ linearly independent?

C22† In the vector space of polynomials P_3, determine if the set S is linearly independent or linearly dependent.

$$S = \left\{2 + x - 3x^2 - 8x^3,\ 1 + x + x^2 + 5x^3,\ 3 - 4x^2 - 7x^3\right\}$$

C23† Determine if the set $S = \{(3,\ 1),\ (7,\ 3)\}$ is linearly independent in the crazy vector space C (Example CVS).

C24† In the vector space of real-valued functions $F = \{\, f \mid f : \mathbb{R} \to \mathbb{R}\}$, determine if the following set S is linearly independent.

$$S = \left\{\sin^2 x, \cos^2 x, 2\right\}$$

C25† Let

$$S = \left\{ \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix},\ \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix},\ \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \right\}$$

1. Determine if S spans $M_{2,2}$.
2. Determine if S is linearly independent.

C26† Let

$$S = \left\{ \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix},\ \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix},\ \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix},\ \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},\ \begin{bmatrix} 1 & 4 \\ 0 & 3 \end{bmatrix} \right\}$$

1. Determine if S spans $M_{2,2}$.
2. Determine if S is linearly independent.

C30 In Example LIM32, find another nontrivial relation of linear dependence on the linearly dependent set of 3×2 matrices, S.

C40† Determine if the set $T = \left\{x^2 - x + 5,\ 4x^3 - x^2 + 5x,\ 3x + 2\right\}$ spans the vector space of polynomials with degree 4 or less, P_4.

C41† The set W is a subspace of M_{22}, the vector space of all 2×2 matrices. Prove that S is a spanning set for W.

$$W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| 2a - 3b + 4c - d = 0 \right\} \qquad S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},\ \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix},\ \begin{bmatrix} 0 & 0 \\ 1 & 4 \end{bmatrix} \right\}$$

C42† Determine if the set $S = \{(3,\ 1),\ (7,\ 3)\}$ spans the crazy vector space C (Example CVS).

M10† Halfway through Example SSP4, we need to show that the system of equations

$$\mathcal{LS}\left(\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -8 \\ 0 & 1 & -6 & 24 \\ 1 & -4 & 12 & -32 \\ -2 & 4 & -8 & 16 \end{bmatrix},\ \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix}\right)$$

is consistent for every choice of the vector of constants satisfying $16a+8b+4c+2d+e=0$.

Express the column space of the coefficient matrix of this system as a null space, using Theorem FS. From this use Theorem CSCS to establish that the system is always consistent. Notice that this approach removes from Example SSP4 the need to row-reduce a symbolic matrix.

T20† Suppose that S is a finite linearly independent set of vectors from the vector space V. Let T be any subset of S. Prove that T is linearly independent.

T40 Prove the following variant of Theorem EMMVP that has a weaker hypothesis: Suppose that $C = \{\mathbf{u}_1,\,\mathbf{u}_2,\,\mathbf{u}_3,\,\ldots,\,\mathbf{u}_p\}$ is a linearly independent spanning set for $\mathbb{C}^n$. Suppose also that A and B are $m\times n$ matrices such that $A\mathbf{u}_i = B\mathbf{u}_i$ for every $1 \le i \le n$. Then $A = B$.

Can you weaken the hypothesis even further while still preserving the conclusion?

T50† Suppose that V is a vector space and $\mathbf{u},\,\mathbf{v} \in V$ are two vectors in V. Use the definition of linear independence to prove that $S = \{\mathbf{u},\,\mathbf{v}\}$ is a linearly dependent set if and only if one of the two vectors is a scalar multiple of the other. Prove this directly in the context of an abstract vector space (V), without simply giving an upgraded version of Theorem DLDS for the special case of just two vectors.

T51† Carefully formulate the converse of Theorem VRRB and provide a proof.

Section B
Bases

A basis of a vector space is one of the most useful concepts in linear algebra. It often provides a concise, finite description of an infinite vector space.

Subsection B
Bases

We now have all the tools in place to define a basis of a vector space.

Definition B Basis
Suppose V is a vector space. Then a subset $S \subseteq V$ is a **basis** of V if it is linearly independent and spans V. □

So, a basis is a linearly independent spanning set for a vector space. The requirement that the set spans V insures that S has enough raw material to build V, while the linear independence requirement insures that we do not have any more raw material than we need. As we shall see soon in Section D, a basis is a minimal spanning set.

You may have noticed that we used the term basis for some of the titles of previous theorems (e.g. Theorem BNS, Theorem BCS, Theorem BRS) and if you review each of these theorems you will see that their conclusions provide linearly independent spanning sets for sets that we now recognize as subspaces of $\mathbb{C}^m$. Examples associated with these theorems include Example NSLIL, Example CSOCD and Example IAS. As we will see, these three theorems will continue to be powerful tools, even in the setting of more general vector spaces.

Furthermore, the archetypes contain an abundance of bases. For each coefficient matrix of a system of equations, and for each archetype defined simply as a matrix, there is a basis for the null space, *three* bases for the column space, and a basis for the row space. For this reason, our subsequent examples will concentrate on bases for vector spaces other than $\mathbb{C}^m$.

Notice that Definition B does not preclude a vector space from having many bases, and this is the case, as hinted above by the statement that the archetypes contain three bases for the column space of a matrix. More generally, we can grab any basis for a vector space, multiply any one basis vector by a nonzero scalar and create a slightly different set that is still a basis. For "important" vector spaces, it will be convenient to have a collection of "nice" bases. When a vector space has a single particularly nice basis, it is sometimes called the **standard basis** though there is nothing precise enough about this term to allow us to define it formally — it is a question of style. Here are some nice bases for important vector spaces.

Theorem SUVB Standard Unit Vectors are a Basis
The set of standard unit vectors for $\mathbb{C}^m$ (Definition SUV), $B = \{\mathbf{e}_i \mid 1 \leq i \leq m\}$ is a basis for the vector space $\mathbb{C}^m$.

Proof. We must show that the set B is both linearly independent and a spanning set for $\mathbb{C}^m$. First, the vectors in B are, by Definition SUV, the columns of the identity matrix, which we know is nonsingular (since it row-reduces to the identity matrix, Theorem NMRRI). And the columns of a nonsingular matrix are linearly independent by Theorem NMLIC.

Suppose we grab an arbitrary vector from $\mathbb{C}^m$, say

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_m \end{bmatrix}.$$

Can we write $\mathbf{v}$ as a linear combination of the vectors in B? Yes, and quite simply.

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_m \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + v_m \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3 + \cdots + v_m\mathbf{e}_m$$

This shows that $\mathbb{C}^m \subseteq \langle B\rangle$, which is sufficient to show that B is a spanning set for $\mathbb{C}^m$. ■

Example BP Bases for P_n

The vector space of polynomials with degree at most n, P_n, has the basis

$$B = \left\{1,\, x,\, x^2,\, x^3,\, \ldots,\, x^n\right\}.$$

Another nice basis for P_n is

$$C = \left\{1,\, 1+x,\, 1+x+x^2,\, 1+x+x^2+x^3,\, \ldots,\, 1+x+x^2+x^3+\cdots+x^n\right\}.$$

Checking that each of B and C is a linearly independent spanning set are good exercises. △

Example BM A basis for the vector space of matrices

In the vector space M_{mn} of matrices (Example VSM) define the matrices $B_{k\ell}$, $1 \leq k \leq m$, $1 \leq \ell \leq n$ by

$$[B_{k\ell}]_{ij} = \begin{cases} 1 & \text{if } k = i,\, \ell = j \\ 0 & \text{otherwise} \end{cases}$$

So these matrices have entries that are all zeros, with the exception of a lone entry that is one. The set of all mn of them,

$$B = \{\, B_{k\ell} \mid 1 \leq k \leq m,\ 1 \leq \ell \leq n\}$$

forms a basis for M_{mn}. See Exercise B.M20. △

The bases described above will often be convenient ones to work with. However a basis does not have to obviously look like a basis.

Example BSP4 A basis for a subspace of P_4

In Example SSP4 we showed that

$$S = \left\{x-2,\, x^2-4x+4,\, x^3-6x^2+12x-8,\, x^4-8x^3+24x^2-32x+16\right\}$$

is a spanning set for $W = \{\, p(x) \mid p \in P_4,\ p(2) = 0\}$. We will now show that S is also linearly independent in W. Begin with a relation of linear dependence,

$$\begin{aligned}
&0 + 0x + 0x^2 + 0x^3 + 0x^4 \\
&\qquad = \alpha_1\left(x-2\right) + \alpha_2\left(x^2-4x+4\right) + \alpha_3\left(x^3-6x^2+12x-8\right) \\
&\qquad\quad + \alpha_4\left(x^4-8x^3+24x^2-32x+16\right) \\
&\qquad = \alpha_4x^4 + \left(\alpha_3-8\alpha_4\right)x^3 + \left(\alpha_2-6\alpha_3+24\alpha_4\right)x^2 \\
&\qquad\quad + \left(\alpha_1-4\alpha_2+12\alpha_3-32\alpha_4\right)x + \left(-2\alpha_1+4\alpha_2-8\alpha_3+16\alpha_4\right)
\end{aligned}$$

Equating coefficients (vector equality in P_4) gives the homogeneous system of five equations in four variables,

$$\begin{aligned} \alpha_4 &= 0 \\ \alpha_3 - 8\alpha_4 &= 0 \\ \alpha_2 - 6\alpha_3 + 24\alpha_4 &= 0 \\ \alpha_1 - 4\alpha_2 + 12\alpha_3 - 32\alpha_4 &= 0 \\ -2\alpha_1 + 4\alpha_2 - 8\alpha_3 + 16\alpha_4 &= 0 \end{aligned}$$

We form the coefficient matrix, and row-reduce to obtain a matrix in reduced row-echelon form

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

With *only* the trivial solution to this homogeneous system, we conclude that only scalars that will form a relation of linear dependence are the trivial ones, and therefore the set S is linearly independent (Definition LI). Finally, S has earned the right to be called a basis for W (Definition B). △

Example BSM22 A basis for a subspace of M_{22}
In Example SSM22 we discovered that

$$Q = \left\{ \begin{bmatrix} -3 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \right\}$$

is a spanning set for the subspace

$$Z = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \, a + 3b - c - 5d = 0, \; -2a - 6b + 3c + 14d = 0 \right\}$$

of the vector space of all 2×2 matrices, M_{22}. If we can also determine that Q is linearly independent in Z (or in M_{22}), then it will qualify as a basis for Z.

Let us begin with a relation of linear dependence.

$$\begin{aligned} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} &= \alpha_1 \begin{bmatrix} -3 & 1 \\ 0 & 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -3\alpha_1 + \alpha_2 & \alpha_1 \\ -4\alpha_2 & \alpha_2 \end{bmatrix} \end{aligned}$$

Using our definition of matrix equality (Definition ME) we equate entries and get a homogeneous system of four equations in two variables,

$$\begin{aligned} -3\alpha_1 + \alpha_2 &= 0 \\ \alpha_1 &= 0 \\ -4\alpha_2 &= 0 \\ \alpha_2 &= 0 \end{aligned}$$

We could row-reduce the coefficient matrix of this homogeneous system, but it is not necessary. The second and fourth equations tell us that $\alpha_1 = 0$, $\alpha_2 = 0$ is the *only* solution to this homogeneous system. This qualifies the set Q as being linearly independent, since the only relation of linear dependence is trivial (Definition LI). Therefore Q is a basis for Z (Definition B). △

Example BC Basis for the crazy vector space
In Example LIC and Example SSC we determined that the set $R = \{(1, 0), (6, 3)\}$

from the crazy vector space, C (Example CVS), is linearly independent and is a spanning set for C. By Definition B we see that R is a basis for C. △

We have seen that several of the sets associated with a matrix are subspaces of vector spaces of column vectors. Specifically these are the null space (Theorem NSMS), column space (Theorem CSMS), row space (Theorem RSMS) and left null space (Theorem LNSMS). As subspaces they are vector spaces (Definition S) and it is natural to ask about bases for these vector spaces. Theorem BNS, Theorem BCS, Theorem BRS each have conclusions that provide linearly independent spanning sets for (respectively) the null space, column space, and row space. Notice that each of these theorems contains the word "basis" in its title, even though we did not know the precise meaning of the word at the time. To find a basis for a left null space we can use the definition of this subspace as a null space (Definition LNS) and apply Theorem BNS. Or Theorem FS tells us that the left null space can be expressed as a row space and we can then use Theorem BRS.

Theorem BS is another early result that provides a linearly independent spanning set (i.e. a basis) as its conclusion. If a vector space of column vectors can be expressed as a span of a set of column vectors, then Theorem BS can be employed in a straightforward manner to quickly yield a basis.

Subsection BSCV
Bases for Spans of Column Vectors

We have seen several examples of bases in different vector spaces. In this subsection, and the next (Subsection B.BNM), we will consider building bases for $\mathbb{C}^m$ and its subspaces.

Suppose we have a subspace of $\mathbb{C}^m$ that is expressed as the span of a set of vectors, S, and S is not necessarily linearly independent, or perhaps not very attractive. Theorem REMRS says that row-equivalent matrices have identical row spaces, while Theorem BRS says the nonzero rows of a matrix in reduced row-echelon form are a basis for the row space. These theorems together give us a great computational tool for quickly finding a basis for a subspace that is expressed originally as a span.

Example RSB Row space basis
When we first defined the span of a set of column vectors, in Example SCAD we looked at the set

$$W = \left\langle\left\{\begin{bmatrix}2\\-3\\1\end{bmatrix},\,\begin{bmatrix}1\\4\\1\end{bmatrix},\,\begin{bmatrix}7\\-5\\4\end{bmatrix},\,\begin{bmatrix}-7\\-6\\-5\end{bmatrix}\right\}\right\rangle$$

with an eye towards realizing W as the span of a smaller set. By building relations of linear dependence (though we did not know them by that name then) we were able to remove two vectors and write W as the span of the other two vectors. These two remaining vectors formed a linearly independent set, even though we did not know that at the time.

Now we know that W is a subspace and must have a basis. Consider the matrix, C, whose rows are the vectors in the spanning set for W,

$$C = \begin{bmatrix}2 & -3 & 1\\1 & 4 & 1\\7 & -5 & 4\\-7 & -6 & -5\end{bmatrix}$$

Then, by Definition RSM, the row space of C will be W, $\mathcal{R}(C) = W$. Theorem BRS tells us that if we row-reduce C, the nonzero rows of the row-equivalent matrix in reduced row-echelon form will be a basis for $\mathcal{R}(C)$, and hence a basis for W. Let us do it — C row-reduces to

$$\begin{bmatrix} \boxed{1} & 0 & \frac{7}{11} \\ 0 & \boxed{1} & \frac{1}{11} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

If we convert the two nonzero rows to column vectors then we have a basis,

$$B = \left\{ \begin{bmatrix} 1 \\ 0 \\ \frac{7}{11} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \frac{1}{11} \end{bmatrix} \right\}$$

and

$$W = \left\langle \left\{ \begin{bmatrix} 1 \\ 0 \\ \frac{7}{11} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \frac{1}{11} \end{bmatrix} \right\} \right\rangle$$

For aesthetic reasons, we might wish to multiply each vector in B by 11, which will not change the spanning or linear independence properties of B as a basis. Then we can also write

$$W = \left\langle \left\{ \begin{bmatrix} 11 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 11 \\ 1 \end{bmatrix} \right\} \right\rangle$$

△

Example IAS provides another example of this flavor, though now we can notice that X is a subspace, and that the resulting set of three vectors is a basis. This is such a powerful technique that we should do one more example.

Example RS Reducing a span
In Example RSC5 we began with a set of $n = 4$ vectors from $\mathbb{C}^5$,

$$R = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -7 \\ 6 \\ -11 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 2 \\ 1 \\ 6 \end{bmatrix} \right\}$$

and defined $V = \langle R \rangle$. Our goal in that problem was to find a relation of linear dependence on the vectors in R, solve the resulting equation for one of the vectors, and re-express V as the span of a set of three vectors.

Here is another way to accomplish something similar. The row space of the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 & 2 \\ 2 & 1 & 3 & 1 & 2 \\ 0 & -7 & 6 & -11 & -2 \\ 4 & 1 & 2 & 1 & 6 \end{bmatrix}$$

is equal to $\langle R \rangle$. By Theorem BRS we can row-reduce this matrix, ignore any zero rows, and use the nonzero rows as column vectors that are a basis for the row space of A. Row-reducing A creates the matrix

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{17} & \frac{30}{17} \\ 0 & 1 & 0 & \frac{25}{17} & -\frac{2}{17} \\ 0 & 0 & 1 & -\frac{2}{17} & -\frac{8}{17} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So

$$\left\{\begin{bmatrix}1\\0\\0\\-\frac{1}{17}\\\frac{30}{17}\end{bmatrix},\ \begin{bmatrix}0\\1\\0\\\frac{25}{17}\\-\frac{2}{17}\end{bmatrix},\ \begin{bmatrix}0\\0\\1\\-\frac{2}{17}\\-\frac{8}{17}\end{bmatrix}\right\}$$

is a basis for V. Our theorem tells us this is a basis, there is no need to verify that the subspace spanned by three vectors (rather than four) is the identical subspace, and there is no need to verify that we have reached the limit in reducing the set, since the set of three vectors is guaranteed to be linearly independent. △

Subsection BNM Bases and Nonsingular Matrices

A quick source of diverse bases for $\mathbb{C}^m$ is the set of columns of a nonsingular matrix.

Theorem CNMB Columns of Nonsingular Matrix are a Basis
Suppose that A is a square matrix of size m. Then the columns of A are a basis of $\mathbb{C}^m$ if and only if A is nonsingular.

Proof. ($\Rightarrow$) Suppose that the columns of A are a basis for $\mathbb{C}^m$. Then Definition B says the set of columns is linearly independent. Theorem NMLIC then says that A is nonsingular.

($\Leftarrow$) Suppose that A is nonsingular. Then by Theorem NMLIC this set of columns is linearly independent. Theorem CSNM says that for a nonsingular matrix, $\mathcal{C}(A)=\mathbb{C}^m$. This is equivalent to saying that the columns of A are a spanning set for the vector space $\mathbb{C}^m$. As a linearly independent spanning set, the columns of A qualify as a basis for $\mathbb{C}^m$ (Definition B). ■

Example CABAK Columns as Basis, Archetype K
Archetype K is the 5×5 matrix

$$K=\begin{bmatrix}10 & 18 & 24 & 24 & -12\\ 12 & -2 & -6 & 0 & -18\\ -30 & -21 & -23 & -30 & 39\\ 27 & 30 & 36 & 37 & -30\\ 18 & 24 & 30 & 30 & -20\end{bmatrix}$$

which is row-equivalent to the 5×5 identity matrix I_5. So by Theorem NMRRI, K is nonsingular. Then Theorem CNMB says the set

$$\left\{\begin{bmatrix}10\\12\\-30\\27\\18\end{bmatrix},\ \begin{bmatrix}18\\-2\\-21\\30\\24\end{bmatrix},\ \begin{bmatrix}24\\-6\\-23\\36\\30\end{bmatrix},\ \begin{bmatrix}24\\0\\-30\\37\\30\end{bmatrix},\ \begin{bmatrix}-12\\-18\\39\\-30\\-20\end{bmatrix}\right\}$$

is a (novel) basis of $\mathbb{C}^5$. △

Perhaps we should view the fact that the standard unit vectors are a basis (Theorem SUVB) as just a simple corollary of Theorem CNMB? (See Proof Technique LC.)

With a new equivalence for a nonsingular matrix, we can update our list of equivalences.

Theorem NME5 Nonsingular Matrix Equivalences, Round 5
Suppose that A is a square matrix of size n. The following are equivalent.

1. *A is nonsingular.*

2. *A row-reduces to the identity matrix.*

3. *The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.*

4. *The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.*

5. *The columns of A are a linearly independent set.*

6. *A is invertible.*

7. *The column space of A is $\mathbb{C}^n$, $\mathcal{C}(A) = \mathbb{C}^n$.*

8. *The columns of A are a basis for $\mathbb{C}^n$.*

Proof. With a new equivalence for a nonsingular matrix in Theorem CNMB we can expand Theorem NME4. ■

Subsection OBC Orthonormal Bases and Coordinates

We learned about orthogonal sets of vectors in $\mathbb{C}^m$ back in Section O, and we also learned that orthogonal sets are automatically linearly independent (Theorem OSLI). When an orthogonal set also spans a subspace of $\mathbb{C}^m$, then the set is a basis. And when the set is orthonormal, then the set is an incredibly nice basis. We will back up this claim with a theorem, but first consider how you might manufacture such a set.

Suppose that W is a subspace of $\mathbb{C}^m$ with basis B. Then B spans W and is a linearly independent set of nonzero vectors. We can apply the Gram-Schmidt Procedure (Theorem GSP) and obtain a linearly independent set T such that $\langle T\rangle = \langle B\rangle = W$ and T is orthogonal. In other words, T is a basis for W, and is an orthogonal set. By scaling each vector of T to norm 1, we can convert T into an orthonormal set, without destroying the properties that make it a basis of W. In short, we can convert any basis into an orthonormal basis. Example GSTV, followed by Example ONTV, illustrates this process.

Unitary matrices (Definition UM) are another good source of orthonormal bases (and vice versa). Suppose that Q is a unitary matrix of size n. Then the n columns of Q form an orthonormal set (Theorem CUMOS) that is therefore linearly independent (Theorem OSLI). Since Q is invertible (Theorem UMI), we know Q is nonsingular (Theorem NI), and then the columns of Q span $\mathbb{C}^n$ (Theorem CSNM). So the columns of a unitary matrix of size n are an orthonormal basis for $\mathbb{C}^n$.

Why all the fuss about orthonormal bases? Theorem VRRB told us that any vector in a vector space could be written, uniquely, as a linear combination of basis vectors. For an orthonormal basis, finding the scalars for this linear combination is extremely easy, and this is the content of the next theorem. Furthermore, with vectors written this way (as linear combinations of the elements of an orthonormal set) certain computations and analysis become much easier. Here is the promised theorem.

Theorem COB Coordinates and Orthonormal Bases
Suppose that $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_p\}$ is an orthonormal basis of the subspace W of $\mathbb{C}^m$. For any $\mathbf{w} \in W$,

$$\mathbf{w} = \langle \mathbf{v}_1, \mathbf{w}\rangle \mathbf{v}_1 + \langle \mathbf{v}_2, \mathbf{w}\rangle \mathbf{v}_2 + \langle \mathbf{v}_3, \mathbf{w}\rangle \mathbf{v}_3 + \cdots + \langle \mathbf{v}_p, \mathbf{w}\rangle \mathbf{v}_p$$

Proof. Because B is a basis of W, Theorem VRRB tells us that we can write $\mathbf{w}$ uniquely as a linear combination of the vectors in B. So it is not this aspect of the conclusion that makes this theorem interesting. What is interesting is that the particular scalars are so easy to compute. No need to solve big systems of equations

— just do an inner product of $\mathbf{w}$ with $\mathbf{v}_i$ to arrive at the coefficient of $\mathbf{v}_i$ in the linear combination.

So begin the proof by writing $\mathbf{w}$ as a linear combination of the vectors in B, using unknown scalars,

$$\mathbf{w} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \cdots + a_p\mathbf{v}_p$$

and compute,

$$\begin{aligned}
\langle \mathbf{v}_i,\, \mathbf{w}\rangle &= \left\langle \mathbf{v}_i,\, \sum_{k=1}^{p} a_k\mathbf{v}_k \right\rangle && \text{Theorem VRRB}\\
&= \sum_{k=1}^{p} \langle \mathbf{v}_i,\, a_k\mathbf{v}_k\rangle && \text{Theorem IPVA}\\
&= \sum_{k=1}^{p} a_k \langle \mathbf{v}_i,\, \mathbf{v}_k\rangle && \text{Theorem IPSM}\\
&= a_i \langle \mathbf{v}_i,\, \mathbf{v}_i\rangle + \sum_{\substack{k=1\\k\neq i}}^{p} a_k \langle \mathbf{v}_i,\, \mathbf{v}_k\rangle && \text{Property C}\\
&= a_i(1) + \sum_{\substack{k=1\\k\neq i}}^{p} a_k(0) && \text{Definition ONS}\\
&= a_i
\end{aligned}$$

So the (unique) scalars for the linear combination are indeed the inner products advertised in the conclusion of the theorem's statement. ■

Example CROB4 Coordinatization relative to an orthonormal basis, $\mathbb{C}^4$
The set

$$\{\mathbf{x}_1,\,\mathbf{x}_2,\,\mathbf{x}_3,\,\mathbf{x}_4\} = \left\{ \begin{bmatrix}1+i\\1\\1-i\\i\end{bmatrix},\, \begin{bmatrix}1+5i\\6+5i\\-7-i\\1-6i\end{bmatrix},\, \begin{bmatrix}-7+34i\\-8-23i\\-10+22i\\30+13i\end{bmatrix},\, \begin{bmatrix}-2-4i\\6+i\\4+3i\\6-i\end{bmatrix} \right\}$$

was proposed, and partially verified, as an orthogonal set in Example AOS. Let us scale each vector to norm 1, so as to form an orthonormal set in $\mathbb{C}^4$. Then by Theorem OSLI the set will be linearly independent, and by Theorem NME5 the set will be a basis for $\mathbb{C}^4$. So, once scaled to norm 1, the adjusted set will be an orthonormal basis of $\mathbb{C}^4$. The norms are,

$$\|\mathbf{x}_1\| = \sqrt{6} \qquad \|\mathbf{x}_2\| = \sqrt{174} \qquad \|\mathbf{x}_3\| = \sqrt{3451} \qquad \|\mathbf{x}_4\| = \sqrt{119}$$

So an orthonormal basis is

$$\begin{aligned}
B &= \{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3,\,\mathbf{v}_4\}\\
&= \left\{ \frac{1}{\sqrt{6}}\begin{bmatrix}1+i\\1\\1-i\\i\end{bmatrix},\, \frac{1}{\sqrt{174}}\begin{bmatrix}1+5i\\6+5i\\-7-i\\1-6i\end{bmatrix},\, \frac{1}{\sqrt{3451}}\begin{bmatrix}-7+34i\\-8-23i\\-10+22i\\30+13i\end{bmatrix},\, \frac{1}{\sqrt{119}}\begin{bmatrix}-2-4i\\6+i\\4+3i\\6-i\end{bmatrix} \right\}
\end{aligned}$$

Now, to illustrate Theorem COB, choose any vector from $\mathbb{C}^4$, say $\mathbf{w} = \begin{bmatrix}2\\-3\\1\\4\end{bmatrix}$, and compute

$$\langle \mathbf{w},\, \mathbf{v}_1\rangle = \frac{-5i}{\sqrt{6}} \qquad\qquad \langle \mathbf{w},\, \mathbf{v}_2\rangle = \frac{-19+30i}{\sqrt{174}}$$

$$\langle \mathbf{w},\, \mathbf{v}_3\rangle = \frac{120-211i}{\sqrt{3451}} \qquad\qquad \langle \mathbf{w},\, \mathbf{v}_4\rangle = \frac{6+12i}{\sqrt{119}}$$

Then Theorem COB guarantees that

$$\begin{bmatrix}2\\-3\\1\\4\end{bmatrix} = \frac{-5i}{\sqrt{6}}\left(\frac{1}{\sqrt{6}}\begin{bmatrix}1+i\\1\\1-i\\i\end{bmatrix}\right) + \frac{-19+30i}{\sqrt{174}}\left(\frac{1}{\sqrt{174}}\begin{bmatrix}1+5i\\6+5i\\-7-i\\1-6i\end{bmatrix}\right)$$
$$+\frac{120-211i}{\sqrt{3451}}\left(\frac{1}{\sqrt{3451}}\begin{bmatrix}-7+34i\\-8-23i\\-10+22i\\30+13i\end{bmatrix}\right) + \frac{6+12i}{\sqrt{119}}\left(\frac{1}{\sqrt{119}}\begin{bmatrix}-2-4i\\6+i\\4+3i\\6-i\end{bmatrix}\right)$$

as you might want to check (if you have unlimited patience). △

A slightly less intimidating example follows, in three dimensions and with just real numbers.

Example CROB3 Coordinatization relative to an orthonormal basis, $\mathbb{C}^3$
The set

$$\{\mathbf{x}_1,\, \mathbf{x}_2,\, \mathbf{x}_3\} = \left\{\begin{bmatrix}1\\2\\1\end{bmatrix},\, \begin{bmatrix}-1\\0\\1\end{bmatrix},\, \begin{bmatrix}2\\1\\1\end{bmatrix}\right\}$$

is a linearly independent set, which the Gram-Schmidt Process (Theorem GSP) converts to an orthogonal set, and which can then be converted to the orthonormal set,

$$B = \{\mathbf{v}_1,\, \mathbf{v}_2,\, \mathbf{v}_3\} = \left\{\frac{1}{\sqrt{6}}\begin{bmatrix}1\\2\\1\end{bmatrix},\, \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix},\, \frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\1\end{bmatrix}\right\}$$

which is therefore an orthonormal basis of $\mathbb{C}^3$. With three vectors in $\mathbb{C}^3$, all with real number entries, the inner product (Definition IP) reduces to the usual "dot product" (or scalar product) and the orthogonal pairs of vectors can be interpreted as perpendicular pairs of directions. So the vectors in B serve as replacements for our usual 3-D axes, or the usual 3-D unit vectors $\vec{i}, \vec{j}$ and $\vec{k}$. We would like to decompose arbitrary vectors into "components" in the directions of each of these basis vectors. It is Theorem COB that tells us how to do this.

Suppose that we choose $\mathbf{w} = \begin{bmatrix}2\\-1\\5\end{bmatrix}$. Compute

$$\langle \mathbf{w},\, \mathbf{v}_1\rangle = \frac{5}{\sqrt{6}} \qquad \langle \mathbf{w},\, \mathbf{v}_2\rangle = \frac{3}{\sqrt{2}} \qquad \langle \mathbf{w},\, \mathbf{v}_3\rangle = \frac{8}{\sqrt{3}}$$

then Theorem COB guarantees that

$$\begin{bmatrix}2\\-1\\5\end{bmatrix} = \frac{5}{\sqrt{6}}\left(\frac{1}{\sqrt{6}}\begin{bmatrix}1\\2\\1\end{bmatrix}\right) + \frac{3}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix}\right) + \frac{8}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\1\end{bmatrix}\right)$$

which you should be able to check easily, even if you do not have much patience. △

Not only do the columns of a unitary matrix form an orthonormal basis, but there is a deeper connection between orthonormal bases and unitary matrices. Informally, the next theorem says that if we transform each vector of an orthonormal basis by multiplying it by a unitary matrix, then the resulting set will be another orthonormal basis. And more remarkably, any matrix with this property must be unitary! As an equivalence (Proof Technique E) we could take this as our defining property of a unitary matrix, though it might not have the same utility as Definition UM.

Theorem UMCOB Unitary Matrices Convert Orthonormal Bases

Let A be an $n \times n$ matrix and $B = \{\mathbf{x}_1,\, \mathbf{x}_2,\, \mathbf{x}_3,\, \ldots,\, \mathbf{x}_n\}$ be an orthonormal basis of $\mathbb{C}^n$. Define

$$C = \{A\mathbf{x}_1,\, A\mathbf{x}_2,\, A\mathbf{x}_3,\, \ldots,\, A\mathbf{x}_n\}$$

Then A is a unitary matrix if and only if C is an orthonormal basis of $\mathbb{C}^n$.

Proof. ($\Rightarrow$) Assume A is a unitary matrix and establish several facts about C. First we check that C is an orthonormal set (Definition ONS). By Theorem UMPIP, for $i \neq j$,

$$\langle A\mathbf{x}_i,\, A\mathbf{x}_j\rangle = \langle \mathbf{x}_i,\, \mathbf{x}_j\rangle = 0$$

Similarly, Theorem UMPIP also gives, for $1 \leq i \leq n$,

$$\|A\mathbf{x}_i\| = \|\mathbf{x}_i\| = 1$$

As C is an orthogonal set (Definition OSV), Theorem OSLI yields the linear independence of C. Having established that the column vectors on C form a linearly independent set, a matrix whose columns are the vectors of C is nonsingular (Theorem NMLIC), and hence these vectors form a basis of $\mathbb{C}^n$ by Theorem CNMB.

($\Leftarrow$) Now assume that C is an orthonormal set. Let $\mathbf{y}$ be an arbitrary vector from $\mathbb{C}^n$. Since B spans $\mathbb{C}^n$, there are scalars, $a_1,\, a_2,\, a_3,\, \ldots,\, a_n$, such that

$$\mathbf{y} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + a_3\mathbf{x}_3 + \cdots + a_n\mathbf{x}_n$$

Now

$$\begin{aligned}
A^*A\mathbf{y} &= \sum_{i=1}^{n} \langle \mathbf{x}_i,\, A^*A\mathbf{y}\rangle \mathbf{x}_i && \text{Theorem COB}\\
&= \sum_{i=1}^{n} \left\langle \mathbf{x}_i,\, A^*A\sum_{j=1}^{n} a_j\mathbf{x}_j \right\rangle \mathbf{x}_i && \text{Definition SSVS}\\
&= \sum_{i=1}^{n} \left\langle \mathbf{x}_i,\, \sum_{j=1}^{n} A^*Aa_j\mathbf{x}_j \right\rangle \mathbf{x}_i && \text{Theorem MMDAA}\\
&= \sum_{i=1}^{n} \left\langle \mathbf{x}_i,\, \sum_{j=1}^{n} a_jA^*A\mathbf{x}_j \right\rangle \mathbf{x}_i && \text{Theorem MMSMM}\\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} \langle \mathbf{x}_i,\, a_jA^*A\mathbf{x}_j\rangle \mathbf{x}_i && \text{Theorem IPVA}\\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} a_j\langle \mathbf{x}_i,\, A^*A\mathbf{x}_j\rangle \mathbf{x}_i && \text{Theorem IPSM}\\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} a_j\langle A\mathbf{x}_i,\, A\mathbf{x}_j\rangle \mathbf{x}_i && \text{Theorem AIP}\\
&= \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} a_j\langle A\mathbf{x}_i,\, A\mathbf{x}_j\rangle \mathbf{x}_i + \sum_{\ell=1}^{n} a_\ell\langle A\mathbf{x}_\ell,\, A\mathbf{x}_\ell\rangle \mathbf{x}_\ell && \text{Property C}\\
&= \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} a_j(0)\mathbf{x}_i + \sum_{\ell=1}^{n} a_\ell(1)\mathbf{x}_\ell && \text{Definition ONS}\\
&= \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} \mathbf{0} + \sum_{\ell=1}^{n} a_\ell\mathbf{x}_\ell && \text{Theorem ZSSM}
\end{aligned}$$

$$= \sum_{\ell=1}^{n} a_\ell \mathbf{x}_\ell \qquad \text{Property Z}$$
$$= \mathbf{y}$$
$$= I_n \mathbf{y} \qquad \text{Theorem MMIM}$$

Since the choice of $\mathbf{y}$ was arbitrary, Theorem EMMVP tells us that $A^* A = I_n$, so A is unitary (Definition UM). ■

Reading Questions

1. The matrix below is nonsingular. What can you now say about its columns?

$$A = \begin{bmatrix} -3 & 0 & 1 \\ 1 & 2 & 1 \\ 5 & 1 & 6 \end{bmatrix}$$

2. Write the vector $\mathbf{w} = \begin{bmatrix} 6 \\ 6 \\ 15 \end{bmatrix}$ as a linear combination of the columns of the matrix A above. How many ways are there to answer this question?

3. Why is an orthonormal basis desirable?

Exercises

C10† Find a basis for $\langle S \rangle$, where

$$S = \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 1 \\ 3 \end{bmatrix} \right\}.$$

C11† Find a basis for the subspace W of $\mathbb{C}^4$,

$$W = \left\{ \begin{bmatrix} a + b - 2c \\ a + b - 2c + d \\ -2a + 2b + 4c - d \\ b + d \end{bmatrix} \middle| \, a, b, c, d \in \mathbb{C} \right\}$$

C12† Find a basis for the vector space T of lower triangular 3×3 matrices; that is, matrices of the form $\begin{bmatrix} * & 0 & 0 \\ * & * & 0 \\ * & * & * \end{bmatrix}$ where an asterisk represents any complex number.

C13† Find a basis for the subspace Q of P_2, $Q = \left\{ p(x) = a + bx + cx^2 \,\middle|\, p(0) = 0 \right\}$.

C14† Find a basis for the subspace R of P_2, $R = \left\{ p(x) = a + bx + cx^2 \,\middle|\, p'(0) = 0 \right\}$, where p' denotes the derivative.

C40† From Example RSB, form an arbitrary (and nontrivial) linear combination of the four vectors in the original spanning set for W. So the result of this computation is of course an element of W. As such, this vector should be a linear combination of the basis vectors in B. Find the (unique) scalars that provide this linear combination. Repeat with another linear combination of the original four vectors.

C80 Prove that $\{(1,\, 2),\, (2,\, 3)\}$ is a basis for the crazy vector space C (Example CVS).

M20† In Example BM provide the verifications (linear independence and spanning) to show that B is a basis of M_{mn}.

T50† Theorem UMCOB says that unitary matrices are characterized as those matrices that "carry" orthonormal bases to orthonormal bases. This problem asks you to prove a similar result: nonsingular matrices are characterized as those matrices that "carry" bases to bases.

More precisely, suppose that A is a square matrix of size n and $B = \{\mathbf{x}_1,\, \mathbf{x}_2,\, \mathbf{x}_3,\, \ldots,\, \mathbf{x}_n\}$

is a basis of $\mathbb{C}^n$. Prove that A is nonsingular if and only if $C = \{A\mathbf{x}_1,\, A\mathbf{x}_2,\, A\mathbf{x}_3,\, \ldots,\, A\mathbf{x}_n\}$ is a basis of $\mathbb{C}^n$. (See also Exercise PD.T33, Exercise MR.T20.)

T51[†] Use the result of Exercise B.T50 to build a very concise proof of Theorem CNMB. (Hint: make a judicious choice for the basis B.)

Section D
Dimension

Almost every vector space we have encountered has been infinite in size (an exception is Example VSS). But some are bigger and richer than others. Dimension, once suitably defined, will be a measure of the size of a vector space, and a useful tool for studying its properties. You probably already have a rough notion of what a mathematical definition of dimension might be — try to forget these imprecise ideas and go with the new ones given here.

Subsection D
Dimension

Definition D Dimension
Suppose that V is a vector space and $\{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3,\,\ldots,\,\mathbf{v}_t\}$ is a basis of V. Then the **dimension** of V is defined by $\dim(V) = t$. If V has no finite bases, we say V has infinite dimension. □

This is a very simple definition, which belies its power. Grab a basis, any basis, and count up the number of vectors it contains. That is the dimension. However, this simplicity causes a problem. Given a vector space, you and I could each construct different bases — remember that a vector space might have many bases. And what if your basis and my basis had different sizes? Applying Definition D we would arrive at different numbers! With our current knowledge about vector spaces, we would have to say that dimension is not "well-defined." Fortunately, there is a theorem that will correct this problem.

In a strictly logical progression, the next two theorems would *precede* the definition of dimension. Many subsequent theorems will trace their lineage back to the following fundamental result.

Theorem SSLD Spanning Sets and Linear Dependence
Suppose that $S = \{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3,\,\ldots,\,\mathbf{v}_t\}$ is a finite set of vectors which spans the vector space V. Then any set of $t+1$ or more vectors from V is linearly dependent.

Proof. We want to prove that any set of $t+1$ or more vectors from V is linearly dependent. So we will begin with a totally arbitrary set of vectors from V, $R = \{\mathbf{u}_1,\,\mathbf{u}_2,\,\mathbf{u}_3,\,\ldots,\,\mathbf{u}_m\}$, where $m > t$. We will now construct a nontrivial relation of linear dependence on R.

Each vector $\mathbf{u}_1,\,\mathbf{u}_2,\,\mathbf{u}_3,\,\ldots,\,\mathbf{u}_m$ can be written as a linear combination of the vectors $\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3,\,\ldots,\,\mathbf{v}_t$ since S is a spanning set of V. This means there exist scalars a_{ij}, $1 \leq i \leq t$, $1 \leq j \leq m$, so that

$$
\begin{aligned}
\mathbf{u}_1 &= a_{11}\mathbf{v}_1 + a_{21}\mathbf{v}_2 + a_{31}\mathbf{v}_3 + \cdots + a_{t1}\mathbf{v}_t\\
\mathbf{u}_2 &= a_{12}\mathbf{v}_1 + a_{22}\mathbf{v}_2 + a_{32}\mathbf{v}_3 + \cdots + a_{t2}\mathbf{v}_t\\
\mathbf{u}_3 &= a_{13}\mathbf{v}_1 + a_{23}\mathbf{v}_2 + a_{33}\mathbf{v}_3 + \cdots + a_{t3}\mathbf{v}_t\\
&\vdots\\
\mathbf{u}_m &= a_{1m}\mathbf{v}_1 + a_{2m}\mathbf{v}_2 + a_{3m}\mathbf{v}_3 + \cdots + a_{tm}\mathbf{v}_t
\end{aligned}
$$

Now we form, unmotivated, the homogeneous system of t equations in the m variables, $x_1,\,x_2,\,x_3,\,\ldots,\,x_m$, where the coefficients are the just-discovered scalars a_{ij},

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1m}x_m &= 0\\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2m}x_m &= 0\\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3m}x_m &= 0
\end{aligned}
$$

$$\vdots$$
$$a_{t1}x_1 + a_{t2}x_2 + a_{t3}x_3 + \cdots + a_{tm}x_m = 0$$

This is a homogeneous system with more variables than equations (our hypothesis is expressed as $m > t$), so by Theorem HMVEI there are infinitely many solutions. Choose a nontrivial solution and denote it by $x_1 = c_1$, $x_2 = c_2$, $x_3 = c_3$, …, $x_m = c_m$. As a solution to the homogeneous system, we then have

$$\begin{aligned}
a_{11}c_1 + a_{12}c_2 + a_{13}c_3 + \cdots + a_{1m}c_m &= 0\\
a_{21}c_1 + a_{22}c_2 + a_{23}c_3 + \cdots + a_{2m}c_m &= 0\\
a_{31}c_1 + a_{32}c_2 + a_{33}c_3 + \cdots + a_{3m}c_m &= 0\\
&\vdots\\
a_{t1}c_1 + a_{t2}c_2 + a_{t3}c_3 + \cdots + a_{tm}c_m &= 0
\end{aligned}$$

As a collection of nontrivial scalars, $c_1, c_2, c_3, \ldots, c_m$ will provide the nontrivial relation of linear dependence we desire,

$$\begin{aligned}
&c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + \cdots + c_m\mathbf{u}_m\\
&= c_1\left(a_{11}\mathbf{v}_1 + a_{21}\mathbf{v}_2 + a_{31}\mathbf{v}_3 + \cdots + a_{t1}\mathbf{v}_t\right) && \text{Definition SSVS}\\
&\quad + c_2\left(a_{12}\mathbf{v}_1 + a_{22}\mathbf{v}_2 + a_{32}\mathbf{v}_3 + \cdots + a_{t2}\mathbf{v}_t\right)\\
&\quad + c_3\left(a_{13}\mathbf{v}_1 + a_{23}\mathbf{v}_2 + a_{33}\mathbf{v}_3 + \cdots + a_{t3}\mathbf{v}_t\right)\\
&\qquad\vdots\\
&\quad + c_m\left(a_{1m}\mathbf{v}_1 + a_{2m}\mathbf{v}_2 + a_{3m}\mathbf{v}_3 + \cdots + a_{tm}\mathbf{v}_t\right)\\
&= c_1a_{11}\mathbf{v}_1 + c_1a_{21}\mathbf{v}_2 + c_1a_{31}\mathbf{v}_3 + \cdots + c_1a_{t1}\mathbf{v}_t && \text{Property DVA}\\
&\quad + c_2a_{12}\mathbf{v}_1 + c_2a_{22}\mathbf{v}_2 + c_2a_{32}\mathbf{v}_3 + \cdots + c_2a_{t2}\mathbf{v}_t\\
&\quad + c_3a_{13}\mathbf{v}_1 + c_3a_{23}\mathbf{v}_2 + c_3a_{33}\mathbf{v}_3 + \cdots + c_3a_{t3}\mathbf{v}_t\\
&\qquad\vdots\\
&\quad + c_ma_{1m}\mathbf{v}_1 + c_ma_{2m}\mathbf{v}_2 + c_ma_{3m}\mathbf{v}_3 + \cdots + c_ma_{tm}\mathbf{v}_t\\
&= \left(c_1a_{11} + c_2a_{12} + c_3a_{13} + \cdots + c_ma_{1m}\right)\mathbf{v}_1 && \text{Property DSA}\\
&\quad + \left(c_1a_{21} + c_2a_{22} + c_3a_{23} + \cdots + c_ma_{2m}\right)\mathbf{v}_2\\
&\quad + \left(c_1a_{31} + c_2a_{32} + c_3a_{33} + \cdots + c_ma_{3m}\right)\mathbf{v}_3\\
&\qquad\vdots\\
&\quad + \left(c_1a_{t1} + c_2a_{t2} + c_3a_{t3} + \cdots + c_ma_{tm}\right)\mathbf{v}_t\\
&= \left(a_{11}c_1 + a_{12}c_2 + a_{13}c_3 + \cdots + a_{1m}c_m\right)\mathbf{v}_1 && \text{Property CMCN}\\
&\quad + \left(a_{21}c_1 + a_{22}c_2 + a_{23}c_3 + \cdots + a_{2m}c_m\right)\mathbf{v}_2\\
&\quad + \left(a_{31}c_1 + a_{32}c_2 + a_{33}c_3 + \cdots + a_{3m}c_m\right)\mathbf{v}_3\\
&\qquad\vdots\\
&\quad + \left(a_{t1}c_1 + a_{t2}c_2 + a_{t3}c_3 + \cdots + a_{tm}c_m\right)\mathbf{v}_t\\
&= 0\mathbf{v}_1 + 0\mathbf{v}_2 + 0\mathbf{v}_3 + \cdots + 0\mathbf{v}_t && c_j \text{ as solution}\\
&= \mathbf{0} + \mathbf{0} + \mathbf{0} + \cdots + \mathbf{0} && \text{Theorem ZSSM}\\
&= \mathbf{0} && \text{Property Z}
\end{aligned}$$

That does it. R has been undeniably shown to be a linearly dependent set.

The proof just given has some monstrous expressions in it, mostly owing to the double subscripts present. Now is a great opportunity to show the value of a more compact notation. We will rewrite the key steps of the previous proof using summation notation, resulting in a more economical presentation, and even greater insight into the key aspects of the proof. So here is an alternate proof — study it carefully.

Alternate Proof: We want to prove that any set of $t+1$ or more vectors from V is linearly dependent. So we will begin with a totally arbitrary set of vectors from V, $R = \{\mathbf{u}_j \mid 1 \leq j \leq m\}$, where $m > t$. We will now construct a nontrivial relation of linear dependence on R.

Each vector $\mathbf{u_j}$, $1 \leq j \leq m$ can be written as a linear combination of $\mathbf{v}_i$, $1 \leq i \leq t$ since S is a spanning set of V. This means there are scalars a_{ij}, $1 \leq i \leq t$, $1 \leq j \leq m$, so that

$$\mathbf{u}_j = \sum_{i=1}^{t} a_{ij}\mathbf{v}_i \qquad\qquad 1 \leq j \leq m$$

Now we form, unmotivated, the homogeneous system of t equations in the m variables, x_j, $1 \leq j \leq m$, where the coefficients are the just-discovered scalars a_{ij},

$$\sum_{j=1}^{m} a_{ij}x_j = 0 \qquad\qquad 1 \leq i \leq t$$

This is a homogeneous system with more variables than equations (our hypothesis is expressed as $m > t$), so by Theorem HMVEI there are infinitely many solutions. Choose one of these solutions that is not trivial and denote it by $x_j = c_j$, $1 \leq j \leq m$. As a solution to the homogeneous system, we then have $\sum_{j=1}^{m} a_{ij}c_j = 0$ for $1 \leq i \leq t$. As a collection of nontrivial scalars, c_j, $1 \leq j \leq m$, will provide the nontrivial relation of linear dependence we desire,

$$\begin{aligned}
\sum_{j=1}^{m} c_j\mathbf{u}_j &= \sum_{j=1}^{m} c_j\left(\sum_{i=1}^{t} a_{ij}\mathbf{v}_i\right) && \text{Definition SSVS}\\
&= \sum_{j=1}^{m}\sum_{i=1}^{t} c_j a_{ij}\mathbf{v}_i && \text{Property DVA}\\
&= \sum_{i=1}^{t}\sum_{j=1}^{m} c_j a_{ij}\mathbf{v}_i && \text{Property CMCN}\\
&= \sum_{i=1}^{t}\sum_{j=1}^{m} a_{ij} c_j\mathbf{v}_i && \text{Commutativity in } \mathbb{C}\\
&= \sum_{i=1}^{t}\left(\sum_{j=1}^{m} a_{ij} c_j\right)\mathbf{v}_i && \text{Property DSA}\\
&= \sum_{i=1}^{t} 0\mathbf{v}_i && c_j \text{ as solution}\\
&= \sum_{i=1}^{t} \mathbf{0} && \text{Theorem ZSSM}\\
&= \mathbf{0} && \text{Property Z}
\end{aligned}$$

That does it. R has been undeniably shown to be a linearly dependent set. ■

Notice how the swap of the two summations is so much easier in the third step above, as opposed to all the rearranging and regrouping that takes place in the previous proof. And using only about half the space. And there are no ellipses (…).

Theorem SSLD can be viewed as a generalization of Theorem MVSLD. We know that $\mathbb{C}^m$ has a basis with m vectors in it (Theorem SUVB), so it is a set of m vectors that spans $\mathbb{C}^m$. By Theorem SSLD, any set of more than m vectors from $\mathbb{C}^m$ will be linearly dependent. But this is exactly the conclusion we have in Theorem MVSLD. Maybe this is not a total shock, as the proofs of both theorems rely heavily on Theorem HMVEI. The beauty of Theorem SSLD is that it applies in any vector space. We illustrate the generality of this theorem, and hint at its power, in the next example.

Example LDP4 Linearly dependent set in P_4
In Example SSP4 we showed that

$$S = \left\{x - 2,\, x^2 - 4x + 4,\, x^3 - 6x^2 + 12x - 8,\, x^4 - 8x^3 + 24x^2 - 32x + 16\right\}$$

is a spanning set for $W = \{p(x) |\, p \in P_4,\ p(2) = 0\}$. So we can apply Theorem SSLD to W with $t = 4$. Here is a set of five vectors from W, as you may check by verifying that each is a polynomial of degree 4 or less and has $x = 2$ as a root,

$$T = \{p_1, p_2, p_3, p_4, p_5\} \subseteq W$$

$$\begin{aligned}
p_1 &= x^4 - 2x^3 + 2x^2 - 8x + 8\\
p_2 &= -x^3 + 6x^2 - 5x - 6\\
p_3 &= 2x^4 - 5x^3 + 5x^2 - 7x + 2\\
p_4 &= -x^4 + 4x^3 - 7x^2 + 6x\\
p_5 &= 4x^3 - 9x^2 + 5x - 6
\end{aligned}$$

By Theorem SSLD we conclude that T is linearly dependent, with no further computations. △

Theorem SSLD is indeed powerful, but our main purpose in proving it right now was to make sure that our definition of dimension (Definition D) is well-defined. Here is the theorem.

Theorem BIS Bases have Identical Sizes
Suppose that V is a vector space with a finite basis B and a second basis C. Then B and C have the same size.

Proof. Suppose that C has more vectors than B. (Allowing for the possibility that C is infinite, we can replace C by a subset that has more vectors than B.) As a basis, B is a spanning set for V (Definition B), so Theorem SSLD says that C is linearly dependent. However, this contradicts the fact that as a basis C is linearly independent (Definition B). So C must also be a finite set, with size less than, or equal to, that of B.

Suppose that B has more vectors than C. As a basis, C is a spanning set for V (Definition B), so Theorem SSLD says that B is linearly dependent. However, this contradicts the fact that as a basis B is linearly independent (Definition B). So C cannot be strictly smaller than B.

The only possibility left for the sizes of B and C is for them to be equal. ■

Theorem BIS tells us that if we find one finite basis in a vector space, then they all have the same size. This (finally) makes Definition D unambiguous.

Subsection DVS
Dimension of Vector Spaces

We can now collect the dimension of some common, and not so common, vector spaces.

Theorem DCM Dimension of $\mathbb{C}^m$
The dimension of $\mathbb{C}^m$ (Example VSCV) is m.

Proof. Theorem SUVB provides a basis with m vectors. ■

Theorem DP Dimension of P_n
The dimension of P_n (Example VSP) is $n+1$.

Proof. Example BP provides *two* bases with $n+1$ vectors. Take your pick. ■

Theorem DM Dimension of M_{mn}
The dimension of M_{mn} (Example VSM) is mn.

Proof. Example BM provides a basis with mn vectors. ■

Example DSM22 Dimension of a subspace of M_{22}
It should now be plausible that

$$Z = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \, 2a + b + 3c + 4d = 0,\ -a + 3b - 5c - 2d = 0 \right\}$$

is a subspace of the vector space M_{22} (Example VSM). (It is.) To find the dimension of Z we must first find a basis, though any old basis will do.

First concentrate on the conditions relating a, b, c and d. They form a homogeneous system of two equations in four variables with coefficient matrix

$$\begin{bmatrix} 2 & 1 & 3 & 4 \\ -1 & 3 & -5 & -2 \end{bmatrix}$$

We can row-reduce this matrix to obtain

$$\begin{bmatrix} \boxed{1} & 0 & 2 & 2 \\ 0 & \boxed{1} & -1 & 0 \end{bmatrix}$$

Rewrite the two equations represented by each row of this matrix, expressing the dependent variables (a and b) in terms of the free variables (c and d), and we obtain,

$$a = -2c - 2d$$
$$b = c$$

We can now write a typical entry of Z strictly in terms of c and d, and we can decompose the result,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -2c - 2d & c \\ c & d \end{bmatrix} = \begin{bmatrix} -2c & c \\ c & 0 \end{bmatrix} + \begin{bmatrix} -2d & 0 \\ 0 & d \end{bmatrix} = c \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$$

This equation says that an arbitrary matrix in Z can be written as a linear combination of the two vectors in

$$S = \left\{ \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

so we know that

$$Z = \langle S \rangle = \left\langle \left\{ \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \right\} \right\rangle$$

Are these two matrices (vectors) also linearly independent? Begin with a relation of linear dependence on S,

$$a_1 \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} + a_2 \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} = \mathcal{O}$$

$$\begin{bmatrix} -2a_1 - 2a_2 & a_1 \\ a_1 & a_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

From the equality of the two entries in the last row, we conclude that $a_1 = 0$, $a_2 = 0$. Thus the only possible relation of linear dependence is the trivial one, and therefore S is linearly independent (Definition LI). So S is a basis for V (Definition B). Finally, we can conclude that $\dim(Z) = 2$ (Definition D) since S has two elements. △

Example DSP4 Dimension of a subspace of P_4
In Example BSP4 we showed that

$$S = \left\{x - 2,\, x^2 - 4x + 4,\, x^3 - 6x^2 + 12x - 8,\, x^4 - 8x^3 + 24x^2 - 32x + 16\right\}$$

is a basis for $W = \{p(x) \mid p \in P_4,\ p(2) = 0\}$. Thus, the dimension of W is four, $\dim(W) = 4$.

Note that $\dim(P_4) = 5$ by Theorem DP, so W is a subspace of dimension 4 within the vector space P_4 of dimension 5, illustrating the upcoming Theorem PSSD. △

Example DC Dimension of the crazy vector space
In Example BC we determined that the set $R = \{(1,\, 0),\, (6,\, 3)\}$ from the crazy vector space, C (Example CVS), is a basis for C. By Definition D we see that C has dimension 2, $\dim(C) = 2$. △

It is possible for a vector space to have no finite bases, in which case we say it has infinite dimension. Many of the best examples of this are vector spaces of functions, which lead to constructions like Hilbert spaces. We will focus exclusively on finite-dimensional vector spaces. OK, one infinite-dimensional example, and *then* we will focus exclusively on finite-dimensional vector spaces.

Example VSPUD Vector space of polynomials with unbounded degree
Define the set P by

$$P = \{p \mid p(x) \text{ is a polynomial in } x\}$$

Our operations will be the same as those defined for P_n (Example VSP).

With no restrictions on the possible degrees of our polynomials, any finite set that is a candidate for spanning P will come up short. We will give a proof by contradiction (Proof Technique CD). To this end, suppose that the dimension of P is finite, say $\dim(P) = n$.

The set $T = \left\{1,\, x,\, x^2,\, \ldots,\, x^n\right\}$ is a linearly independent set (check this!) containing $n + 1$ polynomials from P. However, a basis of P will be a spanning set of P containing n vectors. This situation is a contradiction of Theorem SSLD, so our assumption that P has finite dimension is false. Thus, we say $\dim(P) = \infty$. △

Subsection RNM
Rank and Nullity of a Matrix

For any matrix, we have seen that we can associate several subspaces — the null space (Theorem NSMS), the column space (Theorem CSMS), row space (Theorem RSMS) and the left null space (Theorem LNSMS). As vector spaces, each of these has a dimension, and for the null space and column space, they are important enough to warrant names.

Definition NOM Nullity Of a Matrix
Suppose that A is an $m \times n$ matrix. Then the **nullity** of A is the dimension of the null space of A, $n(A) = \dim(\mathcal{N}(A))$. □

Definition ROM Rank Of a Matrix
Suppose that A is an $m \times n$ matrix. Then the **rank** of A is the dimension of the column space of A, $r(A) = \dim(\mathcal{C}(A))$. □

Example RNM Rank and nullity of a matrix
Let us compute the rank and nullity of

$$A = \begin{bmatrix} 2 & -4 & -1 & 3 & 2 & 1 & -4 \\ 1 & -2 & 0 & 0 & 4 & 0 & 1 \\ -2 & 4 & 1 & 0 & -5 & -4 & -8 \\ 1 & -2 & 1 & 1 & 6 & 1 & -3 \\ 2 & -4 & -1 & 1 & 4 & -2 & -1 \\ -1 & 2 & 3 & -1 & 6 & 3 & -1 \end{bmatrix}$$

To do this, we will first row-reduce the matrix since that will help us determine bases for the null space and column space.

$$\begin{bmatrix} \boxed{1} & -2 & 0 & 0 & 4 & 0 & 1 \\ 0 & 0 & \boxed{1} & 0 & 3 & 0 & -2 \\ 0 & 0 & 0 & \boxed{1} & -1 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From this row-equivalent matrix in reduced row-echelon form we record $D = \{1, 3, 4, 6\}$ and $F = \{2, 5, 7\}$.

For each index in D, Theorem BCS creates a single basis vector. In total the basis will have 4 vectors, so the column space of A will have dimension 4 and we write $r(A) = 4$.

For each index in F, Theorem BNS creates a single basis vector. In total the basis will have 3 vectors, so the null space of A will have dimension 3 and we write $n(A) = 3$. △

There were no accidents or coincidences in the previous example — with the row-reduced version of a matrix in hand, the rank and nullity are easy to compute.

Theorem CRN Computing Rank and Nullity
Suppose that A is an $m \times n$ matrix and B is a row-equivalent matrix in reduced row-echelon form. Let r denote the number of pivot columns (or the number of nonzero rows). Then $r(A) = r$ and $n(A) = n - r$.

Proof. Theorem BCS provides a basis for the column space by choosing columns of A that that have the same indices as the pivot columns of B. In the analysis of B, each leading 1 provides one nonzero row and one pivot column. So there are r column vectors in a basis for $\mathcal{C}(A)$.

Theorem BNS provides a basis for the null space by creating basis vectors of the null space of A from entries of B, one basis vector for each column that is *not* a pivot column. So there are $n - r$ column vectors in a basis for $n(A)$. ■

Every archetype (Archetypes) that involves a matrix lists its rank and nullity. You may have noticed as you studied the archetypes that the larger the column space is the smaller the null space is. A simple corollary states this trade-off succinctly. (See Proof Technique LC.)

Theorem RPNC Rank Plus Nullity is Columns
Suppose that A is an $m \times n$ matrix. Then $r(A) + n(A) = n$.

Proof. Let r be the number of nonzero rows in a row-equivalent matrix in reduced row-echelon form. By Theorem CRN,

$$r(A) + n(A) = r + (n - r) = n$$

When we first introduced r as our standard notation for the number of nonzero rows in a matrix in reduced row-echelon form you might have thought r stood for "rows." Not really — it stands for "rank"!

Subsection RNNM
Rank and Nullity of a Nonsingular Matrix

Let us take a look at the rank and nullity of a square matrix.

Example RNSM Rank and nullity of a square matrix
The matrix

$$E = \begin{bmatrix} 0 & 4 & -1 & 2 & 2 & 3 & 1 \\ 2 & -2 & 1 & -1 & 0 & -4 & -3 \\ -2 & -3 & 9 & -3 & 9 & -1 & 9 \\ -3 & -4 & 9 & 4 & -1 & 6 & -2 \\ -3 & -4 & 6 & -2 & 5 & 9 & -4 \\ 9 & -3 & 8 & -2 & -4 & 2 & 4 \\ 8 & 2 & 2 & 9 & 3 & 0 & 9 \end{bmatrix}$$

is row-equivalent to the matrix in reduced row-echelon form,

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

With $n = 7$ columns and $r = 7$ nonzero rows Theorem CRN tells us the rank is $r(E) = 7$ and the nullity is $n(E) = 7 - 7 = 0$. △

The value of either the nullity or the rank are enough to characterize a nonsingular matrix.

Theorem RNNM Rank and Nullity of a Nonsingular Matrix
Suppose that A is a square matrix of size n. The following are equivalent.

1. *A is nonsingular.*

2. *The rank of A is n, $r(A) = n$.*

3. *The nullity of A is zero, $n(A) = 0$.*

Proof. ($1 \Rightarrow 2$) Theorem CSNM says that if A is nonsingular then $\mathcal{C}(A) = \mathbb{C}^n$. If $\mathcal{C}(A) = \mathbb{C}^n$, then the column space has dimension n by Theorem DCM, so the rank of A is n.

($2 \Rightarrow 3$) Suppose $r(A) = n$. Then Theorem RPNC gives

$$\begin{aligned} n(A) &= n - r(A) && \text{Theorem RPNC} \\ &= n - n && \text{Hypothesis} \\ &= 0 \end{aligned}$$

($3 \Rightarrow 1$) Suppose $n(A) = 0$, so a basis for the null space of A is the empty set. This implies that $\mathcal{N}(A) = \{\mathbf{0}\}$ and Theorem NMTNS says A is nonsingular. ■

With a new equivalence for a nonsingular matrix, we can update our list of equivalences (Theorem NME5) which now becomes a list requiring double digits to number.

Theorem NME6 Nonsingular Matrix Equivalences, Round 6
Suppose that A is a square matrix of size n. The following are equivalent.

1. *A is nonsingular.*

2. *A row-reduces to the identity matrix.*
3. *The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.*
4. *The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.*
5. *The columns of A are a linearly independent set.*
6. *A is invertible.*
7. *The column space of A is $\mathbb{C}^n$, $\mathcal{C}(A) = \mathbb{C}^n$.*
8. *The columns of A are a basis for $\mathbb{C}^n$.*
9. *The rank of A is n, $r(A) = n$.*
10. *The nullity of A is zero, $n(A) = 0$.*

Proof. Building on Theorem NME5 we can add two of the statements from Theorem RNNM. ■

Reading Questions

1. What is the dimension of the vector space P_6, the set of all polynomials of degree 6 or less?
2. How are the rank and nullity of a matrix related?
3. Explain why we might say that a nonsingular matrix has "full rank."

Exercises

C20 The archetypes listed below are matrices, or systems of equations with coefficient matrices. For each, compute the nullity and rank of the matrix. This information is listed for each archetype (along with the number of columns in the matrix, so as to illustrate Theorem RPNC), and notice how it could have been computed immediately after the determination of the sets D and F associated with the reduced row-echelon form of the matrix.
Archetype A, Archetype B, Archetype C, Archetype D/Archetype E, Archetype F, Archetype G/Archetype H, Archetype I, Archetype J, Archetype K, Archetype L

C21† Find the dimension of the subspace $W = \left\{ \begin{bmatrix} a+b \\ a+c \\ a+d \\ d \end{bmatrix} \middle| a, b, c, d \in \mathbb{C} \right\}$ of $\mathbb{C}^4$.

C22† Find the dimension of the subspace $W = \left\{ a + bx + cx^2 + dx^3 \middle| a + b + c + d = 0 \right\}$ of P_3.

C23† Find the dimension of the subspace $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a + b = c, b + c = d, c + d = a \right\}$ of $M_{2,2}$.

C30† For the matrix A below, compute the dimension of the null space of A, $\dim(\mathcal{N}(A))$.

$$A = \begin{bmatrix} 2 & -1 & -3 & 11 & 9 \\ 1 & 2 & 1 & -7 & -3 \\ 3 & 1 & -3 & 6 & 8 \\ 2 & 1 & 2 & -5 & -3 \end{bmatrix}$$

C31† The set W below is a subspace of $\mathbb{C}^4$. Find the dimension of W.

$$W = \left\langle \left\{ \begin{bmatrix} 2 \\ -3 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -4 \\ -3 \\ 2 \\ 5 \end{bmatrix} \right\} \right\rangle$$

C35† Find the rank and nullity of the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

C36† Find the rank and nullity of the matrix $A = \begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 1 & 3 & 2 & 0 & 4 \\ 1 & 2 & 1 & 1 & 1 \end{bmatrix}$.

C37† Find the rank and nullity of the matrix $A = \begin{bmatrix} 3 & 2 & 1 & 1 & 1 \\ 2 & 3 & 0 & 1 & 1 \\ -1 & 1 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 & -1 \end{bmatrix}$.

C40 In Example LDP4 we determined that the set of five polynomials, T, is linearly dependent by a simple invocation of Theorem SSLD. Prove that T is linearly dependent from scratch, beginning with Definition LI.

M20† M_{22} is the vector space of 2×2 matrices. Let S_{22} denote the set of all 2×2 symmetric matrices. That is

$$S_{22} = \left\{ A \in M_{22} | \, A^t = A \right\}$$

1. Show that S_{22} is a subspace of M_{22}.
2. Exhibit a basis for S_{22} and prove that it has the required properties.
3. What is the dimension of S_{22}?

M21† A 2×2 matrix B is upper triangular if $[B]_{21} = 0$. Let UT_2 be the set of all 2×2 upper triangular matrices. Then UT_2 is a subspace of the vector space of all 2×2 matrices, M_{22} (you may assume this). Determine the dimension of UT_2 providing *all* of the necessary justifications for your answer.

Section PD
Properties of Dimension

Once the dimension of a vector space is known, then the determination of whether or not a set of vectors is linearly independent, or if it spans the vector space, can often be much easier. In this section we will state a workhorse theorem and then apply it to the column space and row space of a matrix. It will also help us describe a super-basis for $\mathbb{C}^m$.

Subsection GT
Goldilocks' Theorem

We begin with a useful theorem that we will need later, and in the proof of the main theorem in this subsection. This theorem says that we can extend linearly independent sets, one vector at a time, by adding vectors from outside the span of the linearly independent set, all the while preserving the linear independence of the set.

Theorem ELIS Extending Linearly Independent Sets
Suppose V is a vector space and S is a linearly independent set of vectors from V. Suppose $\mathbf{w}$ is a vector such that $\mathbf{w} \not\in \langle S\rangle$. Then the set $S' = S \cup \{\mathbf{w}\}$ is linearly independent.

Proof. Suppose $S = \{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3,\,\ldots,\,\mathbf{v}_m\}$ and begin with a relation of linear dependence on S',

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \cdots + a_m\mathbf{v}_m + a_{m+1}\mathbf{w} = \mathbf{0}.$$

There are two cases to consider. First suppose that $a_{m+1} = 0$. Then the relation of linear dependence on S' becomes

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \cdots + a_m\mathbf{v}_m = \mathbf{0}.$$

and by the linear independence of the set S, we conclude that $a_1 = a_2 = a_3 = \cdots = a_m = 0$. So all of the scalars in the relation of linear dependence on S' are zero.

In the second case, suppose that $a_{m+1} \neq 0$. Then the relation of linear dependence on S' becomes

$$\begin{aligned} a_{m+1}\mathbf{w} &= -a_1\mathbf{v}_1 - a_2\mathbf{v}_2 - a_3\mathbf{v}_3 - \cdots - a_m\mathbf{v}_m \\ \mathbf{w} &= -\frac{a_1}{a_{m+1}}\mathbf{v}_1 - \frac{a_2}{a_{m+1}}\mathbf{v}_2 - \frac{a_3}{a_{m+1}}\mathbf{v}_3 - \cdots - \frac{a_m}{a_{m+1}}\mathbf{v}_m \end{aligned}$$

This equation expresses $\mathbf{w}$ as a linear combination of the vectors in S, contrary to the assumption that $\mathbf{w} \not\in \langle S\rangle$, so this case leads to a contradiction.

The first case yielded only a trivial relation of linear dependence on S' and the second case led to a contradiction. So S' is a linearly independent set since any relation of linear dependence is trivial. ■

In the story *Goldilocks and the Three Bears*, the young girl Goldilocks visits the empty house of the three bears while out walking in the woods. One bowl of porridge is too hot, the other too cold, the third is just right. One chair is too hard, one too soft, the third is just right. So it is with sets of vectors — some are too big (linearly dependent), some are too small (they do not span), and some are just right (bases). Here is Goldilocks' Theorem.

Theorem G Goldilocks
Suppose that V is a vector space of dimension t. Let $S = \{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3,\,\ldots,\,\mathbf{v}_m\}$ be a set of vectors from V. Then

1. *If $m > t$, then S is linearly dependent.*

2. *If $m < t$, then S does not span V.*

3. *If $m = t$ and S is linearly independent, then S spans V.*

4. *If $m = t$ and S spans V, then S is linearly independent.*

Proof. Let B be a basis of V. Since $\dim(V) = t$, Definition B and Theorem BIS imply that B is a linearly independent set of t vectors that spans V.

1. Suppose to the contrary that S is linearly independent. Then B is a smaller set of vectors that spans V. This contradicts Theorem SSLD.

2. Suppose to the contrary that S does span V. Then B is a larger set of vectors that is linearly independent. This contradicts Theorem SSLD.

3. Suppose to the contrary that S does not span V. Then we can choose a vector $\mathbf{w}$ such that $\mathbf{w} \in V$ and $\mathbf{w} \notin \langle S \rangle$. By Theorem ELIS, the set $S' = S \cup \{\mathbf{w}\}$ is again linearly independent. Then S' is a set of $m + 1 = t + 1$ vectors that are linearly independent, while B is a set of t vectors that span V. This contradicts Theorem SSLD.

4. Suppose to the contrary that S is linearly dependent. Then by Theorem DLDS (which can be upgraded, with no changes in the proof, to the setting of a general vector space), there is a vector in S, say $\mathbf{v}_k$ that is equal to a linear combination of the other vectors in S. Let $S' = S \setminus \{\mathbf{v}_k\}$, the set of "other" vectors in S. Then it is easy to show that $V = \langle S \rangle = \langle S' \rangle$. So S' is a set of $m - 1 = t - 1$ vectors that spans V, while B is a set of t linearly independent vectors in V. This contradicts Theorem SSLD.

■

There is a tension in the construction of a basis. Make a set too big and you will end up with relations of linear dependence among the vectors. Make a set too small and you will not have enough raw material to span the entire vector space. Make a set just the right size (the dimension) and you only need to have linear independence or spanning, and you get the other property for free. These roughly-stated ideas are made precise by Theorem G.

The structure and proof of this theorem also deserve comment. The hypotheses seem innocuous. We presume we know the dimension of the vector space in hand, then we mostly just look at the size of the set S. From this we get big conclusions about spanning and linear independence. Each of the four proofs relies on ultimately contradicting Theorem SSLD, so in a way we could think of this entire theorem as a corollary of Theorem SSLD. (See Proof Technique LC.) The proofs of the third and fourth parts parallel each other in style: introduce $\mathbf{w}$ using Theorem ELIS or toss $\mathbf{v}_k$ using Theorem DLDS. Then obtain a contradiction to Theorem SSLD.

Theorem G is useful in both concrete examples and as a tool in other proofs. We will use it often to bypass verifying linear independence or spanning.

Example BPR Bases for P_n, reprised
In Example BP we claimed that

$$B = \left\{1,\, x,\, x^2,\, x^3,\, \ldots,\, x^n\right\}$$
$$C = \left\{1,\, 1 + x,\, 1 + x + x^2,\, 1 + x + x^2 + x^3,\, \ldots,\, 1 + x + x^2 + x^3 + \cdots + x^n\right\}.$$

were both bases for P_n (Example VSP). Suppose we had first verified that B was a basis, so we would then know that $\dim(P_n) = n + 1$. The size of C is $n + 1$, the right size to be a basis. We could then verify that C is linearly independent. We would not have to make any special efforts to prove that C spans P_n, since Theorem G would allow us to conclude this property of C directly. Then we would be able to say that C is a basis of P_n also. △

Example BDM22 Basis by dimension in M_{22}

In Example DSM22 we showed that

$$B = \left\{ \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix},\, \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

is a basis for the subspace Z of M_{22} (Example VSM) given by

$$Z = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle|\, 2a + b + 3c + 4d = 0,\ -a + 3b - 5c - d = 0 \right\}$$

This tells us that $\dim(Z) = 2$. In this example we will find another basis. We can construct two new matrices in Z by forming linear combinations of the matrices in B.

$$2 \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} + (-3) \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & -3 \end{bmatrix}$$
$$3 \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} + 1 \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -8 & 3 \\ 3 & 1 \end{bmatrix}$$

Then the set

$$C = \left\{ \begin{bmatrix} 2 & 2 \\ 2 & -3 \end{bmatrix},\, \begin{bmatrix} -8 & 3 \\ 3 & 1 \end{bmatrix} \right\}$$

has the right size to be a basis of Z. Let us see if it is a linearly independent set. The relation of linear dependence

$$a_1 \begin{bmatrix} 2 & 2 \\ 2 & -3 \end{bmatrix} + a_2 \begin{bmatrix} -8 & 3 \\ 3 & 1 \end{bmatrix} = \mathcal{O}$$
$$\begin{bmatrix} 2a_1 - 8a_2 & 2a_1 + 3a_2 \\ 2a_1 + 3a_2 & -3a_1 + a_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

leads to the homogeneous system of equations whose coefficient matrix

$$\begin{bmatrix} 2 & -8 \\ 2 & 3 \\ 2 & 3 \\ -3 & 1 \end{bmatrix}$$

row-reduces to

$$\begin{bmatrix} \boxed{1} & 0 \\ 0 & \boxed{1} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So with $a_1 = a_2 = 0$ as the only solution, the set is linearly independent. Now we can apply Theorem G to see that C also spans Z and therefore is a second basis for Z. △

Example SVP4 Sets of vectors in P_4

In Example BSP4 we showed that

$$B = \left\{x - 2,\, x^2 - 4x + 4,\, x^3 - 6x^2 + 12x - 8,\, x^4 - 8x^3 + 24x^2 - 32x + 16\right\}$$

is a basis for $W = \{p(x) |\, p \in P_4,\ p(2) = 0\}$. So $\dim(W) = 4$.

The set

$$\left\{3x^2 - 5x - 2,\, 2x^2 - 7x + 6,\, x^3 - 2x^2 + x - 2\right\}$$

is a subset of W (check this) and it happens to be linearly independent (check this, too). However, by Theorem G it cannot span W.

The set

$$\left\{3x^2 - 5x - 2,\, 2x^2 - 7x + 6,\, x^3 - 2x^2 + x - 2,\, -x^4 + 2x^3 + 5x^2 - 10x,\, x^4 - 16\right\}$$

is another subset of W (check this) and Theorem G tells us that it must be linearly dependent.

The set

$$\{x-2,\, x^2-2x,\, x^3-2x^2,\, x^4-2x^3\}$$

is a third subset of W (check this) and is linearly independent (check this). Since it has the right size to be a basis, and is linearly independent, Theorem G tells us that it also spans W, and therefore is a basis of W. △

A simple consequence of Theorem G is the observation that a proper subspace has strictly smaller dimension that its parent vector space. Hopefully this may seem intuitively obvious, but it still requires proof, and we will cite this result later.

Theorem PSSD Proper Subspaces have Smaller Dimension
Suppose that U and V are subspaces of the vector space W, such that $U \subsetneq V$. Then $\dim(U) < \dim(V)$.

Proof. Suppose that $\dim(U) = m$ and $\dim(V) = t$. Then U has a basis B of size m. If $m > t$, then by Theorem G, B is linearly dependent, which is a contradiction. If $m = t$, then by Theorem G, B spans V. Then $U = \langle B\rangle = V$, also a contradiction. All that remains is that $m < t$, which is the desired conclusion. ■

The final theorem of this subsection is an extremely powerful tool for establishing the equality of two sets that are subspaces. Notice that the hypotheses include the equality of two integers (dimensions) while the conclusion is the equality of two sets (subspaces). It is the extra "structure" of a vector space and its dimension that makes possible this huge leap from an integer equality to a set equality.

Theorem EDYES Equal Dimensions Yields Equal Subspaces
Suppose that U and V are subspaces of the vector space W, such that $U \subseteq V$ and $\dim(U) = \dim(V)$. Then $U = V$.

Proof. We give a proof by contradiction (Proof Technique CD). Suppose to the contrary that $U \neq V$. Since $U \subseteq V$, there must be a vector $\mathbf{v}$ such that $\mathbf{v} \in V$ and $\mathbf{v} \notin U$. Let $B = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_t\}$ be a basis for U. Then, by Theorem ELIS, the set $C = B \cup \{\mathbf{v}\} = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_t,\, \mathbf{v}\}$ is a linearly independent set of $t+1$ vectors in V. However, by hypothesis, V has the same dimension as U (namely t) and therefore Theorem G says that C is too big to be linearly independent. This contradiction shows that $U = V$. ■

Subsection RT
Ranks and Transposes

We now prove one of the most surprising theorems about matrices. Notice the paucity of hypotheses compared to the precision of the conclusion.

Theorem RMRT Rank of a Matrix is the Rank of the Transpose
Suppose A is an $m \times n$ matrix. Then $r(A) = r(A^t)$.

Proof. Suppose we row-reduce A to the matrix B in reduced row-echelon form, and B has r nonzero rows. The quantity r tells us three things about B: the number of leading 1's, the number of nonzero rows and the number of pivot columns. For this proof we will be interested in the latter two.

Theorem BRS and Theorem BCS each has a conclusion that provides a basis, for the row space and the column space, respectively. In each case, these bases contain r vectors. This observation makes the following go.

$$r(A) = \dim(\mathcal{C}(A)) \qquad \text{Definition ROM}$$

$$
\begin{aligned}
&= r && \text{Theorem BCS}\\
&= \dim\left(\mathcal{R}(A)\right) && \text{Theorem BRS}\\
&= \dim\left(\mathcal{C}\left(A^t\right)\right) && \text{Theorem CSRST}\\
&= r\left(A^t\right) && \text{Definition ROM}
\end{aligned}
$$

Jacob Linenthal helped with this proof. ■

This says that the row space and the column space of a matrix have the same dimension, which should be very surprising. It does *not* say that column space and the row space are identical. Indeed, if the matrix is not square, then the sizes (number of slots) of the vectors in each space are different, so the sets are not even comparable.

It is not hard to construct by yourself examples of matrices that illustrate Theorem RMRT, since it applies equally well to *any* matrix. Grab a matrix, row-reduce it, count the nonzero rows or the number of pivot columns. That is the rank. Transpose the matrix, row-reduce that, count the nonzero rows or the pivot columns. That is the rank of the transpose. The theorem says the two will be equal. Every time. Here is an example anyway.

Example RRTI Rank, rank of transpose, Archetype I
Archetype I has a 4×7 coefficient matrix which row-reduces to

$$
\begin{bmatrix}
\boxed{1} & 4 & 0 & 0 & 2 & 1 & -3\\
0 & 0 & \boxed{1} & 0 & 1 & -3 & 5\\
0 & 0 & 0 & \boxed{1} & 2 & -6 & 6\\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

so the rank is 3. Row-reducing the transpose yields

$$
\begin{bmatrix}
\boxed{1} & 0 & 0 & -\frac{31}{7}\\
0 & \boxed{1} & 0 & \frac{12}{7}\\
0 & 0 & \boxed{1} & \frac{13}{7}\\
0 & 0 & 0 & 0\\
0 & 0 & 0 & 0\\
0 & 0 & 0 & 0\\
0 & 0 & 0 & 0
\end{bmatrix}.
$$

demonstrating that the rank of the transpose is also 3. △

Subsection DFS Dimension of Four Subspaces

That the rank of a matrix equals the rank of its transpose is a fundamental and surprising result. However, applying Theorem FS we can easily determine the dimension of all four fundamental subspaces associated with a matrix.

Theorem DFS Dimensions of Four Subspaces
Suppose that A is an $m \times n$ matrix, and B is a row-equivalent matrix in reduced row-echelon form with r nonzero rows. Then

1. $\dim\left(\mathcal{N}(A)\right) = n - r$
2. $\dim\left(\mathcal{C}(A)\right) = r$
3. $\dim\left(\mathcal{R}(A)\right) = r$
4. $\dim\left(\mathcal{L}(A)\right) = m - r$

Proof. If A row-reduces to a matrix in reduced row-echelon form with r nonzero rows, then the matrix C of extended echelon form (Definition EEF) will be an $r \times n$ matrix in reduced row-echelon form with no zero rows and r pivot columns (Theorem PEEF). Similarly, the matrix L of extended echelon form (Definition EEF) will be an $m - r \times m$ matrix in reduced row-echelon form with no zero rows and $m - r$ pivot columns (Theorem PEEF).

$$\begin{aligned} \dim(\mathcal{N}(A)) &= \dim(\mathcal{N}(C)) && \text{Theorem FS}\\ &= n - r && \text{Theorem BNS} \end{aligned}$$

$$\begin{aligned} \dim(\mathcal{C}(A)) &= \dim(\mathcal{N}(L)) && \text{Theorem FS}\\ &= m - (m - r) && \text{Theorem BNS}\\ &= r \end{aligned}$$

$$\begin{aligned} \dim(\mathcal{R}(A)) &= \dim(\mathcal{R}(C)) && \text{Theorem FS}\\ &= r && \text{Theorem BRS} \end{aligned}$$

$$\begin{aligned} \dim(\mathcal{L}(A)) &= \dim(\mathcal{R}(L)) && \text{Theorem FS}\\ &= m - r && \text{Theorem BRS} \end{aligned}$$

■

There are many different ways to state and prove this result, and indeed, the equality of the dimensions of the column space and row space is just a slight expansion of Theorem RMRT. However, we have restricted our techniques to applying Theorem FS and then determining dimensions with bases provided by Theorem BNS and Theorem BRS. This provides an appealing symmetry to the results and the proof.

Reading Questions

1. Why does Theorem G have the title it does?
2. Why is Theorem RMRT so surprising ?
3. Row-reduce the matrix A to reduced row-echelon form. Without any further computations, compute the dimensions of the four subspaces, (a) $\mathcal{N}(A)$, (b) $\mathcal{C}(A)$, (c) $\mathcal{R}(A)$ and (d) $\mathcal{L}(A)$.

$$A = \begin{bmatrix} 1 & -1 & 2 & 8 & 5\\ 1 & 1 & 1 & 4 & -1\\ 0 & 2 & -3 & -8 & -6\\ 2 & 0 & 1 & 8 & 4 \end{bmatrix}$$

Exercises

C10 Example SVP4 leaves several details for the reader to check. Verify these five claims.

C40† Determine if the set $T = \left\{x^2 - x + 5,\, 4x^3 - x^2 + 5x,\, 3x + 2\right\}$ spans the vector space of polynomials with degree 4 or less, P_4. (Compare the solution to this exercise with Solution LISS.C40.)

M50 Mimic Definition DS and construct a reasonable definition of $V = U_1 \oplus U_2 \oplus U_3 \oplus \ldots \oplus U_m$.

T05 Trivially, if U and V are two subspaces of W with $U = V$, then $\dim(U) = \dim(V)$. Combine this fact, Theorem PSSD, and Theorem EDYES all into one grand combined theorem. You might look to Theorem PIP for stylistic inspiration. (Notice this problem does not ask you to prove anything. It just asks you to roll up three theorems into one compact, logically equivalent statement.)

T10 Prove the following theorem, which could be viewed as a reformulation of parts (3) and (4) of Theorem G, or more appropriately as a corollary of Theorem G (Proof Technique LC).

Suppose V is a vector space and S is a subset of V such that the number of vectors in S equals the dimension of V. Then S is linearly independent if and only if S spans V.

T15 Suppose that A is an $m \times n$ matrix and let $\min(m,\,n)$ denote the minimum of m and n. Prove that $r\left(A\right) \leq \min(m,\,n)$. (If m and n are two numbers, then $\min(m,\,n)$ stands for the number that is the smaller of the two. For example $\min(4,\,6) = 4$.)

T20† Suppose that A is an $m \times n$ matrix and $\mathbf{b} \in \mathbb{C}^m$. Prove that the linear system $\mathcal{LS}(A,\,\mathbf{b})$ is consistent if and only if $r\left(A\right) = r\left(\left[\,A\,|\,\mathbf{b}\right]\right)$.

T25 Suppose that V is a vector space with finite dimension. Let W be any subspace of V. Prove that W has finite dimension.

T33† Part of Exercise B.T50 is the half of the proof where we assume the matrix A is nonsingular and prove that a set is a basis. In Solution B.T50 we proved directly that the set was both linearly independent and a spanning set. Shorten this part of the proof by applying Theorem G. Be careful, there is one subtlety.

T60† Suppose that W is a vector space with dimension 5, and U and V are subspaces of W, each of dimension 3. Prove that $U \cap V$ contains a nonzero vector. State a more general result.

Chapter D
Determinants

The determinant is a function that takes a square matrix as an input and produces a scalar as an output. So unlike a vector space, it is not an algebraic structure. However, it has many beneficial properties for studying vector spaces, matrices and systems of equations, so it is hard to ignore (though some have tried). While the properties of a determinant can be very useful, they are also complicated to prove.

Section DM
Determinant of a Matrix

Before we define the determinant of a matrix, we take a slight detour to introduce elementary matrices. These will bring us back to the beginning of the course and our old friend, row operations.

Subsection EM
Elementary Matrices

Elementary matrices are very simple, as you might have suspected from their name. Their purpose is to effect row operations (Definition RO) on a matrix through matrix multiplication (Definition MM). Their definitions look much more complicated than they really are, so be sure to skip over them on your first reading and head right for the explanation that follows and the first example.

Definition ELEM Elementary Matrices

1. For $i \neq j$, $E_{i,j}$ is the square matrix of size n with

$$[E_{i,j}]_{k\ell} = \begin{cases} 0 & k \neq i, k \neq j, \ell \neq k \\ 1 & k \neq i, k \neq j, \ell = k \\ 0 & k = i, \ell \neq j \\ 1 & k = i, \ell = j \\ 0 & k = j, \ell \neq i \\ 1 & k = j, \ell = i \end{cases}$$

2. For $\alpha \neq 0$, $E_i(\alpha)$ is the square matrix of size n with

$$[E_i(\alpha)]_{k\ell} = \begin{cases} 0 & k \neq i, \ell \neq k \\ 1 & k \neq i, \ell = k \\ \alpha & k = i, \ell = i \end{cases}$$

3. For $i \neq j$, $E_{i,j}(\alpha)$ is the square matrix of size n with

$$\left[E_{i,j}(\alpha)\right]_{k\ell} = \begin{cases} 0 & k \neq j, \ell \neq k \\ 1 & k \neq j, \ell = k \\ 0 & k = j, \ell \neq i, \ell \neq j \\ 1 & k = j, \ell = j \\ \alpha & k = j, \ell = i \end{cases}$$

□

Again, these matrices are not as complicated as their definitions suggest, since they are just small perturbations of the $n \times n$ identity matrix (Definition IM). $E_{i,j}$ is the identity matrix with rows (or columns) i and j trading places, $E_i(\alpha)$ is the identity matrix where the diagonal entry in row i and column i has been replaced by α, and $E_{i,j}(\alpha)$ is the identity matrix where the entry in row j and column i has been replaced by α. (Yes, those subscripts look backwards in the description of $E_{i,j}(\alpha)$). Notice that our notation makes no reference to the size of the elementary matrix, since this will always be apparent from the context, or unimportant.

The *raison d'etre* for elementary matrices is to "do" row operations on matrices with matrix multiplication. So here is an example where we will both see some elementary matrices and see how they accomplish row operations when used with matrix multiplication.

Example EMRO Elementary matrices and row operations
We will perform a sequence of row operations (Definition RO) on the 3×4 matrix A, while also multiplying the matrix on the left by the appropriate 3×3 elementary matrix.

$$A = \begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 3 & 2 & 4 \\ 5 & 0 & 3 & 1 \end{bmatrix}$$

$$R_1 \leftrightarrow R_3: \begin{bmatrix} 5 & 0 & 3 & 1 \\ 1 & 3 & 2 & 4 \\ 2 & 1 & 3 & 1 \end{bmatrix} \qquad E_{1,3}: \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 3 & 2 & 4 \\ 5 & 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 3 & 1 \\ 1 & 3 & 2 & 4 \\ 2 & 1 & 3 & 1 \end{bmatrix}$$

$$2R_2: \begin{bmatrix} 5 & 0 & 3 & 1 \\ 2 & 6 & 4 & 8 \\ 2 & 1 & 3 & 1 \end{bmatrix} \qquad E_2(2): \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 3 & 1 \\ 1 & 3 & 2 & 4 \\ 2 & 1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 3 & 1 \\ 2 & 6 & 4 & 8 \\ 2 & 1 & 3 & 1 \end{bmatrix}$$

$$2R_3 + R_1: \begin{bmatrix} 9 & 2 & 9 & 3 \\ 2 & 6 & 4 & 8 \\ 2 & 1 & 3 & 1 \end{bmatrix} \qquad E_{3,1}(2): \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 3 & 1 \\ 2 & 6 & 4 & 8 \\ 2 & 1 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 2 & 9 & 3 \\ 2 & 6 & 4 & 8 \\ 2 & 1 & 3 & 1 \end{bmatrix}$$

△

The next three theorems establish that each elementary matrix effects a row operation via matrix multiplication.

Theorem EMDRO Elementary Matrices Do Row Operations
Suppose that A is an $m \times n$ matrix, and B is a matrix of the same size that is obtained from A by a single row operation (Definition RO). Then there is an elementary matrix of size m that will convert A to B via matrix multiplication on the left. More precisely,

1. *If the row operation swaps rows i and j, then $B = E_{i,j}A$.*

2. *If the row operation multiplies row i by α, then $B = E_i(\alpha)A$.*

3. *If the row operation multiplies row i by α and adds the result to row j, then $B = E_{i,j}(\alpha)A$.*

Proof. In each of the three conclusions, performing the row operation on A will create the matrix B where only one or two rows will have changed. So we will establish the equality of the matrix entries row by row, first for the unchanged rows, then for the changed rows, showing in each case that the result of the matrix product is the same as the result of the row operation. Here we go.

Row k of the product $E_{i,j}A$, where $k \neq i$, $k \neq j$, is unchanged from A,

$$\begin{aligned}
\left[E_{i,j}A\right]_{k\ell} &= \sum_{p=1}^{n} \left[E_{i,j}\right]_{kp} \left[A\right]_{p\ell} && \text{Theorem EMP}\\
&= \left[E_{i,j}\right]_{kk} \left[A\right]_{k\ell} + \sum_{\substack{p=1\\p\neq k}}^{n} \left[E_{i,j}\right]_{kp} \left[A\right]_{p\ell} && \text{Property CACN}\\
&= 1\left[A\right]_{k\ell} + \sum_{\substack{p=1\\p\neq k}}^{n} 0\left[A\right]_{p\ell} && \text{Definition ELEM}\\
&= \left[A\right]_{k\ell}
\end{aligned}$$

Row i of the product $E_{i,j}A$ is row j of A,

$$\begin{aligned}
\left[E_{i,j}A\right]_{i\ell} &= \sum_{p=1}^{n} \left[E_{i,j}\right]_{ip} \left[A\right]_{p\ell} && \text{Theorem EMP}\\
&= \left[E_{i,j}\right]_{ij} \left[A\right]_{j\ell} + \sum_{\substack{p=1\\p\neq j}}^{n} \left[E_{i,j}\right]_{ip} \left[A\right]_{p\ell} && \text{Property CACN}\\
&= 1\left[A\right]_{j\ell} + \sum_{\substack{p=1\\p\neq j}}^{n} 0\left[A\right]_{p\ell} && \text{Definition ELEM}\\
&= \left[A\right]_{j\ell}
\end{aligned}$$

Row j of the product $E_{i,j}A$ is row i of A,

$$\begin{aligned}
\left[E_{i,j}A\right]_{j\ell} &= \sum_{p=1}^{n} \left[E_{i,j}\right]_{jp} \left[A\right]_{p\ell} && \text{Theorem EMP}\\
&= \left[E_{i,j}\right]_{ji} \left[A\right]_{i\ell} + \sum_{\substack{p=1\\p\neq i}}^{n} \left[E_{i,j}\right]_{jp} \left[A\right]_{p\ell} && \text{Property CACN}\\
&= 1\left[A\right]_{i\ell} + \sum_{\substack{p=1\\p\neq i}}^{n} 0\left[A\right]_{p\ell} && \text{Definition ELEM}\\
&= \left[A\right]_{i\ell}
\end{aligned}$$

So the matrix product $E_{i,j}A$ is the same as the row operation that swaps rows i and j.

Row k of the product $E_i(\alpha)A$, where $k \neq i$, is unchanged from A,

$$\begin{aligned}
\left[E_i(\alpha)A\right]_{k\ell} &= \sum_{p=1}^{n} \left[E_i(\alpha)\right]_{kp} \left[A\right]_{p\ell} && \text{Theorem EMP}\\
&= \left[E_i(\alpha)\right]_{kk} \left[A\right]_{k\ell} + \sum_{\substack{p=1\\p\neq k}}^{n} \left[E_i(\alpha)\right]_{kp} \left[A\right]_{p\ell} && \text{Property CACN}
\end{aligned}$$

$$= 1\left[A\right]_{k\ell} + \sum_{\substack{p=1\\p\neq k}}^{n} 0\left[A\right]_{p\ell} \qquad \text{Definition ELEM}$$

$$= \left[A\right]_{k\ell}$$

Row i of the product $E_i(\alpha)A$ is α times row i of A,

$$\begin{aligned}
\left[E_i(\alpha)A\right]_{i\ell} &= \sum_{p=1}^{n} \left[E_i(\alpha)\right]_{ip}\left[A\right]_{p\ell} && \text{Theorem EMP}\\
&= \left[E_i(\alpha)\right]_{ii}\left[A\right]_{i\ell} + \sum_{\substack{p=1\\p\neq i}}^{n} \left[E_i(\alpha)\right]_{ip}\left[A\right]_{p\ell} && \text{Property CACN}\\
&= \alpha\left[A\right]_{i\ell} + \sum_{\substack{p=1\\p\neq i}}^{n} 0\left[A\right]_{p\ell} && \text{Definition ELEM}\\
&= \alpha\left[A\right]_{i\ell}
\end{aligned}$$

So the matrix product $E_i(\alpha)A$ is the same as the row operation that swaps multiplies row i by α.

Row k of the product $E_{i,j}(\alpha)A$, where $k \neq j$, is unchanged from A,

$$\begin{aligned}
\left[E_{i,j}(\alpha)A\right]_{k\ell} &= \sum_{p=1}^{n} \left[E_{i,j}(\alpha)\right]_{kp}\left[A\right]_{p\ell} && \text{Theorem EMP}\\
&= \left[E_{i,j}(\alpha)\right]_{kk}\left[A\right]_{k\ell} + \sum_{\substack{p=1\\p\neq k}}^{n} \left[E_{i,j}(\alpha)\right]_{kp}\left[A\right]_{p\ell} && \text{Property CACN}\\
&= 1\left[A\right]_{k\ell} + \sum_{\substack{p=1\\p\neq k}}^{n} 0\left[A\right]_{p\ell} && \text{Definition ELEM}\\
&= \left[A\right]_{k\ell}
\end{aligned}$$

Row j of the product $E_{i,j}(\alpha)A$, is α times row i of A and then added to row j of A,

$$\begin{aligned}
\left[E_{i,j}(\alpha)A\right]_{j\ell} &= \sum_{p=1}^{n} \left[E_{i,j}(\alpha)\right]_{jp}\left[A\right]_{p\ell} && \text{Theorem EMP}\\
&= \left[E_{i,j}(\alpha)\right]_{jj}\left[A\right]_{j\ell} + \\
&\qquad \left[E_{i,j}(\alpha)\right]_{ji}\left[A\right]_{i\ell} + \sum_{\substack{p=1\\p\neq j,i}}^{n} \left[E_{i,j}(\alpha)\right]_{jp}\left[A\right]_{p\ell} && \text{Property CACN}\\
&= 1\left[A\right]_{j\ell} + \alpha\left[A\right]_{i\ell} + \sum_{\substack{p=1\\p\neq j,i}}^{n} 0\left[A\right]_{p\ell} && \text{Definition ELEM}\\
&= \left[A\right]_{j\ell} + \alpha\left[A\right]_{i\ell}
\end{aligned}$$

So the matrix product $E_{i,j}(\alpha)A$ is the same as the row operation that multiplies row i by α and adds the result to row j. ■

Later in this section we will need two facts about elementary matrices.

Theorem EMN Elementary Matrices are Nonsingular
If E is an elementary matrix, then E is nonsingular.

Proof. We show that we can row-reduce each elementary matrix to the identity matrix. Given an elementary matrix of the form $E_{i,j}$, perform the row operation that swaps row j with row i. Given an elementary matrix of the form $E_i(\alpha)$, with $\alpha \neq 0$, perform the row operation that multiplies row i by $1/\alpha$. Given an elementary matrix of the form $E_{i,j}(\alpha)$, with $\alpha \neq 0$, perform the row operation that multiplies row i by $-\alpha$ and adds it to row j. In each case, the result of the single row operation is the identity matrix. So each elementary matrix is row-equivalent to the identity matrix, and by Theorem NMRRI is nonsingular. ■

Notice that we have now made use of the nonzero restriction on α in the definition of $E_i(\alpha)$. One more key property of elementary matrices.

Theorem NMPEM Nonsingular Matrices are Products of Elementary Matrices
Suppose that A is a nonsingular matrix. Then there exists elementary matrices $E_1, E_2, E_3, \ldots, E_t$ so that $A = E_1E_2E_3 \ldots E_t$.

Proof. Since A is nonsingular, it is row-equivalent to the identity matrix by Theorem NMRRI, so there is a sequence of t row operations that converts I to A. For each of these row operations, form the associated elementary matrix from Theorem EMDRO and denote these matrices by $E_1, E_2, E_3, \ldots, E_t$. Applying the first row operation to I yields the matrix E_1I. The second row operation yields $E_2(E_1I)$, and the third row operation creates $E_3E_2E_1I$. The result of the full sequence of t row operations will yield A, so

$$A = E_t \ldots E_3E_2E_1I = E_t \ldots E_3E_2E_1$$

Other than the cosmetic matter of re-indexing these elementary matrices in the opposite order, this is the desired result. ■

Subsection DD
Definition of the Determinant

We will now turn to the definition of a determinant and do some sample computations. The definition of the determinant function is **recursive**, that is, the determinant of a large matrix is defined in terms of the determinant of smaller matrices. To this end, we will make a few definitions.

Definition SM SubMatrix
Suppose that A is an $m\times n$ matrix. Then the **submatrix** $A(i|j)$ is the $(m-1)\times(n-1)$ matrix obtained from A by removing row i and column j. □

Example SS Some submatrices
For the matrix

$$A = \begin{bmatrix} 1 & -2 & 3 & 9 \\ 4 & -2 & 0 & 1 \\ 3 & 5 & 2 & 1 \end{bmatrix}$$

we have the submatrices

$$A(2|3) = \begin{bmatrix} 1 & -2 & 9 \\ 3 & 5 & 1 \end{bmatrix} \qquad A(3|1) = \begin{bmatrix} -2 & 3 & 9 \\ -2 & 0 & 1 \end{bmatrix}$$

△

Definition DM Determinant of a Matrix
Suppose A is a square matrix. Then its **determinant**, $\det(A) = |A|$, is an element of $\mathbb{C}$ defined recursively by:

1. If A is a 1×1 matrix, then $\det(A) = [A]_{11}$.

2. If A is a matrix of size n with $n \geq 2$, then

$$\det(A) = [A]_{11} \det(A(1|1)) - [A]_{12} \det(A(1|2)) + [A]_{13} \det(A(1|3)) - [A]_{14} \det(A(1|4)) + \cdots + (-1)^{n+1} [A]_{1n} \det(A(1|n))$$

□

So to compute the determinant of a 5×5 matrix we must build 5 submatrices, each of size 4. To compute the determinants of each the 4×4 matrices we need to create 4 submatrices each, these now of size 3 and so on. To compute the determinant of a 10×10 matrix would require computing the determinant of $10! = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 = 3,628,800$ 1×1 matrices. Fortunately there are better ways. However this does suggest an excellent computer programming exercise to write a recursive procedure to compute a determinant.

Let us compute the determinant of a reasonably sized matrix by hand.

Example D33M Determinant of a 3×3 matrix
Suppose that we have the 3×3 matrix

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 4 & 1 & 6 \\ -3 & -1 & 2 \end{bmatrix}$$

Then

$$\begin{aligned} \det(A) = |A| &= \begin{vmatrix} 3 & 2 & -1 \\ 4 & 1 & 6 \\ -3 & -1 & 2 \end{vmatrix} \\ &= 3 \begin{vmatrix} 1 & 6 \\ -1 & 2 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ -3 & 2 \end{vmatrix} + (-1) \begin{vmatrix} 4 & 1 \\ -3 & -1 \end{vmatrix} \\ &= 3(1|2| - 6|-1|) - 2(4|2| - 6|-3|) - (4|-1| - 1|-3|) \\ &= 3(1(2) - 6(-1)) - 2(4(2) - 6(-3)) - (4(-1) - 1(-3)) \\ &= 24 - 52 + 1 \\ &= -27 \end{aligned}$$

△

In practice it is a bit silly to decompose a 2×2 matrix down into a couple of 1×1 matrices and then compute the exceedingly easy determinant of these puny matrices. So here is a simple theorem.

Theorem DMST Determinant of Matrices of Size Two
Suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. *Then* $\det(A) = ad - bc$.

Proof. Applying Definition DM,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = a|d| - b|c| = ad - bc$$

■

Do you recall seeing the expression $ad - bc$ before? (Hint: Theorem TTMI)

Subsection CD
Computing Determinants

There are a variety of ways to compute the determinant. We will establish first that we can choose to mimic our definition of the determinant, but by using matrix entries and submatrices based on a row other than the first one.

Theorem DER Determinant Expansion about Rows
Suppose that A is a square matrix of size n. Then for $1 \leq i \leq n$

$$\begin{aligned}\det(A) &= (-1)^{i+1}[A]_{i1}\det(A(i|1)) + (-1)^{i+2}[A]_{i2}\det(A(i|2))\\ &\quad + (-1)^{i+3}[A]_{i3}\det(A(i|3)) + \cdots + (-1)^{i+n}[A]_{in}\det(A(i|n))\end{aligned}$$

which is known as ***expansion*** *about row i.*

Proof. First, the statement of the theorem coincides with Definition DM when $i = 1$, so throughout, we need only consider $i > 1$.

Given the recursive definition of the determinant, it should be no surprise that we will use induction for this proof (Proof Technique I). When $n = 1$, there is nothing to prove since there is but one row. When $n = 2$, we just examine expansion about the second row,

$$\begin{aligned}&(-1)^{2+1}[A]_{21}\det(A(2|1)) + (-1)^{2+2}[A]_{22}\det(A(2|2))\\ &\quad = -[A]_{21}[A]_{12} + [A]_{22}[A]_{11} && \text{Definition DM}\\ &\quad = [A]_{11}[A]_{22} - [A]_{12}[A]_{21}\\ &\quad = \det(A) && \text{Theorem DMST}\end{aligned}$$

So the theorem is true for matrices of size $n = 1$ and $n = 2$. Now assume the result is true for all matrices of size $n - 1$ as we derive an expression for expansion about row i for a matrix of size n. We will abuse our notation for a submatrix slightly, so $A(i_1, i_2|j_1, j_2)$ will denote the matrix formed by removing rows i_1 and i_2, along with removing columns j_1 and j_2. Also, as we take a determinant of a submatrix, we will need to “jump up” the index of summation partway through as we “skip over” a missing column. To do this smoothly we will set

$$\epsilon_{\ell j} = \begin{cases} 0 & \ell < j \\ 1 & \ell > j \end{cases}$$

Now,

$$\begin{aligned}&\det(A)\\ &= \sum_{j=1}^{n}(-1)^{1+j}[A]_{1j}\det(A(1|j)) && \text{Definition DM}\\ &= \sum_{j=1}^{n}(-1)^{1+j}[A]_{1j}\sum_{\substack{1\leq\ell\leq n\\ \ell\neq j}}(-1)^{i-1+\ell-\epsilon_{\ell j}}[A]_{i\ell}\det(A(1,i|j,\ell)) && \text{Induction}\\ &= \sum_{j=1}^{n}\sum_{\substack{1\leq\ell\leq n\\ \ell\neq j}}(-1)^{j+i+\ell-\epsilon_{\ell j}}[A]_{1j}[A]_{i\ell}\det(A(1,i|j,\ell)) && \text{Property DCN}\\ &= \sum_{\ell=1}^{n}\sum_{\substack{1\leq j\leq n\\ j\neq\ell}}(-1)^{j+i+\ell-\epsilon_{\ell j}}[A]_{1j}[A]_{i\ell}\det(A(1,i|j,\ell)) && \text{Property CACN}\\ &= \sum_{\ell=1}^{n}(-1)^{i+\ell}[A]_{i\ell}\sum_{\substack{1\leq j\leq n\\ j\neq\ell}}(-1)^{j-\epsilon_{\ell j}}[A]_{1j}\det(A(1,i|j,\ell)) && \text{Property DCN}\\ &= \sum_{\ell=1}^{n}(-1)^{i+\ell}[A]_{i\ell}\sum_{\substack{1\leq j\leq n\\ j\neq\ell}}(-1)^{\epsilon_{\ell j}+j}[A]_{1j}\det(A(i,1|\ell,j)) && 2\epsilon_{\ell j}\text{ is even}\end{aligned}$$

$$= \sum_{\ell=1}^{n} (-1)^{i+\ell} [A]_{i\ell} \det(A(i|\ell)) \qquad \text{Definition DM}$$

■

We can also obtain a formula that computes a determinant by expansion about a column, but this will be simpler if we first prove a result about the interplay of determinants and transposes. Notice how the following proof makes use of the ability to compute a determinant by expanding about *any* row.

Theorem DT Determinant of the Transpose
Suppose that A is a square matrix. Then $\det(A^t) = \det(A)$.

Proof. With our definition of the determinant (Definition DM) and theorems like Theorem DER, using induction (Proof Technique I) is a natural approach to proving properties of determinants. And so it is here. Let n be the size of the matrix A, and we will use induction on n.

For $n = 1$, the transpose of a matrix is identical to the original matrix, so vacuously, the determinants are equal.

Now assume the result is true for matrices of size $n-1$. Then,

$$\begin{aligned}
\det\left(A^t\right) &= \frac{1}{n} \sum_{i=1}^{n} \det\left(A^t\right) \\
&= \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} (-1)^{i+j} \left[A^t\right]_{ij} \det\left(A^t(i|j)\right) && \text{Theorem DER} \\
&= \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} (-1)^{i+j} [A]_{ji} \det\left(A^t(i|j)\right) && \text{Definition TM} \\
&= \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} (-1)^{i+j} [A]_{ji} \det\left((A(j|i))^t\right) && \text{Definition TM} \\
&= \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} (-1)^{i+j} [A]_{ji} \det(A(j|i)) && \text{Induction Hypothesis} \\
&= \frac{1}{n} \sum_{j=1}^{n} \sum_{i=1}^{n} (-1)^{j+i} [A]_{ji} \det(A(j|i)) && \text{Property CACN} \\
&= \frac{1}{n} \sum_{j=1}^{n} \det(A) && \text{Theorem DER} \\
&= \det(A)
\end{aligned}$$

■

Now we can easily get the result that a determinant can be computed by expansion about any column as well.

Theorem DEC Determinant Expansion about Columns
Suppose that A is a square matrix of size n. Then for $1 \leq j \leq n$

$$\begin{aligned}
\det(A) = (-1)^{1+j} [A]_{1j} \det(A(1|j)) + (-1)^{2+j} [A]_{2j} \det(A(2|j)) \\
+ (-1)^{3+j} [A]_{3j} \det(A(3|j)) + \cdots + (-1)^{n+j} [A]_{nj} \det(A(n|j))
\end{aligned}$$

which is known as ***expansion*** *about column j.*

Proof.

$$\det(A) = \det\left(A^t\right) \qquad \text{Theorem DT}$$

$$= \sum_{i=1}^{n} (-1)^{j+i} \left[A^t\right]_{ji} \det\left(A^t\left(j|i\right)\right) \qquad \text{Theorem DER}$$

$$= \sum_{i=1}^{n} (-1)^{j+i} \left[A^t\right]_{ji} \det\left(\left(A\left(i|j\right)\right)^t\right) \qquad \text{Definition TM}$$

$$= \sum_{i=1}^{n} (-1)^{j+i} \left[A^t\right]_{ji} \det\left(A\left(i|j\right)\right) \qquad \text{Theorem DT}$$

$$= \sum_{i=1}^{n} (-1)^{i+j} \left[A\right]_{ij} \det\left(A\left(i|j\right)\right) \qquad \text{Definition TM}$$

■

That the determinant of an $n \times n$ matrix can be computed in $2n$ different (albeit similar) ways is nothing short of remarkable. For the doubters among us, we will do an example, computing a 4×4 matrix in two different ways.

Example TCSD Two computations, same determinant
Let

$$A = \begin{bmatrix} -2 & 3 & 0 & 1 \\ 9 & -2 & 0 & 1 \\ 1 & 3 & -2 & -1 \\ 4 & 1 & 2 & 6 \end{bmatrix}$$

Then expanding about the fourth row (Theorem DER with $i = 4$) yields,

$$\begin{aligned} |A| &= (4)(-1)^{4+1} \begin{vmatrix} 3 & 0 & 1 \\ -2 & 0 & 1 \\ 3 & -2 & -1 \end{vmatrix} + (1)(-1)^{4+2} \begin{vmatrix} -2 & 0 & 1 \\ 9 & 0 & 1 \\ 1 & -2 & -1 \end{vmatrix} \\ &\quad + (2)(-1)^{4+3} \begin{vmatrix} -2 & 3 & 1 \\ 9 & -2 & 1 \\ 1 & 3 & -1 \end{vmatrix} + (6)(-1)^{4+4} \begin{vmatrix} -2 & 3 & 0 \\ 9 & -2 & 0 \\ 1 & 3 & -2 \end{vmatrix} \\ &= (-4)(10) + (1)(-22) + (-2)(61) + 6(46) = 92 \end{aligned}$$

Expanding about column 3 (Theorem DEC with $j = 3$) gives

$$\begin{aligned} |A| &= (0)(-1)^{1+3} \begin{vmatrix} 9 & -2 & 1 \\ 1 & 3 & -1 \\ 4 & 1 & 6 \end{vmatrix} + (0)(-1)^{2+3} \begin{vmatrix} -2 & 3 & 1 \\ 1 & 3 & -1 \\ 4 & 1 & 6 \end{vmatrix} + \\ &\quad (-2)(-1)^{3+3} \begin{vmatrix} -2 & 3 & 1 \\ 9 & -2 & 1 \\ 4 & 1 & 6 \end{vmatrix} + (2)(-1)^{4+3} \begin{vmatrix} -2 & 3 & 1 \\ 9 & -2 & 1 \\ 1 & 3 & -1 \end{vmatrix} \\ &= 0 + 0 + (-2)(-107) + (-2)(61) = 92 \end{aligned}$$

Notice how much easier the second computation was. By choosing to expand about the third column, we have two entries that are zero, so two 3×3 determinants need not be computed at all! △

When a matrix has all zeros above (or below) the diagonal, exploiting the zeros by expanding about the proper row or column makes computing a determinant insanely easy.

Example DUTM Determinant of an upper triangular matrix
Suppose that

$$T = \begin{bmatrix} 2 & 3 & -1 & 3 & 3 \\ 0 & -1 & 5 & 2 & -1 \\ 0 & 0 & 3 & 9 & 2 \\ 0 & 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

We will compute the determinant of this 5×5 matrix by consistently expanding about the first column for each submatrix that arises and does not have a zero entry multiplying it.

$$\begin{aligned}
\det(T) &= \begin{vmatrix} 2 & 3 & -1 & 3 & 3 \\ 0 & -1 & 5 & 2 & -1 \\ 0 & 0 & 3 & 9 & 2 \\ 0 & 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{vmatrix} \\
&= 2(-1)^{1+1} \begin{vmatrix} -1 & 5 & 2 & -1 \\ 0 & 3 & 9 & 2 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 5 \end{vmatrix} \\
&= 2(-1)(-1)^{1+1} \begin{vmatrix} 3 & 9 & 2 \\ 0 & -1 & 3 \\ 0 & 0 & 5 \end{vmatrix} \\
&= 2(-1)(3)(-1)^{1+1} \begin{vmatrix} -1 & 3 \\ 0 & 5 \end{vmatrix} \\
&= 2(-1)(3)(-1)(-1)^{1+1} \left|5\right| \\
&= 2(-1)(3)(-1)(5) = 30
\end{aligned}$$

△

When you consult other texts in your study of determinants, you may run into the terms "minor" and "cofactor," especially in a discussion centered on expansion about rows and columns. We have chosen not to make these definitions formally since we have been able to get along without them. However, informally, a **minor** is a determinant of a submatrix, specifically $\det(A(i|j))$ and is usually referenced as the minor of $[A]_{ij}$. A **cofactor** is a signed minor, specifically the cofactor of $[A]_{ij}$ is $(-1)^{i+j}\det(A(i|j))$.

Reading Questions

1. Construct the elementary matrix that will effect the row operation $-6R_2 + R_3$ on a 4×7 matrix.

2. Compute the determinant of the matrix

$$\begin{bmatrix} 2 & 3 & -1 \\ 3 & 8 & 2 \\ 4 & -1 & -3 \end{bmatrix}$$

3. Compute the determinant of the matrix

$$\begin{bmatrix} 3 & 9 & -2 & 4 & 2 \\ 0 & 1 & 4 & -2 & 7 \\ 0 & 0 & -2 & 5 & 2 \\ 0 & 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

Exercises

C21† Doing the computations by hand, find the determinant of the matrix below.

$$\begin{bmatrix} 1 & 3 \\ 6 & 2 \end{bmatrix}$$

C22† Doing the computations by hand, find the determinant of the matrix below.

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$$

C23† Doing the computations by hand, find the determinant of the matrix below.

$$\begin{bmatrix} 1 & 3 & 2 \\ 4 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}$$

C24† Doing the computations by hand, find the determinant of the matrix below.

$$\begin{bmatrix} -2 & 3 & -2 \\ -4 & -2 & 1 \\ 2 & 4 & 2 \end{bmatrix}$$

C25† Doing the computations by hand, find the determinant of the matrix below.

$$\begin{bmatrix} 3 & -1 & 4 \\ 2 & 5 & 1 \\ 2 & 0 & 6 \end{bmatrix}$$

C26† Doing the computations by hand, find the determinant of the matrix A.

$$A = \begin{bmatrix} 2 & 0 & 3 & 2 \\ 5 & 1 & 2 & 4 \\ 3 & 0 & 1 & 2 \\ 5 & 3 & 2 & 1 \end{bmatrix}$$

C27† Doing the computations by hand, find the determinant of the matrix A.

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 2 & -1 & 1 \\ 2 & 1 & 3 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

C28† Doing the computations by hand, find the determinant of the matrix A.

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & -1 & -1 & 1 \\ 2 & 5 & 3 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix}$$

C29† Doing the computations by hand, find the determinant of the matrix A.

$$A = \begin{bmatrix} 2 & 3 & 0 & 2 & 1 \\ 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

C30† Doing the computations by hand, find the determinant of the matrix A.

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 & 1 \\ 2 & 1 & 2 & -1 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 1 & 0 & 3 & 1 & 1 \\ 2 & 1 & 1 & 2 & 1 \end{bmatrix}$$

M10† Find a value of k so that the matrix $A = \begin{bmatrix} 2 & 4 \\ 3 & k \end{bmatrix}$ has $\det(A) = 0$, or explain why it is not possible.

M11† Find a value of k so that the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 2 & 3 & k \end{bmatrix}$ has $\det(A) = 0$, or explain why it is not possible.

M15† Given the matrix $B = \begin{bmatrix} 2-x & 1 \\ 4 & 2-x \end{bmatrix}$, find all values of x that are solutions of $\det(B) = 0$.

M16† Given the matrix $B = \begin{bmatrix} 4-x & -4 & -4 \\ 2 & -2-x & -4 \\ 3 & -3 & -4-x \end{bmatrix}$, find all values of x that are solutions of $\det(B) = 0$.

M30 The two matrices below are row-equivalent. How would you confirm this? Since the matrices are row-equivalent, there is a sequence of row operations that converts X into Y, which would be a product of elementary matrices, M, such that $MX = Y$. Find M. (This approach could be used to find the "9 scalars" of the very early Exercise RREF.M40.)
Hint: Compute the extended echelon form for both matrices, and then use the property from Theorem PEEF that reads $B = JA$, where A is the original matrix, B is the echelon form of the matrix and J is a nonsingular matrix obtained from extended echelon form. Combine the two square matrices in the right way to obtain M.

$$X = \begin{bmatrix} -1 & 3 & 1 & -2 & 8 \\ -1 & 3 & 2 & -1 & 4 \\ 2 & -4 & -3 & 2 & -7 \\ -2 & 5 & 3 & -2 & 8 \end{bmatrix} \qquad Y = \begin{bmatrix} -1 & 2 & 2 & 0 & 0 \\ -3 & 6 & 8 & -1 & 1 \\ 0 & 1 & -2 & -2 & 9 \\ -1 & 4 & -3 & -3 & 16 \end{bmatrix}$$

Section PDM
Properties of Determinants of Matrices

We have seen how to compute the determinant of a matrix, and the incredible fact that we can perform expansion about *any* row *or* column to make this computation. In this largely theoretical section, we will state and prove several more intriguing properties about determinants. Our main goal will be the two results in Theorem SMZD and Theorem DRMM, but more specifically, we will see how the value of a determinant will allow us to gain insight into the various properties of a square matrix.

Subsection DRO
Determinants and Row Operations

We start easy with a straightforward theorem whose proof presages the style of subsequent proofs in this subsection.

Theorem DZRC Determinant with Zero Row or Column
Suppose that A is a square matrix with a row where every entry is zero, or a column where every entry is zero. Then $\det(A) = 0$.

Proof. Suppose that A is a square matrix of size n and row i has every entry equal to zero. We compute $\det(A)$ via expansion about row i.

$$\begin{aligned}
\det(A) &= \sum_{j=1}^{n} (-1)^{i+j} [A]_{ij} \det(A(i|j)) && \text{Theorem DER} \\
&= \sum_{j=1}^{n} (-1)^{i+j}\, 0 \det(A(i|j)) && \text{Row } i \text{ is zeros} \\
&= \sum_{j=1}^{n} 0 = 0
\end{aligned}$$

The proof for the case of a zero column is entirely similar, or could be derived from an application of Theorem DT employing the transpose of the matrix. ■

Theorem DRCS Determinant for Row or Column Swap
Suppose that A is a square matrix. Let B be the square matrix obtained from A by interchanging the location of two rows, or interchanging the location of two columns. Then $\det(B) = -\det(A)$.

Proof. Begin with the special case where A is a square matrix of size n and we form B by swapping *adjacent* rows i and $i+1$ for some $1 \leq i \leq n-1$. Notice that the assumption about swapping adjacent rows means that $B(i+1|j) = A(i|j)$ for all $1 \leq j \leq n$, and $[B]_{i+1,j} = [A]_{ij}$ for all $1 \leq j \leq n$. We compute $\det(B)$ via expansion about row $i+1$.

$$\begin{aligned}
\det(B) &= \sum_{j=1}^{n} (-1)^{(i+1)+j} [B]_{i+1,j} \det(B(i+1|j)) && \text{Theorem DER} \\
&= \sum_{j=1}^{n} (-1)^{(i+1)+j} [A]_{ij} \det(A(i|j)) && \text{Hypothesis} \\
&= \sum_{j=1}^{n} (-1)^{1}(-1)^{i+j} [A]_{ij} \det(A(i|j))
\end{aligned}$$

$$= (-1)\sum_{j=1}^{n}(-1)^{i+j}\left[A\right]_{ij}\det\left(A\left(i|j\right)\right)$$
$$= -\det\left(A\right) \qquad \text{Theorem DER}$$

So the result holds for the special case where we swap adjacent rows of the matrix. As any computer scientist knows, we can accomplish *any* rearrangement of an ordered list by swapping adjacent elements. This principle can be demonstrated by naïve sorting algorithms such as "bubble sort." In any event, we do not need to discuss every possible reordering, we just need to consider a swap of two rows, say rows s and t with $1 \leq s < t \leq n$.

Begin with row s, and repeatedly swap it with each row just below it, including row t and stopping there. This will total $t - s$ swaps. Now swap the former row t, which currently lives in row $t - 1$, with each row above it, stopping when it becomes row s. This will total another $t - s - 1$ swaps. In this way, we create B through a sequence of $2(t - s) - 1$ swaps of adjacent rows, each of which adjusts $\det\left(A\right)$ by a multiplicative factor of -1. So

$$\det\left(B\right) = (-1)^{2(t-s)-1}\det\left(A\right) = \left((-1)^2\right)^{t-s}(-1)^{-1}\det\left(A\right) = -\det\left(A\right)$$

as desired.

The proof for the case of swapping two columns is entirely similar, or could be derived from an application of Theorem DT employing the transpose of the matrix. ■

So Theorem DRCS tells us the effect of the first row operation (Definition RO) on the determinant of a matrix. Here is the effect of the second row operation.

Theorem DRCM Determinant for Row or Column Multiples
Suppose that A is a square matrix. Let B be the square matrix obtained from A by multiplying a single row by the scalar α, or by multiplying a single column by the scalar α. Then $\det\left(B\right) = \alpha\det\left(A\right)$.

Proof. Suppose that A is a square matrix of size n and we form the square matrix B by multiplying each entry of row i of A by α. Notice that the other rows of A and B are equal, so $A\left(i|j\right) = B\left(i|j\right)$, for all $1 \leq j \leq n$. We compute $\det\left(B\right)$ via expansion about row i.

$$\det\left(B\right) = \sum_{j=1}^{n}(-1)^{i+j}\left[B\right]_{ij}\det\left(B\left(i|j\right)\right) \qquad \text{Theorem DER}$$
$$= \sum_{j=1}^{n}(-1)^{i+j}\left[B\right]_{ij}\det\left(A\left(i|j\right)\right) \qquad \text{Hypothesis}$$
$$= \sum_{j=1}^{n}(-1)^{i+j}\alpha\left[A\right]_{ij}\det\left(A\left(i|j\right)\right) \qquad \text{Hypothesis}$$
$$= \alpha\sum_{j=1}^{n}(-1)^{i+j}\left[A\right]_{ij}\det\left(A\left(i|j\right)\right)$$
$$= \alpha\det\left(A\right) \qquad \text{Theorem DER}$$

The proof for the case of a multiple of a column is entirely similar, or could be derived from an application of Theorem DT employing the transpose of the matrix. ■

Let us go for understanding the effect of all three row operations. But first we need an intermediate result, but it is an easy one.

Theorem DERC Determinant with Equal Rows or Columns
Suppose that A is a square matrix with two equal rows, or two equal columns. Then $\det(A) = 0$.

Proof. Suppose that A is a square matrix of size n where the two rows s and t are equal. Form the matrix B by swapping rows s and t. Notice that as a consequence of our hypothesis, $A = B$. Then

$$\begin{aligned}
\det(A) &= \frac{1}{2}(\det(A) + \det(A)) \\
&= \frac{1}{2}(\det(A) - \det(B)) && \text{Theorem DRCS} \\
&= \frac{1}{2}(\det(A) - \det(A)) && \text{Hypothesis, } A = B \\
&= \frac{1}{2}(0) = 0
\end{aligned}$$

The proof for the case of two equal columns is entirely similar, or could be derived from an application of Theorem DT employing the transpose of the matrix. ∎

Now explain the third row operation. Here we go.

Theorem DRCMA Determinant for Row or Column Multiples and Addition
Suppose that A is a square matrix. Let B be the square matrix obtained from A by multiplying a row by the scalar α and then adding it to another row, or by multiplying a column by the scalar α and then adding it to another column. Then $\det(B) = \det(A)$.

Proof. Suppose that A is a square matrix of size n. Form the matrix B by multiplying row s by α and adding it to row t. Let C be the auxiliary matrix where we replace row t of A by row s of A. Notice that $A(t|j) = B(t|j) = C(t|j)$ for all $1 \leq j \leq n$. We compute the determinant of B by expansion about row t.

$$\begin{aligned}
\det(B) &= \sum_{j=1}^{n} (-1)^{t+j} [B]_{tj} \det(B(t|j)) && \text{Theorem DER} \\
&= \sum_{j=1}^{n} (-1)^{t+j} \left(\alpha [A]_{sj} + [A]_{tj}\right) \det(B(t|j)) && \text{Hypothesis} \\
&= \sum_{j=1}^{n} (-1)^{t+j} \alpha [A]_{sj} \det(B(t|j)) \\
&\quad + \sum_{j=1}^{n} (-1)^{t+j} [A]_{tj} \det(B(t|j)) \\
&= \alpha \sum_{j=1}^{n} (-1)^{t+j} [A]_{sj} \det(B(t|j)) \\
&\quad + \sum_{j=1}^{n} (-1)^{t+j} [A]_{tj} \det(B(t|j)) \\
&= \alpha \sum_{j=1}^{n} (-1)^{t+j} [C]_{tj} \det(C(t|j)) \\
&\quad + \sum_{j=1}^{n} (-1)^{t+j} [A]_{tj} \det(A(t|j)) \\
&= \alpha \det(C) + \det(A) && \text{Theorem DER} \\
&= \alpha\, 0 + \det(A) = \det(A) && \text{Theorem DERC}
\end{aligned}$$

The proof for the case of adding a multiple of a column is entirely similar, or could be derived from an application of Theorem DT employing the transpose of the matrix. ■

Is this what you expected? We could argue that the third row operation is the most popular, and yet it has no effect whatsoever on the determinant of a matrix! We can exploit this, along with our understanding of the other two row operations, to provide another approach to computing a determinant. We'll explain this in the context of an example.

Example DRO Determinant by row operations
Suppose we desire the determinant of the 4×4 matrix

$$A = \begin{bmatrix} 2 & 0 & 2 & 3 \\ 1 & 3 & -1 & 1 \\ -1 & 1 & -1 & 2 \\ 3 & 5 & 4 & 0 \end{bmatrix}$$

We will perform a sequence of row operations on this matrix, shooting for an upper triangular matrix, whose determinant will be simply the product of its diagonal entries. For each row operation, we will track the effect on the determinant via Theorem DRCS, Theorem DRCM, Theorem DRCMA.

$$A = \begin{bmatrix} 2 & 0 & 2 & 3 \\ 1 & 3 & -1 & 1 \\ -1 & 1 & -1 & 2 \\ 3 & 5 & 4 & 0 \end{bmatrix} \qquad \det(A)$$

$$\xrightarrow{R_1 \leftrightarrow R_2} A_1 = \begin{bmatrix} 1 & 3 & -1 & 1 \\ 2 & 0 & 2 & 3 \\ -1 & 1 & -1 & 2 \\ 3 & 5 & 4 & 0 \end{bmatrix} \qquad = -\det(A_1) \qquad \text{Theorem DRCS}$$

$$\xrightarrow{-2R_1+R_2} A_2 = \begin{bmatrix} 1 & 3 & -1 & 1 \\ 0 & -6 & 4 & 1 \\ -1 & 1 & -1 & 2 \\ 3 & 5 & 4 & 0 \end{bmatrix} \qquad = -\det(A_2) \qquad \text{Theorem DRCMA}$$

$$\xrightarrow{1R_1+R_3} A_3 = \begin{bmatrix} 1 & 3 & -1 & 1 \\ 0 & -6 & 4 & 1 \\ 0 & 4 & -2 & 3 \\ 3 & 5 & 4 & 0 \end{bmatrix} \qquad = -\det(A_3) \qquad \text{Theorem DRCMA}$$

$$\xrightarrow{-3R_1+R_4} A_4 = \begin{bmatrix} 1 & 3 & -1 & 1 \\ 0 & -6 & 4 & 1 \\ 0 & 4 & -2 & 3 \\ 0 & -4 & 7 & -3 \end{bmatrix} \qquad = -\det(A_4) \qquad \text{Theorem DRCMA}$$

$$\xrightarrow{1R_3+R_2} A_5 = \begin{bmatrix} 1 & 3 & -1 & 1 \\ 0 & -2 & 2 & 4 \\ 0 & 4 & -2 & 3 \\ 0 & -4 & 7 & -3 \end{bmatrix} \qquad = -\det(A_5) \qquad \text{Theorem DRCMA}$$

$$\xrightarrow{-\frac{1}{2}R_2} A_6 = \begin{bmatrix} 1 & 3 & -1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 4 & -2 & 3 \\ 0 & -4 & 7 & -3 \end{bmatrix} \qquad = 2\det(A_6) \qquad \text{Theorem DRCM}$$

$$\xrightarrow{-4R_2+R_3} A_7 = \begin{bmatrix} 1 & 3 & -1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 2 & 11 \\ 0 & -4 & 7 & -3 \end{bmatrix} \qquad = 2\det(A_7) \qquad \text{Theorem DRCMA}$$

$$\xrightarrow{4R_2+R_4} A_8 = \begin{bmatrix} 1 & 3 & -1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 2 & 11 \\ 0 & 0 & 3 & -11 \end{bmatrix} = 2\det(A_8) \qquad \text{Theorem DRCMA}$$

$$\xrightarrow{-1R_3+R_4} A_9 = \begin{bmatrix} 1 & 3 & -1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 2 & 11 \\ 0 & 0 & 1 & -22 \end{bmatrix} = 2\det(A_9) \qquad \text{Theorem DRCMA}$$

$$\xrightarrow{-2R_4+R_3} A_{10} = \begin{bmatrix} 1 & 3 & -1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 55 \\ 0 & 0 & 1 & -22 \end{bmatrix} = 2\det(A_{10}) \qquad \text{Theorem DRCMA}$$

$$\xrightarrow{R_3\leftrightarrow R_4} A_{11} = \begin{bmatrix} 1 & 3 & -1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & -22 \\ 0 & 0 & 0 & 55 \end{bmatrix} = -2\det(A_{11}) \qquad \text{Theorem DRCS}$$

$$\xrightarrow{\frac{1}{55}R_4} A_{12} = \begin{bmatrix} 1 & 3 & -1 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & -22 \\ 0 & 0 & 0 & 1 \end{bmatrix} = -110\det(A_{12}) \qquad \text{Theorem DRCM}$$

The matrix A_{12} is upper triangular, so expansion about the first column (repeatedly) will result in $\det(A_{12}) = (1)(1)(1)(1) = 1$ (see Example DUTM) and thus, $\det(A) = -110(1) = -110$.

Notice that our sequence of row operations was somewhat *ad hoc*, such as the transformation to A_5. We could have been even more methodical, and strictly followed the process that converts a matrix to reduced row-echelon form (Theorem REMEF), eventually achieving the same numerical result with a final matrix that equaled the 4×4 identity matrix. Notice too that we could have stopped with A_8, since at this point we could compute $\det(A_8)$ by two expansions about first columns, followed by a simple determinant of a 2×2 matrix (Theorem DMST).

The beauty of this approach is that computationally we should already have written a procedure to convert matrices to reduced-row echelon form, so all we need to do is track the multiplicative changes to the determinant as the algorithm proceeds. Further, for a square matrix of size n this approach requires on the order of n^3 multiplications, while a recursive application of expansion about a row or column (Theorem DER, Theorem DEC) will require in the vicinity of $(n-1)(n!)$ multiplications. So even for very small matrices, a computational approach utilizing row operations will have superior run-time. Tracking, and controlling, the effects of round-off errors is another story, best saved for a numerical linear algebra course. △

Subsection DROEM
Determinants, Row Operations, Elementary Matrices

As a final preparation for our two most important theorems about determinants, we prove a handful of facts about the interplay of row operations and matrix multiplication with elementary matrices with regard to the determinant. But first, a simple, but crucial, fact about the identity matrix.

Theorem DIM Determinant of the Identity Matrix
For every $n \geq 1$, $\det(I_n) = 1$.

Proof. It may be overkill, but this is a good situation to run through a proof by induction on n (Proof Technique I). Is the result true when $n=1$? Yes,

$$\begin{aligned}\det(I_1) &= [I_1]_{11} && \text{Definition DM}\\ &= 1 && \text{Definition IM}\end{aligned}$$

Now assume the theorem is true for the identity matrix of size $n-1$ and investigate the determinant of the identity matrix of size n with expansion about row 1,

$$\begin{aligned}\det(I_n) &= \sum_{j=1}^{n}(-1)^{1+j}\,[I_n]_{1j}\det(I_n(1|j)) && \text{Definition DM}\\ &= (-1)^{1+1}\,[I_n]_{11}\det(I_n(1|1))\\ &\quad+\sum_{j=2}^{n}(-1)^{1+j}\,[I_n]_{1j}\det(I_n(1|j))\\ &= 1\det(I_{n-1})+\sum_{j=2}^{n}(-1)^{1+j}\,0\det(I_n(1|j)) && \text{Definition IM}\\ &= 1(1)+\sum_{j=2}^{n}0=1 && \text{Induction Hypothesis}\end{aligned}$$

■

Theorem DEM Determinants of Elementary Matrices
For the three possible versions of an elementary matrix (Definition ELEM) we have the determinants,

1. $\det(E_{i,j})=-1$
2. $\det(E_i(\alpha))=\alpha$
3. $\det(E_{i,j}(\alpha))=1$

Proof. Swapping rows i and j of the identity matrix will create $E_{i,j}$ (Definition ELEM), so

$$\begin{aligned}\det(E_{i,j}) &= -\det(I_n) && \text{Theorem DRCS}\\ &= -1 && \text{Theorem DIM}\end{aligned}$$

Multiplying row i of the identity matrix by α will create $E_i(\alpha)$ (Definition ELEM), so

$$\begin{aligned}\det(E_i(\alpha)) &= \alpha\det(I_n) && \text{Theorem DRCM}\\ &= \alpha(1)=\alpha && \text{Theorem DIM}\end{aligned}$$

Multiplying row i of the identity matrix by α and adding to row j will create $E_{i,j}(\alpha)$ (Definition ELEM), so

$$\begin{aligned}\det(E_{i,j}(\alpha)) &= \det(I_n) && \text{Theorem DRCMA}\\ &= 1 && \text{Theorem DIM}\end{aligned}$$

■

Theorem DEMMM Determinants, Elementary Matrices, Matrix Multiplication
Suppose that A is a square matrix of size n and E is any elementary matrix of size n. Then

$$\det(EA) = \det(E)\det(A)$$

Proof. The proof procedes in three parts, one for each type of elementary matrix, with each part very similar to the other two.

First, let B be the matrix obtained from A by swapping rows i and j,

$$\begin{aligned}\det(E_{i,j}A) &= \det(B) && \text{Theorem EMDRO}\\ &= -\det(A) && \text{Theorem DRCS}\\ &= \det(E_{i,j})\det(A) && \text{Theorem DEM}\end{aligned}$$

Second, let B be the matrix obtained from A by multiplying row i by α,

$$\begin{aligned}\det(E_i(\alpha)A) &= \det(B) && \text{Theorem EMDRO}\\ &= \alpha\det(A) && \text{Theorem DRCM}\\ &= \det(E_i(\alpha))\det(A) && \text{Theorem DEM}\end{aligned}$$

Third, let B be the matrix obtained from A by multiplying row i by α and adding to row j,

$$\begin{aligned}\det(E_{i,j}(\alpha)A) &= \det(B) && \text{Theorem EMDRO}\\ &= \det(A) && \text{Theorem DRCMA}\\ &= \det(E_{i,j}(\alpha))\det(A) && \text{Theorem DEM}\end{aligned}$$

Since the desired result holds for each variety of elementary matrix individually, we are done. ■

Subsection DNMMM
Determinants, Nonsingular Matrices, Matrix Multiplication

If you asked someone with substantial experience working with matrices about the value of the determinant, they'd be likely to quote the following theorem as the first thing to come to mind.

Theorem SMZD Singular Matrices have Zero Determinants
Let A be a square matrix. Then A is singular if and only if $\det(A) = 0$.

Proof. Rather than jumping into the two halves of the equivalence, we first establish a few items. Let B be the unique square matrix that is row-equivalent to A and in reduced row-echelon form (Theorem REMEF, Theorem RREFU). For each of the row operations that converts B into A, there is an elementary matrix E_i which effects the row operation by matrix multiplication (Theorem EMDRO). Repeated applications of Theorem EMDRO allow us to write

$$A = E_sE_{s-1}\dots E_2E_1B$$

Then

$$\begin{aligned}\det(A) &= \det(E_sE_{s-1}\dots E_2E_1B)\\ &= \det(E_s)\det(E_{s-1})\dots\det(E_2)\det(E_1)\det(B) && \text{Theorem DEMMM}\end{aligned}$$

From Theorem DEM we can infer that the determinant of an elementary matrix is never zero (note the ban on $\alpha = 0$ for $E_i(\alpha)$ in Definition ELEM). So the product on the right is composed of nonzero scalars, with the possible exception of $\det(B)$. More precisely, we can argue that $\det(A) = 0$ if and only if $\det(B) = 0$. With this established, we can take up the two halves of the equivalence.

($\Rightarrow$) If A is singular, then by Theorem NMRRI, B cannot be the identity matrix. Because (1) the number of pivot columns is equal to the number of nonzero rows, (2) not every column is a pivot column, and (3) B is square, we see that B must have a zero row. By Theorem DZRC the determinant of B is zero, and by the above, we conclude that the determinant of A is zero.

($\Leftarrow$) We will prove the contrapositive (Proof Technique CP). So assume A is nonsingular, then by Theorem NMRRI, B is the identity matrix and Theorem DIM tells us that $\det(B) = 1 \neq 0$. With the argument above, we conclude that the determinant of A is nonzero as well. ■

For the case of 2×2 matrices you might compare the application of Theorem SMZD with the combination of the results stated in Theorem DMST and Theorem TTMI.

Example ZNDAB Zero and nonzero determinant, Archetypes A and B
The coefficient matrix in Archetype A has a zero determinant (check this!) while the coefficient matrix Archetype B has a nonzero determinant (check this, too). These matrices are singular and nonsingular, respectively. This is exactly what Theorem SMZD says, and continues our list of contrasts between these two archetypes. △

Since Theorem SMZD is an equivalence (Proof Technique E) we can expand on our growing list of equivalences about nonsingular matrices. The addition of the condition $\det(A) \neq 0$ is one of the best motivations for learning about determinants.

Theorem NME7 Nonsingular Matrix Equivalences, Round 7
Suppose that A is a square matrix of size n. The following are equivalent.

1. *A is nonsingular.*
2. *A row-reduces to the identity matrix.*
3. *The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.*
4. *The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.*
5. *The columns of A are a linearly independent set.*
6. *A is invertible.*
7. *The column space of A is $\mathbb{C}^n$, $\mathcal{C}(A) = \mathbb{C}^n$.*
8. *The columns of A are a basis for $\mathbb{C}^n$.*
9. *The rank of A is n, $r(A) = n$.*
10. *The nullity of A is zero, $n(A) = 0$.*
11. *The determinant of A is nonzero, $\det(A) \neq 0$.*

Proof. Theorem SMZD says A is singular if and only if $\det(A) = 0$. If we negate each of these statements, we arrive at two contrapositives that we can combine as the equivalence, A is nonsingular if and only if $\det(A) \neq 0$. This allows us to add a new statement to the list found in Theorem NME6. ■

Computationally, row-reducing a matrix is the most efficient way to determine if a matrix is nonsingular, though the effect of using division in a computer can lead to round-off errors that confuse small quantities with critical zero quantities. Conceptually, the determinant may seem the most efficient way to determine if a matrix is nonsingular. The definition of a determinant uses just addition, subtraction and multiplication, so division is never a problem. And the final test is easy: is the determinant zero or not? However, the number of operations involved in computing a determinant by the definition very quickly becomes so excessive as to be impractical.

Now for the *coup de grace*. We will generalize Theorem DEMMM to the case of *any* two square matrices. You may recall thinking that matrix multiplication was defined in a needlessly complicated manner. For sure, the definition of a determinant seems even stranger. (Though Theorem SMZD might be forcing you to reconsider.) Read the statement of the next theorem and contemplate how nicely matrix multiplication and determinants play with each other.

Theorem DRMM Determinant Respects Matrix Multiplication
Suppose that A and B are square matrices of the same size. Then $\det(AB) = \det(A)\det(B)$.

Proof. This proof is constructed in two cases. First, suppose that A is singular. Then $\det(A) = 0$ by Theorem SMZD. By the contrapositive of Theorem NPNT, AB is singular as well. So by a second application of Theorem SMZD, $\det(AB) = 0$. Putting it all together

$$\det(AB) = 0 = 0\det(B) = \det(A)\det(B)$$

as desired.

For the second case, suppose that A is nonsingular. By Theorem NMPEM there are elementary matrices E_1, E_2, E_3, …, E_s such that $A = E_1E_2E_3\dots E_s$. Then

$$\begin{aligned}
\det(AB) &= \det(E_1E_2E_3\dots E_sB) \\
&= \det(E_1)\det(E_2)\det(E_3)\dots\det(E_s)\det(B) && \text{Theorem DEMMM} \\
&= \det(E_1E_2E_3\dots E_s)\det(B) && \text{Theorem DEMMM} \\
&= \det(A)\det(B)
\end{aligned}$$

■

It is amazing that matrix multiplication and the determinant interact this way. Might it also be true that $\det(A + B) = \det(A) + \det(B)$? (Exercise PDM.M30)

Reading Questions

1. Consider the two matrices below, and suppose you already have computed $\det(A) = -120$. What is $\det(B)$? Why?

$$A = \begin{bmatrix} 0 & 8 & 3 & -4 \\ -1 & 2 & -2 & 5 \\ -2 & 8 & 4 & 3 \\ 0 & -4 & 2 & -3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 8 & 3 & -4 \\ 0 & -4 & 2 & -3 \\ -2 & 8 & 4 & 3 \\ -1 & 2 & -2 & 5 \end{bmatrix}$$

2. State the theorem that allows us to make yet another extension to our NMEx series of theorems.

3. What is amazing about the interaction between matrix multiplication and the determinant?

Exercises

C30 Each of the archetypes below is a system of equations with a square coefficient matrix, or is a square matrix itself. Compute the determinant of each matrix, noting how Theorem SMZD indicates when the matrix is singular or nonsingular.

Archetype A, Archetype B, Archetype F, Archetype K, Archetype L

M20† Construct a 3×3 nonsingular matrix and call it A. Then, for each entry of the matrix, compute the corresponding cofactor, and create a new 3×3 matrix full of these cofactors by placing the cofactor of an entry in the same location as the entry it was based on. Once complete, call this matrix C. Compute AC^t. Any observations? Repeat with a new matrix, or perhaps with a 4×4 matrix.

M30 Construct an example to show that the following statement is not true for all square matrices A and B of the same size: $\det(A+B) = \det(A) + \det(B)$.

T10 Theorem NPNT says that if the product of square matrices AB is nonsingular, then the individual matrices A and B are nonsingular also. Construct a new proof of this result making use of theorems about determinants of matrices.

T15 Use Theorem DRCM to prove Theorem DZRC as a corollary. (See Proof Technique LC.)

T20 Suppose that A is a square matrix of size n and $\alpha \in \mathbb{C}$ is a scalar. Prove that $\det(\alpha A) = \alpha^n \det(A)$.

T25 Employ Theorem DT to construct the second half of the proof of Theorem DRCM (the portion about a multiple of a column).

Chapter E
Eigenvalues

When we have a square matrix of size n, A, and we multiply it by a vector $\mathbf{x}$ from $\mathbb{C}^n$ to form the matrix-vector product (Definition MVP), the result is another vector in $\mathbb{C}^n$. So we can adopt a functional view of this computation — the act of multiplying by a square matrix is a function that converts one vector ($\mathbf{x}$) into another one ($A\mathbf{x}$) of the same size. For some vectors, this seemingly complicated computation is really no more complicated than scalar multiplication. The vectors vary according to the choice of A, so the question is to determine, for an individual choice of A, if there are any such vectors, and if so, which ones. It happens in a variety of situations that these vectors (and the scalars that go along with them) are of special interest.

We will be solving polynomial equations in this chapter, which raises the specter of complex numbers as roots. This distinct possibility is our main reason for entertaining the complex numbers throughout the course. You might be moved to revisit Section CNO and Section O.

Section EE
Eigenvalues and Eigenvectors

In this section, we will define the eigenvalues and eigenvectors of a matrix, and see how to compute them. More theoretical properties will be taken up in the next section.

Subsection EEM
Eigenvalues and Eigenvectors of a Matrix

We start with the principal definition for this chapter.

Definition EEM Eigenvalues and Eigenvectors of a Matrix
Suppose that A is a square matrix of size n, $\mathbf{x} \neq \mathbf{0}$ is a vector in $\mathbb{C}^n$, and λ is a scalar in $\mathbb{C}$. Then we say $\mathbf{x}$ is an **eigenvector** of A with **eigenvalue** λ if

$$A\mathbf{x} = \lambda\mathbf{x}$$

□

Before going any further, perhaps we should convince you that such things ever happen at all. Understand the next example, but do not concern yourself with where the pieces come from. We will have methods soon enough to be able to discover these eigenvectors ourselves.

Example SEE Some eigenvalues and eigenvectors

Consider the matrix

$$A = \begin{bmatrix} 204 & 98 & -26 & -10 \\ -280 & -134 & 36 & 14 \\ 716 & 348 & -90 & -36 \\ -472 & -232 & 60 & 28 \end{bmatrix}$$

and the vectors

$$\mathbf{x} = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 5 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} -3 \\ 4 \\ -10 \\ 4 \end{bmatrix} \qquad \mathbf{z} = \begin{bmatrix} -3 \\ 7 \\ 0 \\ 8 \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} 1 \\ -1 \\ 4 \\ 0 \end{bmatrix}$$

Then

$$A\mathbf{x} = \begin{bmatrix} 204 & 98 & -26 & -10 \\ -280 & -134 & 36 & 14 \\ 716 & 348 & -90 & -36 \\ -472 & -232 & 60 & 28 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ 8 \\ 20 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ -1 \\ 2 \\ 5 \end{bmatrix} = 4\mathbf{x}$$

so $\mathbf{x}$ is an eigenvector of A with eigenvalue $\lambda = 4$.

Also,

$$A\mathbf{y} = \begin{bmatrix} 204 & 98 & -26 & -10 \\ -280 & -134 & 36 & 14 \\ 716 & 348 & -90 & -36 \\ -472 & -232 & 60 & 28 \end{bmatrix} \begin{bmatrix} -3 \\ 4 \\ -10 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 0 \begin{bmatrix} -3 \\ 4 \\ -10 \\ 4 \end{bmatrix} = 0\mathbf{y}$$

so $\mathbf{y}$ is an eigenvector of A with eigenvalue $\lambda = 0$.

Also,

$$A\mathbf{z} = \begin{bmatrix} 204 & 98 & -26 & -10 \\ -280 & -134 & 36 & 14 \\ 716 & 348 & -90 & -36 \\ -472 & -232 & 60 & 28 \end{bmatrix} \begin{bmatrix} -3 \\ 7 \\ 0 \\ 8 \end{bmatrix} = \begin{bmatrix} -6 \\ 14 \\ 0 \\ 16 \end{bmatrix} = 2 \begin{bmatrix} -3 \\ 7 \\ 0 \\ 8 \end{bmatrix} = 2\mathbf{z}$$

so $\mathbf{z}$ is an eigenvector of A with eigenvalue $\lambda = 2$.

Also,

$$A\mathbf{w} = \begin{bmatrix} 204 & 98 & -26 & -10 \\ -280 & -134 & 36 & 14 \\ 716 & 348 & -90 & -36 \\ -472 & -232 & 60 & 28 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 8 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \\ 4 \\ 0 \end{bmatrix} = 2\mathbf{w}$$

so $\mathbf{w}$ is an eigenvector of A with eigenvalue $\lambda = 2$.

So we have demonstrated four eigenvectors of A. Are there more? Yes, any nonzero scalar multiple of an eigenvector is again an eigenvector. In this example, set $\mathbf{u} = 30\mathbf{x}$. Then

$$\begin{aligned} A\mathbf{u} &= A(30\mathbf{x}) \\ &= 30A\mathbf{x} && \text{Theorem MMSMM} \\ &= 30(4\mathbf{x}) && \mathbf{x} \text{ an eigenvector of } A \\ &= 4(30\mathbf{x}) && \text{Property SMAM} \\ &= 4\mathbf{u} \end{aligned}$$

so that $\mathbf{u}$ is also an eigenvector of A for the same eigenvalue, $\lambda = 4$.

The vectors $\mathbf{z}$ and $\mathbf{w}$ are both eigenvectors of A for the same eigenvalue $\lambda = 2$, yet this is not as simple as the two vectors just being scalar multiples of each other (they are not). Look what happens when we add them together, to form $\mathbf{v} = \mathbf{z} + \mathbf{w}$, and multiply by A,

$$\begin{aligned} A\mathbf{v} &= A(\mathbf{z} + \mathbf{w}) \\ &= A\mathbf{z} + A\mathbf{w} && \text{Theorem MMDAA} \end{aligned}$$

$$
\begin{aligned}
&= 2\mathbf{z} + 2\mathbf{w} && \mathbf{z},\ \mathbf{w} \text{ eigenvectors of } A\\
&= 2(\mathbf{z} + \mathbf{w}) && \text{Property DVAC}\\
&= 2\mathbf{v}
\end{aligned}
$$

so that $\mathbf{v}$ is also an eigenvector of A for the eigenvalue $\lambda = 2$. So it would appear that the set of eigenvectors that are associated with a fixed eigenvalue is closed under the vector space operations of $\mathbb{C}^n$. Hmmm.

The vector $\mathbf{y}$ is an eigenvector of A for the eigenvalue $\lambda = 0$, so we can use Theorem ZSSM to write $A\mathbf{y} = 0\mathbf{y} = \mathbf{0}$. But this also means that $\mathbf{y} \in \mathcal{N}(A)$. There would appear to be a connection here also. △

Example SEE hints at a number of intriguing properties, and there are many more. We will explore the general properties of eigenvalues and eigenvectors in Section PEE, but in this section we will concern ourselves with the question of actually computing eigenvalues and eigenvectors. First we need a bit of background material on polynomials and matrices.

Subsection PM
Polynomials and Matrices

A polynomial is a combination of powers, multiplication by scalar coefficients, and addition (with subtraction just being the inverse of addition). We never have occasion to divide when computing the value of a polynomial. So it is with matrices. We can add and subtract matrices, we can multiply matrices by scalars, and we can form powers of square matrices by repeated applications of matrix multiplication. We do not normally divide matrices (though sometimes we can multiply by an inverse). If a matrix is square, all the operations constituting a polynomial will preserve the size of the matrix. So it is natural to consider evaluating a polynomial with a matrix, effectively replacing the variable of the polynomial by a matrix. We will demonstrate with an example.

Example PM Polynomial of a matrix
Let

$$
p(x) = 14 + 19x - 3x^2 - 7x^3 + x^4 \qquad D = \begin{bmatrix} -1 & 3 & 2 \\ 1 & 0 & -2 \\ -3 & 1 & 1 \end{bmatrix}
$$

and we will compute $p(D)$.

First, the necessary powers of D. Notice that D^0 is defined to be the multiplicative identity, I_3, as will be the case in general.

$$
\begin{aligned}
D^0 &= I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\\
D^1 &= D = \begin{bmatrix} -1 & 3 & 2 \\ 1 & 0 & -2 \\ -3 & 1 & 1 \end{bmatrix}\\
D^2 &= DD^1 = \begin{bmatrix} -1 & 3 & 2 \\ 1 & 0 & -2 \\ -3 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & 2 \\ 1 & 0 & -2 \\ -3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -1 & -6 \\ 5 & 1 & 0 \\ 1 & -8 & -7 \end{bmatrix}\\
D^3 &= DD^2 = \begin{bmatrix} -1 & 3 & 2 \\ 1 & 0 & -2 \\ -3 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & -1 & -6 \\ 5 & 1 & 0 \\ 1 & -8 & -7 \end{bmatrix} = \begin{bmatrix} 19 & -12 & -8 \\ -4 & 15 & 8 \\ 12 & -4 & 11 \end{bmatrix}\\
D^4 &= DD^3 = \begin{bmatrix} -1 & 3 & 2 \\ 1 & 0 & -2 \\ -3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 19 & -12 & -8 \\ -4 & 15 & 8 \\ 12 & -4 & 11 \end{bmatrix} = \begin{bmatrix} -7 & 49 & 54 \\ -5 & -4 & -30 \\ -49 & 47 & 43 \end{bmatrix}
\end{aligned}
$$

Then

$$\begin{aligned} p(D) &= 14 + 19D - 3D^2 - 7D^3 + D^4 \\ &= 14\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} + 19\begin{bmatrix}-1 & 3 & 2\\1 & 0 & -2\\-3 & 1 & 1\end{bmatrix} - 3\begin{bmatrix}-2 & -1 & -6\\5 & 1 & 0\\1 & -8 & -7\end{bmatrix} \\ &\quad - 7\begin{bmatrix}19 & -12 & -8\\-4 & 15 & 8\\12 & -4 & 11\end{bmatrix} + \begin{bmatrix}-7 & 49 & 54\\-5 & -4 & -30\\-49 & 47 & 43\end{bmatrix} \\ &= \begin{bmatrix}-139 & 193 & 166\\27 & -98 & -124\\-193 & 118 & 20\end{bmatrix}\end{aligned}$$

Notice that $p(x)$ factors as

$$p(x) = 14 + 19x - 3x^2 - 7x^3 + x^4 = (x-2)(x-7)(x+1)^2$$

Because D commutes with itself ($DD = DD$), we can use distributivity of matrix multiplication across matrix addition (Theorem MMDAA) without being careful with any of the matrix products, and just as easily evaluate $p(D)$ using the factored form of $p(x)$,

$$\begin{aligned} p(D) &= 14 + 19D - 3D^2 - 7D^3 + D^4 = (D - 2I_3)(D - 7I_3)(D + I_3)^2 \\ &= \begin{bmatrix}-3 & 3 & 2\\1 & -2 & -2\\-3 & 1 & -1\end{bmatrix}\begin{bmatrix}-8 & 3 & 2\\1 & -7 & -2\\-3 & 1 & -6\end{bmatrix}\begin{bmatrix}0 & 3 & 2\\1 & 1 & -2\\-3 & 1 & 2\end{bmatrix}^2 \\ &= \begin{bmatrix}-139 & 193 & 166\\27 & -98 & -124\\-193 & 118 & 20\end{bmatrix}\end{aligned}$$

This example is not meant to be too profound. It *is* meant to show you that it is natural to evaluate a polynomial with a matrix, and that the factored form of the polynomial is as good as (or maybe better than) the expanded form. And do not forget that constant terms in polynomials are really multiples of the identity matrix when we are evaluating the polynomial with a matrix. △

Subsection EEE
Existence of Eigenvalues and Eigenvectors

Before we embark on computing eigenvalues and eigenvectors, we will prove that every matrix has at least one eigenvalue (and an eigenvector to go with it). Later, in Theorem MNEM, we will determine the maximum number of eigenvalues a matrix may have.

The determinant (Definition DM) will be a powerful tool in Subsection EE.CEE when it comes time to compute eigenvalues. However, it is possible, with some more advanced machinery, to compute eigenvalues without ever making use of the determinant. Sheldon Axler does just that in his book, *Linear Algebra Done Right*. Here and now, we give Axler's "determinant-free" proof that every matrix has an eigenvalue. The result is not too startling, but the proof is most enjoyable.

Theorem EMHE Every Matrix Has an Eigenvalue
Suppose A is a square matrix. Then A has at least one eigenvalue.

Proof. Suppose that A has size n, and choose $\mathbf{x}$ as *any* nonzero vector from $\mathbb{C}^n$. (Notice how much latitude we have in our choice of $\mathbf{x}$. Only the zero vector is off-limits.) Consider the set

$$S = \left\{\mathbf{x},\ A\mathbf{x},\ A^2\mathbf{x},\ A^3\mathbf{x},\ \dots,\ A^n\mathbf{x}\right\}$$

This is a set of $n+1$ vectors from $\mathbb{C}^n$, so by Theorem MVSLD, S is linearly dependent. Let $a_0,\ a_1,\ a_2,\ \dots,\ a_n$ be a collection of $n+1$ scalars from $\mathbb{C}$, not all zero, that provide a relation of linear dependence on S. In other words,

$$a_0\mathbf{x} + a_1A\mathbf{x} + a_2A^2\mathbf{x} + a_3A^3\mathbf{x} + \cdots + a_nA^n\mathbf{x} = \mathbf{0}$$

Some of the a_i are nonzero. Suppose that just $a_0 \neq 0$, and $a_1 = a_2 = a_3 = \cdots = a_n = 0$. Then $a_0\mathbf{x} = \mathbf{0}$ and by Theorem SMEZV, either $a_0 = 0$ or $\mathbf{x} = \mathbf{0}$, which are both contradictions. So $a_i \neq 0$ for some $i \geq 1$. Let m be the largest integer such that $a_m \neq 0$. From this discussion we know that $m \geq 1$. We can also assume that $a_m = 1$, for if not, replace each a_i by a_i/a_m to obtain scalars that serve equally well in providing a relation of linear dependence on S.

Define the polynomial

$$p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_mx^m$$

Because we have consistently used $\mathbb{C}$ as our set of scalars (rather than $\mathbb{R}$), we know that we can factor $p(x)$ into linear factors of the form $(x - b_i)$, where $b_i \in \mathbb{C}$. So there are scalars, $b_1,\ b_2,\ b_3,\ \dots,\ b_m$, from $\mathbb{C}$ so that,

$$p(x) = (x - b_m)(x - b_{m-1})\cdots(x - b_3)(x - b_2)(x - b_1)$$

Put it all together and

$$\begin{aligned}
\mathbf{0} &= a_0\mathbf{x} + a_1A\mathbf{x} + a_2A^2\mathbf{x} + \cdots + a_nA^n\mathbf{x} \\
&= a_0\mathbf{x} + a_1A\mathbf{x} + a_2A^2\mathbf{x} + \cdots + a_mA^m\mathbf{x} && a_i = 0 \text{ for } i > m \\
&= \left(a_0I_n + a_1A + a_2A^2 + \cdots + a_mA^m\right)\mathbf{x} && \text{Theorem MMDAA} \\
&= p(A)\mathbf{x} && \text{Definition of } p(x) \\
&= (A - b_mI_n)(A - b_{m-1}I_n)\cdots(A - b_2I_n)(A - b_1I_n)\mathbf{x}
\end{aligned}$$

Let k be the smallest integer such that

$$(A - b_kI_n)(A - b_{k-1}I_n)\cdots(A - b_2I_n)(A - b_1I_n)\mathbf{x} = \mathbf{0}.$$

From the preceding equation, we know that $k \leq m$. Define the vector $\mathbf{z}$ by

$$\mathbf{z} = (A - b_{k-1}I_n)\cdots(A - b_2I_n)(A - b_1I_n)\mathbf{x}$$

Notice that by the definition of k, the vector $\mathbf{z}$ must be nonzero. In the case where $k = 1$, we understand that $\mathbf{z}$ is defined by $\mathbf{z} = \mathbf{x}$, and $\mathbf{z}$ is still nonzero. Now

$$(A - b_kI_n)\mathbf{z} = (A - b_kI_n)(A - b_{k-1}I_n)\cdots(A - b_3I_n)(A - b_2I_n)(A - b_1I_n)\mathbf{x} = \mathbf{0}$$

which allows us to write

$$\begin{aligned}
A\mathbf{z} &= (A + \mathcal{O})\mathbf{z} && \text{Property ZM} \\
&= (A - b_kI_n + b_kI_n)\mathbf{z} && \text{Property AIM} \\
&= (A - b_kI_n)\mathbf{z} + b_kI_n\mathbf{z} && \text{Theorem MMDAA} \\
&= \mathbf{0} + b_kI_n\mathbf{z} && \text{Defining property of } \mathbf{z} \\
&= b_kI_n\mathbf{z} && \text{Property ZM} \\
&= b_k\mathbf{z} && \text{Theorem MMIM}
\end{aligned}$$

Since $\mathbf{z} \neq \mathbf{0}$, this equation says that $\mathbf{z}$ is an eigenvector of A for the eigenvalue $\lambda = b_k$ (Definition EEM), so we have shown that any square matrix A does have at least one eigenvalue. ■

The proof of Theorem EMHE is constructive (it contains an unambiguous procedure that leads to an eigenvalue), but it is not meant to be practical. We will illustrate the theorem with an example, the purpose being to provide a companion for studying the proof and not to suggest this is the best procedure for computing an eigenvalue.

Example CAEHW Computing an eigenvalue the hard way
This example illustrates the proof of Theorem EMHE, and so will employ the same notation as the proof — look there for full explanations. It is *not* meant to be an example of a reasonable computational approach to finding eigenvalues and eigenvectors. OK, warnings in place, here we go.

Consider the matrix A, and choose the vector $\mathbf{x}$,

$$A = \begin{bmatrix} -7 & -1 & 11 & 0 & -4 \\ 4 & 1 & 0 & 2 & 0 \\ -10 & -1 & 14 & 0 & -4 \\ 8 & 2 & -15 & -1 & 5 \\ -10 & -1 & 16 & 0 & -6 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} 3 \\ 0 \\ 3 \\ -5 \\ 4 \end{bmatrix}$$

It is important to notice that the choice of $\mathbf{x}$ could be *anything*, so long as it is *not* the zero vector. We have not chosen $\mathbf{x}$ totally at random, but so as to make our illustration of the theorem as general as possible. You could replicate this example with your own choice and the computations are guaranteed to be reasonable, provided you have a computational tool that will factor a fifth degree polynomial for you.

The set

$$\begin{aligned} S &= \left\{\mathbf{x},\, A\mathbf{x},\, A^2\mathbf{x},\, A^3\mathbf{x},\, A^4\mathbf{x},\, A^5\mathbf{x}\right\} \\ &= \left\{ \begin{bmatrix} 3 \\ 0 \\ 3 \\ -5 \\ 4 \end{bmatrix}, \begin{bmatrix} -4 \\ 2 \\ -4 \\ 4 \\ -6 \end{bmatrix}, \begin{bmatrix} 6 \\ -6 \\ 6 \\ -2 \\ 10 \end{bmatrix}, \begin{bmatrix} -10 \\ 14 \\ -10 \\ -2 \\ -18 \end{bmatrix}, \begin{bmatrix} 18 \\ -30 \\ 18 \\ 10 \\ 34 \end{bmatrix}, \begin{bmatrix} -34 \\ 62 \\ -34 \\ -26 \\ -66 \end{bmatrix} \right\} \end{aligned}$$

is guaranteed to be linearly dependent, as it has six vectors from $\mathbb{C}^5$ (Theorem MVSLD).

We will search for a nontrivial relation of linear dependence by solving a homogeneous system of equations whose coefficient matrix has the vectors of S as columns through row operations,

$$\begin{bmatrix} 3 & -4 & 6 & -10 & 18 & -34 \\ 0 & 2 & -6 & 14 & -30 & 62 \\ 3 & -4 & 6 & -10 & 18 & -34 \\ -5 & 4 & -2 & -2 & 10 & -26 \\ 4 & -6 & 10 & -18 & 34 & -66 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -2 & 6 & -14 & 30 \\ 0 & \boxed{1} & -3 & 7 & -15 & 31 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

There are four free variables for describing solutions to this homogeneous system, so we have our pick of solutions. The most expedient choice would be to set $x_3 = 1$ and $x_4 = x_5 = x_6 = 0$. However, we will again opt to maximize the generality of our illustration of Theorem EMHE and choose $x_3 = -8$, $x_4 = -3$, $x_5 = 1$ and $x_6 = 0$. This leads to a solution with $x_1 = 16$ and $x_2 = 12$.

This relation of linear dependence then says that

$$\begin{aligned} \mathbf{0} &= 16\mathbf{x} + 12A\mathbf{x} - 8A^2\mathbf{x} - 3A^3\mathbf{x} + A^4\mathbf{x} + 0A^5\mathbf{x} \\ \mathbf{0} &= \left(16 + 12A - 8A^2 - 3A^3 + A^4\right)\mathbf{x} \end{aligned}$$

So we define $p(x) = 16 + 12x - 8x^2 - 3x^3 + x^4$, and as advertised in the proof of Theorem EMHE, we have a polynomial of degree $m = 4 > 1$ such that $p(A)\mathbf{x} = \mathbf{0}$. Now we need to factor $p(x)$ over $\mathbb{C}$. If you made your own choice of $\mathbf{x}$ at the start, this is where you might have a fifth degree polynomial, and where you might need to use a computational tool to find roots and factors. We have

$$p(x) = 16 + 12x - 8x^2 - 3x^3 + x^4 = (x-4)(x+2)(x-2)(x+1)$$

So we know that

$$\mathbf{0} = p(A)\mathbf{x} = (A - 4I_5)(A + 2I_5)(A - 2I_5)(A + 1I_5)\mathbf{x}$$

We apply one factor at a time, until we get the zero vector, so as to determine the value of k described in the proof of Theorem EMHE,

$$(A + 1I_5)\mathbf{x} = \begin{bmatrix} -6 & -1 & 11 & 0 & -4 \\ 4 & 2 & 0 & 2 & 0 \\ -10 & -1 & 15 & 0 & -4 \\ 8 & 2 & -15 & 0 & 5 \\ -10 & -1 & 16 & 0 & -5 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 3 \\ -5 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -1 \\ -1 \\ -2 \end{bmatrix}$$

$$(A - 2I_5)(A + 1I_5)\mathbf{x} = \begin{bmatrix} -9 & -1 & 11 & 0 & -4 \\ 4 & -1 & 0 & 2 & 0 \\ -10 & -1 & 12 & 0 & -4 \\ 8 & 2 & -15 & -3 & 5 \\ -10 & -1 & 16 & 0 & -8 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ -1 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ -8 \\ 4 \\ 4 \\ 8 \end{bmatrix}$$

$$(A + 2I_5)(A - 2I_5)(A + 1I_5)\mathbf{x} = \begin{bmatrix} -5 & -1 & 11 & 0 & -4 \\ 4 & 3 & 0 & 2 & 0 \\ -10 & -1 & 16 & 0 & -4 \\ 8 & 2 & -15 & 1 & 5 \\ -10 & -1 & 16 & 0 & -4 \end{bmatrix} \begin{bmatrix} 4 \\ -8 \\ 4 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

So $k = 3$ and

$$\mathbf{z} = (A - 2I_5)(A + 1I_5)\mathbf{x} = \begin{bmatrix} 4 \\ -8 \\ 4 \\ 4 \\ 8 \end{bmatrix}$$

is an eigenvector of A for the eigenvalue $\lambda = -2$, as you can check by doing the computation $A\mathbf{z}$. If you work through this example with your own choice of the vector $\mathbf{x}$ (strongly recommended) then the eigenvalue you will find may be different, but will be in the set $\{3,\ 0,\ 1,\ -1,\ -2\}$. See Exercise EE.M60 for a suggested starting vector. △

Subsection CEE
Computing Eigenvalues and Eigenvectors

Fortunately, we need not rely on the procedure of Theorem EMHE each time we need an eigenvalue. It is the determinant, and specifically Theorem SMZD, that provides the main tool for computing eigenvalues. Here is an informal sequence of equivalences that is the key to determining the eigenvalues and eigenvectors of a matrix,

$$A\mathbf{x} = \lambda\mathbf{x} \iff A\mathbf{x} - \lambda I_n\mathbf{x} = \mathbf{0} \iff (A - \lambda I_n)\,\mathbf{x} = \mathbf{0}$$

So, for an eigenvalue λ and associated eigenvector $\mathbf{x} \neq \mathbf{0}$, the vector $\mathbf{x}$ will be a nonzero element of the null space of $A - \lambda I_n$, while the matrix $A - \lambda I_n$ will be singular and therefore have zero determinant. These ideas are made precise in Theorem EMRCP and Theorem EMNS, but for now this brief discussion should suffice as motivation for the following definition and example.

Definition CP Characteristic Polynomial
Suppose that A is a square matrix of size n. Then the **characteristic polynomial** of A is the polynomial $p_A(x)$ defined by

$$p_A(x) = \det(A - xI_n)$$

□

Example CPMS3 Characteristic polynomial of a matrix, size 3
Consider

$$F = \begin{bmatrix} -13 & -8 & -4 \\ 12 & 7 & 4 \\ 24 & 16 & 7 \end{bmatrix}$$

Then

$$\begin{aligned}
p_F(x) &= \det(F - xI_3) \\
&= \begin{vmatrix} -13-x & -8 & -4 \\ 12 & 7-x & 4 \\ 24 & 16 & 7-x \end{vmatrix} && \text{Definition CP} \\
&= (-13-x)\begin{vmatrix} 7-x & 4 \\ 16 & 7-x \end{vmatrix} + (-8)(-1)\begin{vmatrix} 12 & 4 \\ 24 & 7-x \end{vmatrix} && \text{Definition DM} \\
&\quad + (-4)\begin{vmatrix} 12 & 7-x \\ 24 & 16 \end{vmatrix} \\
&= (-13-x)((7-x)(7-x) - 4(16)) && \text{Theorem DMST} \\
&\quad + (-8)(-1)(12(7-x) - 4(24)) \\
&\quad + (-4)(12(16) - (7-x)(24)) \\
&= 3 + 5x + x^2 - x^3 \\
&= -(x-3)(x+1)^2
\end{aligned}$$

△

The characteristic polynomial is our main computational tool for finding eigenvalues, and will sometimes be used to aid us in determining the properties of eigenvalues.

Theorem EMRCP Eigenvalues of a Matrix are Roots of Characteristic Polynomials
Suppose A is a square matrix. Then λ is an eigenvalue of A if and only if $p_A(\lambda) = 0$.

Proof. Suppose A has size n.

$$\begin{aligned}
&\lambda \text{ is an eigenvalue of } A \\
\iff & \text{there exists } \mathbf{x} \neq \mathbf{0} \text{ so that } A\mathbf{x} = \lambda\mathbf{x} && \text{Definition EEM} \\
\iff & \text{there exists } \mathbf{x} \neq \mathbf{0} \text{ so that } A\mathbf{x} - \lambda\mathbf{x} = \mathbf{0} \\
\iff & \text{there exists } \mathbf{x} \neq \mathbf{0} \text{ so that } A\mathbf{x} - \lambda I_n\mathbf{x} = \mathbf{0} && \text{Theorem MMIM} \\
\iff & \text{there exists } \mathbf{x} \neq \mathbf{0} \text{ so that } (A - \lambda I_n)\mathbf{x} = \mathbf{0} && \text{Theorem MMDAA} \\
\iff & A - \lambda I_n \text{ is singular} && \text{Definition NM} \\
\iff & \det(A - \lambda I_n) = 0 && \text{Theorem SMZD} \\
\iff & p_A(\lambda) = 0 && \text{Definition CP}
\end{aligned}$$

■

Example EMS3 Eigenvalues of a matrix, size 3
In Example CPMS3 we found the characteristic polynomial of

$$F = \begin{bmatrix} -13 & -8 & -4 \\ 12 & 7 & 4 \\ 24 & 16 & 7 \end{bmatrix}$$

to be $p_F(x) = -(x-3)(x+1)^2$. Factored, we can find all of its roots easily, they are $x = 3$ and $x = -1$. By Theorem EMRCP, $\lambda = 3$ and $\lambda = -1$ are both eigenvalues of F, and these are the only eigenvalues of F. We have found them all. △

Let us now turn our attention to the computation of eigenvectors.

Definition EM Eigenspace of a Matrix
Suppose that A is a square matrix and λ is an eigenvalue of A. Then the **eigenspace**

of A for λ, $\mathcal{E}_A(\lambda)$, is the set of all the eigenvectors of A for λ, together with the inclusion of the zero vector. □

Example SEE hinted that the set of eigenvectors for a single eigenvalue might have some closure properties, and with the addition of the one eigenvector that is never an eigenvector, $\mathbf{0}$, we indeed get a whole subspace.

Theorem EMS Eigenspace for a Matrix is a Subspace
Suppose A is a square matrix of size n and λ is an eigenvalue of A. Then the eigenspace $\mathcal{E}_A(\lambda)$ is a subspace of the vector space $\mathbb{C}^n$.

Proof. We will check the three conditions of Theorem TSS. First, Definition EM explicitly includes the zero vector in $\mathcal{E}_A(\lambda)$, so the set is nonempty.

Suppose that $\mathbf{x}, \mathbf{y} \in \mathcal{E}_A(\lambda)$, that is, $\mathbf{x}$ and $\mathbf{y}$ are two eigenvectors of A for λ. Then

$$\begin{aligned} A(\mathbf{x}+\mathbf{y}) &= A\mathbf{x} + A\mathbf{y} && \text{Theorem MMDAA} \\ &= \lambda\mathbf{x} + \lambda\mathbf{y} && \mathbf{x}, \mathbf{y} \text{ eigenvectors of } A \\ &= \lambda(\mathbf{x}+\mathbf{y}) && \text{Property DVAC} \end{aligned}$$

So either $\mathbf{x}+\mathbf{y} = \mathbf{0}$, or $\mathbf{x}+\mathbf{y}$ is an eigenvector of A for λ (Definition EEM). So, in either event, $\mathbf{x}+\mathbf{y} \in \mathcal{E}_A(\lambda)$, and we have additive closure.

Suppose that $\alpha \in \mathbb{C}$, and that $\mathbf{x} \in \mathcal{E}_A(\lambda)$, that is, $\mathbf{x}$ is an eigenvector of A for λ. Then

$$\begin{aligned} A(\alpha\mathbf{x}) &= \alpha(A\mathbf{x}) && \text{Theorem MMSMM} \\ &= \alpha\lambda\mathbf{x} && \mathbf{x} \text{ an eigenvector of } A \\ &= \lambda(\alpha\mathbf{x}) && \text{Property SMAC} \end{aligned}$$

So either $\alpha\mathbf{x} = \mathbf{0}$, or $\alpha\mathbf{x}$ is an eigenvector of A for λ (Definition EEM). So, in either event, $\alpha\mathbf{x} \in \mathcal{E}_A(\lambda)$, and we have scalar closure.

With the three conditions of Theorem TSS met, we know $\mathcal{E}_A(\lambda)$ is a subspace. ■

Theorem EMS tells us that an eigenspace is a subspace (and hence a vector space in its own right). Our next theorem tells us how to quickly construct this subspace.

Theorem EMNS Eigenspace of a Matrix is a Null Space
Suppose A is a square matrix of size n and λ is an eigenvalue of A. Then

$$\mathcal{E}_A(\lambda) = \mathcal{N}(A - \lambda I_n)$$

Proof. The conclusion of this theorem is an equality of sets, so normally we would follow the advice of Definition SE. However, in this case we can construct a sequence of equivalences which will together provide the two subset inclusions we need. First, notice that $\mathbf{0} \in \mathcal{E}_A(\lambda)$ by Definition EM and $\mathbf{0} \in \mathcal{N}(A - \lambda I_n)$ by Theorem HSC. Now consider any nonzero vector $\mathbf{x} \in \mathbb{C}^n$,

$$\begin{aligned} \mathbf{x} \in \mathcal{E}_A(\lambda) &\iff A\mathbf{x} = \lambda\mathbf{x} && \text{Definition EM} \\ &\iff A\mathbf{x} - \lambda\mathbf{x} = \mathbf{0} \\ &\iff A\mathbf{x} - \lambda I_n\mathbf{x} = \mathbf{0} && \text{Theorem MMIM} \\ &\iff (A - \lambda I_n)\mathbf{x} = \mathbf{0} && \text{Theorem MMDAA} \\ &\iff \mathbf{x} \in \mathcal{N}(A - \lambda I_n) && \text{Definition NSM} \end{aligned}$$

■

You might notice the close parallels (and differences) between the proofs of Theorem EMRCP and Theorem EMNS. Since Theorem EMNS describes the set of all the eigenvectors of A as a null space we can use techniques such as Theorem BNS to provide concise descriptions of eigenspaces. Theorem EMNS also provides a trivial proof for Theorem EMS.

Example ESMS3 Eigenspaces of a matrix, size 3
Example CPMS3 and Example EMS3 describe the characteristic polynomial and eigenvalues of the 3×3 matrix

$$F = \begin{bmatrix} -13 & -8 & -4 \\ 12 & 7 & 4 \\ 24 & 16 & 7 \end{bmatrix}$$

We will now take each eigenvalue in turn and compute its eigenspace. To do this, we row-reduce the matrix $F - \lambda I_3$ in order to determine solutions to the homogeneous system $\mathcal{LS}(F - \lambda I_3, \mathbf{0})$ and then express the eigenspace as the null space of $F - \lambda I_3$ (Theorem EMNS). Theorem BNS then tells us how to write the null space as the span of a basis.

$$\lambda = 3 \qquad F - 3I_3 = \begin{bmatrix} -16 & -8 & -4 \\ 12 & 4 & 4 \\ 24 & 16 & 4 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & \frac{1}{2} \\ 0 & \boxed{1} & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{E}_F(3) = \mathcal{N}(F - 3I_3) = \left\langle \left\{ \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} \right\} \right\rangle = \left\langle \left\{ \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right\} \right\rangle$$

$$\lambda = -1 \quad F + 1I_3 = \begin{bmatrix} -12 & -8 & -4 \\ 12 & 8 & 4 \\ 24 & 16 & 8 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{E}_F(-1) = \mathcal{N}(F + 1I_3) = \left\langle \left\{ \begin{bmatrix} -\frac{2}{3} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix} \right\} \right\rangle = \left\langle \left\{ \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} \right\} \right\rangle$$

Eigenspaces in hand, we can easily compute eigenvectors by forming nontrivial linear combinations of the basis vectors describing each eigenspace. In particular, notice that we can "pretty up" our basis vectors by using scalar multiples to clear out fractions. △

Subsection ECEE
Examples of Computing Eigenvalues and Eigenvectors

There are no theorems in this section, just a selection of examples meant to illustrate the range of possibilities for the eigenvalues and eigenvectors of a matrix. These examples can all be done by hand, though the computation of the characteristic polynomial would be very time-consuming and error-prone. It can also be difficult to factor an arbitrary polynomial, though if we were to suggest that most of our eigenvalues are going to be integers, then it can be easier to hunt for roots. These examples are meant to look similar to a concatenation of Example CPMS3, Example EMS3 and Example ESMS3. First, we will sneak in a pair of definitions so we can illustrate them throughout this sequence of examples.

Definition AME Algebraic Multiplicity of an Eigenvalue
Suppose that A is a square matrix and λ is an eigenvalue of A. Then the **algebraic multiplicity** of λ, $\alpha_A(\lambda)$, is the highest power of $(x-\lambda)$ that divides the characteristic polynomial, $p_A(x)$. □

Since an eigenvalue λ is a root of the characteristic polynomial, there is always a factor of $(x - \lambda)$, and the algebraic multiplicity is just the power of this factor in a factorization of $p_A(x)$. So in particular, $\alpha_A(\lambda) \geq 1$. Compare the definition of algebraic multiplicity with the next definition.

Definition GME Geometric Multiplicity of an Eigenvalue
Suppose that A is a square matrix and λ is an eigenvalue of A. Then the **geometric multiplicity** of λ, $\gamma_A(\lambda)$, is the dimension of the eigenspace $\mathcal{E}_A(\lambda)$. □

Every eigenvalue must have at least one eigenvector, so the associated eigenspace cannot be trivial, and so $\gamma_A(\lambda) \geq 1$.

Example EMMS4 Eigenvalue multiplicities, matrix of size 4
Consider the matrix

$$B = \begin{bmatrix} -2 & 1 & -2 & -4 \\ 12 & 1 & 4 & 9 \\ 6 & 5 & -2 & -4 \\ 3 & -4 & 5 & 10 \end{bmatrix}$$

then

$$p_B(x) = 8 - 20x + 18x^2 - 7x^3 + x^4 = (x-1)(x-2)^3$$

So the eigenvalues are $\lambda = 1,\ 2$ with algebraic multiplicities $\alpha_B(1) = 1$ and $\alpha_B(2) = 3$.

Computing eigenvectors,

$$\lambda = 1 \qquad B - 1I_4 = \begin{bmatrix} -3 & 1 & -2 & -4 \\ 12 & 0 & 4 & 9 \\ 6 & 5 & -3 & -4 \\ 3 & -4 & 5 & 9 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & \frac{1}{3} & 0 \\ 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{E}_B(1) = \mathcal{N}(B - 1I_4) = \left\langle\left\{\begin{bmatrix} -\frac{1}{3} \\ 1 \\ 1 \\ 0 \end{bmatrix}\right\}\right\rangle = \left\langle\left\{\begin{bmatrix} -1 \\ 3 \\ 3 \\ 0 \end{bmatrix}\right\}\right\rangle$$

$$\lambda = 2 \qquad B - 2I_4 = \begin{bmatrix} -4 & 1 & -2 & -4 \\ 12 & -1 & 4 & 9 \\ 6 & 5 & -4 & -4 \\ 3 & -4 & 5 & 8 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 1/2 \\ 0 & \boxed{1} & 0 & -1 \\ 0 & 0 & \boxed{1} & 1/2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{E}_B(2) = \mathcal{N}(B - 2I_4) = \left\langle\left\{\begin{bmatrix} -\frac{1}{2} \\ 1 \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right\}\right\rangle = \left\langle\left\{\begin{bmatrix} -1 \\ 2 \\ -1 \\ 2 \end{bmatrix}\right\}\right\rangle$$

So each eigenspace has dimension 1 and so $\gamma_B(1) = 1$ and $\gamma_B(2) = 1$. This example is of interest because of the discrepancy between the two multiplicities for $\lambda = 2$. In many of our examples the algebraic and geometric multiplicities will be equal for all of the eigenvalues (as it was for $\lambda = 1$ in this example), so keep this example in mind. We will have some explanations for this phenomenon later (see Example NDMS4). △

Example ESMS4 Eigenvalues, symmetric matrix of size 4
Consider the matrix

$$C = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

then

$$p_C(x) = -3 + 4x + 2x^2 - 4x^3 + x^4 = (x-3)(x-1)^2(x+1)$$

So the eigenvalues are $\lambda = 3,\ 1,\ -1$ with algebraic multiplicities $\alpha_C(3) = 1$, $\alpha_C(1) = 2$ and $\alpha_C(-1) = 1$.

Computing eigenvectors,

$$\lambda = 3 \qquad C - 3I_4 = \begin{bmatrix} -2 & 0 & 1 & 1 \\ 0 & -2 & 1 & 1 \\ 1 & 1 & -2 & 0 \\ 1 & 1 & 0 & -2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & -1 \\ 0 & \boxed{1} & 0 & -1 \\ 0 & 0 & \boxed{1} & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{E}_C(3) = \mathcal{N}(C - 3I_4) = \left\langle \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\} \right\rangle$$

$$\lambda = 1 \qquad C - 1I_4 = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 1 & 0 & 0 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{E}_C(1) = \mathcal{N}(C - 1I_4) = \left\langle \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix} \right\} \right\rangle$$

$$\lambda = -1 \qquad C + 1I_4 = \begin{bmatrix} 2 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 1 \\ 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & \boxed{1} & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{E}_C(-1) = \mathcal{N}(C + 1I_4) = \left\langle \left\{ \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\} \right\rangle$$

So the eigenspace dimensions yield geometric multiplicities $\gamma_C(3) = 1$, $\gamma_C(1) = 2$ and $\gamma_C(-1) = 1$, the same as for the algebraic multiplicities. This example is of interest because A is a symmetric matrix, and will be the subject of Theorem HMRE. △

Example HMEM5 High multiplicity eigenvalues, matrix of size 5
Consider the matrix

$$E = \begin{bmatrix} 29 & 14 & 2 & 6 & -9 \\ -47 & -22 & -1 & -11 & 13 \\ 19 & 10 & 5 & 4 & -8 \\ -19 & -10 & -3 & -2 & 8 \\ 7 & 4 & 3 & 1 & -3 \end{bmatrix}$$

then

$$p_E(x) = -16 + 16x + 8x^2 - 16x^3 + 7x^4 - x^5 = -(x-2)^4(x+1)$$

So the eigenvalues are $\lambda = 2, -1$ with algebraic multiplicities $\alpha_E(2) = 4$ and $\alpha_E(-1) = 1$.

Computing eigenvectors,

$$\lambda = 2$$

$$E - 2I_5 = \begin{bmatrix} 27 & 14 & 2 & 6 & -9 \\ -47 & -24 & -1 & -11 & 13 \\ 19 & 10 & 3 & 4 & -8 \\ -19 & -10 & -3 & -4 & 8 \\ 7 & 4 & 3 & 1 & -5 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 1 & 0 \\ 0 & \boxed{1} & 0 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & \boxed{1} & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{E}_E(2) = \mathcal{N}(E - 2I_5) = \left\langle\left\{\begin{bmatrix}-1\\ \frac{3}{2}\\ 0\\ 1\\ 0\end{bmatrix}, \begin{bmatrix}0\\ \frac{1}{2}\\ 1\\ 0\\ 1\end{bmatrix}\right\}\right\rangle = \left\langle\left\{\begin{bmatrix}-2\\ 3\\ 0\\ 2\\ 0\end{bmatrix}, \begin{bmatrix}0\\ 1\\ 2\\ 0\\ 2\end{bmatrix}\right\}\right\rangle$$

$\lambda = -1$

$$E + 1I_5 = \begin{bmatrix}30 & 14 & 2 & 6 & -9\\ -47 & -21 & -1 & -11 & 13\\ 19 & 10 & 6 & 4 & -8\\ -19 & -10 & -3 & -1 & 8\\ 7 & 4 & 3 & 1 & -2\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}\boxed{1} & 0 & 0 & 2 & 0\\ 0 & \boxed{1} & 0 & -4 & 0\\ 0 & 0 & \boxed{1} & 1 & 0\\ 0 & 0 & 0 & 0 & \boxed{1}\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

$$\mathcal{E}_E(-1) = \mathcal{N}(E + 1I_5) = \left\langle\left\{\begin{bmatrix}-2\\ 4\\ -1\\ 1\\ 0\end{bmatrix}\right\}\right\rangle$$

So the eigenspace dimensions yield geometric multiplicities $\gamma_E(2) = 2$ and $\gamma_E(-1) = 1$. This example is of interest because $\lambda = 2$ has such a large algebraic multiplicity, which is also not equal to its geometric multiplicity. △

Example CEMS6 Complex eigenvalues, matrix of size 6
Consider the matrix

$$F = \begin{bmatrix}-59 & -34 & 41 & 12 & 25 & 30\\ 1 & 7 & -46 & -36 & -11 & -29\\ -233 & -119 & 58 & -35 & 75 & 54\\ 157 & 81 & -43 & 21 & -51 & -39\\ -91 & -48 & 32 & -5 & 32 & 26\\ 209 & 107 & -55 & 28 & -69 & -50\end{bmatrix}$$

then

$$\begin{aligned} p_F(x) &= -50 + 55x + 13x^2 - 50x^3 + 32x^4 - 9x^5 + x^6\\ &= (x-2)(x+1)(x^2 - 4x + 5)^2\\ &= (x-2)(x+1)((x-(2+i))(x-(2-i)))^2\\ &= (x-2)(x+1)(x-(2+i))^2(x-(2-i))^2 \end{aligned}$$

So the eigenvalues are $\lambda = 2,\ -1, 2+i,\ 2-i$ with algebraic multiplicities $\alpha_F(2) = 1$, $\alpha_F(-1) = 1$, $\alpha_F(2+i) = 2$ and $\alpha_F(2-i) = 2$.

We compute eigenvectors, noting that the last two basis vectors are each a scalar multiple of what Theorem BNS will provide,

$\lambda = 2 \qquad F - 2I_6 =$

$$\begin{bmatrix}-61 & -34 & 41 & 12 & 25 & 30\\ 1 & 5 & -46 & -36 & -11 & -29\\ -233 & -119 & 56 & -35 & 75 & 54\\ 157 & 81 & -43 & 19 & -51 & -39\\ -91 & -48 & 32 & -5 & 30 & 26\\ 209 & 107 & -55 & 28 & -69 & -52\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}\boxed{1} & 0 & 0 & 0 & 0 & \frac{1}{5}\\ 0 & \boxed{1} & 0 & 0 & 0 & 0\\ 0 & 0 & \boxed{1} & 0 & 0 & \frac{3}{5}\\ 0 & 0 & 0 & \boxed{1} & 0 & -\frac{1}{5}\\ 0 & 0 & 0 & 0 & \boxed{1} & \frac{4}{5}\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

$$\mathcal{E}_F(2) = \mathcal{N}(F - 2I_6) = \left\langle\left\{\begin{bmatrix}-\frac{1}{5}\\0\\-\frac{3}{5}\\\frac{1}{5}\\-\frac{4}{5}\\1\end{bmatrix}\right\}\right\rangle = \left\langle\left\{\begin{bmatrix}-1\\0\\-3\\1\\-4\\5\end{bmatrix}\right\}\right\rangle$$

$\lambda = -1 \qquad F + 1I_6 =$

$$\begin{bmatrix}-58 & -34 & 41 & 12 & 25 & 30\\1 & 8 & -46 & -36 & -11 & -29\\-233 & -119 & 59 & -35 & 75 & 54\\157 & 81 & -43 & 22 & -51 & -39\\-91 & -48 & 32 & -5 & 33 & 26\\209 & 107 & -55 & 28 & -69 & -49\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix}\boxed{1} & 0 & 0 & 0 & 0 & \frac{1}{2}\\0 & \boxed{1} & 0 & 0 & 0 & -\frac{3}{2}\\0 & 0 & \boxed{1} & 0 & 0 & \frac{1}{2}\\0 & 0 & 0 & \boxed{1} & 0 & 0\\0 & 0 & 0 & 0 & \boxed{1} & -\frac{1}{2}\\0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

$$\mathcal{E}_F(-1) = \mathcal{N}(F + I_6) = \left\langle\left\{\begin{bmatrix}-\frac{1}{2}\\\frac{3}{2}\\-\frac{1}{2}\\0\\\frac{1}{2}\\1\end{bmatrix}\right\}\right\rangle = \left\langle\left\{\begin{bmatrix}-1\\3\\-1\\0\\1\\2\end{bmatrix}\right\}\right\rangle$$

$\lambda = 2 + i$

$$F - (2+i)I_6 = \begin{bmatrix}-61-i & -34 & 41 & 12 & 25 & 30\\1 & 5-i & -46 & -36 & -11 & -29\\-233 & -119 & 56-i & -35 & 75 & 54\\157 & 81 & -43 & 19-i & -51 & -39\\-91 & -48 & 32 & -5 & 30-i & 26\\209 & 107 & -55 & 28 & -69 & -52-i\end{bmatrix}$$

$$\xrightarrow{\text{RREF}} \begin{bmatrix}\boxed{1} & 0 & 0 & 0 & 0 & \frac{1}{5}(7+i)\\0 & \boxed{1} & 0 & 0 & 0 & \frac{1}{5}(-9-2i)\\0 & 0 & \boxed{1} & 0 & 0 & 1\\0 & 0 & 0 & \boxed{1} & 0 & -1\\0 & 0 & 0 & 0 & \boxed{1} & 1\\0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

$$\mathcal{E}_F(2+i) = \mathcal{N}(F - (2+i)I_6) = \left\langle\left\{\begin{bmatrix}-7-i\\9+2i\\-5\\5\\-5\\5\end{bmatrix}\right\}\right\rangle$$

$\lambda = 2 - i$

$$F - (2-i)I_6 = \begin{bmatrix}-61+i & -34 & 41 & 12 & 25 & 30\\1 & 5+i & -46 & -36 & -11 & -29\\-233 & -119 & 56+i & -35 & 75 & 54\\157 & 81 & -43 & 19+i & -51 & -39\\-91 & -48 & 32 & -5 & 30+i & 26\\209 & 107 & -55 & 28 & -69 & -52+i\end{bmatrix}$$

$$\xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 0 & \frac{1}{5}(7-i) \\ 0 & \boxed{1} & 0 & 0 & 0 & \frac{1}{5}(-9+2i) \\ 0 & 0 & \boxed{1} & 0 & 0 & 1 \\ 0 & 0 & 0 & \boxed{1} & 0 & -1 \\ 0 & 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{E}_F(2-i) = \mathcal{N}(F-(2-i)I_6) = \left\langle \left\{ \begin{bmatrix} -7+i \\ 9-2i \\ -5 \\ 5 \\ -5 \\ 5 \end{bmatrix} \right\} \right\rangle$$

Eigenspace dimensions yield geometric multiplicities of $\gamma_F(2) = 1$, $\gamma_F(-1) = 1$, $\gamma_F(2+i) = 1$ and $\gamma_F(2-i) = 1$. This example demonstrates some of the possibilities for the appearance of complex eigenvalues, even when all the entries of the matrix are real. Notice how all the numbers in the analysis of $\lambda = 2-i$ are conjugates of the corresponding number in the analysis of $\lambda = 2+i$. This is the content of the upcoming Theorem ERMCP. △

Example DEMS5 Distinct eigenvalues, matrix of size 5
Consider the matrix

$$H = \begin{bmatrix} 15 & 18 & -8 & 6 & -5 \\ 5 & 3 & 1 & -1 & -3 \\ 0 & -4 & 5 & -4 & -2 \\ -43 & -46 & 17 & -14 & 15 \\ 26 & 30 & -12 & 8 & -10 \end{bmatrix}$$

then

$$p_H(x) = -6x + x^2 + 7x^3 - x^4 - x^5 = x(x-2)(x-1)(x+1)(x+3)$$

So the eigenvalues are $\lambda = 2, 1, 0, -1, -3$ with algebraic multiplicities $\alpha_H(2) = 1$, $\alpha_H(1) = 1$, $\alpha_H(0) = 1$, $\alpha_H(-1) = 1$ and $\alpha_H(-3) = 1$.

Computing eigenvectors,

$\lambda = 2$

$$H - 2I_5 = \begin{bmatrix} 13 & 18 & -8 & 6 & -5 \\ 5 & 1 & 1 & -1 & -3 \\ 0 & -4 & 3 & -4 & -2 \\ -43 & -46 & 17 & -16 & 15 \\ 26 & 30 & -12 & 8 & -12 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 0 & -1 \\ 0 & \boxed{1} & 0 & 0 & 1 \\ 0 & 0 & \boxed{1} & 0 & 2 \\ 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{E}_H(2) = \mathcal{N}(H - 2I_5) = \left\langle \left\{ \begin{bmatrix} 1 \\ -1 \\ -2 \\ -1 \\ 1 \end{bmatrix} \right\} \right\rangle$$

$\lambda = 1$

$$H - 1I_5 = \begin{bmatrix} 14 & 18 & -8 & 6 & -5 \\ 5 & 2 & 1 & -1 & -3 \\ 0 & -4 & 4 & -4 & -2 \\ -43 & -46 & 17 & -15 & 15 \\ 26 & 30 & -12 & 8 & -11 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{E}_{H}\left(1\right)=\mathcal{N}(H-1I_5)=\left\langle\left\{\begin{bmatrix}\frac{1}{2}\\0\\-\frac{1}{2}\\-1\\1\end{bmatrix}\right\}\right\rangle=\left\langle\left\{\begin{bmatrix}1\\0\\-1\\-2\\2\end{bmatrix}\right\}\right\rangle$$

$\lambda=0$

$$H-0I_5=\begin{bmatrix}15&18&-8&6&-5\\5&3&1&-1&-3\\0&-4&5&-4&-2\\-43&-46&17&-14&15\\26&30&-12&8&-10\end{bmatrix}\xrightarrow{\text{RREF}}\begin{bmatrix}\boxed{1}&0&0&0&1\\0&\boxed{1}&0&0&-2\\0&0&\boxed{1}&0&-2\\0&0&0&\boxed{1}&0\\0&0&0&0&0\end{bmatrix}$$

$$\mathcal{E}_{H}\left(0\right)=\mathcal{N}(H-0I_5)=\left\langle\left\{\begin{bmatrix}-1\\2\\2\\0\\1\end{bmatrix}\right\}\right\rangle$$

$\lambda=-1$

$$H+1I_5=\begin{bmatrix}16&18&-8&6&-5\\5&4&1&-1&-3\\0&-4&6&-4&-2\\-43&-46&17&-13&15\\26&30&-12&8&-9\end{bmatrix}\xrightarrow{\text{RREF}}\begin{bmatrix}\boxed{1}&0&0&0&-1/2\\0&\boxed{1}&0&0&0\\0&0&\boxed{1}&0&0\\0&0&0&\boxed{1}&1/2\\0&0&0&0&0\end{bmatrix}$$

$$\mathcal{E}_{H}\left(-1\right)=\mathcal{N}(H+1I_5)=\left\langle\left\{\begin{bmatrix}\frac{1}{2}\\0\\0\\-\frac{1}{2}\\1\end{bmatrix}\right\}\right\rangle=\left\langle\left\{\begin{bmatrix}1\\0\\0\\-1\\2\end{bmatrix}\right\}\right\rangle$$

$\lambda=-3$

$$H+3I_5=\begin{bmatrix}18&18&-8&6&-5\\5&6&1&-1&-3\\0&-4&8&-4&-2\\-43&-46&17&-11&15\\26&30&-12&8&-7\end{bmatrix}\xrightarrow{\text{RREF}}\begin{bmatrix}\boxed{1}&0&0&0&-1\\0&\boxed{1}&0&0&\frac{1}{2}\\0&0&\boxed{1}&0&1\\0&0&0&\boxed{1}&2\\0&0&0&0&0\end{bmatrix}$$

$$\mathcal{E}_{H}\left(-3\right)=\mathcal{N}(H+3I_5)=\left\langle\left\{\begin{bmatrix}1\\-\frac{1}{2}\\-1\\-2\\1\end{bmatrix}\right\}\right\rangle=\left\langle\left\{\begin{bmatrix}-2\\1\\2\\4\\-2\end{bmatrix}\right\}\right\rangle$$

So the eigenspace dimensions yield geometric multiplicities $\gamma_H\left(2\right)=1$, $\gamma_H\left(1\right)=1$, $\gamma_H\left(0\right)=1$, $\gamma_H\left(-1\right)=1$ and $\gamma_H\left(-3\right)=1$, identical to the algebraic multiplicities. This example is of interest for two reasons. First, $\lambda=0$ is an eigenvalue, illustrating the upcoming Theorem SMZE. Second, all the eigenvalues are distinct, yielding algebraic and geometric multiplicities of 1 for each eigenvalue, illustrating Theorem DED. △

Reading Questions

1. Suppose A is the 2×2 matrix

$$A = \begin{bmatrix} -5 & 8 \\ -4 & 7 \end{bmatrix}$$

   Find the eigenvalues of A.

2. For each eigenvalue of A, find the corresponding eigenspace.

3. For the polynomial $p(x) = 3x^2 - x + 2$ and A from above, compute $p(A)$.

Exercises

C10† Find the characteristic polynomial of the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

C11† Find the characteristic polynomial of the matrix $A = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$.

C12† Find the characteristic polynomial of the matrix $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{bmatrix}$.

C19† Find the eigenvalues, eigenspaces, algebraic multiplicities and geometric multiplicities for the matrix below. It is possible to do all these computations by hand, and it would be instructive to do so.

$$C = \begin{bmatrix} -1 & 2 \\ -6 & 6 \end{bmatrix}$$

C20† Find the eigenvalues, eigenspaces, algebraic multiplicities and geometric multiplicities for the matrix below. It is possible to do all these computations by hand, and it would be instructive to do so.

$$B = \begin{bmatrix} -12 & 30 \\ -5 & 13 \end{bmatrix}$$

C21† The matrix A below has $\lambda = 2$ as an eigenvalue. Find the geometric multiplicity of $\lambda = 2$ using your calculator only for row-reducing matrices.

$$A = \begin{bmatrix} 18 & -15 & 33 & -15 \\ -4 & 8 & -6 & 6 \\ -9 & 9 & -16 & 9 \\ 5 & -6 & 9 & -4 \end{bmatrix}$$

C22† Without using a calculator, find the eigenvalues of the matrix B.

$$B = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$$

C23† Find the eigenvalues, eigenspaces, algebraic and geometric multiplicities for $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

C24† Find the eigenvalues, eigenspaces, algebraic and geometric multiplicities for $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$.

C25† Find the eigenvalues, eigenspaces, algebraic and geometric multiplicities for the 3×3 identity matrix I_3. Do your results make sense?

C26† For matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$, the characteristic polynomial of A is $p_A(x) = (4 - x)(1 - x)^2$. Find the eigenvalues and corresponding eigenspaces of A.

C27† For matrix $A = \begin{bmatrix} 0 & 4 & -1 & 1 \\ -2 & 6 & -1 & 1 \\ -2 & 8 & -1 & -1 \\ -2 & 8 & -3 & 1 \end{bmatrix}$, the characteristic polynomial of A is $p_A(x) = (x+2)(x-2)^2(x-4)$. Find the eigenvalues and corresponding eigenspaces of A.

M60† Repeat Example CAEHW by choosing $\mathbf{x} = \begin{bmatrix} 0 \\ 8 \\ 2 \\ 1 \\ 2 \end{bmatrix}$ and then arrive at an eigenvalue and eigenvector of the matrix A. The hard way.

T10† A matrix A is idempotent if $A^2 = A$. Show that the only possible eigenvalues of an idempotent matrix are $\lambda = 0$ and $\lambda = 1$. Then give an example of a matrix that is idempotent and has both of these two values as eigenvalues.

T15† The characteristic polynomial of the square matrix A is usually defined as $r_A(x) = \det(xI_n - A)$. Find a specific relationship between our characteristic polynomial, $p_A(x)$, and $r_A(x)$, give a proof of your relationship, and use this to explain why Theorem EMRCP can remain essentially unchanged with either definition. Explain the advantages of each definition over the other. (Computing with both definitions, for a 2×2 and a 3×3 matrix, might be a good way to start.)

T20† Suppose that λ and ρ are two different eigenvalues of the square matrix A. Prove that the intersection of the eigenspaces for these two eigenvalues is trivial. That is, $\mathcal{E}_A(\lambda) \cap \mathcal{E}_A(\rho) = \{\mathbf{0}\}$.

Section PEE
Properties of Eigenvalues and Eigenvectors

The previous section introduced eigenvalues and eigenvectors, and concentrated on their existence and determination. This section will be more about theorems, and the various properties eigenvalues and eigenvectors enjoy. Like a good 4×100 meter relay, we will lead-off with one of our better theorems and save the very best for the anchor leg.

Subsection BPE
Basic Properties of Eigenvalues

Theorem EDELI Eigenvectors with Distinct Eigenvalues are Linearly Independent
Suppose that A is an $n \times n$ square matrix and $S = \{\mathbf{x}_1,\, \mathbf{x}_2,\, \mathbf{x}_3,\, \ldots,\, \mathbf{x}_p\}$ is a set of eigenvectors with eigenvalues $\lambda_1,\, \lambda_2,\, \lambda_3,\, \ldots,\, \lambda_p$ such that $\lambda_i \neq \lambda_j$ whenever $i \neq j$. Then S is a linearly independent set.

Proof. If $p = 1$, then the set $S = \{\mathbf{x}_1\}$ is linearly independent since eigenvectors are nonzero (Definition EEM), so assume for the remainder that $p \geq 2$.

We will prove this result by contradiction (Proof Technique CD). Suppose to the contrary that S is a linearly dependent set. Define $S_i = \{\mathbf{x}_1,\, \mathbf{x}_2,\, \mathbf{x}_3,\, \ldots,\, \mathbf{x}_i\}$ and let k be an integer such that $S_{k-1} = \{\mathbf{x}_1,\, \mathbf{x}_2,\, \mathbf{x}_3,\, \ldots,\, \mathbf{x}_{k-1}\}$ is linearly independent and $S_k = \{\mathbf{x}_1,\, \mathbf{x}_2,\, \mathbf{x}_3,\, \ldots,\, \mathbf{x}_k\}$ is linearly dependent. We have to ask if there is even such an integer k? First, since eigenvectors are nonzero, the set $\{\mathbf{x}_1\}$ is linearly independent. Since we are assuming that $S = S_p$ is linearly dependent, there must be an integer k, $2 \leq k \leq p$, where the sets S_i transition from linear independence to linear dependence (and stay that way). In other words, $\mathbf{x}_k$ is the vector with the smallest index that is a linear combination of just vectors with smaller indices.

Since $\{\mathbf{x}_1,\, \mathbf{x}_2,\, \mathbf{x}_3,\, \ldots,\, \mathbf{x}_k\}$ is a linearly dependent set there must be scalars, $a_1,\, a_2,\, a_3,\, \ldots,\, a_k$, not all zero (Definition LI), so that

$$\mathbf{0} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + a_3\mathbf{x}_3 + \cdots + a_k\mathbf{x}_k$$

Then,

$$\begin{aligned}
\mathbf{0} &= (A - \lambda_k I_n)\,\mathbf{0} && \text{Theorem ZVSM}\\
&= (A - \lambda_k I_n)\,(a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + a_3\mathbf{x}_3 + \cdots + a_k\mathbf{x}_k) && \text{Definition RLD}\\
&= (A - \lambda_k I_n)\,a_1\mathbf{x}_1 + \cdots + (A - \lambda_k I_n)\,a_k\mathbf{x}_k && \text{Theorem MMDAA}\\
&= a_1\,(A - \lambda_k I_n)\,\mathbf{x}_1 + \cdots + a_k\,(A - \lambda_k I_n)\,\mathbf{x}_k && \text{Theorem MMSMM}\\
&= a_1\,(A\mathbf{x}_1 - \lambda_k I_n\mathbf{x}_1) + \cdots + a_k\,(A\mathbf{x}_k - \lambda_k I_n\mathbf{x}_k) && \text{Theorem MMDAA}\\
&= a_1\,(A\mathbf{x}_1 - \lambda_k\mathbf{x}_1) + \cdots + a_k\,(A\mathbf{x}_k - \lambda_k\mathbf{x}_k) && \text{Theorem MMIM}\\
&= a_1\,(\lambda_1\mathbf{x}_1 - \lambda_k\mathbf{x}_1) + \cdots + a_k\,(\lambda_k\mathbf{x}_k - \lambda_k\mathbf{x}_k) && \text{Definition EEM}\\
&= a_1\,(\lambda_1 - \lambda_k)\,\mathbf{x}_1 + \cdots + a_k\,(\lambda_k - \lambda_k)\,\mathbf{x}_k && \text{Theorem MMDAA}\\
&= a_1\,(\lambda_1 - \lambda_k)\,\mathbf{x}_1 + \cdots + a_{k-1}\,(\lambda_{k-1} - \lambda_k)\,\mathbf{x}_{k-1} + a_k\,(0)\,\mathbf{x}_k && \text{Property AICN}\\
&= a_1\,(\lambda_1 - \lambda_k)\,\mathbf{x}_1 + \cdots + a_{k-1}\,(\lambda_{k-1} - \lambda_k)\,\mathbf{x}_{k-1} + \mathbf{0} && \text{Theorem ZSSM}\\
&= a_1\,(\lambda_1 - \lambda_k)\,\mathbf{x}_1 + \cdots + a_{k-1}\,(\lambda_{k-1} - \lambda_k)\,\mathbf{x}_{k-1} && \text{Property Z}
\end{aligned}$$

This equation is a relation of linear dependence on the linearly independent set $\{\mathbf{x}_1,\, \mathbf{x}_2,\, \mathbf{x}_3,\, \ldots,\, \mathbf{x}_{k-1}\}$, so the scalars must all be zero. That is, $a_i\,(\lambda_i - \lambda_k) = 0$ for $1 \leq i \leq k-1$. However, we have the hypothesis that the eigenvalues are distinct, so $\lambda_i \neq \lambda_k$ for $1 \leq i \leq k-1$. Thus $a_i = 0$ for $1 \leq i \leq k-1$.

This reduces the original relation of linear dependence on $\{\mathbf{x}_1,\, \mathbf{x}_2,\, \mathbf{x}_3,\, \ldots,\, \mathbf{x}_k\}$ to the simpler equation $a_k\mathbf{x}_k = \mathbf{0}$. By Theorem SMEZV we conclude that $a_k = 0$ or $\mathbf{x}_k = \mathbf{0}$. Eigenvectors are never the zero vector (Definition EEM), so $a_k = 0$. So all of the scalars a_i, $1 \leq i \leq k$ are zero, contradicting their introduction as the scalars creating a nontrivial relation of linear dependence on the set $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_k\}$. With a contradiction in hand, we conclude that S must be linearly independent. ■

There is a simple connection between the eigenvalues of a matrix and whether or not the matrix is nonsingular.

Theorem SMZE Singular Matrices have Zero Eigenvalues
Suppose A is a square matrix. Then A is singular if and only if $\lambda = 0$ is an eigenvalue of A.

Proof. We have the following equivalences:

$$\begin{aligned} A \text{ is singular} &\iff \text{there exists } \mathbf{x} \neq \mathbf{0},\ A\mathbf{x} = \mathbf{0} && \text{Definition NM} \\ &\iff \text{there exists } \mathbf{x} \neq \mathbf{0},\ A\mathbf{x} = 0\mathbf{x} && \text{Theorem ZSSM} \\ &\iff \lambda = 0 \text{ is an eigenvalue of } A && \text{Definition EEM} \end{aligned}$$

■

With an equivalence about singular matrices we can update our list of equivalences about nonsingular matrices.

Theorem NME8 Nonsingular Matrix Equivalences, Round 8
Suppose that A is a square matrix of size n. The following are equivalent.

1. *A is nonsingular.*
2. *A row-reduces to the identity matrix.*
3. *The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.*
4. *The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.*
5. *The columns of A are a linearly independent set.*
6. *A is invertible.*
7. *The column space of A is $\mathbb{C}^n$, $\mathcal{C}(A) = \mathbb{C}^n$.*
8. *The columns of A are a basis for $\mathbb{C}^n$.*
9. *The rank of A is n, $r(A) = n$.*
10. *The nullity of A is zero, $n(A) = 0$.*
11. *The determinant of A is nonzero, $\det(A) \neq 0$.*
12. *$\lambda = 0$ is not an eigenvalue of A.*

Proof. The equivalence of the first and last statements is Theorem SMZE, reformulated by negating each statement in the equivalence. So we are able to improve on Theorem NME7 with this addition. ■

Certain changes to a matrix change its eigenvalues in a predictable way.

Theorem ESMM Eigenvalues of a Scalar Multiple of a Matrix
Suppose A is a square matrix and λ is an eigenvalue of A. Then $\alpha\lambda$ is an eigenvalue of αA.

Proof. Let $\mathbf{x}\neq\mathbf{0}$ be one eigenvector of A for λ. Then

$$\begin{aligned}(\alpha A)\,\mathbf{x} &= \alpha\,(A\mathbf{x}) && \text{Theorem MMSMM}\\ &= \alpha\,(\lambda\mathbf{x}) && \mathbf{x}\text{ eigenvector of }A\\ &= (\alpha\lambda)\,\mathbf{x} && \text{Property SMAC}\end{aligned}$$

So $\mathbf{x}\neq\mathbf{0}$ is an eigenvector of αA for the eigenvalue $\alpha\lambda$. ■

Unfortunately, there are not parallel theorems about the sum or product of arbitrary matrices. But we can prove a similar result for powers of a matrix.

Theorem EOMP Eigenvalues Of Matrix Powers
Suppose A is a square matrix, λ is an eigenvalue of A, and $s\geq 0$ is an integer. Then λ^s is an eigenvalue of A^s.

Proof. Let $\mathbf{x}\neq\mathbf{0}$ be one eigenvector of A for λ. Suppose A has size n. Then we proceed by induction on s (Proof Technique I). First, for $s=0$,

$$\begin{aligned}A^s\mathbf{x} &= A^0\mathbf{x}\\ &= I_n\mathbf{x}\\ &= \mathbf{x} && \text{Theorem MMIM}\\ &= 1\mathbf{x} && \text{Property OC}\\ &= \lambda^0\mathbf{x}\\ &= \lambda^s\mathbf{x}\end{aligned}$$

so λ^s is an eigenvalue of A^s in this special case. If we assume the theorem is true for s, then we find

$$\begin{aligned}A^{s+1}\mathbf{x} &= A^sA\mathbf{x}\\ &= A^s\,(\lambda\mathbf{x}) && \mathbf{x}\text{ eigenvector of }A\text{ for }\lambda\\ &= \lambda\,(A^s\mathbf{x}) && \text{Theorem MMSMM}\\ &= \lambda\,(\lambda^s\mathbf{x}) && \text{Induction hypothesis}\\ &= (\lambda\lambda^s)\,\mathbf{x} && \text{Property SMAC}\\ &= \lambda^{s+1}\mathbf{x}\end{aligned}$$

So $\mathbf{x}\neq\mathbf{0}$ is an eigenvector of A^{s+1} for λ^{s+1}, and induction tells us the theorem is true for all $s\geq 0$. ■

While we cannot prove that the sum of two arbitrary matrices behaves in any reasonable way with regard to eigenvalues, we can work with the sum of dissimilar powers of the *same* matrix. We have already seen two connections between eigenvalues and polynomials, in the proof of Theorem EMHE and the characteristic polynomial (Definition CP). Our next theorem strengthens this connection.

Theorem EPM Eigenvalues of the Polynomial of a Matrix
Suppose A is a square matrix and λ is an eigenvalue of A. Let $q(x)$ be a polynomial in the variable x. Then $q(\lambda)$ is an eigenvalue of the matrix $q(A)$.

Proof. Let $\mathbf{x}\neq\mathbf{0}$ be one eigenvector of A for λ, and write $q(x)=a_0+a_1x+a_2x^2+\cdots+a_mx^m$. Then

$$\begin{aligned}q(A)\mathbf{x} &= \left(a_0A^0+a_1A^1+a_2A^2+\cdots+a_mA^m\right)\mathbf{x}\\ &= (a_0A^0)\mathbf{x}+(a_1A^1)\mathbf{x}+(a_2A^2)\mathbf{x}+\cdots+(a_mA^m)\mathbf{x} && \text{Theorem MMDAA}\\ &= a_0(A^0\mathbf{x})+a_1(A^1\mathbf{x})+a_2(A^2\mathbf{x})+\cdots+a_m(A^m\mathbf{x}) && \text{Theorem MMSMM}\\ &= a_0(\lambda^0\mathbf{x})+a_1(\lambda^1\mathbf{x})+a_2(\lambda^2\mathbf{x})+\cdots+a_m(\lambda^m\mathbf{x}) && \text{Theorem EOMP}\end{aligned}$$

$$
\begin{aligned}
&= (a_0\lambda^0)\mathbf{x} + (a_1\lambda^1)\mathbf{x} + (a_2\lambda^2)\mathbf{x} + \cdots + (a_m\lambda^m)\mathbf{x} && \text{Property SMAC}\\
&= \left(a_0\lambda^0 + a_1\lambda^1 + a_2\lambda^2 + \cdots + a_m\lambda^m\right)\mathbf{x} && \text{Property DSAC}\\
&= q(\lambda)\mathbf{x}
\end{aligned}
$$

So $\mathbf{x} \neq 0$ is an eigenvector of $q(A)$ for the eigenvalue $q(\lambda)$. ■

Example BDE Building desired eigenvalues
In Example ESMS4 the 4×4 symmetric matrix

$$C = \begin{bmatrix} 1 & 0 & 1 & 1\\ 0 & 1 & 1 & 1\\ 1 & 1 & 1 & 0\\ 1 & 1 & 0 & 1 \end{bmatrix}$$

is shown to have the three eigenvalues $\lambda = 3,\ 1,\ -1$. Suppose we wanted a 4×4 matrix that has the three eigenvalues $\lambda = 4,\ 0,\ -2$. We can employ Theorem EPM by finding a polynomial that converts 3 to 4, 1 to 0, and -1 to -2. Such a polynomial is called an **interpolating polynomial**, and in this example we can use

$$r(x) = \frac{1}{4}x^2 + x - \frac{5}{4}$$

We will not discuss how to concoct this polynomial, but a text on numerical analysis should provide the details. For now, simply verify that $r(3) = 4$, $r(1) = 0$ and $r(-1) = -2$.

Now compute

$$
\begin{aligned}
r(C) &= \frac{1}{4}C^2 + C - \frac{5}{4}I_4\\
&= \frac{1}{4}\begin{bmatrix} 3 & 2 & 2 & 2\\ 2 & 3 & 2 & 2\\ 2 & 2 & 3 & 2\\ 2 & 2 & 2 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 & 1\\ 0 & 1 & 1 & 1\\ 1 & 1 & 1 & 0\\ 1 & 1 & 0 & 1 \end{bmatrix} - \frac{5}{4}\begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 3 & 3\\ 1 & 1 & 3 & 3\\ 3 & 3 & 1 & 1\\ 3 & 3 & 1 & 1 \end{bmatrix}
\end{aligned}
$$

Theorem EPM tells us that if $r(x)$ transforms the eigenvalues in the desired manner, then $r(C)$ will have the desired eigenvalues. You can check this by computing the eigenvalues of $r(C)$ directly. Furthermore, notice that the multiplicities are the same, and the eigenspaces of C and $r(C)$ are identical. △

Inverses and transposes also behave predictably with regard to their eigenvalues.

Theorem EIM Eigenvalues of the Inverse of a Matrix
Suppose A is a square nonsingular matrix and λ is an eigenvalue of A. Then λ^{-1} is an eigenvalue of the matrix A^{-1}.

Proof. Notice that since A is assumed nonsingular, A^{-1} exists by Theorem NI, but more importantly, $\lambda^{-1} = 1/\lambda$ does not involve division by zero since Theorem SMZE prohibits this possibility.

Let $\mathbf{x} \neq \mathbf{0}$ be one eigenvector of A for λ. Suppose A has size n. Then

$$
\begin{aligned}
A^{-1}\mathbf{x} &= A^{-1}(1\mathbf{x}) && \text{Property OC}\\
&= A^{-1}(\frac{1}{\lambda}\lambda\mathbf{x}) && \text{Property MICN}\\
&= \frac{1}{\lambda}A^{-1}(\lambda\mathbf{x}) && \text{Theorem MMSMM}\\
&= \frac{1}{\lambda}A^{-1}(A\mathbf{x}) && \text{Definition EEM}\\
&= \frac{1}{\lambda}(A^{-1}A)\mathbf{x} && \text{Theorem MMA}\\
&= \frac{1}{\lambda}I_n\mathbf{x} && \text{Definition MI}
\end{aligned}
$$

$$= \frac{1}{\lambda}\mathbf{x} \qquad \text{Theorem MMIM}$$

So $\mathbf{x} \neq 0$ is an eigenvector of A^{-1} for the eigenvalue $\frac{1}{\lambda}$. ■

The proofs of the theorems above have a similar style to them. They all begin by grabbing an eigenvalue-eigenvector pair and adjusting it in some way to reach the desired conclusion. You should add this to your toolkit as a general approach to proving theorems about eigenvalues.

So far we have been able to reserve the characteristic polynomial for strictly computational purposes. However, sometimes a theorem about eigenvalues can be proved easily by employing the characteristic polynomial (rather than using an eigenvalue-eigenvector pair). The next theorem is an example of this.

Theorem ETM Eigenvalues of the Transpose of a Matrix
Suppose A is a square matrix and λ is an eigenvalue of A. Then λ is an eigenvalue of the matrix A^t.

Proof. Suppose A has size n. Then

$$\begin{aligned}
p_A(x) &= \det(A - xI_n) && \text{Definition CP}\\
&= \det\left((A - xI_n)^t\right) && \text{Theorem DT}\\
&= \det\left(A^t - (xI_n)^t\right) && \text{Theorem TMA}\\
&= \det\left(A^t - xI_n^t\right) && \text{Theorem TMSM}\\
&= \det\left(A^t - xI_n\right) && \text{Definition IM}\\
&= p_{A^t}(x) && \text{Definition CP}
\end{aligned}$$

So A and A^t have the same characteristic polynomial, and by Theorem EMRCP, their eigenvalues are identical and have equal algebraic multiplicities. Notice that what we have proved here is a bit stronger than the stated conclusion in the theorem. ■

If a matrix has only real entries, then the computation of the characteristic polynomial (Definition CP) will result in a polynomial with coefficients that are real numbers. Complex numbers could result as roots of this polynomial, but they are roots of quadratic factors with real coefficients, and as such, come in conjugate pairs. The next theorem proves this, and a bit more, without mentioning the characteristic polynomial.

Theorem ERMCP Eigenvalues of Real Matrices come in Conjugate Pairs
Suppose A is a square matrix with real entries and $\mathbf{x}$ is an eigenvector of A for the eigenvalue λ. Then $\overline{\mathbf{x}}$ is an eigenvector of A for the eigenvalue $\overline{\lambda}$.

Proof.

$$\begin{aligned}
A\overline{\mathbf{x}} &= \overline{A}\overline{\mathbf{x}} && A \text{ has real entries}\\
&= \overline{A\mathbf{x}} && \text{Theorem MMCC}\\
&= \overline{\lambda\mathbf{x}} && \mathbf{x} \text{ eigenvector of } A\\
&= \overline{\lambda}\overline{\mathbf{x}} && \text{Theorem CRSM}
\end{aligned}$$

So $\overline{\mathbf{x}}$ is an eigenvector of A for the eigenvalue $\overline{\lambda}$. ■

This phenomenon is amply illustrated in Example CEMS6, where the four complex eigenvalues come in two pairs, and the two basis vectors of the eigenspaces are complex conjugates of each other. Theorem ERMCP can be a time-saver for computing eigenvalues and eigenvectors of real matrices with complex eigenvalues, since the conjugate eigenvalue and eigenspace can be inferred from the theorem rather than computed.

Subsection ME
Multiplicities of Eigenvalues

A polynomial of degree n will have exactly n roots. From this fact about polynomial equations we can say more about the algebraic multiplicities of eigenvalues.

Theorem DCP Degree of the Characteristic Polynomial
Suppose that A is a square matrix of size n. Then the characteristic polynomial of A, $p_A(x)$, has degree n.

Proof. We will prove a more general result by induction (Proof Technique I). Then the theorem will be true as a special case. We will carefully state this result as a proposition indexed by m, $m \geq 1$.

$P(m)$: Suppose that A is an $m \times m$ matrix whose entries are complex numbers or linear polynomials in the variable x of the form $c - x$, where c is a complex number. Suppose further that there are exactly k entries that contain x and that no row or column contains more than one such entry. Then, when $k = m$, $\det(A)$ is a polynomial in x of degree m, with leading coefficient ± 1, and when $k < m$, $\det(A)$ is a polynomial in x of degree k or less.

Base Case: Suppose A is a 1×1 matrix. Then its determinant is equal to the lone entry (Definition DM). When $k = m = 1$, the entry is of the form $c - x$, a polynomial in x of degree $m = 1$ with leading coefficient -1. When $k < m$, then $k = 0$ and the entry is simply a complex number, a polynomial of degree $0 \leq k$. So $P(1)$ is true.

Induction Step: Assume $P(m)$ is true, and that A is an $(m+1) \times (m+1)$ matrix with k entries of the form $c - x$. There are two cases to consider.

Suppose $k = m + 1$. Then every row and every column will contain an entry of the form $c - x$. Suppose that for the first row, this entry is in column t. Compute the determinant of A by an expansion about this first row (Definition DM). The term associated with entry t of this row will be of the form

$$(c - x)(-1)^{1+t} \det(A(1|t))$$

The submatrix $A(1|t)$ is an $m \times m$ matrix with $k = m$ terms of the form $c - x$, no more than one per row or column. By the induction hypothesis, $\det(A(1|t))$ will be a polynomial in x of degree m with coefficient ± 1. So this entire term is then a polynomial of degree $m + 1$ with leading coefficient ± 1.

The remaining terms (which constitute the sum that is the determinant of A) are products of complex numbers from the first row with cofactors built from submatrices that lack the first row of A and lack some column of A, other than column t. As such, these submatrices are $m \times m$ matrices with $k = m - 1 < m$ entries of the form $c - x$, no more than one per row or column. Applying the induction hypothesis, we see that these terms are polynomials in x of degree $m - 1$ or less. Adding the single term from the entry in column t with all these others, we see that $\det(A)$ is a polynomial in x of degree $m + 1$ and leading coefficient ± 1.

The second case occurs when $k < m + 1$. Now there is a row of A that does not contain an entry of the form $c - x$. We consider the determinant of A by expanding about this row (Theorem DER), whose entries are all complex numbers. The cofactors employed are built from submatrices that are $m \times m$ matrices with either k or $k - 1$ entries of the form $c - x$, no more than one per row or column. In either case, $k \leq m$, and we can apply the induction hypothesis to see that the determinants computed for the cofactors are all polynomials of degree k or less. Summing these contributions to the determinant of A yields a polynomial in x of degree k or less, as desired.

Definition CP tells us that the characteristic polynomial of an $n \times n$ matrix is the determinant of a matrix having exactly n entries of the form $c - x$, no more than one per row or column. As such we can apply $P(n)$ to see that the characteristic polynomial has degree n. ■

Theorem NEM Number of Eigenvalues of a Matrix
Suppose that $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k$ are the distinct eigenvalues of a square matrix A of size n. Then

$$\sum_{i=1}^{k} \alpha_A(\lambda_i) = n$$

Proof. By the definition of the algebraic multiplicity (Definition AME), we can factor the characteristic polynomial as

$$p_A(x) = c(x - \lambda_1)^{\alpha_A(\lambda_1)}(x - \lambda_2)^{\alpha_A(\lambda_2)}(x - \lambda_3)^{\alpha_A(\lambda_3)} \cdots (x - \lambda_k)^{\alpha_A(\lambda_k)}$$

where c is a nonzero constant. (We could prove that $c = (-1)^n$, but we do not need that specificity right now. See Exercise PEE.T30) The left-hand side is a polynomial of degree n by Theorem DCP and the right-hand side is a polynomial of degree $\sum_{i=1}^{k} \alpha_A(\lambda_i)$. So the equality of the polynomials' degrees gives the equality $\sum_{i=1}^{k} \alpha_A(\lambda_i) = n$. ■

Theorem ME Multiplicities of an Eigenvalue
Suppose that A is a square matrix of size n and λ is an eigenvalue. Then

$$1 \leq \gamma_A(\lambda) \leq \alpha_A(\lambda) \leq n$$

Proof. Since λ is an eigenvalue of A, there is an eigenvector of A for λ, $\mathbf{x}$. Then $\mathbf{x} \in \mathcal{E}_A(\lambda)$, so $\gamma_A(\lambda) \geq 1$, since we can extend $\{\mathbf{x}\}$ into a basis of $\mathcal{E}_A(\lambda)$ (Theorem ELIS).

To show $\gamma_A(\lambda) \leq \alpha_A(\lambda)$ is the most involved portion of this proof. To this end, let $g = \gamma_A(\lambda)$ and let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_g$ be a basis for the eigenspace of λ, $\mathcal{E}_A(\lambda)$. Construct another $n - g$ vectors, $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots, \mathbf{y}_{n-g}$, so that

$$\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_g, \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots, \mathbf{y}_{n-g}\}$$

is a basis of $\mathbb{C}^n$. This can be done by repeated applications of Theorem ELIS.

Finally, define a matrix S by

$$S = [\mathbf{x}_1|\mathbf{x}_2|\mathbf{x}_3|\ldots|\mathbf{x}_g|\mathbf{y}_1|\mathbf{y}_2|\mathbf{y}_3|\ldots|\mathbf{y}_{n-g}] = [\mathbf{x}_1|\mathbf{x}_2|\mathbf{x}_3|\ldots|\mathbf{x}_g|R]$$

where R is an $n \times (n - g)$ matrix whose columns are $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots, \mathbf{y}_{n-g}$. The columns of S are linearly independent by design, so S is nonsingular (Theorem NMLIC) and therefore invertible (Theorem NI).

Then,

$$\begin{aligned}
[\mathbf{e}_1|\mathbf{e}_2|\mathbf{e}_3|\ldots|\mathbf{e}_n] &= I_n \\
&= S^{-1}S \\
&= S^{-1}[\mathbf{x}_1|\mathbf{x}_2|\mathbf{x}_3|\ldots|\mathbf{x}_g|R] \\
&= [S^{-1}\mathbf{x}_1|S^{-1}\mathbf{x}_2|S^{-1}\mathbf{x}_3|\ldots|S^{-1}\mathbf{x}_g|S^{-1}R]
\end{aligned}$$

So

$$S^{-1}\mathbf{x}_i = \mathbf{e}_i \quad 1 \leq i \leq g \qquad (*)$$

Preparations in place, we compute the characteristic polynomial of A,

$$p_A(x) = \det(A - xI_n) \qquad \text{Definition CP}$$

$$
\begin{aligned}
&= 1 \det(A - xI_n) && \text{Property OCN}\\
&= \det(I_n)\det(A - xI_n) && \text{Definition DM}\\
&= \det\left(S^{-1}S\right)\det(A - xI_n) && \text{Definition MI}\\
&= \det\left(S^{-1}\right)\det(S)\det(A - xI_n) && \text{Theorem DRMM}\\
&= \det\left(S^{-1}\right)\det(A - xI_n)\det(S) && \text{Property CMCN}\\
&= \det\left(S^{-1}(A - xI_n)S\right) && \text{Theorem DRMM}\\
&= \det\left(S^{-1}AS - S^{-1}xI_nS\right) && \text{Theorem MMDAA}\\
&= \det\left(S^{-1}AS - xS^{-1}I_nS\right) && \text{Theorem MMSMM}\\
&= \det\left(S^{-1}AS - xS^{-1}S\right) && \text{Theorem MMIM}\\
&= \det\left(S^{-1}AS - xI_n\right) && \text{Definition MI}\\
&= p_{S^{-1}AS}(x) && \text{Definition CP}
\end{aligned}
$$

What can we learn then about the matrix $S^{-1}AS$?

$$
\begin{aligned}
S^{-1}AS &= S^{-1}A[\mathbf{x}_1|\mathbf{x}_2|\mathbf{x}_3|\dots|\mathbf{x}_g|R]\\
&= S^{-1}[A\mathbf{x}_1|A\mathbf{x}_2|A\mathbf{x}_3|\dots|A\mathbf{x}_g|AR] && \text{Definition MM}\\
&= S^{-1}[\lambda\mathbf{x}_1|\lambda\mathbf{x}_2|\lambda\mathbf{x}_3|\dots|\lambda\mathbf{x}_g|AR] && \text{Definition EEM}\\
&= [S^{-1}\lambda\mathbf{x}_1|S^{-1}\lambda\mathbf{x}_2|S^{-1}\lambda\mathbf{x}_3|\dots|S^{-1}\lambda\mathbf{x}_g|S^{-1}AR] && \text{Definition MM}\\
&= [\lambda S^{-1}\mathbf{x}_1|\lambda S^{-1}\mathbf{x}_2|\lambda S^{-1}\mathbf{x}_3|\dots|\lambda S^{-1}\mathbf{x}_g|S^{-1}AR] && \text{Theorem MMSMM}\\
&= [\lambda\mathbf{e}_1|\lambda\mathbf{e}_2|\lambda\mathbf{e}_3|\dots|\lambda\mathbf{e}_g|S^{-1}AR] && S^{-1}S = I_n,\ ((*)\text{ above})
\end{aligned}
$$

Now imagine computing the characteristic polynomial of A by computing the characteristic polynomial of $S^{-1}AS$ using the form just obtained. The first g columns of $S^{-1}AS$ are all zero, save for a λ on the diagonal. So if we compute the determinant by expanding about the first column, successively, we will get successive factors of $(\lambda - x)$. More precisely, let T be the square matrix of size $n - g$ that is formed from the last $n - g$ rows and last $n - g$ columns of $S^{-1}AR$. Then

$$p_A(x) = p_{S^{-1}AS}(x) = (\lambda - x)^g p_T(x).$$

This says that $(x - \lambda)$ is a factor of the characteristic polynomial *at least* g times, so the algebraic multiplicity of λ as an eigenvalue of A is greater than or equal to g (Definition AME). In other words,

$$\gamma_A(\lambda) = g \leq \alpha_A(\lambda)$$

as desired.

Theorem NEM says that the sum of the algebraic multiplicities for *all* the eigenvalues of A is equal to n. Since the algebraic multiplicity is a positive quantity, no single algebraic multiplicity can exceed n without the sum of all of the algebraic multiplicities doing the same. ■

Theorem MNEM Maximum Number of Eigenvalues of a Matrix
Suppose that A is a square matrix of size n. Then A cannot have more than n distinct eigenvalues.

Proof. Suppose that A has k distinct eigenvalues, $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_k$. Then

$$
\begin{aligned}
k &= \sum_{i=1}^{k} 1\\
&\leq \sum_{i=1}^{k} \alpha_A(\lambda_i) && \text{Theorem ME}\\
&= n && \text{Theorem NEM}
\end{aligned}
$$

■

Subsection EHM
Eigenvalues of Hermitian Matrices

Recall that a matrix is Hermitian (or self-adjoint) if $A = A^*$ (Definition HM). In the case where A is a matrix whose entries are all real numbers, being Hermitian is identical to being symmetric (Definition SYM). Keep this in mind as you read the next two theorems. Their hypotheses could be changed to "suppose A is a real symmetric matrix."

Theorem HMRE Hermitian Matrices have Real Eigenvalues
Suppose that A is a Hermitian matrix and λ is an eigenvalue of A. Then $\lambda \in \mathbb{R}$.

Proof. Let $\mathbf{x} \neq \mathbf{0}$ be one eigenvector of A for the eigenvalue λ. Then by Theorem PIP we know $\langle \mathbf{x},\, \mathbf{x}\rangle \neq 0$. So

$$\begin{aligned}
\lambda &= \frac{1}{\langle \mathbf{x},\, \mathbf{x}\rangle}\lambda \langle \mathbf{x},\, \mathbf{x}\rangle && \text{Property MICN}\\
&= \frac{1}{\langle \mathbf{x},\, \mathbf{x}\rangle} \langle \mathbf{x},\, \lambda\mathbf{x}\rangle && \text{Theorem IPSM}\\
&= \frac{1}{\langle \mathbf{x},\, \mathbf{x}\rangle} \langle \mathbf{x},\, A\mathbf{x}\rangle && \text{Definition EEM}\\
&= \frac{1}{\langle \mathbf{x},\, \mathbf{x}\rangle} \langle A\mathbf{x},\, \mathbf{x}\rangle && \text{Theorem HMIP}\\
&= \frac{1}{\langle \mathbf{x},\, \mathbf{x}\rangle} \langle \lambda\mathbf{x},\, \mathbf{x}\rangle && \text{Definition EEM}\\
&= \frac{1}{\langle \mathbf{x},\, \mathbf{x}\rangle}\overline{\lambda} \langle \mathbf{x},\, \mathbf{x}\rangle && \text{Theorem IPSM}\\
&= \overline{\lambda} && \text{Property MICN}
\end{aligned}$$

If a complex number is equal to its conjugate, then it has a complex part equal to zero, and therefore is a real number. ■

Notice the appealing symmetry to the justifications given for the steps of this proof. In the center is the ability to pitch a Hermitian matrix from one side of the inner product to the other.

Look back and compare Example ESMS4 and Example CEMS6. In Example CEMS6 the matrix has only real entries, yet the characteristic polynomial has roots that are complex numbers, and so the matrix has complex eigenvalues. However, in Example ESMS4, the matrix has only real entries, but is also symmetric, and hence Hermitian. So by Theorem HMRE, we were guaranteed eigenvalues that are real numbers.

In many physical problems, a matrix of interest will be real and symmetric, or Hermitian. Then if the eigenvalues are to represent physical quantities of interest, Theorem HMRE guarantees that these values will not be complex numbers.

The eigenvectors of a Hermitian matrix also enjoy a pleasing property that we will exploit later.

Theorem HMOE Hermitian Matrices have Orthogonal Eigenvectors
Suppose that A is a Hermitian matrix and $\mathbf{x}$ and $\mathbf{y}$ are two eigenvectors of A for different eigenvalues. Then $\mathbf{x}$ and $\mathbf{y}$ are orthogonal vectors.

Proof. Let $\mathbf{x}$ be an eigenvector of A for λ and let $\mathbf{y}$ be an eigenvector of A for a different eigenvalue ρ. So we have $\lambda - \rho \neq 0$. Then

$$\begin{aligned}
\langle \mathbf{x},\,\mathbf{y}\rangle &= \frac{1}{\lambda-\rho}\left(\lambda-\rho\right)\langle \mathbf{x},\,\mathbf{y}\rangle && \text{Property MICN}\\
&= \frac{1}{\lambda-\rho}\left(\lambda\langle \mathbf{x},\,\mathbf{y}\rangle - \rho\langle \mathbf{x},\,\mathbf{y}\rangle\right) && \text{Property DCN}\\
&= \frac{1}{\lambda-\rho}\left(\langle \overline{\lambda}\mathbf{x},\,\mathbf{y}\rangle - \langle \mathbf{x},\,\rho\mathbf{y}\rangle\right) && \text{Theorem IPSM}\\
&= \frac{1}{\lambda-\rho}\left(\langle \lambda\mathbf{x},\,\mathbf{y}\rangle - \langle \mathbf{x},\,\rho\mathbf{y}\rangle\right) && \text{Theorem HMRE}\\
&= \frac{1}{\lambda-\rho}\left(\langle A\mathbf{x},\,\mathbf{y}\rangle - \langle \mathbf{x},\,A\mathbf{y}\rangle\right) && \text{Definition EEM}\\
&= \frac{1}{\lambda-\rho}\left(\langle A\mathbf{x},\,\mathbf{y}\rangle - \langle A\mathbf{x},\,\mathbf{y}\rangle\right) && \text{Theorem HMIP}\\
&= \frac{1}{\lambda-\rho}\left(0\right) && \text{Property AICN}\\
&= 0
\end{aligned}$$

This equality says that $\mathbf{x}$ and $\mathbf{y}$ are orthogonal vectors (Definition OV). ■

Notice again how the key step in this proof is the fundamental property of a Hermitian matrix (Theorem HMIP) — the ability to swap A across the two arguments of the inner product. We will build on these results and continue to see some more interesting properties in Section OD.

Reading Questions

1. How can you identify a nonsingular matrix just by looking at its eigenvalues?
2. How many different eigenvalues may a square matrix of size n have?
3. What is amazing about the eigenvalues of a Hermitian matrix and why is it amazing?

Exercises

T10† Suppose that A is a square matrix. Prove that the constant term of the characteristic polynomial of A is equal to the determinant of A.

T20† Suppose that A is a square matrix. Prove that a single vector may not be an eigenvector of A for two different eigenvalues.

T22 Suppose that U is a unitary matrix with eigenvalue λ. Prove that λ has modulus 1, i.e. $|\lambda| = 1$. This says that all of the eigenvalues of a unitary matrix lie on the unit circle of the complex plane.

T30 Theorem DCP tells us that the characteristic polynomial of a square matrix of size n has degree n. By suitably augmenting the proof of Theorem DCP prove that the coefficient of x^n in the characteristic polynomial is $(-1)^n$.

T50† Theorem EIM says that if λ is an eigenvalue of the nonsingular matrix A, then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}. Write an alternate proof of this theorem using the characteristic polynomial and without making reference to an eigenvector of A for λ.

Section SD
Similarity and Diagonalization

This section's topic will perhaps seem out of place at first, but we will make the connection soon with eigenvalues and eigenvectors. This is also our first look at one of the central ideas of Chapter R.

Subsection SM
Similar Matrices

The notion of matrices being "similar" is a lot like saying two matrices are row-equivalent. Two similar matrices are not equal, but they share many important properties. This section, and later sections in Chapter R will be devoted in part to discovering just what these common properties are.

First, the main definition for this section.

Definition SIM Similar Matrices
Suppose A and B are two square matrices of size n. Then A and B are **similar** if there exists a nonsingular matrix of size n, S, such that $A = S^{-1}BS$. □

We will say "A is similar to B via S" when we want to emphasize the role of S in the relationship between A and B. Also, it does not matter if we say A is similar to B, or B is similar to A. If one statement is true then so is the other, as can be seen by using S^{-1} in place of S (see Theorem SER for the careful proof). Finally, we will refer to $S^{-1}BS$ as a **similarity transformation** when we want to emphasize the way S changes B. OK, enough about language, let us build a few examples.

Example SMS5 Similar matrices of size 5
If you wondered if there are examples of similar matrices, then it will not be hard to convince you they exist. Define

$$B = \begin{bmatrix} -4 & 1 & -3 & -2 & 2 \\ 1 & 2 & -1 & 3 & -2 \\ -4 & 1 & 3 & 2 & 2 \\ -3 & 4 & -2 & -1 & -3 \\ 3 & 1 & -1 & 1 & -4 \end{bmatrix} \qquad S = \begin{bmatrix} 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & -1 & -2 & -1 \\ 1 & 3 & -1 & 1 & 1 \\ -2 & -3 & 3 & 1 & -2 \\ 1 & 3 & -1 & 2 & 1 \end{bmatrix}$$

Check that S is nonsingular and then compute

$$\begin{aligned} A &= S^{-1}BS \\ &= \begin{bmatrix} 10 & 1 & 0 & 2 & -5 \\ -1 & 0 & 1 & 0 & 0 \\ 3 & 0 & 2 & 1 & -3 \\ 0 & 0 & -1 & 0 & 1 \\ -4 & -1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} -4 & 1 & -3 & -2 & 2 \\ 1 & 2 & -1 & 3 & -2 \\ -4 & 1 & 3 & 2 & 2 \\ -3 & 4 & -2 & -1 & -3 \\ 3 & 1 & -1 & 1 & -4 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 & 1 & 1 \\ 0 & 1 & -1 & -2 & -1 \\ 1 & 3 & -1 & 1 & 1 \\ -2 & -3 & 3 & 1 & -2 \\ 1 & 3 & -1 & 2 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -10 & -27 & -29 & -80 & -25 \\ -2 & 6 & 6 & 10 & -2 \\ -3 & 11 & -9 & -14 & -9 \\ -1 & -13 & 0 & -10 & -1 \\ 11 & 35 & 6 & 49 & 19 \end{bmatrix} \end{aligned}$$

So by this construction, we know that A and B are similar. △

Let us do that again.

Example SMS3 Similar matrices of size 3
Define

$$B = \begin{bmatrix} -13 & -8 & -4 \\ 12 & 7 & 4 \\ 24 & 16 & 7 \end{bmatrix} \qquad S = \begin{bmatrix} 1 & 1 & 2 \\ -2 & -1 & -3 \\ 1 & -2 & 0 \end{bmatrix}$$

Check that S is nonsingular and then compute

$$\begin{aligned} A &= S^{-1}BS \\ &= \begin{bmatrix} -6 & -4 & -1 \\ -3 & -2 & -1 \\ 5 & 3 & 1 \end{bmatrix} \begin{bmatrix} -13 & -8 & -4 \\ 12 & 7 & 4 \\ 24 & 16 & 7 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ -2 & -1 & -3 \\ 1 & -2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} \end{aligned}$$

So by this construction, we know that A and B are similar. But before we move on, look at how pleasing the form of A is. Not convinced? Then consider that several computations related to A are especially easy. For example, in the spirit of Example DUTM, $\det(A) = (-1)(3)(-1) = 3$. Similarly, the characteristic polynomial is straightforward to compute by hand, $p_A(x) = (-1-x)(3-x)(-1-x) = -(x-3)(x+1)^2$ and since the result is already factored, the eigenvalues are transparently $\lambda = 3, -1$. Finally, the eigenvectors of A are just the standard unit vectors (Definition SUV). △

Subsection PSM
Properties of Similar Matrices

Similar matrices share many properties and it is these theorems that justify the choice of the word "similar." First we will show that similarity is an **equivalence relation**. Equivalence relations are important in the study of various algebras and can always be regarded as a kind of weak version of equality. Sort of alike, but not quite equal. The notion of two matrices being row-equivalent is an example of an equivalence relation we have been working with since the beginning of the course (see Exercise RREF.T11). Row-equivalent matrices are not equal, but they are a lot alike. For example, row-equivalent matrices have the same rank. Formally, an equivalence relation requires three conditions hold: reflexive, symmetric and transitive. We will illustrate these as we prove that similarity is an equivalence relation.

Theorem SER Similarity is an Equivalence Relation
Suppose A, B and C are square matrices of size n. Then

1. *A is similar to A. (Reflexive)*
2. *If A is similar to B, then B is similar to A. (Symmetric)*
3. *If A is similar to B and B is similar to C, then A is similar to C. (Transitive)*

Proof. To see that A is similar to A, we need only demonstrate a nonsingular matrix that effects a similarity transformation of A to A. I_n is nonsingular (since it row-reduces to the identity matrix, Theorem NMRRI), and

$$I_n^{-1}AI_n = I_nAI_n = A$$

If we assume that A is similar to B, then we know there is a nonsingular matrix S so that $A = S^{-1}BS$ by Definition SIM. By Theorem MIMI, S^{-1} is invertible, and by Theorem NI is therefore nonsingular. So

$$\begin{aligned} (S^{-1})^{-1}A(S^{-1}) &= SAS^{-1} && \text{Theorem MIMI} \\ &= SS^{-1}BSS^{-1} && \text{Definition SIM} \\ &= \left(SS^{-1}\right)B\left(SS^{-1}\right) && \text{Theorem MMA} \\ &= I_nBI_n && \text{Definition MI} \\ &= B && \text{Theorem MMIM} \end{aligned}$$

and we see that B is similar to A.

Assume that A is similar to B, and B is similar to C. This gives us the existence of two nonsingular matrices, S and R, such that $A = S^{-1}BS$ and $B = R^{-1}CR$, by Definition SIM. (Notice how we have to assume $S \neq R$, as will usually be the case.) Since S and R are invertible, so too RS is invertible by Theorem SS and then nonsingular by Theorem NI. Now

$$\begin{aligned}
(RS)^{-1}C(RS) &= S^{-1}R^{-1}CRS && \text{Theorem SS}\\
&= S^{-1}\left(R^{-1}CR\right)S && \text{Theorem MMA}\\
&= S^{-1}BS && \text{Definition SIM}\\
&= A
\end{aligned}$$

so A is similar to C via the nonsingular matrix RS. ■

Here is another theorem that tells us exactly what sorts of properties similar matrices share.

Theorem SMEE Similar Matrices have Equal Eigenvalues
Suppose A and B are similar matrices. Then the characteristic polynomials of A and B are equal, that is, $p_A(x) = p_B(x)$.

Proof. Let n denote the size of A and B. Since A and B are similar, there exists a nonsingular matrix S, such that $A = S^{-1}BS$ (Definition SIM). Then

$$\begin{aligned}
p_A(x) &= \det\left(A - xI_n\right) && \text{Definition CP}\\
&= \det\left(S^{-1}BS - xI_n\right) && \text{Definition SIM}\\
&= \det\left(S^{-1}BS - xS^{-1}I_nS\right) && \text{Theorem MMIM}\\
&= \det\left(S^{-1}BS - S^{-1}xI_nS\right) && \text{Theorem MMSMM}\\
&= \det\left(S^{-1}\left(B - xI_n\right)S\right) && \text{Theorem MMDAA}\\
&= \det\left(S^{-1}\right)\det\left(B - xI_n\right)\det\left(S\right) && \text{Theorem DRMM}\\
&= \det\left(S^{-1}\right)\det\left(S\right)\det\left(B - xI_n\right) && \text{Property CMCN}\\
&= \det\left(S^{-1}S\right)\det\left(B - xI_n\right) && \text{Theorem DRMM}\\
&= \det\left(I_n\right)\det\left(B - xI_n\right) && \text{Definition MI}\\
&= 1\det\left(B - xI_n\right) && \text{Definition DM}\\
&= p_B(x) && \text{Definition CP}
\end{aligned}$$

■

So similar matrices not only have the same *set* of eigenvalues, the algebraic multiplicities of these eigenvalues will also be the same. However, be careful with this theorem. It is tempting to think the converse is true, and argue that if two matrices have the same eigenvalues, then they are similar. Not so, as the following example illustrates.

Example EENS Equal eigenvalues, not similar
Define

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and check that

$$p_A(x) = p_B(x) = 1 - 2x + x^2 = (x-1)^2$$

and so A and B have equal characteristic polynomials. If the converse of Theorem SMEE were true, then A and B would be similar. Suppose this is the case. More precisely, suppose there is a nonsingular matrix S so that $A = S^{-1}BS$.

Then

$$A = S^{-1}BS = S^{-1}I_2S = S^{-1}S = I_2$$

Clearly $A \neq I_2$ and this contradiction tells us that the converse of Theorem SMEE is false. △

Subsection D Diagonalization

Good things happen when a matrix is similar to a diagonal matrix. For example, the eigenvalues of the matrix are the entries on the diagonal of the diagonal matrix. And it can be a much simpler matter to compute high powers of the matrix. Diagonalizable matrices are also of interest in more abstract settings. Here are the relevant definitions, then our main theorem for this section.

Definition DIM Diagonal Matrix
Suppose that A is a square matrix. Then A is a **diagonal matrix** if $[A]_{ij} = 0$ whenever $i \neq j$. □

Definition DZM Diagonalizable Matrix
Suppose A is a square matrix. Then A is **diagonalizable** if A is similar to a diagonal matrix. □

Example DAB Diagonalization of Archetype B
Archetype B has a 3×3 coefficient matrix

$$B = \begin{bmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{bmatrix}$$

and is similar to a diagonal matrix, as can be seen by the following computation with the nonsingular matrix S,

$$\begin{aligned} S^{-1}BS &= \begin{bmatrix} -5 & -3 & -2 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} -5 & -3 & -2 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & -1 & -1 \\ 2 & 3 & 1 \\ -1 & -2 & 1 \end{bmatrix} \begin{bmatrix} -7 & -6 & -12 \\ 5 & 5 & 7 \\ 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} -5 & -3 & -2 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \end{aligned}$$

△

Example SMS3 provides yet another example of a matrix that is subjected to a similarity transformation and the result is a diagonal matrix. Alright, just how would we find the magic matrix S that can be used in a similarity transformation to produce a diagonal matrix? Before you read the statement of the next theorem, you might study the eigenvalues and eigenvectors of Archetype B and compute the eigenvalues and eigenvectors of the matrix in Example SMS3.

Theorem DC Diagonalization Characterization
Suppose A is a square matrix of size n. Then A is diagonalizable if and only if there exists a linearly independent set S that contains n eigenvectors of A.

Proof. ($\Leftarrow$) Let $S = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots, \mathbf{x}_n\}$ be a linearly independent set of eigenvectors of A for the eigenvalues $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$. Recall Definition SUV and define

$$R = [\mathbf{x}_1|\mathbf{x}_2|\mathbf{x}_3|\ldots|\mathbf{x}_n]$$

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix} = [\lambda_1\mathbf{e}_1|\lambda_2\mathbf{e}_2|\lambda_3\mathbf{e}_3|\ldots|\lambda_n\mathbf{e}_n]$$

The columns of R are the vectors of the linearly independent set S and so by Theorem NMLIC the matrix R is nonsingular. By Theorem NI we know R^{-1} exists.

$$\begin{aligned} R^{-1}AR &= R^{-1}A\,[\mathbf{x}_1|\mathbf{x}_2|\mathbf{x}_3|\ldots|\mathbf{x}_n] \\ &= R^{-1}[A\mathbf{x}_1|A\mathbf{x}_2|A\mathbf{x}_3|\ldots|A\mathbf{x}_n] && \text{Definition MM} \\ &= R^{-1}[\lambda_1\mathbf{x}_1|\lambda_2\mathbf{x}_2|\lambda_3\mathbf{x}_3|\ldots|\lambda_n\mathbf{x}_n] && \text{Definition EEM} \\ &= R^{-1}[\lambda_1R\mathbf{e}_1|\lambda_2R\mathbf{e}_2|\lambda_3R\mathbf{e}_3|\ldots|\lambda_nR\mathbf{e}_n] && \text{Definition MVP} \\ &= R^{-1}[R(\lambda_1\mathbf{e}_1)|R(\lambda_2\mathbf{e}_2)|R(\lambda_3\mathbf{e}_3)|\ldots|R(\lambda_n\mathbf{e}_n)] && \text{Theorem MMSMM} \\ &= R^{-1}R[\lambda_1\mathbf{e}_1|\lambda_2\mathbf{e}_2|\lambda_3\mathbf{e}_3|\ldots|\lambda_n\mathbf{e}_n] && \text{Definition MM} \\ &= I_nD && \text{Definition MI} \\ &= D && \text{Theorem MMIM} \end{aligned}$$

This says that A is similar to the diagonal matrix D via the nonsingular matrix R. Thus A is diagonalizable (Definition DZM).

($\Rightarrow$) Suppose that A is diagonalizable, so there is a nonsingular matrix of size n

$$T = [\mathbf{y}_1|\mathbf{y}_2|\mathbf{y}_3|\ldots|\mathbf{y}_n]$$

and a diagonal matrix (recall Definition SUV)

$$E = \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix} = [d_1\mathbf{e}_1|d_2\mathbf{e}_2|d_3\mathbf{e}_3|\ldots|d_n\mathbf{e}_n]$$

such that $T^{-1}AT = E$.

Then consider,

$$\begin{aligned} &[A\mathbf{y}_1|A\mathbf{y}_2|A\mathbf{y}_3|\ldots|A\mathbf{y}_n] \\ &\quad = A\,[\mathbf{y}_1|\mathbf{y}_2|\mathbf{y}_3|\ldots|\mathbf{y}_n] && \text{Definition MM} \\ &\quad = AT \\ &\quad = I_nAT && \text{Theorem MMIM} \\ &\quad = TT^{-1}AT && \text{Definition MI} \\ &\quad = TE \\ &\quad = T[d_1\mathbf{e}_1|d_2\mathbf{e}_2|d_3\mathbf{e}_3|\ldots|d_n\mathbf{e}_n] \\ &\quad = [T(d_1\mathbf{e}_1)|T(d_2\mathbf{e}_2)|T(d_3\mathbf{e}_3)|\ldots|T(d_n\mathbf{e}_n)] && \text{Definition MM} \\ &\quad = [d_1T\mathbf{e}_1|d_2T\mathbf{e}_2|d_3T\mathbf{e}_3|\ldots|d_nT\mathbf{e}_n] && \text{Definition MM} \\ &\quad = [d_1\mathbf{y}_1|d_2\mathbf{y}_2|d_3\mathbf{y}_3|\ldots|d_n\mathbf{y}_n] && \text{Definition MVP} \end{aligned}$$

This equality of matrices (Definition ME) allows us to conclude that the individual columns are equal vectors (Definition CVE). That is, $A\mathbf{y}_i = d_i\mathbf{y}_i$ for $1 \leq i \leq n$. In other words, $\mathbf{y}_i$ is an eigenvector of A for the eigenvalue d_i, $1 \leq i \leq n$. (Why does $\mathbf{y}_i \neq \mathbf{0}$?). Because T is nonsingular, the set containing T's columns, $S = \{\mathbf{y}_1,\,\mathbf{y}_2,\,\mathbf{y}_3,\,\ldots,\,\mathbf{y}_n\}$, is a linearly independent set (Theorem NMLIC). So the set S has all the required properties. ■

Notice that the proof of Theorem DC is constructive. To diagonalize a matrix, we need only locate n linearly independent eigenvectors. Then we can construct a nonsingular matrix using the eigenvectors as columns (R) so that $R^{-1}AR$ is a diagonal matrix (D). The entries on the diagonal of D will be the eigenvalues of the eigenvectors used to create R, *in the same order* as the eigenvectors appear in R. We illustrate this by **diagonalizing** some matrices.

Example DMS3 Diagonalizing a matrix of size 3
Consider the matrix

$$F = \begin{bmatrix} -13 & -8 & -4 \\ 12 & 7 & 4 \\ 24 & 16 & 7 \end{bmatrix}$$

of Example CPMS3, Example EMS3 and Example ESMS3. F's eigenvalues and eigenspaces are

$$\lambda = 3 \qquad \mathcal{E}_F(3) = \left\langle\left\{\begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}\right\}\right\rangle$$

$$\lambda = -1 \qquad \mathcal{E}_F(-1) = \left\langle\left\{\begin{bmatrix} -\frac{2}{3} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix}\right\}\right\rangle$$

Define the matrix S to be the 3×3 matrix whose columns are the three basis vectors in the eigenspaces for F,

$$S = \begin{bmatrix} -\frac{1}{2} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{2} & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Check that S is nonsingular (row-reduces to the identity matrix, Theorem NMRRI or has a nonzero determinant, Theorem SMZD). Then the three columns of S are a linearly independent set (Theorem NMLIC). By Theorem DC we now know that F is diagonalizable. Furthermore, the construction in the proof of Theorem DC tells us that if we apply the matrix S to F in a similarity transformation, the result will be a diagonal matrix with the eigenvalues of F on the diagonal. The eigenvalues appear on the diagonal of the matrix in the same order as the eigenvectors appear in S. So,

$$\begin{aligned} S^{-1}FS &= \begin{bmatrix} -\frac{1}{2} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{2} & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -13 & -8 & -4 \\ 12 & 7 & 4 \\ 24 & 16 & 7 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{2} & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 4 & 2 \\ -3 & -1 & -1 \\ -6 & -4 & -1 \end{bmatrix} \begin{bmatrix} -13 & -8 & -4 \\ 12 & 7 & 4 \\ 24 & 16 & 7 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} & -\frac{2}{3} & -\frac{1}{3} \\ \frac{1}{2} & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \end{aligned}$$

Note that the above computations can be viewed two ways. The proof of Theorem DC tells us that the four matrices (F, S, F^{-1} and the diagonal matrix) *will* interact the way we have written the equation. Or as an example, we can actually *perform* the computations to verify what the theorem predicts. △

The dimension of an eigenspace can be no larger than the algebraic multiplicity of the eigenvalue by Theorem ME. When every eigenvalue's eigenspace is this large, then we can diagonalize the matrix, and only then. Three examples we have seen so far in this section, Example SMS5, Example DAB and Example DMS3, illustrate the diagonalization of a matrix, with varying degrees of detail about just how the diagonalization is achieved. However, in each case, you can verify that the geometric and algebraic multiplicities are equal for every eigenvalue. This is the substance of the next theorem.

Theorem DMFE Diagonalizable Matrices have Full Eigenspaces
Suppose A is a square matrix. Then A is diagonalizable if and only if $\gamma_A(\lambda) = \alpha_A(\lambda)$ for every eigenvalue λ of A.

Proof. Suppose A has size n and k distinct eigenvalues, $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_k$. Let $S_i = \left\{\mathbf{x}_{i1}, \mathbf{x}_{i2}, \mathbf{x}_{i3}, \ldots, \mathbf{x}_{i\gamma_A(\lambda_i)}\right\}$, denote a basis for the eigenspace of λ_i, $\mathcal{E}_A(\lambda_i)$, for $1 \leq i \leq k$. Then

$$S = S_1 \cup S_2 \cup S_3 \cup \cdots \cup S_k$$

is a set of eigenvectors for A. A vector cannot be an eigenvector for two different eigenvalues (see Exercise EE.T20) so $S_i \cap S_j = \emptyset$ whenever $i \neq j$. In other words, S is a disjoint union of S_i, $1 \leq i \leq k$.

($\Leftarrow$) The size of S is

$$\begin{aligned}
|S| &= \sum_{i=1}^{k} \gamma_A(\lambda_i) && S \text{ disjoint union of } S_i\\
&= \sum_{i=1}^{k} \alpha_A(\lambda_i) && \text{Hypothesis}\\
&= n && \text{Theorem NEM}
\end{aligned}$$

We next show that S is a linearly independent set. So we will begin with a relation of linear dependence on S, using doubly-subscripted scalars and eigenvectors,

$$\begin{aligned}
\mathbf{0} = &\left(a_{11}\mathbf{x}_{11} + a_{12}\mathbf{x}_{12} + \cdots + a_{1\gamma_A(\lambda_1)}\mathbf{x}_{1\gamma_A(\lambda_1)}\right) +\\
&\left(a_{21}\mathbf{x}_{21} + a_{22}\mathbf{x}_{22} + \cdots + a_{2\gamma_A(\lambda_2)}\mathbf{x}_{2\gamma_A(\lambda_2)}\right) +\\
&\left(a_{31}\mathbf{x}_{31} + a_{32}\mathbf{x}_{32} + \cdots + a_{3\gamma_A(\lambda_3)}\mathbf{x}_{3\gamma_A(\lambda_3)}\right) +\\
&\quad\vdots\\
&\left(a_{k1}\mathbf{x}_{k1} + a_{k2}\mathbf{x}_{k2} + \cdots + a_{k\gamma_A(\lambda_k)}\mathbf{x}_{k\gamma_A(\lambda_k)}\right)
\end{aligned}$$

Define the vectors $\mathbf{y}_i$, $1 \leq i \leq k$ by

$$\begin{aligned}
\mathbf{y}_1 &= \left(a_{11}\mathbf{x}_{11} + a_{12}\mathbf{x}_{12} + a_{13}\mathbf{x}_{13} + \cdots + a_{\gamma_A(1\lambda_1)}\mathbf{x}_{1\gamma_A(\lambda_1)}\right)\\
\mathbf{y}_2 &= \left(a_{21}\mathbf{x}_{21} + a_{22}\mathbf{x}_{22} + a_{23}\mathbf{x}_{23} + \cdots + a_{\gamma_A(2\lambda_2)}\mathbf{x}_{2\gamma_A(\lambda_2)}\right)\\
\mathbf{y}_3 &= \left(a_{31}\mathbf{x}_{31} + a_{32}\mathbf{x}_{32} + a_{33}\mathbf{x}_{33} + \cdots + a_{\gamma_A(3\lambda_3)}\mathbf{x}_{3\gamma_A(\lambda_3)}\right)\\
&\quad\vdots\\
\mathbf{y}_k &= \left(a_{k1}\mathbf{x}_{k1} + a_{k2}\mathbf{x}_{k2} + a_{k3}\mathbf{x}_{k3} + \cdots + a_{\gamma_A(k\lambda_k)}\mathbf{x}_{k\gamma_A(\lambda_k)}\right)
\end{aligned}$$

Then the relation of linear dependence becomes

$$\mathbf{0} = \mathbf{y}_1 + \mathbf{y}_2 + \mathbf{y}_3 + \cdots + \mathbf{y}_k$$

Since the eigenspace $\mathcal{E}_A(\lambda_i)$ is closed under vector addition and scalar multiplication, $\mathbf{y}_i \in \mathcal{E}_A(\lambda_i)$, $1 \leq i \leq k$. Thus, for each i, the vector $\mathbf{y}_i$ is an eigenvector of A for λ_i, or is the zero vector. Recall that sets of eigenvectors whose eigenvalues are distinct form a linearly independent set by Theorem EDELI. Should any (or some) $\mathbf{y}_i$ be nonzero, the previous equation would provide a nontrivial relation of linear dependence on a set of eigenvectors with distinct eigenvalues, contradicting Theorem EDELI. Thus $\mathbf{y}_i = \mathbf{0}$, $1 \leq i \leq k$.

Each of the k equations, $\mathbf{y}_i = \mathbf{0}$, is a relation of linear dependence on the corresponding set S_i, a set of basis vectors for the eigenspace $\mathcal{E}_A(\lambda_i)$, which is therefore linearly independent. From these relations of linear dependence on linearly independent sets we conclude that the scalars are all zero, more precisely, $a_{ij} = 0$,

$1 \leq j \leq \gamma_A(\lambda_i)$ for $1 \leq i \leq k$. This establishes that our original relation of linear dependence on S has only the trivial relation of linear dependence, and hence S is a linearly independent set.

We have determined that S is a set of n linearly independent eigenvectors for A, and so by Theorem DC is diagonalizable.

($\Rightarrow$) Now we assume that A is diagonalizable. Aiming for a contradiction (Proof Technique CD), suppose that there is at least one eigenvalue, say λ_t, such that $\gamma_A(\lambda_t) \neq \alpha_A(\lambda_t)$. By Theorem ME we must have $\gamma_A(\lambda_t) < \alpha_A(\lambda_t)$, and $\gamma_A(\lambda_i) \leq \alpha_A(\lambda_i)$ for $1 \leq i \leq k$, $i \neq t$.

Since A is diagonalizable, Theorem DC guarantees a set of n linearly independent vectors, all of which are eigenvectors of A. Let n_i denote the number of eigenvectors in S that are eigenvectors for λ_i, and recall that a vector cannot be an eigenvector for two different eigenvalues (Exercise EE.T20). S is a linearly independent set, so the subset S_i containing the n_i eigenvectors for λ_i must also be linearly independent. Because the eigenspace $\mathcal{E}_A(\lambda_i)$ has dimension $\gamma_A(\lambda_i)$ and S_i is a linearly independent subset in $\mathcal{E}_A(\lambda_i)$, Theorem G tells us that $n_i \leq \gamma_A(\lambda_i)$, for $1 \leq i \leq k$.

Putting all these facts together gives,

$$\begin{aligned}
n &= n_1 + n_2 + n_3 + \cdots + n_t + \cdots + n_k && \text{Definition SU}\\
&\leq \gamma_A(\lambda_1) + \gamma_A(\lambda_2) + \gamma_A(\lambda_3) + \cdots + \gamma_A(\lambda_t) + \cdots + \gamma_A(\lambda_k) && \text{Theorem G}\\
&< \alpha_A(\lambda_1) + \alpha_A(\lambda_2) + \alpha_A(\lambda_3) + \cdots + \alpha_A(\lambda_t) + \cdots + \alpha_A(\lambda_k) && \text{Theorem ME}\\
&= n && \text{Theorem NEM}
\end{aligned}$$

This is a contradiction (we cannot have $n < n$!) and so our assumption that some eigenspace had less than full dimension was false. ■

Example SEE, Example CAEHW, Example ESMS3, Example ESMS4, Example DEMS5, Archetype B, Archetype F, Archetype K and Archetype L are all examples of matrices that are diagonalizable and that illustrate Theorem DMFE. While we have provided many examples of matrices that are diagonalizable, especially among the archetypes, there are many matrices that are not diagonalizable. Here is one now.

Example NDMS4 A non-diagonalizable matrix of size 4
In Example EMMS4 the matrix

$$B = \begin{bmatrix} -2 & 1 & -2 & -4 \\ 12 & 1 & 4 & 9 \\ 6 & 5 & -2 & -4 \\ 3 & -4 & 5 & 10 \end{bmatrix}$$

was determined to have characteristic polynomial

$$p_B(x) = (x-1)(x-2)^3$$

and an eigenspace for $\lambda = 2$ of

$$\mathcal{E}_B(2) = \left\langle \left\{ \begin{bmatrix} -\frac{1}{2} \\ 1 \\ -\frac{1}{2} \\ 1 \end{bmatrix} \right\} \right\rangle$$

So the geometric multiplicity of $\lambda = 2$ is $\gamma_B(2) = 1$, while the algebraic multiplicity is $\alpha_B(2) = 3$. By Theorem DMFE, the matrix B is not diagonalizable. △

Archetype A is the lone archetype with a square matrix that is not diagonalizable, as the algebraic and geometric multiplicities of the eigenvalue $\lambda = 0$ differ. Example HMEM5 is another example of a matrix that cannot be diagonalized due to the difference between the geometric and algebraic multiplicities of $\lambda = 2$, as is Example CEMS6 which has two complex eigenvalues, each with differing multiplicities. Likewise, Example EMMS4 has an eigenvalue with different algebraic and geometric multiplicities and so cannot be diagonalized.

Theorem DED Distinct Eigenvalues implies Diagonalizable
Suppose A is a square matrix of size n with n distinct eigenvalues. Then A is diagonalizable.

Proof. Let $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ denote the n distinct eigenvalues of A. Then by Theorem NEM we have $n = \sum_{i=1}^{n} \alpha_A(\lambda_i)$, which implies that $\alpha_A(\lambda_i) = 1$, $1 \leq i \leq n$. From Theorem ME it follows that $\gamma_A(\lambda_i) = 1$, $1 \leq i \leq n$. So $\gamma_A(\lambda_i) = \alpha_A(\lambda_i)$, $1 \leq i \leq n$ and Theorem DMFE says A is diagonalizable. ■

Example DEHD Distinct eigenvalues, hence diagonalizable
In Example DEMS5 the matrix

$$H = \begin{bmatrix} 15 & 18 & -8 & 6 & -5 \\ 5 & 3 & 1 & -1 & -3 \\ 0 & -4 & 5 & -4 & -2 \\ -43 & -46 & 17 & -14 & 15 \\ 26 & 30 & -12 & 8 & -10 \end{bmatrix}$$

has characteristic polynomial

$$p_H(x) = x(x-2)(x-1)(x+1)(x+3)$$

and so is a 5×5 matrix with 5 distinct eigenvalues.

By Theorem DED we know H must be diagonalizable. But just for practice, we exhibit a diagonalization. The matrix S contains eigenvectors of H as columns, one from each eigenspace, guaranteeing linear independent columns and thus the nonsingularity of S. Notice that we are using the versions of the eigenvectors from Example DEMS5 that have integer entries. The diagonal matrix has the eigenvalues of H in the same order that their respective eigenvectors appear as the columns of S. With these matrices, verify computationally that $S^{-1}HS = D$.

$$S = \begin{bmatrix} 2 & 1 & -1 & 1 & 1 \\ -1 & 0 & 2 & 0 & -1 \\ -2 & 0 & 2 & -1 & -2 \\ -4 & -1 & 0 & -2 & -1 \\ 2 & 2 & 1 & 2 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} -3 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Note that there are many different ways to diagonalize H. We could replace eigenvectors by nonzero scalar multiples, or we could rearrange the order of the eigenvectors as the columns of S (which would subsequently reorder the eigenvalues along the diagonal of D). △

Archetype B is another example of a matrix that has as many distinct eigenvalues as its size, and is hence diagonalizable by Theorem DED.

Powers of a diagonal matrix are easy to compute, and when a matrix is diagonalizable, it is almost as easy. We could state a theorem here perhaps, but we will settle instead for an example that makes the point just as well.

Example HPDM High power of a diagonalizable matrix
Suppose that

$$A = \begin{bmatrix} 19 & 0 & 6 & 13 \\ -33 & -1 & -9 & -21 \\ 21 & -4 & 12 & 21 \\ -36 & 2 & -14 & -28 \end{bmatrix}$$

and we wish to compute A^{20}. Normally this would require 19 matrix multiplications, but since A is diagonalizable, we can simplify the computations substantially.

First, we diagonalize A. With

$$S = \begin{bmatrix} 1 & -1 & 2 & -1 \\ -2 & 3 & -3 & 3 \\ 1 & 1 & 3 & 3 \\ -2 & 1 & -4 & 0 \end{bmatrix}$$

we find

$$\begin{aligned} D &= S^{-1}AS \\ &= \begin{bmatrix} -6 & 1 & -3 & -6 \\ 0 & 2 & -2 & -3 \\ 3 & 0 & 1 & 2 \\ -1 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 19 & 0 & 6 & 13 \\ -33 & -1 & -9 & -21 \\ 21 & -4 & 12 & 21 \\ -36 & 2 & -14 & -28 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 & -1 \\ -2 & 3 & -3 & 3 \\ 1 & 1 & 3 & 3 \\ -2 & 1 & -4 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Now we find an alternate expression for A^{20},

$$\begin{aligned} A^{20} &= AAA\dots A \\ &= I_nAI_nAI_nAI_n\dots I_nAI_n \\ &= \left(SS^{-1}\right)A\left(SS^{-1}\right)A\left(SS^{-1}\right)A\left(SS^{-1}\right)\dots\left(SS^{-1}\right)A\left(SS^{-1}\right) \\ &= S\left(S^{-1}AS\right)\left(S^{-1}AS\right)\left(S^{-1}AS\right)\dots\left(S^{-1}AS\right)S^{-1} \\ &= SDDD\dots DS^{-1} \\ &= SD^{20}S^{-1} \end{aligned}$$

and since D is a diagonal matrix, powers are much easier to compute,

$$\begin{aligned} &= S\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{20} S^{-1} \\ &= S\begin{bmatrix} (-1)^{20} & 0 & 0 & 0 \\ 0 & (0)^{20} & 0 & 0 \\ 0 & 0 & (2)^{20} & 0 \\ 0 & 0 & 0 & (1)^{20} \end{bmatrix} S^{-1} \\ &= \begin{bmatrix} 1 & -1 & 2 & -1 \\ -2 & 3 & -3 & 3 \\ 1 & 1 & 3 & 3 \\ -2 & 1 & -4 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1048576 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -6 & 1 & -3 & -6 \\ 0 & 2 & -2 & -3 \\ 3 & 0 & 1 & 2 \\ -1 & -1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 6291451 & 2 & 2097148 & 4194297 \\ -9437175 & -5 & -3145719 & -6291441 \\ 9437175 & -2 & 3145728 & 6291453 \\ -12582900 & -2 & -4194298 & -8388596 \end{bmatrix} \end{aligned}$$

Notice how we effectively replaced the twentieth power of A by the twentieth power of D, and how a high power of a diagonal matrix is just a collection of powers of scalars on the diagonal. The price we pay for this simplification is the need to diagonalize the matrix (by computing eigenvalues and eigenvectors) and finding the inverse of the matrix of eigenvectors. And we still need to do two matrix products. But the higher the power, the greater the savings. △

Subsection FS Fibonacci Sequences

Example FSCF Fibonacci sequence, closed form

The **Fibonacci sequence** is a sequence of integers defined recursively by

$$a_0 = 0 \qquad a_1 = 1 \qquad a_{n+1} = a_n + a_{n-1}, \quad n \geq 1$$

So the initial portion of the sequence is 0, 1, 1, 2, 3, 5, 8, 13, 21, ... In this subsection we will illustrate an application of eigenvalues and diagonalization through the determination of a closed-form expression for an arbitrary term of this sequence.

To begin, verify that for any $n \geq 1$ the recursive statement above establishes the truth of the statement

$$\begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} a_{n-1} \\ a_n \end{bmatrix}$$

Let A denote this 2×2 matrix. Through repeated applications of the statement above we have

$$\begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix} = A \begin{bmatrix} a_{n-1} \\ a_n \end{bmatrix} = A^2 \begin{bmatrix} a_{n-2} \\ a_{n-1} \end{bmatrix} = A^3 \begin{bmatrix} a_{n-3} \\ a_{n-2} \end{bmatrix} = \cdots = A^n \begin{bmatrix} a_0 \\ a_1 \end{bmatrix}$$

In preparation for working with this high power of A, not unlike in Example HPDM, we will diagonalize A. The characteristic polynomial of A is $p_A(x) = x^2 - x - 1$, with roots (the eigenvalues of A by Theorem EMRCP)

$$\rho = \frac{1 + \sqrt{5}}{2} \qquad \delta = \frac{1 - \sqrt{5}}{2}$$

With two distinct eigenvalues, Theorem DED implies that A is diagonalizable. It will be easier to compute with these eigenvalues once you confirm the following properties (all but the last can be derived from the fact that ρ and δ are roots of the characteristic polynomial, in a factored or unfactored form)

$$\rho + \delta = 1 \qquad \rho\delta = -1 \qquad 1 + \rho = \rho^2 \qquad 1 + \delta = \delta^2 \qquad \rho - \delta = \sqrt{5}$$

Then eigenvectors of A (for ρ and δ, respectively) are

$$\begin{bmatrix} 1 \\ \rho \end{bmatrix} \qquad \begin{bmatrix} 1 \\ \delta \end{bmatrix}$$

which can be easily confirmed, as we demonstrate for the eigenvector for ρ,

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ \rho \end{bmatrix} = \begin{bmatrix} \rho \\ 1 + \rho \end{bmatrix} = \begin{bmatrix} \rho \\ \rho^2 \end{bmatrix} = \rho \begin{bmatrix} 1 \\ \rho \end{bmatrix}$$

From the proof of Theorem DC we know A can be diagonalized by a matrix S with these eigenvectors as columns, giving $D = S^{-1}AS$. We list S, S^{-1} and the diagonal matrix D,

$$S = \begin{bmatrix} 1 & 1 \\ \rho & \delta \end{bmatrix} \qquad S^{-1} = \frac{1}{\rho - \delta} \begin{bmatrix} -\delta & 1 \\ \rho & -1 \end{bmatrix} \qquad D = \begin{bmatrix} \rho & 0 \\ 0 & \delta \end{bmatrix}$$

OK, we have everything in place now. The main step in the following is to replace A by SDS^{-1}. Here we go,

$$\begin{aligned} \begin{bmatrix} a_n \\ a_{n+1} \end{bmatrix} &= A^n \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \\ &= \left(SDS^{-1}\right)^n \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \\ &= SDS^{-1}SDS^{-1}SDS^{-1} \cdots SDS^{-1} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \\ &= SDDD \cdots DS^{-1} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} \end{aligned}$$

$$
\begin{aligned}
&= SD^nS^{-1}\begin{bmatrix}a_0\\a_1\end{bmatrix}\\
&= \begin{bmatrix}1&1\\\rho&\delta\end{bmatrix}\begin{bmatrix}\rho&0\\0&\delta\end{bmatrix}^n\frac{1}{\rho-\delta}\begin{bmatrix}-\delta&1\\\rho&-1\end{bmatrix}\begin{bmatrix}a_0\\a_1\end{bmatrix}\\
&= \frac{1}{\rho-\delta}\begin{bmatrix}1&1\\\rho&\delta\end{bmatrix}\begin{bmatrix}\rho^n&0\\0&\delta^n\end{bmatrix}\begin{bmatrix}-\delta&1\\\rho&-1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\\
&= \frac{1}{\rho-\delta}\begin{bmatrix}1&1\\\rho&\delta\end{bmatrix}\begin{bmatrix}\rho^n&0\\0&\delta^n\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}\\
&= \frac{1}{\rho-\delta}\begin{bmatrix}1&1\\\rho&\delta\end{bmatrix}\begin{bmatrix}\rho^n\\-\delta^n\end{bmatrix}\\
&= \frac{1}{\rho-\delta}\begin{bmatrix}\rho^n-\delta^n\\\rho^{n+1}-\delta^{n+1}\end{bmatrix}
\end{aligned}
$$

Performing the scalar multiplication and equating the first entries of the two vectors, we arrive at the closed form expression

$$
\begin{aligned}
a_n &= \frac{1}{\rho-\delta}\left(\rho^n-\delta^n\right)\\
&= \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right)\\
&= \frac{1}{2^n\sqrt{5}}\left(\left(1+\sqrt{5}\right)^n-\left(1-\sqrt{5}\right)^n\right)
\end{aligned}
$$

Notice that it does not matter whether we use the equality of the first or second entries of the vectors, we will arrive at the same formula, once in terms of n and again in terms of $n+1$. Also, our definition clearly describes a sequence that will only contain integers, yet the presence of the irrational number $\sqrt{5}$ might make us suspicious. But no, our expression for a^n will always yield an integer!

The Fibonacci sequence, and generalizations of it, have been extensively studied (Fibonacci lived in the 12th and 13th centuries). There are many ways to derive the closed-form expression we just found, and our approach may not be the most efficient route. But it is a nice demonstration of how diagonalization can be used to solve a problem outside the field of linear algebra. △

We close this section with a comment about an important upcoming theorem that we prove in Chapter R. A consequence of Theorem OD is that every Hermitian matrix (Definition HM) is diagonalizable (Definition DZM), and the similarity transformation that accomplishes the diagonalization uses a unitary matrix (Definition UM). This means that for every Hermitian matrix of size n there is a basis of $\mathbb{C}^n$ that is composed entirely of eigenvectors for the matrix and also forms an orthonormal set (Definition ONS). Notice that for matrices with only real entries, we only need the hypothesis that the matrix is symmetric (Definition SYM) to reach this conclusion (Example ESMS4). Can you imagine a prettier basis for use with a matrix? I cannot.

These results in Section OD explain much of our recurring interest in orthogonality, and make the section a high point in your study of linear algebra. A precise statement of this diagonalization result applies to a slightly broader class of matrices, known as "normal" matrices (Definition NRML), which are matrices that commute with their adjoints. With this expanded category of matrices, the result becomes an equivalence (Proof Technique E). See Theorem OD and Theorem OBNM in Section OD for all the details.

Reading Questions

1. What is an equivalence relation?

2. State a condition that is equivalent to a matrix being diagonalizable, but is not the definition.

3. Find a diagonal matrix similar to

$$A = \begin{bmatrix} -5 & 8 \\ -4 & 7 \end{bmatrix}$$

Exercises

C20† Consider the matrix A below. First, show that A is diagonalizable by computing the geometric multiplicities of the eigenvalues and quoting the relevant theorem. Second, find a diagonal matrix D and a nonsingular matrix S so that $S^{-1}AS = D$. (See Exercise EE.C20 for some of the necessary computations.)

$$A = \begin{bmatrix} 18 & -15 & 33 & -15 \\ -4 & 8 & -6 & 6 \\ -9 & 9 & -16 & 9 \\ 5 & -6 & 9 & -4 \end{bmatrix}$$

C21† Determine if the matrix A below is diagonalizable. If the matrix is diagonalizable, then find a diagonal matrix D that is similar to A, and provide the invertible matrix S that performs the similarity transformation. You should use your calculator to find the eigenvalues of the matrix, but try only using the row-reducing function of your calculator to assist with finding eigenvectors.

$$A = \begin{bmatrix} 1 & 9 & 9 & 24 \\ -3 & -27 & -29 & -68 \\ 1 & 11 & 13 & 26 \\ 1 & 7 & 7 & 18 \end{bmatrix}$$

C22† Consider the matrix A below. Find the eigenvalues of A using a calculator and use these to construct the characteristic polynomial of A, $p_A(x)$. State the algebraic multiplicity of each eigenvalue. Find all of the eigenspaces for A by computing expressions for null spaces, only using your calculator to row-reduce matrices. State the geometric multiplicity of each eigenvalue. Is A diagonalizable? If not, explain why. If so, find a diagonal matrix D that is similar to A.

$$A = \begin{bmatrix} 19 & 25 & 30 & 5 \\ -23 & -30 & -35 & -5 \\ 7 & 9 & 10 & 1 \\ -3 & -4 & -5 & -1 \end{bmatrix}$$

T15† Suppose that A and B are similar matrices. Prove that A^3 and B^3 are similar matrices. Generalize.

T16† Suppose that A and B are similar matrices, with A nonsingular. Prove that B is nonsingular, and that A^{-1} is similar to B^{-1}.

T17† Suppose that B is a nonsingular matrix. Prove that AB is similar to BA.

Chapter LT
Linear Transformations

In the next linear algebra course you take, the first lecture might be a reminder about what a vector space is (Definition VS), their ten properties, basic theorems and then some examples. The second lecture would likely be all about linear transformations. While it may seem we have waited a long time to present what must be a central topic, in truth we have already been working with linear transformations for some time.

Functions are important objects in the study of calculus, but have been absent from this course until now (well, not really, it just seems that way). In your study of more advanced mathematics it is nearly impossible to escape the use of functions — they are as fundamental as sets are.

Section LT
Linear Transformations

Early in Chapter VS we prefaced the definition of a vector space with the comment that it was "one of the two most important definitions in the entire course." Here comes the other. Any capsule summary of linear algebra would have to describe the subject as the interplay of linear transformations and vector spaces. Here we go.

Subsection LT
Linear Transformations

Definition LT Linear Transformation
A **linear transformation**, $T\colon U \to V$, is a function that carries elements of the vector space U (called the **domain**) to the vector space V (called the **codomain**), and which has two additional properties

1. $T(\mathbf{u}_1 + \mathbf{u}_2) = T(\mathbf{u}_1) + T(\mathbf{u}_2)$ for all $\mathbf{u}_1,\, \mathbf{u}_2 \in U$
2. $T(\alpha\mathbf{u}) = \alpha T(\mathbf{u})$ for all $\mathbf{u} \in U$ and all $\alpha \in \mathbb{C}$

□

The two defining conditions in the definition of a linear transformation should "feel linear," whatever that means. Conversely, these two conditions could be taken as *exactly* what it means *to be* linear. As every vector space property derives from vector addition and scalar multiplication, so too, every property of a linear transformation derives from these two defining properties. While these conditions may be reminiscent of how we test subspaces, they really are quite different, so do not confuse the two.

Here are two diagrams that convey the essence of the two defining properties of a linear transformation. In each case, begin in the upper left-hand corner, and follow the arrows around the rectangle to the lower-right hand corner, taking two different routes and doing the indicated operations labeled on the arrows. There are two results there. For a linear transformation these two expressions are always equal.

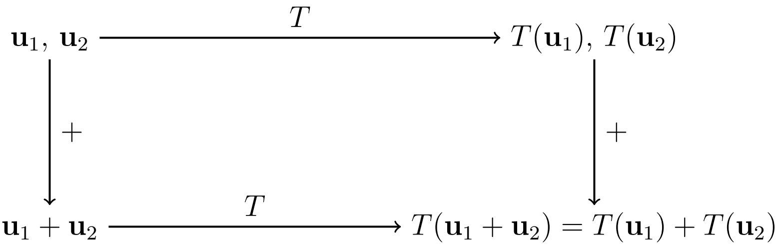

Diagram DLTA: Definition of Linear Transformation, Additive

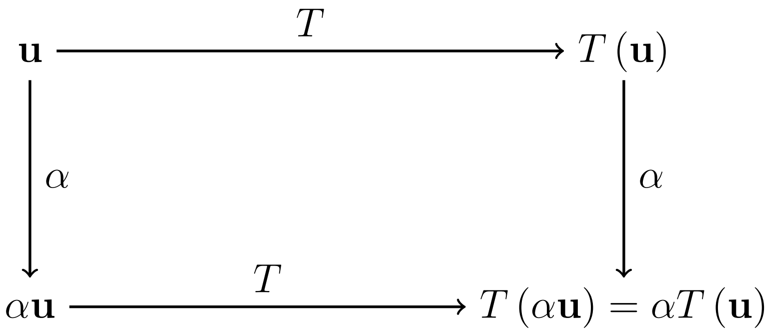

Diagram DLTM: Definition of Linear Transformation, Multiplicative

A couple of words about notation. T is the *name* of the linear transformation, and should be used when we want to discuss the function as a whole. $T\left(\mathbf{u}\right)$ is how we talk about the output of the function, it is a vector in the vector space V. When we write $T\left(\mathbf{x}+\mathbf{y}\right)=T\left(\mathbf{x}\right)+T\left(\mathbf{y}\right)$, the plus sign on the left is the operation of vector addition in the vector space U, since $\mathbf{x}$ and $\mathbf{y}$ are elements of U. The plus sign on the right is the operation of vector addition in the vector space V, since $T\left(\mathbf{x}\right)$ and $T\left(\mathbf{y}\right)$ are elements of the vector space V. These two instances of vector addition might be wildly different.

Let us examine several examples and begin to form a catalog of known linear transformations to work with.

Example ALT A linear transformation
Define $T\colon\mathbb{C}^3\to\mathbb{C}^2$ by describing the output of the function for a generic input with the formula

$$T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)=\begin{bmatrix}2x_1+x_3\\-4x_2\end{bmatrix}$$

and check the two defining properties.

$$\begin{aligned}T\left(\mathbf{x}+\mathbf{y}\right)&=T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix}\right)\\&=T\left(\begin{bmatrix}x_1+y_1\\x_2+y_2\\x_3+y_3\end{bmatrix}\right)\\&=\begin{bmatrix}2(x_1+y_1)+(x_3+y_3)\\-4(x_2+y_2)\end{bmatrix}\end{aligned}$$

$$
\begin{aligned}
&= \begin{bmatrix} (2x_1 + x_3) + (2y_1 + y_3) \\ -4x_2 + (-4)y_2 \end{bmatrix} \\
&= \begin{bmatrix} 2x_1 + x_3 \\ -4x_2 \end{bmatrix} + \begin{bmatrix} 2y_1 + y_3 \\ -4y_2 \end{bmatrix} \\
&= T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) + T\left(\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) \\
&= T(\mathbf{x}) + T(\mathbf{y})
\end{aligned}
$$

and

$$
\begin{aligned}
T(\alpha\mathbf{x}) &= T\left(\alpha\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) \\
&= T\left(\begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \alpha x_3 \end{bmatrix}\right) \\
&= \begin{bmatrix} 2(\alpha x_1) + (\alpha x_3) \\ -4(\alpha x_2) \end{bmatrix} \\
&= \begin{bmatrix} \alpha(2x_1 + x_3) \\ \alpha(-4x_2) \end{bmatrix} \\
&= \alpha \begin{bmatrix} 2x_1 + x_3 \\ -4x_2 \end{bmatrix} \\
&= \alpha T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) \\
&= \alpha T(\mathbf{x})
\end{aligned}
$$

So by Definition LT, T is a linear transformation. △

It can be just as instructive to look at functions that are *not* linear transformations. Since the defining conditions must be true for *all* vectors and scalars, it is enough to find just one situation where the properties fail.

Example NLT Not a linear transformation
Define $S\colon \mathbb{C}^3 \to \mathbb{C}^3$ by

$$
S\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 4x_1 + 2x_2 \\ 0 \\ x_1 + 3x_3 - 2 \end{bmatrix}
$$

This function "looks" linear, but consider

$$
3\,S\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) = 3\begin{bmatrix} 8 \\ 0 \\ 8 \end{bmatrix} = \begin{bmatrix} 24 \\ 0 \\ 24 \end{bmatrix}
$$

while

$$
S\left(3\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) = S\left(\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}\right) = \begin{bmatrix} 24 \\ 0 \\ 28 \end{bmatrix}
$$

So the second required property fails for the choice of $\alpha = 3$ and $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and by Definition LT, S is not a linear transformation. It is just about as easy to find an example where the first defining property fails (try it!). Notice that it is the "-2" in the third component of the definition of S that prevents the function from being a linear transformation. △

Example LTPM Linear transformation, polynomials to matrices
Define a linear transformation $T\colon P_3 \to M_{22}$ by

$$T\left(a + bx + cx^2 + dx^3\right) = \begin{bmatrix} a+b & a-2c \\ d & b-d \end{bmatrix}$$

We verify the two defining conditions of a linear transformation.

$$\begin{aligned}
T(\mathbf{x}+\mathbf{y}) &= T\left((a_1 + b_1x + c_1x^2 + d_1x^3) + (a_2 + b_2x + c_2x^2 + d_2x^3)\right) \\
&= T\left((a_1+a_2) + (b_1+b_2)x + (c_1+c_2)x^2 + (d_1+d_2)x^3\right) \\
&= \begin{bmatrix} (a_1+a_2)+(b_1+b_2) & (a_1+a_2)-2(c_1+c_2) \\ d_1+d_2 & (b_1+b_2)-(d_1+d_2) \end{bmatrix} \\
&= \begin{bmatrix} (a_1+b_1)+(a_2+b_2) & (a_1-2c_1)+(a_2-2c_2) \\ d_1+d_2 & (b_1-d_1)+(b_2-d_2) \end{bmatrix} \\
&= \begin{bmatrix} a_1+b_1 & a_1-2c_1 \\ d_1 & b_1-d_1 \end{bmatrix} + \begin{bmatrix} a_2+b_2 & a_2-2c_2 \\ d_2 & b_2-d_2 \end{bmatrix} \\
&= T\left(a_1 + b_1x + c_1x^2 + d_1x^3\right) + T\left(a_2 + b_2x + c_2x^2 + d_2x^3\right) \\
&= T(\mathbf{x}) + T(\mathbf{y})
\end{aligned}$$

and

$$\begin{aligned}
T(\alpha\mathbf{x}) &= T\left(\alpha(a + bx + cx^2 + dx^3)\right) \\
&= T\left((\alpha a) + (\alpha b)x + (\alpha c)x^2 + (\alpha d)x^3\right) \\
&= \begin{bmatrix} (\alpha a)+(\alpha b) & (\alpha a)-2(\alpha c) \\ \alpha d & (\alpha b)-(\alpha d) \end{bmatrix} \\
&= \begin{bmatrix} \alpha(a+b) & \alpha(a-2c) \\ \alpha d & \alpha(b-d) \end{bmatrix} \\
&= \alpha \begin{bmatrix} a+b & a-2c \\ d & b-d \end{bmatrix} \\
&= \alpha T\left(a + bx + cx^2 + dx^3\right) \\
&= \alpha T(\mathbf{x})
\end{aligned}$$

So by Definition LT, T is a linear transformation. △

Example LTPP Linear transformation, polynomials to polynomials
Define a function $S\colon P_4 \to P_5$ by

$$S(p(x)) = (x-2)p(x)$$

Then

$$\begin{aligned}
S(p(x)+q(x)) &= (x-2)(p(x)+q(x)) \\
&= (x-2)p(x) + (x-2)q(x) = S(p(x)) + S(q(x)) \\
S(\alpha p(x)) &= (x-2)(\alpha p(x)) = (x-2)\alpha p(x) = \alpha(x-2)p(x) = \alpha S(p(x))
\end{aligned}$$

So by Definition LT, S is a linear transformation. △

Linear transformations have many amazing properties, which we will investigate through the next few sections. However, as a taste of things to come, here is a theorem we can prove now and put to use immediately.

Theorem LTTZZ Linear Transformations Take Zero to Zero
Suppose $T\colon U \to V$ is a linear transformation. Then $T(\mathbf{0}) = \mathbf{0}$.

Proof. The two zero vectors in the conclusion of the theorem are different. The first is from U while the second is from V. We will subscript the zero vectors in this proof to highlight the distinction. Think about your objects. (This proof is contributed by Mark Shoemaker).

$$\begin{aligned} T(\mathbf{0}_U) &= T(0\mathbf{0}_U) && \text{Theorem ZSSM in } U \\ &= 0T(\mathbf{0}_U) && \text{Definition LT} \\ &= \mathbf{0}_V && \text{Theorem ZSSM in } V \end{aligned}$$

■

Return to Example NLT and compute $S\left(\begin{bmatrix}0\\0\\0\end{bmatrix}\right) = \begin{bmatrix}0\\0\\-2\end{bmatrix}$ to quickly see again that S is not a linear transformation, while in Example LTPM compute

$$S\left(0 + 0x + 0x^2 + 0x^3\right) = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$$

as an example of Theorem LTTZZ at work.

Subsection LTC
Linear Transformation Cartoons

Throughout this chapter, and Chapter R, we will include drawings of linear transformations. We will call them "cartoons," not because they are humorous, but because they will only expose a portion of the truth. A Bugs Bunny cartoon might give us some insights on human nature, but the rules of physics and biology are routinely (and grossly) violated. So it will be with our **linear transformation cartoons**. Here is our first, followed by a guide to help you understand how these are meant to describe fundamental truths about linear transformations, while simultaneously violating other truths.

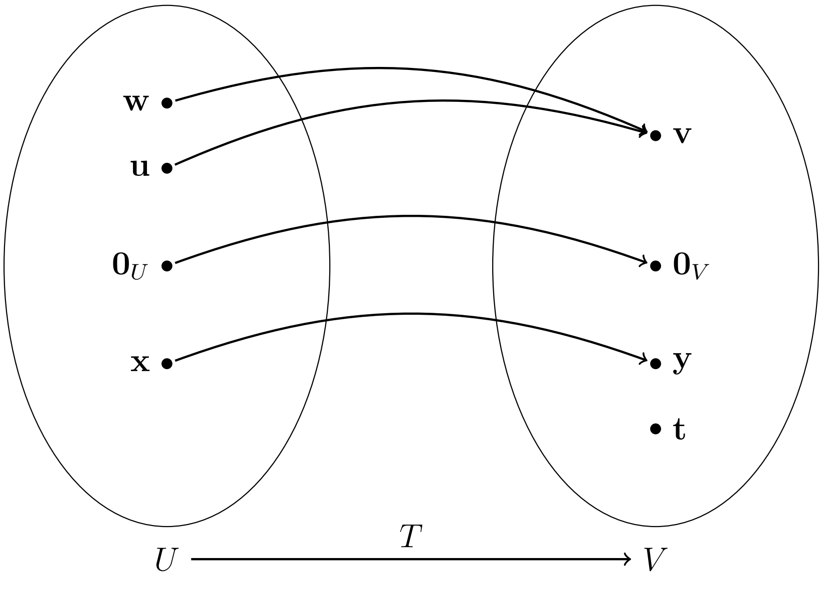

Diagram GLT: General Linear Transformation

Here we picture a linear transformation $T\colon U \to V$, where this information will be consistently displayed along the bottom edge. The ovals are meant to represent the vector spaces, in this case U, the domain, on the left and V, the codomain, on the right. Of course, vector spaces are typically infinite sets, so you will have to imagine that characteristic of these sets. A small dot inside of an oval will represent a vector within that vector space, sometimes with a name, sometimes not (in this case every vector has a name). The sizes of the ovals are meant to be proportional to the dimensions of the vector spaces. However, when we make no assumptions about the dimensions, we will draw the ovals as the same size, as we have done here (which is not meant to suggest that the dimensions have to be equal).

To convey that the linear transformation associates a certain input with a certain output, we will draw an arrow from the input to the output. So, for example, in this cartoon we suggest that $T(\mathbf{x}) = \mathbf{y}$. Nothing in the definition of a linear transformation prevents two different inputs being sent to the same output and we see this in $T(\mathbf{u}) = \mathbf{v} = T(\mathbf{w})$. Similarly, an output may not have any input being sent its way, as illustrated by no arrow pointing at $\mathbf{t}$. In this cartoon, we have captured the essence of our one general theorem about linear transformations, Theorem LTTZZ, $T(\mathbf{0}_U) = \mathbf{0}_V$. On occasion we might include this basic fact when it is relevant, at other times maybe not. Note that the definition of a linear transformation requires that it be a function, so every element of the domain should be associated with some element of the codomain. This will be reflected by never having an element of the domain without an arrow originating there.

These cartoons are of course no substitute for careful definitions and proofs, but they can be a handy way to think about the various properties we will be studying.

Subsection MLT
Matrices and Linear Transformations

If you give me a matrix, then I can quickly build you a linear transformation. Always. First a motivating example and then the theorem.

Example LTM Linear transformation from a matrix
Let

$$A = \begin{bmatrix} 3 & -1 & 8 & 1 \\ 2 & 0 & 5 & -2 \\ 1 & 1 & 3 & -7 \end{bmatrix}$$

and define a function $P\colon \mathbb{C}^4 \to \mathbb{C}^3$ by

$$P(\mathbf{x}) = A\mathbf{x}$$

So we are using an old friend, the matrix-vector product (Definition MVP) as a way to convert a vector with 4 components into a vector with 3 components. Applying Definition MVP allows us to write the defining formula for P in a slightly different form,

$$P(\mathbf{x}) = A\mathbf{x} = \begin{bmatrix} 3 & -1 & 8 & 1 \\ 2 & 0 & 5 & -2 \\ 1 & 1 & 3 & -7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} 8 \\ 5 \\ 3 \end{bmatrix} + x_4\begin{bmatrix} 1 \\ -2 \\ -7 \end{bmatrix}$$

So we recognize the action of the function P as using the components of the vector (x_1, x_2, x_3, x_4) as scalars to form the output of P as a linear combination of the four columns of the matrix A, which are all members of $\mathbb{C}^3$, so the result is a vector in $\mathbb{C}^3$. We can rearrange this expression further, using our definitions of operations in $\mathbb{C}^3$ (Section VO).

$$\begin{aligned} P(\mathbf{x}) &= A\mathbf{x} && \text{Definition of } P \\ &= x_1\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + x_2\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} 8 \\ 5 \\ 3 \end{bmatrix} + x_4\begin{bmatrix} 1 \\ -2 \\ -7 \end{bmatrix} && \text{Definition MVP} \end{aligned}$$

$$= \begin{bmatrix} 3x_1 \\ 2x_1 \\ x_1 \end{bmatrix} + \begin{bmatrix} -x_2 \\ 0 \\ x_2 \end{bmatrix} + \begin{bmatrix} 8x_3 \\ 5x_3 \\ 3x_3 \end{bmatrix} + \begin{bmatrix} x_4 \\ -2x_4 \\ -7x_4 \end{bmatrix} \qquad \text{Definition CVSM}$$

$$= \begin{bmatrix} 3x_1 - x_2 + 8x_3 + x_4 \\ 2x_1 + 5x_3 - 2x_4 \\ x_1 + x_2 + 3x_3 - 7x_4 \end{bmatrix} \qquad \text{Definition CVA}$$

You might recognize this final expression as being similar in style to some previous examples (Example ALT) and some linear transformations defined in the archetypes (Archetype M through Archetype R). But the expression that says the output of this linear transformation is a linear combination of the columns of A is probably the most powerful way of thinking about examples of this type.

Almost forgot — we should verify that P is indeed a linear transformation. This is easy with two matrix properties from Section MM.

$$\begin{aligned} P(\mathbf{x}+\mathbf{y}) &= A(\mathbf{x}+\mathbf{y}) && \text{Definition of } P \\ &= A\mathbf{x} + A\mathbf{y} && \text{Theorem MMDAA} \\ &= P(\mathbf{x}) + P(\mathbf{y}) && \text{Definition of } P \end{aligned}$$

and

$$\begin{aligned} P(\alpha\mathbf{x}) &= A(\alpha\mathbf{x}) && \text{Definition of } P \\ &= \alpha(A\mathbf{x}) && \text{Theorem MMSMM} \\ &= \alpha P(\mathbf{x}) && \text{Definition of } P \end{aligned}$$

So by Definition LT, P is a linear transformation. △

So the multiplication of a vector by a matrix “transforms” the input vector into an output vector, possibly of a different size, by performing a linear combination. And this transformation happens in a “linear” fashion. This “functional” view of the matrix-vector product is the most important shift you can make right now in how you think about linear algebra. Here is the theorem, whose proof is very nearly an exact copy of the verification in the last example.

Theorem MBLT Matrices Build Linear Transformations
Suppose that A is an $m \times n$ matrix. Define a function $T\colon \mathbb{C}^n \to \mathbb{C}^m$ by $T(\mathbf{x}) = A\mathbf{x}$. Then T is a linear transformation.

Proof.

$$\begin{aligned} T(\mathbf{x}+\mathbf{y}) &= A(\mathbf{x}+\mathbf{y}) && \text{Definition of } T \\ &= A\mathbf{x} + A\mathbf{y} && \text{Theorem MMDAA} \\ &= T(\mathbf{x}) + T(\mathbf{y}) && \text{Definition of } T \end{aligned}$$

and

$$\begin{aligned} T(\alpha\mathbf{x}) &= A(\alpha\mathbf{x}) && \text{Definition of } T \\ &= \alpha(A\mathbf{x}) && \text{Theorem MMSMM} \\ &= \alpha T(\mathbf{x}) && \text{Definition of } T \end{aligned}$$

So by Definition LT, T is a linear transformation. ■

So Theorem MBLT gives us a rapid way to construct linear transformations. Grab an $m \times n$ matrix A, define $T(\mathbf{x}) = A\mathbf{x}$ and Theorem MBLT tells us that T is a linear transformation from $\mathbb{C}^n$ to $\mathbb{C}^m$, without any further checking.

We can turn Theorem MBLT around. You give me a linear transformation and I will give you a matrix.

Example MFLT Matrix from a linear transformation
Define the function $R\colon \mathbb{C}^3 \to \mathbb{C}^4$ by

$$R\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}2x_1 - 3x_2 + 4x_3\\ x_1 + x_2 + x_3\\ -x_1 + 5x_2 - 3x_3\\ x_2 - 4x_3\end{bmatrix}$$

You could verify that R is a linear transformation by applying the definition, but we will instead massage the expression defining a typical output until we recognize the form of a known class of linear transformations.

$$\begin{aligned}
R\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) &= \begin{bmatrix}2x_1 - 3x_2 + 4x_3\\ x_1 + x_2 + x_3\\ -x_1 + 5x_2 - 3x_3\\ x_2 - 4x_3\end{bmatrix}\\
&= \begin{bmatrix}2x_1\\x_1\\-x_1\\0\end{bmatrix} + \begin{bmatrix}-3x_2\\x_2\\5x_2\\x_2\end{bmatrix} + \begin{bmatrix}4x_3\\x_3\\-3x_3\\-4x_3\end{bmatrix} && \text{Definition CVA}\\
&= x_1\begin{bmatrix}2\\1\\-1\\0\end{bmatrix} + x_2\begin{bmatrix}-3\\1\\5\\1\end{bmatrix} + x_3\begin{bmatrix}4\\1\\-3\\-4\end{bmatrix} && \text{Definition CVSM}\\
&= \begin{bmatrix}2 & -3 & 4\\1 & 1 & 1\\-1 & 5 & -3\\0 & 1 & -4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} && \text{Definition MVP}
\end{aligned}$$

So if we define the matrix

$$B = \begin{bmatrix}2 & -3 & 4\\1 & 1 & 1\\-1 & 5 & -3\\0 & 1 & -4\end{bmatrix}$$

then $R(\mathbf{x}) = B\mathbf{x}$. By Theorem MBLT, we can easily recognize R as a linear transformation since it has the form described in the hypothesis of the theorem. △

Example MFLT was not an accident. Consider any one of the archetypes where both the domain and codomain are sets of column vectors (Archetype M through Archetype R) and you should be able to mimic the previous example. Here is the theorem, which is notable since it is our first occasion to use the full power of the defining properties of a linear transformation when our hypothesis includes a linear transformation.

Theorem MLTCV Matrix of a Linear Transformation, Column Vectors
Suppose that $T\colon \mathbb{C}^n \to \mathbb{C}^m$ is a linear transformation. Then there is an $m \times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.

Proof. The conclusion says a certain matrix exists. What better way to prove something exists than to actually build it? So our proof will be constructive (Proof Technique C), and the procedure that we will use abstractly in the proof can be used concretely in specific examples.

Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \ldots, \mathbf{e}_n$ be the columns of the identity matrix of size n, I_n (Definition SUV). Evaluate the linear transformation T with each of these standard unit vectors as an input, and record the result. In other words, define n vectors in $\mathbb{C}^m$, $\mathbf{A}_i$, $1 \leq i \leq n$ by

$$\mathbf{A}_i = T(\mathbf{e}_i)$$

Then package up these vectors as the columns of a matrix

$$A = [\mathbf{A}_1|\mathbf{A}_2|\mathbf{A}_3|\ldots|\mathbf{A}_n]$$

Does A have the desired properties? First, A is clearly an $m \times n$ matrix. Then

$$\begin{aligned}
T(\mathbf{x}) &= T(I_n\mathbf{x}) && \text{Theorem MMIM}\\
&= T([\mathbf{e}_1|\mathbf{e}_2|\mathbf{e}_3|\ldots|\mathbf{e}_n]\,\mathbf{x}) && \text{Definition SUV}\\
&= T([\mathbf{x}]_1\,\mathbf{e}_1 + [\mathbf{x}]_2\,\mathbf{e}_2 + [\mathbf{x}]_3\,\mathbf{e}_3 + \cdots + [\mathbf{x}]_n\,\mathbf{e}_n) && \text{Definition MVP}\\
&= T([\mathbf{x}]_1\,\mathbf{e}_1) + T([\mathbf{x}]_2\,\mathbf{e}_2) + T([\mathbf{x}]_3\,\mathbf{e}_3) + \cdots + T([\mathbf{x}]_n\,\mathbf{e}_n) && \text{Definition LT}\\
&= [\mathbf{x}]_1\,T(\mathbf{e}_1) + [\mathbf{x}]_2\,T(\mathbf{e}_2) + [\mathbf{x}]_3\,T(\mathbf{e}_3) + \cdots + [\mathbf{x}]_n\,T(\mathbf{e}_n) && \text{Definition LT}\\
&= [\mathbf{x}]_1\,\mathbf{A}_1 + [\mathbf{x}]_2\,\mathbf{A}_2 + [\mathbf{x}]_3\,\mathbf{A}_3 + \cdots + [\mathbf{x}]_n\,\mathbf{A}_n && \text{Definition of } \mathbf{A}_i\\
&= A\mathbf{x} && \text{Definition MVP}
\end{aligned}$$

as desired. ■

So if we were to restrict our study of linear transformations to those where the domain and codomain are both vector spaces of column vectors (Definition VSCV), every matrix leads to a linear transformation of this type (Theorem MBLT), while every such linear transformation leads to a matrix (Theorem MLTCV). So matrices and linear transformations are fundamentally the same. We call the matrix A of Theorem MLTCV the **matrix representation** of T.

We have defined linear transformations for more general vector spaces than just $\mathbb{C}^m$. Can we extend this correspondence between linear transformations and matrices to more general linear transformations (more general domains and codomains)? Yes, and this is the main theme of Chapter R. Stay tuned. For now, let us illustrate Theorem MLTCV with an example.

Example MOLT Matrix of a linear transformation
Suppose $S\colon \mathbb{C}^3 \to \mathbb{C}^4$ is defined by

$$S\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}3x_1 - 2x_2 + 5x_3\\ x_1 + x_2 + x_3\\ 9x_1 - 2x_2 + 5x_3\\ 4x_2\end{bmatrix}$$

Then

$$\mathbf{C}_1 = S(\mathbf{e_1}) = S\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}3\\1\\9\\0\end{bmatrix}$$

$$\mathbf{C}_2 = S(\mathbf{e_2}) = S\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}-2\\1\\-2\\4\end{bmatrix}$$

$$\mathbf{C}_3 = S(\mathbf{e_3}) = S\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}5\\1\\5\\0\end{bmatrix}$$

so define

$$C = [C_1|C_2|C_3] = \begin{bmatrix}3 & -2 & 5\\ 1 & 1 & 1\\ 9 & -2 & 5\\ 0 & 4 & 0\end{bmatrix}$$

and Theorem MLTCV guarantees that $S(\mathbf{x}) = C\mathbf{x}$.

As an illuminating exercise, let $\mathbf{z} = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$ and compute $S(\mathbf{z})$ two different ways. First, return to the definition of S and evaluate $S(\mathbf{z})$ directly. Then do the matrix-vector product $C\mathbf{z}$. In both cases you should obtain the vector $S(\mathbf{z}) = \begin{bmatrix} 27 \\ 2 \\ 39 \\ -12 \end{bmatrix}$.

△

Subsection LTLC
Linear Transformations and Linear Combinations

It is the interaction between linear transformations and linear combinations that lies at the heart of many of the important theorems of linear algebra. The next theorem distills the essence of this. The proof is not deep, the result is hardly startling, but it will be referenced frequently. We have already passed by one occasion to employ it, in the proof of Theorem MLTCV. Paraphrasing, this theorem says that we can "push" linear transformations "down into" linear combinations, or "pull" linear transformations "up out" of linear combinations. We will have opportunities to both push and pull.

Theorem LTLC Linear Transformations and Linear Combinations
Suppose that $T\colon U \to V$ *is a linear transformation,* $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_t$ *are vectors from* U *and* $a_1, a_2, a_3, \ldots, a_t$ *are scalars from* $\mathbb{C}$. *Then*

$$T(a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \cdots + a_t\mathbf{u}_t) = a_1T(\mathbf{u}_1) + a_2T(\mathbf{u}_2) + a_3T(\mathbf{u}_3) + \cdots + a_tT(\mathbf{u}_t)$$

Proof.

$$\begin{aligned} &T(a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \cdots + a_t\mathbf{u}_t) \\ &\quad= T(a_1\mathbf{u}_1) + T(a_2\mathbf{u}_2) + T(a_3\mathbf{u}_3) + \cdots + T(a_t\mathbf{u}_t) && \text{Definition LT} \\ &\quad= a_1T(\mathbf{u}_1) + a_2T(\mathbf{u}_2) + a_3T(\mathbf{u}_3) + \cdots + a_tT(\mathbf{u}_t) && \text{Definition LT} \end{aligned}$$

■

Some authors, especially in more advanced texts, take the conclusion of Theorem LTLC as the defining condition of a linear transformation. This has the appeal of being a single condition, rather than the two-part condition of Definition LT. (See Exercise LT.T20).

Our next theorem says, informally, that it is enough to know how a linear transformation behaves for inputs from any basis of the domain, and *all* the other outputs are described by a linear combination of these few values. Again, the statement of the theorem, and its proof, are not remarkable, but the insight that goes along with it is very fundamental.

Theorem LTDB Linear Transformation Defined on a Basis
Suppose U *is a vector space with basis* $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$ *and the vector space* V *contains the vectors* $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n$ *(which may not be distinct). Then there is a unique linear transformation,* $T\colon U \to V$, *such that* $T(\mathbf{u}_i) = \mathbf{v}_i$, $1 \leq i \leq n$.

Proof. To prove the existence of T, we construct a function and show that it is a linear transformation (Proof Technique C). Suppose $\mathbf{w} \in U$ is an arbitrary element of the domain. Then by Theorem VRRB there are unique scalars $a_1, a_2, a_3, \ldots, a_n$ such that

$$\mathbf{w} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \cdots + a_n\mathbf{u}_n$$

Then *define* the function T by

$$T(\mathbf{w}) = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \cdots + a_n\mathbf{v}_n$$

It should be clear that T behaves as required for n inputs from B. Since the scalars provided by Theorem VRRB are unique, there is no ambiguity in this definition, and T qualifies as a function with domain U and codomain V (i.e. T is well-defined). But is T a linear transformation as well?

Let $\mathbf{x} \in U$ be a second element of the domain, and suppose the scalars provided by Theorem VRRB (relative to B) are $b_1, b_2, b_3, \ldots, b_n$. Then

$$\begin{aligned}
T(\mathbf{w}+\mathbf{x}) &= T(a_1\mathbf{u}_1 + \cdots + a_n\mathbf{u}_n + b_1\mathbf{u}_1 + \cdots + b_n\mathbf{u}_n) \\
&= T((a_1+b_1)\mathbf{u}_1 + \cdots + (a_n+b_n)\mathbf{u}_n) && \text{Definition VS} \\
&= (a_1+b_1)\mathbf{v}_1 + \cdots + (a_n+b_n)\mathbf{v}_n && \text{Definition of } T \\
&= a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n + b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n && \text{Definition VS} \\
&= T(\mathbf{w}) + T(\mathbf{x})
\end{aligned}$$

Let $\alpha \in \mathbb{C}$ be any scalar. Then

$$\begin{aligned}
T(\alpha\mathbf{w}) &= T(\alpha(a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \cdots + a_n\mathbf{u}_n)) \\
&= T(\alpha a_1\mathbf{u}_1 + \alpha a_2\mathbf{u}_2 + \alpha a_3\mathbf{u}_3 + \cdots + \alpha a_n\mathbf{u}_n) && \text{Definition VS} \\
&= \alpha a_1\mathbf{v}_1 + \alpha a_2\mathbf{v}_2 + \alpha a_3\mathbf{v}_3 + \cdots + \alpha a_n\mathbf{v}_n && \text{Definition of } T \\
&= \alpha(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \cdots + a_n\mathbf{v}_n) && \text{Definition VS} \\
&= \alpha T(\mathbf{w})
\end{aligned}$$

So by Definition LT, T is a linear transformation.

Is T unique (among all linear transformations that take the $\mathbf{u}_i$ to the $\mathbf{v}_i$)? Applying Proof Technique U, we posit the existence of a second linear transformation, $S\colon U \to V$ such that $S(\mathbf{u}_i) = \mathbf{v}_i$, $1 \leq i \leq n$. Again, let $\mathbf{w} \in U$ represent an arbitrary element of U and let $a_1, a_2, a_3, \ldots, a_n$ be the scalars provided by Theorem VRRB (relative to B). We have,

$$\begin{aligned}
T(\mathbf{w}) &= T(a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \cdots + a_n\mathbf{u}_n) && \text{Theorem VRRB} \\
&= a_1T(\mathbf{u}_1) + a_2T(\mathbf{u}_2) + a_3T(\mathbf{u}_3) + \cdots + a_nT(\mathbf{u}_n) && \text{Theorem LTLC} \\
&= a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + \cdots + a_n\mathbf{v}_n && \text{Definition of } T \\
&= a_1S(\mathbf{u}_1) + a_2S(\mathbf{u}_2) + a_3S(\mathbf{u}_3) + \cdots + a_nS(\mathbf{u}_n) && \text{Definition of } S \\
&= S(a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \cdots + a_n\mathbf{u}_n) && \text{Theorem LTLC} \\
&= S(\mathbf{w}) && \text{Theorem VRRB}
\end{aligned}$$

So the output of T and S agree on every input, which means they are equal as functions, $T = S$. So T is unique. ■

You might recall facts from analytic geometry, such as "any two points determine a line" and "any three non-collinear points determine a parabola." Theorem LTDB has much of the same feel. By specifying the n outputs for inputs from a basis, an entire linear transformation is determined. The analogy is not perfect, but the style of these facts are not very dissimilar from Theorem LTDB.

Notice that the statement of Theorem LTDB asserts the *existence* of a linear transformation with certain properties, while the proof shows us exactly how to define the desired linear transformation. The next two examples show how to compute values of linear transformations that we create this way.

Example LTDB1 Linear transformation defined on a basis
Consider the linear transformation $T\colon \mathbb{C}^3 \to \mathbb{C}^2$ that is required to have the following

three values,

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}2\\1\end{bmatrix}\qquad T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}-1\\4\end{bmatrix}\qquad T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}6\\0\end{bmatrix}$$

Because

$$B=\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\,\begin{bmatrix}0\\1\\0\end{bmatrix},\,\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$$

is a basis for $\mathbb{C}^3$ (Theorem SUVB), Theorem LTDB says there is a unique linear transformation T that behaves this way.

How do we compute other values of T? Consider the input

$$\mathbf{w}=\begin{bmatrix}2\\-3\\1\end{bmatrix}=(2)\begin{bmatrix}1\\0\\0\end{bmatrix}+(-3)\begin{bmatrix}0\\1\\0\end{bmatrix}+(1)\begin{bmatrix}0\\0\\1\end{bmatrix}$$

Then

$$T(\mathbf{w})=(2)\begin{bmatrix}2\\1\end{bmatrix}+(-3)\begin{bmatrix}-1\\4\end{bmatrix}+(1)\begin{bmatrix}6\\0\end{bmatrix}=\begin{bmatrix}13\\-10\end{bmatrix}$$

Doing it again,

$$\mathbf{x}=\begin{bmatrix}5\\2\\-3\end{bmatrix}=(5)\begin{bmatrix}1\\0\\0\end{bmatrix}+(2)\begin{bmatrix}0\\1\\0\end{bmatrix}+(-3)\begin{bmatrix}0\\0\\1\end{bmatrix}$$

so

$$T(\mathbf{x})=(5)\begin{bmatrix}2\\1\end{bmatrix}+(2)\begin{bmatrix}-1\\4\end{bmatrix}+(-3)\begin{bmatrix}6\\0\end{bmatrix}=\begin{bmatrix}-10\\13\end{bmatrix}$$

Any other value of T could be computed in a similar manner. So rather than being given a *formula* for the outputs of T, the *requirement* that T behave in a certain way for the inputs chosen from a basis of the domain, is as sufficient as a formula for computing any value of the function. You might notice some parallels between this example and Example MOLT or Theorem MLTCV. △

Example LTDB2 Linear transformation defined on a basis
Consider the linear transformation $R\colon \mathbb{C}^3\to\mathbb{C}^2$ with the three values,

$$R\left(\begin{bmatrix}1\\2\\1\end{bmatrix}\right)=\begin{bmatrix}5\\-1\end{bmatrix}\qquad R\left(\begin{bmatrix}-1\\5\\1\end{bmatrix}\right)=\begin{bmatrix}0\\4\end{bmatrix}\qquad R\left(\begin{bmatrix}3\\1\\4\end{bmatrix}\right)=\begin{bmatrix}2\\3\end{bmatrix}$$

You can check that

$$D=\left\{\begin{bmatrix}1\\2\\1\end{bmatrix},\,\begin{bmatrix}-1\\5\\1\end{bmatrix},\,\begin{bmatrix}3\\1\\4\end{bmatrix}\right\}$$

is a basis for $\mathbb{C}^3$ (make the vectors the columns of a square matrix and check that the matrix is nonsingular, Theorem CNMB). By Theorem LTDB we know there is a unique linear transformation R with the three specified outputs. However, we have to work just a bit harder to take an input vector and express it as a linear combination of the vectors in D.

For example, consider,

$$\mathbf{y}=\begin{bmatrix}8\\-3\\5\end{bmatrix}$$

Then we must first write $\mathbf{y}$ as a linear combination of the vectors in D and solve for the unknown scalars, to arrive at

$$\mathbf{y} = \begin{bmatrix} 8 \\ -3 \\ 5 \end{bmatrix} = (3)\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + (-2)\begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix} + (1)\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$$

Then the proof of Theorem LTDB gives us

$$R(\mathbf{y}) = (3)\begin{bmatrix} 5 \\ -1 \end{bmatrix} + (-2)\begin{bmatrix} 0 \\ 4 \end{bmatrix} + (1)\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 17 \\ -8 \end{bmatrix}$$

Any other value of R could be computed in a similar manner. △

Here is a third example of a linear transformation defined by its action on a basis, only with more abstract vector spaces involved.

Example LTDB3 Linear transformation defined on a basis
The set $W = \{p(x) \in P_3 \mid p(1) = 0, p(3) = 0\} \subseteq P_3$ is a subspace of the vector space of polynomials P_3. This subspace has $C = \left\{3 - 4x + x^2,\ 12 - 13x + x^3\right\}$ as a basis (check this!). Suppose we consider the linear transformation $S\colon P_3 \to M_{22}$ with values

$$S\left(3 - 4x + x^2\right) = \begin{bmatrix} 1 & -3 \\ 2 & 0 \end{bmatrix} \qquad S\left(12 - 13x + x^3\right) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

By Theorem LTDB we know there is a unique linear transformation with these two values. To illustrate a sample computation of S, consider $q(x) = 9-6x-5x^2+2x^3$. Verify that $q(x)$ is an element of W (does it have roots at $x = 1$ and $x = 3$?), then find the scalars needed to write it as a linear combination of the basis vectors in C. Because

$$q(x) = 9 - 6x - 5x^2 + 2x^3 = (-5)(3 - 4x + x^2) + (2)(12 - 13x + x^3)$$

The proof of Theorem LTDB gives us

$$S(q) = (-5)\begin{bmatrix} 1 & -3 \\ 2 & 0 \end{bmatrix} + (2)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -5 & 17 \\ -8 & 0 \end{bmatrix}$$

And all the other outputs of S could be computed in the same manner. Every output of S will have a zero in the second row, second column. Can you see why this is so? △

Informally, we can describe Theorem LTDB by saying "it is enough to know what a linear transformation does to a basis (of the domain)."

Subsection PI
Pre-Images

The definition of a function requires that for each input in the domain there is *exactly* one output in the codomain. However, the correspondence does not have to behave the other way around. An output from the codomain could have many different inputs from the domain which the transformation sends to that output, or there could be no inputs at all which the transformation sends to that output. To formalize our discussion of this aspect of linear transformations, we define the pre-image.

Definition PI Pre-Image
Suppose that $T\colon U \to V$ is a linear transformation. For each $\mathbf{v}$, define the **pre-image** of $\mathbf{v}$ to be the subset of U given by

$$T^{-1}(\mathbf{v}) = \{\mathbf{u} \in U \mid T(\mathbf{u}) = \mathbf{v}\}$$

□

In other words, $T^{-1}(\mathbf{v})$ is the set of all those vectors in the domain U that get "sent" to the vector $\mathbf{v}$.

Example SPIAS Sample pre-images, Archetype S
Archetype S is the linear transformation defined by

$$T\colon \mathbb{C}^3 \to M_{22}, \quad T\left(\begin{bmatrix} a\\ b\\ c \end{bmatrix}\right) = \begin{bmatrix} a-b & 2a+2b+c\\ 3a+b+c & -2a-6b-2c \end{bmatrix}$$

We could compute a pre-image for every element of the codomain M_{22}. However, even in a free textbook, we do not have the room to do that, so we will compute just two.

Choose

$$\mathbf{v} = \begin{bmatrix} 2 & 1\\ 3 & 2 \end{bmatrix} \in M_{22}$$

for no particular reason. What is $T^{-1}(\mathbf{v})$? Suppose $\mathbf{u} = \begin{bmatrix} u_1\\ u_2\\ u_3 \end{bmatrix} \in T^{-1}(\mathbf{v})$. The condition that $T(\mathbf{u}) = \mathbf{v}$ becomes

$$\begin{bmatrix} 2 & 1\\ 3 & 2 \end{bmatrix} = \mathbf{v} = T(\mathbf{u}) = T\left(\begin{bmatrix} u_1\\ u_2\\ u_3 \end{bmatrix}\right) = \begin{bmatrix} u_1-u_2 & 2u_1+2u_2+u_3\\ 3u_1+u_2+u_3 & -2u_1-6u_2-2u_3 \end{bmatrix}$$

Using matrix equality (Definition ME), we arrive at a system of four equations in the three unknowns u_1, u_2, u_3 with an augmented matrix that we can row-reduce in the hunt for solutions,

$$\begin{bmatrix} 1 & -1 & 0 & 2\\ 2 & 2 & 1 & 1\\ 3 & 1 & 1 & 3\\ -2 & -6 & -2 & 2 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & \frac{1}{4} & \frac{5}{4}\\ 0 & \boxed{1} & \frac{1}{4} & -\frac{3}{4}\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We recognize this system as having infinitely many solutions described by the single free variable u_3. Eventually obtaining the vector form of the solutions (Theorem VFSLS), we can describe the preimage precisely as,

$$\begin{aligned} T^{-1}(\mathbf{v}) &= \left\{\mathbf{u} \in \mathbb{C}^3 \,\middle|\, T(\mathbf{u}) = \mathbf{v}\right\}\\ &= \left\{\begin{bmatrix} u_1\\ u_2\\ u_3 \end{bmatrix} \,\middle|\, u_1 = \frac{5}{4} - \frac{1}{4}u_3,\ u_2 = -\frac{3}{4} - \frac{1}{4}u_3\right\}\\ &= \left\{\begin{bmatrix} \frac{5}{4} - \frac{1}{4}u_3\\ -\frac{3}{4} - \frac{1}{4}u_3\\ u_3 \end{bmatrix} \,\middle|\, u_3 \in \mathbb{C}\right\}\\ &= \left\{\begin{bmatrix} \frac{5}{4}\\ -\frac{3}{4}\\ 0 \end{bmatrix} + u_3\begin{bmatrix} -\frac{1}{4}\\ -\frac{1}{4}\\ 1 \end{bmatrix} \,\middle|\, u_3 \in \mathbb{C}\right\}\\ &= \begin{bmatrix} \frac{5}{4}\\ -\frac{3}{4}\\ 0 \end{bmatrix} + \left\langle\left\{\begin{bmatrix} -\frac{1}{4}\\ -\frac{1}{4}\\ 1 \end{bmatrix}\right\}\right\rangle \end{aligned}$$

This last line is merely a suggestive way of describing the set on the previous line. You might create three or four vectors in the preimage, and evaluate T with each. Was the result what you expected? For a hint of things to come, you might try evaluating T with just the lone vector in the spanning set above. What was the result? Now take a look back at Theorem PSPHS. Hmmmm.

OK, let us compute another preimage, but with a different outcome this time.

Choose

$$\mathbf{v} = \begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} \in M_{22}$$

What is $T^{-1}(\mathbf{v})$? Suppose $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \in T^{-1}(\mathbf{v})$. That $T(\mathbf{u}) = \mathbf{v}$ becomes

$$\begin{bmatrix} 1 & 1 \\ 2 & 4 \end{bmatrix} = \mathbf{v} = T(\mathbf{u}) = T\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} u_1 - u_2 & 2u_1 + 2u_2 + u_3 \\ 3u_1 + u_2 + u_3 & -2u_1 - 6u_2 - 2u_3 \end{bmatrix}$$

Using matrix equality (Definition ME), we arrive at a system of four equations in the three unknowns u_1, u_2, u_3 with an augmented matrix that we can row-reduce in the hunt for solutions,

$$\begin{bmatrix} 1 & -1 & 0 & 1 \\ 2 & 2 & 1 & 1 \\ 3 & 1 & 1 & 2 \\ -2 & -6 & -2 & 4 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & \frac{1}{4} & 0 \\ 0 & \boxed{1} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

By Theorem RCLS we recognize this system as inconsistent. So no vector $\mathbf{u}$ is a member of $T^{-1}(\mathbf{v})$ and so

$$T^{-1}(\mathbf{v}) = \emptyset$$

△

The preimage is just a set, it is almost never a subspace of U (you might think about just when $T^{-1}(\mathbf{v})$ is a subspace, see Exercise ILT.T10). We will describe its properties going forward, and it will be central to the main ideas of this chapter.

Subsection NLTFO New Linear Transformations From Old

We can combine linear transformations in natural ways to create new linear transformations. So we will define these combinations and then prove that the results really are still linear transformations. First the sum of two linear transformations.

Definition LTA Linear Transformation Addition
Suppose that $T\colon U \to V$ and $S\colon U \to V$ are two linear transformations with the same domain and codomain. Then their **sum** is the function $T + S\colon U \to V$ whose outputs are defined by

$$(T + S)(\mathbf{u}) = T(\mathbf{u}) + S(\mathbf{u})$$

□

Notice that the first plus sign in the definition is the operation being defined, while the second one is the vector addition in V. (Vector addition in U will appear just now in the proof that $T + S$ is a linear transformation.) Definition LTA only provides a function. It would be nice to know that when the constituents (T, S) are linear transformations, then so too is $T + S$.

Theorem SLTLT Sum of Linear Transformations is a Linear Transformation
Suppose that $T\colon U \to V$ and $S\colon U \to V$ are two linear transformations with the same domain and codomain. Then $T + S\colon U \to V$ is a linear transformation.

Proof. We simply check the defining properties of a linear transformation (Definition LT). This is a good place to consistently ask yourself which objects are being combined with which operations.

$$(T + S)(\mathbf{x} + \mathbf{y}) = T(\mathbf{x} + \mathbf{y}) + S(\mathbf{x} + \mathbf{y}) \qquad \text{Definition LTA}$$

$$
\begin{aligned}
&= T(\mathbf{x}) + T(\mathbf{y}) + S(\mathbf{x}) + S(\mathbf{y}) && \text{Definition LT}\\
&= T(\mathbf{x}) + S(\mathbf{x}) + T(\mathbf{y}) + S(\mathbf{y}) && \text{Property C in } V\\
&= (T+S)(\mathbf{x}) + (T+S)(\mathbf{y}) && \text{Definition LTA}
\end{aligned}
$$

and

$$
\begin{aligned}
(T+S)(\alpha\mathbf{x}) &= T(\alpha\mathbf{x}) + S(\alpha\mathbf{x}) && \text{Definition LTA}\\
&= \alpha T(\mathbf{x}) + \alpha S(\mathbf{x}) && \text{Definition LT}\\
&= \alpha\left(T(\mathbf{x}) + S(\mathbf{x})\right) && \text{Property DVA in } V\\
&= \alpha(T+S)(\mathbf{x}) && \text{Definition LTA}
\end{aligned}
$$

■

Example STLT Sum of two linear transformations
Suppose that $T\colon \mathbb{C}^2 \to \mathbb{C}^3$ and $S\colon \mathbb{C}^2 \to \mathbb{C}^3$ are defined by

$$
T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = \begin{bmatrix}x_1+2x_2\\3x_1-4x_2\\5x_1+2x_2\end{bmatrix} \qquad S\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = \begin{bmatrix}4x_1-x_2\\x_1+3x_2\\-7x_1+5x_2\end{bmatrix}
$$

Then by Definition LTA, we have

$$
\begin{aligned}
(T+S)\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) &= T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) + S\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)\\
&= \begin{bmatrix}x_1+2x_2\\3x_1-4x_2\\5x_1+2x_2\end{bmatrix} + \begin{bmatrix}4x_1-x_2\\x_1+3x_2\\-7x_1+5x_2\end{bmatrix} = \begin{bmatrix}5x_1+x_2\\4x_1-x_2\\-2x_1+7x_2\end{bmatrix}
\end{aligned}
$$

and by Theorem SLTLT we know $T+S$ is also a linear transformation from $\mathbb{C}^2$ to $\mathbb{C}^3$. △

Definition LTSM Linear Transformation Scalar Multiplication
Suppose that $T\colon U \to V$ is a linear transformation and $\alpha \in \mathbb{C}$. Then the **scalar multiple** is the function $\alpha T\colon U \to V$ whose outputs are defined by

$$
(\alpha T)(\mathbf{u}) = \alpha T(\mathbf{u})
$$

□

Given that T is a linear transformation, it would be nice to know that αT is also a linear transformation.

Theorem MLTLT Multiple of a Linear Transformation is a Linear Transformation
Suppose that $T\colon U \to V$ is a linear transformation and $\alpha \in \mathbb{C}$. Then $(\alpha T)\colon U \to V$ is a linear transformation.

Proof. We simply check the defining properties of a linear transformation (Definition LT). This is another good place to consistently ask yourself which objects are being combined with which operations.

$$
\begin{aligned}
(\alpha T)(\mathbf{x}+\mathbf{y}) &= \alpha\left(T(\mathbf{x}+\mathbf{y})\right) && \text{Definition LTSM}\\
&= \alpha\left(T(\mathbf{x}) + T(\mathbf{y})\right) && \text{Definition LT}\\
&= \alpha T(\mathbf{x}) + \alpha T(\mathbf{y}) && \text{Property DVA in } V\\
&= (\alpha T)(\mathbf{x}) + (\alpha T)(\mathbf{y}) && \text{Definition LTSM}
\end{aligned}
$$

and

$$
\begin{aligned}
(\alpha T)(\beta\mathbf{x}) &= \alpha T(\beta\mathbf{x}) && \text{Definition LTSM}\\
&= \alpha\left(\beta T(\mathbf{x})\right) && \text{Definition LT}
\end{aligned}
$$

$$
\begin{aligned}
&= (\alpha\beta)\, T(\mathbf{x}) && \text{Property SMA in } V \\
&= (\beta\alpha)\, T(\mathbf{x}) && \text{Commutativity in } \mathbb{C} \\
&= \beta\, (\alpha T(\mathbf{x})) && \text{Property SMA in } V \\
&= \beta\, ((\alpha T)(\mathbf{x})) && \text{Definition LTSM}
\end{aligned}
$$

■

Example SMLT Scalar multiple of a linear transformation
Suppose that $T\colon \mathbb{C}^4 \to \mathbb{C}^3$ is defined by

$$T\left(\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}\right) = \begin{bmatrix}x_1+2x_2-x_3+2x_4\\x_1+5x_2-3x_3+x_4\\-2x_1+3x_2-4x_3+2x_4\end{bmatrix}$$

For the sake of an example, choose $\alpha = 2$, so by Definition LTSM, we have

$$\alpha T\left(\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}\right) = 2T\left(\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}\right) = 2\begin{bmatrix}x_1+2x_2-x_3+2x_4\\x_1+5x_2-3x_3+x_4\\-2x_1+3x_2-4x_3+2x_4\end{bmatrix}$$

$$= \begin{bmatrix}2x_1+4x_2-2x_3+4x_4\\2x_1+10x_2-6x_3+2x_4\\-4x_1+6x_2-8x_3+4x_4\end{bmatrix}$$

and by Theorem MLTLT we know $2T$ is also a linear transformation from $\mathbb{C}^4$ to $\mathbb{C}^3$.
△

Now, let us imagine we have two vector spaces, U and V, and we collect every possible linear transformation from U to V into one big set, and call it $\mathcal{LT}(U, V)$. Definition LTA and Definition LTSM tell us how we can "add" and "scalar multiply" two elements of $\mathcal{LT}(U, V)$. Theorem SLTLT and Theorem MLTLT tell us that if we do these operations, then the resulting functions are linear transformations that are also in $\mathcal{LT}(U, V)$. Hmmmm, sounds like a vector space to me! A set of objects, an addition and a scalar multiplication. Why not?

Theorem VSLT Vector Space of Linear Transformations
Suppose that U and V are vector spaces. Then the set of all linear transformations from U to V, $\mathcal{LT}(U, V)$, is a vector space when the operations are those given in Definition LTA and Definition LTSM.

Proof. Theorem SLTLT and Theorem MLTLT provide two of the ten properties in Definition VS. However, we still need to verify the remaining eight properties. By and large, the proofs are straightforward and rely on concocting the obvious object, or by reducing the question to the same vector space property in the vector space V.

The zero vector is of some interest, though. What linear transformation would we add to any other linear transformation, so as to keep the second one unchanged? The answer is $Z\colon U \to V$ defined by $Z(\mathbf{u}) = \mathbf{0}_V$ for every $\mathbf{u} \in U$. Notice how we do not need to know any of the specifics about U and V to make this definition of Z. ■

Definition LTC Linear Transformation Composition
Suppose that $T\colon U \to V$ and $S\colon V \to W$ are linear transformations. Then the **composition** of S and T is the function $(S \circ T)\colon U \to W$ whose outputs are defined by

$$(S \circ T)(\mathbf{u}) = S(T(\mathbf{u}))$$

□

Given that T and S are linear transformations, it would be nice to know that $S \circ T$ is also a linear transformation.

Theorem CLTLT Composition of Linear Transformations is a Linear Transformation
Suppose that $T\colon U \to V$ and $S\colon V \to W$ are linear transformations. Then $(S \circ T)\colon U \to W$ is a linear transformation.

Proof. We simply check the defining properties of a linear transformation (Definition LT).

$$\begin{aligned}
(S \circ T)(\mathbf{x} + \mathbf{y}) &= S(T(\mathbf{x} + \mathbf{y})) && \text{Definition LTC} \\
&= S(T(\mathbf{x}) + T(\mathbf{y})) && \text{Definition LT for } T \\
&= S(T(\mathbf{x})) + S(T(\mathbf{y})) && \text{Definition LT for } S \\
&= (S \circ T)(\mathbf{x}) + (S \circ T)(\mathbf{y}) && \text{Definition LTC}
\end{aligned}$$

and

$$\begin{aligned}
(S \circ T)(\alpha\mathbf{x}) &= S(T(\alpha\mathbf{x})) && \text{Definition LTC} \\
&= S(\alpha T(\mathbf{x})) && \text{Definition LT for } T \\
&= \alpha S(T(\mathbf{x})) && \text{Definition LT for } S \\
&= \alpha (S \circ T)(\mathbf{x}) && \text{Definition LTC}
\end{aligned}$$

■

Example CTLT Composition of two linear transformations
Suppose that $T\colon \mathbb{C}^2 \to \mathbb{C}^4$ and $S\colon \mathbb{C}^4 \to \mathbb{C}^3$ are defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 \\ 3x_1 - 4x_2 \\ 5x_1 + 2x_2 \\ 6x_1 - 3x_2 \end{bmatrix} \qquad S\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - x_2 + x_3 - x_4 \\ 5x_1 - 3x_2 + 8x_3 - 2x_4 \\ -4x_1 + 3x_2 - 4x_3 + 5x_4 \end{bmatrix}$$

Then by Definition LTC

$$\begin{aligned}
(S \circ T)\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) &= S\left(T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)\right) = S\left(\begin{bmatrix} x_1 + 2x_2 \\ 3x_1 - 4x_2 \\ 5x_1 + 2x_2 \\ 6x_1 - 3x_2 \end{bmatrix}\right) \\
&= \begin{bmatrix} 2(x_1 + 2x_2) - (3x_1 - 4x_2) + (5x_1 + 2x_2) - (6x_1 - 3x_2) \\ 5(x_1 + 2x_2) - 3(3x_1 - 4x_2) + 8(5x_1 + 2x_2) - 2(6x_1 - 3x_2) \\ -4(x_1 + 2x_2) + 3(3x_1 - 4x_2) - 4(5x_1 + 2x_2) + 5(6x_1 - 3x_2) \end{bmatrix} \\
&= \begin{bmatrix} -2x_1 + 13x_2 \\ 24x_1 + 44x_2 \\ 15x_1 - 43x_2 \end{bmatrix}
\end{aligned}$$

and by Theorem CLTLT $S \circ T$ is a linear transformation from $\mathbb{C}^2$ to $\mathbb{C}^3$. △

Here is an interesting exercise that will presage an important result later. In Example STLT compute (via Theorem MLTCV) the matrix of T, S and $T + S$. Do you see a relationship between these three matrices?

In Example SMLT compute (via Theorem MLTCV) the matrix of T and $2T$. Do you see a relationship between these two matrices?

Here is the tough one. In Example CTLT compute (via Theorem MLTCV) the matrix of T, S and $S \circ T$. Do you see a relationship between these three matrices???

Reading Questions

1. Is the function below a linear transformation? Why or why not?

$$T\colon \mathbb{C}^3 \to \mathbb{C}^2, \quad T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 - x_2 + x_3 \\ 8x_2 - 6 \end{bmatrix}$$

2. Determine the matrix representation of the linear transformation S below.

$$S\colon \mathbb{C}^2 \to \mathbb{C}^3, \quad S\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + 5x_2 \\ 8x_1 - 3x_2 \\ -4x_1 \end{bmatrix}$$

3. Theorem LTLC has a fairly simple proof. Yet the result itself is very powerful. Comment on why we might say this.

Exercises

C15 The archetypes below are all linear transformations whose domains and codomains are vector spaces of column vectors (Definition VSCV). For each one, compute the matrix representation described in the proof of Theorem MLTCV.

Archetype M, Archetype N, Archetype O, Archetype P, Archetype Q, Archetype R

C16† Find the matrix representation of $T\colon \mathbb{C}^3 \to \mathbb{C}^4$, $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 3x + 2y + z \\ x + y + z \\ x - 3y \\ 2x + 3y + z \end{bmatrix}$.

C20† Let $\mathbf{w} = \begin{bmatrix} -3 \\ 1 \\ 4 \end{bmatrix}$. Referring to Example MOLT, compute $S(\mathbf{w})$ two different ways. First use the definition of S, then compute the matrix-vector product $C\mathbf{w}$ (Definition MVP).

C25† Define the linear transformation

$$T\colon \mathbb{C}^3 \to \mathbb{C}^2, \quad T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - x_2 + 5x_3 \\ -4x_1 + 2x_2 - 10x_3 \end{bmatrix}$$

Verify that T is a linear transformation.

C26† Verify that the function below is a linear transformation.

$$T\colon P_2 \to \mathbb{C}^2, \quad T\left(a + bx + cx^2\right) = \begin{bmatrix} 2a - b \\ b + c \end{bmatrix}$$

C30† Define the linear transformation

$$T\colon \mathbb{C}^3 \to \mathbb{C}^2, \quad T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - x_2 + 5x_3 \\ -4x_1 + 2x_2 - 10x_3 \end{bmatrix}$$

Compute the preimages, $T^{-1}\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right)$ and $T^{-1}\left(\begin{bmatrix} 4 \\ -8 \end{bmatrix}\right)$.

C31† For the linear transformation S compute the pre-images.

$$S\colon \mathbb{C}^3 \to \mathbb{C}^3, \quad S\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a - 2b - c \\ 3a - b + 2c \\ a + b + 2c \end{bmatrix}$$

$$S^{-1}\left(\begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}\right) \qquad S^{-1}\left(\begin{bmatrix} -5 \\ 5 \\ 7 \end{bmatrix}\right)$$

C40† If $T\colon \mathbb{C}^2 \to \mathbb{C}^2$ satisfies $T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ and $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, find $T\left(\begin{bmatrix} 4 \\ 3 \end{bmatrix}\right)$.

C41† If $T\colon\mathbb{C}^2\to\mathbb{C}^3$ satisfies $T\left(\begin{bmatrix}2\\3\end{bmatrix}\right)=\begin{bmatrix}2\\2\\1\end{bmatrix}$ and $T\left(\begin{bmatrix}3\\4\end{bmatrix}\right)=\begin{bmatrix}-1\\0\\2\end{bmatrix}$, find the matrix representation of T.

C42† Define $T\colon M_{2,2}\to\mathbb{R}$ by $T\left(\begin{bmatrix}a&b\\c&d\end{bmatrix}\right)=a+b+c-d$. Find the pre-image $T^{-1}(3)$.

C43† Define $T\colon P_3\to P_2$ by $T\left(a+bx+cx^2+dx^3\right)=b+2cx+3dx^2$. Find the pre-image of $\mathbf{0}$. Does this linear transformation seem familiar?

M10† Define two linear transformations, $T\colon\mathbb{C}^4\to\mathbb{C}^3$ and $S\colon\mathbb{C}^3\to\mathbb{C}^2$ by

$$S\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)=\begin{bmatrix}x_1-2x_2+3x_3\\5x_1+4x_2+2x_3\end{bmatrix}\qquad T\left(\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}\right)=\begin{bmatrix}-x_1+3x_2+x_3+9x_4\\2x_1+x_3+7x_4\\4x_1+2x_2+x_3+2x_4\end{bmatrix}$$

Using the proof of Theorem MLTCV compute the matrix representations of the three linear transformations T, S and $S\circ T$. Discover and comment on the relationship between these three matrices.

M60 Suppose U and V are vector spaces and define a function $Z\colon U\to V$ by $T(\mathbf{u})=\mathbf{0}_V$ for every $\mathbf{u}\in U$. Prove that Z is a (stupid) linear transformation. (See Exercise ILT.M60, Exercise SLT.M60, Exercise IVLT.M60.)

T20 Use the conclusion of Theorem LTLC to motivate a new definition of a linear transformation. Then prove that your new definition is equivalent to Definition LT. (Proof Technique D and Proof Technique E might be helpful if you are not sure what you are being asked to prove here.)

Theorem SER established three properties of matrix similarity that are collectively known as the defining properties of an “equivalence relation”. Exercises T30 and T31 extend this idea to linear transformations.

T30 Suppose that $T\colon U\to V$ is a linear transformation. Say that two vectors from U, $\mathbf{x}$ and $\mathbf{y}$, are **related** exactly when $T(\mathbf{x})=T(\mathbf{y})$ in V. Prove the three properties of an equivalence relation on U: (a) for any $\mathbf{x}\in U$, $\mathbf{x}$ is related to $\mathbf{x}$, (b) if $\mathbf{x}$ is related to $\mathbf{y}$, then $\mathbf{y}$ is related to $\mathbf{x}$, and (c) if $\mathbf{x}$ is related to $\mathbf{y}$ and $\mathbf{y}$ is related to $\mathbf{z}$, then $\mathbf{x}$ is related to $\mathbf{z}$.

T31† Equivalence relations always create a partition of the set they are defined on, via a construction called equivalence classes. For the relation in the previous problem, the equivalence classes are the pre-images. Prove directly that the collection of pre-images partition U by showing that (a) every $\mathbf{x}\in U$ is contained in some pre-image, and that (b) any two different pre-images do not have any elements in common.

Section ILT
Injective Linear Transformations

Some linear transformations possess one, or both, of two key properties, which go by the names injective and surjective. We will see that they are closely related to ideas like linear independence and spanning, and subspaces like the null space and the column space. In this section we will define an injective linear transformation and analyze the resulting consequences. The next section will do the same for the surjective property. In the final section of this chapter we will see what happens when we have the two properties simultaneously.

Subsection ILT
Injective Linear Transformations

As usual, we lead with a definition.

Definition ILT Injective Linear Transformation
Suppose $T\colon U \to V$ is a linear transformation. Then T is **injective** if whenever $T(\mathbf{x}) = T(\mathbf{y})$, then $\mathbf{x} = \mathbf{y}$. □

Given an arbitrary function, it is possible for two different inputs to yield the same output (think about the function $f(x) = x^2$ and the inputs $x = 3$ and $x = -3$). For an injective function, this never happens. If we have equal outputs ($T(\mathbf{x}) = T(\mathbf{y})$) then we must have achieved those equal outputs by employing equal inputs ($\mathbf{x} = \mathbf{y}$). Some authors prefer the term **one-to-one** where we use injective, and we will sometimes refer to an injective linear transformation as an **injection**.

Subsection EILT
Examples of Injective Linear Transformations

It is perhaps most instructive to examine a linear transformation that is not injective first.

Example NIAQ Not injective, Archetype Q
Archetype Q is the linear transformation

$$T\colon \mathbb{C}^5 \to \mathbb{C}^5, \quad T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}\right) = \begin{bmatrix} -2x_1 + 3x_2 + 3x_3 - 6x_4 + 3x_5 \\ -16x_1 + 9x_2 + 12x_3 - 28x_4 + 28x_5 \\ -19x_1 + 7x_2 + 14x_3 - 32x_4 + 37x_5 \\ -21x_1 + 9x_2 + 15x_3 - 35x_4 + 39x_5 \\ -9x_1 + 5x_2 + 7x_3 - 16x_4 + 16x_5 \end{bmatrix}$$

Notice that for

$$\mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ -1 \\ 2 \\ 4 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} 4 \\ 7 \\ 0 \\ 5 \\ 7 \end{bmatrix}$$

we have

$$T\left(\begin{bmatrix} 1 \\ 3 \\ -1 \\ 2 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 55 \\ 72 \\ 77 \\ 31 \end{bmatrix} \qquad T\left(\begin{bmatrix} 4 \\ 7 \\ 0 \\ 5 \\ 7 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 55 \\ 72 \\ 77 \\ 31 \end{bmatrix}$$

So we have two vectors from the domain, $\mathbf{x} \neq \mathbf{y}$, yet $T(\mathbf{x}) = T(\mathbf{y})$, in violation of Definition ILT. This is another example where you should not concern yourself with how **x** and **y** were selected, as this will be explained shortly. However, do understand *why* these two vectors provide enough evidence to conclude that T is not injective.△

Here is a cartoon of a non-injective linear transformation. Notice that the central feature of this cartoon is that $T(\mathbf{u}) = \mathbf{v} = T(\mathbf{w})$. Even though this happens again with some unnamed vectors, it only takes one occurrence to destroy the possibility of injectivity. Note also that the two vectors displayed in the bottom of V have no bearing, either way, on the injectivity of T.

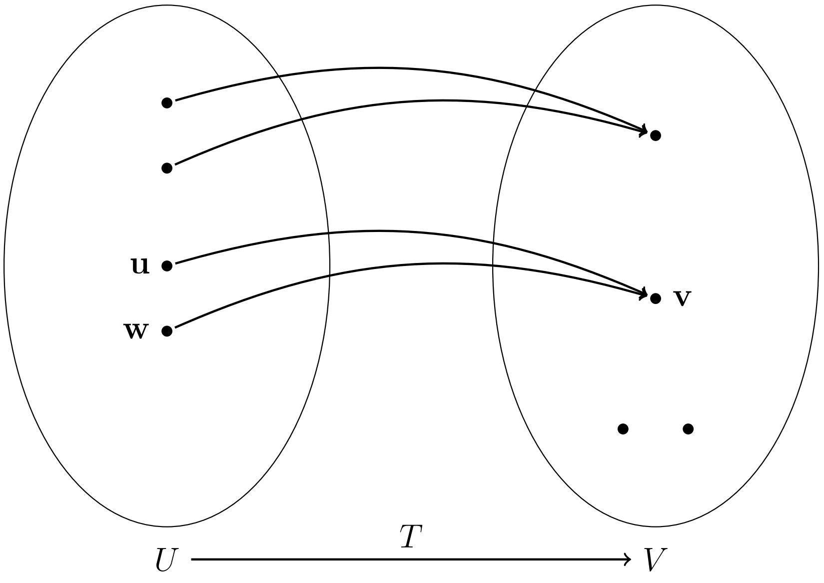

Diagram NILT: Non-Injective Linear Transformation

To show that a linear transformation is not injective, it is enough to find a single pair of inputs that get sent to the identical output, as in Example NIAQ. However, to show that a linear transformation is injective we must establish that this coincidence of outputs *never* occurs. Here is an example that shows how to establish this.

Example IAR Injective, Archetype R
Archetype R is the linear transformation

$$T\colon \mathbb{C}^5 \to \mathbb{C}^5, \quad T\left(\begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4\\ x_5 \end{bmatrix}\right) = \begin{bmatrix} -65x_1 + 128x_2 + 10x_3 - 262x_4 + 40x_5\\ 36x_1 - 73x_2 - x_3 + 151x_4 - 16x_5\\ -44x_1 + 88x_2 + 5x_3 - 180x_4 + 24x_5\\ 34x_1 - 68x_2 - 3x_3 + 140x_4 - 18x_5\\ 12x_1 - 24x_2 - x_3 + 49x_4 - 5x_5 \end{bmatrix}$$

To establish that R is injective we must begin with the assumption that $T(\mathbf{x}) = T(\mathbf{y})$ and somehow arrive at the conclusion that $\mathbf{x} = \mathbf{y}$. Here we go,

$$\begin{aligned} \begin{bmatrix} 0\\0\\0\\0\\0 \end{bmatrix} &= T(\mathbf{x}) - T(\mathbf{y})\\ &= T\left(\begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4\\ x_5 \end{bmatrix}\right) - T\left(\begin{bmatrix} y_1\\ y_2\\ y_3\\ y_4\\ y_5 \end{bmatrix}\right) \end{aligned}$$

$$= \begin{bmatrix} -65x_1 + 128x_2 + 10x_3 - 262x_4 + 40x_5 \\ 36x_1 - 73x_2 - x_3 + 151x_4 - 16x_5 \\ -44x_1 + 88x_2 + 5x_3 - 180x_4 + 24x_5 \\ 34x_1 - 68x_2 - 3x_3 + 140x_4 - 18x_5 \\ 12x_1 - 24x_2 - x_3 + 49x_4 - 5x_5 \end{bmatrix}$$

$$- \begin{bmatrix} -65y_1 + 128y_2 + 10y_3 - 262y_4 + 40y_5 \\ 36y_1 - 73y_2 - y_3 + 151y_4 - 16y_5 \\ -44y_1 + 88y_2 + 5y_3 - 180y_4 + 24y_5 \\ 34y_1 - 68y_2 - 3y_3 + 140y_4 - 18y_5 \\ 12y_1 - 24y_2 - y_3 + 49y_4 - 5y_5 \end{bmatrix}$$

$$= \begin{bmatrix} -65(x_1 - y_1) + 128(x_2 - y_2) + 10(x_3 - y_3) - 262(x_4 - y_4) + 40(x_5 - y_5) \\ 36(x_1 - y_1) - 73(x_2 - y_2) - (x_3 - y_3) + 151(x_4 - y_4) - 16(x_5 - y_5) \\ -44(x_1 - y_1) + 88(x_2 - y_2) + 5(x_3 - y_3) - 180(x_4 - y_4) + 24(x_5 - y_5) \\ 34(x_1 - y_1) - 68(x_2 - y_2) - 3(x_3 - y_3) + 140(x_4 - y_4) - 18(x_5 - y_5) \\ 12(x_1 - y_1) - 24(x_2 - y_2) - (x_3 - y_3) + 49(x_4 - y_4) - 5(x_5 - y_5) \end{bmatrix}$$

$$= \begin{bmatrix} -65 & 128 & 10 & -262 & 40 \\ 36 & -73 & -1 & 151 & -16 \\ -44 & 88 & 5 & -180 & 24 \\ 34 & -68 & -3 & 140 & -18 \\ 12 & -24 & -1 & 49 & -5 \end{bmatrix} \begin{bmatrix} x_1 - y_1 \\ x_2 - y_2 \\ x_3 - y_3 \\ x_4 - y_4 \\ x_5 - y_5 \end{bmatrix}$$

Now we recognize that we have a homogeneous system of 5 equations in 5 variables (the terms $x_i - y_i$ are the variables), so we row-reduce the coefficient matrix to

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

So the only solution is the trivial solution

$$x_1 - y_1 = 0 \qquad x_2 - y_2 = 0 \qquad x_3 - y_3 = 0 \qquad x_4 - y_4 = 0 \qquad x_5 - y_5 = 0$$

and we conclude that indeed $\mathbf{x} = \mathbf{y}$. By Definition ILT, T is injective. △

Here is the cartoon for an injective linear transformation. It is meant to suggest that we never have two inputs associated with a single output. Again, the two lonely vectors at the bottom of V have no bearing either way on the injectivity of T.

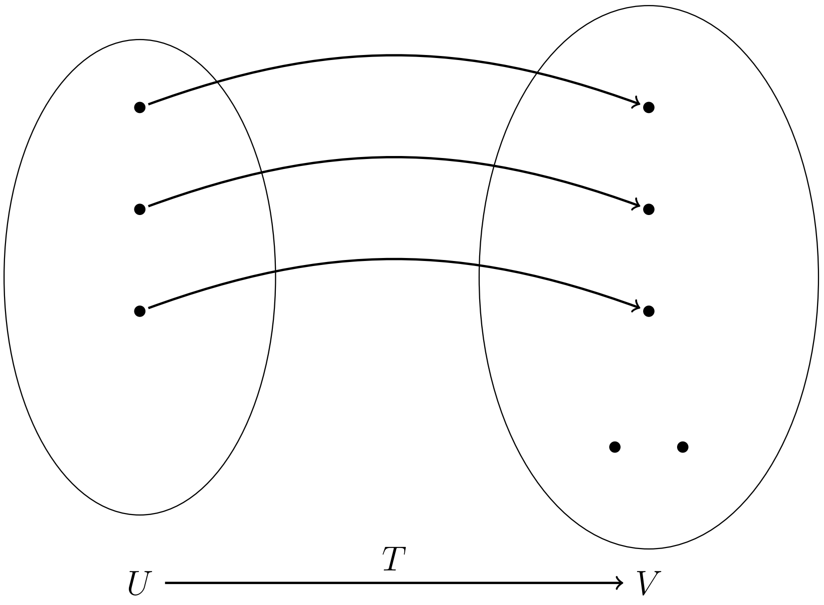

Diagram ILT: Injective Linear Transformation

Let us now examine an injective linear transformation between abstract vector spaces.

Example IAV Injective, Archetype V
Archetype V is defined by

$$T\colon P_3 \to M_{22}, \quad T\left(a + bx + cx^2 + dx^3\right) = \begin{bmatrix} a+b & a-2c \\ d & b-d \end{bmatrix}$$

To establish that the linear transformation is injective, begin by supposing that two polynomial inputs yield the same output matrix,

$$T\left(a_1 + b_1x + c_1x^2 + d_1x^3\right) = T\left(a_2 + b_2x + c_2x^2 + d_2x^3\right)$$

Then

$$\begin{aligned}
\mathcal{O} &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
&= T\left(a_1 + b_1x + c_1x^2 + d_1x^3\right) - T\left(a_2 + b_2x + c_2x^2 + d_2x^3\right) && \text{Hypothesis} \\
&= T\left((a_1 + b_1x + c_1x^2 + d_1x^3) - (a_2 + b_2x + c_2x^2 + d_2x^3)\right) && \text{Definition LT} \\
&= T\left((a_1 - a_2) + (b_1 - b_2)x + (c_1 - c_2)x^2 + (d_1 - d_2)x^3\right) && \text{Operations in } P_3 \\
&= \begin{bmatrix} (a_1 - a_2) + (b_1 - b_2) & (a_1 - a_2) - 2(c_1 - c_2) \\ (d_1 - d_2) & (b_1 - b_2) - (d_1 - d_2) \end{bmatrix} && \text{Definition of } T
\end{aligned}$$

This single matrix equality translates to the homogeneous system of equations in the variables $a_i - b_i$,

$$\begin{aligned}
(a_1 - a_2) + (b_1 - b_2) &= 0 \\
(a_1 - a_2) - 2(c_1 - c_2) &= 0 \\
(d_1 - d_2) &= 0 \\
(b_1 - b_2) - (d_1 - d_2) &= 0
\end{aligned}$$

This system of equations can be rewritten as the matrix equation

$$\begin{bmatrix}1 & 1 & 0 & 0\\1 & 0 & -2 & 0\\0 & 0 & 0 & 1\\0 & 1 & 0 & -1\end{bmatrix}\begin{bmatrix}(a_1-a_2)\\(b_1-b_2)\\(c_1-c_2)\\(d_1-d_2)\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

Since the coefficient matrix is nonsingular (check this) the only solution is trivial, i.e.

$$a_1-a_2=0 \qquad b_1-b_2=0 \qquad c_1-c_2=0 \qquad d_1-d_2=0$$

so that

$$a_1=a_2 \qquad b_1=b_2 \qquad c_1=c_2 \qquad d_1=d_2$$

so the two inputs must be equal polynomials. By Definition ILT, T is injective. △

Subsection KLT
Kernel of a Linear Transformation

For a linear transformation $T\colon U\to V$, the kernel is a subset of the domain U. Informally, it is the set of all inputs that the transformation sends to the zero vector of the codomain. It will have some natural connections with the null space of a matrix, so we will keep the same notation, and if you think about your objects, then there should be little confusion. Here is the careful definition.

Definition KLT Kernel of a Linear Transformation
Suppose $T\colon U\to V$ is a linear transformation. Then the **kernel** of T is the set

$$\mathcal{K}(T)=\{\mathbf{u}\in U|\,T(\mathbf{u})=\mathbf{0}\}$$

□

Notice that the kernel of T is just the preimage of $\mathbf{0}$, $T^{-1}(\mathbf{0})$ (Definition PI). Here is an example.

Example NKAO Nontrivial kernel, Archetype O
Archetype O is the linear transformation

$$T\colon\mathbb{C}^3\to\mathbb{C}^5,\quad T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right)=\begin{bmatrix}-x_1+x_2-3x_3\\-x_1+2x_2-4x_3\\x_1+x_2+x_3\\2x_1+3x_2+x_3\\x_1+2x_3\end{bmatrix}$$

To determine the elements of $\mathbb{C}^3$ in $\mathcal{K}(T)$, find those vectors $\mathbf{u}$ such that $T(\mathbf{u})=\mathbf{0}$, that is,

$$T(\mathbf{u})=\mathbf{0}$$
$$\begin{bmatrix}-u_1+u_2-3u_3\\-u_1+2u_2-4u_3\\u_1+u_2+u_3\\2u_1+3u_2+u_3\\u_1+2u_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}$$

Vector equality (Definition CVE) leads us to a homogeneous system of 5 equations in the variables u_i,

$$\begin{aligned}-u_1+u_2-3u_3&=0\\-u_1+2u_2-4u_3&=0\\u_1+u_2+u_3&=0\\2u_1+3u_2+u_3&=0\end{aligned}$$

$$u_1 + 2u_3 = 0$$

Row-reducing the coefficient matrix gives

$$\begin{bmatrix} \boxed{1} & 0 & 2 \\ 0 & \boxed{1} & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The kernel of T is the set of solutions to this homogeneous system of equations, which by Theorem BNS can be expressed as

$$\mathcal{K}(T) = \left\langle\left\{\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}\right\}\right\rangle$$

△

We know that the span of a set of vectors is always a subspace (Theorem SSS), so the kernel computed in Example NKAO is also a subspace. This is no accident, the kernel of a linear transformation is *always* a subspace.

Theorem KLTS Kernel of a Linear Transformation is a Subspace
Suppose that $T: U \to V$ is a linear transformation. Then the kernel of T, $\mathcal{K}(T)$, is a subspace of U.

Proof. We can apply the three-part test of Theorem TSS. First $T(\mathbf{0}_U) = \mathbf{0}_V$ by Theorem LTTZZ, so $\mathbf{0}_U \in \mathcal{K}(T)$ and we know that the kernel is nonempty.

Suppose we assume that $\mathbf{x}$, $\mathbf{y} \in \mathcal{K}(T)$. Is $\mathbf{x} + \mathbf{y} \in \mathcal{K}(T)$?

$$\begin{aligned} T(\mathbf{x} + \mathbf{y}) &= T(\mathbf{x}) + T(\mathbf{y}) && \text{Definition LT} \\ &= \mathbf{0} + \mathbf{0} && \mathbf{x},\ \mathbf{y} \in \mathcal{K}(T) \\ &= \mathbf{0} && \text{Property Z} \end{aligned}$$

This qualifies $\mathbf{x} + \mathbf{y}$ for membership in $\mathcal{K}(T)$. So we have additive closure.

Suppose we assume that $\alpha \in \mathbb{C}$ and $\mathbf{x} \in \mathcal{K}(T)$. Is $\alpha\mathbf{x} \in \mathcal{K}(T)$?

$$\begin{aligned} T(\alpha\mathbf{x}) &= \alpha T(\mathbf{x}) && \text{Definition LT} \\ &= \alpha\mathbf{0} && \mathbf{x} \in \mathcal{K}(T) \\ &= \mathbf{0} && \text{Theorem ZVSM} \end{aligned}$$

This qualifies $\alpha\mathbf{x}$ for membership in $\mathcal{K}(T)$. So we have scalar closure and Theorem TSS tells us that $\mathcal{K}(T)$ is a subspace of U. ■

Let us compute another kernel, now that we know in advance that it will be a subspace.

Example TKAP Trivial kernel, Archetype P
Archetype P is the linear transformation

$$T: \mathbb{C}^3 \to \mathbb{C}^5, \quad T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -x_1 + x_2 + x_3 \\ -x_1 + 2x_2 + 2x_3 \\ x_1 + x_2 + 3x_3 \\ 2x_1 + 3x_2 + x_3 \\ -2x_1 + x_2 + 3x_3 \end{bmatrix}$$

To determine the elements of $\mathbb{C}^3$ in $\mathcal{K}(T)$, find those vectors $\mathbf{u}$ such that $T(\mathbf{u}) = \mathbf{0}$, that is,

$$T(\mathbf{u}) = \mathbf{0}$$

$$\begin{bmatrix} -u_1 + u_2 + u_3 \\ -u_1 + 2u_2 + 2u_3 \\ u_1 + u_2 + 3u_3 \\ 2u_1 + 3u_2 + u_3 \\ -2u_1 + u_2 + 3u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Vector equality (Definition CVE) leads us to a homogeneous system of 5 equations in the variables u_i,

$$\begin{aligned} -u_1 + u_2 + u_3 &= 0 \\ -u_1 + 2u_2 + 2u_3 &= 0 \\ u_1 + u_2 + 3u_3 &= 0 \\ 2u_1 + 3u_2 + u_3 &= 0 \\ -2u_1 + u_2 + 3u_3 &= 0 \end{aligned}$$

Row-reducing the coefficient matrix gives

$$\begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The kernel of T is the set of solutions to this homogeneous system of equations, which is simply the trivial solution $\mathbf{u} = \mathbf{0}$, so

$$\mathcal{K}(T) = \{\mathbf{0}\} = \langle \{\ \} \rangle$$

△

Our next theorem says that if a preimage is a nonempty set then we can construct it by picking any one element and adding on elements of the kernel.

Theorem KPI Kernel and Pre-Image
Suppose $T\colon U \to V$ is a linear transformation and $\mathbf{v} \in V$. If the preimage $T^{-1}(\mathbf{v})$ is nonempty, and $\mathbf{u} \in T^{-1}(\mathbf{v})$ then

$$T^{-1}(\mathbf{v}) = \{\mathbf{u} + \mathbf{z} |\, \mathbf{z} \in \mathcal{K}(T)\} = \mathbf{u} + \mathcal{K}(T)$$

Proof. Let $M = \{\mathbf{u} + \mathbf{z} |\, \mathbf{z} \in \mathcal{K}(T)\}$. First, we show that $M \subseteq T^{-1}(\mathbf{v})$. Suppose that $\mathbf{w} \in M$, so $\mathbf{w}$ has the form $\mathbf{w} = \mathbf{u} + \mathbf{z}$, where $\mathbf{z} \in \mathcal{K}(T)$. Then

$$\begin{aligned} T(\mathbf{w}) &= T(\mathbf{u} + \mathbf{z}) \\ &= T(\mathbf{u}) + T(\mathbf{z}) && \text{Definition LT} \\ &= \mathbf{v} + \mathbf{0} && \mathbf{u} \in T^{-1}(\mathbf{v}),\ \mathbf{z} \in \mathcal{K}(T) \\ &= \mathbf{v} && \text{Property Z} \end{aligned}$$

which qualifies $\mathbf{w}$ for membership in the preimage of $\mathbf{v}$, $\mathbf{w} \in T^{-1}(\mathbf{v})$.

For the opposite inclusion, suppose $\mathbf{x} \in T^{-1}(\mathbf{v})$. Then,

$$\begin{aligned} T(\mathbf{x} - \mathbf{u}) &= T(\mathbf{x}) - T(\mathbf{u}) && \text{Definition LT} \\ &= \mathbf{v} - \mathbf{v} && \mathbf{x}, \mathbf{u} \in T^{-1}(\mathbf{v}) \\ &= \mathbf{0} \end{aligned}$$

This qualifies $\mathbf{x} - \mathbf{u}$ for membership in the kernel of T, $\mathcal{K}(T)$. So there is a vector $\mathbf{z} \in \mathcal{K}(T)$ such that $\mathbf{x} - \mathbf{u} = \mathbf{z}$. Rearranging this equation gives $\mathbf{x} = \mathbf{u} + \mathbf{z}$ and so $\mathbf{x} \in M$. So $T^{-1}(\mathbf{v}) \subseteq M$ and we see that $M = T^{-1}(\mathbf{v})$, as desired. ■

This theorem, and its proof, should remind you very much of Theorem PSPHS. Additionally, you might go back and review Example SPIAS. Can you tell now which is the only preimage to be a subspace?

Here is the cartoon which describes the "many-to-one" behavior of a typical linear transformation. Presume that $T(\mathbf{u}_i) = \mathbf{v}_i$, for $i = 1, 2, 3$, and as guaranteed by Theorem LTTZZ, $T(\mathbf{0}_U) = \mathbf{0}_V$. Then four pre-images are depicted, each labeled slightly different. $T^{-1}(\mathbf{v}_2)$ is the most general, employing Theorem KPI to provide two equal descriptions of the set. The most unusual is $T^{-1}(\mathbf{0}_V)$ which is equal to the kernel, $\mathcal{K}(T)$, and hence is a subspace (by Theorem KLTS). The subdivisions of the domain, U, are meant to suggest the partioning of the domain by the collection of pre-images. It also suggests that each pre-image is of similar size or structure, since each is a "shifted" copy of the kernel. Notice that we cannot speak of the dimension of a pre-image, since it is almost never a subspace. Also notice that $\mathbf{x}$, $\mathbf{y} \in V$ are elements of the codomain with empty pre-images.

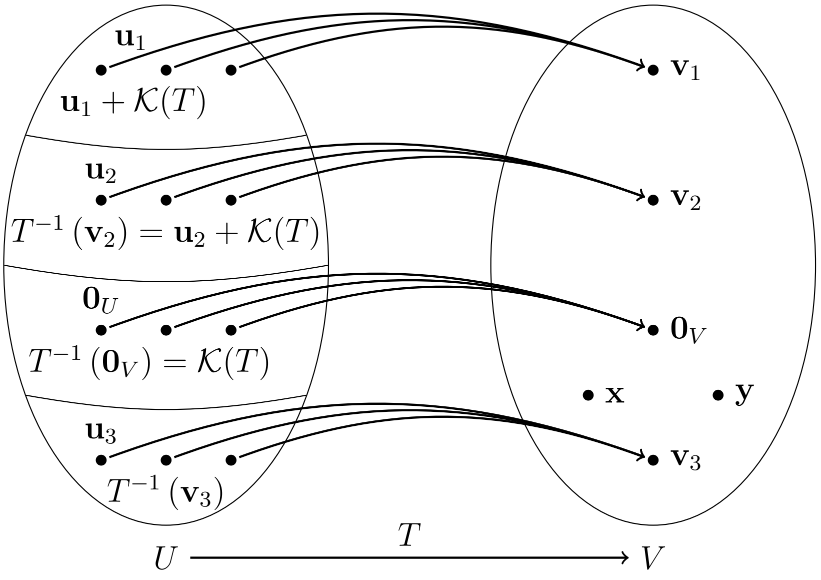

Diagram KPI: Kernel and Pre-Image

The next theorem is one we will cite frequently, as it characterizes injections by the size of the kernel.

Theorem KILT Kernel of an Injective Linear Transformation
Suppose that $T\colon U \to V$ is a linear transformation. Then T is injective if and only if the kernel of T is trivial, $\mathcal{K}(T) = \{\mathbf{0}\}$.

Proof. ($\Rightarrow$) We assume T is injective and we need to establish that two sets are equal (Definition SE). Since the kernel is a subspace (Theorem KLTS), $\{\mathbf{0}\} \subseteq \mathcal{K}(T)$. To establish the opposite inclusion, suppose $\mathbf{x} \in \mathcal{K}(T)$.

$$\begin{aligned} T(\mathbf{x}) &= \mathbf{0} && \text{Definition KLT} \\ &= T(\mathbf{0}) && \text{Theorem LTTZZ} \end{aligned}$$

We can apply Definition ILT to conclude that $\mathbf{x} = \mathbf{0}$. Therefore $\mathcal{K}(T) \subseteq \{\mathbf{0}\}$ and by Definition SE, $\mathcal{K}(T) = \{\mathbf{0}\}$.

($\Leftarrow$) To establish that T is injective, appeal to Definition ILT and begin with the assumption that $T(\mathbf{x}) = T(\mathbf{y})$. Then

$$\begin{aligned} T(\mathbf{x} - \mathbf{y}) &= T(\mathbf{x}) - T(\mathbf{y}) && \text{Definition LT} \\ &= \mathbf{0} && \text{Hypothesis} \end{aligned}$$

So $\mathbf{x} - \mathbf{y} \in \mathcal{K}(T)$ by Definition KLT and with the hypothesis that the kernel is trivial we conclude that $\mathbf{x}-\mathbf{y}=\mathbf{0}$. Then

$$\mathbf{y}=\mathbf{y}+\mathbf{0}=\mathbf{y}+(\mathbf{x}-\mathbf{y})=\mathbf{x}$$

thus establishing that T is injective by Definition ILT. ■

You might begin to think about how Diagram KPI would change if the linear transformation is injective, which would make the kernel trivial by Theorem KILT.

Example NIAQR Not injective, Archetype Q, revisited
We are now in a position to revisit our first example in this section, Example NIAQ. In that example, we showed that Archetype Q is not injective by constructing two vectors, which when used to evaluate the linear transformation provided the same output, thus violating Definition ILT. Just where did those two vectors come from?

The key is the vector

$$\mathbf{z}=\begin{bmatrix}3\\4\\1\\3\\3\end{bmatrix}$$

which you can check is an element of $\mathcal{K}(T)$ for Archetype Q. Choose a vector $\mathbf{x}$ at random, and then compute $\mathbf{y}=\mathbf{x}+\mathbf{z}$ (verify this computation back in Example NIAQ). Then

$$\begin{aligned}
T(\mathbf{y}) &= T(\mathbf{x}+\mathbf{z}) \\
&= T(\mathbf{x})+T(\mathbf{z}) && \text{Definition LT} \\
&= T(\mathbf{x})+\mathbf{0} && \mathbf{z}\in\mathcal{K}(T) \\
&= T(\mathbf{x}) && \text{Property Z}
\end{aligned}$$

Whenever the kernel of a linear transformation is nontrivial, we can employ this device and conclude that the linear transformation is not injective. This is another way of viewing Theorem KILT. For an injective linear transformation, the kernel is trivial and our only choice for $\mathbf{z}$ is the zero vector, which will not help us create two *different* inputs for T that yield identical outputs. For every one of the archetypes that is not injective, there is an example presented of exactly this form. △

Example NIAO Not injective, Archetype O
In Example NKAO the kernel of Archetype O was determined to be

$$\left\langle\left\{\begin{bmatrix}-2\\1\\1\end{bmatrix}\right\}\right\rangle$$

a subspace of $\mathbb{C}^3$ with dimension 1. Since the kernel is not trivial, Theorem KILT tells us that T is not injective. △

Example IAP Injective, Archetype P
In Example TKAP it was shown that the linear transformation in Archetype P has a trivial kernel. So by Theorem KILT, T is injective. △

Subsection ILTLI
Injective Linear Transformations and Linear Independence

There is a connection between injective linear transformations and linearly independent sets that we will make precise in the next two theorems. However, more informally, we can get a feel for this connection when we think about how each property is defined. A set of vectors is linearly independent if the **only** relation of linear dependence is the trivial one. A linear transformation is injective if the **only** way two input vectors can produce the same output is in the trivial way, when both input vectors are equal.

Theorem ILTLI Injective Linear Transformations and Linear Independence
Suppose that $T\colon U \to V$ *is an injective linear transformation and*

$$S = \{\mathbf{u}_1,\,\mathbf{u}_2,\,\mathbf{u}_3,\,\ldots,\,\mathbf{u}_t\}$$

is a linearly independent subset of U*. Then*

$$R = \{T(\mathbf{u}_1),\,T(\mathbf{u}_2),\,T(\mathbf{u}_3),\,\ldots,\,T(\mathbf{u}_t)\}$$

is a linearly independent subset of V*.*

Proof. Begin with a relation of linear dependence on R (Definition RLD, Definition LI),

$$\begin{aligned}
a_1T(\mathbf{u}_1) + a_2T(\mathbf{u}_2) + a_3T(\mathbf{u}_3) + \ldots + a_tT(\mathbf{u}_t) &= \mathbf{0} \\
T(a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \cdots + a_t\mathbf{u}_t) &= \mathbf{0} && \text{Theorem LTLC} \\
a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \cdots + a_t\mathbf{u}_t &\in \mathcal{K}(T) && \text{Definition KLT} \\
a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \cdots + a_t\mathbf{u}_t &\in \{\mathbf{0}\} && \text{Theorem KILT} \\
a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \cdots + a_t\mathbf{u}_t &= \mathbf{0} && \text{Definition SET}
\end{aligned}$$

Since this is a relation of linear dependence on the linearly independent set S, we can conclude that

$$a_1 = 0 \qquad a_2 = 0 \qquad a_3 = 0 \qquad \ldots \qquad a_t = 0$$

and this establishes that R is a linearly independent set. ■

Theorem ILTB Injective Linear Transformations and Bases
Suppose that $T\colon U \to V$ *is a linear transformation and*

$$B = \{\mathbf{u}_1,\,\mathbf{u}_2,\,\mathbf{u}_3,\,\ldots,\,\mathbf{u}_m\}$$

is a basis of U*. Then* T *is injective if and only if*

$$C = \{T(\mathbf{u}_1),\,T(\mathbf{u}_2),\,T(\mathbf{u}_3),\,\ldots,\,T(\mathbf{u}_m)\}$$

is a linearly independent subset of V*.*

Proof. ($\Rightarrow$) Assume T is injective. Since B is a basis, we know B is linearly independent (Definition B). Then Theorem ILTLI says that C is a linearly independent subset of V.

($\Leftarrow$) Assume that C is linearly independent. To establish that T is injective, we will show that the kernel of T is trivial (Theorem KILT). Suppose that $\mathbf{u} \in \mathcal{K}(T)$. As an element of U, we can write $\mathbf{u}$ as a linear combination of the basis vectors in B (uniquely). So there are are scalars, $a_1,\,a_2,\,a_3,\,\ldots,\,a_m$, such that

$$\mathbf{u} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \cdots + a_m\mathbf{u}_m$$

Then,

$$\begin{aligned}
\mathbf{0} &= T(\mathbf{u}) && \text{Definition KLT} \\
&= T(a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \cdots + a_m\mathbf{u}_m) && \text{Definition SSVS} \\
&= a_1T(\mathbf{u}_1) + a_2T(\mathbf{u}_2) + a_3T(\mathbf{u}_3) + \cdots + a_mT(\mathbf{u}_m) && \text{Theorem LTLC}
\end{aligned}$$

This is a relation of linear dependence (Definition RLD) on the linearly independent set C, so the scalars are all zero: $a_1 = a_2 = a_3 = \cdots = a_m = 0$. Then

$$\begin{aligned}
\mathbf{u} &= a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \cdots + a_m\mathbf{u}_m \\
&= 0\mathbf{u}_1 + 0\mathbf{u}_2 + 0\mathbf{u}_3 + \cdots + 0\mathbf{u}_m && \text{Theorem ZSSM} \\
&= \mathbf{0} + \mathbf{0} + \mathbf{0} + \cdots + \mathbf{0} && \text{Theorem ZSSM}
\end{aligned}$$

$$= \mathbf{0} \qquad \text{Property Z}$$

Since $\mathbf{u}$ was chosen as an arbitrary vector from $\mathcal{K}(T)$, we have $\mathcal{K}(T) = \{\mathbf{0}\}$ and Theorem KILT tells us that T is injective. ■

Subsection ILTD
Injective Linear Transformations and Dimension

Theorem ILTD Injective Linear Transformations and Dimension
Suppose that $T\colon U \to V$ is an injective linear transformation. Then $\dim(U) \leq \dim(V)$.

Proof. Suppose to the contrary that $m = \dim(U) > \dim(V) = t$. Let B be a basis of U, which will then contain m vectors. Apply T to each element of B to form a set C that is a subset of V. By Theorem ILTB, C is linearly independent and therefore must contain m distinct vectors. So we have found a set of m linearly independent vectors in V, a vector space of dimension t, with $m > t$. However, this contradicts Theorem G, so our assumption is false and $\dim(U) \leq \dim(V)$. ■

Example NIDAU Not injective by dimension, Archetype U
The linear transformation in Archetype U is

$$T\colon M_{23} \to \mathbb{C}^4, \quad T\left(\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\right) = \begin{bmatrix} a + 2b + 12c - 3d + e + 6f \\ 2a - b - c + d - 11f \\ a + b + 7c + 2d + e - 3f \\ a + 2b + 12c + 5e - 5f \end{bmatrix}$$

Since $\dim(M_{23}) = 6 > 4 = \dim\left(\mathbb{C}^4\right)$, T cannot be injective for then T would violate Theorem ILTD. △

Notice that the previous example made no use of the actual formula defining the function. Merely a comparison of the dimensions of the domain and codomain are enough to conclude that the linear transformation is not injective. Archetype M and Archetype N are two more examples of linear transformations that have "big" domains and "small" codomains, resulting in "collisions" of outputs and thus are non-injective linear transformations.

Subsection CILT
Composition of Injective Linear Transformations

In Subsection LT.NLTFO we saw how to combine linear transformations to build new linear transformations, specifically, how to build the composition of two linear transformations (Definition LTC). It will be useful later to know that the composition of injective linear transformations is again injective, so we prove that here.

Theorem CILTI Composition of Injective Linear Transformations is Injective
Suppose that $T\colon U \to V$ and $S\colon V \to W$ are injective linear transformations. Then $(S \circ T)\colon U \to W$ is an injective linear transformation.

Proof. That the composition is a linear transformation was established in Theorem CLTLT, so we need only establish that the composition is injective. Applying Definition ILT, choose $\mathbf{x}$, $\mathbf{y}$ from U. Then if $(S \circ T)(\mathbf{x}) = (S \circ T)(\mathbf{y})$,

$$\begin{aligned}
&\Rightarrow & S(T(\mathbf{x})) &= S(T(\mathbf{y})) && \text{Definition LTC} \\
&\Rightarrow & T(\mathbf{x}) &= T(\mathbf{y}) && \text{Definition ILT for } S \\
&\Rightarrow & \mathbf{x} &= \mathbf{y} && \text{Definition ILT for } T
\end{aligned}$$

■

Reading Questions

1. Suppose $T\colon \mathbb{C}^8 \to \mathbb{C}^5$ is a linear transformation. Why is T not injective?
2. Describe the kernel of an injective linear transformation.
3. Theorem KPI should remind you of Theorem PSPHS. Why do we say this?

Exercises

C10 Each archetype below is a linear transformation. Compute the kernel for each.

Archetype M, Archetype N, Archetype O, Archetype P, Archetype Q, Archetype R, Archetype S, Archetype T, Archetype U, Archetype V, Archetype W, Archetype X

C20† The linear transformation $T\colon \mathbb{C}^4 \to \mathbb{C}^3$ is not injective. Find two inputs $\mathbf{x},\,\mathbf{y} \in \mathbb{C}^4$ that yield the same output (that is $T(\mathbf{x}) = T(\mathbf{y})$).

$$T\left(\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}\right) = \begin{bmatrix}2x_1 + x_2 + x_3\\ -x_1 + 3x_2 + x_3 - x_4\\ 3x_1 + x_2 + 2x_3 - 2x_4\end{bmatrix}$$

C25† Define the linear transformation

$$T\colon \mathbb{C}^3 \to \mathbb{C}^2, \quad T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}2x_1 - x_2 + 5x_3\\ -4x_1 + 2x_2 - 10x_3\end{bmatrix}$$

Find a basis for the kernel of T, $\mathcal{K}(T)$. Is T injective?

C26† Let $A = \begin{bmatrix}1 & 2 & 3 & 1 & 0\\ 2 & -1 & 1 & 0 & 1\\ 1 & 2 & -1 & -2 & 1\\ 1 & 3 & 2 & 1 & 2\end{bmatrix}$ and let $T\colon \mathbb{C}^5 \to \mathbb{C}^4$ be given by $T(\mathbf{x}) = A\mathbf{x}$. Is T injective? (Hint: No calculation is required.)

C27† Let $T\colon \mathbb{C}^3 \to \mathbb{C}^3$ be given by $T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}2x + y + z\\ x - y + 2z\\ x + 2y - z\end{bmatrix}$. Find $\mathcal{K}(T)$. Is T injective?

C28† Let $A = \begin{bmatrix}1 & 2 & 3 & 1\\ 2 & -1 & 1 & 0\\ 1 & 2 & -1 & -2\\ 1 & 3 & 2 & 1\end{bmatrix}$ and let $T\colon \mathbb{C}^4 \to \mathbb{C}^4$ be given by $T(\mathbf{x}) = A\mathbf{x}$. Find $\mathcal{K}(T)$. Is T injective?

C29† Let $A = \begin{bmatrix}1 & 2 & 1 & 1\\ 2 & 1 & 1 & 0\\ 1 & 2 & 1 & 2\\ 1 & 2 & 1 & 1\end{bmatrix}$ and let $T\colon \mathbb{C}^4 \to \mathbb{C}^4$ be given by $T(\mathbf{x}) = A\mathbf{x}$. Find $\mathcal{K}(T)$. Is T injective?

C30† Let $T : M_{2,2} \to P_2$ be given by $T\left(\begin{bmatrix}a & b\\ c & d\end{bmatrix}\right) = (a+b) + (a+c)x + (a+d)x^2$. Is T injective? Find $\mathcal{K}(T)$.

C31† Given that the linear transformation $T\colon \mathbb{C}^3 \to \mathbb{C}^3$, $T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}2x + y\\ 2y + z\\ x + 2z\end{bmatrix}$ is injective, show directly that $\{T(\mathbf{e}_1),\, T(\mathbf{e}_2),\, T(\mathbf{e}_3)\}$ is a linearly independent set.

C32† Given that the linear transformation $T\colon \mathbb{C}^2 \to \mathbb{C}^3$, $T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}x + y\\ 2x + y\\ x + 2y\end{bmatrix}$ is injective, show directly that $\{T(\mathbf{e}_1),\, T(\mathbf{e}_2)\}$ is a linearly independent set.

C33† Given that the linear transformation $T\colon\mathbb{C}^3\to\mathbb{C}^5$, $T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right)=\begin{bmatrix}1&3&2\\0&1&1\\1&2&1\\1&0&1\\3&1&2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}$ is injective, show directly that $\{T(\mathbf{e}_1), T(\mathbf{e}_2), T(\mathbf{e}_3)\}$ is a linearly independent set.

C40† Show that the linear transformation R is not injective by finding two different elements of the domain, $\mathbf{x}$ and $\mathbf{y}$, such that $R(\mathbf{x}) = R(\mathbf{y})$. ($S_{22}$ is the vector space of symmetric 2×2 matrices.)

$$R\colon S_{22}\to P_1 \quad R\left(\begin{bmatrix}a&b\\b&c\end{bmatrix}\right)=(2a-b+c)+(a+b+2c)x$$

M60 Suppose U and V are vector spaces. Define the function $Z\colon U\to V$ by $T(\mathbf{u})=\mathbf{0}_V$ for every $\mathbf{u}\in U$. Then by Exercise LT.M60, Z is a linear transformation. Formulate a condition on U that is equivalent to Z being an injective linear transformation. In other words, fill in the blank to complete the following statement (and then give a proof): Z is injective if and only if U is . (See Exercise SLT.M60, Exercise IVLT.M60.)

T10† Suppose $T\colon U\to V$ is a linear transformation. For which vectors $\mathbf{v}\in V$ is $T^{-1}(\mathbf{v})$ a subspace of U?

T15† Suppose that that $T\colon U\to V$ and $S\colon V\to W$ are linear transformations. Prove the following relationship between kernels.

$$\mathcal{K}(T)\subseteq\mathcal{K}(S\circ T)$$

T20† Suppose that A is an $m\times n$ matrix. Define the linear transformation T by

$$T\colon\mathbb{C}^n\to\mathbb{C}^m,\quad T(\mathbf{x})=A\mathbf{x}$$

Prove that the kernel of T equals the null space of A, $\mathcal{K}(T)=\mathcal{N}(A)$.

Section SLT
Surjective Linear Transformations

The companion to an injection is a surjection. Surjective linear transformations are closely related to spanning sets and ranges. So as you read this section reflect back on Section ILT and note the parallels and the contrasts. In the next section, Section IVLT, we will combine the two properties.

Subsection SLT
Surjective Linear Transformations

As usual, we lead with a definition.

Definition SLT Surjective Linear Transformation
Suppose $T\colon U \to V$ is a linear transformation. Then T is **surjective** if for every $\mathbf{v} \in V$ there exists a $\mathbf{u} \in U$ so that $T(\mathbf{u}) = \mathbf{v}$. □

Given an arbitrary function, it is possible for there to be an element of the codomain that is not an output of the function (think about the function $y = f(x) = x^2$ and the codomain element $y = -3$). For a surjective function, this never happens. If we choose any element of the codomain ($\mathbf{v} \in V$) then there must be an input from the domain ($\mathbf{u} \in U$) which will create the output when used to evaluate the linear transformation ($T(\mathbf{u}) = \mathbf{v}$). Some authors prefer the term **onto** where we use surjective, and we will sometimes refer to a surjective linear transformation as a **surjection**.

Subsection ESLT
Examples of Surjective Linear Transformations

It is perhaps most instructive to examine a linear transformation that is not surjective first.

Example NSAQ Not surjective, Archetype Q
Archetype Q is the linear transformation

$$T\colon \mathbb{C}^5 \to \mathbb{C}^5, \quad T\left(\begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4\\ x_5 \end{bmatrix}\right) = \begin{bmatrix} -2x_1 + 3x_2 + 3x_3 - 6x_4 + 3x_5\\ -16x_1 + 9x_2 + 12x_3 - 28x_4 + 28x_5\\ -19x_1 + 7x_2 + 14x_3 - 32x_4 + 37x_5\\ -21x_1 + 9x_2 + 15x_3 - 35x_4 + 39x_5\\ -9x_1 + 5x_2 + 7x_3 - 16x_4 + 16x_5 \end{bmatrix}$$

We will demonstrate that

$$\mathbf{v} = \begin{bmatrix} -1\\ 2\\ 3\\ -1\\ 4 \end{bmatrix}$$

is an unobtainable element of the codomain. Suppose to the contrary that $\mathbf{u}$ is an element of the domain such that $T(\mathbf{u}) = \mathbf{v}$.

Then

$$\begin{bmatrix} -1\\ 2\\ 3\\ -1\\ 4 \end{bmatrix} = \mathbf{v} = T(\mathbf{u}) = T\left(\begin{bmatrix} u_1\\ u_2\\ u_3\\ u_4\\ u_5 \end{bmatrix}\right)$$

$$= \begin{bmatrix} -2u_1 + 3u_2 + 3u_3 - 6u_4 + 3u_5 \\ -16u_1 + 9u_2 + 12u_3 - 28u_4 + 28u_5 \\ -19u_1 + 7u_2 + 14u_3 - 32u_4 + 37u_5 \\ -21u_1 + 9u_2 + 15u_3 - 35u_4 + 39u_5 \\ -9u_1 + 5u_2 + 7u_3 - 16u_4 + 16u_5 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 3 & 3 & -6 & 3 \\ -16 & 9 & 12 & -28 & 28 \\ -19 & 7 & 14 & -32 & 37 \\ -21 & 9 & 15 & -35 & 39 \\ -9 & 5 & 7 & -16 & 16 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix}$$

Now we recognize the appropriate input vector **u** as a solution to a linear system of equations. Form the augmented matrix of the system, and row-reduce to

$$\begin{bmatrix} \boxed{1} & 0 & 0 & 0 & -1 & 0 \\ 0 & \boxed{1} & 0 & 0 & -\frac{4}{3} & 0 \\ 0 & 0 & \boxed{1} & 0 & -\frac{1}{3} & 0 \\ 0 & 0 & 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

With a leading 1 in the last column, Theorem RCLS tells us the system is inconsistent. From the absence of any solutions we conclude that no such vector **u** exists, and by Definition SLT, T is not surjective.

Again, do not concern yourself with how **v** was selected, as this will be explained shortly. However, do understand *why* this vector provides enough evidence to conclude that T is not surjective. △

Here is a cartoon of a non-surjective linear transformation. Notice that the central feature of this cartoon is that the vector $\mathbf{v} \in V$ does not have an arrow pointing to it, implying there is no $\mathbf{u} \in U$ such that $T(\mathbf{u}) = \mathbf{v}$. Even though this happens again with a second unnamed vector in V, it only takes one occurrence to destroy the possibility of surjectivity.

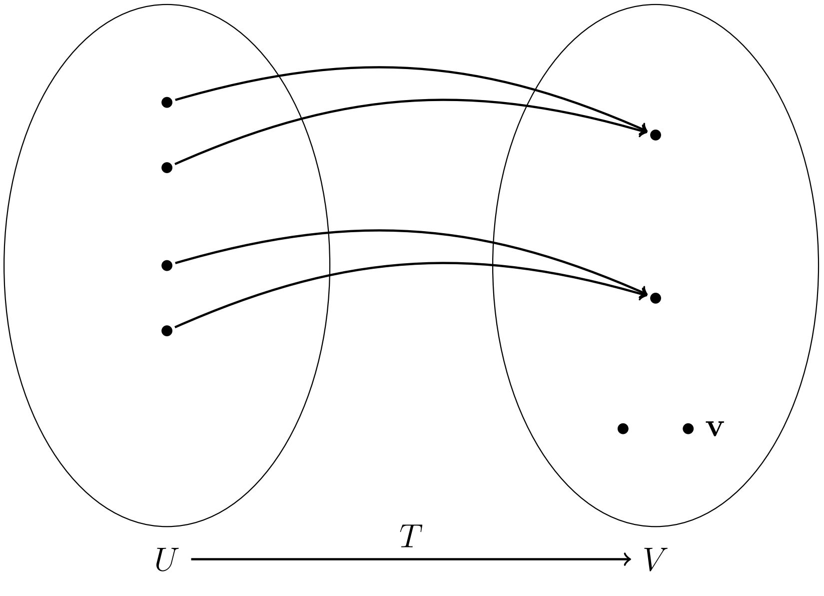

Diagram NSLT: Non-Surjective Linear Transformation

To show that a linear transformation is not surjective, it is enough to find a single element of the codomain that is never created by any input, as in Example NSAQ. However, to show that a linear transformation is surjective we must establish that *every* element of the codomain occurs as an output of the linear transformation for some appropriate input.

Example SAR Surjective, Archetype R
Archetype R is the linear transformation

$$T\colon \mathbb{C}^5 \to \mathbb{C}^5, \quad T\left(\begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4\\ x_5 \end{bmatrix}\right) = \begin{bmatrix} -65x_1 + 128x_2 + 10x_3 - 262x_4 + 40x_5\\ 36x_1 - 73x_2 - x_3 + 151x_4 - 16x_5\\ -44x_1 + 88x_2 + 5x_3 - 180x_4 + 24x_5\\ 34x_1 - 68x_2 - 3x_3 + 140x_4 - 18x_5\\ 12x_1 - 24x_2 - x_3 + 49x_4 - 5x_5 \end{bmatrix}$$

To establish that R is surjective we must begin with a totally arbitrary element of the codomain, $\mathbf{v}$ and somehow find an input vector $\mathbf{u}$ such that $T(\mathbf{u}) = \mathbf{v}$. We desire,

$$T(\mathbf{u}) = \mathbf{v}$$

$$\begin{bmatrix} -65u_1 + 128u_2 + 10u_3 - 262u_4 + 40u_5\\ 36u_1 - 73u_2 - u_3 + 151u_4 - 16u_5\\ -44u_1 + 88u_2 + 5u_3 - 180u_4 + 24u_5\\ 34u_1 - 68u_2 - 3u_3 + 140u_4 - 18u_5\\ 12u_1 - 24u_2 - u_3 + 49u_4 - 5u_5 \end{bmatrix} = \begin{bmatrix} v_1\\ v_2\\ v_3\\ v_4\\ v_5 \end{bmatrix}$$

$$\begin{bmatrix} -65 & 128 & 10 & -262 & 40\\ 36 & -73 & -1 & 151 & -16\\ -44 & 88 & 5 & -180 & 24\\ 34 & -68 & -3 & 140 & -18\\ 12 & -24 & -1 & 49 & -5 \end{bmatrix} \begin{bmatrix} u_1\\ u_2\\ u_3\\ u_4\\ u_5 \end{bmatrix} = \begin{bmatrix} v_1\\ v_2\\ v_3\\ v_4\\ v_5 \end{bmatrix}$$

We recognize this equation as a system of equations in the variables u_i, but our vector of constants contains symbols. In general, we would have to row-reduce the augmented matrix by hand, due to the symbolic final column. However, in this particular example, the 5×5 coefficient matrix is nonsingular and so has an inverse (Theorem NI, Definition MI).

$$\begin{bmatrix} -65 & 128 & 10 & -262 & 40\\ 36 & -73 & -1 & 151 & -16\\ -44 & 88 & 5 & -180 & 24\\ 34 & -68 & -3 & 140 & -18\\ 12 & -24 & -1 & 49 & -5 \end{bmatrix}^{-1} = \begin{bmatrix} -47 & 92 & 1 & -181 & -14\\ 27 & -55 & \frac{7}{2} & \frac{221}{2} & 11\\ -32 & 64 & -1 & -126 & -12\\ 25 & -50 & \frac{3}{2} & \frac{199}{2} & 9\\ 9 & -18 & \frac{1}{2} & \frac{71}{2} & 4 \end{bmatrix}$$

so we find that

$$\begin{bmatrix} u_1\\ u_2\\ u_3\\ u_4\\ u_5 \end{bmatrix} = \begin{bmatrix} -47 & 92 & 1 & -181 & -14\\ 27 & -55 & \frac{7}{2} & \frac{221}{2} & 11\\ -32 & 64 & -1 & -126 & -12\\ 25 & -50 & \frac{3}{2} & \frac{199}{2} & 9\\ 9 & -18 & \frac{1}{2} & \frac{71}{2} & 4 \end{bmatrix} \begin{bmatrix} v_1\\ v_2\\ v_3\\ v_4\\ v_5 \end{bmatrix}$$

$$= \begin{bmatrix} -47v_1 + 92v_2 + v_3 - 181v_4 - 14v_5\\ 27v_1 - 55v_2 + \frac{7}{2}v_3 + \frac{221}{2}v_4 + 11v_5\\ -32v_1 + 64v_2 - v_3 - 126v_4 - 12v_5\\ 25v_1 - 50v_2 + \frac{3}{2}v_3 + \frac{199}{2}v_4 + 9v_5\\ 9v_1 - 18v_2 + \frac{1}{2}v_3 + \frac{71}{2}v_4 + 4v_5 \end{bmatrix}$$

This establishes that if we are given *any* output vector $\mathbf{v}$, we can use its components in this final expression to formulate a vector $\mathbf{u}$ such that $T(\mathbf{u}) = \mathbf{v}$. So by Definition SLT we now know that T is surjective. You might try to verify this condition in its full generality (i.e. evaluate T with this final expression and see if you get $\mathbf{v}$ as the result), or test it more specifically for some numerical vector $\mathbf{v}$ (see Exercise SLT.C20). △

Here is the cartoon for a surjective linear transformation. It is meant to suggest that for every output in V there is *at least one* input in U that is sent to the output. (Even though we have depicted several inputs sent to each output.) The key feature of this cartoon is that there are no vectors in V without an arrow pointing to them.

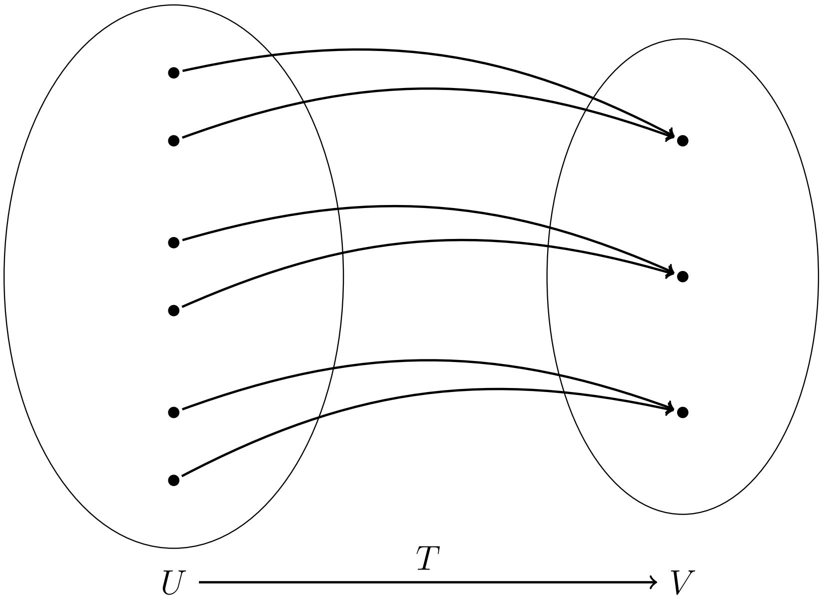

Diagram SLT: Surjective Linear Transformation

Let us now examine a surjective linear transformation between abstract vector spaces.

Example SAV Surjective, Archetype V
Archetype V is defined by

$$T\colon P_3 \to M_{22}, \quad T\left(a + bx + cx^2 + dx^3\right) = \begin{bmatrix} a+b & a-2c \\ d & b-d \end{bmatrix}$$

To establish that the linear transformation is surjective, begin by choosing an arbitrary output. In this example, we need to choose an arbitrary 2×2 matrix, say

$$\mathbf{v} = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

and we would like to find an input polynomial

$$\mathbf{u} = a + bx + cx^2 + dx^3$$

so that $T(\mathbf{u}) = \mathbf{v}$. So we have,

$$\begin{aligned} \begin{bmatrix} x & y \\ z & w \end{bmatrix} &= \mathbf{v} \\ &= T(\mathbf{u}) \\ &= T\left(a + bx + cx^2 + dx^3\right) \\ &= \begin{bmatrix} a+b & a-2c \\ d & b-d \end{bmatrix} \end{aligned}$$

Matrix equality leads us to the system of four equations in the four unknowns, x, y, z, w,

$$\begin{aligned} a + b &= x \\ a - 2c &= y \\ d &= z \end{aligned}$$

$$b - d = w$$

which can be rewritten as a matrix equation,

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

The coefficient matrix is nonsingular, hence it has an inverse,

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

so we have

$$\begin{aligned} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} &= \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \\ &= \begin{bmatrix} x - z - w \\ z + w \\ \frac{1}{2}(x - y - z - w) \\ z \end{bmatrix} \end{aligned}$$

So the input polynomial $\mathbf{u} = (x-z-w)+(z+w)x+\frac{1}{2}(x-y-z-w)x^2+zx^3$ will yield the output matrix $\mathbf{v}$, no matter what form $\mathbf{v}$ takes. This means by Definition SLT that T is surjective. All the same, let us do a concrete demonstration and evaluate T with $\mathbf{u}$,

$$\begin{aligned} T(\mathbf{u}) &= T\left((x - z - w) + (z + w)x + \frac{1}{2}(x - y - z - w)x^2 + zx^3\right) \\ &= \begin{bmatrix} (x - z - w) + (z + w) & (x - z - w) - 2(\frac{1}{2}(x - y - z - w)) \\ z & (z + w) - z \end{bmatrix} \\ &= \begin{bmatrix} x & y \\ z & w \end{bmatrix} \\ &= \mathbf{v} \end{aligned}$$

△

Subsection RLT
Range of a Linear Transformation

For a linear transformation $T\colon U \to V$, the range is a subset of the codomain V. Informally, it is the set of all outputs that the transformation creates when fed every possible input from the domain. It will have some natural connections with the column space of a matrix, so we will keep the same notation, and if you think about your objects, then there should be little confusion. Here is the careful definition.

Definition RLT Range of a Linear Transformation
Suppose $T\colon U \to V$ is a linear transformation. Then the **range** of T is the set

$$\mathcal{R}(T) = \{T(\mathbf{u}) \mid \mathbf{u} \in U\}$$

□

Example RAO Range, Archetype O

Archetype O is the linear transformation

$$T\colon \mathbb{C}^3 \to \mathbb{C}^5, \quad T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}-x_1 + x_2 - 3x_3\\ -x_1 + 2x_2 - 4x_3\\ x_1 + x_2 + x_3\\ 2x_1 + 3x_2 + x_3\\ x_1 + 2x_3\end{bmatrix}$$

To determine the elements of $\mathbb{C}^5$ in $\mathcal{R}(T)$, find those vectors $\mathbf{v}$ such that $T(\mathbf{u}) = \mathbf{v}$ for some $\mathbf{u} \in \mathbb{C}^3$,

$$\begin{aligned}\mathbf{v} &= T(\mathbf{u})\\ &= \begin{bmatrix}-u_1 + u_2 - 3u_3\\ -u_1 + 2u_2 - 4u_3\\ u_1 + u_2 + u_3\\ 2u_1 + 3u_2 + u_3\\ u_1 + 2u_3\end{bmatrix}\\ &= \begin{bmatrix}-u_1\\-u_1\\u_1\\2u_1\\u_1\end{bmatrix} + \begin{bmatrix}u_2\\2u_2\\u_2\\3u_2\\0\end{bmatrix} + \begin{bmatrix}-3u_3\\-4u_3\\u_3\\u_3\\2u_3\end{bmatrix}\\ &= u_1\begin{bmatrix}-1\\-1\\1\\2\\1\end{bmatrix} + u_2\begin{bmatrix}1\\2\\1\\3\\0\end{bmatrix} + u_3\begin{bmatrix}-3\\-4\\1\\1\\2\end{bmatrix}\end{aligned}$$

This says that every output of T (in other words, the vector $\mathbf{v}$) can be written as a linear combination of the three vectors

$$\begin{bmatrix}-1\\-1\\1\\2\\1\end{bmatrix} \qquad \begin{bmatrix}1\\2\\1\\3\\0\end{bmatrix} \qquad \begin{bmatrix}-3\\-4\\1\\1\\2\end{bmatrix}$$

using the scalars u_1, u_2, u_3. Furthermore, since $\mathbf{u}$ can be any element of $\mathbb{C}^3$, every such linear combination is an output. This means that

$$\mathcal{R}(T) = \left\langle\left\{\begin{bmatrix}-1\\-1\\1\\2\\1\end{bmatrix},\, \begin{bmatrix}1\\2\\1\\3\\0\end{bmatrix},\, \begin{bmatrix}-3\\-4\\1\\1\\2\end{bmatrix}\right\}\right\rangle$$

The three vectors in this spanning set for $\mathcal{R}(T)$ form a linearly dependent set (check this!). So we can find a more economical presentation by any of the various methods from Section CRS and Section FS. We will place the vectors into a matrix as rows, row-reduce, toss out zero rows and appeal to Theorem BRS, so we can describe the range of T with a basis,

$$\mathcal{R}(T) = \left\langle\left\{\begin{bmatrix}1\\0\\-3\\-7\\-2\end{bmatrix},\, \begin{bmatrix}0\\1\\2\\5\\1\end{bmatrix}\right\}\right\rangle$$

△

We know that the span of a set of vectors is always a subspace (Theorem SSS), so the range computed in Example RAO is also a subspace. This is no accident, the range of a linear transformation is *always* a subspace.

Theorem RLTS Range of a Linear Transformation is a Subspace
Suppose that $T\colon U \to V$ is a linear transformation. Then the range of T, $\mathcal{R}(T)$, is a subspace of V.

Proof. We can apply the three-part test of Theorem TSS. First, $\mathbf{0}_U \in U$ and $T(\mathbf{0}_U) = \mathbf{0}_V$ by Theorem LTTZZ, so $\mathbf{0}_V \in \mathcal{R}(T)$ and we know that the range is nonempty.

Suppose we assume that $\mathbf{x},\,\mathbf{y} \in \mathcal{R}(T)$. Is $\mathbf{x}+\mathbf{y} \in \mathcal{R}(T)$? If $\mathbf{x},\,\mathbf{y} \in \mathcal{R}(T)$ then we know there are vectors $\mathbf{w},\,\mathbf{z} \in U$ such that $T(\mathbf{w}) = \mathbf{x}$ and $T(\mathbf{z}) = \mathbf{y}$. Because U is a vector space, additive closure (Property AC) implies that $\mathbf{w}+\mathbf{z} \in U$.

Then

$$\begin{aligned} T(\mathbf{w}+\mathbf{z}) &= T(\mathbf{w}) + T(\mathbf{z}) && \text{Definition LT}\\ &= \mathbf{x}+\mathbf{y} && \text{Definition of } \mathbf{w} \text{ and } \mathbf{z}\end{aligned}$$

So we have found an input, $\mathbf{w}+\mathbf{z}$, which when fed into T creates $\mathbf{x}+\mathbf{y}$ as an output. This qualifies $\mathbf{x}+\mathbf{y}$ for membership in $\mathcal{R}(T)$. So we have additive closure.

Suppose we assume that $\alpha \in \mathbb{C}$ and $\mathbf{x} \in \mathcal{R}(T)$. Is $\alpha\mathbf{x} \in \mathcal{R}(T)$? If $\mathbf{x} \in \mathcal{R}(T)$, then there is a vector $\mathbf{w} \in U$ such that $T(\mathbf{w}) = \mathbf{x}$. Because U is a vector space, scalar closure implies that $\alpha\mathbf{w} \in U$. Then

$$\begin{aligned} T(\alpha\mathbf{w}) &= \alpha T(\mathbf{w}) && \text{Definition LT}\\ &= \alpha\mathbf{x} && \text{Definition of } \mathbf{w}\end{aligned}$$

So we have found an input $(\alpha\mathbf{w})$ which when fed into T creates $\alpha\mathbf{x}$ as an output. This qualifies $\alpha\mathbf{x}$ for membership in $\mathcal{R}(T)$. So we have scalar closure and Theorem TSS tells us that $\mathcal{R}(T)$ is a subspace of V. ∎

Let us compute another range, now that we know in advance that it will be a subspace.

Example FRAN Full range, Archetype N
Archetype N is the linear transformation

$$T\colon \mathbb{C}^5 \to \mathbb{C}^3, \quad T\left(\begin{bmatrix} x_1\\ x_2\\ x_3\\ x_4\\ x_5\end{bmatrix}\right) = \begin{bmatrix} 2x_1 + x_2 + 3x_3 - 4x_4 + 5x_5\\ x_1 - 2x_2 + 3x_3 - 9x_4 + 3x_5\\ 3x_1 + 4x_3 - 6x_4 + 5x_5\end{bmatrix}$$

To determine the elements of $\mathbb{C}^3$ in $\mathcal{R}(T)$, find those vectors $\mathbf{v}$ such that $T(\mathbf{u}) = \mathbf{v}$ for some $\mathbf{u} \in \mathbb{C}^5$,

$$\begin{aligned}\mathbf{v} &= T(\mathbf{u})\\ &= \begin{bmatrix} 2u_1 + u_2 + 3u_3 - 4u_4 + 5u_5\\ u_1 - 2u_2 + 3u_3 - 9u_4 + 3u_5\\ 3u_1 + 4u_3 - 6u_4 + 5u_5\end{bmatrix}\\ &= \begin{bmatrix} 2u_1\\ u_1\\ 3u_1\end{bmatrix} + \begin{bmatrix} u_2\\ -2u_2\\ 0\end{bmatrix} + \begin{bmatrix} 3u_3\\ 3u_3\\ 4u_3\end{bmatrix} + \begin{bmatrix} -4u_4\\ -9u_4\\ -6u_4\end{bmatrix} + \begin{bmatrix} 5u_5\\ 3u_5\\ 5u_5\end{bmatrix}\\ &= u_1\begin{bmatrix} 2\\ 1\\ 3\end{bmatrix} + u_2\begin{bmatrix} 1\\ -2\\ 0\end{bmatrix} + u_3\begin{bmatrix} 3\\ 3\\ 4\end{bmatrix} + u_4\begin{bmatrix} -4\\ -9\\ -6\end{bmatrix} + u_5\begin{bmatrix} 5\\ 3\\ 5\end{bmatrix}\end{aligned}$$

This says that every output of T (in other words, the vector $\mathbf{v}$) can be written as a linear combination of the five vectors

$$\begin{bmatrix} 2\\ 1\\ 3\end{bmatrix} \qquad \begin{bmatrix} 1\\ -2\\ 0\end{bmatrix} \qquad \begin{bmatrix} 3\\ 3\\ 4\end{bmatrix} \qquad \begin{bmatrix} -4\\ -9\\ -6\end{bmatrix} \qquad \begin{bmatrix} 5\\ 3\\ 5\end{bmatrix}$$

using the scalars u_1, u_2, u_3, u_4, u_5. Furthermore, since $\mathbf{u}$ can be any element of $\mathbb{C}^5$, every such linear combination is an output. This means that

$$\mathcal{R}(T) = \left\langle\left\{\begin{bmatrix}2\\1\\3\end{bmatrix}, \begin{bmatrix}1\\-2\\0\end{bmatrix}, \begin{bmatrix}3\\3\\4\end{bmatrix}, \begin{bmatrix}-4\\-9\\-6\end{bmatrix}, \begin{bmatrix}5\\3\\5\end{bmatrix}\right\}\right\rangle$$

The five vectors in this spanning set for $\mathcal{R}(T)$ form a linearly dependent set (Theorem MVSLD). So we can find a more economical presentation by any of the various methods from Section CRS and Section FS. We will place the vectors into a matrix as rows, row-reduce, toss out zero rows and appeal to Theorem BRS, so we can describe the range of T with a (nice) basis,

$$\mathcal{R}(T) = \left\langle\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}\right\rangle = \mathbb{C}^3$$

△

In contrast to injective linear transformations having small (trivial) kernels (Theorem KILT), surjective linear transformations have large ranges, as indicated in the next theorem.

Theorem RSLT Range of a Surjective Linear Transformation
Suppose that $T\colon U \to V$ is a linear transformation. Then T is surjective if and only if the range of T equals the codomain, $\mathcal{R}(T) = V$.

Proof. ($\Rightarrow$) By Definition RLT, we know that $\mathcal{R}(T) \subseteq V$. To establish the reverse inclusion, assume $\mathbf{v} \in V$. Then since T is surjective (Definition SLT), there exists a vector $\mathbf{u} \in U$ so that $T(\mathbf{u}) = \mathbf{v}$. However, the existence of $\mathbf{u}$ gains $\mathbf{v}$ membership in $\mathcal{R}(T)$, so $V \subseteq \mathcal{R}(T)$. Thus, $\mathcal{R}(T) = V$.

($\Leftarrow$) To establish that T is surjective, choose $\mathbf{v} \in V$. Since we are assuming that $\mathcal{R}(T) = V$, $\mathbf{v} \in \mathcal{R}(T)$. This says there is a vector $\mathbf{u} \in U$ so that $T(\mathbf{u}) = \mathbf{v}$, i.e. T is surjective. ■

Example NSAQR Not surjective, Archetype Q, revisited
We are now in a position to revisit our first example in this section, Example NSAQ. In that example, we showed that Archetype Q is not surjective by constructing a vector in the codomain where no element of the domain could be used to evaluate the linear transformation to create the output, thus violating Definition SLT. Just where did this vector come from?

The short answer is that the vector

$$\mathbf{v} = \begin{bmatrix}-1\\2\\3\\-1\\4\end{bmatrix}$$

was constructed to lie outside of the range of T. How was this accomplished? First, the range of T is given by

$$\mathcal{R}(T) = \left\langle\left\{\begin{bmatrix}1\\0\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\\-1\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\-1\end{bmatrix}, \begin{bmatrix}0\\0\\0\\1\\2\end{bmatrix}\right\}\right\rangle$$

Suppose an element of the range $\mathbf{v}^*$ has its first 4 components equal to -1, 2, 3,

-1, in that order. Then to be an element of $\mathcal{R}(T)$, we would have

$$\mathbf{v}^* = (-1)\begin{bmatrix}1\\0\\0\\0\\1\end{bmatrix} + (2)\begin{bmatrix}0\\1\\0\\0\\-1\end{bmatrix} + (3)\begin{bmatrix}0\\0\\1\\0\\-1\end{bmatrix} + (-1)\begin{bmatrix}0\\0\\0\\1\\2\end{bmatrix} = \begin{bmatrix}-1\\2\\3\\-1\\-8\end{bmatrix}$$

So the only vector in the range with these first four components specified, must have -8 in the fifth component. To set the fifth component to any other value (say, 4) will result in a vector ($\mathbf{v}$ in Example NSAQ) outside of the range. Any attempt to find an input for T that will produce $\mathbf{v}$ as an output will be doomed to failure.

Whenever the range of a linear transformation is not the whole codomain, we can employ this device and conclude that the linear transformation is not surjective. This is another way of viewing Theorem RSLT. For a surjective linear transformation, the range is all of the codomain and there is no choice for a vector $\mathbf{v}$ that lies in V, yet not in the range. For every one of the archetypes that is not surjective, there is an example presented of exactly this form. △

Example NSAO Not surjective, Archetype O
In Example RAO the range of Archetype O was determined to be

$$\mathcal{R}(T) = \left\langle\left\{\begin{bmatrix}1\\0\\-3\\-7\\-2\end{bmatrix}, \begin{bmatrix}0\\1\\2\\5\\1\end{bmatrix}\right\}\right\rangle$$

a subspace of dimension 2 in $\mathbb{C}^5$. Since $\mathcal{R}(T) \neq \mathbb{C}^5$, Theorem RSLT says T is not surjective. △

Example SAN Surjective, Archetype N
The range of Archetype N was computed in Example FRAN to be

$$\mathcal{R}(T) = \left\langle\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}\right\rangle$$

Since the basis for this subspace is the set of standard unit vectors for $\mathbb{C}^3$ (Theorem SUVB), we have $\mathcal{R}(T) = \mathbb{C}^3$ and by Theorem RSLT, T is surjective. △

Subsection SSSLT
Spanning Sets and Surjective Linear Transformations

Just as injective linear transformations are allied with linear independence (Theorem ILTLI, Theorem ILTB), surjective linear transformations are allied with spanning sets.

Theorem SSRLT Spanning Set for Range of a Linear Transformation
Suppose that $T\colon U \to V$ *is a linear transformation and*

$$S = \{\mathbf{u}_1,\,\mathbf{u}_2,\,\mathbf{u}_3,\,\ldots,\,\mathbf{u}_t\}$$

spans U*. Then*

$$R = \{T(\mathbf{u}_1),\,T(\mathbf{u}_2),\,T(\mathbf{u}_3),\,\ldots,\,T(\mathbf{u}_t)\}$$

spans $\mathcal{R}(T)$*.*

Proof. We need to establish that $\mathcal{R}(T) = \langle R\rangle$, a set equality. First we establish that $\mathcal{R}(T) \subseteq \langle R\rangle$. To this end, choose $\mathbf{v} \in \mathcal{R}(T)$. Then there exists a vector $\mathbf{u} \in U$, such that $T(\mathbf{u}) = \mathbf{v}$ (Definition RLT). Because S spans U there are scalars,

$a_1, a_2, a_3, \ldots, a_t$, such that

$$\mathbf{u} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \cdots + a_t\mathbf{u}_t$$

Then

$$\begin{aligned}
\mathbf{v} &= T(\mathbf{u}) && \text{Definition RLT}\\
&= T(a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \cdots + a_t\mathbf{u}_t) && \text{Definition SSVS}\\
&= a_1T(\mathbf{u}_1) + a_2T(\mathbf{u}_2) + a_3T(\mathbf{u}_3) + \ldots + a_tT(\mathbf{u}_t) && \text{Theorem LTLC}
\end{aligned}$$

which establishes that $\mathbf{v} \in \langle R\rangle$ (Definition SS). So $\mathcal{R}(T) \subseteq \langle R\rangle$.

To establish the opposite inclusion, choose an element of the span of R, say $\mathbf{v} \in \langle R\rangle$. Then there are scalars $b_1, b_2, b_3, \ldots, b_t$ so that

$$\begin{aligned}
\mathbf{v} &= b_1T(\mathbf{u}_1) + b_2T(\mathbf{u}_2) + b_3T(\mathbf{u}_3) + \cdots + b_tT(\mathbf{u}_t) && \text{Definition SS}\\
&= T(b_1\mathbf{u}_1 + b_2\mathbf{u}_2 + b_3\mathbf{u}_3 + \cdots + b_t\mathbf{u}_t) && \text{Theorem LTLC}
\end{aligned}$$

This demonstrates that $\mathbf{v}$ is an output of the linear transformation T, so $\mathbf{v} \in \mathcal{R}(T)$. Therefore $\langle R\rangle \subseteq \mathcal{R}(T)$, so we have the set equality $\mathcal{R}(T) = \langle R\rangle$ (Definition SE). In other words, R spans $\mathcal{R}(T)$ (Definition SSVS). ■

Theorem SSRLT provides an easy way to begin the construction of a basis for the range of a linear transformation, since the construction of a spanning set requires simply evaluating the linear transformation on a spanning set of the domain. In practice the best choice for a spanning set of the domain would be as small as possible, in other words, a basis. The resulting spanning set for the codomain may not be linearly independent, so to find a basis for the range might require tossing out redundant vectors from the spanning set. Here is an example.

Example BRLT A basis for the range of a linear transformation
Define the linear transformation $T\colon M_{22} \to P_2$ by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a + 2b + 8c + d) + (-3a + 2b + 5d)\,x + (a + b + 5c)\,x^2$$

A convenient spanning set for M_{22} is the basis

$$S = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

So by Theorem SSRLT, a spanning set for $\mathcal{R}(T)$ is

$$\begin{aligned}
R &= \left\{T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right), T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right), T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right), T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)\right\}\\
&= \left\{1 - 3x + x^2,\ 2 + 2x + x^2,\ 8 + 5x^2,\ 1 + 5x\right\}
\end{aligned}$$

The set R is not linearly independent, so if we desire a basis for $\mathcal{R}(T)$, we need to eliminate some redundant vectors. Two particular relations of linear dependence on R are

$$\begin{aligned}
(-2)(1 - 3x + x^2) + (-3)(2 + 2x + x^2) + (8 + 5x^2) &= 0 + 0x + 0x^2 = \mathbf{0}\\
(1 - 3x + x^2) + (-1)(2 + 2x + x^2) + (1 + 5x) &= 0 + 0x + 0x^2 = \mathbf{0}
\end{aligned}$$

These, individually, allow us to remove $8 + 5x^2$ and $1 + 5x$ from R without destroying the property that R spans $\mathcal{R}(T)$. The two remaining vectors are linearly independent (check this!), so we can write

$$\mathcal{R}(T) = \left\langle\left\{1 - 3x + x^2,\ 2 + 2x + x^2\right\}\right\rangle$$

and see that $\dim(\mathcal{R}(T)) = 2$. △

Elements of the range are precisely those elements of the codomain with nonempty preimages.

Theorem RPI Range and Pre-Image
Suppose that $T\colon U \to V$ is a linear transformation. Then

$$\mathbf{v} \in \mathcal{R}(T) \text{ if and only if } T^{-1}(\mathbf{v}) \neq \emptyset$$

Proof. ($\Rightarrow$) If $\mathbf{v} \in \mathcal{R}(T)$, then there is a vector $\mathbf{u} \in U$ such that $T(\mathbf{u}) = \mathbf{v}$. This qualifies $\mathbf{u}$ for membership in $T^{-1}(\mathbf{v})$, and thus the preimage of $\mathbf{v}$ is not empty.

($\Leftarrow$) Suppose the preimage of $\mathbf{v}$ is not empty, so we can choose a vector $\mathbf{u} \in U$ such that $T(\mathbf{u}) = \mathbf{v}$. Then $\mathbf{v} \in \mathcal{R}(T)$. ■

Now would be a good time to return to Diagram KPI which depicted the pre-images of a non-surjective linear transformation. The vectors $\mathbf{x}$, $\mathbf{y} \in V$ were elements of the codomain whose pre-images were empty, as we expect for a non-surjective linear transformation from the characterization in Theorem RPI.

Theorem SLTB Surjective Linear Transformations and Bases
Suppose that $T\colon U \to V$ is a linear transformation and

$$B = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_m\}$$

is a basis of U. Then T is surjective if and only if

$$C = \{T(\mathbf{u}_1),\, T(\mathbf{u}_2),\, T(\mathbf{u}_3),\, \ldots,\, T(\mathbf{u}_m)\}$$

is a spanning set for V.

Proof. ($\Rightarrow$) Assume T is surjective. Since B is a basis, we know B is a spanning set of U (Definition B). Then Theorem SSRLT says that C spans $\mathcal{R}(T)$. But the hypothesis that T is surjective means $V = \mathcal{R}(T)$ (Theorem RSLT), so C spans V.

($\Leftarrow$) Assume that C spans V. To establish that T is surjective, we will show that every element of V is an output of T for some input (Definition SLT). Suppose that $\mathbf{v} \in V$. As an element of V, we can write $\mathbf{v}$ as a linear combination of the spanning set C. So there are scalars, $b_1,\, b_2,\, b_3,\, \ldots,\, b_m$, such that

$$\mathbf{v} = b_1 T(\mathbf{u}_1) + b_2 T(\mathbf{u}_2) + b_3 T(\mathbf{u}_3) + \cdots + b_m T(\mathbf{u}_m)$$

Now define the vector $\mathbf{u} \in U$ by

$$\mathbf{u} = b_1\mathbf{u}_1 + b_2\mathbf{u}_2 + b_3\mathbf{u}_3 + \cdots + b_m\mathbf{u}_m$$

Then

$$\begin{aligned} T(\mathbf{u}) &= T(b_1\mathbf{u}_1 + b_2\mathbf{u}_2 + b_3\mathbf{u}_3 + \cdots + b_m\mathbf{u}_m) \\ &= b_1 T(\mathbf{u}_1) + b_2 T(\mathbf{u}_2) + b_3 T(\mathbf{u}_3) + \cdots + b_m T(\mathbf{u}_m) && \text{Theorem LTLC} \\ &= \mathbf{v} \end{aligned}$$

So, given any choice of a vector $\mathbf{v} \in V$, we can design an input $\mathbf{u} \in U$ to produce $\mathbf{v}$ as an output of T. Thus, by Definition SLT, T is surjective. ■

Subsection SLTD
Surjective Linear Transformations and Dimension

Theorem SLTD Surjective Linear Transformations and Dimension
Suppose that $T\colon U \to V$ is a surjective linear transformation. Then $\dim(U) \geq \dim(V)$.

Proof. Suppose to the contrary that $m = \dim(U) < \dim(V) = t$. Let B be a basis of U, which will then contain m vectors. Apply T to each element of B to form a set C that is a subset of V. By Theorem SLTB, C is a spanning set of V with m or fewer vectors. So we have a set of m or fewer vectors that span V, a vector space of dimension t, with $m < t$. However, this contradicts Theorem G, so our assumption is false and $\dim(U) \geq \dim(V)$. ■

Example NSDAT Not surjective by dimension, Archetype T
The linear transformation in Archetype T is

$$T\colon P_4 \to P_5, \quad T\left(p(x)\right) = (x-2)p(x)$$

Since $\dim\left(P_4\right) = 5 < 6 = \dim\left(P_5\right)$, T cannot be surjective for then it would violate Theorem SLTD. △

Notice that the previous example made no use of the actual formula defining the function. Merely a comparison of the dimensions of the domain and codomain are enough to conclude that the linear transformation is not surjective. Archetype O and Archetype P are two more examples of linear transformations that have "small" domains and "big" codomains, resulting in an inability to create all possible outputs and thus they are non-surjective linear transformations.

Subsection CSLT
Composition of Surjective Linear Transformations

In Subsection LT.NLTFO we saw how to combine linear transformations to build new linear transformations, specifically, how to build the composition of two linear transformations (Definition LTC). It will be useful later to know that the composition of surjective linear transformations is again surjective, so we prove that here.

Theorem CSLTS Composition of Surjective Linear Transformations is Surjective
Suppose that $T\colon U \to V$ and $S\colon V \to W$ are surjective linear transformations. Then $(S \circ T)\colon U \to W$ is a surjective linear transformation.

Proof. That the composition is a linear transformation was established in Theorem CLTLT, so we need only establish that the composition is surjective. Applying Definition SLT, choose $\mathbf{w} \in W$.

Because S is surjective, there must be a vector $\mathbf{v} \in V$, such that $S\left(\mathbf{v}\right) = \mathbf{w}$. With the existence of $\mathbf{v}$ established, that T is surjective guarantees a vector $\mathbf{u} \in U$ such that $T\left(\mathbf{u}\right) = \mathbf{v}$. Now,

$$\begin{aligned}
(S \circ T)\left(\mathbf{u}\right) &= S\left(T\left(\mathbf{u}\right)\right) && \text{Definition LTC}\\
&= S\left(\mathbf{v}\right) && \text{Definition of } \mathbf{u}\\
&= \mathbf{w} && \text{Definition of } \mathbf{v}
\end{aligned}$$

This establishes that any element of the codomain ($\mathbf{w}$) can be created by evaluating $S \circ T$ with the right input ($\mathbf{u}$). Thus, by Definition SLT, $S \circ T$ is surjective. ■

Reading Questions

1. Suppose $T\colon \mathbb{C}^5 \to \mathbb{C}^8$ is a linear transformation. Why is T not surjective?
2. What is the relationship between a surjective linear transformation and its range?
3. There are many similarities and differences between injective and surjective linear transformations. Compare and contrast these two different types of linear transformations. (This means going well beyond just stating their definitions.)

Exercises

C10 Each archetype below is a linear transformation. Compute the range for each.

Archetype M, Archetype N, Archetype O, Archetype P, Archetype Q, Archetype R, Archetype S, Archetype T, Archetype U, Archetype V, Archetype W, Archetype X

C20 Example SAR concludes with an expression for a vector $\mathbf{u} \in \mathbb{C}^5$ that we believe will create the vector $\mathbf{v} \in \mathbb{C}^5$ when used to evaluate T. That is, $T\left(\mathbf{u}\right) = \mathbf{v}$. Verify this assertion by actually evaluating T with $\mathbf{u}$. If you do not have the patience to push around all these symbols, try choosing a numerical instance of $\mathbf{v}$, compute $\mathbf{u}$, and then compute $T(\mathbf{u})$, which should result in $\mathbf{v}$.

C22† The linear transformation $S\colon \mathbb{C}^4 \to \mathbb{C}^3$ is not surjective. Find an output $\mathbf{w} \in \mathbb{C}^3$ that has an empty pre-image (that is $S^{-1}(\mathbf{w}) = \emptyset$.)

$$S\left(\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}\right) = \begin{bmatrix}2x_1 + x_2 + 3x_3 - 4x_4\\x_1 + 3x_2 + 4x_3 + 3x_4\\-x_1 + 2x_2 + x_3 + 7x_4\end{bmatrix}$$

C23† Determine whether or not the following linear transformation $T\colon \mathbb{C}^5 \to P_3$ is surjective:

$$T\left(\begin{bmatrix}a\\b\\c\\d\\e\end{bmatrix}\right) = a + (b+c)x + (c+d)x^2 + (d+e)x^3$$

C24† Determine whether or not the linear transformation $T\colon P_3 \to \mathbb{C}^5$ below is surjective:

$$T\left(a + bx + cx^2 + dx^3\right) = \begin{bmatrix}a+b\\b+c\\c+d\\a+c\\b+d\end{bmatrix}.$$

C25† Define the linear transformation

$$T\colon \mathbb{C}^3 \to \mathbb{C}^2, \quad T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}2x_1 - x_2 + 5x_3\\-4x_1 + 2x_2 - 10x_3\end{bmatrix}$$

Find a basis for the range of T, $\mathcal{R}(T)$. Is T surjective?

C26† Let $T\colon \mathbb{C}^3 \to \mathbb{C}^3$ be given by $T\left(\begin{bmatrix}a\\b\\c\end{bmatrix}\right) = \begin{bmatrix}a+b+2c\\2c\\a+b+c\end{bmatrix}$. Find a basis of $\mathcal{R}(T)$. Is T surjective?

C27† Let $T\colon \mathbb{C}^3 \to \mathbb{C}^4$ be given by $T\left(\begin{bmatrix}a\\b\\c\end{bmatrix}\right) = \begin{bmatrix}a+b-c\\a-b+c\\-a+b+c\\a+b+c\end{bmatrix}$. Find a basis of $\mathcal{R}(T)$. Is T surjective?

C28† Let $T\colon \mathbb{C}^4 \to M_{2,2}$ be given by $T\left(\begin{bmatrix}a\\b\\c\\d\end{bmatrix}\right) = \begin{bmatrix}a+b & a+b+c\\a+b+c & a+d\end{bmatrix}$. Find a basis of $\mathcal{R}(T)$. Is T surjective?

C29† Let $T\colon P_2 \to P_4$ be given by $T(p(x)) = x^2p(x)$. Find a basis of $\mathcal{R}(T)$. Is T surjective?

C30† Let $T\colon P_4 \to P_3$ be given by $T(p(x)) = p'(x)$, where $p'(x)$ is the derivative. Find a basis of $\mathcal{R}(T)$. Is T surjective?

C40† Show that the linear transformation T is not surjective by finding an element of the codomain, $\mathbf{v}$, such that there is no vector $\mathbf{u}$ with $T(\mathbf{u}) = \mathbf{v}$.

$$T\colon \mathbb{C}^3 \to \mathbb{C}^3, \quad T\left(\begin{bmatrix}a\\b\\c\end{bmatrix}\right) = \begin{bmatrix}2a+3b-c\\2b-2c\\a-b+2c\end{bmatrix}$$

M60 Suppose U and V are vector spaces. Define the function $Z\colon U \to V$ by $T(\mathbf{u}) = \mathbf{0}_V$ for every $\mathbf{u} \in U$. Then by Exercise LT.M60, Z is a linear transformation. Formulate a condition on V that is equivalent to Z being an surjective linear transformation. In other words, fill in the blank to complete the following statement (and then give a proof): Z is surjective if and only if V is . (See Exercise ILT.M60, Exercise IVLT.M60.)

T15[†] Suppose that $T\colon U \to V$ and $S\colon V \to W$ are linear transformations. Prove the following relationship between ranges.

$$\mathcal{R}(S \circ T) \subseteq \mathcal{R}(S)$$

T20[†] Suppose that A is an $m \times n$ matrix. Define the linear transformation T by

$$T\colon \mathbb{C}^n \to \mathbb{C}^m, \quad T(\mathbf{x}) = A\mathbf{x}$$

Prove that the range of T equals the column space of A, $\mathcal{R}(T) = \mathcal{C}(A)$.

Section IVLT
Invertible Linear Transformations

In this section we will conclude our introduction to linear transformations by bringing together the twin properties of injectivity and surjectivity and consider linear transformations with both of these properties.

Subsection IVLT
Invertible Linear Transformations

One preliminary definition, and then we will have our main definition for this section.

Definition IDLT Identity Linear Transformation
The **identity linear transformation** on the vector space W is defined as

$$I_W\colon W \to W, \qquad I_W(\mathbf{w}) = \mathbf{w}$$

□

Informally, I_W is the "do-nothing" function. You should check that I_W is really a linear transformation, as claimed, and then compute its kernel and range to see that it is both injective and surjective. All of these facts should be straightforward to verify (Exercise IVLT.T05). With this in hand we can make our main definition.

Definition IVLT Invertible Linear Transformations
Suppose that $T\colon U \to V$ is a linear transformation. If there is a function $S\colon V \to U$ such that

$$S \circ T = I_U \qquad\qquad T \circ S = I_V$$

then T is **invertible**. In this case, we call S the **inverse** of T and write $S = T^{-1}$. □

Informally, a linear transformation T is invertible if there is a companion linear transformation, S, which "undoes" the action of T. When the two linear transformations are applied consecutively (composition), in either order, the result is to have no real effect. It is entirely analogous to squaring a positive number and then taking its (positive) square root.

Here is an example of a linear transformation that is invertible. As usual at the beginning of a section, do not be concerned with where S came from, just understand how it illustrates Definition IVLT.

Example AIVLT An invertible linear transformation
Archetype V is the linear transformation

$$T\colon P_3 \to M_{22}, \quad T\left(a + bx + cx^2 + dx^3\right) = \begin{bmatrix} a+b & a-2c \\ d & b-d \end{bmatrix}$$

Define the function $S\colon M_{22} \to P_3$ defined by

$$S\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a-c-d) + (c+d)x + \frac{1}{2}(a-b-c-d)x^2 + cx^3$$

Then

$$\begin{aligned}
&(T \circ S)\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\\
&\quad = T\left(S\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\right)\\
&\quad = T\left((a-c-d) + (c+d)x + \frac{1}{2}(a-b-c-d)x^2 + cx^3\right)\\
&\quad = \begin{bmatrix} (a-c-d)+(c+d) & (a-c-d)-2(\frac{1}{2}(a-b-c-d)) \\ c & (c+d)-c \end{bmatrix}
\end{aligned}$$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$= I_{M_{22}}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

and

$$\begin{aligned}
&(S \circ T)\left(a + bx + cx^2 + dx^3\right) \\
&= S\left(T\left(a + bx + cx^2 + dx^3\right)\right) \\
&= S\left(\begin{bmatrix} a+b & a-2c \\ d & b-d \end{bmatrix}\right) \\
&= ((a+b) - d - (b-d)) + (d + (b-d))x \\
&+ \left(\frac{1}{2}((a+b) - (a-2c) - d - (b-d))\right)x^2 + (d)x^3 \\
&= a + bx + cx^2 + dx^3 \\
&= I_{P_3}\left(a + bx + cx^2 + dx^3\right)
\end{aligned}$$

For now, understand why these computations show that T is invertible, and that $S = T^{-1}$. Maybe even be amazed by how S works so perfectly in concert with T! We will see later just how to arrive at the correct form of S (when it is possible).△

It can be as instructive to study a linear transformation that is not invertible.

Example ANILT A non-invertible linear transformation
Consider the linear transformation $T\colon \mathbb{C}^3 \to M_{22}$ defined by

$$T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a-b & 2a+2b+c \\ 3a+b+c & -2a-6b-2c \end{bmatrix}$$

Suppose we were to search for an inverse function $S\colon M_{22} \to \mathbb{C}^3$.

First verify that the 2×2 matrix $A = \begin{bmatrix} 5 & 3 \\ 8 & 2 \end{bmatrix}$ is not in the range of T. This will amount to finding an input to T, $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$, such that

$$\begin{aligned}
a - b &= 5 \\
2a + 2b + c &= 3 \\
3a + b + c &= 8 \\
-2a - 6b - 2c &= 2
\end{aligned}$$

As this system of equations is inconsistent, there is no input column vector, and $A \notin \mathcal{R}(T)$. How should we define $S(A)$? Note that

$$T(S(A)) = (T \circ S)(A) = I_{M_{22}}(A) = A$$

So any definition we would provide for $S(A)$ must then be a column vector that T sends to A and we would have $A \in \mathcal{R}(T)$, contrary to the definition of T. This is enough to see that there is no function S that will allow us to conclude that T is invertible, since we cannot provide a consistent definition for $S(A)$ if we assume T is invertible.

Even though we now know that T is not invertible, let us not leave this example just yet. Check that

$$T\left(\begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 3 & 2 \\ 5 & 2 \end{bmatrix} = B \qquad T\left(\begin{bmatrix} 0 \\ -3 \\ 8 \end{bmatrix}\right) = \begin{bmatrix} 3 & 2 \\ 5 & 2 \end{bmatrix} = B$$

How would we define $S(B)$?

$$S(B) = S\left(T\left(\begin{bmatrix}1\\-2\\4\end{bmatrix}\right)\right) = (S \circ T)\left(\begin{bmatrix}1\\-2\\4\end{bmatrix}\right) = I_{\mathbb{C}^3}\left(\begin{bmatrix}1\\-2\\4\end{bmatrix}\right) = \begin{bmatrix}1\\-2\\4\end{bmatrix}$$

or

$$S(B) = S\left(T\left(\begin{bmatrix}0\\-3\\8\end{bmatrix}\right)\right) = (S \circ T)\left(\begin{bmatrix}0\\-3\\8\end{bmatrix}\right) = I_{\mathbb{C}^3}\left(\begin{bmatrix}0\\-3\\8\end{bmatrix}\right) = \begin{bmatrix}0\\-3\\8\end{bmatrix}$$

Which definition should we provide for $S(B)$? Both are necessary. But then S is not a function. So we have a second reason to know that there is no function S that will allow us to conclude that T is invertible. It happens that there are infinitely many column vectors that S would have to take to B. Construct the kernel of T,

$$\mathcal{K}(T) = \left\langle\left\{\begin{bmatrix}-1\\-1\\4\end{bmatrix}\right\}\right\rangle$$

Now choose either of the two inputs used above for T and add to it a scalar multiple of the basis vector for the kernel of T. For example,

$$\mathbf{x} = \begin{bmatrix}1\\-2\\4\end{bmatrix} + (-2)\begin{bmatrix}-1\\-1\\4\end{bmatrix} = \begin{bmatrix}3\\0\\-4\end{bmatrix}$$

then verify that $T(\mathbf{x}) = B$. Practice creating a few more inputs for T that would be sent to B, and see why it is hopeless to think that we could ever provide a reasonable definition for $S(B)$! There is a "whole subspace's worth" of values that $S(B)$ would have to take on. △

In Example ANILT you may have noticed that T is not surjective, since the matrix A was not in the range of T. And T is not injective since there are two different input column vectors that T sends to the matrix B. Linear transformations T that are not surjective lead to putative inverse functions S that are undefined on inputs outside of the range of T. Linear transformations T that are not injective lead to putative inverse functions S that are multiply-defined on each of their inputs. We will formalize these ideas in Theorem ILTIS.

But first notice in Definition IVLT that we only require the inverse (when it exists) to be a function. When it does exist, it too is a linear transformation.

Theorem ILTLT Inverse of a Linear Transformation is a Linear Transformation
Suppose that $T\colon U \to V$ is an invertible linear transformation. Then the function $T^{-1}\colon V \to U$ is a linear transformation.

Proof. We work through verifying Definition LT for T^{-1}, using the fact that T is a linear transformation to obtain the second equality in each half of the proof. To this end, suppose $\mathbf{x}, \mathbf{y} \in V$ and $\alpha \in \mathbb{C}$.

$$\begin{aligned}
T^{-1}(\mathbf{x}+\mathbf{y}) &= T^{-1}\left(T\left(T^{-1}(\mathbf{x})\right) + T\left(T^{-1}(\mathbf{y})\right)\right) && \text{Definition IVLT}\\
&= T^{-1}\left(T\left(T^{-1}(\mathbf{x}) + T^{-1}(\mathbf{y})\right)\right) && \text{Definition LT}\\
&= T^{-1}(\mathbf{x}) + T^{-1}(\mathbf{y}) && \text{Definition IVLT}
\end{aligned}$$

Now check the second defining property of a linear transformation for T^{-1},

$$\begin{aligned}
T^{-1}(\alpha\mathbf{x}) &= T^{-1}\left(\alpha T\left(T^{-1}(\mathbf{x})\right)\right) && \text{Definition IVLT}\\
&= T^{-1}\left(T\left(\alpha T^{-1}(\mathbf{x})\right)\right) && \text{Definition LT}\\
&= \alpha T^{-1}(\mathbf{x}) && \text{Definition IVLT}
\end{aligned}$$

So T^{-1} fulfills the requirements of Definition LT and is therefore a linear transformation. ■

So when T has an inverse, T^{-1} is also a linear transformation. Furthermore, T^{-1} is an invertible linear transformation and *its* inverse is what you might expect.

Theorem IILT Inverse of an Invertible Linear Transformation
Suppose that $T\colon U \rightarrow V$ is an invertible linear transformation. Then T^{-1} is an invertible linear transformation and $\left(T^{-1}\right)^{-1} = T$.

Proof. Because T is invertible, Definition IVLT tells us there is a function $T^{-1}\colon V \rightarrow U$ such that

$$T^{-1} \circ T = I_U \qquad\qquad T \circ T^{-1} = I_V$$

Additionally, Theorem ILTLT tells us that T^{-1} is more than just a function, it is a linear transformation. Now view these two statements as properties of the linear transformation T^{-1}. In light of Definition IVLT, they together say that T^{-1} is invertible (let T play the role of S in the statement of the definition). Furthermore, the inverse of T^{-1} is then T, i.e. $\left(T^{-1}\right)^{-1} = T$. ■

Subsection IV
Invertibility

We now know what an inverse linear transformation is, but just which linear transformations have inverses? Here is a theorem we have been preparing for all chapter long.

Theorem ILTIS Invertible Linear Transformations are Injective and Surjective
Suppose $T\colon U \rightarrow V$ is a linear transformation. Then T is invertible if and only if T is injective and surjective.

Proof. ($\Rightarrow$) Since T is presumed invertible, we can employ its inverse, T^{-1} (Definition IVLT). To see that T is injective, suppose $\mathbf{x}, \mathbf{y} \in U$ and assume that $T(\mathbf{x}) = T(\mathbf{y})$,

$$\begin{aligned}
\mathbf{x} &= I_U(\mathbf{x}) && \text{Definition IDLT}\\
&= \left(T^{-1} \circ T\right)(\mathbf{x}) && \text{Definition IVLT}\\
&= T^{-1}\left(T(\mathbf{x})\right) && \text{Definition LTC}\\
&= T^{-1}\left(T(\mathbf{y})\right) && \text{Definition ILT}\\
&= \left(T^{-1} \circ T\right)(\mathbf{y}) && \text{Definition LTC}\\
&= I_U(\mathbf{y}) && \text{Definition IVLT}\\
&= \mathbf{y} && \text{Definition IDLT}
\end{aligned}$$

So by Definition ILT T is injective.

To check that T is surjective, suppose $\mathbf{v} \in V$. Then $T^{-1}(\mathbf{v})$ is a vector in U. Compute

$$\begin{aligned}
T\left(T^{-1}(\mathbf{v})\right) &= \left(T \circ T^{-1}\right)(\mathbf{v}) && \text{Definition LTC}\\
&= I_V(\mathbf{v}) && \text{Definition IVLT}\\
&= \mathbf{v} && \text{Definition IDLT}
\end{aligned}$$

So there is an element from U, when used as an input to T (namely $T^{-1}(\mathbf{v})$) that produces the desired output, $\mathbf{v}$, and hence T is surjective by Definition SLT.

($\Leftarrow$) Now assume that T is both injective and surjective. We will build a function $S\colon V \rightarrow U$ that will establish that T is invertible. To this end, choose any $\mathbf{v} \in V$. Since T is surjective, Theorem RSLT says $\mathcal{R}(T) = V$, so we have $\mathbf{v} \in \mathcal{R}(T)$. Theorem

RPI says that the pre-image of $\mathbf{v}$, $T^{-1}(\mathbf{v})$, is nonempty. So we can choose a vector from the pre-image of $\mathbf{v}$, say $\mathbf{u}$. In other words, there exists $\mathbf{u} \in T^{-1}(\mathbf{v})$.

Since $T^{-1}(\mathbf{v})$ is nonempty, Theorem KPI then says that

$$T^{-1}(\mathbf{v}) = \{\mathbf{u} + \mathbf{z} | \mathbf{z} \in \mathcal{K}(T)\}$$

However, because T is injective, by Theorem KILT the kernel is trivial, $\mathcal{K}(T) = \{\mathbf{0}\}$. So the pre-image is a set with just one element, $T^{-1}(\mathbf{v}) = \{\mathbf{u}\}$. Now we can define S by $S(\mathbf{v}) = \mathbf{u}$. This is the key to this half of this proof. Normally the preimage of a vector from the codomain might be an empty set, or an infinite set. But surjectivity requires that the preimage not be empty, and then injectivity limits the preimage to a singleton. Since our choice of $\mathbf{v}$ was arbitrary, we know that every pre-image for T is a set with a single element. This allows us to construct S as a *function*. Now that it is defined, verifying that it is the inverse of T will be easy. Here we go.

Choose $\mathbf{u} \in U$. Define $\mathbf{v} = T(\mathbf{u})$. Then $T^{-1}(\mathbf{v}) = \{\mathbf{u}\}$, so that $S(\mathbf{v}) = \mathbf{u}$ and,

$$(S \circ T)(\mathbf{u}) = S(T(\mathbf{u})) = S(\mathbf{v}) = \mathbf{u} = I_U(\mathbf{u})$$

and since our choice of $\mathbf{u}$ was arbitrary we have function equality, $S \circ T = I_U$.

Now choose $\mathbf{v} \in V$. Define $\mathbf{u}$ to be the single vector in the set $T^{-1}(\mathbf{v})$, in other words, $\mathbf{u} = S(\mathbf{v})$. Then $T(\mathbf{u}) = \mathbf{v}$, so

$$(T \circ S)(\mathbf{v}) = T(S(\mathbf{v})) = T(\mathbf{u}) = \mathbf{v} = I_V(\mathbf{v})$$

and since our choice of $\mathbf{v}$ was arbitrary we have function equality, $T \circ S = I_V$. ■

When a linear transformation is both injective and surjective, the pre-image of any element of the codomain is a set of size one (a "singleton"). This fact allowed us to *construct* the inverse linear transformation in one half of the proof of Theorem ILTIS (see Proof Technique C) and is illustrated in the following cartoon. This should remind you of the very general Diagram KPI which was used to illustrate Theorem KPI about pre-images, only now we have an invertible linear transformation which is therefore surjective and injective (Theorem ILTIS). As a surjective linear transformation, there are no vectors depicted in the codomain, V, that have empty pre-images. More importantly, as an injective linear transformation, the kernel is trivial (Theorem KILT), so each pre-image is a single vector. This makes it possible to "turn around" all the arrows to create the inverse linear transformation T^{-1}.

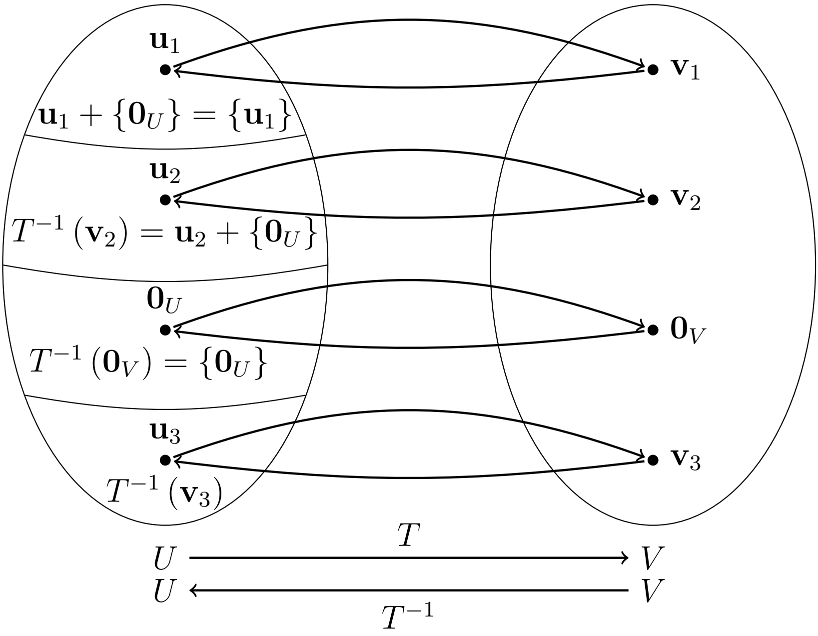

Diagram IVLT: Invertible Linear Transformation

Many will call an injective and surjective function a **bijective** function or just a **bijection**. Theorem ILTIS tells us that this is just a synonym for the term invertible (which we will use exclusively).

We can follow the constructive approach of the proof of Theorem ILTIS to construct the inverse of a specific linear transformation, as the next example shows.

Example CIVLT Computing the Inverse of a Linear Transformations
Consider the linear transformation $T\colon S_{22}\to P_2$ defined by

$$T\left(\begin{bmatrix} a & b\\ b & c\end{bmatrix}\right) = (a+b+c) + (-a+2c)\,x + (2a+3b+6c)\,x^2$$

T is invertible, which you are able to verify, perhaps by determining that the kernel of T is trivial and the range of T is all of P_2. This will be easier once we have Theorem RPNDD, which appears later in this section.

By Theorem ILTIS we know T^{-1} exists, and it will be critical shortly to realize that T^{-1} is automatically known to be a linear transformation as well (Theorem ILTLT). To determine the complete behavior of $T^{-1}\colon P_2\to S_{22}$ we can simply determine its action on a basis for the domain, P_2. This is the substance of Theorem LTDB, and an excellent example of its application. Choose any basis of P_2, the simpler the better, such as $B=\left\{1,\,x,\,x^2\right\}$. Values of T^{-1} for these three basis elements will be the single elements of their preimages. In turn, we have

$T^{-1}\left(1\right):$

$$T\left(\begin{bmatrix} a & b\\ b & c\end{bmatrix}\right) = 1+0x+0x^2$$

$$\begin{bmatrix} 1 & 1 & 1 & 1\\ -1 & 0 & 2 & 0\\ 2 & 3 & 6 & 0\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & -6\\ 0 & 1 & 0 & 10\\ 0 & 0 & 1 & -3\end{bmatrix}$$

(preimage) $$T^{-1}\left(1\right) = \left\{\begin{bmatrix} -6 & 10\\ 10 & -3\end{bmatrix}\right\}$$

(function) $$T^{-1}\left(1\right) = \begin{bmatrix} -6 & 10\\ 10 & -3\end{bmatrix}$$

$T^{-1}\left(x\right):$

$$T\left(\begin{bmatrix} a & b\\ b & c\end{bmatrix}\right) = 0+1x+0x^2$$

$$\begin{bmatrix} 1 & 1 & 1 & 0\\ -1 & 0 & 2 & 1\\ 2 & 3 & 6 & 0\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & -3\\ 0 & 1 & 0 & 4\\ 0 & 0 & 1 & -1\end{bmatrix}$$

(preimage) $$T^{-1}\left(x\right) = \left\{\begin{bmatrix} -3 & 4\\ 4 & -1\end{bmatrix}\right\}$$

(function) $$T^{-1}\left(x\right) = \begin{bmatrix} -3 & 4\\ 4 & -1\end{bmatrix}$$

$T^{-1}\left(x^2\right):$

$$T\left(\begin{bmatrix} a & b\\ b & c\end{bmatrix}\right) = 0+0x+1x^2$$

$$\begin{bmatrix} 1 & 1 & 1 & 0\\ -1 & 0 & 2 & 0\\ 2 & 3 & 6 & 1\end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 2\\ 0 & 1 & 0 & -3\\ 0 & 0 & 1 & 1\end{bmatrix}$$

(preimage) $$T^{-1}\left(x^2\right) = \left\{\begin{bmatrix} 2 & -3\\ -3 & 1\end{bmatrix}\right\}$$

$$\text{(function)} \qquad T^{-1}\left(x^2\right) = \begin{bmatrix} 2 & -3 \\ -3 & 1 \end{bmatrix}$$

Theorem LTDB says, informally, "it is enough to know what a linear transformation does to a basis." Formally, we have the outputs of T^{-1} for a basis, so by Theorem LTDB there is a unique linear transformation with these outputs. So we put this information to work. The key step here is that we can convert any element of P_2 into a linear combination of the elements of the basis B (Theorem VRRB). We are after a "formula" for the value of T^{-1} on a generic element of P_2, say $p + qx + rx^2$.

$$\begin{aligned}
T^{-1}\left(p + qx + rx^2\right) &= T^{-1}\left(p(1) + q(x) + r(x^2)\right) && \text{Theorem VRRB}\\
&= pT^{-1}(1) + qT^{-1}(x) + rT^{-1}\left(x^2\right) && \text{Theorem LTLC}\\
&= p\begin{bmatrix} -6 & 10 \\ 10 & -3 \end{bmatrix} + q\begin{bmatrix} -3 & 4 \\ 4 & -1 \end{bmatrix} + r\begin{bmatrix} 2 & -3 \\ -3 & 1 \end{bmatrix}\\
&= \begin{bmatrix} -6p - 3q + 2r & 10p + 4q - 3r \\ 10p + 4q - 3r & -3p - q + r \end{bmatrix}
\end{aligned}$$

Notice how a linear combination in the domain of T^{-1} has been translated into a linear combination in the codomain of T^{-1} since we know T^{-1} is a linear transformation by Theorem ILTLT.

Also, notice how the augmented matrices used to determine the three pre-images could be combined into one calculation of a matrix in extended echelon form, reminiscent of a procedure we know for computing the inverse of a matrix (see Example CMI). Hmmmm. △

We will make frequent use of the characterization of invertible linear transformations provided by Theorem ILTIS. The next theorem is a good example of this, and we will use it often, too.

Theorem CIVLT Composition of Invertible Linear Transformations
Suppose that $T\colon U \to V$ and $S\colon V \to W$ are invertible linear transformations. Then the composition, $(S \circ T)\colon U \to W$ is an invertible linear transformation.

Proof. Since S and T are both linear transformations, $S \circ T$ is also a linear transformation by Theorem CLTLT. Since S and T are both invertible, Theorem ILTIS says that S and T are both injective and surjective. Then Theorem CILTI says $S \circ T$ is injective, and Theorem CSLTS says $S \circ T$ is surjective. Now apply the "other half" of Theorem ILTIS and conclude that $S \circ T$ is invertible. ■

When a composition is invertible, the inverse is easy to construct.

Theorem ICLT Inverse of a Composition of Linear Transformations
Suppose that $T\colon U \to V$ and $S\colon V \to W$ are invertible linear transformations. Then $S \circ T$ is invertible and $(S \circ T)^{-1} = T^{-1} \circ S^{-1}$.

Proof. Compute, for all $\mathbf{w} \in W$

$$\begin{aligned}
\left((S \circ T) \circ \left(T^{-1} \circ S^{-1}\right)\right)(\mathbf{w}) &= S\left(T\left(T^{-1}\left(S^{-1}(\mathbf{w})\right)\right)\right)\\
&= S\left(I_V\left(S^{-1}(\mathbf{w})\right)\right) && \text{Definition IVLT}\\
&= S\left(S^{-1}(\mathbf{w})\right) && \text{Definition IDLT}\\
&= \mathbf{w} && \text{Definition IVLT}\\
&= I_W(\mathbf{w}) && \text{Definition IDLT}
\end{aligned}$$

So $(S \circ T) \circ \left(T^{-1} \circ S^{-1}\right) = I_W$, and also

$$\begin{aligned}
\left(\left(T^{-1} \circ S^{-1}\right) \circ (S \circ T)\right)(\mathbf{u}) &= T^{-1}\left(S^{-1}\left(S\left(T(\mathbf{u})\right)\right)\right)\\
&= T^{-1}\left(I_V\left(T(\mathbf{u})\right)\right) && \text{Definition IVLT}
\end{aligned}$$

$$= T^{-1}(T(\mathbf{u})) \qquad \text{Definition IDLT}$$
$$= \mathbf{u} \qquad \text{Definition IVLT}$$
$$= I_U(\mathbf{u}) \qquad \text{Definition IDLT}$$

so $\left(T^{-1} \circ S^{-1}\right) \circ (S \circ T) = I_U$.

By Definition IVLT, $S \circ T$ is invertible and $(S \circ T)^{-1} = T^{-1} \circ S^{-1}$. ■

Notice that this theorem not only establishes *what* the inverse of $S \circ T$ *is*, it also duplicates the conclusion of Theorem CIVLT and also establishes the invertibility of $S \circ T$. But somehow, the proof of Theorem CIVLT is a nicer way to get this property.

Does Theorem ICLT remind you of the flavor of any theorem we have seen about matrices? (Hint: Think about getting dressed.) Hmmmm.

Subsection SI
Structure and Isomorphism

A vector space is defined (Definition VS) as a set of objects ("vectors") endowed with a definition of vector addition (+) and a definition of scalar multiplication (written with juxtaposition). Many of our definitions about vector spaces involve linear combinations (Definition LC), such as the span of a set (Definition SS) and linear independence (Definition LI). Other definitions are built up from these ideas, such as bases (Definition B) and dimension (Definition D). The defining properties of a linear transformation require that a function "respect" the operations of the two vector spaces that are the domain and the codomain (Definition LT). Finally, an invertible linear transformation is one that can be "undone" — it has a companion that reverses its effect. In this subsection we are going to begin to roll all these ideas into one.

A vector space has "structure" derived from definitions of the two operations and the requirement that these operations interact in ways that satisfy the ten properties of Definition VS. When two different vector spaces have an invertible linear transformation defined between them, then we can translate questions about linear combinations (spans, linear independence, bases, dimension) from the first vector space to the second. The answers obtained in the second vector space can then be translated back, via the inverse linear transformation, and interpreted in the setting of the first vector space. We say that these invertible linear transformations "preserve structure." And we say that the two vector spaces are "structurally the same." The precise term is "isomorphic," from Greek meaning "of the same form." Let us begin to try to understand this important concept.

Definition IVS Isomorphic Vector Spaces
Two vector spaces U and V are **isomorphic** if there exists an invertible linear transformation T with domain U and codomain V, $T\colon U \to V$. In this case, we write $U \cong V$, and the linear transformation T is known as an **isomorphism** between U and V. □

A few comments on this definition. First, be careful with your language (Proof Technique L). Two vector spaces are isomorphic, or not. It is a yes/no situation and the term only applies to a pair of vector spaces. Any invertible linear transformation can be called an isomorphism, it is a term that applies to functions. Second, given a pair of vector spaces there might be several different isomorphisms between the two vector spaces. But it only takes the existence of one to call the pair isomorphic. Third, U isomorphic to V, or V isomorphic to U? It does not matter, since the inverse linear transformation will provide the needed isomorphism in the "opposite" direction. Being "isomorphic to" is an equivalence relation on the set of all vector spaces (see Theorem SER for a reminder about equivalence relations).

Example IVSAV Isomorphic vector spaces, Archetype V
Archetype V is a linear transformation from P_3 to M_{22},

$$T\colon P_3 \to M_{22},\quad T\left(a + bx + cx^2 + dx^3\right) = \begin{bmatrix} a+b & a-2c \\ d & b-d \end{bmatrix}$$

Since it is injective and surjective, Theorem ILTIS tells us that it is an invertible linear transformation. By Definition IVS we say P_3 and M_{22} are isomorphic.

At a basic level, the term "isomorphic" is nothing more than a codeword for the presence of an invertible linear transformation. However, it is also a description of a powerful idea, and this power only becomes apparent in the course of studying examples and related theorems. In this example, we are led to believe that there is nothing "structurally" different about P_3 and M_{22}. In a certain sense they are the same. Not equal, but the same. One is as good as the other. One is just as interesting as the other.

Here is an extremely basic application of this idea. Suppose we want to compute the following linear combination of polynomials in P_3,

$$5(2 + 3x - 4x^2 + 5x^3) + (-3)(3 - 5x + 3x^2 + x^3)$$

Rather than doing it straight-away (which is very easy), we will apply the transformation T to convert into a linear combination of matrices, and then compute in M_{22} according to the definitions of the vector space operations there (Example VSM),

$$\begin{aligned} &T\left(5(2 + 3x - 4x^2 + 5x^3) + (-3)(3 - 5x + 3x^2 + x^3)\right) \\ &= 5T\left(2 + 3x - 4x^2 + 5x^3\right) + (-3)T\left(3 - 5x + 3x^2 + x^3\right) && \text{Theorem LTLC} \\ &= 5\begin{bmatrix} 5 & 10 \\ 5 & -2 \end{bmatrix} + (-3)\begin{bmatrix} -2 & -3 \\ 1 & -6 \end{bmatrix} && \text{Definition of } T \\ &= \begin{bmatrix} 31 & 59 \\ 22 & 8 \end{bmatrix} && \text{Operations in } M_{22} \end{aligned}$$

Now we will translate our answer back to P_3 by applying T^{-1}, which we demonstrated in Example AIVLT,

$$T^{-1}\colon M_{22} \to P_3,\quad T^{-1}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a-c-d) + (c+d)x + \frac{1}{2}(a-b-c-d)x^2 + cx^3$$

We compute,

$$T^{-1}\left(\begin{bmatrix} 31 & 59 \\ 22 & 8 \end{bmatrix}\right) = 1 + 30x - 29x^2 + 22x^3$$

which is, as expected, exactly what we would have computed for the original linear combination had we just used the definitions of the operations in P_3 (Example VSP). Notice this is meant only as an *illustration* and not a suggested route for doing this particular computation. △

In Example IVSAV we avoided a computation in P_3 by a conversion of the computation to a new vector space, M_{22}, via an invertible linear transformation (also known as an isomorphism). Here is a diagram meant to to illustrate the more general situation of two vector spaces, U and V, and an invertible linear transformation, T. The diagram is simply about a sum of two vectors from U, rather than a more involved linear combination. It should remind you of Diagram DLTA.

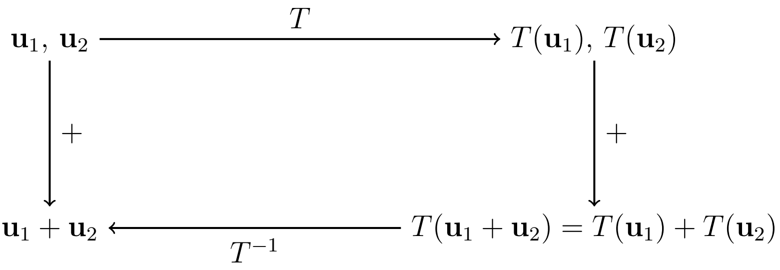

Diagram AIVS: Addition in Isomorphic Vector Spaces

To understand this diagram, begin in the upper-left corner, and by going straight down we can compute the sum of the two vectors using the addition for the vector space U. The more circuitous alternative, in the spirit of Example IVSAV, is to begin in the upper-left corner and then proceed clockwise around the other three sides of the rectangle. Notice that the vector addition is accomplished using the addition in the vector space V. Then, because T is a linear transformation, we can say that the result of $T(\mathbf{u}_1) + T(\mathbf{u}_2)$ is equal to $T(\mathbf{u}_1 + \mathbf{u}_2)$. Then the key feature is to recognize that applying T^{-1} obviously converts the second version of this result into the sum in the lower-left corner. So there are two routes to the sum $\mathbf{u}_1 + \mathbf{u}_2$, each employing an addition from a different vector space, but one is "direct" and the other is "roundabout". You might try designing a similar diagram for the case of scalar multiplication (see Diagram DLTM) or for a full linear combination.

Checking the dimensions of two vector spaces can be a quick way to establish that they are not isomorphic. Here is the theorem.

Theorem IVSED Isomorphic Vector Spaces have Equal Dimension
Suppose U and V are isomorphic vector spaces. Then $\dim(U) = \dim(V)$.

Proof. If U and V are isomorphic, there is an invertible linear transformation $T: U \to V$ (Definition IVS). T is injective by Theorem ILTIS and so by Theorem ILTD, $\dim(U) \leq \dim(V)$. Similarly, T is surjective by Theorem ILTIS and so by Theorem SLTD, $\dim(U) \geq \dim(V)$. The net effect of these two inequalities is that $\dim(U) = \dim(V)$. ■

The contrapositive of Theorem IVSED says that if U and V have different dimensions, then they are not isomorphic. Dimension is the simplest "structural" characteristic that will allow you to distinguish non-isomorphic vector spaces. For example P_6 is not isomorphic to M_{34} since their dimensions (7 and 12, respectively) are not equal. With tools developed in Section VR we will be able to establish that the converse of Theorem IVSED is true. Think about that one for a moment.

Subsection RNLT
Rank and Nullity of a Linear Transformation

Just as a matrix has a rank and a nullity, so too do linear transformations. And just like the rank and nullity of a matrix are related (they sum to the number of columns, Theorem RPNC) the rank and nullity of a linear transformation are related. Here are the definitions and theorems, see the Archetypes (Archetypes) for loads of examples.

Definition ROLT Rank Of a Linear Transformation
Suppose that $T: U \to V$ is a linear transformation. Then the **rank** of T, $r(T)$, is the dimension of the range of T,

$$r(T) = \dim(\mathcal{R}(T))$$

□

Definition NOLT Nullity Of a Linear Transformation
Suppose that $T\colon U \to V$ is a linear transformation. Then the **nullity** of T, $n(T)$, is the dimension of the kernel of T,

$$n(T) = \dim(\mathcal{K}(T))$$

□

Here are two quick theorems.

Theorem ROSLT Rank Of a Surjective Linear Transformation
Suppose that $T\colon U \to V$ is a linear transformation. Then the rank of T is the dimension of V, $r(T) = \dim(V)$, if and only if T is surjective.

Proof. By Theorem RSLT, T is surjective if and only if $\mathcal{R}(T) = V$. Applying Definition ROLT, $\mathcal{R}(T) = V$ if and only if $r(T) = \dim(\mathcal{R}(T)) = \dim(V)$. ■

Theorem NOILT Nullity Of an Injective Linear Transformation
Suppose that $T\colon U \to V$ is a linear transformation. Then the nullity of T is zero, $n(T) = 0$, if and only if T is injective.

Proof. By Theorem KILT, T is injective if and only if $\mathcal{K}(T) = \{\mathbf{0}\}$. Applying Definition NOLT, $\mathcal{K}(T) = \{\mathbf{0}\}$ if and only if $n(T) = 0$. ■

Just as injectivity and surjectivity come together in invertible linear transformations, there is a clear relationship between rank and nullity of a linear transformation. If one is big, the other is small.

Theorem RPNDD Rank Plus Nullity is Domain Dimension
Suppose that $T\colon U \to V$ is a linear transformation. Then

$$r(T) + n(T) = \dim(U)$$

Proof. Let $r = r(T)$ and $s = n(T)$. Suppose that $R = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_r\} \subseteq V$ is a basis of the range of T, $\mathcal{R}(T)$, and $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_s\} \subseteq U$ is a basis of the kernel of T, $\mathcal{K}(T)$. Note that R and S are possibly empty, which means that some of the sums in this proof are "empty" and are equal to the zero vector.

Because the elements of R are all in the range of T, each must have a nonempty pre-image by Theorem RPI. Choose vectors $\mathbf{w}_i \in U$, $1 \leq i \leq r$ such that $\mathbf{w}_i \in T^{-1}(\mathbf{v}_i)$. So $T(\mathbf{w}_i) = \mathbf{v}_i$, $1 \leq i \leq r$. Consider the set

$$B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_s, \mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \ldots, \mathbf{w}_r\}$$

We claim that B is a basis for U.

To establish linear independence for B, begin with a relation of linear dependence on B. So suppose there are scalars $a_1, a_2, a_3, \ldots, a_s$ and $b_1, b_2, b_3, \ldots, b_r$

$$\mathbf{0} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \cdots + a_s\mathbf{u}_s + b_1\mathbf{w}_1 + b_2\mathbf{w}_2 + b_3\mathbf{w}_3 + \cdots + b_r\mathbf{w}_r$$

Then

$$\begin{aligned}
\mathbf{0} &= T(\mathbf{0}) && \text{Theorem LTTZZ}\\
&= T(a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \cdots + a_s\mathbf{u}_s +\\
&\qquad b_1\mathbf{w}_1 + b_2\mathbf{w}_2 + b_3\mathbf{w}_3 + \cdots + b_r\mathbf{w}_r) && \text{Definition LI}\\
&= a_1T(\mathbf{u}_1) + a_2T(\mathbf{u}_2) + a_3T(\mathbf{u}_3) + \cdots + a_sT(\mathbf{u}_s) +\\
&\qquad b_1T(\mathbf{w}_1) + b_2T(\mathbf{w}_2) + b_3T(\mathbf{w}_3) + \cdots + b_rT(\mathbf{w}_r) && \text{Theorem LTLC}\\
&= a_1\mathbf{0} + a_2\mathbf{0} + a_3\mathbf{0} + \cdots + a_s\mathbf{0} +\\
&\qquad b_1T(\mathbf{w}_1) + b_2T(\mathbf{w}_2) + b_3T(\mathbf{w}_3) + \cdots + b_rT(\mathbf{w}_r) && \text{Definition KLT}\\
&= \mathbf{0} + \mathbf{0} + \mathbf{0} + \cdots + \mathbf{0} +
\end{aligned}$$

$$\begin{aligned}
&b_1T(\mathbf{w}_1) + b_2T(\mathbf{w}_2) + b_3T(\mathbf{w}_3) + \cdots + b_rT(\mathbf{w}_r) && \text{Theorem ZVSM}\\
&= b_1T(\mathbf{w}_1) + b_2T(\mathbf{w}_2) + b_3T(\mathbf{w}_3) + \cdots + b_rT(\mathbf{w}_r) && \text{Property Z}\\
&= b_1\mathbf{v}_1 + b_2\mathbf{v}_2 + b_3\mathbf{v}_3 + \cdots + b_r\mathbf{v}_r && \text{Definition PI}
\end{aligned}$$

This is a relation of linear dependence on R (Definition RLD), and since R is a linearly independent set (Definition LI), we see that $b_1 = b_2 = b_3 = \ldots = b_r = 0$. Then the original relation of linear dependence on B becomes

$$\begin{aligned}
\mathbf{0} &= a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \cdots + a_s\mathbf{u}_s + 0\mathbf{w}_1 + 0\mathbf{w}_2 + \ldots + 0\mathbf{w}_r\\
&= a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \cdots + a_s\mathbf{u}_s + \mathbf{0} + \mathbf{0} + \ldots + \mathbf{0} && \text{Theorem ZSSM}\\
&= a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \cdots + a_s\mathbf{u}_s && \text{Property Z}
\end{aligned}$$

But this is again a relation of linear independence (Definition RLD), now on the set S. Since S is linearly independent (Definition LI), we have $a_1 = a_2 = a_3 = \ldots = a_r = 0$. Since we now know that all the scalars in the relation of linear dependence on B must be zero, we have established the linear independence of S through Definition LI.

To now establish that B spans U, choose an arbitrary vector $\mathbf{u} \in U$. Then $T(\mathbf{u}) \in R(T)$, so there are scalars $c_1, c_2, c_3, \ldots, c_r$ such that

$$T(\mathbf{u}) = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdots + c_r\mathbf{v}_r$$

Use the scalars $c_1, c_2, c_3, \ldots, c_r$ to define a vector $\mathbf{y} \in U$,

$$\mathbf{y} = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 + \cdots + c_r\mathbf{w}_r$$

Then

$$\begin{aligned}
T(\mathbf{u} - \mathbf{y}) &= T(\mathbf{u}) - T(\mathbf{y}) && \text{Theorem LTLC}\\
&= T(\mathbf{u}) - T(c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 + \cdots + c_r\mathbf{w}_r) && \text{Substitution}\\
&= T(\mathbf{u}) - (c_1T(\mathbf{w}_1) + c_2T(\mathbf{w}_2) + \cdots + c_rT(\mathbf{w}_r)) && \text{Theorem LTLC}\\
&= T(\mathbf{u}) - (c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 + \cdots + c_r\mathbf{v}_r) && \mathbf{w}_i \in T^{-1}(\mathbf{v}_i)\\
&= T(\mathbf{u}) - T(\mathbf{u}) && \text{Substitution}\\
&= \mathbf{0} && \text{Property AI}
\end{aligned}$$

So the vector $\mathbf{u} - \mathbf{y}$ is sent to the zero vector by T and hence is an element of the kernel of T. As such it can be written as a linear combination of the basis vectors for $\mathcal{K}(T)$, the elements of the set S. So there are scalars $d_1, d_2, d_3, \ldots, d_s$ such that

$$\mathbf{u} - \mathbf{y} = d_1\mathbf{u}_1 + d_2\mathbf{u}_2 + d_3\mathbf{u}_3 + \cdots + d_s\mathbf{u}_s$$

Then

$$\begin{aligned}
\mathbf{u} &= (\mathbf{u} - \mathbf{y}) + \mathbf{y}\\
&= d_1\mathbf{u}_1 + d_2\mathbf{u}_2 + d_3\mathbf{u}_3 + \cdots + d_s\mathbf{u}_s + c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + c_3\mathbf{w}_3 + \cdots + c_r\mathbf{w}_r
\end{aligned}$$

This says that for any vector, $\mathbf{u}$, from U, there exist scalars ($d_1, d_2, d_3, \ldots, d_s, c_1, c_2, c_3, \ldots, c_r$) that form $\mathbf{u}$ as a linear combination of the vectors in the set B. In other words, B spans U (Definition SS).

So B is a basis (Definition B) of U with $s + r$ vectors, and thus

$$\dim(U) = s + r = n(T) + r(T)$$

as desired. ■

Theorem RPNC said that the rank and nullity of a matrix sum to the number of columns of the matrix. This result is now an easy consequence of Theorem RPNDD when we consider the linear transformation $T\colon \mathbb{C}^n \to \mathbb{C}^m$ defined with the $m \times n$ matrix A by $T(\mathbf{x}) = A\mathbf{x}$. The range and kernel of T are identical to the column space and null space of the matrix A (Exercise ILT.T20, Exercise SLT.T20), so the rank and nullity of the matrix A are identical to the rank and nullity of the linear transformation T. The dimension of the domain of T is the dimension of $\mathbb{C}^n$, exactly the number of columns for the matrix A.

This theorem can be especially useful in determining basic properties of linear transformations. For example, suppose that $T\colon \mathbb{C}^6 \to \mathbb{C}^6$ is a linear transformation and you are able to quickly establish that the kernel is trivial. Then $n(T) = 0$. First this means that T is injective by Theorem NOILT. Also, Theorem RPNDD becomes

$$6 = \dim\left(\mathbb{C}^6\right) = r(T) + n(T) = r(T) + 0 = r(T)$$

So the rank of T is equal to the rank of the codomain, and by Theorem ROSLT we know T is surjective. Finally, we know T is invertible by Theorem ILTIS. So from the determination that the kernel is trivial, and consideration of various dimensions, the theorems of this section allow us to conclude the existence of an inverse linear transformation for T. Similarly, Theorem RPNDD can be used to provide alternative proofs for Theorem ILTD, Theorem SLTD and Theorem IVSED. It would be an interesting exercise to construct these proofs.

It would be instructive to study the archetypes that are linear transformations and see how many of their properties can be deduced just from considering only the dimensions of the domain and codomain. Then add in just knowledge of either the nullity or rank, and see how much more you can learn about the linear transformation. The table preceding all of the archetypes (Archetypes) could be a good place to start this analysis.

Subsection SLELT
Systems of Linear Equations and Linear Transformations

This subsection does not really belong in this section, or any other section, for that matter. It is just the right time to have a discussion about the connections between the central topic of linear algebra, linear transformations, and our motivating topic from Chapter SLE, systems of linear equations. We will discuss several theorems we have seen already, but we will also make some forward-looking statements that will be justified in Chapter R.

Archetype D and Archetype E are ideal examples to illustrate connections with linear transformations. Both have the same coefficient matrix,

$$D = \begin{bmatrix} 2 & 1 & 7 & -7 \\ -3 & 4 & -5 & -6 \\ 1 & 1 & 4 & -5 \end{bmatrix}$$

To apply the *theory* of linear transformations to these two archetypes, employ the matrix-vector product (Definition MVP) and define the linear transformation,

$$T\colon \mathbb{C}^4 \to \mathbb{C}^3, \quad T(\mathbf{x}) = D\mathbf{x} = x_1 \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 7 \\ -5 \\ 4 \end{bmatrix} + x_4 \begin{bmatrix} -7 \\ -6 \\ -5 \end{bmatrix}$$

Theorem MBLT tells us that T is indeed a linear transformation. Archetype D asks for solutions to $\mathcal{LS}(D, \mathbf{b})$, where $\mathbf{b} = \begin{bmatrix} 8 \\ -12 \\ -4 \end{bmatrix}$. In the language of linear transformations this is equivalent to asking for $T^{-1}(\mathbf{b})$. In the language of vectors and matrices it asks for a linear combination of the four columns of D that will equal $\mathbf{b}$. One solution listed is $\mathbf{w} = \begin{bmatrix} 7 \\ 8 \\ 1 \\ 3 \end{bmatrix}$. With a nonempty preimage, Theorem KPI tells us that the complete solution set of the linear system is the preimage of $\mathbf{b}$,

$$\mathbf{w} + \mathcal{K}(T) = \{\mathbf{w} + \mathbf{z} \mid \mathbf{z} \in \mathcal{K}(T)\}$$

The kernel of the linear transformation T is exactly the null space of the matrix D (see Exercise ILT.T20), so this approach to the solution set should be reminiscent of Theorem PSPHS. The kernel of the linear transformation is the preimage of the zero vector, exactly equal to the solution set of the homogeneous system $\mathcal{LS}(D,\,\mathbf{0})$. Since D has a null space of dimension two, every preimage (and in particular the preimage of $\mathbf{b}$) is as "big" as a subspace of dimension two (but is not a subspace).

Archetype E is identical to Archetype D but with a different vector of constants, $\mathbf{d} = \begin{bmatrix}2\\3\\2\end{bmatrix}$. We can use the same linear transformation T to discuss this system of equations since the coefficient matrix is identical. Now the set of solutions to $\mathcal{LS}(D,\,\mathbf{d})$ is the pre-image of $\mathbf{d}$, $T^{-1}(\mathbf{d})$. However, the vector $\mathbf{d}$ is not in the range of the linear transformation (nor is it in the column space of the matrix, since these two sets are equal by Exercise SLT.T20). So the empty pre-image is equivalent to the inconsistency of the linear system.

These two archetypes each have three equations in four variables, so either the resulting linear systems are inconsistent, or they are consistent and application of Theorem CMVEI tells us that the system has infinitely many solutions. Considering these same parameters for the linear transformation, the dimension of the domain, $\mathbb{C}^4$, is four, while the codomain, $\mathbb{C}^3$, has dimension three. Then

$$\begin{aligned}
n\left(T\right) &= \dim\left(\mathbb{C}^4\right) - r\left(T\right) && \text{Theorem RPNDD}\\
&= 4 - \dim\left(\mathcal{R}(T)\right) && \text{Definition ROLT}\\
&\geq 4 - 3 && \mathcal{R}(T)\text{ subspace of }\mathbb{C}^3\\
&= 1
\end{aligned}$$

So the kernel of T is nontrivial simply by considering the dimensions of the domain (number of variables) and the codomain (number of equations). Pre-images of elements of the codomain that are not in the range of T are empty (inconsistent systems). For elements of the codomain that are in the range of T (consistent systems), Theorem KPI tells us that the pre-images are built from the kernel, and with a nontrivial kernel, these pre-images are infinite (infinitely many solutions).

When do systems of equations have unique solutions? Consider the system of linear equations $\mathcal{LS}(C,\,\mathbf{f})$ and the linear transformation $S\left(\mathbf{x}\right) = C\mathbf{x}$. If S has a trivial kernel, then pre-images will either be empty or be finite sets with single elements. Correspondingly, the coefficient matrix C will have a trivial null space and solution sets will either be empty (inconsistent) or contain a single solution (unique solution). Should the matrix be square and have a trivial null space then we recognize the matrix as being nonsingular. A square matrix means that the corresponding linear transformation, T, has equal-sized domain and codomain. With a nullity of zero, T is injective, and also Theorem RPNDD tells us that rank of T is equal to the dimension of the domain, which in turn is equal to the dimension of the codomain. In other words, T is surjective. Injective and surjective, and Theorem ILTIS tells us that T is invertible. Just as we can use the inverse of the coefficient matrix to find the unique solution of any linear system with a nonsingular coefficient matrix (Theorem SNCM), we can use the inverse of the linear transformation to construct the unique element of any pre-image (proof of Theorem ILTIS).

The executive summary of this discussion is that to every coefficient matrix of a system of linear equations we can associate a natural linear transformation. Solution sets for systems with this coefficient matrix are preimages of elements of the codomain of the linear transformation. For every theorem about systems of linear equations there is an analogue about linear transformations. The theory of linear transformations provides all the tools to recreate the theory of solutions to linear systems of equations.

We will continue this adventure in Chapter R.

Reading Questions

1. What conditions allow us to **easily** determine if a linear transformation is invertible?
2. What does it mean to say two vector spaces are isomorphic? Both technically, and informally?
3. How do linear transformations relate to systems of linear equations?

Exercises

C10 The archetypes below are linear transformations of the form $T\colon U \to V$ that are invertible. For each, the inverse linear transformation is given explicitly as part of the archetype's description. Verify for each linear transformation that

$$T^{-1} \circ T = I_U \qquad\qquad T \circ T^{-1} = I_V$$

Archetype R, Archetype V, Archetype W

C20† Determine if the linear transformation $T\colon P_2 \to M_{22}$ is (a) injective, (b) surjective, (c) invertible.

$$T\left(a + bx + cx^2\right) = \begin{bmatrix} a + 2b - 2c & 2a + 2b \\ -a + b - 4c & 3a + 2b + 2c \end{bmatrix}$$

C21† Determine if the linear transformation $S\colon P_3 \to M_{22}$ is (a) injective, (b) surjective, (c) invertible.

$$S\left(a + bx + cx^2 + dx^3\right) = \begin{bmatrix} -a + 4b + c + 2d & 4a - b + 6c - d \\ a + 5b - 2c + 2d & a + 2c + 5d \end{bmatrix}$$

C25 For each linear transformation below: (a) Find the matrix representation of T, (b) Calculate $n(T)$, (c) Calculate $r(T)$, (d) Graph the image in either $\mathbb{R}^2$ or $\mathbb{R}^3$ as appropriate, (e) How many dimensions are lost?, and (f) How many dimensions are preserved?

1. $T\colon \mathbb{C}^3 \to \mathbb{C}^3$ given by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ x \\ x \end{bmatrix}$
2. $T\colon \mathbb{C}^3 \to \mathbb{C}^3$ given by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$
3. $T\colon \mathbb{C}^3 \to \mathbb{C}^2$ given by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ x \end{bmatrix}$
4. $T\colon \mathbb{C}^3 \to \mathbb{C}^2$ given by $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$
5. $T\colon \mathbb{C}^2 \to \mathbb{C}^3$ given by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$
6. $T\colon \mathbb{C}^2 \to \mathbb{C}^3$ given by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ x + y \end{bmatrix}$

C50† Consider the linear transformation $S\colon M_{12} \to P_1$ from the set of 1×2 matrices to the set of polynomials of degree at most 1, defined by

$$S\left(\begin{bmatrix} a & b \end{bmatrix}\right) = (3a + b) + (5a + 2b)x$$

Prove that S is invertible. Then show that the linear transformation

$$R\colon P_1 \to M_{12}, \quad R\left(r + sx\right) = \begin{bmatrix} (2r - s) & (-5r + 3s) \end{bmatrix}$$

is the inverse of S, that is $S^{-1} = R$.

M30† The linear transformation S below is invertible. Find a formula for the inverse linear transformation, S^{-1}.

$$S\colon P_1 \to M_{1,2}, \quad S\left(a + bx\right) = \begin{bmatrix} 3a + b & 2a + b \end{bmatrix}$$

M31† The linear transformation $R\colon M_{12} \to M_{21}$ is invertible. Determine a formula for the inverse linear transformation $R^{-1}\colon M_{21} \to M_{12}$.

$$R\left(\begin{bmatrix} a & b \end{bmatrix}\right) = \begin{bmatrix} a + 3b \\ 4a + 11b \end{bmatrix}$$

M50 Rework Example CIVLT, only in place of the basis B for P_2, choose instead to use the basis $C = \left\{1,\, 1 + x,\, 1 + x + x^2\right\}$. This will complicate writing a generic element of the domain of T^{-1} as a linear combination of the basis elements, and the algebra will be a bit messier, but in the end you should obtain the same formula for T^{-1}. The inverse linear transformation is what it is, and the choice of a particular basis should not influence the outcome.

M60 Suppose U and V are vector spaces. Define the function $Z\colon U \to V$ by $T\left(\mathbf{u}\right) = \mathbf{0}_V$ for every $\mathbf{u} \in U$. Then by Exercise LT.M60, Z is a linear transformation. Formulate a condition on U and V that is equivalent to Z being an invertible linear transformation. In other words, fill in the blank to complete the following statement (and then give a proof): Z is invertible if and only if U and V are . (See Exercise ILT.M60, Exercise SLT.M60, Exercise MR.M60.)

T05 Prove that the identity linear transformation (Definition IDLT) is both injective and surjective, and hence invertible.

T15† Suppose that $T\colon U \to V$ is a surjective linear transformation and $\dim\left(U\right) = \dim\left(V\right)$. Prove that T is injective.

T16 Suppose that $T\colon U \to V$ is an injective linear transformation and $\dim\left(U\right) = \dim\left(V\right)$. Prove that T is surjective.

T30† Suppose that U and V are isomorphic vector spaces. Prove that there are infinitely many isomorphisms between U and V.

T40† Suppose $T\colon U \to V$ and $S\colon V \to W$ are linear transformations and $\dim\left(U\right) = \dim\left(V\right) = \dim\left(W\right)$. Suppose that $S \circ T$ is invertible. Prove that S and T are individually invertible (this could be construed as a converse of Theorem CIVLT).

Chapter R
Representations

Previous work with linear transformations may have convinced you that we can convert most questions about linear transformations into questions about systems of equations or properties of subspaces of $\mathbb{C}^m$. In this section we begin to make these vague notions precise. We have used the word "representation" prior, but it will get a heavy workout in this chapter. In many ways, everything we have studied so far was in preparation for this chapter.

Section VR
Vector Representations

You may have noticed that many questions about elements of abstract vector spaces eventually become questions about column vectors or systems of equations. Example SM32 would be an example of this. We will make this vague idea more precise in this section.

Subsection VR
Vector Representation

We begin by establishing an invertible linear transformation between any vector space V of dimension m and $\mathbb{C}^m$. This will allow us to "go back and forth" between the two vector spaces, no matter how abstract the definition of V might be.

Definition VR Vector Representation
Suppose that V is a vector space with a basis $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$. Define a function $\rho_B\colon V \to \mathbb{C}^n$ as follows. For $\mathbf{w} \in V$ define the column vector $\rho_B(\mathbf{w}) \in \mathbb{C}^n$ by

$$\mathbf{w} = [\rho_B(\mathbf{w})]_1 \mathbf{v}_1 + [\rho_B(\mathbf{w})]_2 \mathbf{v}_2 + [\rho_B(\mathbf{w})]_3 \mathbf{v}_3 + \cdots + [\rho_B(\mathbf{w})]_n \mathbf{v}_n$$

□

This definition looks more complicated that it really is, though the form above will be useful in proofs. Simply stated, given $\mathbf{w} \in V$, we write $\mathbf{w}$ as a linear combination of the basis elements of B. It is key to realize that Theorem VRRB guarantees that we can do this for every $\mathbf{w}$, and furthermore this expression as a linear combination is unique. The resulting scalars are just the entries of the vector $\rho_B(\mathbf{w})$. This discussion should convince you that ρ_B is "well-defined" as a function. We can determine a precise output for any input. Now we want to establish that ρ_B is a function with additional properties — it is a linear transformation.

Theorem VRLT Vector Representation is a Linear Transformation
The function ρ_B (Definition VR) is a linear transformation.

Proof. We will take a novel approach in this proof. We will construct another function, which we will easily determine is a linear transformation, and then show that this second function is really ρ_B in disguise. Here we go.

Since B is a basis, we can define $T\colon V \to \mathbb{C}^n$ to be the unique linear transformation such that $T(\mathbf{v}_i) = \mathbf{e}_i$, $1 \leq i \leq n$, as guaranteed by Theorem LTDB, and where the $\mathbf{e}_i$ are the standard unit vectors (Definition SUV). Then suppose for an arbitrary $\mathbf{w} \in V$ we have,

$$\begin{aligned}
\left[T(\mathbf{w})\right]_i &= \left[T\left(\sum_{j=1}^{n} \left[\rho_B(\mathbf{w})\right]_j \mathbf{v}_j\right)\right]_i && \text{Definition VR}\\
&= \left[\sum_{j=1}^{n} \left[\rho_B(\mathbf{w})\right]_j T(\mathbf{v}_j)\right]_i && \text{Theorem LTLC}\\
&= \left[\sum_{j=1}^{n} \left[\rho_B(\mathbf{w})\right]_j \mathbf{e}_j\right]_i\\
&= \sum_{j=1}^{n} \left[\left[\rho_B(\mathbf{w})\right]_j \mathbf{e}_j\right]_i && \text{Definition CVA}\\
&= \sum_{j=1}^{n} \left[\rho_B(\mathbf{w})\right]_j \left[\mathbf{e}_j\right]_i && \text{Definition CVSM}\\
&= \left[\rho_B(\mathbf{w})\right]_i \left[\mathbf{e}_i\right]_i + \sum_{\substack{j=1\\ j\neq i}}^{n} \left[\rho_B(\mathbf{w})\right]_j \left[\mathbf{e}_j\right]_i && \text{Property CC}\\
&= \left[\rho_B(\mathbf{w})\right]_i (1) + \sum_{\substack{j=1\\ j\neq i}}^{n} \left[\rho_B(\mathbf{w})\right]_j (0) && \text{Definition SUV}\\
&= \left[\rho_B(\mathbf{w})\right]_i
\end{aligned}$$

As column vectors, Definition CVE implies that $T(\mathbf{w}) = \rho_B(\mathbf{w})$. Since $\mathbf{w}$ was an arbitrary element of V, as functions $T = \rho_B$. Now, since T is known to be a linear transformation, it must follow that ρ_B is also a linear transformation. ■

The proof of Theorem VRLT provides an alternate definition of vector representation relative to a basis B that we could state as a corollary (Proof Technique LC): ρ_B is the unique linear transformation that takes B to the standard unit basis.

Example VRC4 Vector representation in $\mathbb{C}^4$

Consider the vector $\mathbf{y} \in \mathbb{C}^4$

$$\mathbf{y} = \begin{bmatrix} 6\\ 14\\ 6\\ 7 \end{bmatrix}$$

We will find several vector representations of $\mathbf{y}$ in this example. Notice that $\mathbf{y}$ never changes, but the *representations* of $\mathbf{y}$ do change. One basis for $\mathbb{C}^4$ is

$$B = \{\mathbf{u}_1,\,\mathbf{u}_2,\,\mathbf{u}_3,\,\mathbf{u}_4\} = \left\{\begin{bmatrix} -2\\ 1\\ 2\\ -3 \end{bmatrix},\,\begin{bmatrix} 3\\ -6\\ 2\\ -4 \end{bmatrix},\,\begin{bmatrix} 1\\ 2\\ 0\\ 5 \end{bmatrix},\,\begin{bmatrix} 4\\ 3\\ 1\\ 6 \end{bmatrix}\right\}$$

as can be seen by making these vectors the columns of a matrix, checking that the matrix is nonsingular and applying Theorem CNMB. To find $\rho_B(\mathbf{y})$, we need to find scalars, a_1, a_2, a_3, a_4 such that

$$\mathbf{y} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + a_4\mathbf{u}_4$$

By Theorem SLSLC the desired scalars are a solution to the linear system of equations with a coefficient matrix whose columns are the vectors in B and with a vector of constants $\mathbf{y}$. With a nonsingular coefficient matrix, the solution is unique, but this is no surprise as this is the content of Theorem VRRB. This unique solution is

$$a_1 = 2 \qquad a_2 = -1 \qquad a_3 = -3 \qquad a_4 = 4$$

Then by Definition VR, we have

$$\rho_B(\mathbf{y}) = \begin{bmatrix} 2 \\ -1 \\ -3 \\ 4 \end{bmatrix}$$

Suppose now that we construct a representation of $\mathbf{y}$ relative to another basis of $\mathbb{C}^4$,

$$C = \left\{ \begin{bmatrix} -15 \\ 9 \\ -4 \\ -2 \end{bmatrix}, \begin{bmatrix} 16 \\ -14 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} -26 \\ 14 \\ -6 \\ -3 \end{bmatrix}, \begin{bmatrix} 14 \\ -13 \\ 4 \\ 6 \end{bmatrix} \right\}$$

As with B, it is easy to check that C is a basis. Writing $\mathbf{y}$ as a linear combination of the vectors in C leads to solving a system of four equations in the four unknown scalars with a nonsingular coefficient matrix. The unique solution can be expressed as

$$\mathbf{y} = \begin{bmatrix} 6 \\ 14 \\ 6 \\ 7 \end{bmatrix} = (-28)\begin{bmatrix} -15 \\ 9 \\ -4 \\ -2 \end{bmatrix} + (-8)\begin{bmatrix} 16 \\ -14 \\ 5 \\ 2 \end{bmatrix} + 11\begin{bmatrix} -26 \\ 14 \\ -6 \\ -3 \end{bmatrix} + 0\begin{bmatrix} 14 \\ -13 \\ 4 \\ 6 \end{bmatrix}$$

so that Definition VR gives

$$\rho_C(\mathbf{y}) = \begin{bmatrix} -28 \\ -8 \\ 11 \\ 0 \end{bmatrix}$$

We often perform representations relative to standard bases, but for vectors in $\mathbb{C}^m$ this is a little silly. Let us find the vector representation of $\mathbf{y}$ relative to the standard basis (Theorem SUVB),

$$D = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$$

Then, without any computation, we can check that

$$\mathbf{y} = \begin{bmatrix} 6 \\ 14 \\ 6 \\ 7 \end{bmatrix} = 6\mathbf{e}_1 + 14\mathbf{e}_2 + 6\mathbf{e}_3 + 7\mathbf{e}_4$$

so by Definition VR,

$$\rho_D(\mathbf{y}) = \begin{bmatrix} 6 \\ 14 \\ 6 \\ 7 \end{bmatrix}$$

which is not very exciting. Notice however that the *order* in which we place the vectors in the basis is critical to the representation. Let us keep the standard unit vectors as our basis, but rearrange the order we place them in the basis. So a fourth basis is

$$E = \{\mathbf{e}_3,\, \mathbf{e}_4,\, \mathbf{e}_2,\, \mathbf{e}_1\}$$

Then,

$$\mathbf{y} = \begin{bmatrix} 6 \\ 14 \\ 6 \\ 7 \end{bmatrix} = 6\mathbf{e}_3 + 7\mathbf{e}_4 + 14\mathbf{e}_2 + 6\mathbf{e}_1$$

so by Definition VR,

$$\rho_E(\mathbf{y}) = \begin{bmatrix} 6 \\ 7 \\ 14 \\ 6 \end{bmatrix}$$

So for every possible basis of $\mathbb{C}^4$ we could construct a different representation of $\mathbf{y}$. △

Vector representations are most interesting for vector spaces that are not $\mathbb{C}^m$.

Example VRP2 Vector representations in P_2
Consider the vector $\mathbf{u} = 15 + 10x - 6x^2 \in P_2$ from the vector space of polynomials with degree at most 2 (Example VSP). A nice basis for P_2 is

$$B = \left\{1,\, x,\, x^2\right\}$$

so that

$$\mathbf{u} = 15 + 10x - 6x^2 = 15(1) + 10(x) + (-6)(x^2)$$

so by Definition VR

$$\rho_B(\mathbf{u}) = \begin{bmatrix} 15 \\ 10 \\ -6 \end{bmatrix}$$

Another nice basis for P_2 is

$$C = \left\{1,\, 1 + x,\, 1 + x + x^2\right\}$$

so that now it takes a bit of computation to determine the scalars for the representation. We want a_1, a_2, a_3 so that

$$15 + 10x - 6x^2 = a_1(1) + a_2(1 + x) + a_3(1 + x + x^2)$$

Performing the operations in P_2 on the right-hand side, and equating coefficients, gives the three equations in the three unknown scalars,

$$\begin{aligned} 15 &= a_1 + a_2 + a_3 \\ 10 &= a_2 + a_3 \\ -6 &= a_3 \end{aligned}$$

The coefficient matrix of this sytem is nonsingular, leading to a unique solution (no surprise there, see Theorem VRRB),

$$a_1 = 5 \qquad a_2 = 16 \qquad a_3 = -6$$

so by Definition VR

$$\rho_C(\mathbf{u}) = \begin{bmatrix} 5 \\ 16 \\ -6 \end{bmatrix}$$

While we often form vector representations relative to "nice" bases, nothing prevents us from forming representations relative to "nasty" bases. For example, the set

$$D = \left\{-2 - x + 3x^2,\, 1 - 2x^2,\, 5 + 4x + x^2\right\}$$

can be verified as a basis of P_2 by checking linear independence with Definition LI and then arguing that 3 vectors from P_2, a vector space of dimension 3 (Theorem DP), must also be a spanning set (Theorem G).

Now we desire scalars a_1, a_2, a_3 so that

$$15 + 10x - 6x^2 = a_1(-2 - x + 3x^2) + a_2(1 - 2x^2) + a_3(5 + 4x + x^2)$$

Performing the operations in P_2 on the right-hand side, and equating coefficients, gives the three equations in the three unknown scalars,

$$\begin{aligned} 15 &= -2a_1 + a_2 + 5a_3 \\ 10 &= -a_1 + 4a_3 \\ -6 &= 3a_1 - 2a_2 + a_3 \end{aligned}$$

The coefficient matrix of this sytem is nonsingular, leading to a unique solution (no surprise there, see Theorem VRRB),

$$a_1 = -2 \qquad a_2 = 1 \qquad a_3 = 2$$

so by Definition VR

$$\rho_D(\mathbf{u}) = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

△

Theorem VRI Vector Representation is Injective
The function ρ_B (Definition VR) is an injective linear transformation.

Proof. We will appeal to Theorem KILT. Suppose U is a vector space of dimension n, so vector representation is $\rho_B\colon U \to \mathbb{C}^n$. Let $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$ be the basis of U used in the definition of ρ_B. Suppose $\mathbf{u} \in \mathcal{K}(\rho_B)$. We write $\mathbf{u}$ as a linear combination of the vectors in the basis B where the scalars are the components of the vector representation, $\rho_B(\mathbf{u})$.

$$\begin{aligned} \mathbf{u} &= [\rho_B(\mathbf{u})]_1 \mathbf{u}_1 + [\rho_B(\mathbf{u})]_2 \mathbf{u}_2 + \cdots + [\rho_B(\mathbf{u})]_n \mathbf{u}_n && \text{Definition VR} \\ &= [\mathbf{0}]_1 \mathbf{u}_1 + [\mathbf{0}]_2 \mathbf{u}_2 + \cdots + [\mathbf{0}]_n \mathbf{u}_n && \text{Definition KLT} \\ &= 0\mathbf{u}_1 + 0\mathbf{u}_2 + \cdots + 0\mathbf{u}_n && \text{Definition ZCV} \\ &= \mathbf{0} + \mathbf{0} + \cdots + \mathbf{0} && \text{Theorem ZSSM} \\ &= \mathbf{0} && \text{Property Z} \end{aligned}$$

Thus an arbitrary vector, $\mathbf{u}$, from the kernel ,$\mathcal{K}(\rho_B)$, must equal the zero vector of U. So $\mathcal{K}(\rho_B) = \{\mathbf{0}\}$ and by Theorem KILT, ρ_B is injective. ■

Theorem VRS Vector Representation is Surjective
The function ρ_B (Definition VR) is a surjective linear transformation.

Proof. We will appeal to Theorem RSLT. Suppose U is a vector space of dimension n, so vector representation is $\rho_B\colon U \to \mathbb{C}^n$. Let $B = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_n\}$ be the basis of U used in the definition of ρ_B. Suppose $\mathbf{v} \in \mathbb{C}^n$. Define the vector $\mathbf{u}$ by

$$\mathbf{u} = [\mathbf{v}]_1 \mathbf{u}_1 + [\mathbf{v}]_2 \mathbf{u}_2 + [\mathbf{v}]_3 \mathbf{u}_3 + \cdots + [\mathbf{v}]_n \mathbf{u}_n$$

Then for $1 \leq i \leq n$, by Definition VR,

$$[\rho_B(\mathbf{u})]_i = [\rho_B([\mathbf{v}]_1 \mathbf{u}_1 + [\mathbf{v}]_2 \mathbf{u}_2 + [\mathbf{v}]_3 \mathbf{u}_3 + \cdots + [\mathbf{v}]_n \mathbf{u}_n)]_i = [\mathbf{v}]_i$$

so the entries of vectors $\rho_B(\mathbf{u})$ and $\mathbf{v}$ are equal and Definition CVE yields the vector equality $\rho_B(\mathbf{u}) = \mathbf{v}$. This demonstrates that $\mathbf{v} \in \mathcal{R}(\rho_B)$, so $\mathbb{C}^n \subseteq \mathcal{R}(\rho_B)$. Since

$\mathcal{R}(\rho_B) \subseteq \mathbb{C}^n$ by Definition RLT, we have $\mathcal{R}(\rho_B) = \mathbb{C}^n$ and Theorem RSLT says ρ_B is surjective. ■

We will have many occasions later to employ the inverse of vector representation, so we will record the fact that vector representation is an invertible linear transformation.

Theorem VRILT Vector Representation is an Invertible Linear Transformation
The function ρ_B (Definition VR) is an invertible linear transformation.

Proof. The function ρ_B (Definition VR) is a linear transformation (Theorem VRLT) that is injective (Theorem VRI) and surjective (Theorem VRS) with domain V and codomain $\mathbb{C}^n$. By Theorem ILTIS we then know that ρ_B is an invertible linear transformation. ■

Informally, we will refer to the application of ρ_B as **coordinatizing** a vector, while the application of ρ_B^{-1} will be referred to as **un-coordinatizing** a vector.

Subsection CVS
Characterization of Vector Spaces

Limiting our attention to vector spaces with finite dimension, we now describe every possible vector space. All of them. Really.

Theorem CFDVS Characterization of Finite Dimensional Vector Spaces
Suppose that V is a vector space with dimension n. Then V is isomorphic to $\mathbb{C}^n$.

Proof. Since V has dimension n we can find a basis of V of size n (Definition D) which we will call B. The linear transformation ρ_B is an invertible linear transformation from V to $\mathbb{C}^n$, so by Definition IVS, we have that V and $\mathbb{C}^n$ are isomorphic. ■

Theorem CFDVS is the first of several surprises in this chapter, though it might be a bit demoralizing too. It says that there really are not all that many different (finite dimensional) vector spaces, and none are really any more complicated than $\mathbb{C}^n$. Hmmm. The following examples should make this point.

Example TIVS Two isomorphic vector spaces
The vector space of polynomials with degree 8 or less, P_8, has dimension 9 (Theorem DP). By Theorem CFDVS, P_8 is isomorphic to $\mathbb{C}^9$. △

Example CVSR Crazy vector space revealed
The crazy vector space, C of Example CVS, has dimension 2 by Example DC. By Theorem CFDVS, C is isomorphic to $\mathbb{C}^2$. Hmmmm. Not really so crazy after all? △

Example ASC A subspace characterized
In Example DSP4 we determined that a certain subspace W of P_4 has dimension 4. By Theorem CFDVS, W is isomorphic to $\mathbb{C}^4$. △

Theorem IFDVS Isomorphism of Finite Dimensional Vector Spaces
Suppose U and V are both finite-dimensional vector spaces. Then U and V are isomorphic if and only if $\dim(U) = \dim(V)$.

Proof. ($\Rightarrow$) This is just the statement proved in Theorem IVSED.

($\Leftarrow$) This is the advertised converse of Theorem IVSED. We will assume U and V have equal dimension and discover that they are isomorphic vector spaces. Let n be the common dimension of U and V. Then by Theorem CFDVS there are isomorphisms $T\colon U \to \mathbb{C}^n$ and $S\colon V \to \mathbb{C}^n$.

T is therefore an invertible linear transformation by Definition IVS. Similarly, S is an invertible linear transformation, and so S^{-1} is an invertible linear transformation (Theorem IILT). The composition of invertible linear transformations is again invertible (Theorem CIVLT) so the composition of S^{-1} with T is invertible. Then

$\left(S^{-1} \circ T\right): U \rightarrow V$ is an invertible linear transformation from U to V and Definition IVS says U and V are isomorphic. ■

Example MIVS Multiple isomorphic vector spaces
$\mathbb{C}^{10}$, P_9, $M_{2,5}$ and $M_{5,2}$ are all vector spaces and each has dimension 10. By Theorem IFDVS each is isomorphic to any other.

The subspace of $M_{4,4}$ that contains all the symmetric matrices (Definition SYM) has dimension 10, so this subspace is also isomorphic to each of the four vector spaces above. △

Subsection CP
Coordinatization Principle

With ρ_B available as an invertible linear transformation, we can translate between vectors in a vector space U of dimension m and $\mathbb{C}^m$. Furthermore, as a linear transformation, ρ_B respects the addition and scalar multiplication in U, while ρ_B^{-1} respects the addition and scalar multiplication in $\mathbb{C}^m$. Since our definitions of linear independence, spans, bases and dimension are all built up from linear combinations, we will finally be able to translate fundamental properties between abstract vector spaces (U) and concrete vector spaces ($\mathbb{C}^m$).

Theorem CLI Coordinatization and Linear Independence
Suppose that U is a vector space with a basis B of size n. Then

$$S = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_k\}$$

is a linearly independent subset of U if and only if

$$R = \{\rho_B(\mathbf{u}_1),\, \rho_B(\mathbf{u}_2),\, \rho_B(\mathbf{u}_3),\, \ldots,\, \rho_B(\mathbf{u}_k)\}$$

is a linearly independent subset of $\mathbb{C}^n$.

Proof. The linear transformation ρ_B is an isomorphism between U and $\mathbb{C}^n$ (Theorem VRILT). As an invertible linear transformation, ρ_B is an injective linear transformation (Theorem ILTIS), and ρ_B^{-1} is also an injective linear transformation (Theorem IILT, Theorem ILTIS).

($\Rightarrow$) Since ρ_B is an injective linear transformation and S is linearly independent, Theorem ILTLI says that R is linearly independent.

($\Leftarrow$) If we apply ρ_B^{-1} to each element of R, we will create the set S. Since we are assuming R is linearly independent and ρ_B^{-1} is injective, Theorem ILTLI says that S is linearly independent. ■

Theorem CSS Coordinatization and Spanning Sets
Suppose that U is a vector space with a basis B of size n. Then

$$\mathbf{u} \in \langle\{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_k\}\rangle$$

if and only if

$$\rho_B(\mathbf{u}) \in \langle\{\rho_B(\mathbf{u}_1),\, \rho_B(\mathbf{u}_2),\, \rho_B(\mathbf{u}_3),\, \ldots,\, \rho_B(\mathbf{u}_k)\}\rangle$$

Proof. ($\Rightarrow$) Suppose $\mathbf{u} \in \langle\{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_k\}\rangle$. Then we know there are scalars, $a_1,\, a_2,\, a_3,\, \ldots,\, a_k$, such that

$$\mathbf{u} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \cdots + a_k\mathbf{u}_k$$

Then, by Theorem LTLC,

$$\begin{aligned}\rho_B(\mathbf{u}) &= \rho_B(a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \cdots + a_k\mathbf{u}_k)\\ &= a_1\rho_B(\mathbf{u}_1) + a_2\rho_B(\mathbf{u}_2) + a_3\rho_B(\mathbf{u}_3) + \cdots + a_k\rho_B(\mathbf{u}_k)\end{aligned}$$

which says that $\rho_B(\mathbf{u}) \in \langle\{\rho_B(\mathbf{u}_1),\, \rho_B(\mathbf{u}_2),\, \rho_B(\mathbf{u}_3),\, \ldots,\, \rho_B(\mathbf{u}_k)\}\rangle$.

($\Leftarrow$) Suppose that $\rho_B(\mathbf{u}) \in \langle\{\rho_B(\mathbf{u}_1), \rho_B(\mathbf{u}_2), \rho_B(\mathbf{u}_3), \ldots, \rho_B(\mathbf{u}_k)\}\rangle$. Then there are scalars $b_1, b_2, b_3, \ldots, b_k$ such that

$$\rho_B(\mathbf{u}) = b_1\rho_B(\mathbf{u}_1) + b_2\rho_B(\mathbf{u}_2) + b_3\rho_B(\mathbf{u}_3) + \cdots + b_k\rho_B(\mathbf{u}_k)$$

Recall that ρ_B is invertible (Theorem VRILT), so

$$\begin{aligned}
\mathbf{u} &= I_U(\mathbf{u}) && \text{Definition IDLT}\\
&= \left(\rho_B^{-1} \circ \rho_B\right)(\mathbf{u}) && \text{Definition IVLT}\\
&= \rho_B^{-1}(\rho_B(\mathbf{u})) && \text{Definition LTC}\\
&= \rho_B^{-1}(b_1\rho_B(\mathbf{u}_1) + b_2\rho_B(\mathbf{u}_2) + \cdots + b_k\rho_B(\mathbf{u}_k))\\
&= b_1\rho_B^{-1}(\rho_B(\mathbf{u}_1)) + b_2\rho_B^{-1}(\rho_B(\mathbf{u}_2)) + \cdots + b_k\rho_B^{-1}(\rho_B(\mathbf{u}_k)) && \text{Theorem LTLC}\\
&= b_1I_U(\mathbf{u}_1) + b_2I_U(\mathbf{u}_2) + \cdots + b_kI_U(\mathbf{u}_k) && \text{Definition IVLT}\\
&= b_1\mathbf{u}_1 + b_2\mathbf{u}_2 + b_3\mathbf{u}_3 + \cdots + b_k\mathbf{u}_k && \text{Definition IDLT}
\end{aligned}$$

which says that $\mathbf{u} \in \langle\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_k\}\rangle$. ■

Here is a fairly simple example that illustrates a very, very important idea.

Example CP2 Coordinatizing in P_2
In Example VRP2 we needed to know that

$$D = \left\{-2 - x + 3x^2,\ 1 - 2x^2,\ 5 + 4x + x^2\right\}$$

is a basis for P_2. With Theorem CLI and Theorem CSS this task is much easier.

First, choose a known basis for P_2, a basis that forms vector representations easily. We will choose

$$B = \left\{1,\ x,\ x^2\right\}$$

Now, form the subset of $\mathbb{C}^3$ that is the result of applying ρ_B to each element of D,

$$\begin{aligned}
F &= \left\{\rho_B\left(-2 - x + 3x^2\right),\ \rho_B\left(1 - 2x^2\right),\ \rho_B\left(5 + 4x + x^2\right)\right\}\\
&= \left\{\begin{bmatrix}-2\\-1\\3\end{bmatrix},\ \begin{bmatrix}1\\0\\-2\end{bmatrix},\ \begin{bmatrix}5\\4\\1\end{bmatrix}\right\}
\end{aligned}$$

and ask if F is a linearly independent spanning set for $\mathbb{C}^3$. This is easily seen to be the case by forming a matrix A whose columns are the vectors of F, row-reducing A to the identity matrix I_3, and then using the nonsingularity of A to assert that F is a basis for $\mathbb{C}^3$ (Theorem CNMB). Now, since F is a basis for $\mathbb{C}^3$, Theorem CLI and Theorem CSS tell us that D is also a basis for P_2. △

Example CP2 illustrates the broad notion that computations in abstract vector spaces can be reduced to computations in $\mathbb{C}^m$. You may have noticed this phenomenon as you worked through examples in Chapter VS or Chapter LT employing vector spaces of matrices or polynomials. These computations seemed to invariably result in systems of equations or the like from Chapter SLE, Chapter V and Chapter M. It is vector representation, ρ_B, that allows us to make this connection formal and precise.

Knowing that vector representation allows us to translate questions about linear combinations, linear independence and spans from general vector spaces to $\mathbb{C}^m$ allows us to prove a great many theorems about how to translate other properties. Rather than prove these theorems, each of the same style as the other, we will offer some general guidance about how to best employ Theorem VRLT, Theorem CLI and Theorem CSS. This comes in the form of a "principle": a basic truth, but most definitely not a theorem (hence, no proof).

The Coordinatization Principle

Suppose that U is a vector space with a basis B of size n. Then any question about U, or its elements, which ultimately depends on the vector addition or scalar multiplication in U, or depends on linear independence or spanning, may be translated into the same question in $\mathbb{C}^n$ by application of the linear transformation ρ_B to the relevant vectors. Once the question is answered in $\mathbb{C}^n$, the answer may be translated back to U through application of the inverse linear transformation ρ_B^{-1} (if necessary).

Example CM32 Coordinatization in M_{32}
This is a simple example of the Coordinatization Principle, depending only on the fact that coordinatizing is an invertible linear transformation (Theorem VRILT). Suppose we have a linear combination to perform in M_{32}, the vector space of 3×2 matrices, but we are adverse to doing the operations of M_{32} (Definition MA, Definition MSM). More specifically, suppose we are faced with the computation

$$6\begin{bmatrix}3 & 7\\ -2 & 4\\ 0 & -3\end{bmatrix} + 2\begin{bmatrix}-1 & 3\\ 4 & 8\\ -2 & 5\end{bmatrix}$$

We choose a nice basis for M_{32} (or a nasty basis if we are so inclined),

$$B = \left\{\begin{bmatrix}1 & 0\\ 0 & 0\\ 0 & 0\end{bmatrix},\ \begin{bmatrix}0 & 0\\ 1 & 0\\ 0 & 0\end{bmatrix},\ \begin{bmatrix}0 & 0\\ 0 & 0\\ 1 & 0\end{bmatrix},\ \begin{bmatrix}0 & 1\\ 0 & 0\\ 0 & 0\end{bmatrix},\ \begin{bmatrix}0 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix},\ \begin{bmatrix}0 & 0\\ 0 & 0\\ 0 & 1\end{bmatrix}\right\}$$

and apply ρ_B to each vector in the linear combination. This gives us a new computation, now in the vector space $\mathbb{C}^6$, which we can compute with operations in $\mathbb{C}^6$ (Definition CVA, Definition CVSM),

$$6\begin{bmatrix}3\\ -2\\ 0\\ 7\\ 4\\ -3\end{bmatrix} + 2\begin{bmatrix}-1\\ 4\\ -2\\ 3\\ 8\\ 5\end{bmatrix} = \begin{bmatrix}16\\ -4\\ -4\\ 48\\ 40\\ -8\end{bmatrix}$$

We are after the result of a computation in M_{32}, so we now can apply ρ_B^{-1} to obtain a 3×2 matrix,

$$16\begin{bmatrix}1 & 0\\ 0 & 0\\ 0 & 0\end{bmatrix} + (-4)\begin{bmatrix}0 & 0\\ 1 & 0\\ 0 & 0\end{bmatrix} + (-4)\begin{bmatrix}0 & 0\\ 0 & 0\\ 1 & 0\end{bmatrix} + 48\begin{bmatrix}0 & 1\\ 0 & 0\\ 0 & 0\end{bmatrix} + 40\begin{bmatrix}0 & 0\\ 0 & 1\\ 0 & 0\end{bmatrix} + (-8)\begin{bmatrix}0 & 0\\ 0 & 0\\ 0 & 1\end{bmatrix}$$
$$= \begin{bmatrix}16 & 48\\ -4 & 40\\ -4 & -8\end{bmatrix}$$

which is exactly the matrix we would have computed had we just performed the matrix operations in the first place. So this was not meant to be an *easier* way to compute a linear combination of two matrices, just a *different* way. △

Reading Questions

1. The vector space of 3×5 matrices, $M_{3,5}$ is isomorphic to what fundamental vector space?

2. A basis for $\mathbb{C}^3$ is

$$B = \left\{\begin{bmatrix}1\\ 2\\ -1\end{bmatrix},\ \begin{bmatrix}3\\ -1\\ 2\end{bmatrix},\ \begin{bmatrix}1\\ 1\\ 1\end{bmatrix}\right\}$$

Compute $\rho_B\left(\begin{bmatrix}5\\ 8\\ -1\end{bmatrix}\right)$.

3. What is the first "surprise," and why is it surprising?

Exercises

C10† In the vector space $\mathbb{C}^3$, compute the vector representation $\rho_B(\mathbf{v})$ for the basis B and vector $\mathbf{v}$ below.

$$B = \left\{ \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix} \right\} \qquad \mathbf{v} = \begin{bmatrix} 11 \\ 5 \\ 8 \end{bmatrix}$$

C20† Rework Example CM32 replacing the basis B by the basis

$$C = \left\{ \begin{bmatrix} -14 & -9 \\ 10 & 10 \\ -6 & -2 \end{bmatrix}, \begin{bmatrix} -7 & -4 \\ 5 & 5 \\ -3 & -1 \end{bmatrix}, \begin{bmatrix} -3 & -1 \\ 0 & -2 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -7 & -4 \\ 3 & 2 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 4 & 2 \\ -3 & -3 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -1 & -2 \\ 1 & 1 \end{bmatrix} \right\}$$

M10 Prove that the set S below is a basis for the vector space of 2×2 matrices, M_{22}. Do this by choosing a natural basis for M_{22} and coordinatizing the elements of S with respect to this basis. Examine the resulting set of column vectors from $\mathbb{C}^4$ and apply the Coordinatization Principle.

$$S = \left\{ \begin{bmatrix} 33 & 99 \\ 78 & -9 \end{bmatrix}, \begin{bmatrix} -16 & -47 \\ -36 & 2 \end{bmatrix}, \begin{bmatrix} 10 & 27 \\ 17 & 3 \end{bmatrix}, \begin{bmatrix} -2 & -7 \\ -6 & 4 \end{bmatrix} \right\}$$

M20† The set $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a basis of the vector space P_3, polynomials with degree 3 or less. Therefore ρ_B is a linear transformation, according to Theorem VRLT. Find a "formula" for ρ_B. In other words, find an expression for $\rho_B\left(a + bx + cx^2 + dx^3\right)$.

$$\mathbf{v}_1 = 1 - 5x - 22x^2 + 3x^3 \qquad \mathbf{v}_2 = -2 + 11x + 49x^2 - 8x^3$$
$$\mathbf{v}_3 = -1 + 7x + 33x^2 - 8x^3 \qquad \mathbf{v}_4 = -1 + 4x + 16x^2 + x^3$$

Section MR
Matrix Representations

We have seen that linear transformations whose domain and codomain are vector spaces of columns vectors have a close relationship with matrices (Theorem MBLT, Theorem MLTCV). In this section, we will extend the relationship between matrices and linear transformations to the setting of linear transformations between abstract vector spaces.

Subsection MR
Matrix Representations

This is a fundamental definition.

Definition MR Matrix Representation
Suppose that $T\colon U \to V$ is a linear transformation, $B = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_n\}$ is a basis for U of size n, and C is a basis for V of size m. Then the **matrix representation** of T relative to B and C is the $m \times n$ matrix,

$$M^T_{B,C} = \left[\rho_C\left(T\left(\mathbf{u}_1\right)\right)\middle|\, \rho_C\left(T\left(\mathbf{u}_2\right)\right)\middle|\, \rho_C\left(T\left(\mathbf{u}_3\right)\right)\middle|\ldots\middle|\rho_C\left(T\left(\mathbf{u}_n\right)\right)\right]$$

□

Example OLTTR One linear transformation, three representations
Consider the linear transformation, $S\colon P_3 \to M_{22}$, given by

$$S\left(a+bx+cx^2+dx^3\right) = \begin{bmatrix} 3a+7b-2c-5d & 8a+14b-2c-11d \\ -4a-8b+2c+6d & 12a+22b-4c-17d \end{bmatrix}$$

First, we build a representation relative to the bases,

$$B = \left\{1+2x+x^2-x^3,\, 1+3x+x^2+x^3,\, -1-2x+2x^3,\, 2+3x+2x^2-5x^3\right\}$$
$$C = \left\{\begin{bmatrix}1&1\\1&2\end{bmatrix},\, \begin{bmatrix}2&3\\2&5\end{bmatrix},\, \begin{bmatrix}-1&-1\\0&-2\end{bmatrix},\, \begin{bmatrix}-1&-4\\-2&-4\end{bmatrix}\right\}$$

We evaluate S with each element of the basis for the domain, B, and coordinatize the result relative to the vectors in the basis for the codomain, C. Notice here how we take elements of vector spaces and decompose them into linear combinations of basis elements as the key step in constructing coordinatizations of vectors. There is a system of equations involved almost every time, but we will omit these details since this should be a routine exercise at this stage.

$$\rho_C\left(S\left(1+2x+x^2-x^3\right)\right) = \rho_C\left(\begin{bmatrix}20&45\\-24&69\end{bmatrix}\right)$$
$$= \rho_C\left((-90)\begin{bmatrix}1&1\\1&2\end{bmatrix} + 37\begin{bmatrix}2&3\\2&5\end{bmatrix} + (-40)\begin{bmatrix}-1&-1\\0&-2\end{bmatrix} + 4\begin{bmatrix}-1&-4\\-2&-4\end{bmatrix}\right) = \begin{bmatrix}-90\\37\\-40\\4\end{bmatrix}$$
$$\rho_C\left(S\left(1+3x+x^2+x^3\right)\right) = \rho_C\left(\begin{bmatrix}17&37\\-20&57\end{bmatrix}\right)$$
$$= \rho_C\left((-72)\begin{bmatrix}1&1\\1&2\end{bmatrix} + 29\begin{bmatrix}2&3\\2&5\end{bmatrix} + (-34)\begin{bmatrix}-1&-1\\0&-2\end{bmatrix} + 3\begin{bmatrix}-1&-4\\-2&-4\end{bmatrix}\right) = \begin{bmatrix}-72\\29\\-34\\3\end{bmatrix}$$
$$\rho_C\left(S\left(-1-2x+2x^3\right)\right) = \rho_C\left(\begin{bmatrix}-27&-58\\32&-90\end{bmatrix}\right)$$

$$= \rho_C\left(114\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix} + (-46)\begin{bmatrix}2 & 3\\2 & 5\end{bmatrix} + 54\begin{bmatrix}-1 & -1\\0 & -2\end{bmatrix} + (-5)\begin{bmatrix}-1 & -4\\-2 & -4\end{bmatrix}\right) = \begin{bmatrix}114\\-46\\54\\-5\end{bmatrix}$$

$$\rho_C\left(S\left(2+3x+2x^2-5x^3\right)\right) = \rho_C\left(\begin{bmatrix}48 & 109\\-58 & 167\end{bmatrix}\right)$$

$$= \rho_C\left(-220\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix} + 91\begin{bmatrix}2 & 3\\2 & 5\end{bmatrix} + (-96)\begin{bmatrix}-1 & -1\\0 & -2\end{bmatrix} + 10\begin{bmatrix}-1 & -4\\-2 & -4\end{bmatrix}\right) = \begin{bmatrix}-220\\91\\-96\\10\end{bmatrix}$$

Thus, employing Definition MR

$$M^S_{B,C} = \begin{bmatrix}-90 & -72 & 114 & -220\\37 & 29 & -46 & 91\\-40 & -34 & 54 & -96\\4 & 3 & -5 & 10\end{bmatrix}$$

Often we use "nice" bases to build matrix representations and the work involved is much easier. Suppose we take bases

$$D = \left\{1,\, x,\, x^2,\, x^3\right\} \qquad E = \left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}$$

The evaluation of S at the elements of D is easy and coordinatization relative to E can be done on sight,

$$\rho_E\left(S\left(1\right)\right) = \rho_E\left(\begin{bmatrix}3 & 8\\-4 & 12\end{bmatrix}\right)$$

$$= \rho_E\left(3\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + 8\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} + (-4)\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} + 12\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right) = \begin{bmatrix}3\\8\\-4\\12\end{bmatrix}$$

$$\rho_E\left(S\left(x\right)\right) = \rho_E\left(\begin{bmatrix}7 & 14\\-8 & 22\end{bmatrix}\right)$$

$$= \rho_E\left(7\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + 14\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} + (-8)\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} + 22\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right) = \begin{bmatrix}7\\14\\-8\\22\end{bmatrix}$$

$$\rho_E\left(S\left(x^2\right)\right) = \rho_E\left(\begin{bmatrix}-2 & -2\\2 & -4\end{bmatrix}\right)$$

$$= \rho_E\left((-2)\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + (-2)\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} + 2\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} + (-4)\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right) = \begin{bmatrix}-2\\-2\\2\\-4\end{bmatrix}$$

$$\rho_E\left(S\left(x^3\right)\right) = \rho_E\left(\begin{bmatrix}-5 & -11\\6 & -17\end{bmatrix}\right)$$

$$= \rho_E\left((-5)\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + (-11)\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} + 6\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} + (-17)\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right)$$

$$= \begin{bmatrix}-5\\-11\\6\\-17\end{bmatrix}$$

So the matrix representation of S relative to D and E is

$$M^S_{D,E} = \begin{bmatrix} 3 & 7 & -2 & -5 \\ 8 & 14 & -2 & -11 \\ -4 & -8 & 2 & 6 \\ 12 & 22 & -4 & -17 \end{bmatrix}$$

One more time, but now let us use bases

$$F = \left\{1 + x - x^2 + 2x^3,\ -1 + 2x + 2x^3,\ 2 + x - 2x^2 + 3x^3,\ 1 + x + 2x^3\right\}$$
$$G = \left\{\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ -2 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\right\}$$

and evaluate S with the elements of F, then coordinatize the results relative to G,

$$\rho_G\left(S\left(1 + x - x^2 + 2x^3\right)\right) = \rho_G\left(\begin{bmatrix} 2 & 2 \\ -2 & 4 \end{bmatrix}\right) = \rho_G\left(2\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rho_G\left(S\left(-1 + 2x + 2x^3\right)\right) = \rho_G\left(\begin{bmatrix} 1 & -2 \\ 0 & -2 \end{bmatrix}\right) = \rho_G\left((-1)\begin{bmatrix} -1 & 2 \\ 0 & 2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

$$\rho_G\left(S\left(2 + x - 2x^2 + 3x^3\right)\right) = \rho_G\left(\begin{bmatrix} 2 & 1 \\ -2 & 3 \end{bmatrix}\right) = \rho_G\left(\begin{bmatrix} 2 & 1 \\ -2 & 3 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\rho_G\left(S\left(1 + x + 2x^3\right)\right) = \rho_G\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = \rho_G\left(0\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

So we arrive at an especially economical matrix representation,

$$M^S_{F,G} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

△

We may choose to use whatever terms we want when we make a definition. Some are arbitrary, while others make sense, but only in light of subsequent theorems. Matrix representation is in the latter category. We begin with a linear transformation and produce a matrix. So what? Here is the theorem that *justifies* the term "matrix representation."

Theorem FTMR Fundamental Theorem of Matrix Representation
Suppose that $T\colon U \to V$ is a linear transformation, B is a basis for U, C is a basis for V and $M^T_{B,C}$ is the matrix representation of T relative to B and C. Then, for any $\mathbf{u} \in U$,

$$\rho_C\left(T\left(\mathbf{u}\right)\right) = M^T_{B,C}\left(\rho_B\left(\mathbf{u}\right)\right)$$

or equivalently

$$T\left(\mathbf{u}\right) = \rho_C^{-1}\left(M^T_{B,C}\left(\rho_B\left(\mathbf{u}\right)\right)\right)$$

Proof. Let $B = \{\mathbf{u}_1,\, \mathbf{u}_2,\, \mathbf{u}_3,\, \ldots,\, \mathbf{u}_n\}$ be the basis of U. Since $\mathbf{u} \in U$, there are scalars $a_1,\, a_2,\, a_3,\, \ldots,\, a_n$ such that

$$\mathbf{u} = a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \cdots + a_n\mathbf{u}_n$$

Then,

$$
\begin{aligned}
&M^T_{B,C}\rho_B(\mathbf{u}) \\
&= [\rho_C(T(\mathbf{u}_1))|\,\rho_C(T(\mathbf{u}_2))|\,\rho_C(T(\mathbf{u}_3))|\dots|\rho_C(T(\mathbf{u}_n))]\,\rho_B(\mathbf{u}) && \text{Definition MR} \\
&= [\rho_C(T(\mathbf{u}_1))|\,\rho_C(T(\mathbf{u}_2))|\,\rho_C(T(\mathbf{u}_3))|\dots|\rho_C(T(\mathbf{u}_n))] \begin{bmatrix} a_1\\ a_2\\ a_3\\ \vdots\\ a_n \end{bmatrix} && \text{Definition VR} \\
&= a_1\rho_C(T(\mathbf{u}_1)) + a_2\rho_C(T(\mathbf{u}_2)) + \cdots + a_n\rho_C(T(\mathbf{u}_n)) && \text{Definition MVP} \\
&= \rho_C(a_1T(\mathbf{u}_1) + a_2T(\mathbf{u}_2) + a_3T(\mathbf{u}_3) + \cdots + a_nT(\mathbf{u}_n)) && \text{Theorem LTLC} \\
&= \rho_C(T(a_1\mathbf{u}_1 + a_2\mathbf{u}_2 + a_3\mathbf{u}_3 + \cdots + a_n\mathbf{u}_n)) && \text{Theorem LTLC} \\
&= \rho_C(T(\mathbf{u}))
\end{aligned}
$$

The alternative conclusion is obtained as

$$
\begin{aligned}
T(\mathbf{u}) &= I_V(T(\mathbf{u})) && \text{Definition IDLT} \\
&= \left(\rho_C^{-1}\circ\rho_C\right)(T(\mathbf{u})) && \text{Definition IVLT} \\
&= \rho_C^{-1}(\rho_C(T(\mathbf{u}))) && \text{Definition LTC} \\
&= \rho_C^{-1}\left(M^T_{B,C}(\rho_B(\mathbf{u}))\right)
\end{aligned}
$$

■

This theorem says that we can apply T to $\mathbf{u}$ and coordinatize the result relative to C in V, or we can first coordinatize $\mathbf{u}$ relative to B in U, then multiply by the matrix representation. Either way, the result is the same. So the effect of a linear transformation can always be accomplished by a matrix-vector product (Definition MVP). That is important enough to say again. The effect of a linear transformation is a matrix-vector product.

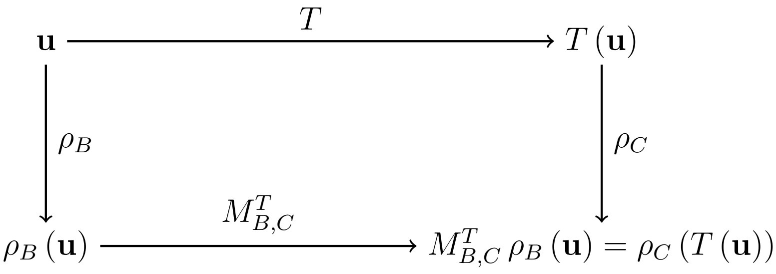

Diagram FTMR: Fundamental Theorem of Matrix Representations

The alternative conclusion of this result might be even more striking. It says that to effect a linear transformation (T) of a vector ($\mathbf{u}$), coordinatize the input (with ρ_B), do a matrix-vector product (with $M^T_{B,C}$), and un-coordinatize the result (with ρ_C^{-1}). So, absent some bookkeeping about vector representations, a linear transformation *is* a matrix. To adjust the diagram, we "reverse" the arrow on the right, which means inverting the vector representation ρ_C on V. Now we can go directly across the top of the diagram, computing the linear transformation between the abstract vector spaces. Or, we can around the other three sides, using vector representation, a matrix-vector product, followed by un-coordinatization.

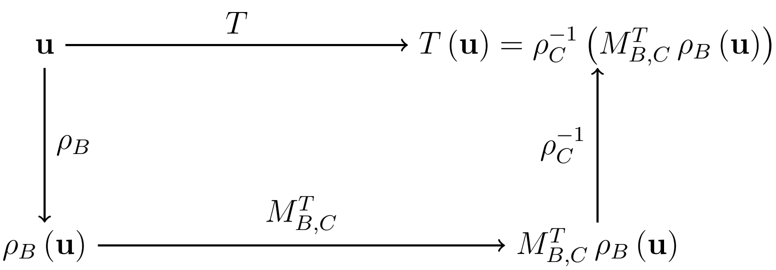

Diagram FTMRA: Fundamental Theorem of Matrix Representations (Alternate)

Here is an example to illustrate how the "action" of a linear transformation can be effected by matrix multiplication.

Example ALTMM A linear transformation as matrix multiplication
In Example OLTTR we found three representations of the linear transformation S. In this example, we will compute a single output of S in four different ways. First "normally," then three times over using Theorem FTMR.

Choose $p(x) = 3-x+2x^2-5x^3$, for no particular reason. Then the straightforward application of S to $p(x)$ yields

$$\begin{aligned}
S(p(x)) &= S\left(3-x+2x^2-5x^3\right)\\
&= \begin{bmatrix} 3(3)+7(-1)-2(2)-5(-5) & 8(3)+14(-1)-2(2)-11(-5)\\ -4(3)-8(-1)+2(2)+6(-5) & 12(3)+22(-1)-4(2)-17(-5)\end{bmatrix}\\
&= \begin{bmatrix} 23 & 61\\ -30 & 91\end{bmatrix}
\end{aligned}$$

Now use the representation of S relative to the bases B and C and Theorem FTMR. Note that we will employ the following linear combination in moving from the second line to the third,

$$\begin{aligned}
3-x+2x^2-5x^3 = 48(1+2x+x^2-x^3)+(-20)(1+3x+x^2+x^3)+\\
(-1)(-1-2x+2x^3)+(-13)(2+3x+2x^2-5x^3)
\end{aligned}$$

$$\begin{aligned}
S(p(x)) &= \rho_C^{-1}\left(M_{B,C}^{S}\rho_B(p(x))\right)\\
&= \rho_C^{-1}\left(M_{B,C}^{S}\rho_B\left(3-x+2x^2-5x^3\right)\right)\\
&= \rho_C^{-1}\left(M_{B,C}^{S}\begin{bmatrix}48\\-20\\-1\\-13\end{bmatrix}\right)\\
&= \rho_C^{-1}\left(\begin{bmatrix}-90 & -72 & 114 & -220\\ 37 & 29 & -46 & 91\\ -40 & -34 & 54 & -96\\ 4 & 3 & -5 & 10\end{bmatrix}\begin{bmatrix}48\\-20\\-1\\-13\end{bmatrix}\right)\\
&= \rho_C^{-1}\left(\begin{bmatrix}-134\\59\\-46\\7\end{bmatrix}\right)\\
&= (-134)\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}+59\begin{bmatrix}2 & 3\\2 & 5\end{bmatrix}+(-46)\begin{bmatrix}-1 & -1\\0 & -2\end{bmatrix}+7\begin{bmatrix}-1 & -4\\-2 & -4\end{bmatrix}\\
&= \begin{bmatrix}23 & 61\\-30 & 91\end{bmatrix}
\end{aligned}$$

Again, but now with "nice" bases like D and E, and the computations are more transparent.

$$\begin{aligned}
S(p(x)) &= \rho_E^{-1}\left(M_{D,E}^S \rho_D(p(x))\right)\\
&= \rho_E^{-1}\left(M_{D,E}^S \rho_D\left(3 - x + 2x^2 - 5x^3\right)\right)\\
&= \rho_E^{-1}\left(M_{D,E}^S \rho_D\left(3(1) + (-1)(x) + 2(x^2) + (-5)(x^3)\right)\right)\\
&= \rho_E^{-1}\left(M_{D,E}^S \begin{bmatrix}3\\-1\\2\\-5\end{bmatrix}\right)\\
&= \rho_E^{-1}\left(\begin{bmatrix}3 & 7 & -2 & -5\\ 8 & 14 & -2 & -11\\ -4 & -8 & 2 & 6\\ 12 & 22 & -4 & -17\end{bmatrix}\begin{bmatrix}3\\-1\\2\\-5\end{bmatrix}\right)\\
&= \rho_E^{-1}\left(\begin{bmatrix}23\\61\\-30\\91\end{bmatrix}\right)\\
&= 23\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + 61\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} + (-30)\begin{bmatrix}0 & 0\\1 & 0\end{bmatrix} + 91\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\\
&= \begin{bmatrix}23 & 61\\-30 & 91\end{bmatrix}
\end{aligned}$$

OK, last time, now with the bases F and G. The coordinatizations will take some work this time, but the matrix-vector product (Definition MVP) (which is the actual action of the linear transformation) will be especially easy, given the diagonal nature of the matrix representation, $M_{F,G}^S$. Here we go,

$$\begin{aligned}
S(p(x)) &= \rho_G^{-1}\left(M_{F,G}^S \rho_F(p(x))\right)\\
&= \rho_G^{-1}\left(M_{F,G}^S \rho_F\left(3 - x + 2x^2 - 5x^3\right)\right)\\
&= \rho_G^{-1}\left(M_{F,G}^S \rho_F\left(32(1 + x - x^2 + 2x^3) - 7(-1 + 2x + 2x^3)\right.\right.\\
&\qquad\qquad\left.\left.-17(2 + x - 2x^2 + 3x^3) - 2(1 + x + 2x^3)\right)\right)\\
&= \rho_G^{-1}\left(M_{F,G}^S \begin{bmatrix}32\\-7\\-17\\-2\end{bmatrix}\right)\\
&= \rho_G^{-1}\left(\begin{bmatrix}2 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}32\\-7\\-17\\-2\end{bmatrix}\right)\\
&= \rho_G^{-1}\left(\begin{bmatrix}64\\7\\-17\\0\end{bmatrix}\right)\\
&= 64\begin{bmatrix}1 & 1\\-1 & 2\end{bmatrix} + 7\begin{bmatrix}-1 & 2\\0 & 2\end{bmatrix} + (-17)\begin{bmatrix}2 & 1\\-2 & 3\end{bmatrix} + 0\begin{bmatrix}1 & 1\\0 & 2\end{bmatrix}\\
&= \begin{bmatrix}23 & 61\\-30 & 91\end{bmatrix}
\end{aligned}$$

This example is not meant to necessarily illustrate that any one of these four computations is simpler than the others. Instead, it is meant to illustrate the many different ways we can arrive at the same result, with the last three all employing a matrix representation to effect the linear transformation. △

We will use Theorem FTMR frequently in the next few sections. A typical application will feel like the linear transformation T "commutes" with a vector representation, ρ_C, and as it does the transformation morphs into a matrix, $M^T_{B,C}$, while the vector representation changes to a new basis, ρ_B. Or vice-versa.

Subsection NRFO New Representations from Old

In Subsection LT.NLTFO we built new linear transformations from other linear transformations. Sums, scalar multiples and compositions. These new linear transformations will have matrix representations as well. How do the new matrix representations relate to the old matrix representations? Here are the three theorems.

Theorem MRSLT Matrix Representation of a Sum of Linear Transformations
Suppose that $T\colon U\to V$ and $S\colon U\to V$ are linear transformations, B is a basis of U and C is a basis of V. Then

$$M^{T+S}_{B,C} = M^T_{B,C} + M^S_{B,C}$$

Proof. Let $\mathbf{x}$ be any vector in $\mathbb{C}^n$. Define $\mathbf{u}\in U$ by $\mathbf{u} = \rho_B^{-1}(\mathbf{x})$, so $\mathbf{x} = \rho_B(\mathbf{u})$. Then,

$$\begin{aligned}
M^{T+S}_{B,C}\mathbf{x} &= M^{T+S}_{B,C}\rho_B(\mathbf{u}) && \text{Substitution}\\
&= \rho_C((T+S)(\mathbf{u})) && \text{Theorem FTMR}\\
&= \rho_C(T(\mathbf{u}) + S(\mathbf{u})) && \text{Definition LTA}\\
&= \rho_C(T(\mathbf{u})) + \rho_C(S(\mathbf{u})) && \text{Definition LT}\\
&= M^T_{B,C}(\rho_B(\mathbf{u})) + M^S_{B,C}(\rho_B(\mathbf{u})) && \text{Theorem FTMR}\\
&= \left(M^T_{B,C} + M^S_{B,C}\right)\rho_B(\mathbf{u}) && \text{Theorem MMDAA}\\
&= \left(M^T_{B,C} + M^S_{B,C}\right)\mathbf{x} && \text{Substitution}
\end{aligned}$$

Since the matrices $M^{T+S}_{B,C}$ and $M^T_{B,C} + M^S_{B,C}$ have equal matrix-vector products for *every* vector in $\mathbb{C}^n$, by Theorem EMMVP they are equal matrices. (Now would be a good time to double-back and study the proof of Theorem EMMVP. You did promise to come back to this theorem sometime, didn't you?) ■

Theorem MRMLT Matrix Representation of a Multiple of a Linear Transformation
Suppose that $T\colon U\to V$ is a linear transformation, $\alpha\in\mathbb{C}$, B is a basis of U and C is a basis of V. Then

$$M^{\alpha T}_{B,C} = \alpha M^T_{B,C}$$

Proof. Let $\mathbf{x}$ be any vector in $\mathbb{C}^n$. Define $\mathbf{u}\in U$ by $\mathbf{u} = \rho_B^{-1}(\mathbf{x})$, so $\mathbf{x} = \rho_B(\mathbf{u})$. Then,

$$\begin{aligned}
M^{\alpha T}_{B,C}\mathbf{x} &= M^{\alpha T}_{B,C}\rho_B(\mathbf{u}) && \text{Substitution}\\
&= \rho_C((\alpha T)(\mathbf{u})) && \text{Theorem FTMR}\\
&= \rho_C(\alpha T(\mathbf{u})) && \text{Definition LTSM}\\
&= \alpha\rho_C(T(\mathbf{u})) && \text{Definition LT}\\
&= \alpha\left(M^T_{B,C}\rho_B(\mathbf{u})\right) && \text{Theorem FTMR}\\
&= \left(\alpha M^T_{B,C}\right)\rho_B(\mathbf{u}) && \text{Theorem MMSMM}\\
&= \left(\alpha M^T_{B,C}\right)\mathbf{x} && \text{Substitution}
\end{aligned}$$

Since the matrices $M^{\alpha T}_{B,C}$ and $\alpha M^T_{B,C}$ have equal matrix-vector products for *every* vector in $\mathbb{C}^n$, by Theorem EMMVP they are equal matrices. ■

The vector space of all linear transformations from U to V is now isomorphic to the vector space of all $m \times n$ matrices.

Theorem MRCLT Matrix Representation of a Composition of Linear Transformations
Suppose that $T\colon U \to V$ and $S\colon V \to W$ are linear transformations, B is a basis of U, C is a basis of V, and D is a basis of W. Then

$$M^{S\circ T}_{B,D} = M^{S}_{C,D} M^{T}_{B,C}$$

Proof. Let $\mathbf{x}$ be any vector in $\mathbb{C}^n$. Define $\mathbf{u} \in U$ by $\mathbf{u} = \rho_B^{-1}(\mathbf{x})$, so $\mathbf{x} = \rho_B(\mathbf{u})$. Then,

$$\begin{aligned}
M^{S\circ T}_{B,D}\mathbf{x} &= M^{S\circ T}_{B,D}\rho_B(\mathbf{u}) && \text{Substitution}\\
&= \rho_D((S\circ T)(\mathbf{u})) && \text{Theorem FTMR}\\
&= \rho_D(S(T(\mathbf{u}))) && \text{Definition LTC}\\
&= M^{S}_{C,D}\rho_C(T(\mathbf{u})) && \text{Theorem FTMR}\\
&= M^{S}_{C,D}\left(M^{T}_{B,C}\rho_B(\mathbf{u})\right) && \text{Theorem FTMR}\\
&= \left(M^{S}_{C,D}M^{T}_{B,C}\right)\rho_B(\mathbf{u}) && \text{Theorem MMA}\\
&= \left(M^{S}_{C,D}M^{T}_{B,C}\right)\mathbf{x} && \text{Substitution}
\end{aligned}$$

Since the matrices $M^{S\circ T}_{B,D}$ and $M^{S}_{C,D}M^{T}_{B,C}$ have equal matrix-vector products for *every* vector in $\mathbb{C}^n$, by Theorem EMMVP they are equal matrices. ■

This is the second great surprise of introductory linear algebra. Matrices are linear transformations (functions, really), and matrix multiplication is function composition! We can form the composition of two linear transformations, then form the matrix representation of the result. Or we can form the matrix representation of each linear transformation separately, then *multiply* the two representations together via Definition MM. In either case, we arrive at the same result.

Example MPMR Matrix product of matrix representations
Consider the two linear transformations,

$$T\colon \mathbb{C}^2 \to P_2 \quad T\left(\begin{bmatrix} a\\ b\end{bmatrix}\right) = (-a+3b) + (2a+4b)x + (a-2b)x^2$$

$$S\colon P_2 \to M_{22} \quad S\left(a+bx+cx^2\right) = \begin{bmatrix} 2a+b+2c & a+4b-c\\ -a+3c & 3a+b+2c\end{bmatrix}$$

and bases for $\mathbb{C}^2$, P_2 and M_{22} (respectively),

$$B = \left\{\begin{bmatrix}3\\1\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}\right\}$$

$$C = \left\{1-2x+x^2,\ -1+3x,\ 2x+3x^2\right\}$$

$$D = \left\{\begin{bmatrix}1 & -2\\ 1 & -1\end{bmatrix}, \begin{bmatrix}1 & -1\\ 1 & 2\end{bmatrix}, \begin{bmatrix}-1 & 2\\ 0 & 0\end{bmatrix}, \begin{bmatrix}2 & -3\\ 2 & 2\end{bmatrix}\right\}$$

Begin by computing the new linear transformation that is the composition of T and S (Definition LTC, Theorem CLTLT), $(S\circ T)\colon \mathbb{C}^2 \to M_{22}$,

$$\begin{aligned}
(S\circ T)\left(\begin{bmatrix}a\\b\end{bmatrix}\right) &= S\left(T\left(\begin{bmatrix}a\\b\end{bmatrix}\right)\right)\\
&= S\left((-a+3b)+(2a+4b)x+(a-2b)x^2\right)\\
&= \begin{bmatrix} 2(-a+3b)+(2a+4b)+2(a-2b) & (-a+3b)+4(2a+4b)-(a-2b)\\ -(-a+3b)+3(a-2b) & 3(-a+3b)+(2a+4b)+2(a-2b)\end{bmatrix}\\
&= \begin{bmatrix} 2a+6b & 6a+21b\\ 4a-9b & a+9b\end{bmatrix}
\end{aligned}$$

Now compute the matrix representations (Definition MR) for each of these three linear transformations (T, S, $S \circ T$), relative to the appropriate bases. First for T,

$$\begin{aligned}
\rho_C\left(T\left(\begin{bmatrix}3\\1\end{bmatrix}\right)\right) &= \rho_C\left(10x + x^2\right)\\
&= \rho_C\left(28(1-2x+x^2) + 28(-1+3x) + (-9)(2x+3x^2)\right)\\
&= \begin{bmatrix}28\\28\\-9\end{bmatrix}\\
\rho_C\left(T\left(\begin{bmatrix}2\\1\end{bmatrix}\right)\right) &= \rho_C\left(1 + 8x\right)\\
&= \rho_C\left(33(1-2x+x^2) + 32(-1+3x) + (-11)(2x+3x^2)\right)\\
&= \begin{bmatrix}33\\32\\-11\end{bmatrix}
\end{aligned}$$

So we have the matrix representation of T,

$$M^T_{B,C} = \begin{bmatrix}28 & 33\\ 28 & 32\\ -9 & -11\end{bmatrix}$$

Now, a representation of S,

$$\begin{aligned}
&\rho_D\left(S\left(1-2x+x^2\right)\right) = \rho_D\left(\begin{bmatrix}2 & -8\\ 2 & 3\end{bmatrix}\right)\\
&\quad= \rho_D\left((-11)\begin{bmatrix}1 & -2\\ 1 & -1\end{bmatrix} + (-21)\begin{bmatrix}1 & -1\\ 1 & 2\end{bmatrix} + 0\begin{bmatrix}-1 & 2\\ 0 & 0\end{bmatrix} + (17)\begin{bmatrix}2 & -3\\ 2 & 2\end{bmatrix}\right)\\
&\quad= \begin{bmatrix}-11\\-21\\0\\17\end{bmatrix}\\
&\rho_D\left(S\left(-1+3x\right)\right) = \rho_D\left(\begin{bmatrix}1 & 11\\ 1 & 0\end{bmatrix}\right)\\
&\quad= \rho_D\left(26\begin{bmatrix}1 & -2\\ 1 & -1\end{bmatrix} + 51\begin{bmatrix}1 & -1\\ 1 & 2\end{bmatrix} + 0\begin{bmatrix}-1 & 2\\ 0 & 0\end{bmatrix} + (-38)\begin{bmatrix}2 & -3\\ 2 & 2\end{bmatrix}\right)\\
&\quad= \begin{bmatrix}26\\51\\0\\-38\end{bmatrix}\\
&\rho_D\left(S\left(2x+3x^2\right)\right) = \rho_D\left(\begin{bmatrix}8 & 5\\ 9 & 8\end{bmatrix}\right)\\
&\quad= \rho_D\left(34\begin{bmatrix}1 & -2\\ 1 & -1\end{bmatrix} + 67\begin{bmatrix}1 & -1\\ 1 & 2\end{bmatrix} + 1\begin{bmatrix}-1 & 2\\ 0 & 0\end{bmatrix} + (-46)\begin{bmatrix}2 & -3\\ 2 & 2\end{bmatrix}\right)\\
&\quad= \begin{bmatrix}34\\67\\1\\-46\end{bmatrix}
\end{aligned}$$

So we have the matrix representation of S,

$$M^S_{C,D} = \begin{bmatrix} -11 & 26 & 34 \\ -21 & 51 & 67 \\ 0 & 0 & 1 \\ 17 & -38 & -46 \end{bmatrix}$$

Finally, a representation of $S \circ T$,

$$\begin{aligned}
\rho_D\left((S \circ T)\left(\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right)\right) &= \rho_D\left(\begin{bmatrix} 12 & 39 \\ 3 & 12 \end{bmatrix}\right) \\
&= \rho_D\left(114\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} + 237\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} + (-9)\begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} + (-174)\begin{bmatrix} 2 & -3 \\ 2 & 2 \end{bmatrix}\right) \\
&= \begin{bmatrix} 114 \\ 237 \\ -9 \\ -174 \end{bmatrix} \\
\rho_D\left((S \circ T)\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)\right) &= \rho_D\left(\begin{bmatrix} 10 & 33 \\ -1 & 11 \end{bmatrix}\right) \\
&= \rho_D\left(95\begin{bmatrix} 1 & -2 \\ 1 & -1 \end{bmatrix} + 202\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix} + (-11)\begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix} + (-149)\begin{bmatrix} 2 & -3 \\ 2 & 2 \end{bmatrix}\right) \\
&= \begin{bmatrix} 95 \\ 202 \\ -11 \\ -149 \end{bmatrix}
\end{aligned}$$

So we have the matrix representation of $S \circ T$,

$$M^{S\circ T}_{B,D} = \begin{bmatrix} 114 & 95 \\ 237 & 202 \\ -9 & -11 \\ -174 & -149 \end{bmatrix}$$

Now, we are all set to verify the conclusion of Theorem MRCLT,

$$\begin{aligned}
M^S_{C,D}M^T_{B,C} &= \begin{bmatrix} -11 & 26 & 34 \\ -21 & 51 & 67 \\ 0 & 0 & 1 \\ 17 & -38 & -46 \end{bmatrix}\begin{bmatrix} 28 & 33 \\ 28 & 32 \\ -9 & -11 \end{bmatrix} \\
&= \begin{bmatrix} 114 & 95 \\ 237 & 202 \\ -9 & -11 \\ -174 & -149 \end{bmatrix} \\
&= M^{S\circ T}_{B,D}
\end{aligned}$$

We have intentionally used nonstandard bases. If you were to choose "nice" bases for the three vector spaces, then the result of the theorem might be rather transparent. But this would still be a worthwhile exercise — give it a go. △

A diagram, similar to ones we have seen earlier, might make the importance of this theorem clearer,

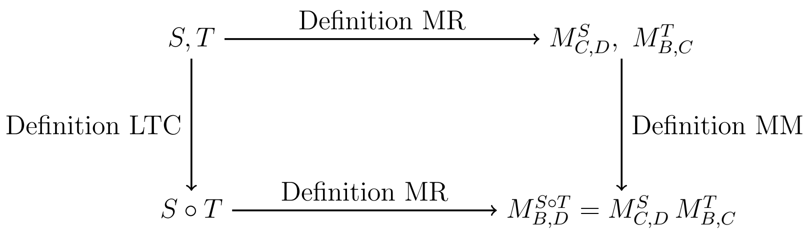

Diagram MRCLT: Matrix Representation and Composition of Linear Transformations

One of our goals in the first part of this book is to make the definition of matrix multiplication (Definition MVP, Definition MM) seem as natural as possible. However, many of us are brought up with an entry-by-entry description of matrix multiplication (Theorem EMP) as the *definition* of matrix multiplication, and then theorems about columns of matrices and linear combinations follow from that definition. With this unmotivated definition, the realization that matrix multiplication is function composition is quite remarkable. It is an interesting exercise to begin with the question, “What is the matrix representation of the composition of two linear transformations?” and then, without using any theorems about matrix multiplication, finally arrive at the entry-by-entry description of matrix multiplication. Try it yourself (Exercise MR.T80).

Subsection PMR Properties of Matrix Representations

It will not be a surprise to discover that the kernel and range of a linear transformation are closely related to the null space and column space of the transformation's matrix representation. Perhaps this idea has been bouncing around in your head already, even before seeing the definition of a matrix representation. However, with a formal definition of a matrix representation (Definition MR), and a fundamental theorem to go with it (Theorem FTMR) we can be formal about the relationship, using the idea of isomorphic vector spaces (Definition IVS). Here are the twin theorems.

Theorem KNSI Kernel and Null Space Isomorphism
Suppose that $T\colon U \to V$ is a linear transformation, B is a basis for U of size n, and C is a basis for V. Then the kernel of T is isomorphic to the null space of $M^T_{B,C}$,

$$\mathcal{K}(T) \cong \mathcal{N}\left(M^T_{B,C}\right)$$

Proof. To establish that two vector spaces are isomorphic, we must find an isomorphism between them, an invertible linear transformation (Definition IVS). The kernel of the linear transformation T, $\mathcal{K}(T)$, is a subspace of U, while the null space of the matrix representation, $\mathcal{N}\left(M^T_{B,C}\right)$ is a subspace of $\mathbb{C}^n$. The function ρ_B is defined as a function from U to $\mathbb{C}^n$, but we can just as well employ the definition of ρ_B as a function from $\mathcal{K}(T)$ to $\mathcal{N}\left(M^T_{B,C}\right)$.

We must first insure that if we choose an input for ρ_B from $\mathcal{K}(T)$ that then the output will be an element of $\mathcal{N}\left(M^T_{B,C}\right)$. So suppose that $\mathbf{u} \in \mathcal{K}(T)$. Then

$$\begin{aligned}
M^T_{B,C}\rho_B(\mathbf{u}) &= \rho_C(T(\mathbf{u})) && \text{Theorem FTMR}\\
&= \rho_C(\mathbf{0}) && \text{Definition KLT}\\
&= \mathbf{0} && \text{Theorem LTTZZ}
\end{aligned}$$

This says that $\rho_B(\mathbf{u}) \in \mathcal{N}\left(M^T_{B,C}\right)$, as desired.

The restriction in the size of the domain and codomain ρ_B will not affect the fact that ρ_B is a linear transformation (Theorem VRLT), nor will it affect the fact that ρ_B is injective (Theorem VRI). Something must be done though to verify that ρ_B is surjective. To this end, appeal to the definition of surjective (Definition SLT), and suppose that we have an element of the codomain, $\mathbf{x} \in \mathcal{N}\left(M^T_{B,C}\right) \subseteq \mathbb{C}^n$ and we wish to find an element of the domain with $\mathbf{x}$ as its image. We now show that the desired element of the domain is $\mathbf{u} = \rho_B^{-1}(\mathbf{x})$. First, verify that $\mathbf{u} \in \mathcal{K}(T)$,

$$\begin{aligned}
T(\mathbf{u}) &= T\left(\rho_B^{-1}(\mathbf{x})\right) \\
&= \rho_C^{-1}\left(M^T_{B,C}\left(\rho_B\left(\rho_B^{-1}(\mathbf{x})\right)\right)\right) && \text{Theorem FTMR} \\
&= \rho_C^{-1}\left(M^T_{B,C}\left(I_{\mathbb{C}^n}(\mathbf{x})\right)\right) && \text{Definition IVLT} \\
&= \rho_C^{-1}\left(M^T_{B,C}\mathbf{x}\right) && \text{Definition IDLT} \\
&= \rho_C^{-1}\left(\mathbf{0}_{\mathbb{C}^n}\right) && \text{Definition KLT} \\
&= \mathbf{0}_V && \text{Theorem LTTZZ}
\end{aligned}$$

Second, verify that the proposed isomorphism, ρ_B, takes $\mathbf{u}$ to $\mathbf{x}$,

$$\begin{aligned}
\rho_B(\mathbf{u}) &= \rho_B\left(\rho_B^{-1}(\mathbf{x})\right) && \text{Substitution} \\
&= I_{\mathbb{C}^n}(\mathbf{x}) && \text{Definition IVLT} \\
&= \mathbf{x} && \text{Definition IDLT}
\end{aligned}$$

With ρ_B demonstrated to be an injective and surjective linear transformation from $\mathcal{K}(T)$ to $\mathcal{N}\left(M^T_{B,C}\right)$, Theorem ILTIS tells us ρ_B is invertible, and so by Definition IVS, we say $\mathcal{K}(T)$ and $\mathcal{N}\left(M^T_{B,C}\right)$ are isomorphic. ■

Example KVMR Kernel via matrix representation
Consider the kernel of the linear transformation, $T\colon M_{22} \to P_2$, given by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (2a - b + c - 5d) + (a + 4b + 5c + 2d)x + (3a - 2b + c - 8d)x^2$$

We will begin with a matrix representation of T relative to the bases for M_{22} and P_2 (respectively),

$$\begin{aligned}
B &= \left\{\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ -1 & -4 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}, \begin{bmatrix} 2 & 5 \\ -2 & -4 \end{bmatrix}\right\} \\
C &= \left\{1 + x + x^2,\ 2 + 3x,\ -1 - 2x^2\right\}
\end{aligned}$$

Then,

$$\begin{aligned}
\rho_C\left(T\left(\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}\right)\right) &= \rho_C\left(4 + 2x + 6x^2\right) \\
&= \rho_C\left(2(1 + x + x^2) + 0(2 + 3x) + (-2)(-1 - 2x^2)\right) \\
&= \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \\
\rho_C\left(T\left(\begin{bmatrix} 1 & 3 \\ -1 & -4 \end{bmatrix}\right)\right) &= \rho_C\left(18 + 28x^2\right) \\
&= \rho_C\left((-24)(1 + x + x^2) + 8(2 + 3x) + (-26)(-1 - 2x^2)\right) \\
&= \begin{bmatrix} -24 \\ 8 \\ -26 \end{bmatrix} \\
\rho_C\left(T\left(\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}\right)\right) &= \rho_C\left(10 + 5x + 15x^2\right) \\
&= \rho_C\left(5(1 + x + x^2) + 0(2 + 3x) + (-5)(-1 - 2x^2)\right)
\end{aligned}$$

$$= \begin{bmatrix} 5 \\ 0 \\ -5 \end{bmatrix}$$

$$\begin{aligned} \rho_C\left(T\left(\begin{bmatrix} 2 & 5 \\ -2 & -4 \end{bmatrix}\right)\right) &= \rho_C\left(17 + 4x + 26x^2\right) \\ &= \rho_C\left((-8)(1 + x + x^2) + (4)(2 + 3x) + (-17)(-1 - 2x^2)\right) \\ &= \begin{bmatrix} -8 \\ 4 \\ -17 \end{bmatrix} \end{aligned}$$

So the matrix representation of T (relative to B and C) is

$$M^T_{B,C} = \begin{bmatrix} 2 & -24 & 5 & -8 \\ 0 & 8 & 0 & 4 \\ -2 & -26 & -5 & -17 \end{bmatrix}$$

We know from Theorem KNSI that the kernel of the linear transformation T is isomorphic to the null space of the matrix representation $M^T_{B,C}$ and by studying the proof of Theorem KNSI we learn that ρ_B is an isomorphism between these null spaces. Rather than trying to compute the kernel of T using definitions and techniques from Chapter LT we will instead analyze the null space of $M^T_{B,C}$ using techniques from way back in Chapter V. First row-reduce $M^T_{B,C}$,

$$\begin{bmatrix} 2 & -24 & 5 & -8 \\ 0 & 8 & 0 & 4 \\ -2 & -26 & -5 & -17 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & \frac{5}{2} & 2 \\ 0 & \boxed{1} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So, by Theorem BNS, a basis for $\mathcal{N}\left(M^T_{B,C}\right)$ is

$$\left\langle\left\{\begin{bmatrix} -\frac{5}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix},\, \begin{bmatrix} -2 \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}\right\}\right\rangle$$

We can now convert this basis of $\mathcal{N}\left(M^T_{B,C}\right)$ into a basis of $\mathcal{K}(T)$ by applying ρ_B^{-1} to each element of the basis,

$$\begin{aligned} \rho_B^{-1}\left(\begin{bmatrix} -\frac{5}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix}\right) &= (-\tfrac{5}{2})\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} + 0\begin{bmatrix} 1 & 3 \\ -1 & -4 \end{bmatrix} + 1\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} + 0\begin{bmatrix} 2 & 5 \\ -2 & -4 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{3}{2} & -3 \\ \frac{5}{2} & \frac{1}{2} \end{bmatrix} \\ \rho_B^{-1}\left(\begin{bmatrix} -2 \\ -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix}\right) &= (-2)\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} + (-\tfrac{1}{2})\begin{bmatrix} 1 & 3 \\ -1 & -4 \end{bmatrix} + 0\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} + 1\begin{bmatrix} 2 & 5 \\ -2 & -4 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix} \end{aligned}$$

So the set

$$\left\{\begin{bmatrix} -\frac{3}{2} & -3 \\ \frac{5}{2} & \frac{1}{2} \end{bmatrix},\, \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}\right\}$$

is a basis for $\mathcal{K}(T)$. Just for fun, you might evaluate T with each of these two basis vectors and verify that the output is the zero polynomial (Exercise MR.C10). △

An entirely similar result applies to the range of a linear transformation and the column space of a matrix representation of the linear transformation.

Theorem RCSI Range and Column Space Isomorphism
Suppose that $T\colon U \to V$ is a linear transformation, B is a basis for U of size n, and C is a basis for V of size m. Then the range of T is isomorphic to the column space of $M^T_{B,C}$,

$$\mathcal{R}(T) \cong \mathcal{C}\left(M^T_{B,C}\right)$$

Proof. To establish that two vector spaces are isomorphic, we must find an isomorphism between them, an invertible linear transformation (Definition IVS). The range of the linear transformation T, $\mathcal{R}(T)$, is a subspace of V, while the column space of the matrix representation, $\mathcal{C}\left(M^T_{B,C}\right)$ is a subspace of $\mathbb{C}^m$. The function ρ_C is defined as a function from V to $\mathbb{C}^m$, but we can just as well employ the definition of ρ_C as a function from $\mathcal{R}(T)$ to $\mathcal{C}\left(M^T_{B,C}\right)$.

We must first insure that if we choose an input for ρ_C from $\mathcal{R}(T)$ that then the output will be an element of $\mathcal{C}\left(M^T_{B,C}\right)$. So suppose that $\mathbf{v} \in \mathcal{R}(T)$. Then there is a vector $\mathbf{u} \in U$, such that $T(\mathbf{u}) = \mathbf{v}$. Consider

$$\begin{aligned} M^T_{B,C}\rho_B(\mathbf{u}) &= \rho_C(T(\mathbf{u})) && \text{Theorem FTMR} \\ &= \rho_C(\mathbf{v}) && \text{Definition RLT} \end{aligned}$$

This says that $\rho_C(\mathbf{v}) \in \mathcal{C}\left(M^T_{B,C}\right)$, as desired.

The restriction in the size of the domain and codomain will not affect the fact that ρ_C is a linear transformation (Theorem VRLT), nor will it affect the fact that ρ_C is injective (Theorem VRI). Something must be done though to verify that ρ_C is surjective. This all gets a bit confusing, since the domain of our isomorphism is the range of the linear transformation, so think about your objects as you go. To establish that ρ_C is surjective, appeal to the definition of a surjective linear transformation (Definition SLT), and suppose that we have an element of the codomain, $\mathbf{y} \in \mathcal{C}\left(M^T_{B,C}\right) \subseteq \mathbb{C}^m$ and we wish to find an element of the domain with $\mathbf{y}$ as its image. Since $\mathbf{y} \in \mathcal{C}\left(M^T_{B,C}\right)$, there exists a vector, $\mathbf{x} \in \mathbb{C}^n$ with $M^T_{B,C}\mathbf{x} = \mathbf{y}$.

We now show that the desired element of the domain is $\mathbf{v} = \rho_C^{-1}(\mathbf{y})$. First, verify that $\mathbf{v} \in \mathcal{R}(T)$ by applying T to $\mathbf{u} = \rho_B^{-1}(\mathbf{x})$,

$$\begin{aligned} T(\mathbf{u}) &= T\left(\rho_B^{-1}(\mathbf{x})\right) \\ &= \rho_C^{-1}\left(M^T_{B,C}\left(\rho_B\left(\rho_B^{-1}(\mathbf{x})\right)\right)\right) && \text{Theorem FTMR} \\ &= \rho_C^{-1}\left(M^T_{B,C}\left(I_{\mathbb{C}^n}(\mathbf{x})\right)\right) && \text{Definition IVLT} \\ &= \rho_C^{-1}\left(M^T_{B,C}\mathbf{x}\right) && \text{Definition IDLT} \\ &= \rho_C^{-1}(\mathbf{y}) && \text{Definition CSM} \\ &= \mathbf{v} && \text{Substitution} \end{aligned}$$

Second, verify that the proposed isomorphism, ρ_C, takes $\mathbf{v}$ to $\mathbf{y}$,

$$\begin{aligned} \rho_C(\mathbf{v}) &= \rho_C\left(\rho_C^{-1}(\mathbf{y})\right) && \text{Substitution} \\ &= I_{\mathbb{C}^m}(\mathbf{y}) && \text{Definition IVLT} \\ &= \mathbf{y} && \text{Definition IDLT} \end{aligned}$$

With ρ_C demonstrated to be an injective and surjective linear transformation from $\mathcal{R}(T)$ to $\mathcal{C}\left(M^T_{B,C}\right)$, Theorem ILTIS tells us ρ_C is invertible, and so by Definition IVS, we say $\mathcal{R}(T)$ and $\mathcal{C}\left(M^T_{B,C}\right)$ are isomorphic. ■

Example RVMR Range via matrix representation
In this example, we will recycle the linear transformation T and the bases B and C of Example KVMR but now we will compute the range of $T\colon M_{22} \to P_2$, given by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (2a - b + c - 5d) + (a + 4b + 5c + 2d)x + (3a - 2b + c - 8d)x^2$$

With bases B and C,

$$B = \left\{ \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ -1 & -4 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}, \begin{bmatrix} 2 & 5 \\ -2 & -4 \end{bmatrix} \right\}$$
$$C = \left\{1 + x + x^2,\ 2 + 3x,\ -1 - 2x^2\right\}$$

we obtain the matrix representation

$$M^T_{B,C} = \begin{bmatrix} 2 & -24 & 5 & -8 \\ 0 & 8 & 0 & 4 \\ -2 & -26 & -5 & -17 \end{bmatrix}$$

We know from Theorem RCSI that the range of the linear transformation T is isomorphic to the column space of the matrix representation $M^T_{B,C}$ and by studying the proof of Theorem RCSI we learn that ρ_C is an isomorphism between these subspaces. Notice that since the range is a subspace of the codomain, we will employ ρ_C as the isomorphism, rather than ρ_B, which was the correct choice for an isomorphism between the null spaces of Example KVMR.

Rather than trying to compute the range of T using definitions and techniques from Chapter LT we will instead analyze the column space of $M^T_{B,C}$ using techniques from way back in Chapter M. First row-reduce $\left(M^T_{B,C}\right)^t$,

$$\begin{bmatrix} 2 & 0 & -2 \\ -24 & 8 & -26 \\ 5 & 0 & -5 \\ -8 & 4 & -17 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & -1 \\ 0 & \boxed{1} & -\frac{25}{4} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now employ Theorem CSRST and Theorem BRS (there are other methods we could choose here to compute the column space, such as Theorem BCS) to obtain the basis for $\mathcal{C}\left(M^T_{B,C}\right)$,

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -\frac{25}{4} \end{bmatrix} \right\}$$

We can now convert this basis of $\mathcal{C}\left(M^T_{B,C}\right)$ into a basis of $\mathcal{R}(T)$ by applying ρ_C^{-1} to each element of the basis,

$$\rho_C^{-1}\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right) = (1 + x + x^2) - (-1 - 2x^2) = 2 + x + 3x^2$$
$$\rho_C^{-1}\left(\begin{bmatrix} 0 \\ 1 \\ -\frac{25}{4} \end{bmatrix}\right) = (2 + 3x) - \frac{25}{4}(-1 - 2x^2) = \frac{33}{4} + 3x + \frac{31}{2}x^2$$

So the set

$$\left\{ 2 + 3x + 3x^2,\ \frac{33}{4} + 3x + \frac{31}{2}x^2 \right\}$$

is a basis for $\mathcal{R}(T)$. △

Theorem KNSI and Theorem RCSI can be viewed as further formal evidence for the Coordinatization Principle, though they are not direct consequences.

Diagram KRI is meant to suggest Theorem KNSI and Theorem RCSI, in addition to their proofs (and so carry the same notation as the statements of these two theorems). The dashed lines indicate a subspace relationship, with the smaller vector space lower down in the diagram. The central square is highly reminiscent of Diagram FTMR. Each of the four vector representations is an isomorphism, so the inverse linear transformation could be depicted with an arrow pointing in the other direction. The four vector spaces across the bottom are familiar from the earliest days of the course, while the four vector spaces across the top are completely abstract. The vector representations that are restrictions (far left and far right) are the functions shown to be invertible representations as the key technique in the proofs of Theorem KNSI and Theorem RCSI. So this diagram could be helpful as you study those two proofs.

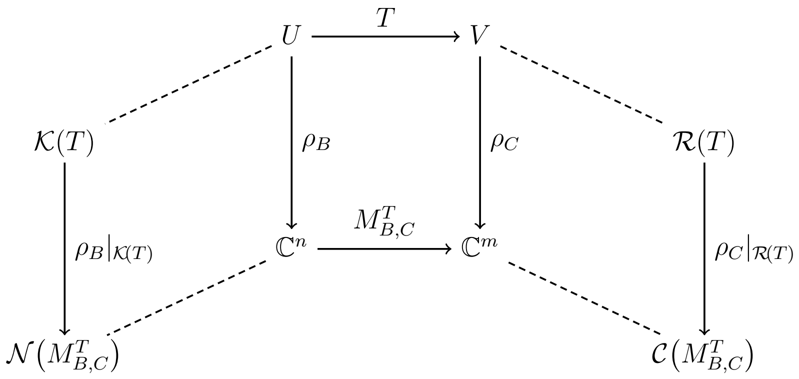

Diagram KRI: Kernel and Range Isomorphisms

Subsection IVLT Invertible Linear Transformations

We have seen, both in theorems and in examples, that questions about linear transformations are often equivalent to questions about matrices. It is the matrix representation of a linear transformation that makes this idea precise. Here is our final theorem that solidifies this connection.

Theorem IMR Invertible Matrix Representations
Suppose that $T\colon U\to V$ is a linear transformation, B is a basis for U and C is a basis for V. Then T is an invertible linear transformation if and only if the matrix representation of T relative to B and C, $M^T_{B,C}$ is an invertible matrix. When T is invertible,

$$M^{T^{-1}}_{C,B} = \left(M^T_{B,C}\right)^{-1}$$

Proof. ($\Rightarrow$) Suppose T is invertible, so the inverse linear transformation $T^{-1}\colon V\to U$ exists (Definition IVLT). Both linear transformations have matrix representations relative to the bases of U and V, namely $M^T_{B,C}$ and $M^{T^{-1}}_{C,B}$ (Definition MR).

Then

$$\begin{aligned}
M^{T^{-1}}_{C,B} M^T_{B,C} &= M^{T^{-1}\circ T}_{B,B} && \text{Theorem MRCLT}\\
&= M^{I_U}_{B,B} && \text{Definition IVLT}\\
&= \left[\rho_B\left(I_U\left(\mathbf{u}_1\right)\right)\middle|\, \rho_B\left(I_U\left(\mathbf{u}_2\right)\right)\middle|\ldots\middle|\rho_B\left(I_U\left(\mathbf{u}_n\right)\right)\right] && \text{Definition MR}\\
&= \left[\rho_B\left(\mathbf{u}_1\right)\middle|\, \rho_B\left(\mathbf{u}_2\right)\middle|\, \rho_B\left(\mathbf{u}_3\right)\middle|\ldots\middle|\rho_B\left(\mathbf{u}_n\right)\right] && \text{Definition IDLT}\\
&= \left[\mathbf{e}_1|\mathbf{e}_2|\mathbf{e}_3|\ldots|\mathbf{e}_n\right] && \text{Definition VR}\\
&= I_n && \text{Definition IM}
\end{aligned}$$

And

$$\begin{aligned}
M^T_{B,C} M^{T^{-1}}_{C,B} &= M^{T\circ T^{-1}}_{C,C} && \text{Theorem MRCLT}\\
&= M^{I_V}_{C,C} && \text{Definition IVLT}\\
&= \left[\rho_C\left(I_V\left(\mathbf{v}_1\right)\right)\middle|\, \rho_C\left(I_V\left(\mathbf{v}_2\right)\right)\middle|\ldots\middle|\rho_C\left(I_V\left(\mathbf{v}_n\right)\right)\right] && \text{Definition MR}\\
&= \left[\rho_C\left(\mathbf{v}_1\right)\middle|\, \rho_C\left(\mathbf{v}_2\right)\middle|\, \rho_C\left(\mathbf{v}_3\right)\middle|\ldots\middle|\rho_C\left(\mathbf{v}_n\right)\right] && \text{Definition IDLT}
\end{aligned}$$

$$= [\mathbf{e}_1|\mathbf{e}_2|\mathbf{e}_3|\dots|\mathbf{e}_n] \qquad \text{Definition VR}$$
$$= I_n \qquad \text{Definition IM}$$

These two equations show that $M^{T}_{B,C}$ and $M^{T^{-1}}_{C,B}$ are inverse matrices (Definition MI) and establish that when T is invertible, then $M^{T^{-1}}_{C,B} = \left(M^{T}_{B,C}\right)^{-1}$.

($\Leftarrow$) Suppose now that $M^{T}_{B,C}$ is an invertible matrix and hence nonsingular (Theorem NI). We compute the nullity of T,

$$\begin{aligned}
n\left(T\right) &= \dim\left(\mathcal{K}(T)\right) && \text{Definition KLT}\\
&= \dim\left(\mathcal{N}\left(M^{T}_{B,C}\right)\right) && \text{Theorem KNSI}\\
&= n\left(M^{T}_{B,C}\right) && \text{Definition NOM}\\
&= 0 && \text{Theorem RNNM}
\end{aligned}$$

So the kernel of T is trivial, and by Theorem KILT, T is injective.

We now compute the rank of T,

$$\begin{aligned}
r\left(T\right) &= \dim\left(\mathcal{R}(T)\right) && \text{Definition RLT}\\
&= \dim\left(\mathcal{C}\left(M^{T}_{B,C}\right)\right) && \text{Theorem RCSI}\\
&= r\left(M^{T}_{B,C}\right) && \text{Definition ROM}\\
&= \dim\left(V\right) && \text{Theorem RNNM}
\end{aligned}$$

Since the dimension of the range of T equals the dimension of the codomain V, by Theorem EDYES, $\mathcal{R}(T) = V$. Which says that T is surjective by Theorem RSLT.

Because T is both injective and surjective, by Theorem ILTIS, T is invertible. ■

By now, the connections between matrices and linear transformations should be starting to become more transparent, and you may have already recognized the invertibility of a matrix as being tantamount to the invertibility of the associated matrix representation. The next example shows how to apply this theorem to the problem of actually building a formula for the inverse of an invertible linear transformation.

Example ILTVR Inverse of a linear transformation via a representation

Consider the linear transformation

$$R\colon P_3 \to M_{22}, \quad R\left(a + bx + cx^2 + x^3\right) = \begin{bmatrix} a+b-c+2d & 2a+3b-2c+3d \\ a+b+2d & -a+b+2c-5d \end{bmatrix}$$

If we wish to quickly find a formula for the inverse of R (presuming it exists), then choosing "nice" bases will work best. So build a matrix representation of R relative to the bases B and C,

$$B = \left\{1,\, x,\, x^2,\, x^3\right\}$$
$$C = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

Then,

$$\rho_C\left(R\left(1\right)\right) = \rho_C\left(\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix}$$

$$\rho_C\left(R\left(x\right)\right) = \rho_C\left(\begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 1 \end{bmatrix}$$

$$\rho_C\left(R\left(x^2\right)\right)=\rho_C\left(\begin{bmatrix}-1 & -2\\ 0 & 2\end{bmatrix}\right)=\begin{bmatrix}-1\\-2\\0\\2\end{bmatrix}$$

$$\rho_C\left(R\left(x^3\right)\right)=\rho_C\left(\begin{bmatrix}2 & 3\\ 2 & -5\end{bmatrix}\right)=\begin{bmatrix}2\\3\\2\\-5\end{bmatrix}$$

So a representation of R is

$$M^R_{B,C}=\begin{bmatrix}1 & 1 & -1 & 2\\ 2 & 3 & -2 & 3\\ 1 & 1 & 0 & 2\\ -1 & 1 & 2 & -5\end{bmatrix}$$

The matrix $M^R_{B,C}$ is invertible (as you can check) so we know for sure that R is invertible by Theorem IMR. Furthermore,

$$M^{R^{-1}}_{C,B}=\left(M^R_{B,C}\right)^{-1}=\begin{bmatrix}1 & 1 & -1 & 2\\ 2 & 3 & -2 & 3\\ 1 & 1 & 0 & 2\\ -1 & 1 & 2 & -5\end{bmatrix}^{-1}=\begin{bmatrix}20 & -7 & -2 & 3\\ -8 & 3 & 1 & -1\\ -1 & 0 & 1 & 0\\ -6 & 2 & 1 & -1\end{bmatrix}$$

We can use this representation of the inverse linear transformation, in concert with Theorem FTMR, to determine an explicit formula for the inverse itself,

$$\begin{aligned}
R^{-1}\left(\begin{bmatrix}a & b\\ c & d\end{bmatrix}\right)&=\rho_B^{-1}\left(M^{R^{-1}}_{C,B}\,\rho_C\left(\begin{bmatrix}a & b\\ c & d\end{bmatrix}\right)\right) && \text{Theorem FTMR}\\
&=\rho_B^{-1}\left(\left(M^R_{B,C}\right)^{-1}\rho_C\left(\begin{bmatrix}a & b\\ c & d\end{bmatrix}\right)\right) && \text{Theorem IMR}\\
&=\rho_B^{-1}\left(\left(M^R_{B,C}\right)^{-1}\begin{bmatrix}a\\b\\c\\d\end{bmatrix}\right) && \text{Definition VR}\\
&=\rho_B^{-1}\left(\begin{bmatrix}20 & -7 & -2 & 3\\ -8 & 3 & 1 & -1\\ -1 & 0 & 1 & 0\\ -6 & 2 & 1 & -1\end{bmatrix}\begin{bmatrix}a\\b\\c\\d\end{bmatrix}\right) && \text{Definition MI}\\
&=\rho_B^{-1}\left(\begin{bmatrix}20a-7b-2c+3d\\ -8a+3b+c-d\\ -a+c\\ -6a+2b+c-d\end{bmatrix}\right) && \text{Definition MVP}\\
&=(20a-7b-2c+3d)+(-8a+3b+c-d)x\\
&\quad+(-a+c)x^2+(-6a+2b+c-d)x^3 && \text{Definition VR}
\end{aligned}$$

△

You might look back at Example AIVLT, where we first witnessed the inverse of a linear transformation and recognize that the inverse (S) was built from using the method of Example ILTVR with a matrix representation of T.

Theorem IMILT Invertible Matrices, Invertible Linear Transformation
Suppose that A is a square matrix of size n and $T\colon\mathbb{C}^n\to\mathbb{C}^n$ is the linear transformation defined by $T(\mathbf{x})=A\mathbf{x}$. Then A is an invertible matrix if and only if T is an invertible linear transformation.

Proof. Choose bases $B=C=\{\mathbf{e}_1,\,\mathbf{e}_2,\,\mathbf{e}_3,\,\dots,\,\mathbf{e}_n\}$ consisting of the standard unit vectors as a basis of $\mathbb{C}^n$ (Theorem SUVB) and build a matrix representation of T

relative to B and C. Then

$$\begin{aligned}\rho_C(T(\mathbf{e}_i)) &= \rho_C(A\mathbf{e}_i)\\ &= \rho_C(\mathbf{A}_i)\\ &= \mathbf{A}_i\end{aligned}$$

So then the matrix representation of T, relative to B and C, is simply $M^T_{B,C} = A$. With this observation, the proof becomes a specialization of Theorem IMR,

$$T \text{ is invertible} \iff M^T_{B,C} \text{ is invertible} \iff A \text{ is invertible}$$

■

This theorem may seem gratuitous. Why state such a special case of Theorem IMR? Because it adds another condition to our NMEx series of theorems, and in some ways it is the most fundamental expression of what it means for a matrix to be nonsingular — the associated linear transformation is invertible. This is our final update.

Theorem NME9 Nonsingular Matrix Equivalences, Round 9
Suppose that A is a square matrix of size n. The following are equivalent.

1. *A is nonsingular.*
2. *A row-reduces to the identity matrix.*
3. *The null space of A contains only the zero vector, $\mathcal{N}(A) = \{\mathbf{0}\}$.*
4. *The linear system $\mathcal{LS}(A, \mathbf{b})$ has a unique solution for every possible choice of $\mathbf{b}$.*
5. *The columns of A are a linearly independent set.*
6. *A is invertible.*
7. *The column space of A is $\mathbb{C}^n$, $\mathcal{C}(A) = \mathbb{C}^n$.*
8. *The columns of A are a basis for $\mathbb{C}^n$.*
9. *The rank of A is n, $r(A) = n$.*
10. *The nullity of A is zero, $n(A) = 0$.*
11. *The determinant of A is nonzero, $\det(A) \neq 0$.*
12. *$\lambda = 0$ is not an eigenvalue of A.*
13. *The linear transformation $T\colon \mathbb{C}^n \to \mathbb{C}^n$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is invertible.*

Proof. By Theorem IMILT, the new addition to this list is equivalent to the statement that A is invertible, so we can expand Theorem NME8. ■

Reading Questions

1. Why does Theorem FTMR deserve the moniker "fundamental"?
2. Find the matrix representation, $M^T_{B,C}$ of the linear transformation

$$T\colon \mathbb{C}^2 \to \mathbb{C}^2, \quad T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right) = \begin{bmatrix}2x_1 - x_2\\3x_1 + 2x_2\end{bmatrix}$$

relative to the bases

$$B = \left\{\begin{bmatrix}2\\3\end{bmatrix}, \begin{bmatrix}-1\\2\end{bmatrix}\right\} \qquad C = \left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right\}$$

3. What is the second "surprise," and why is it surprising?

Exercises

C10 Example KVMR concludes with a basis for the kernel of the linear transformation T. Compute the value of T for each of these two basis vectors. Did you get what you expected?

C20† Compute the matrix representation of T relative to the bases B and C.

$$T\colon P_3 \to \mathbb{C}^3, \quad T\left(a + bx + cx^2 + dx^3\right) = \begin{bmatrix} 2a - 3b + 4c - 2d \\ a + b - c + d \\ 3a + 2c - 3d \end{bmatrix}$$

$$B = \left\{1,\, x,\, x^2,\, x^3\right\} \qquad C = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

C21† Find a matrix representation of the linear transformation T relative to the bases B and C.

$$T\colon P_2 \to \mathbb{C}^2, \quad T\left(p(x)\right) = \begin{bmatrix} p(1) \\ p(3) \end{bmatrix}$$

$$B = \left\{2 - 5x + x^2,\, 1 + x - x^2,\, x^2\right\}$$

$$C = \left\{ \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$$

C22† Let S_{22} be the vector space of 2×2 symmetric matrices. Build the matrix representation of the linear transformation $T\colon P_2 \to S_{22}$ relative to the bases B and C and then use this matrix representation to compute $T\left(3 + 5x - 2x^2\right)$.

$$B = \left\{1,\, 1 + x,\, 1 + x + x^2\right\} \qquad C = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$T\left(a + bx + cx^2\right) = \begin{bmatrix} 2a - b + c & a + 3b - c \\ a + 3b - c & a - c \end{bmatrix}$$

C25† Use a matrix representation to determine if the linear transformation $T\colon P_3 \to M_{22}$ is surjective.

$$T\left(a + bx + cx^2 + dx^3\right) = \begin{bmatrix} -a + 4b + c + 2d & 4a - b + 6c - d \\ a + 5b - 2c + 2d & a + 2c + 5d \end{bmatrix}$$

C30† Find bases for the kernel and range of the linear transformation S below.

$$S\colon M_{22} \to P_2, \quad S\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a + 2b + 5c - 4d) + (3a - b + 8c + 2d)x + (a + b + 4c - 2d)x^2$$

C40† Let S_{22} be the set of 2×2 symmetric matrices. Verify that the linear transformation R is invertible and find R^{-1}.

$$R\colon S_{22} \to P_2, \quad R\left(\begin{bmatrix} a & b \\ b & c \end{bmatrix}\right) = (a - b) + (2a - 3b - 2c)x + (a - b + c)x^2$$

C41† Prove that the linear transformation S is invertible. Then find a formula for the inverse linear transformation, S^{-1}, by employing a matrix inverse.

$$S\colon P_1 \to M_{1,2}, \quad S\left(a + bx\right) = \begin{bmatrix} 3a + b & 2a + b \end{bmatrix}$$

C42† The linear transformation $R\colon M_{12} \to M_{21}$ is invertible. Use a matrix representation to determine a formula for the inverse linear transformation $R^{-1}\colon M_{21} \to M_{12}$.

$$R\left(\begin{bmatrix} a & b \end{bmatrix}\right) = \begin{bmatrix} a + 3b \\ 4a + 11b \end{bmatrix}$$

C50† Use a matrix representation to find a basis for the range of the linear transformation L.

$$L\colon M_{22} \to P_2, \quad T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a + 2b + 4c + d) + (3a + c - 2d)x + (-a + b + 3c + 3d)x^2$$

C51 Use a matrix representation to find a basis for the kernel of the linear transformation L.

$$L\colon M_{22} \to P_2, \quad T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+2b+4c+d) + (3a+c-2d)x + (-a+b+3c+3d)x^2$$

C52† Find a basis for the kernel of the linear transformation $T\colon P_2 \to M_{22}$.

$$T\left(a+bx+cx^2\right) = \begin{bmatrix} a+2b-2c & 2a+2b \\ -a+b-4c & 3a+2b+2c \end{bmatrix}$$

M20† The linear transformation D performs differentiation on polynomials. Use a matrix representation of D to find the rank and nullity of D.

$$D\colon P_n \to P_n, \quad D\left(p(x)\right) = p'(x)$$

M60 Suppose U and V are vector spaces and define a function $Z\colon U \to V$ by $T(\mathbf{u}) = \mathbf{0}_V$ for every $\mathbf{u} \in U$. Then Exercise IVLT.M60 asks you to formulate the theorem: Z is invertible if and only if $U = \{\mathbf{0}_U\}$ and $V = \{\mathbf{0}_V\}$. What would a matrix representation of Z look like in this case? How does Theorem IMR read in this case?

M80 In light of Theorem KNSI and Theorem MRCLT, write a short comparison of Exercise MM.T40 with Exercise ILT.T15.

M81 In light of Theorem RCSI and Theorem MRCLT, write a short comparison of Exercise CRS.T40 with Exercise SLT.T15.

M82 In light of Theorem MRCLT and Theorem IMR, write a short comparison of Theorem SS and Theorem ICLT.

M83 In light of Theorem MRCLT and Theorem IMR, write a short comparison of Theorem NPNT and Exercise IVLT.T40.

T20† Construct a new solution to Exercise B.T50 along the following outline. From the $n \times n$ matrix A, construct the linear transformation $T\colon \mathbb{C}^n \to \mathbb{C}^n$, $T(\mathbf{x}) = A\mathbf{x}$. Use Theorem NI, Theorem IMILT and Theorem ILTIS to translate between the nonsingularity of A and the surjectivity/injectivity of T. Then apply Theorem ILTB and Theorem SLTB to connect these properties with bases.

T60 Create an entirely different proof of Theorem IMILT that relies on Definition IVLT to establish the invertibility of T, and that relies on Definition MI to establish the invertibility of A.

T80† Suppose that $T\colon U \to V$ and $S\colon V \to W$ are linear transformations, and that B, C and D are bases for U, V, and W. Using only Definition MR define matrix representations for T and S. Using these two definitions, and Definition MR, derive a matrix representation for the composition $S \circ T$ in terms of the entries of the matrices $M^T_{B,C}$ and $M^S_{C,D}$. Explain how you would use this result to *motivate a definition* for matrix multiplication that is strikingly similar to Theorem EMP.

Section CB
Change of Basis

We have seen in Section MR that a linear transformation can be represented by a matrix, once we pick bases for the domain and codomain. How does the matrix representation change if we choose different bases? Which bases lead to especially nice representations? From the infinite possibilities, what is the best possible representation? This section will begin to answer these questions. But first we need to define eigenvalues for linear transformations and the change-of-basis matrix.

Subsection EELT
Eigenvalues and Eigenvectors of Linear Transformations

We now define the notion of an eigenvalue and eigenvector of a linear transformation. It should not be too surprising, especially if you remind yourself of the close relationship between matrices and linear transformations.

Definition EELT Eigenvalue and Eigenvector of a Linear Transformation
Suppose that $T\colon V \to V$ is a linear transformation. Then a nonzero vector $\mathbf{v} \in V$ is an **eigenvector** of T for the **eigenvalue** λ if $T(\mathbf{v}) = \lambda\mathbf{v}$. □

We will see shortly the best method for computing the eigenvalues and eigenvectors of a linear transformation, but for now, here are some examples to verify that such things really do exist.

Example ELTBM Eigenvectors of linear transformation between matrices
Consider the linear transformation $T\colon M_{22} \to M_{22}$ defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} -17a + 11b + 8c - 11d & -57a + 35b + 24c - 33d \\ -14a + 10b + 6c - 10d & -41a + 25b + 16c - 23d \end{bmatrix}$$

and the vectors

$$\mathbf{x}_1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{x}_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad \mathbf{x}_3 = \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix} \qquad \mathbf{x}_4 = \begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix}$$

Then compute

$$T(\mathbf{x}_1) = T\left(\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix} = 2\mathbf{x}_1$$

$$T(\mathbf{x}_2) = T\left(\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 2 & 2 \\ 2 & 0 \end{bmatrix} = 2\mathbf{x}_2$$

$$T(\mathbf{x}_3) = T\left(\begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix}\right) = \begin{bmatrix} -1 & -3 \\ -2 & -3 \end{bmatrix} = (-1)\mathbf{x}_3$$

$$T(\mathbf{x}_4) = T\left(\begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix}\right) = \begin{bmatrix} -4 & -12 \\ -2 & -8 \end{bmatrix} = (-2)\mathbf{x}_4$$

So $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, $\mathbf{x}_4$ are eigenvectors of T with eigenvalues (respectively) $\lambda_1 = 2$, $\lambda_2 = 2$, $\lambda_3 = -1$, $\lambda_4 = -2$. △

Here is another.

Example ELTBP Eigenvectors of linear transformation between polynomials
Consider the linear transformation $R\colon P_2 \to P_2$ defined by

$$R\left(a + bx + cx^2\right) = (15a + 8b - 4c) + (-12a - 6b + 3c)x + (24a + 14b - 7c)x^2$$

and the vectors

$$\mathbf{w}_1 = 1 - x + x^2 \qquad \mathbf{w}_2 = x + 2x^2 \qquad \mathbf{w}_3 = 1 + 4x^2$$

Then compute

$$
\begin{aligned}
R(\mathbf{w}_1) &= R\left(1 - x + x^2\right) = 3 - 3x + 3x^2 = 3\mathbf{w}_1\\
R(\mathbf{w}_2) &= R\left(x + 2x^2\right) = 0 + 0x + 0x^2 = 0\mathbf{w}_2\\
R(\mathbf{w}_3) &= R\left(1 + 4x^2\right) = -1 - 4x^2 = (-1)\mathbf{w}_3
\end{aligned}
$$

So $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$ are eigenvectors of R with eigenvalues (respectively) $\lambda_1 = 3$, $\lambda_2 = 0$, $\lambda_3 = -1$. Notice how the eigenvalue $\lambda_2 = 0$ indicates that the eigenvector $\mathbf{w}_2$ is a nontrivial element of the kernel of R, and therefore R is not injective (Exercise CB.T15). △

Of course, these examples are meant only to illustrate the definition of eigenvectors and eigenvalues for linear transformations, and therefore beg the question, "How would I *find* eigenvectors?" We will have an answer before we finish this section. We need one more construction first.

Subsection CBM Change-of-Basis Matrix

Given a vector space, we know we can usually find many different bases for the vector space, some nice, some nasty. If we choose a single vector from this vector space, we can build many different representations of the vector by constructing the representations relative to different bases. How are these different representations related to each other? A change-of-basis matrix answers this question.

Definition CBM Change-of-Basis Matrix
Suppose that V is a vector space, and $I_V : V \to V$ is the identity linear transformation on V. Let $B = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$ and C be two bases of V. Then the **change-of-basis matrix** from B to C is the matrix representation of I_V relative to B and C,

$$
\begin{aligned}
C_{B,C} &= M^{I_V}_{B,C}\\
&= \left[\rho_C(I_V(\mathbf{v}_1))\middle|\, \rho_C(I_V(\mathbf{v}_2))\middle|\, \rho_C(I_V(\mathbf{v}_3))\middle|\ldots\middle|\rho_C(I_V(\mathbf{v}_n))\right]\\
&= \left[\rho_C(\mathbf{v}_1)\middle|\, \rho_C(\mathbf{v}_2)\middle|\, \rho_C(\mathbf{v}_3)\middle|\ldots\middle|\rho_C(\mathbf{v}_n)\right]
\end{aligned}
$$

□

Notice that this definition is primarily about a single vector space (V) and two bases of V (B, C). The linear transformation (I_V) is necessary but not critical. As you might expect, this matrix has something to do with changing bases. Here is the theorem that gives the matrix its name (not the other way around).

Theorem CB Change-of-Basis
Suppose that $\mathbf{v}$ *is a vector in the vector space* V *and* B *and* C *are bases of* V*. Then*

$$\rho_C(\mathbf{v}) = C_{B,C}\,\rho_B(\mathbf{v})$$

Proof.

$$
\begin{aligned}
\rho_C(\mathbf{v}) &= \rho_C(I_V(\mathbf{v})) && \text{Definition IDLT}\\
&= M^{I_V}_{B,C}\rho_B(\mathbf{v}) && \text{Theorem FTMR}\\
&= C_{B,C}\,\rho_B(\mathbf{v}) && \text{Definition CBM}
\end{aligned}
$$

■

So the change-of-basis matrix can be used with matrix multiplication to convert a vector representation of a vector ($\mathbf{v}$) relative to one basis ($\rho_B(\mathbf{v})$) to a representation of the same vector relative to a second basis ($\rho_C(\mathbf{v})$).

Theorem ICBM Inverse of Change-of-Basis Matrix
Suppose that V is a vector space, and B and C are bases of V. Then the change-of-basis matrix $C_{B,C}$ is nonsingular and

$$C_{B,C}^{-1} = C_{C,B}$$

Proof. The linear transformation $I_V\colon V \to V$ is invertible, and its inverse is itself, I_V (check this!). So by Theorem IMR, the matrix $M_{B,C}^{I_V} = C_{B,C}$ is invertible. Theorem NI says an invertible matrix is nonsingular.

Then

$$\begin{aligned}
C_{B,C}^{-1} &= \left(M_{B,C}^{I_V}\right)^{-1} && \text{Definition CBM}\\
&= M_{C,B}^{I_V^{-1}} && \text{Theorem IMR}\\
&= M_{C,B}^{I_V} && \text{Definition IDLT}\\
&= C_{C,B} && \text{Definition CBM}
\end{aligned}$$

■

Example CBP Change of basis with polynomials
The vector space P_4 (Example VSP) has two nice bases (Example BP),

$$\begin{aligned}
B &= \left\{1, x, x^2, x^3, x^4\right\}\\
C &= \left\{1, 1+x, 1+x+x^2, 1+x+x^2+x^3, 1+x+x^2+x^3+x^4\right\}
\end{aligned}$$

To build the change-of-basis matrix between B and C, we must first build a vector representation of each vector in B relative to C,

$$\rho_C(1) = \rho_C((1)(1)) = \begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}$$

$$\rho_C(x) = \rho_C((-1)(1) + (1)(1+x)) = \begin{bmatrix}-1\\1\\0\\0\\0\end{bmatrix}$$

$$\rho_C\left(x^2\right) = \rho_C\left((-1)(1+x) + (1)\left(1+x+x^2\right)\right) = \begin{bmatrix}0\\-1\\1\\0\\0\end{bmatrix}$$

$$\rho_C\left(x^3\right) = \rho_C\left((-1)\left(1+x+x^2\right) + (1)\left(1+x+x^2+x^3\right)\right) = \begin{bmatrix}0\\0\\-1\\1\\0\end{bmatrix}$$

$$\rho_C\left(x^4\right) = \rho_C\left((-1)\left(1+x+x^2+x^3\right) + (1)\left(1+x+x^2+x^3+x^4\right)\right) = \begin{bmatrix}0\\0\\0\\-1\\1\end{bmatrix}$$

Then we package up these vectors as the columns of a matrix,

$$C_{B,C} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Now, to illustrate Theorem CB, consider the vector $\mathbf{u} = 5 - 3x + 2x^2 + 8x^3 - 3x^4$. We can build the representation of $\mathbf{u}$ relative to B easily,

$$\rho_B(\mathbf{u}) = \rho_B\left(5 - 3x + 2x^2 + 8x^3 - 3x^4\right) = \begin{bmatrix} 5 \\ -3 \\ 2 \\ 8 \\ -3 \end{bmatrix}$$

Applying Theorem CB, we obtain a second representation of $\mathbf{u}$, but now relative to C,

$$\begin{aligned} \rho_C(\mathbf{u}) &= C_{B,C}\,\rho_B(\mathbf{u}) && \text{Theorem CB} \\ &= \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \\ 2 \\ 8 \\ -3 \end{bmatrix} \\ &= \begin{bmatrix} 8 \\ -5 \\ -6 \\ 11 \\ -3 \end{bmatrix} && \text{Definition MVP} \end{aligned}$$

We can check our work by unraveling this second representation,

$$\begin{aligned} \mathbf{u} &= \rho_C^{-1}\left(\rho_C(\mathbf{u})\right) && \text{Definition IVLT} \\ &= \rho_C^{-1}\left(\begin{bmatrix} 8 \\ -5 \\ -6 \\ 11 \\ -3 \end{bmatrix}\right) \\ &= 8(1) + (-5)(1 + x) + (-6)(1 + x + x^2) \\ &\quad + (11)(1 + x + x^2 + x^3) + (-3)(1 + x + x^2 + x^3 + x^4) && \text{Definition VR} \\ &= 5 - 3x + 2x^2 + 8x^3 - 3x^4 \end{aligned}$$

The change-of-basis matrix from C to B is actually easier to build. Grab each vector in the basis C and form its representation relative to B

$$\rho_B(1) = \rho_B((1)1) = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rho_B(1 + x) = \rho_B((1)1 + (1)x) = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rho_B\left(1+x+x^2\right)=\rho_B\left((1)1+(1)x+(1)x^2\right)=\begin{bmatrix}1\\1\\1\\0\\0\end{bmatrix}$$

$$\rho_B\left(1+x+x^2+x^3\right)=\rho_B\left((1)1+(1)x+(1)x^2+(1)x^3\right)=\begin{bmatrix}1\\1\\1\\1\\0\end{bmatrix}$$

$$\rho_B\left(1+x+x^2+x^3+x^4\right)=\rho_B\left((1)1+(1)x+(1)x^2+(1)x^3+(1)x^4\right)=\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}$$

Then we package up these vectors as the columns of a matrix,

$$C_{C,B}=\begin{bmatrix}1&1&1&1&1\\0&1&1&1&1\\0&0&1&1&1\\0&0&0&1&1\\0&0&0&0&1\end{bmatrix}$$

We formed two representations of the vector $\mathbf{u}$ above, so we can again provide a check on our computations by converting from the representation of $\mathbf{u}$ relative to C to the representation of $\mathbf{u}$ relative to B,

$$\begin{aligned}\rho_B(\mathbf{u})&=C_{C,B}\rho_C(\mathbf{u}) && \text{Theorem CB}\\&=\begin{bmatrix}1&1&1&1&1\\0&1&1&1&1\\0&0&1&1&1\\0&0&0&1&1\\0&0&0&0&1\end{bmatrix}\begin{bmatrix}8\\-5\\-6\\11\\-3\end{bmatrix}\\&=\begin{bmatrix}5\\-3\\2\\8\\-3\end{bmatrix} && \text{Definition MVP}\end{aligned}$$

One more computation that is either a check on our work, or an illustration of a theorem. The two change-of-basis matrices, $C_{B,C}$ and $C_{C,B}$, should be inverses of each other, according to Theorem ICBM. Here we go,

$$C_{B,C}C_{C,B}=\begin{bmatrix}1&-1&0&0&0\\0&1&-1&0&0\\0&0&1&-1&0\\0&0&0&1&-1\\0&0&0&0&1\end{bmatrix}\begin{bmatrix}1&1&1&1&1\\0&1&1&1&1\\0&0&1&1&1\\0&0&0&1&1\\0&0&0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}$$

△

The computations of the previous example are not meant to present any labor-saving devices, but instead are meant to illustrate the *utility* of the change-of-basis matrix. However, you might have noticed that $C_{C,B}$ was easier to compute than $C_{B,C}$. If you needed $C_{B,C}$, then you could first compute $C_{C,B}$ and then compute its inverse, which by Theorem ICBM, would equal $C_{B,C}$.

Here is another illustrative example. We have been concentrating on working with abstract vector spaces, but all of our theorems and techniques apply just as well to $\mathbb{C}^m$, the vector space of column vectors. We only need to use more complicated bases than the standard unit vectors (Theorem SUVB) to make things interesting.

Example CBCV Change of basis with column vectors
For the vector space $\mathbb{C}^4$ we have the two bases,

$$B = \left\{ \begin{bmatrix}1\\-2\\1\\-2\end{bmatrix}, \begin{bmatrix}-1\\3\\1\\1\end{bmatrix}, \begin{bmatrix}2\\-3\\3\\-4\end{bmatrix}, \begin{bmatrix}-1\\3\\3\\0\end{bmatrix} \right\} \quad C = \left\{ \begin{bmatrix}1\\-6\\-4\\-1\end{bmatrix}, \begin{bmatrix}-4\\8\\-5\\8\end{bmatrix}, \begin{bmatrix}-5\\13\\-2\\9\end{bmatrix}, \begin{bmatrix}3\\-7\\3\\-6\end{bmatrix} \right\}$$

The change-of-basis matrix from B to C requires writing each vector of B as a linear combination the vectors in C,

$$\rho_C\left(\begin{bmatrix}1\\-2\\1\\-2\end{bmatrix}\right) = \rho_C\left((1)\begin{bmatrix}1\\-6\\-4\\-1\end{bmatrix} + (-2)\begin{bmatrix}-4\\8\\-5\\8\end{bmatrix} + (1)\begin{bmatrix}-5\\13\\-2\\9\end{bmatrix} + (-1)\begin{bmatrix}3\\-7\\3\\-6\end{bmatrix}\right) = \begin{bmatrix}1\\-2\\1\\-1\end{bmatrix}$$

$$\rho_C\left(\begin{bmatrix}-1\\3\\1\\1\end{bmatrix}\right) = \rho_C\left((2)\begin{bmatrix}1\\-6\\-4\\-1\end{bmatrix} + (-3)\begin{bmatrix}-4\\8\\-5\\8\end{bmatrix} + (3)\begin{bmatrix}-5\\13\\-2\\9\end{bmatrix} + (0)\begin{bmatrix}3\\-7\\3\\-6\end{bmatrix}\right) = \begin{bmatrix}2\\-3\\3\\0\end{bmatrix}$$

$$\rho_C\left(\begin{bmatrix}2\\-3\\3\\-4\end{bmatrix}\right) = \rho_C\left((1)\begin{bmatrix}1\\-6\\-4\\-1\end{bmatrix} + (-3)\begin{bmatrix}-4\\8\\-5\\8\end{bmatrix} + (1)\begin{bmatrix}-5\\13\\-2\\9\end{bmatrix} + (-2)\begin{bmatrix}3\\-7\\3\\-6\end{bmatrix}\right) = \begin{bmatrix}1\\-3\\1\\-2\end{bmatrix}$$

$$\rho_C\left(\begin{bmatrix}-1\\3\\3\\0\end{bmatrix}\right) = \rho_C\left((2)\begin{bmatrix}1\\-6\\-4\\-1\end{bmatrix} + (-2)\begin{bmatrix}-4\\8\\-5\\8\end{bmatrix} + (4)\begin{bmatrix}-5\\13\\-2\\9\end{bmatrix} + (3)\begin{bmatrix}3\\-7\\3\\-6\end{bmatrix}\right) = \begin{bmatrix}2\\-2\\4\\3\end{bmatrix}$$

Then we package these vectors up as the change-of-basis matrix,

$$C_{B,C} = \begin{bmatrix}1 & 2 & 1 & 2\\-2 & -3 & -3 & -2\\1 & 3 & 1 & 4\\-1 & 0 & -2 & 3\end{bmatrix}$$

Now consider a single (arbitrary) vector $\mathbf{y} = \begin{bmatrix}2\\6\\-3\\4\end{bmatrix}$. First, build the vector representation of $\mathbf{y}$ relative to B. This will require writing $\mathbf{y}$ as a linear combination of the vectors in B,

$$\begin{aligned}\rho_B(\mathbf{y}) &= \rho_B\left(\begin{bmatrix}2\\6\\-3\\4\end{bmatrix}\right)\\ &= \rho_B\left((-21)\begin{bmatrix}1\\-2\\1\\-2\end{bmatrix} + (6)\begin{bmatrix}-1\\3\\1\\1\end{bmatrix} + (11)\begin{bmatrix}2\\-3\\3\\-4\end{bmatrix} + (-7)\begin{bmatrix}-1\\3\\3\\0\end{bmatrix}\right) = \begin{bmatrix}-21\\6\\11\\-7\end{bmatrix}\end{aligned}$$

Now, applying Theorem CB we can convert the representation of $\mathbf{y}$ relative to B into a representation relative to C,

$$\rho_C(\mathbf{y}) = C_{B,C}\,\rho_B(\mathbf{y}) \qquad \text{Theorem CB}$$

$$= \begin{bmatrix} 1 & 2 & 1 & 2 \\ -2 & -3 & -3 & -2 \\ 1 & 3 & 1 & 4 \\ -1 & 0 & -2 & 3 \end{bmatrix} \begin{bmatrix} -21 \\ 6 \\ 11 \\ -7 \end{bmatrix}$$

$$= \begin{bmatrix} -12 \\ 5 \\ -20 \\ -22 \end{bmatrix} \qquad \text{Definition MVP}$$

We could continue further with this example, perhaps by computing the representation of $\mathbf{y}$ relative to the basis C directly as a check on our work (Exercise CB.C20). Or we could choose another vector to play the role of $\mathbf{y}$ and compute two different representations of this vector relative to the two bases B and C. △

Subsection MRS Matrix Representations and Similarity

Here is the main theorem of this section. It looks a bit involved at first glance, but the proof should make you realize it is not all that complicated. In any event, we are more interested in a special case.

Theorem MRCB Matrix Representation and Change of Basis
Suppose that $T\colon U \to V$ is a linear transformation, B and C are bases for U, and D and E are bases for V. Then

$$M^T_{B,D} = C_{E,D} M^T_{C,E} C_{B,C}$$

Proof.

$$\begin{aligned} C_{E,D} M^T_{C,E} C_{B,C} &= M^{I_V}_{E,D} M^T_{C,E} M^{I_U}_{B,C} && \text{Definition CBM} \\ &= M^{I_V}_{E,D} M^{T\circ I_U}_{B,E} && \text{Theorem MRCLT} \\ &= M^{I_V}_{E,D} M^T_{B,E} && \text{Definition IDLT} \\ &= M^{I_V\circ T}_{B,D} && \text{Theorem MRCLT} \\ &= M^T_{B,D} && \text{Definition IDLT} \end{aligned}$$

■

We will be most interested in a special case of this theorem (Theorem SCB), but here is an example that illustrates the full generality of Theorem MRCB.

Example MRCM Matrix representations and change-of-basis matrices
Begin with two vector spaces, S_2, the subspace of M_{22} containing all 2×2 symmetric matrices, and P_3 (Example VSP), the vector space of all polynomials of degree 3 or less. Then define the linear transformation $Q\colon S_2 \to P_3$ by

$$Q\left(\begin{bmatrix} a & b \\ b & c \end{bmatrix}\right) = (5a - 2b + 6c) + (3a - b + 2c)x + (a + 3b - c)x^2 + (-4a + 2b + c)x^3$$

Here are two bases for each vector space, one nice, one nasty. First for S_2,

$$B = \left\{ \begin{bmatrix} 5 & -3 \\ -3 & -2 \end{bmatrix}, \begin{bmatrix} 2 & -3 \\ -3 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \right\} \qquad C = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

and then for P_3,

$$\begin{aligned} D &= \left\{2 + x - 2x^2 + 3x^3,\ -1 - 2x^2 + 3x^3,\ -3 - x + x^3,\ -x^2 + x^3\right\} \\ E &= \left\{1,\ x,\ x^2,\ x^3\right\} \end{aligned}$$

We will begin with a matrix representation of Q relative to C and E. We first find vector representations of the elements of C relative to E,

$$\rho_E\left(Q\left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\right)\right) = \rho_E\left(5+3x+x^2-4x^3\right) = \begin{bmatrix}5\\3\\1\\-4\end{bmatrix}$$

$$\rho_E\left(Q\left(\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\right)\right) = \rho_E\left(-2-x+3x^2+2x^3\right) = \begin{bmatrix}-2\\-1\\3\\2\end{bmatrix}$$

$$\rho_E\left(Q\left(\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right)\right) = \rho_E\left(6+2x-x^2+x^3\right) = \begin{bmatrix}6\\2\\-1\\1\end{bmatrix}$$

So

$$M^Q_{C,E} = \begin{bmatrix}5 & -2 & 6\\3 & -1 & 2\\1 & 3 & -1\\-4 & 2 & 1\end{bmatrix}$$

Now we construct two change-of-basis matrices. First, $C_{B,C}$ requires vector representations of the elements of B, relative to C. Since C is a nice basis, this is straightforward,

$$\rho_C\left(\begin{bmatrix}5 & -3\\-3 & -2\end{bmatrix}\right) = \rho_C\left((5)\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + (-3)\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} + (-2)\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right) = \begin{bmatrix}5\\-3\\-2\end{bmatrix}$$

$$\rho_C\left(\begin{bmatrix}2 & -3\\-3 & 0\end{bmatrix}\right) = \rho_C\left((2)\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + (-3)\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} + (0)\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right) = \begin{bmatrix}2\\-3\\0\end{bmatrix}$$

$$\rho_C\left(\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}\right) = \rho_C\left((1)\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} + (2)\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} + (4)\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right) = \begin{bmatrix}1\\2\\4\end{bmatrix}$$

So

$$C_{B,C} = \begin{bmatrix}5 & 2 & 1\\-3 & -3 & 2\\-2 & 0 & 4\end{bmatrix}$$

The other change-of-basis matrix we will compute is $C_{E,D}$. However, since E is a nice basis (and D is not) we will turn it around and instead compute $C_{D,E}$ and apply Theorem ICBM to use an inverse to compute $C_{E,D}$.

$$\rho_E\left(2+x-2x^2+3x^3\right) = \rho_E\left((2)1+(1)x+(-2)x^2+(3)x^3\right) = \begin{bmatrix}2\\1\\-2\\3\end{bmatrix}$$

$$\rho_E\left(-1-2x^2+3x^3\right) = \rho_E\left((-1)1+(0)x+(-2)x^2+(3)x^3\right) = \begin{bmatrix}-1\\0\\-2\\3\end{bmatrix}$$

$$\rho_E\left(-3-x+x^3\right) = \rho_E\left((-3)1+(-1)x+(0)x^2+(1)x^3\right) = \begin{bmatrix}-3\\-1\\0\\1\end{bmatrix}$$

$$\rho_E\left(-x^2+x^3\right)=\rho_E\left((0)1+(0)x+(-1)x^2+(1)x^3\right)=\begin{bmatrix}0\\0\\-1\\1\end{bmatrix}$$

So, we can package these column vectors up as a matrix to obtain $C_{D,E}$ and then,

$$\begin{aligned}C_{E,D}&=\left(C_{D,E}\right)^{-1} && \text{Theorem ICBM}\\ &=\begin{bmatrix}2&-1&-3&0\\1&0&-1&0\\-2&-2&0&-1\\3&3&1&1\end{bmatrix}^{-1}\\ &=\begin{bmatrix}1&-2&1&1\\-2&5&-1&-1\\1&-3&1&1\\2&-6&-1&0\end{bmatrix}\end{aligned}$$

We are now in a position to apply Theorem MRCB. The matrix representation of Q relative to B and D can be obtained as follows,

$$\begin{aligned}M^Q_{B,D}&=C_{E,D}M^Q_{C,E}C_{B,C} && \text{Theorem MRCB}\\ &=\begin{bmatrix}1&-2&1&1\\-2&5&-1&-1\\1&-3&1&1\\2&-6&-1&0\end{bmatrix}\begin{bmatrix}5&-2&6\\3&-1&2\\1&3&-1\\-4&2&1\end{bmatrix}\begin{bmatrix}5&2&1\\-3&-3&2\\-2&0&4\end{bmatrix}\\ &=\begin{bmatrix}1&-2&1&1\\-2&5&-1&-1\\1&-3&1&1\\2&-6&-1&0\end{bmatrix}\begin{bmatrix}19&16&25\\14&9&9\\-2&-7&3\\-28&-14&4\end{bmatrix}\\ &=\begin{bmatrix}-39&-23&14\\62&34&-12\\-53&-32&5\\-44&-15&-7\end{bmatrix}\end{aligned}$$

Now check our work by computing $M^Q_{B,D}$ directly (Exercise CB.C21). △

Here is a special case of the previous theorem, where we choose U and V to be the same vector space, so the matrix representations and the change-of-basis matrices are all square of the same size.

Theorem SCB Similarity and Change of Basis
Suppose that $T\colon V\to V$ is a linear transformation and B and C are bases of V. Then

$$M^T_{B,B}=C^{-1}_{B,C}M^T_{C,C}C_{B,C}$$

Proof. In the conclusion of Theorem MRCB, replace D by B, and replace E by C,

$$\begin{aligned}M^T_{B,B}&=C_{C,B}M^T_{C,C}C_{B,C} && \text{Theorem MRCB}\\ &=C^{-1}_{B,C}M^T_{C,C}C_{B,C} && \text{Theorem ICBM}\end{aligned}$$

■

This is the third surprise of this chapter. Theorem SCB considers the special case where a linear transformation has the same vector space for the domain and codomain (V). We build a matrix representation of T using the basis B simultaneously for both the domain and codomain ($M^T_{B,B}$), and then we build a second matrix representation of T, now using the basis C for both the domain and codomain ($M^T_{C,C}$). Then these two representations are related via a similarity transformation (Definition SIM) using a change-of-basis matrix ($C_{B,C}$)!

Example MRBE Matrix representation with basis of eigenvectors
We return to the linear transformation $T\colon M_{22} \to M_{22}$ of Example ELTBM defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} -17a + 11b + 8c - 11d & -57a + 35b + 24c - 33d \\ -14a + 10b + 6c - 10d & -41a + 25b + 16c - 23d \end{bmatrix}$$

In Example ELTBM we showcased four eigenvectors of T. We will now put these four vectors in a set,

$$B = \{\mathbf{x}_1,\, \mathbf{x}_2,\, \mathbf{x}_3,\, \mathbf{x}_4\} = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 6 \\ 1 & 4 \end{bmatrix} \right\}$$

Check that B is a basis of M_{22} by first establishing the linear independence of B and then employing Theorem G to get the spanning property easily. Here is a second set of 2×2 matrices, which also forms a basis of M_{22} (Example BM),

$$C = \{\mathbf{y}_1,\, \mathbf{y}_2,\, \mathbf{y}_3,\, \mathbf{y}_4\} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

We can build two matrix representations of T, one relative to B and one relative to C. Each is easy, but for wildly different reasons. In our computation of the matrix representation relative to B we borrow some of our work in Example ELTBM. Here are the representations, then the explanation.

$$\rho_B\left(T\left(\mathbf{x}_1\right)\right) = \rho_B\left(2\mathbf{x}_1\right) = \rho_B\left(2\mathbf{x}_1 + 0\mathbf{x}_2 + 0\mathbf{x}_3 + 0\mathbf{x}_4\right) = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rho_B\left(T\left(\mathbf{x}_2\right)\right) = \rho_B\left(2\mathbf{x}_2\right) = \rho_B\left(0\mathbf{x}_1 + 2\mathbf{x}_2 + 0\mathbf{x}_3 + 0\mathbf{x}_4\right) = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$$\rho_B\left(T\left(\mathbf{x}_3\right)\right) = \rho_B\left((-1)\mathbf{x}_3\right) = \rho_B\left(0\mathbf{x}_1 + 0\mathbf{x}_2 + (-1)\mathbf{x}_3 + 0\mathbf{x}_4\right) = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$$\rho_B\left(T\left(\mathbf{x}_4\right)\right) = \rho_B\left((-2)\mathbf{x}_4\right) = \rho_B\left(0\mathbf{x}_1 + 0\mathbf{x}_2 + 0\mathbf{x}_3 + (-2)\mathbf{x}_4\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \end{bmatrix}$$

So the resulting representation is

$$M^{T}_{B,B} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

Very pretty.

Now for the matrix representation relative to C first compute,

$$\rho_C\left(T\left(\mathbf{y}_1\right)\right) = \rho_C\left(\begin{bmatrix} -17 & -57 \\ -14 & -41 \end{bmatrix}\right)$$

$$= \rho_C\left((-17)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (-57)\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + (-14)\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + (-41)\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} -17 \\ -57 \\ -14 \\ -41 \end{bmatrix}$$

$$\rho_C\left(T\left(\mathbf{y}_2\right)\right) = \rho_C\left(\begin{bmatrix}11 & 35\\ 10 & 25\end{bmatrix}\right)$$

$$= \rho_C\left(11\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + 35\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} + 10\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix} + 25\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right) = \begin{bmatrix}11\\ 35\\ 10\\ 25\end{bmatrix}$$

$$\rho_C\left(T\left(\mathbf{y}_3\right)\right) = \rho_C\left(\begin{bmatrix}8 & 24\\ 6 & 16\end{bmatrix}\right)$$

$$= \rho_C\left(8\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + 24\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} + 6\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix} + 16\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right) = \begin{bmatrix}8\\ 24\\ 6\\ 16\end{bmatrix}$$

$$\rho_C\left(T\left(\mathbf{y}_4\right)\right) = \rho_C\left(\begin{bmatrix}-11 & -33\\ -10 & -23\end{bmatrix}\right)$$

$$= \rho_C\left((-11)\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + (-33)\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} + (-10)\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix} + (-23)\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right) = \begin{bmatrix}-11\\ -33\\ -10\\ -23\end{bmatrix}$$

So the resulting representation is

$$M^T_{C,C} = \begin{bmatrix}-17 & 11 & 8 & -11\\ -57 & 35 & 24 & -33\\ -14 & 10 & 6 & -10\\ -41 & 25 & 16 & -23\end{bmatrix}$$

Not quite as pretty.

The purpose of this example is to illustrate Theorem SCB. This theorem says that the two matrix representations, $M^T_{B,B}$ and $M^T_{C,C}$, of the one linear transformation, T, are related by a similarity transformation using the change-of-basis matrix $C_{B,C}$. Let us compute this change-of-basis matrix. Notice that since C is such a nice basis, this is fairly straightforward,

$$\rho_C\left(\mathbf{x}_1\right) = \rho_C\left(\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}\right) = \rho_C\left(0\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + 1\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} + 0\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix} + 1\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right) = \begin{bmatrix}0\\ 1\\ 0\\ 1\end{bmatrix}$$

$$\rho_C\left(\mathbf{x}_2\right) = \rho_C\left(\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}\right) = \rho_C\left(1\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + 1\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} + 1\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix} + 0\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right) = \begin{bmatrix}1\\ 1\\ 1\\ 0\end{bmatrix}$$

$$\rho_C\left(\mathbf{x}_3\right) = \rho_C\left(\begin{bmatrix}1 & 3\\ 2 & 3\end{bmatrix}\right) = \rho_C\left(1\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + 3\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} + 2\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix} + 3\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right) = \begin{bmatrix}1\\ 3\\ 2\\ 3\end{bmatrix}$$

$$\rho_C\left(\mathbf{x}_4\right) = \rho_C\left(\begin{bmatrix}2 & 6\\ 1 & 4\end{bmatrix}\right) = \rho_C\left(2\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + 6\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} + 1\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix} + 4\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\right) = \begin{bmatrix}2\\ 6\\ 1\\ 4\end{bmatrix}$$

So we have,

$$C_{B,C} = \begin{bmatrix}0 & 1 & 1 & 2\\ 1 & 1 & 3 & 6\\ 0 & 1 & 2 & 1\\ 1 & 0 & 3 & 4\end{bmatrix}$$

Now, according to Theorem SCB we can write,

$$M^T_{B,B} = C^{-1}_{B,C} M^T_{C,C} C_{B,C}$$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 1 & 3 & 6 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 3 & 4 \end{bmatrix}^{-1} \begin{bmatrix} -17 & 11 & 8 & -11 \\ -57 & 35 & 24 & -33 \\ -14 & 10 & 6 & -10 \\ -41 & 25 & 16 & -23 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 1 & 3 & 6 \\ 0 & 1 & 2 & 1 \\ 1 & 0 & 3 & 4 \end{bmatrix}$$

This should look and feel exactly like the process for diagonalizing a matrix, as was described in Section SD. And it is. △

We can now return to the question of computing an eigenvalue or eigenvector of a linear transformation. For a linear transformation of the form $T\colon V \to V$, we know that representations relative to different bases are similar matrices. We also know that similar matrices have equal characteristic polynomials by Theorem SMEE. We will now show that eigenvalues of a linear transformation T are precisely the eigenvalues of *any* matrix representation of T. Since the choice of a different matrix representation leads to a similar matrix, there will be no “new” eigenvalues obtained from this second representation. Similarly, the change-of-basis matrix can be used to show that eigenvectors obtained from one matrix representation will be precisely those obtained from any other representation. So we can determine the eigenvalues and eigenvectors of a linear transformation by forming one matrix representation, using *any* basis we please, and analyzing the matrix in the manner of Chapter E.

Theorem EER Eigenvalues, Eigenvectors, Representations
Suppose that $T\colon V \to V$ is a linear transformation and B is a basis of V. Then $\mathbf{v} \in V$ is an eigenvector of T for the eigenvalue λ if and only if $\rho_B(\mathbf{v})$ is an eigenvector of $M^T_{B,B}$ for the eigenvalue λ.

Proof. ($\Rightarrow$) Assume that $\mathbf{v} \in V$ is an eigenvector of T for the eigenvalue λ. Then

$$\begin{aligned} M^T_{B,B}\rho_B(\mathbf{v}) &= \rho_B(T(\mathbf{v})) && \text{Theorem FTMR} \\ &= \rho_B(\lambda\mathbf{v}) && \text{Definition EELT} \\ &= \lambda\rho_B(\mathbf{v}) && \text{Theorem VRLT} \end{aligned}$$

which by Definition EEM says that $\rho_B(\mathbf{v})$ is an eigenvector of the matrix $M^T_{B,B}$ for the eigenvalue λ.

($\Leftarrow$) Assume that $\rho_B(\mathbf{v})$ is an eigenvector of $M^T_{B,B}$ for the eigenvalue λ. Then

$$\begin{aligned} T(\mathbf{v}) &= \rho_B^{-1}(\rho_B(T(\mathbf{v}))) && \text{Definition IVLT} \\ &= \rho_B^{-1}\left(M^T_{B,B}\rho_B(\mathbf{v})\right) && \text{Theorem FTMR} \\ &= \rho_B^{-1}(\lambda\rho_B(\mathbf{v})) && \text{Definition EEM} \\ &= \lambda\rho_B^{-1}(\rho_B(\mathbf{v})) && \text{Theorem ILTLT} \\ &= \lambda\mathbf{v} && \text{Definition IVLT} \end{aligned}$$

which by Definition EELT says $\mathbf{v}$ is an eigenvector of T for the eigenvalue λ. ■

Subsection CELT
Computing Eigenvectors of Linear Transformations

Theorem EER tells us that the eigenvalues of a linear transformation are the eigenvalues of *any* representation, no matter what the choice of the basis B might be. So we could now unambiguously define items such as the characteristic polynomial of a linear transformation, which we would define as the characteristic polynomial of any matrix representation. We will say that again — eigenvalues, eigenvectors, and characteristic polynomials are intrinsic properties of a linear transformation, independent of the choice of a basis used to construct a matrix representation.

As a practical matter, how does one compute the eigenvalues and eigenvectors of a linear transformation of the form $T\colon V \to V$? Choose a nice basis B for V, one where the vector representations of the values of the linear transformations necessary for the matrix representation are easy to compute. Construct the matrix representation relative to this basis, and find the eigenvalues and eigenvectors of this matrix using the techniques of Chapter E. The resulting eigenvalues of the matrix are precisely the eigenvalues of the linear transformation. The eigenvectors of the matrix are column vectors that need to be converted to vectors in V through application of ρ_B^{-1} (this is part of the content of Theorem EER).

Now consider the case where the matrix representation of a linear transformation is diagonalizable. The n linearly independent eigenvectors that must exist for the matrix (Theorem DC) can be converted (via ρ_B^{-1}) into eigenvectors of the linear transformation. A matrix representation of the linear transformation relative to a basis of eigenvectors will be a diagonal matrix — an especially nice representation! Though we did not know it at the time, the diagonalizations of Section SD were really about finding especially pleasing matrix representations of linear transformations.

Here are some examples.

Example ELTT Eigenvectors of a linear transformation, twice
Consider the linear transformation $S\colon M_{22} \to M_{22}$ defined by

$$S\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} -b-c-3d & -14a-15b-13c+d \\ 18a+21b+19c+3d & -6a-7b-7c-3d \end{bmatrix}$$

To find the eigenvalues and eigenvectors of S we will build a matrix representation and analyze the matrix. Since Theorem EER places no restriction on the choice of the basis B, we may as well use a basis that is easy to work with. So set

$$B = \{\mathbf{x}_1,\,\mathbf{x}_2,\,\mathbf{x}_3,\,\mathbf{x}_4\} = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\,\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\,\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\,\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right\}$$

Then to build the matrix representation of S relative to B compute,

$$\begin{aligned}
\rho_B\left(S\left(\mathbf{x}_1\right)\right) &= \rho_B\left(\begin{bmatrix} 0 & -14 \\ 18 & -6 \end{bmatrix}\right) \\
&= \rho_B\left(0\mathbf{x}_1 + (-14)\mathbf{x}_2 + 18\mathbf{x}_3 + (-6)\mathbf{x}_4\right) = \begin{bmatrix} 0 \\ -14 \\ 18 \\ -6 \end{bmatrix} \\
\rho_B\left(S\left(\mathbf{x}_2\right)\right) &= \rho_B\left(\begin{bmatrix} -1 & -15 \\ 21 & -7 \end{bmatrix}\right) \\
&= \rho_B\left((-1)\mathbf{x}_1 + (-15)\mathbf{x}_2 + 21\mathbf{x}_3 + (-7)\mathbf{x}_4\right) = \begin{bmatrix} -1 \\ -15 \\ 21 \\ -7 \end{bmatrix} \\
\rho_B\left(S\left(\mathbf{x}_3\right)\right) &= \rho_B\left(\begin{bmatrix} -1 & -13 \\ 19 & -7 \end{bmatrix}\right) \\
&= \rho_B\left((-1)\mathbf{x}_1 + (-13)\mathbf{x}_2 + 19\mathbf{x}_3 + (-7)\mathbf{x}_4\right) = \begin{bmatrix} -1 \\ -13 \\ 19 \\ -7 \end{bmatrix} \\
\rho_B\left(S\left(\mathbf{x}_4\right)\right) &= \rho_B\left(\begin{bmatrix} -3 & 1 \\ 3 & -3 \end{bmatrix}\right) \\
&= \rho_B\left((-3)\mathbf{x}_1 + 1\mathbf{x}_2 + 3\mathbf{x}_3 + (-3)\mathbf{x}_4\right) = \begin{bmatrix} -3 \\ 1 \\ 3 \\ -3 \end{bmatrix}
\end{aligned}$$

So by Definition MR we have

$$M = M^S_{B,B} = \begin{bmatrix} 0 & -1 & -1 & -3 \\ -14 & -15 & -13 & 1 \\ 18 & 21 & 19 & 3 \\ -6 & -7 & -7 & -3 \end{bmatrix}$$

Now compute eigenvalues and eigenvectors of the matrix representation of M with the techniques of Section EE. First the characteristic polynomial,

$$p_M(x) = \det(M - xI_4) = x^4 - x^3 - 10x^2 + 4x + 24 = (x-3)(x-2)(x+2)^2$$

We could now make statements about the eigenvalues of M, but in light of Theorem EER we can refer to the eigenvalues of S and mildly abuse (or extend) our notation for multiplicities to write

$$\alpha_S(3) = 1 \qquad \alpha_S(2) = 1 \qquad \alpha_S(-2) = 2$$

Now compute the eigenvectors of M,

$$\lambda = 3 \qquad M - 3I_4 = \begin{bmatrix} -3 & -1 & -1 & -3 \\ -14 & -18 & -13 & 1 \\ 18 & 21 & 16 & 3 \\ -6 & -7 & -7 & -6 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 1 \\ 0 & \boxed{1} & 0 & -3 \\ 0 & 0 & \boxed{1} & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{E}_M(3) = \mathcal{N}(M - 3I_4) = \left\langle \left\{ \begin{bmatrix} -1 \\ 3 \\ -3 \\ 1 \end{bmatrix} \right\} \right\rangle$$

$$\lambda = 2 \qquad M - 2I_4 = \begin{bmatrix} -2 & -1 & -1 & -3 \\ -14 & -17 & -13 & 1 \\ 18 & 21 & 17 & 3 \\ -6 & -7 & -7 & -5 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & 2 \\ 0 & \boxed{1} & 0 & -4 \\ 0 & 0 & \boxed{1} & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{E}_M(2) = \mathcal{N}(M - 2I_4) = \left\langle \left\{ \begin{bmatrix} -2 \\ 4 \\ -3 \\ 1 \end{bmatrix} \right\} \right\rangle$$

$$\lambda = -2 \qquad M - (-2)I_4 = \begin{bmatrix} 2 & -1 & -1 & -3 \\ -14 & -13 & -13 & 1 \\ 18 & 21 & 21 & 3 \\ -6 & -7 & -7 & -1 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} \boxed{1} & 0 & 0 & -1 \\ 0 & \boxed{1} & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{E}_M(-2) = \mathcal{N}(M - (-2)I_4) = \left\langle \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\} \right\rangle$$

According to Theorem EER the eigenvectors just listed as basis vectors for the eigenspaces of M are vector representations (relative to B) of eigenvectors for S. So the application if the inverse function ρ_B^{-1} will convert these column vectors into elements of the vector space M_{22} (2×2 matrices) that are eigenvectors of S. Since ρ_B is an isomorphism (Theorem VRILT), so is ρ_B^{-1}. Applying the inverse function will then preserve linear independence and spanning properties, so with a sweeping application of the Coordinatization Principle and some extensions of our previous notation for eigenspaces and geometric multiplicities, we can write,

$$\rho_B^{-1}\left(\begin{bmatrix} -1 \\ 3 \\ -3 \\ 1 \end{bmatrix} \right) = (-1)\mathbf{x}_1 + 3\mathbf{x}_2 + (-3)\mathbf{x}_3 + 1\mathbf{x}_4 = \begin{bmatrix} -1 & 3 \\ -3 & 1 \end{bmatrix}$$

$$\rho_B^{-1}\left(\begin{bmatrix}-2\\4\\-3\\1\end{bmatrix}\right) = (-2)\mathbf{x}_1 + 4\mathbf{x}_2 + (-3)\mathbf{x}_3 + 1\mathbf{x}_4 = \begin{bmatrix}-2 & 4\\-3 & 1\end{bmatrix}$$
$$\rho_B^{-1}\left(\begin{bmatrix}0\\-1\\1\\0\end{bmatrix}\right) = 0\mathbf{x}_1 + (-1)\mathbf{x}_2 + 1\mathbf{x}_3 + 0\mathbf{x}_4 = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$$
$$\rho_B^{-1}\left(\begin{bmatrix}1\\-1\\0\\1\end{bmatrix}\right) = 1\mathbf{x}_1 + (-1)\mathbf{x}_2 + 0\mathbf{x}_3 + 1\mathbf{x}_4 = \begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}$$

So

$$\mathcal{E}_S(3) = \left\langle\left\{\begin{bmatrix}-1 & 3\\-3 & 1\end{bmatrix}\right\}\right\rangle$$
$$\mathcal{E}_S(2) = \left\langle\left\{\begin{bmatrix}-2 & 4\\-3 & 1\end{bmatrix}\right\}\right\rangle$$
$$\mathcal{E}_S(-2) = \left\langle\left\{\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}, \begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}\right\}\right\rangle$$

with geometric multiplicities given by

$$\gamma_S(3) = 1 \qquad \gamma_S(2) = 1 \qquad \gamma_S(-2) = 2$$

Suppose we now decided to build another matrix representation of S, only now relative to a linearly independent set of eigenvectors of S, such as

$$C = \left\{\begin{bmatrix}-1 & 3\\-3 & 1\end{bmatrix}, \begin{bmatrix}-2 & 4\\-3 & 1\end{bmatrix}, \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}, \begin{bmatrix}1 & -1\\0 & 1\end{bmatrix}\right\}$$

At this point you should have computed enough matrix representations to predict that the result of representing S relative to C will be a diagonal matrix. Computing this representation is an example of how Theorem SCB generalizes the diagonalizations from Section SD. For the record, here is the diagonal representation,

$$M^S_{C,C} = \begin{bmatrix}3 & 0 & 0 & 0\\0 & 2 & 0 & 0\\0 & 0 & -2 & 0\\0 & 0 & 0 & -2\end{bmatrix}$$

Our interest in this example is not necessarily building nice representations, but instead we want to demonstrate how eigenvalues and eigenvectors are an intrinsic property of a linear transformation, independent of any particular representation. To this end, we will repeat the foregoing, but replace B by another basis. We will make this basis different, but not extremely so,

$$D = \{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4\} = \left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}1 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}, \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}\right\}$$

Then to build the matrix representation of S relative to D compute,

$$\rho_D(S(\mathbf{y}_1)) = \rho_D\left(\begin{bmatrix}0 & -14\\18 & -6\end{bmatrix}\right)$$
$$= \rho_D(14\mathbf{y}_1 + (-32)\mathbf{y}_2 + 24\mathbf{y}_3 + (-6)\mathbf{y}_4) = \begin{bmatrix}14\\-32\\24\\-6\end{bmatrix}$$

$$\rho_D\left(S\left(\mathbf{y}_2\right)\right) = \rho_D\left(\begin{bmatrix} -1 & -29 \\ 39 & -13 \end{bmatrix}\right)$$
$$= \rho_D\left(28\mathbf{y}_1 + (-68)\mathbf{y}_2 + 52\mathbf{y}_3 + (-13)\mathbf{y}_4\right) = \begin{bmatrix} 28 \\ -68 \\ 52 \\ -13 \end{bmatrix}$$
$$\rho_D\left(S\left(\mathbf{y}_3\right)\right) = \rho_D\left(\begin{bmatrix} -2 & -42 \\ 58 & -20 \end{bmatrix}\right)$$
$$= \rho_D\left(40\mathbf{y}_1 + (-100)\mathbf{y}_2 + 78\mathbf{y}_3 + (-20)\mathbf{y}_4\right) = \begin{bmatrix} 40 \\ -100 \\ 78 \\ -20 \end{bmatrix}$$
$$\rho_D\left(S\left(\mathbf{y}_4\right)\right) = \rho_D\left(\begin{bmatrix} -5 & -41 \\ 61 & -23 \end{bmatrix}\right)$$
$$= \rho_D\left(36\mathbf{y}_1 + (-102)\mathbf{y}_2 + 84\mathbf{y}_3 + (-23)\mathbf{y}_4\right) = \begin{bmatrix} 36 \\ -102 \\ 84 \\ -23 \end{bmatrix}$$

So by Definition MR we have

$$N = M^S_{D,D} = \begin{bmatrix} 14 & 28 & 40 & 36 \\ -32 & -68 & -100 & -102 \\ 24 & 52 & 78 & 84 \\ -6 & -13 & -20 & -23 \end{bmatrix}$$

Now compute eigenvalues and eigenvectors of the matrix representation of N with the techniques of Section EE. First the characteristic polynomial,

$$p_N\left(x\right) = \det\left(N - xI_4\right) = x^4 - x^3 - 10x^2 + 4x + 24 = (x-3)(x-2)(x+2)^2$$

Of course this is not news. We now know that $M = M^S_{B,B}$ and $N = M^S_{D,D}$ are similar matrices (Theorem SCB). But Theorem SMEE told us long ago that similar matrices have identical characteristic polynomials. Now compute eigenvectors for the matrix representation, which will be different than what we found for M,

$$\lambda = 3 \qquad N - 3I_4 = \begin{bmatrix} 11 & 28 & 40 & 36 \\ -32 & -71 & -100 & -102 \\ 24 & 52 & 75 & 84 \\ -6 & -13 & -20 & -26 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\mathcal{E}_N\left(3\right) = \mathcal{N}(N - 3I_4) = \left\langle\left\{\begin{bmatrix} -4 \\ 6 \\ -4 \\ 1 \end{bmatrix}\right\}\right\rangle$$

$$\lambda = 2 \qquad N - 2I_4 = \begin{bmatrix} 12 & 28 & 40 & 36 \\ -32 & -70 & -100 & -102 \\ 24 & 52 & 76 & 84 \\ -6 & -13 & -20 & -25 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 6 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\mathcal{E}_N\left(2\right) = \mathcal{N}(N - 2I_4) = \left\langle\left\{\begin{bmatrix} -6 \\ 7 \\ -4 \\ 1 \end{bmatrix}\right\}\right\rangle$$

$$\lambda = -2 \quad N - (-2)I_4 = \begin{bmatrix} 16 & 28 & 40 & 36 \\ -32 & -66 & -100 & -102 \\ 24 & 52 & 80 & 84 \\ -6 & -13 & -20 & -21 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & -1 & -3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{E}_N(-2) = \mathcal{N}(N - (-2)I_4) = \left\langle\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}\right\rangle$$

Employing Theorem EER we can apply ρ_D^{-1} to each of the basis vectors of the eigenspaces of N to obtain eigenvectors for S that also form bases for eigenspaces of S,

$$\rho_D^{-1}\left(\begin{bmatrix} -4 \\ 6 \\ -4 \\ 1 \end{bmatrix}\right) = (-4)\mathbf{y}_1 + 6\mathbf{y}_2 + (-4)\mathbf{y}_3 + 1\mathbf{y}_4 = \begin{bmatrix} -1 & 3 \\ -3 & 1 \end{bmatrix}$$

$$\rho_D^{-1}\left(\begin{bmatrix} -6 \\ 7 \\ -4 \\ 1 \end{bmatrix}\right) = (-6)\mathbf{y}_1 + 7\mathbf{y}_2 + (-4)\mathbf{y}_3 + 1\mathbf{y}_4 = \begin{bmatrix} -2 & 4 \\ -3 & 1 \end{bmatrix}$$

$$\rho_D^{-1}\left(\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}\right) = 1\mathbf{y}_1 + (-2)\mathbf{y}_2 + 1\mathbf{y}_3 + 0\mathbf{y}_4 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\rho_D^{-1}\left(\begin{bmatrix} 3 \\ -3 \\ 0 \\ 1 \end{bmatrix}\right) = 3\mathbf{y}_1 + (-3)\mathbf{y}_2 + 0\mathbf{y}_3 + 1\mathbf{y}_4 = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$$

The eigenspaces for the eigenvalues of algebraic multiplicity 1 are exactly as before,

$$\mathcal{E}_S(3) = \left\langle\left\{ \begin{bmatrix} -1 & 3 \\ -3 & 1 \end{bmatrix} \right\}\right\rangle$$

$$\mathcal{E}_S(2) = \left\langle\left\{ \begin{bmatrix} -2 & 4 \\ -3 & 1 \end{bmatrix} \right\}\right\rangle$$

However, the eigenspace for $\lambda = -2$ would at first glance appear to be different. Here are the two eigenspaces for $\lambda = -2$, first the eigenspace obtained from $M = M^S_{B,B}$, then followed by the eigenspace obtained from $M = M^S_{D,D}$.

$$\mathcal{E}_S(-2) = \left\langle\left\{ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \right\}\right\rangle \qquad \mathcal{E}_S(-2) = \left\langle\left\{ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \right\}\right\rangle$$

Subspaces generally have many bases, and that is the situation here. With a careful proof of set equality, you can show that these two eigenspaces are equal sets. The key observation to make such a proof go is that

$$\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

which will establish that the second set is a subset of the first. With equal dimensions, Theorem EDYES will finish the task.

So the eigenvalues of a linear transformation are independent of the matrix representation employed to compute them! △

Another example, this time a bit larger and with complex eigenvalues.

Example CELT Complex eigenvectors of a linear transformation

Consider the linear transformation $Q\colon P_4 \to P_4$ defined by

$$\begin{aligned}
&Q\left(a + bx + cx^2 + dx^3 + ex^4\right)\\
&= (-46a - 22b + 13c + 5d + e) + (117a + 57b - 32c - 15d - 4e)x +\\
&\quad (-69a - 29b + 21c - 7e)x^2 + (159a + 73b - 44c - 13d + 2e)x^3 +\\
&\quad (-195a - 87b + 55c + 10d - 13e)x^4
\end{aligned}$$

Choose a simple basis to compute with, say

$$B = \left\{1,\, x,\, x^2,\, x^3,\, x^4\right\}$$

Then it should be apparent that the matrix representation of Q relative to B is

$$M = M^{Q}_{B,B} = \begin{bmatrix} -46 & -22 & 13 & 5 & 1 \\ 117 & 57 & -32 & -15 & -4 \\ -69 & -29 & 21 & 0 & -7 \\ 159 & 73 & -44 & -13 & 2 \\ -195 & -87 & 55 & 10 & -13 \end{bmatrix}$$

Compute the characteristic polynomial, eigenvalues and eigenvectors according to the techniques of Section EE,

$$\begin{aligned}
p_Q(x) &= -x^5 + 6x^4 - x^3 - 88x^2 + 252x - 208\\
&= -(x-2)^2(x+4)\left(x^2 - 6x + 13\right)\\
&= -(x-2)^2(x+4)\left(x - (3+2i)\right)\left(x - (3-2i)\right)
\end{aligned}$$

$$\alpha_Q(2) = 2 \qquad \alpha_Q(-4) = 1 \qquad \alpha_Q(3+2i) = 1 \qquad \alpha_Q(3-2i) = 1$$

$$\lambda = 2$$

$$M - (2)I_5 = \begin{bmatrix} -48 & -22 & 13 & 5 & 1 \\ 117 & 55 & -32 & -15 & -4 \\ -69 & -29 & 19 & 0 & -7 \\ 159 & 73 & -44 & -15 & 2 \\ -195 & -87 & 55 & 10 & -15 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & -\frac{5}{2} & -\frac{5}{2} \\ 0 & 0 & 1 & -2 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{E}_M(2) = \mathcal{N}(M - (2)I_5) = \left\langle\left\{\begin{bmatrix} -\frac{1}{2} \\ \frac{5}{2} \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ \frac{5}{2} \\ 6 \\ 0 \\ 1 \end{bmatrix}\right\}\right\rangle = \left\langle\left\{\begin{bmatrix} -1 \\ 5 \\ 4 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 12 \\ 0 \\ 2 \end{bmatrix}\right\}\right\rangle$$

$$\lambda = -4$$

$$M - (-4)I_5 = \begin{bmatrix} -42 & -22 & 13 & 5 & 1 \\ 117 & 61 & -32 & -15 & -4 \\ -69 & -29 & 25 & 0 & -7 \\ 159 & 73 & -44 & -9 & 2 \\ -195 & -87 & 55 & 10 & -9 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{E}_M(-4) = \mathcal{N}(M - (-4)I_5) = \left\langle\left\{\begin{bmatrix} -1 \\ 3 \\ 1 \\ 2 \\ 1 \end{bmatrix}\right\}\right\rangle$$

$$\lambda = 3 + 2i$$

$$M - (3+2i)I_5 = \begin{bmatrix} -49-2i & -22 & 13 & 5 & 1 \\ 117 & 54-2i & -32 & -15 & -4 \\ -69 & -29 & 18-2i & 0 & -7 \\ 159 & 73 & -44 & -16-2i & 2 \\ -195 & -87 & 55 & 10 & -16-2i \end{bmatrix}$$

$$\xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{3}{4}+\frac{i}{4} \\ 0 & 1 & 0 & 0 & \frac{7}{4}-\frac{i}{4} \\ 0 & 0 & 1 & 0 & -\frac{1}{2}+\frac{i}{2} \\ 0 & 0 & 0 & 1 & \frac{7}{4}-\frac{i}{4} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{E}_M(3+2i) = \mathcal{N}(M-(3+2i)I_5) = \left\langle\left\{\begin{bmatrix} \frac{3}{4}-\frac{i}{4} \\ -\frac{7}{4}+\frac{i}{4} \\ \frac{1}{2}-\frac{i}{2} \\ -\frac{7}{4}+\frac{i}{4} \\ 1 \end{bmatrix}\right\}\right\rangle = \left\langle\left\{\begin{bmatrix} 3-i \\ -7+i \\ 2-2i \\ -7+i \\ 4 \end{bmatrix}\right\}\right\rangle$$

$$\lambda = 3-2i$$

$$M - (3-2i)I_5 = \begin{bmatrix} -49+2i & -22 & 13 & 5 & 1 \\ 117 & 54+2i & -32 & -15 & -4 \\ -69 & -29 & 18+2i & 0 & -7 \\ 159 & 73 & -44 & -16+2i & 2 \\ -195 & -87 & 55 & 10 & -16+2i \end{bmatrix}$$

$$\xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{3}{4}-\frac{i}{4} \\ 0 & 1 & 0 & 0 & \frac{7}{4}+\frac{i}{4} \\ 0 & 0 & 1 & 0 & -\frac{1}{2}-\frac{i}{2} \\ 0 & 0 & 0 & 1 & \frac{7}{4}+\frac{i}{4} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathcal{E}_M(3-2i) = \mathcal{N}(M-(3-2i)I_5) = \left\langle\left\{\begin{bmatrix} \frac{3}{4}+\frac{i}{4} \\ -\frac{7}{4}-\frac{i}{4} \\ \frac{1}{2}+\frac{i}{2} \\ -\frac{7}{4}-\frac{i}{4} \\ 1 \end{bmatrix}\right\}\right\rangle = \left\langle\left\{\begin{bmatrix} 3+i \\ -7-i \\ 2+2i \\ -7-i \\ 4 \end{bmatrix}\right\}\right\rangle$$

It is straightforward to convert each of these basis vectors for eigenspaces of M back to elements of P_4 by applying the isomorphism ρ_B^{-1},

$$\rho_B^{-1}\left(\begin{bmatrix} -1 \\ 5 \\ 4 \\ 2 \\ 0 \end{bmatrix}\right) = -1 + 5x + 4x^2 + 2x^3$$

$$\rho_B^{-1}\left(\begin{bmatrix} 1 \\ 5 \\ 12 \\ 0 \\ 2 \end{bmatrix}\right) = 1 + 5x + 12x^2 + 2x^4$$

$$\rho_B^{-1}\left(\begin{bmatrix} -1 \\ 3 \\ 1 \\ 2 \\ 1 \end{bmatrix}\right) = -1 + 3x + x^2 + 2x^3 + x^4$$

$$\rho_B^{-1}\left(\begin{bmatrix}3-i\\-7+i\\2-2i\\-7+i\\4\end{bmatrix}\right) = (3-i)+(-7+i)x+(2-2i)x^2+(-7+i)x^3+4x^4$$

$$\rho_B^{-1}\left(\begin{bmatrix}3+i\\-7-i\\2+2i\\-7-i\\4\end{bmatrix}\right) = (3+i)+(-7-i)x+(2+2i)x^2+(-7-i)x^3+4x^4$$

So we apply Theorem EER and the Coordinatization Principle to get the eigenspaces for Q,

$$\begin{aligned}
\mathcal{E}_Q(2) &= \left\langle\left\{-1+5x+4x^2+2x^3,\ 1+5x+12x^2+2x^4\right\}\right\rangle\\
\mathcal{E}_Q(-4) &= \left\langle\left\{-1+3x+x^2+2x^3+x^4\right\}\right\rangle\\
\mathcal{E}_Q(3+2i) &= \left\langle\left\{(3-i)+(-7+i)x+(2-2i)x^2+(-7+i)x^3+4x^4\right\}\right\rangle\\
\mathcal{E}_Q(3-2i) &= \left\langle\left\{(3+i)+(-7-i)x+(2+2i)x^2+(-7-i)x^3+4x^4\right\}\right\rangle
\end{aligned}$$

with geometric multiplicities

$$\gamma_Q(2)=2 \qquad \gamma_Q(-4)=1 \qquad \gamma_Q(3+2i)=1 \qquad \gamma_Q(3-2i)=1$$

△

Reading Questions

1. The change-of-basis matrix is a matrix representation of which linear transformation?
2. Find the change-of-basis matrix, $C_{B,C}$, for the two bases of $\mathbb{C}^2$

$$B=\left\{\begin{bmatrix}2\\3\end{bmatrix},\ \begin{bmatrix}-1\\2\end{bmatrix}\right\} \qquad C=\left\{\begin{bmatrix}1\\0\end{bmatrix},\ \begin{bmatrix}1\\1\end{bmatrix}\right\}$$

3. What is the third "surprise," and why is it surprising?

Exercises

C20 In Example CBCV we computed the vector representation of $\mathbf{y}$ relative to C, $\rho_C(\mathbf{y})$, as an example of Theorem CB. Compute this same representation directly. In other words, apply Definition VR rather than Theorem CB.

C21† Perform a check on Example MRCM by computing $M^Q_{B,D}$ directly. In other words, apply Definition MR rather than Theorem MRCB.

C30† Find a basis for the vector space P_3 composed of eigenvectors of the linear transformation T. Then find a matrix representation of T relative to this basis.

$$T\colon P_3 \to P_3,\quad T\left(a+bx+cx^2+dx^3\right) = (a+c+d)+(b+c+d)x+(a+b+c)x^2+(a+b+d)x^3$$

C40† Let S_{22} be the vector space of 2×2 symmetric matrices. Find a basis C for S_{22} that yields a diagonal matrix representation of the linear transformation R.

$$R\colon S_{22}\to S_{22},\quad R\left(\begin{bmatrix}a & b\\ b & c\end{bmatrix}\right)=\begin{bmatrix}-5a+2b-3c & -12a+5b-6c\\ -12a+5b-6c & 6a-2b+4c\end{bmatrix}$$

C41† Let S_{22} be the vector space of 2×2 symmetric matrices. Find a basis for S_{22} composed of eigenvectors of the linear transformation $Q\colon S_{22}\to S_{22}$.

$$Q\left(\begin{bmatrix}a & b\\ b & c\end{bmatrix}\right)=\begin{bmatrix}25a+18b+30c & -16a-11b-20c\\ -16a-11b-20c & -11a-9b-12c\end{bmatrix}$$

T10† Suppose that $T\colon V \to V$ is an invertible linear transformation with a nonzero eigenvalue λ. Prove that $\dfrac{1}{\lambda}$ is an eigenvalue of T^{-1}.

T15 Suppose that V is a vector space and $T\colon V \to V$ is a linear transformation. Prove that T is injective if and only if $\lambda = 0$ is not an eigenvalue of T.

Section OD
Orthonormal Diagonalization

We have seen in Section SD that under the right conditions a square matrix is similar to a diagonal matrix. We recognize now, via Theorem SCB, that a similarity transformation is a change of basis on a matrix representation. So we can now discuss the choice of a basis used to build a matrix representation, and decide if some bases are better than others for this purpose. This will be the tone of this section. We will also see that every matrix has a reasonably useful matrix representation, and we will discover a new class of diagonalizable linear transformations. First we need some basic facts about triangular matrices.

Subsection TM
Triangular Matrices

An upper, or lower, triangular matrix is exactly what it sounds like it should be, but here are the two relevant definitions.

Definition UTM Upper Triangular Matrix
The $n \times n$ square matrix A is **upper triangular** if $[A]_{ij} = 0$ whenever $i > j$. □

Definition LTM Lower Triangular Matrix
The $n \times n$ square matrix A is **lower triangular** if $[A]_{ij} = 0$ whenever $i < j$. □

Obviously, properties of a lower triangular matrices will have analogues for upper triangular matrices. Rather than stating two very similar theorems, we will say that matrices are "triangular of the same type" as a convenient shorthand to cover both possibilities and then give a proof for just one type.

Theorem PTMT Product of Triangular Matrices is Triangular
Suppose that A and B are square matrices of size n that are triangular of the same type. Then AB is also triangular of that type.

Proof. We prove this for lower triangular matrices and leave the proof for upper triangular matrices to you. Suppose that A and B are both lower triangular. We need only establish that certain entries of the product AB are zero. Suppose that $i < j$, then

$$\begin{aligned}
[AB]_{ij} &= \sum_{k=1}^{n} [A]_{ik} [B]_{kj} && \text{Theorem EMP}\\
&= \sum_{k=1}^{j-1} [A]_{ik} [B]_{kj} + \sum_{k=j}^{n} [A]_{ik} [B]_{kj} && \text{Property AACN}\\
&= \sum_{k=1}^{j-1} [A]_{ik} 0 + \sum_{k=j}^{n} [A]_{ik} [B]_{kj} \qquad k<j, && \text{Definition LTM}\\
&= \sum_{k=1}^{j-1} [A]_{ik} 0 + \sum_{k=j}^{n} 0 [B]_{kj} \qquad i<j\leq k, && \text{Definition LTM}\\
&= \sum_{k=1}^{j-1} 0 + \sum_{k=j}^{n} 0\\
&= 0
\end{aligned}$$

Since $[AB]_{ij} = 0$ whenever $i < j$, by Definition LTM, AB is lower triangular. ■

The inverse of a triangular matrix is triangular, of the same type.

Theorem ITMT Inverse of a Triangular Matrix is Triangular
Suppose that A is a nonsingular matrix of size n that is triangular. Then the inverse of A, A^{-1}, is triangular of the same type. Furthermore, the diagonal entries of A^{-1} are the reciprocals of the corresponding diagonal entries of A. More precisely, $\left[A^{-1}\right]_{ii} = [A]_{ii}^{-1}$.

Proof. We give the proof for the case when A is lower triangular, and leave the case when A is upper triangular for you. Consider the process for computing the inverse of a matrix that is outlined in the proof of Theorem CINM. We augment A with the size n identity matrix, I_n, and row-reduce the $n \times 2n$ matrix to reduced row-echelon form via the algorithm in Theorem REMEF. The proof involves tracking the peculiarities of this process in the case of a lower triangular matrix. Let $M = [A \mid I_n]$.

First, none of the diagonal elements of A are zero. By repeated expansion about the first row, the determinant of a lower triangular matrix can be seen to be the product of the diagonal entries (Theorem DER). If just one of these diagonal elements was zero, then the determinant of A is zero and A is singular by Theorem SMZD. Slightly violating the exact algorithm for row reduction we can form a matrix, M', that is row-equivalent to M, by multiplying row i by the nonzero scalar $[A]_{ii}^{-1}$, for $1 \leq i \leq n$. This sets $[M']_{ii} = 1$ and $[M']_{i,n+1} = [A]_{ii}^{-1}$, and leaves every zero entry of M unchanged.

Let M_j denote the matrix obtained form M' after converting column j to a pivot column. We can convert column j of M_{j-1} into a pivot column with a set of $n-j-1$ row operations of the form $\alpha R_j + R_k$ with $j+1 \leq k \leq n$. The key observation here is that we add multiples of row j only to higher-numbered rows. This means that none of the entries in rows 1 through $j-1$ is changed, and since row j has zeros in columns $j+1$ through n, none of the entries in rows $j+1$ through n is changed in columns $j+1$ through n. The first n columns of M' form a lower triangular matrix with 1's on the diagonal. In its conversion to the identity matrix through this sequence of row operations, it remains lower triangular with 1's on the diagonal.

What happens in columns $n+1$ through $2n$ of M'? These columns began in M as the identity matrix, and in M' each diagonal entry was scaled to a reciprocal of the corresponding diagonal entry of A. Notice that trivially, these final n columns of M' form a lower triangular matrix. Just as we argued for the first n columns, the row operations that convert M_{j-1} into M_j will preserve the lower triangular form in the final n columns and preserve the exact values of the diagonal entries. By Theorem CINM, the final n columns of M_n is the inverse of A, and this matrix has the necessary properties advertised in the conclusion of this theorem. ■

Subsection UTMR
Upper Triangular Matrix Representation

Not every matrix is diagonalizable, but every linear transformation has a matrix representation that is an upper triangular matrix, and the basis that achieves this representation is especially pleasing. Here is the theorem.

Theorem UTMR Upper Triangular Matrix Representation
Suppose that $T\colon V \to V$ is a linear transformation. Then there is a basis B for V such that the matrix representation of T relative to B, $M^T_{B,B}$, is an upper triangular matrix. Each diagonal entry is an eigenvalue of T, and if λ is an eigenvalue of T, then λ occurs $\alpha_T(\lambda)$ times on the diagonal.

Proof. We begin with a proof by induction (Proof Technique I) of the first statement in the conclusion of the theorem. We use induction on the dimension of V to show that if $T\colon V \to V$ is a linear transformation, then there is a basis B for V such that the matrix representation of T relative to B, $M^T_{B,B}$, is an upper triangular matrix.

To start suppose that $\dim(V) = 1$. Choose any nonzero vector $\mathbf{v} \in V$ and realize that $V = \langle\{\mathbf{v}\}\rangle$. Then $T(\mathbf{v}) = \beta\mathbf{v}$ for some $\beta \in \mathbb{C}$, which determines T uniquely (Theorem LTDB). This description of T also gives us a matrix representation relative to the basis $B = \{\mathbf{v}\}$ as the 1×1 matrix with lone entry equal to β. And this matrix representation is upper triangular (Definition UTM).

For the induction step let $\dim(V) = m$, and assume the theorem is true for every linear transformation defined on a vector space of dimension less than m. By Theorem EMHE (suitably converted to the setting of a linear transformation), T has at least one eigenvalue, and we denote this eigenvalue as λ. (We will remark later about how critical this step is.) We now consider properties of the linear transformation $T - \lambda I_V\colon V \to V$.

Let $\mathbf{x}$ be an eigenvector of T for λ. By definition $\mathbf{x} \neq \mathbf{0}$. Then

$$\begin{aligned}
(T - \lambda I_V)(\mathbf{x}) &= T(\mathbf{x}) - \lambda I_V(\mathbf{x}) && \text{Theorem VSLT}\\
&= T(\mathbf{x}) - \lambda\mathbf{x} && \text{Definition IDLT}\\
&= \lambda\mathbf{x} - \lambda\mathbf{x} && \text{Definition EELT}\\
&= \mathbf{0} && \text{Property AI}
\end{aligned}$$

So $T - \lambda I_V$ is not injective, as it has a nontrivial kernel (Theorem KILT). With an application of Theorem RPNDD we bound the rank of $T - \lambda I_V$,

$$r(T - \lambda I_V) = \dim(V) - n(T - \lambda I_V) \leq m - 1$$

Let W be the subspace of V that is the range of $T - \lambda I_V$, $W = \mathcal{R}(T - \lambda I_V)$, and define $k = \dim(W) \leq m - 1$. We define a new linear transformation S, on W,

$$S\colon W \to W, \quad S(\mathbf{w}) = T(\mathbf{w})$$

This does not look we have accomplished much, since the action of S is identical to the action of T. For our purposes this will be a good thing. What is different is the domain and codomain. S is defined on W, a vector space with dimension less than m, and so is susceptible to our induction hypothesis. Verifying that S is really a linear transformation is almost entirely routine, with one exception. Employing T in our definition of S raises the possibility that the outputs of S will not be contained within W (but instead will lie inside V, but outside W). To examine this possibility, suppose that $\mathbf{w} \in W$.

$$\begin{aligned}
S(\mathbf{w}) &= T(\mathbf{w})\\
&= T(\mathbf{w}) + \mathbf{0} && \text{Property Z}\\
&= T(\mathbf{w}) + (\lambda I_V(\mathbf{w}) - \lambda I_V(\mathbf{w})) && \text{Property AI}\\
&= (T(\mathbf{w}) - \lambda I_V(\mathbf{w})) + \lambda I_V(\mathbf{w}) && \text{Property AA}\\
&= (T(\mathbf{w}) - \lambda I_V(\mathbf{w})) + \lambda\mathbf{w} && \text{Definition IDLT}\\
&= (T - \lambda I_V)(\mathbf{w}) + \lambda\mathbf{w} && \text{Theorem VSLT}
\end{aligned}$$

Since W is the range of $T - \lambda I_V$, $(T - \lambda I_V)(\mathbf{w}) \in W$. And by Property SC, $\lambda\mathbf{w} \in W$. Finally, applying Property AC we see by closure that the sum is in W and so we conclude that $S(\mathbf{w}) \in W$. This argument convinces us that it is legitimate to define S as we did with W as the codomain.

S is a linear transformation defined on a vector space with dimension k, less than m, so we can apply the induction hypothesis and conclude that W has a basis, $C = \{\mathbf{w}_1,\, \mathbf{w}_2,\, \mathbf{w}_3,\, \ldots,\, \mathbf{w}_k\}$, such that the matrix representation of S relative to C is an upper triangular matrix.

Beginning with the linearly independent set C, repeatedly apply Theorem ELIS to add vectors to C, maintaining a linearly independent set and spanning ever larger subspaces of V. This process will end with the addition of $m-k$ vectors, which together with C will span all of V. Denote these vectors as $D = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \ldots, \mathbf{u}_{m-k}\}$. Then $B = C \cup D$ is a basis for V, and is the basis we desire for the conclusion of the theorem. So we now consider the matrix representation of T relative to B.

Since the definition of T and S agree on W, the first k columns of $M^T_{B,B}$ will have the upper triangular matrix representation of S in the first k rows. The remaining $m-k$ rows of these first k columns will be all zeros since the outputs of T for basis vectors from C are all contained in W and hence are linear combinations of the basis vectors in C. The situation for T on the basis vectors in D is not quite as pretty, but it is close.

For $1 \leq i \leq m-k$, consider

$$\begin{aligned}
\rho_B\left(T\left(\mathbf{u}_i\right)\right) &= \rho_B\left(T\left(\mathbf{u}_i\right)+\mathbf{0}\right) && \text{Property Z}\\
&= \rho_B\left(T\left(\mathbf{u}_i\right)+\left(\lambda I_V\left(\mathbf{u}_i\right)-\lambda I_V\left(\mathbf{u}_i\right)\right)\right) && \text{Property AI}\\
&= \rho_B\left(\left(T\left(\mathbf{u}_i\right)-\lambda I_V\left(\mathbf{u}_i\right)\right)+\lambda I_V\left(\mathbf{u}_i\right)\right) && \text{Property AA}\\
&= \rho_B\left(\left(T\left(\mathbf{u}_i\right)-\lambda I_V\left(\mathbf{u}_i\right)\right)+\lambda \mathbf{u}_i\right) && \text{Definition IDLT}\\
&= \rho_B\left(\left(T-\lambda I_V\right)\left(\mathbf{u}_i\right)+\lambda \mathbf{u}_i\right) && \text{Theorem VSLT}\\
&= \rho_B\left(a_1\mathbf{w}_1+a_2\mathbf{w}_2+a_3\mathbf{w}_3+\cdots+a_k\mathbf{w}_k+\lambda \mathbf{u}_i\right) && \text{Definition RLT}\\
&= \begin{bmatrix} a_1\\ a_2\\ \vdots\\ a_k\\ 0\\ \vdots\\ 0\\ \lambda\\ 0\\ \vdots\\ 0 \end{bmatrix} && \text{Definition VR}
\end{aligned}$$

In the penultimate equality, we have rewritten an element of the range of $T - \lambda I_V$ as a linear combination of the basis vectors, C, for the range of $T - \lambda I_V$, W, using the scalars $a_1, a_2, a_3, \ldots, a_k$. If we incorporate these $m-k$ column vectors into the matrix representation $M^T_{B,B}$ we find $m-k$ occurrences of λ on the diagonal, and any nonzero entries lying only in the first k rows. Together with the $k \times k$ upper triangular representation in the upper left-hand corner, the entire matrix representation for T is clearly upper triangular. This completes the induction step. So for any linear transformation there is a basis that creates an upper triangular matrix representation.

We have one more statement in the conclusion of the theorem to verify. The eigenvalues of T, and their multiplicities, can be computed with the techniques of Chapter E relative to any matrix representation (Theorem EER). We take this approach with our upper triangular matrix representation $M^T_{B,B}$. Let d_i be the diagonal entry of $M^T_{B,B}$ in row i and column i. Then the characteristic polynomial, computed as a determinant (Definition CP) with repeated expansions about the first column, is

$$p_{M^T_{B,B}}\left(x\right) = \left(d_1 - x\right)\left(d_2 - x\right)\left(d_3 - x\right)\cdots\left(d_m - x\right)$$

The roots of the polynomial equation $p_{M^T_{B,B}}\left(x\right) = 0$ are the eigenvalues of the linear transformation (Theorem EMRCP). So each diagonal entry is an eigenvalue, and is repeated on the diagonal exactly $\alpha_T\left(\lambda\right)$ times (Definition AME). ■

A key step in this proof was the construction of the subspace W with dimension strictly less than that of V. This required an eigenvalue/eigenvector pair, which was guaranteed to us by Theorem EMHE. Digging deeper, the proof of Theorem EMHE requires that we can factor polynomials completely, into linear factors. This will not always happen if our set of scalars is the reals, $\mathbb{R}$. So this is our final explanation of our choice of the complex numbers, $\mathbb{C}$, as our set of scalars. In $\mathbb{C}$ polynomials factor completely, so every matrix has at least one eigenvalue, and an inductive argument will get us to upper triangular matrix representations.

In the case of linear transformations defined on $\mathbb{C}^m$, we can use the inner product (Definition IP) profitably to fine-tune the basis that yields an upper triangular matrix representation. Recall that the adjoint of matrix A (Definition A) is written as A^*.

Theorem OBUTR Orthonormal Basis for Upper Triangular Representation
Suppose that A is a square matrix. Then there is a unitary matrix U, and an upper triangular matrix T, such that

$$U^*AU = T$$

and T has the eigenvalues of A as the entries of the diagonal.

Proof. This theorem is a statement about matrices and similarity. We can convert it to a statement about linear transformations, matrix representations and bases (Theorem SCB). Suppose that A is an $n \times n$ matrix, and define the linear transformation $S\colon \mathbb{C}^n \to \mathbb{C}^n$ by $S(\mathbf{x}) = A\mathbf{x}$. Then Theorem UTMR gives us a basis $B = \{\mathbf{v}_1,\,\mathbf{v}_2,\,\mathbf{v}_3,\,\ldots,\,\mathbf{v}_n\}$ for $\mathbb{C}^n$ such that a matrix representation of S relative to B, $M^S_{B,B}$, is upper triangular.

Now convert the basis B into an orthogonal basis, C, by an application of the Gram-Schmidt procedure (Theorem GSP). This is a messy business computationally, but here we have an excellent illustration of the power of the Gram-Schmidt procedure. We need only be sure that B is linearly independent and spans $\mathbb{C}^n$, and then we know that C is linearly independent, spans $\mathbb{C}^n$ and is also an orthogonal set. We will now consider the matrix representation of S relative to C (rather than B). Write the new basis as $C = \{\mathbf{y}_1,\,\mathbf{y}_2,\,\mathbf{y}_3,\,\ldots,\,\mathbf{y}_n\}$. The application of the Gram-Schmidt procedure creates each vector of C, say $\mathbf{y}_j$, as the difference of $\mathbf{v}_j$ and a linear combination of $\mathbf{y}_1,\,\mathbf{y}_2,\,\mathbf{y}_3,\,\ldots,\,\mathbf{y}_{j-1}$. We are not concerned here with the actual values of the scalars in this linear combination, so we will write

$$\mathbf{y}_j = \mathbf{v}_j - \sum_{k=1}^{j-1} b_{jk}\mathbf{y}_k$$

where the b_{jk} are shorthand for the scalars. The equation above is in a form useful for creating the basis C from B. To better understand the relationship between B and C convert it to read

$$\mathbf{v}_j = \mathbf{y}_j + \sum_{k=1}^{j-1} b_{jk}\mathbf{y}_k$$

In this form, we recognize that the change-of-basis matrix $C_{B,C} = M^{I_{\mathbb{C}^n}}_{B,C}$ (Definition CBM) is an upper triangular matrix. By Theorem SCB we have

$$M^S_{C,C} = C_{B,C} M^S_{B,B} C^{-1}_{B,C}$$

The inverse of an upper triangular matrix is upper triangular (Theorem ITMT), and the product of two upper triangular matrices is again upper triangular (Theorem PTMT). So $M^S_{C,C}$ is an upper triangular matrix.

Now, multiply each vector of C by a nonzero scalar, so that the result has norm 1. In this way we create a new basis D which is an orthonormal set (Definition ONS). Note that the change-of-basis matrix $C_{C,D}$ is a diagonal matrix with nonzero entries equal to the norms of the vectors in C.

Now we can convert our results into the language of matrices. Let E be the basis of $\mathbb{C}^n$ formed with the standard unit vectors (Definition SUV). Then the matrix representation of S relative to E is simply A, $A = M^S_{E,E}$. The change-of-basis matrix $C_{D,E}$ has columns that are simply the vectors in D, the orthonormal basis. As such, Theorem CUMOS tells us that $C_{D,E}$ is a unitary matrix, and by Definition UM has an inverse equal to its adjoint. Write $U = C_{D,E}$. We have

$$
\begin{aligned}
U^*AU &= U^{-1}AU && \text{Theorem UMI} \\
&= C^{-1}_{D,E} M^S_{E,E} C_{D,E} \\
&= M^S_{D,D} && \text{Theorem SCB} \\
&= C_{C,D} M^S_{C,C} C^{-1}_{C,D} && \text{Theorem SCB}
\end{aligned}
$$

The inverse of a diagonal matrix is also a diagonal matrix, and so this final expression is the product of three upper triangular matrices, and so is again upper triangular (Theorem PTMT). Thus the desired upper triangular matrix, T, is the matrix representation of S relative to the orthonormal basis D, $M^S_{D,D}$. ■

Subsection NM
Normal Matrices

Normal matrices comprise a broad class of interesting matrices, many of which we have met already. But they are most interesting since they define exactly which matrices we can diagonalize via a unitary matrix. This is the upcoming Theorem OD. Here is the definition.

Definition NRML Normal Matrix
The square matrix A is normal if $A^*A = AA^*$. □

So a normal matrix commutes with its adjoint. Part of the beauty of this definition is that it includes many other types of matrices. A diagonal matrix will commute with its adjoint, since the adjoint is again diagonal and the entries are just conjugates of the entries of the original diagonal matrix. A Hermitian (self-adjoint) matrix (Definition HM) will trivially commute with its adjoint, since the two matrices are the same. A real, symmetric matrix is Hermitian, so these matrices are also normal. A unitary matrix (Definition UM) has its adjoint as its inverse, and inverses commute (Theorem OSIS), so unitary matrices are normal. Another class of normal matrices is the skew-symmetric matrices. However, these broad descriptions still do not capture all of the normal matrices, as the next example shows.

Example ANM A normal matrix
Let

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Then

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

so we see by Definition NRML that A is normal. However, A is not symmetric (hence, as a real matrix, not Hermitian), not unitary, and not skew-symmetric. △

Subsection OD
Orthonormal Diagonalization

A diagonal matrix is very easy to work with in matrix multiplication (Example HPDM) and an orthonormal basis also has many advantages (Theorem COB). How about converting a matrix to a diagonal matrix through a similarity transformation using a unitary matrix (i.e. build a diagonal matrix representation with an orthonormal matrix)? That'd be fantastic! When can we do this? We can always accomplish this feat when the matrix is normal, and normal matrices are the only ones that behave this way. Here is the theorem.

Theorem OD Orthonormal Diagonalization
Suppose that A is a square matrix. Then there is a unitary matrix U and a diagonal matrix D, with diagonal entries equal to the eigenvalues of A, such that $U^*AU = D$ if and only if A is a normal matrix.

Proof. ($\Rightarrow$) Suppose there is a unitary matrix U that diagonalizes A. We would usally write this condition as $U^*AU = D$, but we will find it convenient in this part of the proof to use our hypothesis in the equivalent form, $A = UDU^*$. Recall that a diagonal matrix is normal, and notice that this observation is at the center of the next sequence of equalities. We check the normality of A,

$$\begin{aligned}
A^*A &= \left(UDU^*\right)^*\left(UDU^*\right) && \text{Hypothesis}\\
&= \left(U^*\right)^* D^*U^*UDU^* && \text{Theorem MMAD}\\
&= UD^*U^*UDU^* && \text{Theorem AA}\\
&= UD^*I_nDU^* && \text{Definition UM}\\
&= UD^*DU^* && \text{Theorem MMIM}\\
&= UDD^*U^* && \text{Definition NRML}\\
&= UDI_nD^*U^* && \text{Theorem MMIM}\\
&= UDU^*UD^*U^* && \text{Definition UM}\\
&= UDU^*\left(U^*\right)^* D^*U^* && \text{Theorem AA}\\
&= \left(UDU^*\right)\left(UDU^*\right)^* && \text{Theorem MMAD}\\
&= AA^* && \text{Hypothesis}
\end{aligned}$$

So by Definition NRML, A is a normal matrix.

($\Leftarrow$) For the converse, suppose that A is a normal matrix. Whether or not A is normal, Theorem OBUTR provides a unitary matrix U and an upper triangular matrix T, whose diagonal entries are the eigenvalues of A, and such that $U^*AU = T$. With the added condition that A is normal, we will determine that the entries of T above the diagonal must be all zero. Here we go.

First notice that Definition UM implies that the inverse of a unitary matrix U is the adjoint, U^*, so the product of these two matrices, in either order, is the identity matrix (Theorem OSIS). We begin by showing that T is normal.

$$\begin{aligned}
T^*T &= \left(U^*AU\right)^*\left(U^*AU\right) && \text{Theorem OBUTR}\\
&= U^*A^*\left(U^*\right)^*U^*AU && \text{Theorem MMAD}\\
&= U^*A^*UU^*AU && \text{Theorem AA}\\
&= U^*A^*I_nAU && \text{Definition UM}\\
&= U^*A^*AU && \text{Theorem MMIM}\\
&= U^*AA^*U && \text{Definition NRML}\\
&= U^*AI_nA^*U && \text{Theorem MMIM}\\
&= U^*AUU^*A^*U && \text{Definition UM}\\
&= U^*AUU^*A^*\left(U^*\right)^* && \text{Theorem AA}\\
&= \left(U^*AU\right)\left(U^*AU\right)^* && \text{Theorem MMAD}\\
&= TT^* && \text{Theorem OBUTR}
\end{aligned}$$

So by Definition NRML, T is a normal matrix.

We can translate the normality of T into the statement $TT^* - T^*T = \mathcal{O}$. We now establish an equality we will use repeatedly. For $1 \leq i \leq n$,

$$\begin{aligned}
0 &= [\mathcal{O}]_{ii} && \text{Definition ZM}\\
&= [TT^* - T^*T]_{ii} && \text{Definition NRML}\\
&= [TT^*]_{ii} - [T^*T]_{ii} && \text{Definition MA}\\
&= \sum_{k=1}^{n} [T]_{ik} [T^*]_{ki} - \sum_{k=1}^{n} [T^*]_{ik} [T]_{ki} && \text{Theorem EMP}\\
&= \sum_{k=1}^{n} [T]_{ik} \overline{[T]_{ik}} - \sum_{k=1}^{n} \overline{[T]_{ki}} [T]_{ki} && \text{Definition A}\\
&= \sum_{k=i}^{n} [T]_{ik} \overline{[T]_{ik}} - \sum_{k=1}^{i} \overline{[T]_{ki}} [T]_{ki} && \text{Definition UTM}\\
&= \sum_{k=i}^{n} \left|[T]_{ik}\right|^2 - \sum_{k=1}^{i} \left|[T]_{ki}\right|^2 && \text{Definition MCN}
\end{aligned}$$

To conclude, we use the above equality repeatedly, beginning with $i = 1$, and discover, row by row, that the entries above the diagonal of T are all zero. The key observation is that a sum of squares can only equal zero when each term of the sum is zero. For $i = 1$ we have

$$0 = \sum_{k=1}^{n} \left|[T]_{1k}\right|^2 - \sum_{k=1}^{1} \left|[T]_{k1}\right|^2 = \sum_{k=2}^{n} \left|[T]_{1k}\right|^2$$

which forces the conclusions

$$[T]_{12} = 0 \qquad [T]_{13} = 0 \qquad [T]_{14} = 0 \qquad \cdots \qquad [T]_{1n} = 0$$

For $i = 2$ we use the same equality, but also incorporate the portion of the above conclusions that says $[T]_{12} = 0$,

$$0 = \sum_{k=2}^{n} \left|[T]_{2k}\right|^2 - \sum_{k=1}^{2} \left|[T]_{k2}\right|^2 = \sum_{k=2}^{n} \left|[T]_{2k}\right|^2 - \sum_{k=2}^{2} \left|[T]_{k2}\right|^2 = \sum_{k=3}^{n} \left|[T]_{2k}\right|^2$$

which forces the conclusions

$$[T]_{23} = 0 \qquad [T]_{24} = 0 \qquad [T]_{25} = 0 \qquad \cdots \qquad [T]_{2n} = 0$$

We can repeat this process for the subsequent values of $i = 3, 4, 5 \ldots, n - 1$. Notice that it is critical we do this in order, since we need to employ portions of each of the previous conclusions about rows having zero entries in order to successfully get the same conclusion for later rows. Eventually, we conclude that all of the nondiagonal entries of T are zero, so the extra assumption of normality forces T to be diagonal. ■

We can rearrange the conclusion of this theorem to read $A = UDU^*$. Recall that a unitary matrix can be viewed as a geometry-preserving transformation (isometry), or more loosely as a rotation of sorts. Then a matrix-vector product, $A\mathbf{x}$, can be viewed instead as a sequence of three transformations. U^* is unitary, and so is a rotation. Since D is diagonal, it just multiplies each entry of a vector by a scalar. Diagonal entries that are positive or negative, with absolute values bigger or smaller than 1 evoke descriptions like reflection, expansion and contraction. Generally we can say that D "stretches" a vector in each component. Final multiplication by U undoes (inverts) the rotation performed by U^*. So a normal matrix is a rotation-stretch-rotation transformation.

The orthonormal basis formed from the columns of U can be viewed as a system of mutually perpendicular axes. The rotation by U^* allows the transformation by A to be replaced by the simple transformation D along these axes, and then D brings the result back to the original coordinate system. For this reason Theorem OD is known as the **Principal Axis Theorem**.

The columns of the unitary matrix in Theorem OD create an especially nice basis for use with the normal matrix. We record this observation as a theorem.

Theorem OBNM Orthonormal Bases and Normal Matrices
Suppose that A is a normal matrix of size n. Then there is an orthonormal basis of $\mathbb{C}^n$ composed of eigenvectors of A.

Proof. Let U be the unitary matrix promised by Theorem OD and let D be the resulting diagonal matrix. The desired set of vectors is formed by collecting the columns of U into a set. Theorem CUMOS says this set of columns is orthonormal. Since U is nonsingular (Theorem UMI), Theorem CNMB says the set is a basis.

Since A is diagonalized by U, the diagonal entries of the matrix D are the eigenvalues of A. An argument exactly like the second half of the proof of Theorem DC shows that each vector of the basis is an eigenvector of A. ■

In a vague way Theorem OBNM is an improvement on Theorem HMOE which said that eigenvectors of a Hermitian matrix for different eigenvalues are always orthogonal. Hermitian matrices are normal and we see that we can find at least one basis where *every* pair of eigenvectors is orthogonal. Notice that this is not a generalization, since Theorem HMOE states a weak result which applies to many (but not all) pairs of eigenvectors, while Theorem OBNM is a seemingly stronger result, but only asserts that there is one collection of eigenvectors with the stronger property.

Given an $n \times n$ matrix A, an orthonormal basis for $\mathbb{C}^n$, comprised of eigenvectors of A is an extremely useful basis to have at the service of the matrix A. Why do we say this? We can consider the vectors of a basis as a preferred set of directions, known as "axes," which taken together might also be called a "coordinate system." The standard basis of Definition SUV could be considered the default, or prototype, coordinate system. When a basis is orthornormal, we can consider the directions to be standardized to have unit length, and we can consider the axes as being mutually perpendicular. But there is more — let us be a bit more formal.

Suppose U is a matrix whose columns are an orthonormal basis of eigenvectors of the $n \times n$ matrix A. So, in particular U is a unitary matrix (Theorem CUMOS). For a vector $\mathbf{x} \in \mathbb{C}^n$, use the notation $\hat{\mathbf{x}}$ for the vector representation of $\mathbf{x}$ relative to the orthonormal basis. So the entries of $\hat{\mathbf{x}}$, used in a linear combination of the columns of U will create $\mathbf{x}$. With Definition MVP, we can write this relationship as

$$U\hat{\mathbf{x}} = \mathbf{x}$$

Since U^* is the inverse of U (Definition UM), we can rearrange this equation as

$$\hat{\mathbf{x}} = U^*\mathbf{x}$$

This says we can easily create the vector representation relative to the orthonormal basis with a matrix-vector product of the adjoint of U. Note that the adjoint is much easier to compute than a matrix inverse, which would be one general way to obtain a vector representation. This is our first observation about coordinatization relative to orthonormal basis. However, we already knew this, as we just have Theorem COB in disguise (see Exercise OD.T20).

We also know that orthonormal bases play nicely with inner products. Theorem UMPIP says unitary matrices preserve inner products (and hence norms). More geometrically, lengths and angles are preserved by multiplication by a unitary matrix. Using our notation, this becomes

$$\langle \mathbf{x},\, \mathbf{y} \rangle = \langle U\hat{\mathbf{x}},\, U\hat{\mathbf{y}} \rangle = \langle \hat{\mathbf{x}},\, \hat{\mathbf{y}} \rangle$$

So we can compute inner products with the original vectors, or with their representations, and obtain the same result. It follows that norms, lengths, and angles can all be computed with the original vectors or with the representations in the new coordinate system based on an orthonormal basis.

So far we have not really said anything new, nor has the matrix A, or its eigenvectors, come into play. We know that a matrix is really a linear transformation, so we express this view of a matrix as a function by writing generically that $A\mathbf{x} = \mathbf{y}$. The matrix U will diagonalize A, creating the diagonal matrix D with diagonal entries equal to the eigenvalues of A. We can write this as $U^*AU = D$ and convert to $U^*A = DU^*$. Then we have

$$\hat{\mathbf{y}} = U^*\mathbf{y} = U^*A\mathbf{x} = DU^*\mathbf{x} = D\hat{\mathbf{x}}$$

So with the coordinatized vectors, the transformation by the matrix A can be accomplished with multiplication by a diagonal matrix D. A moment's thought should convince you that a matrix-vector product with a diagonal matrix is exeedingly simple computationally. Geometrically, this is simply stretching, contracting and/or reflecting in the direction of each basis vector ("axis"). And the multiples used for these changes are the diagonal entries of the diagonal matrix, the eigenvalues of A.

So the new coordinate system (provided by the orthonormal basis of eigenvectors) is a collection of mutually perpendicular unit vectors where inner products are preserved, and the action of the matrix A is described by multiples (eigenvalues) of the entries of the coordinatized versions of the vectors. Nice.

Reading Questions

1. Name three broad classes of normal matrices that we have studied previously. No set that you give should be a subset of another on your list.
2. Compare and contrast Theorem UTMR with Theorem OD.
3. Given an $n \times n$ matrix A, why would you desire an orthonormal basis of $\mathbb{C}^n$ composed entirely of eigenvectors of A?

Exercises

T10 Exercise MM.T35 asked you to show that AA^* is Hermitian. Prove directly that AA^* is a normal matrix.

T20 In the discussion following Theorem OBNM we comment that the equation $\hat{\mathbf{x}} = U^*\mathbf{x}$ is just Theorem COB in disguise. Formulate this observation more formally and prove the equivalence.

T30 For the proof of Theorem PTMT we only show that the product of two lower triangular matrices is again lower triangular. Provide a proof that the product of two upper triangular matrices is again upper triangular. Look to the proof of Theorem PTMT for guidance if you need a hint.

Chapter P
Preliminaries

"Preliminaries" are basic mathematical concepts you are likely to have seen before. So we have collected them near the end as reference material (despite the name). Head back here when you need a refresher, or when a theorem or exercise builds on some of this basic material.

Section CNO
Complex Number Operations

In this section we review some of the basics of working with complex numbers.

Subsection CNA
Arithmetic with complex numbers

A complex number is a linear combination of 1 and $i = \sqrt{-1}$, typically written in the form $a + bi$. Complex numbers can be added, subtracted, multiplied and divided, just like we are used to doing with real numbers, including the restriction on division by zero. We will not define these operations carefully immediately, but instead first illustrate with examples.

Example ACN Arithmetic of complex numbers

$$
\begin{aligned}
(2+5i)+(6-4i) &= (2+6)+(5+(-4))i = 8+i \\
(2+5i)-(6-4i) &= (2-6)+(5-(-4))i = -4+9i \\
(2+5i)(6-4i) &= (2)(6)+(5i)(6)+(2)(-4i)+(5i)(-4i) = 12+30i-8i-20i^2 \\
&= 12+22i-20(-1) = 32+22i
\end{aligned}
$$

Division takes just a bit more care. We multiply the denominator by a complex number chosen to produce a real number and then we can produce a complex number as a result.

$$\frac{2+5i}{6-4i} = \frac{2+5i}{6-4i}\frac{6+4i}{6+4i} = \frac{-8+38i}{52} = -\frac{8}{52} + \frac{38}{52}i = -\frac{2}{13} + \frac{19}{26}i$$

△

In this example, we used $6 + 4i$ to convert the denominator in the fraction to a real number. This number is known as the conjugate, which we define in the next section.

We will often exploit the basic properties of complex number addition, subtraction, multiplication and division, so we will carefully define the two basic operations, together with a definition of equality, and then collect nine basic properties in a theorem.

Definition CNE Complex Number Equality
The complex numbers $\alpha = a + bi$ and $\beta = c + di$ are **equal**, denoted $\alpha = \beta$, if $a = c$ and $b = d$. □

Definition CNA Complex Number Addition
The **sum** of the complex numbers $\alpha = a + bi$ and $\beta = c + di$, denoted $\alpha + \beta$, is $(a + c) + (b + d)i$. □

Definition CNM Complex Number Multiplication
The **product** of the complex numbers $\alpha = a + bi$ and $\beta = c + di$, denoted $\alpha\beta$, is $(ac - bd) + (ad + bc)i$. □

Theorem PCNA Properties of Complex Number Arithmetic
The operations of addition and multiplication of complex numbers have the following properties.

- ACCN Additive Closure, Complex Numbers
  If $\alpha, \beta \in \mathbb{C}$, *then* $\alpha + \beta \in \mathbb{C}$.
- MCCN Multiplicative Closure, Complex Numbers
  If $\alpha, \beta \in \mathbb{C}$, *then* $\alpha\beta \in \mathbb{C}$.
- CACN Commutativity of Addition, Complex Numbers
  For any α, $\beta \in \mathbb{C}$, $\alpha + \beta = \beta + \alpha$.
- CMCN Commutativity of Multiplication, Complex Numbers
  For any α, $\beta \in \mathbb{C}$, $\alpha\beta = \beta\alpha$.
- AACN Additive Associativity, Complex Numbers
  For any α, β, $\gamma \in \mathbb{C}$, $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$.
- MACN Multiplicative Associativity, Complex Numbers
  For any α, β, $\gamma \in \mathbb{C}$, $\alpha\,(\beta\gamma) = (\alpha\beta)\,\gamma$.
- DCN Distributivity, Complex Numbers
  For any α, β, $\gamma \in \mathbb{C}$, $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$.
- ZCN Zero, Complex Numbers
  There is a complex number $0 = 0 + 0i$ *so that for any* $\alpha \in \mathbb{C}$, $0 + \alpha = \alpha$.
- OCN One, Complex Numbers
  There is a complex number $1 = 1 + 0i$ *so that for any* $\alpha \in \mathbb{C}$, $1\alpha = \alpha$.
- AICN Additive Inverse, Complex Numbers
  For every $\alpha \in \mathbb{C}$ *there exists* $-\alpha \in \mathbb{C}$ *so that* $\alpha + (-\alpha) = 0$.
- MICN Multiplicative Inverse, Complex Numbers
  For every $\alpha \in \mathbb{C}$, $\alpha \neq 0$ *there exists* $\frac{1}{\alpha} \in \mathbb{C}$ *so that* $\alpha\left(\frac{1}{\alpha}\right) = 1$.

Proof. We could derive each of these properties of complex numbers with a proof that builds on the identical properties of the real numbers. The only proof that might be at all interesting would be to show Property MICN since we would need to trot out a conjugate. For this property, and especially for the others, we might be tempted to construct proofs of the identical properties for the reals. This would take us way too far afield, so we will draw a line in the sand right here and just agree that these nine fundamental behaviors are true. OK?

Mostly we have stated these nine properties carefully so that we can make reference to them later in other proofs. So we will be linking back here often. ■

Zero and one play special roles, of course, and especially zero. Our first result is one we take for granted, but it requires a proof, derived from our nine properties. You can compare it to its vector space counterparts, Theorem ZSSM and Theorem ZVSM.

Theorem ZPCN Zero Product, Complex Numbers
Suppose $\alpha \in \mathbb{C}$*. Then* $0\alpha = 0$*.*

Proof.

$$\begin{aligned}
0\alpha &= 0\alpha + 0 && \text{Property ZCN}\\
&= 0\alpha + (0\alpha - (0\alpha)) && \text{Property AICN}\\
&= (0\alpha + 0\alpha) - (0\alpha) && \text{Property AACN}\\
&= (0 + 0)\,\alpha - (0\alpha) && \text{Property DCN}\\
&= 0\alpha - (0\alpha) && \text{Property ZCN}\\
&= 0 && \text{Property AICN}
\end{aligned}$$

Our next theorem could be called "cancellation", since it will make that possible. Though you will never see us drawing slashes through parts of products. We will also make very limited use of this result, or its vector space counterpart, Theorem SMEZV.

Theorem ZPZT Zero Product, Zero Terms
Suppose $\alpha, \beta \in \mathbb{C}$*. Then* $\alpha\beta = 0$ *if and only if at least one of* $\alpha = 0$ *or* $\beta = 0$*.*

Proof. ($\Rightarrow$)We conduct the forward argument in two cases. First suppose that $\alpha = 0$. Then we are done. (That was easy.)

For the second case, suppose now that $\alpha \neq 0$. Then

$$\begin{aligned}
\beta &= 1\beta && \text{Property OCN}\\
&= \left(\frac{1}{\alpha}\alpha\right)\beta && \text{Property MICN}\\
&= \frac{1}{\alpha}\left(\alpha\beta\right) && \text{Property MACN}\\
&= \frac{1}{\alpha}0 && \text{Hypothesis}\\
&= 0 && \text{Theorem ZPCN}
\end{aligned}$$

($\Leftarrow$)With two applications of Theorem ZPCN it is easy to see that if one of the scalars is zero, then so is the product. ■

As an equivalence (Proof Technique E), we could restate this result as the contrapositive (Proof Technique CP) by negating each statement, so it would read "$\alpha\beta \neq 0$ if and only if $\alpha \neq 0$ and $\beta \neq 0$." After you have learned more about nonsingular matrices and matrix multiplication, you should compare this result with Theorem NPNT.

Subsection CCN Conjugates of Complex Numbers

Definition CCN Conjugate of a Complex Number
The **conjugate** of the complex number $\alpha = a + bi \in \mathbb{C}$ is the complex number $\overline{\alpha} = a - bi$. □

Example CSCN Conjugate of some complex numbers

$$\overline{2+3i} = 2-3i \qquad \overline{5-4i} = 5+4i \qquad \overline{-3+0i} = -3+0i \qquad \overline{0+0i} = 0+0i$$

△

Notice how the conjugate of a real number leaves the number unchanged. The conjugate enjoys some basic properties that are useful when we work with linear expressions involving addition and multiplication.

Theorem CCRA Complex Conjugation Respects Addition
Suppose that α and β are complex numbers. Then $\overline{\alpha+\beta} = \overline{\alpha} + \overline{\beta}$.

Proof. Let $\alpha = a + bi$ and $\beta = r + si$. Then

$$\overline{\alpha+\beta} = \overline{(a+r)+(b+s)i} = (a+r)-(b+s)i = (a-bi)+(r-si) = \overline{\alpha}+\overline{\beta}$$

■

Theorem CCRM Complex Conjugation Respects Multiplication
Suppose that α and β are complex numbers. Then $\overline{\alpha\beta} = \overline{\alpha}\,\overline{\beta}$.

Proof. Let $\alpha = a + bi$ and $\beta = r + si$. Then

$$\begin{aligned}\overline{\alpha\beta} &= \overline{(ar-bs)+(as+br)i} = (ar-bs)-(as+br)i \\ &= (ar-(-b)(-s))+(a(-s)+(-b)r)i = (a-bi)(r-si) = \overline{\alpha}\,\overline{\beta}\end{aligned}$$

■

Theorem CCT Complex Conjugation Twice
Suppose that α is a complex number. Then $\overline{\overline{\alpha}} = \alpha$.

Proof. Let $\alpha = a + bi$. Then

$$\overline{\overline{\alpha}} = \overline{a-bi} = a-(-bi) = a+bi = \alpha$$

■

Subsection MCN Modulus of a Complex Number

We define one more operation with complex numbers that may be new to you.

Definition MCN Modulus of a Complex Number
The **modulus** of the complex number $\alpha = a+bi \in \mathbb{C}$, is the nonnegative real number

$$|\alpha| = \sqrt{\overline{\alpha}\alpha} = \sqrt{a^2+b^2}.$$

□

Example MSCN Modulus of some complex numbers

$$|2+3i| = \sqrt{13} \qquad |5-4i| = \sqrt{41} \qquad |-3+0i| = 3 \qquad |0+0i| = 0$$

△

The modulus can be interpreted as a version of the absolute value for complex numbers, as is suggested by the notation employed. You can see this in how $|-3| = |-3 + 0i| = 3$. Notice too how the modulus of the complex zero, $0 + 0i$, has value 0.

Section SET
Sets

We will frequently work carefully with sets, so the material in this review section is *very* important. If these topics are new to you, study this section carefully and consider consulting another text for a more comprehensive introduction.

Subsection SET
Sets

Definition SET Set
A **set** is an unordered collection of objects. If S is a set and x is an object that is in the set S, we write $x \in S$. If x is not in S, then we write $x \notin S$. We refer to the objects in a set as its **elements**. □

Hard to get much more basic than that. Notice that the objects in a set can be *anything*, and there is no notion of order among the elements of the set. A set can be finite as well as infinite. A set can contain other sets as its objects. At a primitive level, a set is just a way to break up some class of objects into two groupings: those objects in the set, and those objects not in the set.

Example SETM Set membership
From the set of all possible symbols, construct the following set of three symbols,

$$S = \{\blacksquare, \blacklozenge, \bigstar\}$$

Then the statement $\blacksquare \in S$ is true, while the statement $\blacktriangle \in S$ is false. However, then the statement $\blacktriangle \notin S$ is true. △

A portion of a set is known as a subset. Notice how the following definition uses an implication (if whenever. . . then. . .). Note too how the definition of a subset relies on the definition of a set through the idea of set membership.

Definition SSET Subset
If S and T are two sets, then S is a subset of T, written $S \subseteq T$ if whenever $x \in S$ then $x \in T$. □

If we want to disallow the possibility that S is the same as T, we use the notation $S \subset T$ and we say that S is a **proper subset** of T. We will do an example, but first we will define a special set.

Definition ES Empty Set
The empty set is the set with no elements. It is denoted by $\emptyset$. □

Example SSET Subset
If $S = \{\blacksquare, \blacklozenge, \bigstar\}$, $T = \{\bigstar, \blacklozenge\}$, $R = \{\blacktriangle, \bigstar\}$, then

$$T \subseteq S \qquad R \not\subseteq T \qquad \emptyset \subseteq S$$
$$T \subset S \qquad S \subseteq S \qquad S \not\subset S$$

△

What does it mean for two sets to be equal? They must be the same. Well, that explanation is not really too helpful, is it? How about: If $A \subseteq B$ and $B \subseteq A$, then A equals B. This gives us something to work with, if A is a subset of B, and *vice versa*, then they must really be the same set. We will now make the symbol "=" do double-duty and extend its use to statements like $A = B$, where A and B are sets. Here is the definition, which we will reference often.

Definition SE Set Equality
Two sets, S and T, are equal, if $S \subseteq T$ and $T \subseteq S$. In this case, we write $S = T$. □

Sets are typically written inside of braces, as { }, as we have seen above. However, when sets have more than a few elements, a description will typically have two components. The first is a description of the general type of objects contained in a set, while the second is some sort of restriction on the properties the objects have. Every object in the set must be of the type described in the first part and it must satisfy the restrictions in the second part. Conversely, any object of the proper type for the first part, that also meets the conditions of the second part, will be in the set. These two parts are set off from each other somehow, often with a vertical bar (|) or a colon (:).

I like to think of sets as clubs. The first part is some description of the type of people who *might* belong to the club, the basic objects. For example, a bicycle club would describe its members as being people who like to ride bicycles. The second part is like a membership committee, it restricts the people who are allowed in the club. Continuing with our bicycle club analogy, we might decide to limit ourselves to "serious" riders and only have members who can document having ridden 100 kilometers or more in a single day at least one time.

The restrictions on membership can migrate around some between the first and second part, and there may be several ways to describe the same set of objects. Here is a more mathematical example, employing the set of all integers, $\mathbb{Z}$, to describe the set of even integers.

$$\begin{aligned} E &= \{x \in \mathbb{Z} |\, x \text{ is an even number}\} \\ &= \{x \in \mathbb{Z} |\, 2 \text{ divides } x \text{ evenly}\} \\ &= \{2k |\, k \in \mathbb{Z}\} \end{aligned}$$

Notice how this set tells us that its objects are integer numbers (not, say, matrices or functions, for example) and just those that are even. So we can write that $10 \in E$, while $17 \notin E$ once we check the membership criteria. We also recognize the question

$$\begin{bmatrix} 1 & -3 & 5 \\ 2 & 0 & 3 \end{bmatrix} \in E?$$

as being simply ridiculous.

Subsection SC
Set Cardinality

On occasion, we will be interested in the number of elements in a finite set. Here is the definition and the associated notation.

Definition C Cardinality
Suppose S is a finite set. Then the number of elements in S is called the **cardinality** or **size** of S, and is denoted $|S|$. □

Example CS Cardinality and Size
If $S = \{\blacklozenge, \bigstar, \blacksquare\}$, then $|S| = 3$. △

Subsection SO
Set Operations

In this subsection we define and illustrate the three most common basic ways to manipulate sets to create other sets. Since much of linear algebra is about sets, we will use these often.

Definition SU Set Union
Suppose S and T are sets. Then the **union** of S and T, denoted $S \cup T$, is the set

whose elements are those that are elements of S or of T, or both. More formally,

$$x \in S \cup T \text{ if and only if } x \in S \text{ or } x \in T$$

□

Notice that the use of the word "or" in this definition is meant to be non-exclusive. That is, it allows for x to be an element of both S and T and still qualify for membership in $S \cup T$.

Example SU Set union
If $S = \{\blacklozenge, \bigstar, \blacksquare\}$ and $T = \{\blacklozenge, \bigstar, \blacktriangle\}$ then $S \cup T = \{\blacklozenge, \bigstar, \blacksquare, \blacktriangle\}$. △

Definition SI Set Intersection
Suppose S and T are sets. Then the **intersection** of S and T, denoted $S \cap T$, is the set whose elements are only those that are elements of S and of T. More formally,

$$x \in S \cap T \text{ if and only if } x \in S \text{ and } x \in T$$

□

Example SI Set intersection
If $S = \{\blacklozenge, \bigstar, \blacksquare\}$ and $T = \{\blacklozenge, \bigstar, \blacktriangle\}$ then $S \cap T = \{\blacklozenge, \bigstar\}$. △

The union and intersection of sets are operations that begin with two sets and produce a third, new, set. Our final operation is the set complement, which we usually think of as an operation that takes a single set and creates a second, new, set. However, if you study the definition carefully, you will see that it needs to be computed *relative* to some "universal" set.

Definition SC Set Complement
Suppose S is a set that is a subset of a universal set U. Then the **complement** of S, denoted $\overline{S}$, is the set whose elements are those that are elements of U and not elements of S. More formally,

$$x \in \overline{S} \text{ if and only if } x \in U \text{ and } x \notin S$$

□

Notice that there is nothing at all special about the universal set. This is simply a term that suggests that U contains all of the possible objects we are considering. Often this set will be clear from the context, and we will not think much about it, nor reference it in our notation. In other cases (rarely in our work in this course) the exact nature of the universal set must be made explicit, and reference to it will possibly be carried through in our choice of notation.

Example SC Set complement
If $U = \{\blacklozenge, \bigstar, \blacksquare, \blacktriangle\}$ and $S = \{\blacklozenge, \bigstar, \blacksquare\}$ then $\overline{S} = \{\blacktriangle\}$. △

There are many more natural operations that can be performed on sets, such as an exclusive-or and the symmetric difference. Many of these can be defined in terms of the union, intersection and complement. We will not have much need of them in this course, and so we will not give precise descriptions here in this preliminary section.

There is also an interesting variety of basic results that describe the interplay of these operations with each other. We mention just two as an example, these are known as DeMorgan's Laws.

$$\overline{(S \cup T)} = \overline{S} \cap \overline{T}$$
$$\overline{(S \cap T)} = \overline{S} \cup \overline{T}$$

Besides having an appealing symmetry, we mention these two facts, since constructing the proofs of each is a useful exercise that will require a solid understanding of all but one of the definitions presented in this section. Give it a try.

Reference

Proof Techniques

In this section we collect many short essays designed to help you understand how to read, understand and construct proofs. Some are very factual, while others consist of advice. They appear in the order that they are first needed (or advisable) in the text, and are meant to be self-contained. So you should not think of reading through this section in one sitting as you begin this course. But be sure to head back here for a first reading whenever the text suggests it. Also think about returning to browse at various points during the course, and especially as you struggle with becoming an accomplished mathematician who is comfortable with the difficult process of designing new proofs.

Proof Technique D
Definitions

A definition is a made-up term, used as a kind of shortcut for some typically more complicated idea. For example, we say a whole number is **even** as a shortcut for saying that when we divide the number by two we get a remainder of zero. With a precise definition, we can answer certain questions unambiguously. For example, did you ever wonder if zero was an even number? Now the answer should be clear since we have a precise definition of what we mean by the term even.

A single term might have several possible definitions. For example, we could say that the whole number n is even if there is some whole number k such that $n = 2k$. We say this is an equivalent definition since it categorizes even numbers the same way our first definition does.

Definitions are like two-way streets — we can use a definition to replace something rather complicated by its definition (if it fits) *and* we can replace a definition by its more complicated description. A definition is usually written as some form of an implication, such as "If something-nice-happens, then **blatzo**." However, this also means that "If blatzo, then something-nice-happens," even though this may not be formally stated. This is what we mean when we say a definition is a two-way street — it is really two implications, going in opposite "directions."

Anybody (including you) can make up a definition, so long as it is unambiguous, but the real test of a definition's utility is whether or not it is useful for concisely describing interesting or frequent situations. We will try to restrict our definitions to parts of speech that are nouns (e.g. "matrix") or adjectives (e.g. "nonsingular" matrix), and so avoid definitions that are verbs or adverbs. Therefore our definitions will describe an object (noun) or a property of an object (adjective).

We will talk about theorems later (and especially equivalences). For now, be sure not to confuse the notion of a definition with that of a theorem.

In this book, we will display every new definition carefully set-off from the text, and the term being defined will be written thus: **definition**. Additionally, there is a full list of all the definitions, in order of their appearance located in the reference section of the same name (Definitions). Definitions are critical to doing mathematics and proving theorems, so we have given you lots of ways to locate a definition should you forget its…uh, well, …definition.

Can you formulate a precise definition for what it means for a number to be **odd**? (do not just say it is the opposite of even. Act as if you do not have a definition for even yet.) Can you formulate your definition a second, equivalent, way? Can you employ your definition to test an odd and an even number for "odd-ness"?

Proof Technique T
Theorems

Higher mathematics is about understanding theorems. Reading them, understanding them, applying them, proving them. Every theorem is a shortcut — we prove something in general, and then whenever we find a specific instance covered by the theorem we can immediately say that we know something else about the situation by applying the theorem. In many cases, this new information can be gained with much less effort than if we did not know the theorem.

The first step in understanding a theorem is to realize that the statement of every theorem can be rewritten using statements of the form "If something-happens, then something-else-happens." The "something-happens" part is the **hypothesis** and the "something-else-happens" is the **conclusion**. To understand a theorem, it helps to rewrite its statement using this construction. To apply a theorem, we verify that "something-happens" in a particular instance and immediately conclude that "something-else-happens." To prove a theorem, we must argue based on the assumption that the hypothesis is true, and arrive through the process of logic that the conclusion must then also be true.

Proof Technique L
Language

> Like any science, the language of math must be understood before further study can continue.
>
> Erin Wilson, Student
> September, 2004

Mathematics is a language. It is a way to express complicated ideas clearly, precisely, and unambiguously. Because of this, it can be difficult to read. Read slowly, and have pencil and paper at hand. It will usually be necessary to read something several times. While reading can be difficult, it is even harder to speak mathematics, and so that is the topic of this technique.

"Natural" language, in the present case English, is fraught with ambiguity. Consider the possible meanings of the sentence: The fish is ready to eat. One fish, or two fish? Are the fish hungry, or will the fish be eaten? (See Exercise SSLE.M10, Exercise SSLE.M11, Exercise SSLE.M12, Exercise SSLE.M13.) In your daily interactions with others, give some thought to how many mis-understandings arise from the ambiguity of pronouns, modifiers and objects.

I am going to suggest a simple modification to the way you use language that will make it much, much easier to become proficient at speaking mathematics and eventually it will become second nature. Think of it as a training aid or practice drill you might use when learning to become skilled at a sport.

First, eliminate pronouns from your vocabulary when discussing linear algebra, in class or with your colleagues. Do not use: it, that, those, their or similar sources of confusion. This is the single easiest step you can take to make your oral expression of mathematics clearer to others, and in turn, it will greatly help your own understanding.

Now rid yourself of the word “thing” (or variants like “something”). When you are tempted to use this word realize that there is some object you want to discuss, and we likely have a definition for that object (see the discussion at Proof Technique D). Always “think about your objects” and many aspects of the study of mathematics will get easier. Ask yourself: “Am I working with a set, a number, a function, an operation, a differential equation, or what?” Knowing what an object *is* will allow you to narrow down the procedures you may apply to **it**. If you have studied an object-oriented computer programming language, then you will already have experience identifying objects and thinking carefully about what procedures are allowed to be applied to them.

Third, eliminate the verb “works” (as in “the equation works”) from your vocabulary. This term is used as a substitute when we are not sure just what we are trying to accomplish. Usually we are trying to say that some object fulfills some condition. The condition might even have a definition associated with it, making it even easier to describe.

Last, speak slooooowly and thoughtfully as you try to get by without all these lazy words. It is hard at first, but you will get better with practice. Especially in class, when the pressure is on and all eyes are on you, do not succumb to the temptation to use these weak words. Slow down, we’d all rather wait for a slow, well-formed question or answer than a fast, sloppy, incomprehensible one.

You will find the improvement in your ability to *speak* clearly about complicated ideas will greatly improve your ability to *think* clearly about complicated ideas. And I believe that you cannot think clearly about complicated ideas if you cannot formulate questions or answers clearly in the correct language. This is as applicable to the study of law, economics or philosophy as it is to the study of science or mathematics.

In this spirit, Dupont Hubert has contributed the following quotation, which is widely used in French mathematics courses (and which might be construed as the contrapositive of Proof Technique CP)

> Ce que l’on concoit bien s’enonce clairement,
> Et les mots pour le dire arrivent aisement.

Nicolas Boileau
L’art poetique
Chant I, 1674

which translates as

> Whatever is well conceived is clearly said,
> And the words to say it flow with ease.

So when you come to class, check your pronouns at the door, along with other weak words. And when studying with friends, you might make a game of catching one another using pronouns, “thing,” or “works.” I know I’ll be calling you on it!

Proof Technique GS
Getting Started

“I don’t know how to get started!” is often the lament of the novice proof-builder. Here are a few pieces of advice.

1. As mentioned in Proof Technique T, rewrite the statement of the theorem in an "if-then" form. This will simplify identifying the hypothesis and conclusion, which are referenced in the next few items.

2. Ask yourself what *kind* of statement you are trying to prove. This is always part of your conclusion. Are you being asked to conclude that two numbers are equal, that a function is differentiable or a set is a subset of another? You cannot bring other techniques to bear if you do not know what *type* of conclusion you have.

3. Write down reformulations of your hypotheses. Interpret and translate each definition properly.

4. Write your hypothesis at the top of a sheet of paper and your conclusion at the bottom. See if you can formulate a statement that precedes the conclusion and also implies it. Work down from your hypothesis, and up from your conclusion, and see if you can meet in the middle. When you are finished, rewrite the proof nicely, from hypothesis to conclusion, with verifiable implications giving each subsequent statement.

5. As you work through your proof, think about what kinds of objects your symbols represent. For example, suppose A is a set and $f(x)$ is a real-valued function. Then the expression $A + f$ might make no sense if we have not defined what it means to "add" a set to a function, so we can stop at that point and adjust accordingly. On the other hand we might understand $2f$ to be the function whose rule is described by $(2f)(x) = 2f(x)$. "Think about your objects" means to always verify that your objects and operations are compatible.

Proof Technique C
Constructive Proofs

Conclusions of proofs come in a variety of types. Often a theorem will simply *assert* that something exists. The best way, but not the only way, to show something exists is to actually build it. Such a proof is called **constructive**. The thing to realize about constructive proofs is that the proof itself will contain a procedure that might be used computationally to construct the desired object. If the procedure is not too cumbersome, then the proof itself is as useful as the statement of the theorem.

Proof Technique E
Equivalences

When a theorem uses the phrase "if and only if" (or the abbreviation "iff") it is a shorthand way of saying that two if-then statements are true. So if a theorem says "P if and only if Q," then it is true that "if P, then Q" while it is also true that "if Q, then P." For example, it may be a theorem that "I wear bright yellow knee-high plastic boots if and only if it is raining." This means that I *never* forget to wear my super-duper yellow boots when it is raining *and* I would not be seen in such silly boots *unless* it was raining. You never have one without the other. I have my boots on and it is raining *or* I do not have my boots on and it is dry.

The upshot for proving such theorems is that it is like a 2-for-1 sale, we get to do *two* proofs. Assume P and conclude Q, then start over and assume Q and conclude P. For this reason, "if and only if" is sometimes abbreviated by $\iff$, while proofs indicate which of the two implications is being proved by prefacing each with $\Rightarrow$ or $\Leftarrow$. A carefully written proof will remind the reader which statement is being used as the hypothesis, a quicker version will let the reader deduce it from the direction of the arrow. Tradition dictates we do the "easy" half first, but that is hard for a student to know until you have finished doing both halves! Oh well, if you rewrite your proofs (a good habit), you can then choose to put the easy half first.

Theorems of this type are called "equivalences" or "characterizations," and they are some of the most pleasing results in mathematics. They say that two objects, or two situations, are really the same. You do not have one without the other, like rain and my yellow boots. The more different P and Q seem to be, the more pleasing it is to discover they are really equivalent. And if P describes a very mysterious solution or involves a tough computation, while Q is transparent or involves easy computations, then we have found a great shortcut for better understanding or faster computation. Remember that every theorem really is a shortcut in some form. You will also discover that if proving $P \Rightarrow Q$ is very easy, then proving $Q \Rightarrow P$ is likely to be proportionately harder. Sometimes the two halves are about equally hard. And in rare cases, you can string together a whole sequence of other equivalences to form the one you are after and you do not even need to do two halves. In this case, the argument of one half is just the argument of the other half, but in reverse.

One last thing about equivalences. If you see a statement of a theorem that says two things are "equivalent," translate it first into an "if and only if" statement.

Proof Technique N
Negation

When we construct the contrapositive of a theorem (Proof Technique CP), we need to negate the two statements in the implication. And when we construct a proof by contradiction (Proof Technique CD), we need to negate the conclusion of the theorem. One way to construct a converse (Proof Technique CV) is to simultaneously negate the hypothesis and conclusion of an implication (but remember that this is not guaranteed to be a true statement). So we often have the need to negate statements, and in some situations it can be tricky.

If a statement says that a set is empty, then its negation is the statement that the set is nonempty. That is straightforward. Suppose a statement says "something-happens" for all i, or every i, or any i. Then the negation is that "something-does-not-happen" for at least one value of i. If a statement says that there exists at least one "thing," then the negation is the statement that there is no "thing." If a statement says that a "thing" is unique, then the negation is that there is zero, or more than one, of the "thing."

We are not covering all of the possibilities, but we wish to make the point that logical qualifiers like "there exists" or "for every" must be handled with care when negating statements. Studying the proofs which employ contradiction (as listed in Proof Technique CD) is a good first step towards understanding the range of possibilities.

Proof Technique CP
Contrapositives

The **contrapositive** of an implication $P \Rightarrow Q$ is the implication $\text{not}(Q) \Rightarrow \text{not}(P)$, where "not" means the logical negation, or opposite. An implication is true if and only if its contrapositive is true. In symbols, $(P \Rightarrow Q) \iff (\text{not}(Q) \Rightarrow \text{not}(P))$ is a theorem. Such statements about logic, that are always true, are known as **tautologies**.

For example, it is a theorem that "if a vehicle is a fire truck, then it has big tires and has a siren." (Yes, I'm sure you can conjure up a counterexample, but play along with me anyway.) The contrapositive is "if a vehicle does not have big tires or does not have a siren, then it is not a fire truck." Notice how the "and" became an "or" when we negated the conclusion of the original theorem.

It will frequently happen that it is easier to construct a proof of the contrapositive than of the original implication. If you are having difficulty formulating a proof of some implication, see if the contrapositive is easier for you. The trick is to construct the negation of complicated statements accurately. More on that later.

Proof Technique CV
Converses

The **converse** of the implication $P \Rightarrow Q$ is the implication $Q \Rightarrow P$. There is no guarantee that the truth of these two statements are related. In particular, if an implication has been proven to be a theorem, then do not try to use its converse too, as if it were a theorem. Sometimes the converse is true (and we have an equivalence, see Proof Technique E). But more likely the converse is false, especially if it was not included in the statement of the original theorem.

For example, we have the theorem, "if a vehicle is a fire truck, then it is has big tires and has a siren." The converse is false. The statement that "if a vehicle has big tires and a siren, then it is a fire truck" is false. A police vehicle for use on a sandy public beach would have big tires and a siren, yet is not equipped to fight fires.

We bring this up now, because Theorem CSRN has a tempting converse. Does this theorem say that if $r < n$, then the system is consistent? Definitely not, as Archetype E has $r = 3 < 4 = n$, yet is inconsistent. This example is then said to be a **counterexample** to the converse. Whenever you think a theorem that is an implication might actually be an equivalence, it is good to hunt around for a counterexample that shows the converse to be false (the archetypes, Archetypes, can be a good hunting ground).

Proof Technique CD
Contradiction

Another proof technique is known as "proof by contradiction" and it can be a powerful (and satisfying) approach. Simply put, suppose you wish to prove the implication, "If A, then B." As usual, we assume that A is true, but we also make the additional assumption that B is false. If our original implication is true, then these twin assumptions should lead us to a logical inconsistency. In practice we assume the negation of B to be true (see Proof Technique N). So we argue from the assumptions A and not(B) looking for some obviously false conclusion such as $1 = 6$, or a set is simultaneously empty and nonempty, or a matrix is both nonsingular and singular.

You should be careful about formulating proofs that look like proofs by contradiction, but really are not. This happens when you assume A and not(B) and proceed to give a "normal" and direct proof that B is true by only using the assumption that A is true. Your last step is to then claim that B is true and you then appeal to the assumption that not(B) is true, thus getting the desired contradiction. Instead, you could have avoided the overhead of a proof by contradiction and just run with the direct proof. This stylistic flaw is known, quite graphically, as "setting up the strawman to knock him down."

Here is a simple example of a proof by contradiction. There are direct proofs that are just about as easy, but this will demonstrate the point, while narrowly avoiding knocking down the straw man.

Theorem: If a and b are odd integers, then their product, ab, is odd.

Proof: To begin a proof by contradiction, assume the hypothesis, that a and b are odd. Also assume the negation of the conclusion, in this case, that ab is even.

Then there are integers, j, k, ℓ so that $a = 2j + 1$, $b = 2k + 1$, $ab = 2\ell$. Then

$$\begin{aligned} 0 &= ab - ab \\ &= (2j+1)(2k+1) - (2\ell) \\ &= 4jk + 2j + 2k - 2\ell + 1 \\ &= 2\left(2jk + j + k - \ell\right) + 1 \end{aligned}$$

Again, we do not offer this example as the *best* proof of this fact about even and odd numbers, but rather it is a simple illustration of a proof by contradiction. You can find examples of proofs by contradiction in

Theorem RREFU, Theorem NMUS, Theorem NPNT, Theorem TTMI, Theorem GSP, Theorem ELIS, Theorem EDYES, Theorem EMHE, Theorem EDELI, and Theorem DMFE, in addition to several examples and solutions to exercises.

Proof Technique U
Uniqueness

A theorem will sometimes claim that some object, having some desirable property, is unique. In other words, there should be only one such object. To prove this, a standard technique is to assume there are two such objects and proceed to analyze the consequences. The end result may be a contradiction (Proof Technique CD), or the conclusion that the two allegedly different objects really are equal.

Proof Technique ME
Multiple Equivalences

A very specialized form of a theorem begins with the statement "The following are equivalent...," which is then followed by a list of statements. Informally, this lead-in sometimes gets abbreviated by "TFAE." This formulation means that any two of the statements on the list can be connected with an "if and only if" to form a theorem. So if the list has n statements then, there are $\frac{n(n-1)}{2}$ possible equivalences that can be constructed (and are claimed to be true).

Suppose a theorem of this form has statements denoted as A, B, C, ..., Z. To prove the entire theorem, we can prove $A \Rightarrow B$, $B \Rightarrow C$, $C \Rightarrow D$, ..., $Y \Rightarrow Z$ and finally, $Z \Rightarrow A$. This circular chain of n equivalences would allow us, logically, if not practically, to form any one of the $\frac{n(n-1)}{2}$ possible equivalences by chasing the equivalences around the circle as far as required.

Proof Technique PI
Proving Identities

Many theorems have conclusions that say two objects are equal. Perhaps one object is hard to compute or understand, while the other is easy to compute or understand. This would make for a pleasing theorem. Whether the result is pleasing or not, we take the same approach to formulate a proof. Sometimes we need to employ specialized notions of equality, such as Definition SE or Definition CVE, but in other cases we can string together a list of equalities.

The wrong way to prove an identity is to begin by writing it down and then beating on it until it reduces to an obvious identity. The first flaw is that you would be writing down the statement you wish to prove, as if you already believed it to be true. But more dangerous is the possibility that some of your maneuvers are not reversible. Here is an example. Let us prove that $3 = -3$.

$$3 = -3 \qquad \text{(This is a bad start)}$$

$$\begin{aligned} 3^2 &= (-3)^2 && \text{Square both sides} \\ 9 &= 9 \\ 0 &= 0 && \text{Subtract 9 from both sides} \end{aligned}$$

So because $0 = 0$ is a true statement, does it follow that $3 = -3$ is a true statement? Nope. Of course, we did not really expect a legitimate proof of $3 = -3$, but this attempt should illustrate the dangers of this (incorrect) approach.

What you have just seen in the proof of Theorem VSPCV, and what you will see consistently throughout this text, is proofs of the following form. To prove that $A = D$ we write

$$\begin{aligned} A &= B && \text{Theorem, Definition or Hypothesis justifying } A = B \\ &= C && \text{Theorem, Definition or Hypothesis justifying } B = C \\ &= D && \text{Theorem, Definition or Hypothesis justifying } C = D \end{aligned}$$

In your scratch work exploring possible approaches to proving a theorem you may massage a variety of expressions, sometimes making connections to various bits and pieces, while some parts get abandoned. Once you see a line of attack, rewrite your proof carefully mimicking this style.

Proof Technique DC
Decompositions

Much of your mathematical upbringing, especially once you began a study of algebra, revolved around simplifying expressions — combining like terms, obtaining common denominators so as to add fractions, factoring in order to solve polynomial equations. However, as often as not, we will do the opposite. Many theorems and techniques will revolve around taking some object and "decomposing" it into some combination of other objects, ostensibly in a more complicated fashion. When we say something can "be written as" something else, we mean that the one object can be decomposed into some combination of other objects. This may seem unnatural at first, but results of this type will give us insight into the structure of the original object by exposing its inner workings. An appropriate analogy might be stripping the wallboards away from the interior of a building to expose the structural members supporting the whole building.

Perhaps you have studied integral calculus, or a pre-calculus course, where you learned about partial fractions. This is a technique where a fraction of two polynomials is *decomposed* (written as, expressed as) a sum of simpler fractions. The purpose in calculus is to make finding an antiderivative simpler. For example, you can verify the truth of the expression

$$\frac{12x^5 + 2x^4 - 20x^3 + 66x^2 - 294x + 308}{x^6 + x^5 - 3x^4 + 21x^3 - 52x^2 + 20x - 48} = \frac{5x+2}{x^2 - x + 6} + \frac{3x-7}{x^2+1} + \frac{3}{x+4} + \frac{1}{x-2}$$

In an early course in algebra, you might be expected to combine the four terms on the right over a common denominator to create the "simpler" expression on the left. Going the other way, the partial fraction technique would allow you to systematically *decompose* the fraction of polynomials on the left into the sum of the four (arguably) simpler fractions of polynomials on the right.

This is a major shift in thinking, so come back here often, especially when we say "can be written as", or "can be expressed as," or "can be decomposed as."

Proof Technique I
Induction

"Induction" or "mathematical induction" is a framework for proving statements that are indexed by integers. In other words, suppose you have a statement to prove that is really multiple statements, one for $n = 1$, another for $n = 2$, a third for $n = 3$, and so on. If there is enough similarity between the statements, then you can use a script (the framework) to prove them all at once.

For example, consider the theorem: $1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$ for $n \geq 1$.

This is shorthand for the many statements $1 = \frac{1(1+1)}{2}$, $1+2 = \frac{2(2+1)}{2}$, $1+2+3 = \frac{3(3+1)}{2}$, $1+2+3+4 = \frac{4(4+1)}{2}$, and so on. Forever. You can do the calculations in each of these statements and verify that all four are true. We might not be surprised to learn that the fifth statement is true as well (go ahead and check). However, do we think the theorem is true for $n = 872$? Or $n = 1,234,529$?

To see that these questions are not so ridiculous, consider the following example from Rotman's *Journey into Mathematics*. The statement "$n^2 - n + 41$ is prime" is true for integers $1 \leq n \leq 40$ (check a few). However, when we check $n = 41$ we find $41^2 - 41 + 41 = 41^2$, which is not prime.

So how do we prove infinitely many statements all at once? More formally, let us denote our statements as $P(n)$. Then, if we can prove the two assertions

1. $P(1)$ is true.

2. If $P(k)$ is true, then $P(k+1)$ is true.

then it follows that $P(n)$ is true for all $n \geq 1$. To understand this, I liken the process to climbing an infinitely long ladder with equally spaced rungs. Confronted with such a ladder, suppose I tell you that you are able to step up onto the first rung, and if you are on any particular rung, then you are capable of stepping up to the next rung. It follows that you can climb the ladder as far up as you wish. The first formal assertion above is akin to stepping onto the first rung, and the second formal assertion is akin to assuming that if you are on any one rung then you can always reach the next rung.

In practice, establishing that $P(1)$ is true is called the "base case" and in most cases is straightforward. Establishing that $P(k) \Rightarrow P(k+1)$ is referred to as the "induction step," or in this book (and elsewhere) we will typically refer to the assumption of $P(k)$ as the "induction hypothesis." This is perhaps the most mysterious part of a proof by induction, since we are eventually trying to prove that $P(n)$ is true and it appears we do this by assuming what we are trying to prove (when we assume $P(k)$). We are trying to prove the truth of $P(n)$ (for all n), but in the induction step we establish the truth of an *implication*, $P(k) \Rightarrow P(k+1)$, an "if-then" statement. Sometimes it is even worse, since as you get more comfortable with induction, we often do not bother to use a different letter (k) for the index (n) in the induction step. Notice that the second formal assertion never says that $P(k)$ is true, it simply says that *if* $P(k)$ were true, what might logically follow. We can establish statements like "If I lived on the moon, then I could pole-vault over a bar 12 meters high." This may be a true statement, but it does not say we live on the moon, and indeed we may never live there.

Enough generalities. Let us work an example and prove the theorem above about sums of integers. Formally, our statement is $P(n):\ 1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}$.

Proof: Base Case. $P(1)$ is the statement $1 = \frac{1(1+1)}{2}$, which we see simplifies to the true statement $1 = 1$.

Induction Step: We will assume $P(k)$ is true, and will try to prove $P(k+1)$. Given what we want to accomplish, it is natural to begin by examining the sum of the first $k+1$ integers.

$$
\begin{aligned}
1+2+3+\cdots+(k+1) &= (1+2+3+\cdots+k)+(k+1) \\
&= \frac{k(k+1)}{2}+(k+1) && \text{Induction Hypothesis} \\
&= \frac{k^2+k}{2}+\frac{2k+2}{2} \\
&= \frac{k^2+3k+2}{2} \\
&= \frac{(k+1)(k+2)}{2} \\
&= \frac{(k+1)((k+1)+1)}{2}
\end{aligned}
$$

We then recognize the two ends of this chain of equalities as $P(k+1)$. So, by mathematical induction, the theorem is true for all n.

How do you recognize when to use induction? The first clue is a statement that is really many statements, one for each integer. The second clue would be that you begin a more standard proof and you find yourself using words like "and so on" (as above!) or lots of ellipses (dots) to establish patterns that you are convinced continue on and on forever. However, there are many minor instances where induction might be warranted but we do not bother.

Induction is important enough, and used often enough, that it appears in various variations. The base case sometimes begins with $n=0$, or perhaps an integer greater than n. Some formulate the induction step as $P(k-1) \Rightarrow P(k)$. There is also a "strong form" of induction where we assume all of $P(1)$, $P(2)$, $P(3)$,$\dots$ $P(k)$ as a hypothesis for showing the conclusion $P(k+1)$.

You can find examples of induction in the proofs of Theorem GSP, Theorem DER, Theorem DT, Theorem DIM, Theorem EOMP, Theorem DCP, and Theorem KPLT.

Proof Technique P
Practice

Here is a technique used by many practicing mathematicians when they are teaching themselves new mathematics. As they read a textbook, monograph or research article, they attempt to prove each new theorem themselves, *before* reading the proof. Often the proofs can be very difficult, so it is wise not to spend too much time on each. Maybe limit your losses and try each proof for 10 or 15 minutes. Even if the proof is not found, it is time well-spent. You become more familiar with the definitions involved, and the hypothesis and conclusion of the theorem. When you do work through the proof, it might make more sense, and you will gain added insight about just how to construct a proof.

Proof Technique LC
Lemmas and Corollaries

Theorems often go by different titles. Two of the most popular being "lemma" and "corollary." Before we describe the fine distinctions, be aware that lemmas, corollaries, propositions, claims and facts are all just theorems. And every theorem can be rephrased as an "if-then" statement, or perhaps a pair of "if-then" statements expressed as an equivalence (Proof Technique E).

A lemma is a theorem that is not too interesting in its own right, but is important for proving other theorems. It might be a generalization or abstraction of a key step of several different proofs. For this reason you often hear the phrase “technical lemma” though some might argue that the adjective “technical” is redundant.

A corollary is a theorem that follows very easily from another theorem. For this reason, corollaries frequently do not have proofs. You are expected to easily and quickly see how a previous theorem implies the corollary.

A proposition or fact is really just a codeword for a theorem. A claim might be similar, but some authors like to use claims within a proof to organize key steps. In a similar manner, some long proofs are organized as a series of lemmas.

In order to not confuse the novice, we have just called all our theorems theorems. It is also an organizational convenience. With only theorems and definitions, the theoretical backbone of the course is laid bare in the two lists of Definitions and Theorems.

Archetypes

This section contains definitions and capsule summaries for each archetypical example. Comprehensive and detailed analysis of each can be found in the online supplement.

Archetype A Linear system of three equations, three unknowns. Singular coefficient matrix with dimension 1 null space. Integer eigenvalues and a degenerate eigenspace for coefficient matrix.

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 1 \\ 2x_1 + x_2 + x_3 &= 8 \\ x_1 + x_2 &= 5 \end{aligned}$$

Archetype B System with three equations, three unknowns. Nonsingular coefficient matrix. Distinct integer eigenvalues for coefficient matrix.

$$\begin{aligned} -7x_1 - 6x_2 - 12x_3 &= -33 \\ 5x_1 + 5x_2 + 7x_3 &= 24 \\ x_1 + 4x_3 &= 5 \end{aligned}$$

Archetype C System with three equations, four variables. Consistent. Null space of coefficient matrix has dimension 1.

$$\begin{aligned} 2x_1 - 3x_2 + x_3 - 6x_4 &= -7 \\ 4x_1 + x_2 + 2x_3 + 9x_4 &= -7 \\ 3x_1 + x_2 + x_3 + 8x_4 &= -8 \end{aligned}$$

Archetype D System with three equations, four variables. Consistent. Null space of coefficient matrix has dimension 2. Coefficient matrix identical to that of Archetype E, vector of constants is different.

$$\begin{aligned} 2x_1 + x_2 + 7x_3 - 7x_4 &= 8 \\ -3x_1 + 4x_2 - 5x_3 - 6x_4 &= -12 \\ x_1 + x_2 + 4x_3 - 5x_4 &= 4 \end{aligned}$$

Archetype E System with three equations, four variables. Inconsistent. Null space of coefficient matrix has dimension 2. Coefficient matrix identical to that of Archetype D, constant vector is different.

$$\begin{aligned} 2x_1 + x_2 + 7x_3 - 7x_4 &= 2 \\ -3x_1 + 4x_2 - 5x_3 - 6x_4 &= 3 \\ x_1 + x_2 + 4x_3 - 5x_4 &= 2 \end{aligned}$$

Archetype F System with four equations, four variables. Nonsingular coefficient matrix. Integer eigenvalues, one has "high" multiplicity.

$$\begin{aligned} 33x_1 - 16x_2 + 10x_3 - 2x_4 &= -27 \\ 99x_1 - 47x_2 + 27x_3 - 7x_4 &= -77 \\ 78x_1 - 36x_2 + 17x_3 - 6x_4 &= -52 \\ -9x_1 + 2x_2 + 3x_3 + 4x_4 &= 5 \end{aligned}$$

Archetype G System with five equations, two variables. Consistent. Null space of coefficient matrix has dimension 0. Coefficient matrix identical to that of Archetype

H, constant vector is different.

$$\begin{aligned} 2x_1 + 3x_2 &= 6 \\ -x_1 + 4x_2 &= -14 \\ 3x_1 + 10x_2 &= -2 \\ 3x_1 - x_2 &= 20 \\ 6x_1 + 9x_2 &= 18 \end{aligned}$$

Archetype H System with five equations, two variables. Inconsistent, overdetermined. Null space of coefficient matrix has dimension 0. Coefficient matrix identical to that of Archetype G, constant vector is different.

$$\begin{aligned} 2x_1 + 3x_2 &= 5 \\ -x_1 + 4x_2 &= 6 \\ 3x_1 + 10x_2 &= 2 \\ 3x_1 - x_2 &= -1 \\ 6x_1 + 9x_2 &= 3 \end{aligned}$$

Archetype I System with four equations, seven variables. Consistent. Null space of coefficient matrix has dimension 4.

$$\begin{aligned} x_1 + 4x_2 - x_4 + 7x_6 - 9x_7 &= 3 \\ 2x_1 + 8x_2 - x_3 + 3x_4 + 9x_5 - 13x_6 + 7x_7 &= 9 \\ 2x_3 - 3x_4 - 4x_5 + 12x_6 - 8x_7 &= 1 \\ -x_1 - 4x_2 + 2x_3 + 4x_4 + 8x_5 - 31x_6 + 37x_7 &= 4 \end{aligned}$$

Archetype J System with six equations, nine variables. Consistent. Null space of coefficient matrix has dimension 5.

$$\begin{aligned} x_1 + 2x_2 - 2x_3 + 9x_4 + 3x_5 - 5x_6 - 2x_7 + x_8 + 27x_9 &= -5 \\ 2x_1 + 4x_2 + 3x_3 + 4x_4 - x_5 + 4x_6 + 10x_7 + 2x_8 - 23x_9 &= 18 \\ x_1 + 2x_2 + x_3 + 3x_4 + x_5 + x_6 + 5x_7 + 2x_8 - 7x_9 &= 6 \\ 2x_1 + 4x_2 + 3x_3 + 4x_4 - 7x_5 + 2x_6 + 4x_7 - 11x_9 &= 20 \\ x_1 + 2x_2 + 5x_4 + 2x_5 - 4x_6 + 3x_7 + 8x_8 + 13x_9 &= -4 \\ -3x_1 - 6x_2 - x_3 - 13x_4 + 2x_5 - 5x_6 - 4x_7 + 13x_8 + 10x_9 &= -29 \end{aligned}$$

Archetype K Square matrix of size 5. Nonsingular. 3 distinct eigenvalues, 2 of multiplicity 2.

$$\begin{bmatrix} 10 & 18 & 24 & 24 & -12 \\ 12 & -2 & -6 & 0 & -18 \\ -30 & -21 & -23 & -30 & 39 \\ 27 & 30 & 36 & 37 & -30 \\ 18 & 24 & 30 & 30 & -20 \end{bmatrix}$$

Archetype L Square matrix of size 5. Singular, nullity 2. 2 distinct eigenvalues, each of "high" multiplicity.

$$\begin{bmatrix} -2 & -1 & -2 & -4 & 4 \\ -6 & -5 & -4 & -4 & 6 \\ 10 & 7 & 7 & 10 & -13 \\ -7 & -5 & -6 & -9 & 10 \\ -4 & -3 & -4 & -6 & 6 \end{bmatrix}$$

Archetype M Linear transformation with bigger domain than codomain, so it is guaranteed to not be injective. Happens to not be surjective.

$$T\colon \mathbb{C}^5 \to \mathbb{C}^3, \quad T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 + 3x_3 + 4x_4 + 4x_5 \\ 3x_1 + x_2 + 4x_3 - 3x_4 + 7x_5 \\ x_1 - x_2 - 5x_4 + x_5 \end{bmatrix}$$

Archetype N Linear transformation with domain larger than its codomain, so it is guaranteed to not be injective. Happens to be onto.

$$T\colon \mathbb{C}^5 \to \mathbb{C}^3, \quad T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + x_2 + 3x_3 - 4x_4 + 5x_5 \\ x_1 - 2x_2 + 3x_3 - 9x_4 + 3x_5 \\ 3x_1 + 4x_3 - 6x_4 + 5x_5 \end{bmatrix}$$

Archetype O Linear transformation with a domain smaller than the codomain, so it is guaranteed to not be onto. Happens to not be one-to-one.

$$T\colon \mathbb{C}^3 \to \mathbb{C}^5, \quad T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -x_1 + x_2 - 3x_3 \\ -x_1 + 2x_2 - 4x_3 \\ x_1 + x_2 + x_3 \\ 2x_1 + 3x_2 + x_3 \\ x_1 + 2x_3 \end{bmatrix}$$

Archetype P Linear transformation with a domain smaller that its codomain, so it is guaranteed to not be surjective. Happens to be injective.

$$T\colon \mathbb{C}^3 \to \mathbb{C}^5, \quad T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -x_1 + x_2 + x_3 \\ -x_1 + 2x_2 + 2x_3 \\ x_1 + x_2 + 3x_3 \\ 2x_1 + 3x_2 + x_3 \\ -2x_1 + x_2 + 3x_3 \end{bmatrix}$$

Archetype Q Linear transformation with equal-sized domain and codomain, so it has the potential to be invertible, but in this case is not. Neither injective nor surjective. Diagonalizable, though.

$$T\colon \mathbb{C}^5 \to \mathbb{C}^5, \quad T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}\right) = \begin{bmatrix} -2x_1 + 3x_2 + 3x_3 - 6x_4 + 3x_5 \\ -16x_1 + 9x_2 + 12x_3 - 28x_4 + 28x_5 \\ -19x_1 + 7x_2 + 14x_3 - 32x_4 + 37x_5 \\ -21x_1 + 9x_2 + 15x_3 - 35x_4 + 39x_5 \\ -9x_1 + 5x_2 + 7x_3 - 16x_4 + 16x_5 \end{bmatrix}$$

Archetype R Linear transformation with equal-sized domain and codomain. Injective, surjective, invertible, diagonalizable, the works.

$$T\colon \mathbb{C}^5 \to \mathbb{C}^5, \quad T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}\right) = \begin{bmatrix} -65x_1 + 128x_2 + 10x_3 - 262x_4 + 40x_5 \\ 36x_1 - 73x_2 - x_3 + 151x_4 - 16x_5 \\ -44x_1 + 88x_2 + 5x_3 - 180x_4 + 24x_5 \\ 34x_1 - 68x_2 - 3x_3 + 140x_4 - 18x_5 \\ 12x_1 - 24x_2 - x_3 + 49x_4 - 5x_5 \end{bmatrix}$$

Archetype S Domain is column vectors, codomain is matrices. Domain is dimension 3 and codomain is dimension 4. Not injective, not surjective.

$$T\colon \mathbb{C}^3 \to M_{22}, \quad T\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a - b & 2a + 2b + c \\ 3a + b + c & -2a - 6b - 2c \end{bmatrix}$$

Archetype T Domain and codomain are polynomials. Domain has dimension 5, while codomain has dimension 6. Is injective, can not be surjective.

$$T\colon P_4 \to P_5, \quad T\left(p(x)\right) = (x-2)p(x)$$

Archetype U Domain is matrices, codomain is column vectors. Domain has dimension 6, while codomain has dimension 4. Cannot be injective, is surjective.

$$T\colon M_{23} \to \mathbb{C}^4, \quad T\left(\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}\right) = \begin{bmatrix} a+2b+12c-3d+e+6f \\ 2a-b-c+d-11f \\ a+b+7c+2d+e-3f \\ a+2b+12c+5e-5f \end{bmatrix}$$

Archetype V Domain is polynomials, codomain is matrices. Both domain and codomain have dimension 4. Injective, surjective, invertible. Square matrix representation, but domain and codomain are unequal, so no eigenvalue information.

$$T\colon P_3 \to M_{22}, \quad T\left(a+bx+cx^2+dx^3\right) = \begin{bmatrix} a+b & a-2c \\ d & b-d \end{bmatrix}$$

Archetype W Domain is polynomials, codomain is polynomials. Domain and codomain both have dimension 3. Injective, surjective, invertible, 3 distinct eigenvalues, diagonalizable.

$$\begin{aligned} &T\colon P_2 \to P_2, \\ &T\left(a+bx+cx^2\right) = (19a+6b-4c) + (-24a-7b+4c)\,x + (36a+12b-9c)\,x^2 \end{aligned}$$

Archetype X Domain and codomain are square matrices. Domain and codomain both have dimension 4. Not injective, not surjective, not invertible, 3 distinct eigenvalues, diagonalizable.

$$T\colon M_{22} \to M_{22}, \quad T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} -2a+15b+3c+27d & 10b+6c+18d \\ a-5b-9d & -a-4b-5c-8d \end{bmatrix}$$

Definitions

Theorems

Section PD Properties of Dimension

Section DM Determinant of a Matrix

Section PDM Properties of Determinants of Matrices

Section EE Eigenvalues and Eigenvectors

Section PEE Properties of Eigenvalues and Eigenvectors

Section SD Similarity and Diagonalization

Notation

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