INTERNATIONAL ACADEMY
OF EDUCATION

INTERNATIONAL BUREAU
OF EDUCATION

Effective pedagogy in mathematics

by Glenda Anthony
and Margaret Walshaw

United Nations
Educational, Scientific and
Cultural Organization

International Bureau
of Education

EDUCATIONAL PRACTICES SERIES–19

The International Academy of Education

The International Academy of Education (IAE) is a not-for-profit scientific association that promotes educational research, and its dissemination and implementation. Founded in 1986, the Academy is dedicated to strengthening the contributions of research, solving critical educational problems throughout the world, and providing better communication among policy makers, researchers, and practitioners.

The seat of the Academy is at the Royal Academy of Science, Literature, and Arts in Brussels, Belgium, and its co-ordinating centre is at Curtin University of Technology in Perth, Australia.

The general aim of the IAE is to foster scholarly excellence in all fields of education. Towards this end, the Academy provides timely syntheses of research-based evidence of international importance. The Academy also provides critiques of research and of its evidentiary basis and its application to policy.

The current members of the Board of Directors of the Academy are:

- Monique Boekaerts, University of Leiden, The Netherlands (President);
- Erik De Corte, University of Leuven, Belgium (Past President);
- Barry Fraser, Curtin University of Technology, Australia (Executive Director);
- Jere Brophy, Michigan State University, United States of America;
- Erik Hanushek, Hoover Institute, Stanford University, United States of America;
- Maria de Ibarrola, National Polytechnical Institute, Mexico;
- Denis Phillips, Stanford University, United States of America.

For more information, see the IAE's website at:

http://www.iaoed.org

Series Preface

This booklet about effective mathematics teaching has been prepared for inclusion in the Educational Practices Series developed by the International Academy of Education and distributed by the International Bureau of Education and the Academy. As part of its mission, the Academy provides timely syntheses of research on educational topics of international importance. This is the nineteenth in a series of booklets on educational practices that generally improve learning. It complements an earlier booklet, *Improving Student Achievement in Mathematics*, by Douglas A. Grouws and Kristin J. Cebulla.

This booklet is based on a synthesis of research evidence produced for the New Zealand Ministry of Education's Iterative Best Evidence Synthesis (BES) Programme by Glenda Anthony and Margaret Walshaw. This synthesis, like the others in the series, is intended to be a catalyst for systemic improvement and sustainable development in education. It is electronically available at www.educationcounts.govt.nz/goto/BES. All the BESs have been written using a collaborative approach that involves the writers, teacher unions, principal groups, teacher educators, academics, researchers, policy advisers and other interested groups. To ensure rigour and usefulness, each BES has followed national guidelines developed by the Ministry of Education. Professor Paul Cobb has provided quality assurance for the original synthesis.

Glenda and Margaret are associate professors at Massey University. As directors of the Centre of Excellence for Research in Mathematics Education, they are involved in a wide range of research projects relating to both classroom and teacher education. They are currently engaged in research that focuses on equitable participation practices in classrooms, communication practices, numeracy practices, and teachers as learners. Their research is widely published in peer reviewed journals including *Mathematics Education Research Journal*, *Review of Educational Research*, *Pedagogies: An International Journal*, and *Contemporary Issues in Early Childhood*.

Suggestions or guidelines for practice must always be responsive to the educational and cultural context, and open to continuing evaluation. No. 19 in this Educational Practices Series presents an inquiry model that teachers and teacher educators can use as a tool for adapting and building on the findings of this synthesis in their own particular contexts.

JERE BROPHY
Editor, Michigan State University
United States of America

Previous titles in the "Educational practices" series:

1. Teaching *by Jere Brophy*. 36 p.
2. Parents and learning *by Sam Redding*. 36 p.
3. Effective educational practices *by Herbert J. Walberg and Susan J. Paik*. 24 p.
4. Improving student achievement in mathematics *by Douglas A. Grouws and Kristin J. Cebulla*. 48 p.
5. Tutoring *by Keith Topping*. 36 p.
6. Teaching additional languages *by Elliot L. Judd, Lihua Tan and Herbert J. Walberg*. 24 p.
7. How children learn *by Stella Vosniadou*. 32 p.
8. Preventing behaviour problems: what works *by Sharon L. Foster, Patricia Brennan, Anthony Biglan, Linna Wang and Suad al-Ghaith*. 30 p.
9. Preventing HIV/AIDS in schools *by Inon I. Schenker and Jenny M. Nyirenda*. 32 p.
10. Motivation to learn *by Monique Boekaerts*. 28 p.
11. Academic and social emotional learning *by Maurice J. Elias*. 31 p.
12. Teaching reading *by Elizabeth S. Pang, Angaluki Muaka, Elizabeth B. Bernhardt and Michael L. Kamil*. 23 p.
13. Promoting pre-school language *by John Lybolt and Catherine Gottfred*. 27 p.
14. Teaching speaking, listening and writing *by Trudy Wallace, Winifred E. Stariha and Herbert J. Walberg*. 19 p.
15. Using new media *by Clara Chung-wai Shih and David E. Weekly*. 23 p.
16. Creating a safe and welcoming school *by John E. Mayer*. 27 p.
17. Teaching science *by John R. Staver*. 26 p.
18. Teacher professional learning and development *by Helen Timperley*. 31 p.

These titles can be downloaded from the websites of the IEA (http://www.iaoed.org) or of the IBE (http://www.ibe.unesco.org/publications.htm) or paper copies can be requested from: IBE, Publications Unit, P.O. Box 199, 1211 Geneva 20, Switzerland. Please note that several titles are out of print, but can be downloaded from the IEA and IBE websites.

Table of Contents

This publication was produced in 2009 by the International Academy of Education (IAE), Palais des AcadÈmies, 1, rue Ducale, 1000 Brussels, Belgium, and the International Bureau of Education (IBE), P.O. Box 199, 1211 Geneva 20, Switzerland. It is available free of charge and may be freely reproduced and translated into other languages. Please send a copy of any publication that reproduces this text in whole or in part to the IAE and the IBE. This publication is also available on the Internet. See the "Publications" section, "Educational Practices Series" page at:

http://www.ibe.unesco.org

Printed in 2009 by Gonnet Imprimeur, 01300 Belley, France.

Introduction

This booklet focuses on effective mathematics teaching. Drawing on a wide range of research, it describes the kinds of pedagogical approaches that engage learners and lead to desirable outcomes. The aim of the booklet is to deepen the understanding of practitioners, teacher educators, and policy makers and assist them to optimize opportunities for mathematics learners.

Mathematics is the most international of all curriculum subjects, and mathematical understanding influences decision making in all areas of life—private, social, and civil. Mathematics education is a key to increasing the post-school and citizenship opportunities of young people, but today, as in the past, many students struggle with mathematics and become disaffected as they continually encounter obstacles to engagement. It is imperative, therefore, that we understand what effective mathematics teaching looks like—and what teachers can do to break this pattern.

The principles outlined in this booklet are not stand-alone indicators of best practice: any practice must be understood as nested within a larger network that includes the school, home, community, and wider education system. Teachers will find that some practices are more applicable to their local circumstances than others.

Collectively, the principles found in this booklet are informed by a belief that mathematics pedagogy must:

- be grounded in the general premise that all students have the right to access education and the specific premise that all have the right to access mathematical culture;
- acknowledge that all students, irrespective of age, can develop positive mathematical identities and become powerful mathematical learners;
- be based on interpersonal respect and sensitivity and be responsive to the multiplicity of cultural heritages, thinking processes, and realities typically found in our classrooms;
- be focused on optimising a range of desirable academic outcomes that include conceptual understanding, procedural fluency, strategic competence, and adaptive reasoning;
- be committed to enhancing a range of social outcomes within the mathematics classroom that will contribute to the holistic development of students for productive citizenship.

Suggested Readings: Anthony & Walshaw, 2007; Martin, 2007; National Research Council, 2001.

1. An ethic of care

Caring classroom communities that are focused on mathematical goals help develop students' mathematical identities and proficiencies.

Research findings

Teachers who truly care about their students work hard at developing trusting classroom communities. Equally importantly, they ensure that their classrooms have a strong mathematical focus and that they have high yet realistic expectations about what their students can achieve. In such a climate, students find they are able to think, reason, communicate, reflect upon, and critique the mathematics they encounter; their classroom relationships become a resource for developing their mathematical competencies and identities.

Caring about the development of students' mathematical proficiency

Students want to learn in a harmonious environment. Teachers can help create such an environment by respecting and valuing the mathematics and the cultures that students bring to the classroom. By ensuring safety, teachers make it easier for all their students to get involved. It is important, however, that they avoid the kind of caring relationships that encourage dependency. Rather, they need to promote classroom relationships that allow students to think for themselves, ask questions, and take intellectual risks.

Classroom routines play an important role in developing students' mathematical thinking and reasoning. For example, the everyday practice of inviting students to contribute responses to a mathematical question or problem may do little more than promote cooperation. Teachers need to go further and clarify their expectations about how students can and should contribute, when and in what form, and how others might respond. Teachers who truly care about the development of their students' mathematical proficiency show interest in the ideas they construct and express, no matter how unexpected or unorthodox. By modelling the practice of evaluating ideas, they encourage their students to make thoughtful judgments about the mathematical soundness of the ideas voiced by their classmates. Ideas that are shown to be sound contribute to the shaping of further instruction.

Caring about the development of students' mathematical identities

Teachers are the single most important resource for developing students' mathematical identities. By attending to the differing needs that derive from home environments, languages, capabilities, and perspectives, teachers allow students to develop a positive attitude to mathematics. A positive attitude raises comfort levels and gives students greater confidence in their capacity to learn and to make sense of mathematics.

In the following transcript, students talk about their teacher and the inclusive classroom she has developed—a classroom in which they feel responsibility for themselves and for their own learning.

> She treats you as though you are like … not just a kid. If you say look this is wrong she'll listen to you. If you challenge her she will try and see it your way.
>
> She doesn't regard herself as higher.
>
> She's not bothered about being proven wrong. Most teachers hate being wrong … being proven wrong by students.
>
> It's more like a discussion … you can give answers and say what you think.
>
> We all felt like a family in maths. Does that make sense? Even if we weren't always sending out brotherly/sisterly vibes. Well we got used to each other … so we all worked … We all knew how to work with each other … it was a big group … more like a neighbourhood with loads of different houses.
>
> *Angier & Povey (1999, pp. 153, 157)*

Through her inclusive practices, this particular teacher influenced the way in which students thought of themselves. Confident in their own understandings, they were willing to entertain and assess the validity of new ideas and approaches, including those put forward by their peers. They had developed a belief in themselves as mathematical learners and, as a result, were more inclined to persevere in the face of mathematical challenges.

Suggested Readings: Angier & Povey, 1999; Watson, 2002.

2. Arranging for learning

Effective teachers provide students with opportunities to work both independently and collaboratively to make sense of ideas.

Research findings

When making sense of ideas, students need opportunities to work both independently and collaboratively. At times they need to be able to think and work quietly, away from the demands of the whole class. At times they need to be in pairs or small groups so that they can share ideas and learn with and from others. And at other times they need to be active participants in purposeful, whole-class discussion, where they have the opportunity to clarify their understanding and be exposed to broader interpretations of the mathematical ideas that are the present focus.

Independent thinking time

It can be difficult to grasp a new concept or solve a problem when distracted by the views of others. For this reason, teachers should ensure that all students are given opportunities to think and work quietly by themselves, where they are not required to process the varied, sometimes conflicting perspectives of others.

Whole-class discussion

In whole-class discussion, teachers are the primary resource for nurturing patterns of mathematical reasoning. Teachers manage, facilitate, and monitor student participation and they record students' solutions, emphasising efficient ways of doing this. While ensuring that discussion retains its focus, teachers invite students to explain their solutions to others; they also encourage students to listen to and respect one another, accept and evaluate different viewpoints, and engage in an exchange of thinking and perspectives.

Partners and small groups

Working with partners and in small groups can help students to see themselves as mathematical learners. Such arrangements can often provide the emotional and practical support that students need to clarify the nature of a task and identify possible ways forward. Pairs and small groups are not only useful for enhancing engagement; they also facilitate the exchange and testing of ideas and encourage higher-level thinking. In small, supportive groups, students learn how to make conjectures and engage in mathematical argumentation and validation.

As participants in a group, students require freedom from distraction and space for easy interactions. They need to be reasonably familiar with the focus activity and to be held accountable for the group's work. The teacher is responsible for ensuring that students understand and adhere to the participant roles, which include listening, writing, answering, questioning, and critically assessing. Note how the teacher in the following transcript clarifies expectations:

> I want you to explain to the people in your group how you think you are going to go about working it out. Then I want you to ask if they understand what you are on about and let them ask you questions. Remember in the end you all need to be able to explain how your group did it so think of questions you might be asked and try them out.
>
> Now this group is going to explain and you are going to look at what they do and how they came up with the rule for their pattern. Then as they go along if you are not sure please ask them questions. If you can't make sense of each step remember ask those questions.
>
> *Hunter (2005, pp. 454–455)*

For maximum effectiveness groups should be small—no more than four or five members. When groups include students of varying mathematical achievement, insights come at different levels; these insights will tend to enhance overall understandings.

Suggested Readings: Hunter, 2005; Sfard & Kieran, 2001; Wood, 2002.

3. Building on students' thinking

Effective teachers plan mathematics learning experiences that enable students to build on their existing proficiencies, interests, and experiences.

Research findings

In planning for learning, effective teachers put students' current knowledge and interests at the centre of their instructional decision making. Instead of trying to fix weaknesses and fill gaps, they build on existing proficiencies, adjusting their instruction to meet students' learning needs. Because they view thinking as "understanding in progress", they are able to use their students' thinking as a resource for further learning. Such teachers are responsive both to their students and to the discipline of mathematics.

Connecting learning to what students are thinking

Effective teachers take student competencies as starting points for their planning and their moment-by-moment decision making. Existing competencies, including language, reading and listening skills, ability to cope with complexity, and mathematical reasoning, become resources to build upon. Experientially real tasks are also valuable for advancing understanding. When students can envisage the situations or events in which a problem is embedded, they can use their own experiences and knowledge as a basis for developing context-related strategies that they can later refine into generalized strategies. For example, young children trying to work out how to share three pies among four family members will typically use informal methods that pre-empt formal division procedures.

Because they focus on the thinking that goes on when their students are engaged in tasks, effective teachers are able to pose new questions or design new tasks that will challenge and extend thinking. Consider this problem: *It takes a dragonfly about 2 seconds to fly 18 metres. How long should it take it to fly 110 metres?* Knowing that a student has solved this problem using additive thinking, a teacher might adapt the task so that it is more likely to invite multiplicative reasoning: *How long should it take the dragonfly to fly 1100 metres? or How long should it take a dragonfly to fly 110 metres if it flies about 9 metres in 1 second?*

Using students' misconceptions and errors as building blocks

Learners make mistakes for many reasons, including insufficient time or care. But errors also arise from consistent, alternative interpretations of mathematical ideas that represent the learner's attempts to create meaning. Rather than dismiss such ideas as "wrong thinking", effective teachers view them as a natural and often necessary stage in a learner's conceptual development. For example, young children often transfer the belief that dividing something always makes it smaller to their initial attempts to understand decimal fractions. Effective teachers take such misconceptions and use them as building blocks for developing deeper understandings.

There are many ways in which teachers can provide opportunities for students to learn from their errors. One is to organize discussion that focuses student attention on difficulties that have surfaced. Another is to ask students to share their interpretations or solution strategies so that they can compare and re-evaluate their thinking. Yet another is to pose questions that create tensions that need to be resolved. For example, confronted with the division misconception just referred to, a teacher could ask students to investigate the difference between $10 \div 2$, $2 \div 10$, and $10 \div 0.2$ using diagrams, pictures, or number stories.

Appropriate challenge

By providing appropriate challenge, effective teachers signal their high but realistic expectations. This means building on students' existing thinking and, more often than not, modifying tasks to provide alternative pathways to understanding. For low-achieving students, teachers find ways to reduce the complexity of tasks without falling back on repetition and busywork and without compromising the mathematical integrity of the activity. Modifications include using prompts, reducing the number of steps or variables, simplifying how results are to be represented, reducing the amount of written recording, and using extra thinking tools. Similarly, by putting obstacles in the way of solutions, removing some information, requiring the use of particular representations, or asking for generalizations, teachers can increase the challenge for academically advanced students.

Suggested readings: Carpenter, Fennema, & Franke, 1996; Houssart, 2002; Sullivan, Mousley, & Zevenbergen, 2006.

4. Worthwhile mathematical tasks

Effective teachers understand that the tasks and examples they select influence how students come to view, develop, use, and make sense of mathematics.

Research findings

It is by engaging with tasks that students develop ideas about the nature of mathematics and discover that they have the capacity to make sense of mathematics. Tasks and learning experiences that allow for original thinking about important concepts and relationships encourage students to become proficient doers and learners of mathematics. Tasks should not have a single-minded focus on right answers; they should provide opportunities for students to struggle with ideas and to develop and use an increasingly sophisticated range of mathematical processes (for example, justification, abstraction, and generalization).

Mathematical Focus

Effective teachers design learning experiences and tasks that are based on sound and significant mathematics; they ensure that all students are given tasks that help them improve their understanding in the domain that is currently the focus. Students should not expect that tasks will always involve practising algorithms they have just been taught; rather, they should expect that the tasks they are given will require them to think with and about important mathematical ideas. High-level mathematical thinking involves making use of formulas, algorithms, and procedures in ways that connect to concepts, understandings, and meaning. Tasks that require students to think deeply about mathematical ideas and connections encourage them to think for themselves instead of always relying on their teacher to lead the way. Given such opportunities, students find that mathematics becomes enjoyable and relevant.

Problematic tasks

Through the tasks they pose, teachers send important messages about what doing mathematics involves. Effective teachers set tasks that require students to make and test conjectures, pose problems, look for patterns, and explore alternative solution paths. Open-ended and modelling tasks, in particular, require students to interpret a context and then to make sense of the embedded mathematics. For example, if asked to design a schedule for producing a family meal, students need to interpret information, speculate and present arguments, apply previous learning, and make connections within mathematics and between mathematics and other bodies of knowledge. When working with real-life, complex systems, students learn that doing mathematics consists of more than producing right answers.

Open-ended tasks are ideal for fostering the creative thinking and experimentation that characterize mathematical "play". For example, if asked to explore different ways of showing $^2/_3$, students must engage in such fundamental mathematical practices as investigating, creating, reasoning, and communicating.

Practice activity

Students need opportunities to practice what they are learning, whether it be to improve their computational fluency, problem-solving skills, or conceptual understanding. Skill development can often be incorporated into "doing" mathematics; for example, learning about perimeter and area offers opportunities for students to practice multiplication and fractions. Games can also be a means of developing fluency and automaticity. Instead of using them as time fillers, effective teachers choose and use games because they meet specific mathematical purposes and because they provide appropriate feedback and challenge for all participants.

Suggested readings: Henningsen & Stein, 1997; Watson & De Geest, 2005.

5. Making connections

Effective teachers support students in creating connections between different ways of solving problems, between mathematical representations and topics, and between mathematics and everyday experiences.

Research findings

To make sense of a new concept or skill, students need to be able to connect it to their existing mathematical understandings, in a variety of ways. Tasks that require students to make multiple connections within and across topics help them appreciate the interconnectedness of different mathematical ideas and the relationships that exist between mathematics and real life. When students have opportunities to apply mathematics in everyday contexts, they learn about its value to society and its contribution to other areas of knowledge, and they come to view mathematics as part of their own histories and lives.

Supporting making connections

Effective teachers emphasize links between different mathematical ideas. They make new ideas accessible by progressively introducing modifications that build on students' understandings. A teacher might, for example, introduce "double the 6" as an alternative strategy to "add 6 to 6". Different mathematical patterns and principles can be highlighted by changing the details in a problem set; for example, a sequence of equations, such as $y = 2x + 3$, $y = 2x + 2$, $y = 2x$ and $y = x + 3$, will encourage students to make and test conjectures about the position and slope of the related lines.

The ability to make connections between apparently separate mathematical ideas is crucial for conceptual understanding. While fractions, decimals, percentages, and proportions can be thought of as separate topics, it is important that students are encouraged to see how they are connected by exploring differing representations (for example, $^1/_2 = 50\%$) or solving problems that are situated in everyday contexts (for example, fuel costs for a car trip).

Multiple solutions and representations

Providing students with multiple representations helps develop both their conceptual understandings and their computational flexibility.

Effective teachers give their students opportunities to use an ever-increasing array of representations—and opportunities to translate between them. For example, a student working with different representations of functions (real-life scenarios, graphs, tables, and equations) has different ways of looking at and thinking about relationships between variables.

Tasks that have more than one possible solution strategy can be used to elicit students' own strategies. Effective teachers use whole-class discussion as an opportunity to select and sequence different student approaches with the aim of making explicit links between representations. For example, students may illustrate the solution for 103—28 using an empty number line, a base-ten model, or a notational representation. By sharing solution strategies, students can develop more powerful, fluent, and accurate mathematical thinking.

Connecting to everyday life

When students find they can use mathematics as a tool for solving significant problems in their everyday lives, they begin to view it as relevant and interesting. Effective teachers take care that the contexts they choose do not distract students from the task's mathematical purpose. They make the mathematical connections and goals explicit, to support those students who are inclined to focus on context issues at the expense of the mathematics. They also support students who tend to compartmentalize problems and miss the ideas that connect them.

Suggested readings: Anghileri, 2006; Watson & Mason, 2006.

6. Assessment for learning

Effective teachers use a range of assessment practices to make students′ thinking visible and to support students′ learning.

Research findings

Effective teachers make use of a wide range of formal and informal assessments to monitor learning progress, diagnose learning issues, and determine what they need to do next to further learning. In the course of regular classroom activity, they collect information about how students learn, what they seem to know and be able to do, and what interests them. In this way, they know what is working and what is not, and are able to make informed teaching and learning decisions.

Exploring students' reasoning and probing their understanding

During every lesson, teachers make countless instructional decisions. Moment-by-moment assessment of student progress helps them decide what questions to ask, when to intervene, and how to respond to questions. They can gain a lot from observing students as they work and by talking with them: they can gauge students' understanding, see what strategies they prefer, and listen to the language they use. Effective teachers use this information as a basis for deciding what examples and explanations they will focus on in class discussion.

One-on-one interviews can also provide important insights: a thinking-aloud problem-solving interview will often reveal more about what is going on in a student's mind than a written test. Teachers using interviews for the first time are often surprized with what students know and don't know. Because they challenge their expectations and assumptions, interviews can make teachers more responsive to their students' diverse learning needs.

Teacher Questioning

By asking questions, effective teachers require students to participate in mathematical thinking and problem solving. By allowing sufficient time for students to explore responses in depth and by pressing for explanation and understanding, teachers can ensure that students are productively engaged. Questions are also a powerful means of assessing students' knowledge and exploring their thinking. A key indicator of good questioning is how teachers listen to student responses. Effective teachers pay attention not only to whether an answer is correct, but also to the student's mathematical thinking. They know that a wrong answer might indicate unexpected thinking rather than lack of understanding; equally, a correct answer may be arrived at via faulty thinking.

To explore students' thinking and encourage them to engage at a higher level, teachers can use questions that start at the solution; for example, *If the area of a rectangle is 24 cm^2 and the perimeter is 22 cm, what are its dimensions?* Questions that have a variety of solutions or can be solved in more than one way have the potential to provide valuable insight into student thinking and reasoning.

Feedback

Helpful feedback focuses on the task, not on marks or grades; it explains why something is right or wrong and describes what to do next or suggests strategies for improvement. For example, the feedback, *I want you to go over all of them and write an equals sign in each one* gives a student information that she can use to improve her performance. Effective teachers support students when they are stuck, not by giving full solutions, but by prompting them to search for more information, try another method, or discuss the problem with classmates. In response to a student who says he doesn't understand, a teacher might say: *Well, the first part is just like the last problem. Then we add one more variable. See if you can find out what it is. I'll be back in a few minutes*. This teacher challenges the student to do further thinking before she returns to check on progress.

Self and peer assessment

Effective teachers provide opportunities for students to evaluate their own work. These may include having students design their own test questions, share success criteria, write mathematical journals, or present portfolio evidence of growing understanding. When feedback is used to encourage continued student–student and student–teacher dialogue, self-evaluation becomes a regular part of the learning process and students develop greater self-awareness.

Suggested readings: Steinberg, Empson, & Carpenter, 2004; Wiliam, 2007.

7. Mathematical Communication

Effective teachers are able to facilitate classroom dialogue that is focused on mathematical argumentation.

Research findings

Effective teachers encourage their students to explain and justify their solutions. They ask them to take and defend positions against the contrary mathematical claims of other students. They scaffold student attempts to examine conjectures, disagreements, and counterarguments. With their guidance, students learn how to use mathematical ideas, language, and methods. As attention shifts from procedural rules to making sense of mathematics, students become less preoccupied with finding the answers and more with the thinking that leads to the answers.

Scaffolding attempts at mathematical ways of speaking and thinking

Students need to be taught how to communicate mathematically, give sound mathematical explanations, and justify their solutions. Effective teachers encourage their students to communicate their ideas orally, in writing, and by using a variety of representations.

Revoicing is one way of guiding students in the use of mathematical conventions. Revoicing involves repeating, rephrasing, or expanding on student talk. Teachers can use it (i) to highlight ideas that have come directly from students, (ii) to help develop students' understandings that are implicit in those ideas, (iii) to negotiate meaning with their students, and (iv) to add new ideas, or move discussion in another direction.

Developing skills of mathematical argumentation

To guide students in the ways of mathematical argumentation, effective teachers encourage them to take and defend positions against alternative views; their students become accustomed to listening to the ideas of others and using debate to resolve conflict and arrive at common understandings.

In the following episode, a class has been discussing the claim that fractions can be converted into decimals. Bruno and Gina have been developing the skills of mathematical argumentation during this discussion. The teacher then speaks to the class:

> Teacher: Great, now I hope you're listening because what Gina and Bruno said was very important. Bruno made a conjecture and Gina tested it for him. And based on her tests he revised his conjecture because that's what a conjecture is. It means that you think that you're seeing a pattern so you're gonna come up with a statement that you think is true, but you're not convinced yet. But based on her further evidence, Bruno revised his conjecture. Then he might go back to revise it again, back to what he originally said or to something totally new. But they're doing something important. They're looking for patterns and they're trying to come up with generalizations.
>
> *O'Connor (2001, pp. 155–156)*

This teacher sustained the flow of student ideas, knowing when to step in and out of the discussion, when to press for understanding, when to resolve competing student claims, and when to address misunderstandings or confusion. While the students were learning mathematical argumentation and discovering what makes an argument convincing, she was listening attentively to student ideas and information. Importantly, she withheld her own explanations until they were needed.

Suggested readings: Lobato, Clarke, & Ellis, 2005; O'Connor, 2001; Yackel, Cobb, & Wood, 1998.

8. Mathematical language

Effective teachers shape mathematical language by modelling appropriate terms and communicating their meaning in ways that students understand.

Research findings

Effective teachers foster students' use and understanding of the terminology that is endorsed by the wider mathematical community. They do this by making links between mathematical language, students' intuitive understandings, and the home language. Concepts and technical terms need to be explained and modelled in ways that make sense to students yet are true to the underlying meaning. By carefully distinguishing between terms, teachers make students aware of the variations and subtleties to be found in mathematical language.

Explicit language instruction

Students learn the meaning of mathematical language through explicit "telling" and through modelling. Sometimes, they can be helped to grasp the meaning of a concept through the use of words or symbols that have the same mathematical meaning, for example, "x", "multiply", and "times". Particular care is needed when using words such as "less than", "more", "maybe", and "half", which can have somewhat different meanings in the home. In the following transcript, a teacher holds up two cereal packets, one large and one small, and asks students to describe the difference between them in mathematical terms.

T: Would you say that *those* two are different shapes?

R: They're similar.

T: What does similar mean?

R: Same shape, different sizes.

T: Same shape but different sizes. That's going around in circles isn't it?—We still don't know what you mean by *shape*. What do you mean by shape?

[She gathers three objects: the two cereal packets and the meter ruler. She places the ruler alongside the small cereal packet.]

T: This and this are different shapes, but they're both cuboids.

[She now puts the cereal packets side by side.]

T: This and this are the same shape and different sizes. What makes them the *same* shape?

[One girl refers to a *scaled-down version*. Another to *measuring the sides*—to see if they're in the same *ratio*. Claire picks up their words and emphasizes them.

T: Right. So it's about *ratio* and about *scale*.

Runesson (2005, pp. 75–76)

Multilingual contexts and home language

The teacher should model and use specialized mathematical language in ways that let students grasp it easily. Terms such as "absolute value", "standard deviation", and "very likely" typically do not have equivalents in the language a child uses at home. Where the medium of instruction is different from the home language, children can encounter considerable difficulties with prepositions, word order, logical structures, and conditionals—and the unfamiliar contexts in which problems are situated. Teachers of mathematics are often unaware of the barriers to understanding that students from a different language and culture must overcome. Language (or code) switching, in which the teacher substitutes a home language word, phrase, or sentence for a mathematical concept, can be a useful strategy for helping students grasp underlying meaning.

Suggested readings: Runesson, 2005; Setati & Adler, 2001.

9. Tools and representations

Effective teachers carefully select tools and representations to provide support for students' thinking.

Research findings

Effective teachers draw on a range of representations and tools to support their students' mathematical development. These include the number system itself, algebraic symbolism, graphs, diagrams, models, equations, notations, images, analogies, metaphors, stories, textbooks, and technology. Such tools provide vehicles for representation, communication, reflection, and argumentation. They are most effective when they cease to be external aids, instead becoming integral parts of students' mathematical reasoning. As tools become increasingly invested with meaning, they become increasingly useful for furthering learning.

Thinking with tools

If tools are to offer students "thinking spaces", helping them to organize their mathematical reasoning and support their sense-making, teachers must ensure that the tools they select are used effectively. With the help of an appropriate tool, students can think through a problem or test an idea that their teacher has modelled. For example, ten-frame activities can be used to help students visualize number relationships (e.g., how far a number is from 10) or how a number can be partitioned.

Effective teachers take care when using tools, particularly pre-designed, "concrete" materials such as number lines or ten-frames, to ensure that all students make the intended mathematical sense of them. They do this by explaining how the model is being used, how it represents the ideas under discussion, and how it links to operations, concepts, and symbolic representations.

Communicating with tools

Tools, both representations and virtual manipulatives, are helpful for communicating ideas and thinking that are otherwise difficult to describe, talk about, or write about. Tools do not have to be ready-made; effective teachers acknowledge the value of students generating and using their own representations, whether these be invented notations or graphical, pictorial, tabular, or geometric representations. For example, students can take statistical data and create their own pictorial representations to tell stories well before they acquire formal graphing tools. As they use tools to communicate their ideas, students develop and clarify their own thinking at the same time that they provide their teachers with insight into that thinking.

New technologies

An increasing array of technological tools is available for use in mathematics classrooms. These include calculator and computer applications, presentation technologies such as the interactive whiteboard, mobile technologies such as clickers and data loggers, and the Internet. These dynamic graphical, numerical, and visual applications provide new opportunities for teachers and students to explore and represent mathematical concepts.

With guidance from teachers, technology can support independent inquiry and shared knowledge building. When used for mathematical investigations and modelling activities, technological tools can link the student with the real world, making mathematics more accessible and relevant.

Teachers need to make informed decisions about when and how they use technology to support learning. Effective teachers take time to share with their students the reasoning behind these decisions; they also require them to monitor their own use (including overuse or underuse) of technology. Given the pace of change, teachers need ongoing professional development so that they can use new technologies in ways that advance the mathematical thinking of their students.

Suggested readings: Thomas & Chinnappan, 2008; Zevenbergen & Lerman, 2008.

10. Teacher knowledge

Effective teachers develop and use sound knowledge as a basis for initiating learning and responding to the mathematical needs of all their students.

Research findings

How teachers organize classroom instruction is very much dependent on what they know and believe about mathematics and on what they understand about mathematics teaching and learning. They need knowledge to help them recognize, and then act upon, the teaching opportunities that come up without warning. If they understand the big ideas of mathematics, they can represent mathematics as a coherent and connected system and they can make sense of and manage multiple student viewpoints. Only with substantial content and pedagogical content knowledge can teachers assist students in developing mathematically grounded understandings.

Teacher content knowledge

Effective teachers have a sound grasp of relevant content and how to teach it. They know what the big ideas are that they need to teach. They can think of, model, and use examples and metaphors in ways that advance student thinking. They can critically evaluate students' processes, solutions, and understanding and give appropriate and helpful feedback. They can see the potential in the tasks they set; this, in turn, contributes to sound instructional decision making.

Teacher pedagogical content knowledge

Pedagogical content knowledge is crucial at all levels of mathematics and with all groups of students. Teachers with in-depth knowledge have clear ideas about how to build procedural proficiency and how to extend and challenge student ideas. They use their knowledge to make the multiple decisions about tasks, classroom resources, talk, and actions that feed into or arise out of the learning process. Teachers with limited knowledge tend to structure teaching and learning around discrete concepts instead of creating wider connections between facts, concepts, structures, and practices.

To teach mathematical content effectively, teachers need a grounded understanding of students as learners. With such understanding, they are aware of likely conceptions and misconceptions. They use this awareness to make instructional decisions that strengthen conceptual understanding.

Teacher knowledge in action

As the following transcript illustrates, sound knowledge enables the teacher to listen and question more perceptively, effectively informing her on-the-spot classroom decision making.

> The teacher challenged her year 1–2 class to investigate negative integers.
>
> S: Negative five plus negative five should be negative five.
>
> Teacher: No, because you're adding negative five and negative five, so you start at negative five and how many jumps do you take?
>
> S: Five.
>
> Teacher: Well, you're not going to end up on negative five [points to the negative five on the number line]. So, then negative five. How many jumps do you take?
>
> S: Five.
>
> Teacher: So where are you going to end up?
>
> *Fraivillig, Murphy & Fuson (1999, p. 161)*

Like this teacher, those with sound knowledge are more apt to notice the critical moments when choices or opportunities present themselves. Importantly, given their grasp of mathematical ideas and how to teach, they can adapt and modify their routines to fit the need.

Enhancing teacher knowledge

The development of teacher knowledge is greatly enhanced by efforts within the wider educational community. Teachers need the support of others—particularly material, systems, and human and emotional support. While teachers can learn a great deal by working together with a group of supportive mathematics colleagues, professional development initiatives are often a necessary catalyst for major change.

Suggested readings: Askew, Brown, Rhodes, Johnson, & Wiliam, 1997; Hill, Rowan, & Ball, 2005; Schifter, 2001

Conclusion

Current research findings show that the nature of mathematics teaching significantly affects the nature and outcomes of student learning. This highlights the huge responsibility teachers have for their students' mathematical well-being. In this booklet, we offer ten principles as a starting point for discussing change, innovation, and reform. These principles should be viewed as a whole, not in isolation: teaching is complex, and many interrelated factors have an impact on student learning. The booklet offers ways to address that complexity, and to make mathematics teaching more effective.

Major innovation and genuine reform require aligning the efforts of all those involved in students' mathematical development: teachers, principals, teacher educators, researchers, parents, specialist support services, school boards, policy makers, and the students themselves. Changes need to be negotiated and carried through in classrooms, teams, departments, and faculties, and in teacher education programmes. Innovation and reform must be provided with adequate resources. Schools, communities, and nations need to ensure that their teachers have the knowledge, skills, resources, and incentives to provide students with the very best of learning opportunities. In this way, *all* students will develop their mathematical proficiency. In this way, too, *all* students will have the opportunity to view themselves as powerful learners of mathematics.

References

Anghileri, J. 2006. Scaffolding practices that enhance mathematics learning. *Journal of Mathematics Teacher Education*, no. 9, pp. 33–52.

Angier, C.; Povey, H. 1999. One teacher and a class of school students: Their perception of the culture of their mathematics classroom and its construction. *Educational Review*, vol. 51, no. 2, pp. 147–160.

Anthony, G.; Walshaw, M. 2007. *Effective pedagogy in mathematics/p,ngarau: Best evidence synthesis iteration [BES]*. Wellington: Ministry of Education.

Askew, M. et al. 1997. *Effective teachers of numeracy*. London: Kings College.

Carpenter, T.; Fennema, E.; Franke, M. 1996. Cognitively guided instruction: A knowledge base for reform in primary mathematics instruction. *The Elementary School Journal*, vol. 97, no. 1, pp. 3–20.

Fraivillig, J.; Murphy, L.; Fuson, K. 1999. Advancing children's mathematical thinking in Everyday Mathematics classrooms. *Journal for Research in Mathematics Education*, vol. 30, no. 2, pp. 148–170.

Henningsen, M.; Stein, M. 1997. Mathematical tasks and student cognition: Classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. *Journal for Research in Mathematics Education*, vol. 28, no. 5, pp. 524–549.

Hill, H.; Rowan, B.; Ball, D. 2005. Effects of teachers' mathematical knowledge for teaching on student achievement. *American Education Research Journal*, no. 42, pp. 371–406.

Houssart, J. 2002. Simplification and repetition of mathematical tasks: A recipe for success or failure? *The Journal of Mathematical Behavior*, vol. 21, no. 2, pp. 191–202.

Hunter, R. 2005. Reforming communication in the classroom: One teacher's journey of change. *In*: Clarkson, P. et al., eds. *Building connections: Research, theory and practice* (Proceedings of the 28th annual conference of the Mathematics Education Research Group of Australasia, pp. 451–458). Sydney: MERGA.

Lobato, J.; Clarke, D.; Ellis, A. B. 2005. Initiating and eliciting in teaching: A reformulation of telling. *Journal for Research in Mathematics Education*, vol. 36, no. 2, pp. 101–136.

Martin, T. S., ed. 2007. *Mathematics teaching today: Improving practice, improving student learning*, 2nd ed. Reston, VA: National Council of Teachers of Mathematics.

National Research Council. 2001. *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.

O'Connor, M.C. 2001. "Can any fraction be turned into a decimal?" A case study of a mathematical group discussion. *Educational Studies in Mathematics*, no. 46, pp. 143–185.

Runesson, U. 2005. Beyond discourse and interaction. Variation: A critical aspect for teaching and learning mathematics. *Cambridge Journal of Education*, vol. 35, no. 1, pp. 69–87.

Schifter, D. 2001. Learning to see the invisible. In: Wood, T.; Scott-Nelson, B.; Warfield, J., eds. *Beyond classical pedagogy: Teaching elementary school mathematics* (pp. 109–134). Mahwah, NJ: Lawrence Erlbaum Associates.

Setati, M.; Adler, J. 2001. Code-switching in a senior primary class of secondary-language mathematics learners. *For the Learning of Mathematics*, no. 18, pp. 34–42.

Sfard, A.; Keiran, C. 2001. Cognition as communication: Rethinking learning-by-talking through multi-faceted analysis of students' mathematical interactions. *Mind, Culture, and Activity*, vol. 8, no. 1, pp. 42–76.

Steinberg, R. M.; Empson, S.B.; Carpenter, T.P. 2004. Inquiry into children's mathematical thinking as a means to teacher change. *Journal of Mathematics Teacher Education*, no. 7, pp. 237–267.

Sullivan, P.; Mousley, J.; Zevenbergen, R. 2006. Teacher actions to maximize mathematics learning opportunities in heterogeneous classrooms. *International Journal of Science and Mathematics Education*, vol. 4, no. 1, pp. 117–143.

Thomas, M.; Chinnappan, M. 2008. Teaching and learning with technology: Realising the potential. In: Forgasz, H. et al., eds. *Research in Mathematics Education in Australasia 2004–2007* (pp. 165–193). Rotterdam: Sense Publishers.

Watson, A. 2002. Instances of mathematical thinking among low attaining students in an ordinary secondary classroom. *Journal of Mathematical Behavior*, no. 20, pp. 461–475.

Watson, A.; De Geest, E. 2005. Principled teaching for deep progress: Improving mathematical learning beyond methods and material. *Educational Studies in Mathematics*, no. 58, pp. 209–234.

Watson, A.; Mason, J. 2006. Seeing an exercise as a single mathematical object: Using variation to structure sense-making. *Mathematical Thinking and Learning*, no. 8, pp. 91–111.

Wiliam, D. 2007. Keeping learning on track. In: Lester, F.K., ed. *Second handbook of research on mathematics teaching and learning* (pp. 1053–1098). Charlotte, NC: NCTM & Information Age Publishing.

Wood, T. 2002. What does it mean to teach mathematics differently? In: Barton, B. et al., eds. *Mathematics Education in the South Pacific* (Proceedings of the 25th annual conference of the Mathematics Education Research Group of Australasia, pp. 61–67). Sydney: MERGA.

Yackel, E.; Cobb, P.; Wood, T. 1998. The interactive constitution of mathematical meaning in one second grade classroom: An illustrative example. *Journal of Mathematical Behaviour*, vol. 17, no. 4, pp. 469–488.

Zevenbergen, R.; Lerman, S. 2008. Learning environments using interactive whiteboards: New learning spaces or reproduction of old technologies. *Mathematics Education Research Journal*, vol. 20, no. 1, pp. 107–125.

EDUCATIONAL PRACTICES SERIES–19

The International Bureau of Education–IBE

The IBE was founded in Geneva, Switzerland, as a private, non-governmental organization in 1925. In 1929, under new statutes, it became the first intergovernmental organization in the field of education. Since 1969 the Institute has been an integral part of UNESCO while retaining wide intellectual and functional autonomy.

The mission of the IBE is to function as an international centre for the development of contents and methods of education. It builds networks to share expertise on, and foster national capacities for curriculum change and development in all the regions of the world. It aims to introduce modern approaches in curriculum design and implementation, improve practical skills, and foster international dialogue on educational policies.

The IBE contributes to the attainment of quality Education for All (EFA) mainly through: (a) developing and facilitating a worldwide network and a Community of Practice of curriculum specialists; (b) providing advisory services and technical assistance in response to specific demands for curriculum reform or development; (c) collecting, producing and giving access to a wide range of information resources and materials on education systems, curricula and curriculum development processes from around the world, including online databases (such as *World Data on Education*), thematic studies, publications (such as *Prospects*, the quarterly review of education), national reports, as well as curriculum materials and approaches for HIV & AIDS education at primary and secondary levels through the HIV & AIDS Clearinghouse; and (d) facilitating and fostering international dialogue on educational policies, strategies and reforms among decision-makers and other stakeholders, in particular through the International Conference on Education—organized by the IBE since 1934—, which can be considered one of the main forums for developing world-level policy dialogue between Ministers of Education.

The IBE is governed by a Council composed of representatives of twenty-eight Member States elected by the General Conference of UNESCO. The IBE is proud to be associated with the work of the International Academy of Education and publishes this material in its capacity as a Clearinghouse promoting the exchange of information on educational practices.

Visit the IBE website at: http://www.ibe.unesco.org