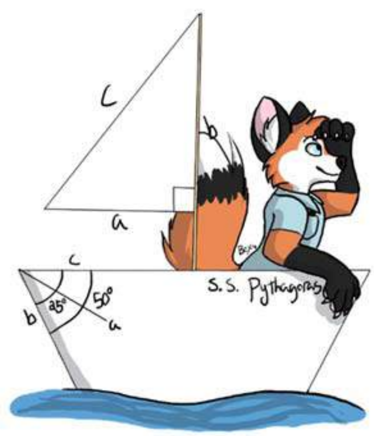
# Math Handbook

# of Formulas, Processes and Tricks

**(www.mathguy.us)**

# Geometry




**Prepared by: Earl L. Whitney, FSA, MAAA**

**Version 4.2**

**August 26, 2023**

# Geometry Handbook
# Table of Contents



Cover art by Rebecca Williams, Twitter handle: @jolteonkitty

# Geometry Handbook
# Table of Contents

**Page** **Description**

# Geometry Handbook
# Table of Contents

**Page Description**

## Useful Websites

**Wolfram Math World** – Perhaps the premier site for mathematics on the Web. This site contains definitions, explanations and examples for elementary and advanced math topics.

mathworld.wolfram.com/

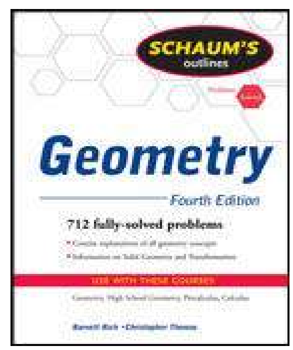
**Mathguy.us** – Developed specifically for math students from Middle School to College, based on the author's extensive experience in professional mathematics in a business setting and in math tutoring. Contains free downloadable handbooks, PC Apps, sample tests, and more.

www.mathguy.us

**Broken Arrow, Oklahoma Standard Geometry Test** – A standardized Geometry test released by the state of Oklahoma. A good way to test your knowledge.

www.baschools.org/pages/uploaded_files/Geometry%20Practice%20Test.pdf

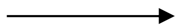
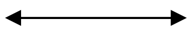
## Schaum’s Outlines




An important student resource for any high school math student is a Schaum’s Outline. Each book in this series provides explanations of the various topics in the course and a substantial number of problems for the student to try. Many of the problems are worked out in the book, so the student can see examples of how they should be solved.

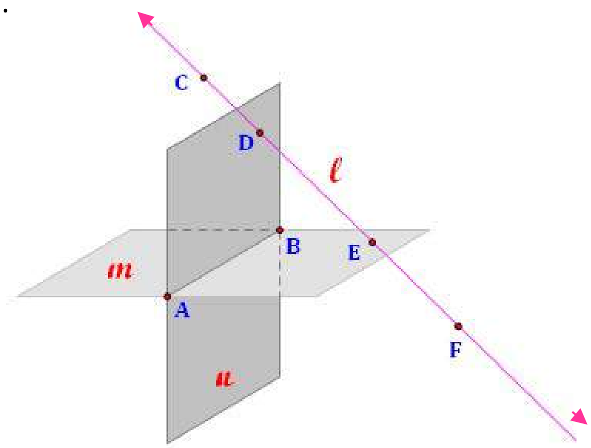
Schaum’s Outlines are available at Amazon.com, Barnes & Noble and other booksellers.

# Geometry
## Points, Lines & Planes

| Item | Illustration | Notation | Definition |
|---|---|---|---|
| **Point** | • | $A$ | A location in space. |
| **Segment** | | $\overline{AB}$ | A straight path that has two endpoints. |
| **Ray** |  | $\overrightarrow{AB}$ | A straight path that has one endpoint and extends infinitely in one direction. |
| **Line** |  | $\ell$ or $\overleftrightarrow{AB}$ | A straight path that extends infinitely in both directions. |
| **Plane** |  | $\boldsymbol{m}$ or $ABD$ (points $A, B, D$ not linear) | A flat surface that extends infinitely in two dimensions. |

**Collinear points** are points that lie on the same line.

**Coplanar points** are points that lie on the same plane.




In the figure at right:

- $A, B, C, D, E$ and $F$ are points.
- $\ell$ is a line
- $m$ and $n$ are planes.

In addition, note that:

- $C, D, E$ and $F$ are **collinear points.**
- $A, B$ and $E$ are **coplanar points.**
- $A, B$ and $D$ are **coplanar points.**
- Ray $\overrightarrow{EF}$ goes off in a southeast direction.
- Ray $\overrightarrow{EC}$ goes off in a northwest direction.
- Together, rays $\overrightarrow{EF}$ and $\overrightarrow{EC}$ make up line $\ell$.
- Line $\ell$ intersects both planes $m$ and $n$.

An **intersection** of geometric shapes is the set of points they share in common.

$\ell$ and $m$ intersect at point $E$.

$\ell$ and $n$ intersect at point $D$.

$m$ and $n$ intersect in line $\overleftrightarrow{AB}$.

**Note:** In geometric figures such as the one above, it is important to remember that, even though planes are drawn with edges, they extend infinitely in the 2 dimensions shown.

# Geometry
# Segments, Rays & Lines

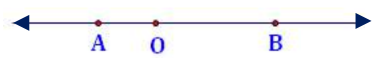
## Some Thoughts About …

### Line Segments


- Line segments are generally named by their endpoints, so the segment at right could be named either $\overline{AB}$ or $\overline{BA}$.
- Segment $\overline{AB}$ contains the two endpoints (A and B) and all points on line $\overleftrightarrow{AB}$ that are between them.
- Congruent segments are segments of equal length.
- A bisector splits a segment into two congruent (equal length) segments.

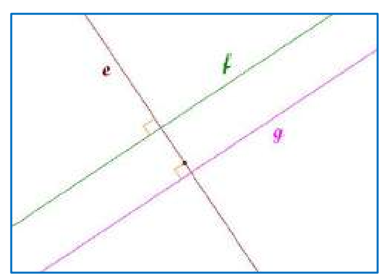
### Rays


- Rays are generally named by their single endpoint, called an initial point, and another point on the ray.
- Ray $\overrightarrow{AB}$ contains its initial point A and all points on line $\overleftrightarrow{AB}$ in the direction of the arrow.
- Rays $\overrightarrow{AB}$ and $\overrightarrow{BA}$ are not the same ray.
- If point O is on line $\overleftrightarrow{AB}$ and is between points A and B, then rays $\overrightarrow{OA}$ and $\overrightarrow{OB}$ are called opposite rays. They have only point O in common, and together they make up line $\overleftrightarrow{AB}$.

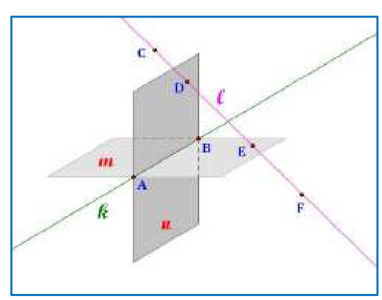



### Lines


- Lines are generally named by either a single script letter (e.g., $\ell$) or by two points on the line (e.g.,. $\overleftrightarrow{AB}$).
- A line extends infinitely in the directions shown by its arrows.
- Lines are parallel if they are in the same plane and they never intersect. Lines $f$ and $g$, at right, are parallel.




- Lines are perpendicular if they intersect at a $90^{o}$ angle. A pair of perpendicular lines is always in the same plane. Lines $f$ and $e$, at right, are perpendicular. Lines $g$ and $e$ are also perpendicular.
- Lines are skew if they are not in the same plane and they never intersect. Lines $k$ and $\ell$, at right, are skew. (Remember this figure is 3-dimensional.)



# Geometry
## Distance Between Points

Distance measures how far apart two things are. The distance between two points can be measured in any number of dimensions, and is defined as the length of the line connecting the two points. Distance is always a positive number.

**1-Dimension** (line segment)

Distance - In one dimension, the distance between two points is determined simply by subtracting the coordinates of the points. If the endpoints are labeled, say $\mathrm{A}$ and $\mathrm{B}$, then the length of segment $\overline{\mathrm{AB}}$ is shown as $\mathrm{AB}$.

**Example 1.1:** In this segment, the length of $\overline{\mathrm{AB}}$, i.e., $\mathrm{AB}$, is calculated as: $5-(-2)=\mathbf{7}$.


Midpoint – the point equidistant from each end of a line segment. That is, the midpoint is halfway from one end of the segment to the other. To obtain the value of the midpoint, add the two end values and divide the result by 2.

**Example 1.2:** The midpoint of segment $\overline{\mathrm{AB}}$ in Example 1.1 is: $\frac{[5+(-2)]}{2}=\mathbf{\frac{3}{2}}$.

### Distances Between Collinear Points

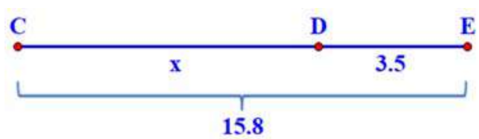
Recall that collinear points are points on the same line.

A common problem in geometry is to split a line segment into parts based on some knowledge about the one or more of the parts.

**Example 1.3:** Find two possible lengths for $\overline{\mathrm{CD}}$ if C, D, and E are collinear, and $\mathrm{CE}=15.8$ cm and $\mathrm{DE}=3.5$ cm.

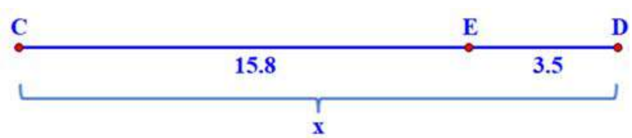
It is helpful to use a line diagram when dealing with midpoint problems. There are two possible line diagrams for this problem: 1) **D** is between **C** and **E**, 2) **E** is between **C** and **D**. In these diagrams, we show distances instead of point values:

Case 1




$x=15.8-3.5=\mathbf{12.3\ cm}$

Case 2



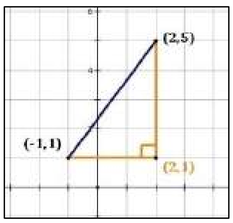

$x=15.8+3.5=\mathbf{19.3\ cm}$

**2-Dimensions**

Distance – In two dimensions, the distance between two points can be calculated by considering the line between them to be the hypotenuse of a right triangle. To determine the length of this line:

- Calculate the difference in the $x$-coordinates of the points
- Calculate the difference in the $y$-coordinates of the points
- Use the Pythagorean Theorem.

This process is illustrated below, using the variable ***"d"*** for distance.




**Example 1.4:** Find the distance between (-1,1) and (2,5). Based on the illustration to the left:

*x-coordinate difference:* $2-(-1)=3.$
*y-coordinate difference:* $5-1=4.$

Then, the distance is calculated using the formula: $d^2=(3^2+4^2)=(9+16)=25$

We get $d^2=25$, so $d=\sqrt{25}=\mathbf{5}$

If we define two points generally as $(x_1, y_1)$ and $(x_2, y_2)$, then the 2-dimensional distance formula would be:

$$\text{distance}=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}.$$

Midpoint – To obtain the value of the midpoint in two or more dimensions, add the corresponding coordinates of the endpoints and divide each result by 2.

If you are given the value of the midpoint and asked for the coordinates of an endpoint, you may choose to calculate a vector, which in this case is simply the difference between two points.

**Example 1.5:** Find the distance between $P(-2,3)$ and $Q(3,15)$.

The formula for the distance between points is: $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

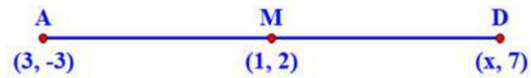
Let point 1 be $P(-2,3)$, and let point 2 be $Q(3,15)$. Then,

$$d=\sqrt{\left(3-(-2)\right)^2+(15-3)^2}=\sqrt{5^2+12^2}=\sqrt{169}=\mathbf{13}$$

Note that 5-12-13 is a Pythagorean Triple.

**Example 1.6:** The midpoint of segment $\overline{\mathrm{AD}}$ is $(1,2)$. Point A has coordinates $(3,-3)$ and point D has coordinates $(x,7)$.

It is helpful to use a line diagram when dealing with midpoint problems. Label the endpoints and midpoint, and identify the coordinates of each:




The difference between points **A** and **M** can be expressed in two dimensions as a vector using "⟨ ⟩" instead of "( )". Let's find the difference (note: "difference" implies subtraction).

| | |
|---|---|
| $(1, \quad 2)$ | Point **M** |
| $-\ (3,-3)$ | Point **A** |
| $\langle -2, \quad 5\rangle$ | Difference vector (difference between the two points) |

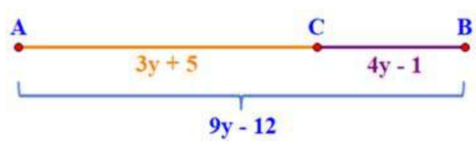
The difference vector can then be applied to the midpoint to get the coordinates of point **D**. If I can get from A to M by moving $\langle -2,5\rangle$, then I can get from M to D by moving $\langle -2,5\rangle$.

| | |
|---|---|
| $(\ \ 1,2)$ | Point **M** |
| $+\ \langle -2,5\rangle$ | Difference vector |
| $(-\mathbf{1},7)$ | Point **D**. Therefore, we conclude that $\boldsymbol{x=-1}$. |

Note that the $y$-value of point **D** in the solution, 7, matches the $y$-value of point **D** in the statement of the problem.

**Example 1.7:** Find the value of $y$ if $\mathrm{AC}=3y+5, \mathrm{CB}=4y-1, \mathrm{AB}=9y-12$, and C lies between A and B.

The line diagram is crucial for this problem. It must be drawn with A and B as endpoints and C between them.




Based on the diagram, we have: $(3y+5)+(4y-1)=9y-12$

$$7y+4=9y-12$$
$$16=2y$$
$$\mathbf{8=y}$$

## Partial Distances and Distance Equations

In order to find a distance part-way between two points, we need to interpolate between the beginning and end points. We must calculate the portion of the distance covered at the desired time, and then interpolate between the start and end points.

Let $k$ be the factor, representing the portion of the total distance that is of interest to us. $k$ is usually given in terms of time, e.g., after 3 hours of a 10-hour journey. In general,

$$k = \frac{\text{elapsed time}}{\text{total time}}.$$

The formula for the interpolation, then, is:

$$\text{desired point} = k \cdot (\text{ending point}) + (1 - k) \cdot (\text{starting point})$$

This interpolation formula works for any number of dimensions, taking each coordinate separately.

**Example 1.8:** A boat begins a journey at location $(2, 5)$ on a grid and heads directly for point $(10, 15)$ on the same grid. It is estimated that the trip will take 10 hours if the boat travels in a straight line. At what point of the grid is the boat after 3 hours?

Start at: $(2, 5)$

End at: $(10, 15)$

3 hours → $k = \frac{3}{10} = 0.3$ of the 10 hour period.

- This is the factor for the endpoint: $(10, 15)$.
- The staring point, $(2, 5)$ gets a factor of $1 - 0.3 = 0.7$. The factors must always add to $1$.

Ordered pair @ $t = 3$ hours is: $(2, 5) \cdot 0.7 + (10, 15) \cdot 0.3 = (\mathbf{4.4, 8.0})$

Note: an alternative method would be to develop separate equations for the $x$-variable and $y$-variable in terms of time, the $t$-variable. These are called parametric equations, and $t$ is the parameter in the equations. For this problem, the parametric equations would be:

$$\text{variable} = \text{start} + (\text{end} - \text{start}) \cdot \left(\frac{t}{\text{period length in years}}\right)$$

$$x = 2 + (10 - 2) \cdot \left(\frac{t}{10}\right) = 2 + 0.8t$$

$$y = 5 + (15 - 5) \cdot \left(\frac{t}{10}\right) = 5 + t$$

Note that the 10 in the denominator of these equations is the length of time, in hours, separating the starting point and the ending point.

Solve for the required ordered pair by substituting $t = 3$ into these equations.

# Geometry

**ADVANCED**

## Distance Formula in "*n*" Dimensions

The distance between two points can be generalized to ***"n"*** dimensions by successive use of the Pythagorean Theorem in multiple dimensions. To move from two dimensions to three dimensions, we start with the two-dimensional formula and apply the Pythagorean Theorem to add the third dimension.

### 3 Dimensions

Consider two 3-dimensional points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$. Consider first the situation where the two z-coordinates are the same. Then, the distance between the points is 2-dimensional, i.e., $\mathrm{d} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

We then add a third dimension using the Pythagorean Theorem:

$$\text{distance}^2 = \mathrm{d}^2 + (\mathrm{z}_2 - \mathrm{z}_1)^2$$

$$\text{distance}^2 = \left(\sqrt{(\mathrm{x}_2 - \mathrm{x}_1)^2 + (\mathrm{y}_2 - \mathrm{y}_1)^2}\right)^2 + (\mathrm{z}_2 - \mathrm{z}_1)^2$$

$$\text{distance}^2 = (\mathrm{x}_2 - \mathrm{x}_1)^2 + (\mathrm{y}_2 - \mathrm{y}_1)^2 + (\mathrm{z}_2 - \mathrm{z}_1)^2$$

And, finally the 3-dimensional difference formula:

$$\text{distance} = \sqrt{(\mathrm{x}_2 - \mathrm{x}_1)^2 + (\mathrm{y}_2 - \mathrm{y}_1)^2 + (\mathrm{z}_2 - \mathrm{z}_1)^2}$$

### n Dimensions

Using the same methodology in "*n*" dimensions, we get the generalized n-dimensional difference formula (where there are *n* terms beneath the radical, one for each dimension):

$$\text{distance} = \sqrt{(\mathrm{x}_2 - \mathrm{x}_1)^2 + (\mathrm{y}_2 - \mathrm{y}_1)^2 + (\mathrm{z}_2 - \mathrm{z}_1)^2 + \cdots + (\mathrm{w}_2 - \mathrm{w}_1)^2}$$

Or, in higher level mathematical notation:
The distance between two points $\mathrm{A} = (\mathrm{a}_1, \mathrm{a}_2, \dots, \mathrm{a}_n)$ and $B = (\mathrm{b}_1, \mathrm{b}_2, \dots, \mathrm{b}_n)$ is

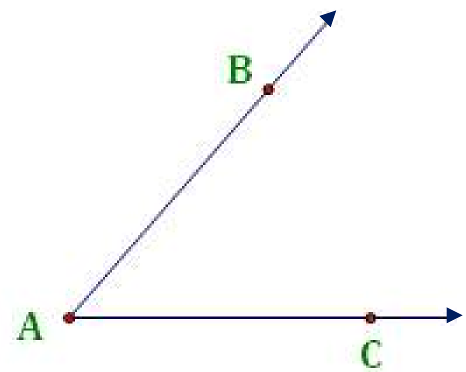
$$d(A, B) = |A - B| = \sqrt{\sum_{i=1}^{n} (a_i - b_i)^2}$$

# Geometry
# Angles

## Parts of an Angle

An angle consists of two rays with a common endpoint (or, initial point).

- Each ray is a side of the angle.
- The common endpoint is called the vertex of the angle.




## Naming Angles

Angles can be named in one of two ways:

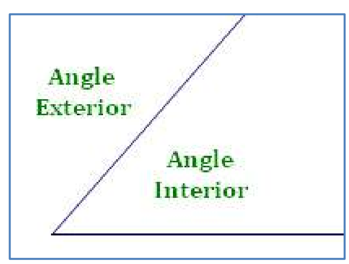
- Point-vertex-point method. In this method, the angle is named from a point on one ray, the vertex, and a point on the other ray. This is the most unambiguous method of naming an angle, and is useful in diagrams with multiple angles sharing the same vertex. In the above figure, the angle shown could be named $\angle BAC$ or $\angle CAB$.
- Vertex method. In cases where it is not ambiguous, an angle can be named based solely on its vertex. In the above figure, the angle could be named $\angle A$.

## Measure of an Angle

There are two conventions for measuring the size of an angle:

- In degrees. The symbol for degrees is $^{o}$. There are $360^{o}$ in a full circle. The angle above measures approximately $360^{0} \div 8 = 45^{0}$ (one-eighth of a circle).
- In radians. There are $2\pi$ radians in a complete circle. The angle above measures approximately $\frac{2\pi}{8} = \frac{\pi}{4}$ radians.

## Some Terms Relating to Angles



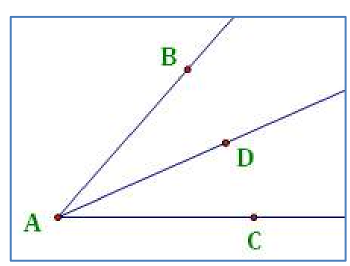

Angle interior is the area between the rays.

Angle exterior is the area not between the rays.

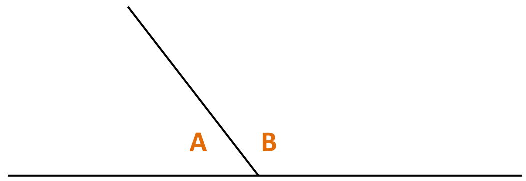
Adjacent angles are angles that share a ray for a side. $\angle BAD$ and $\angle DAC$ in the figure at right are adjacent angles.

Congruent angles are angles with the same measure.

Angle bisector is a ray that divides the angle into two congruent angles. Ray $\overrightarrow{AD}$ bisects $\angle BAC$ in the figure at right.



# Geometry
## Types of Angles

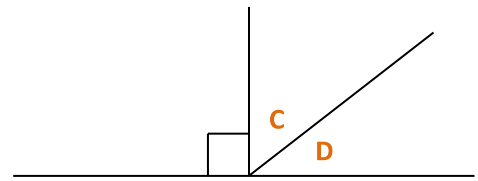



Supplementary Angles

Angles A and B are supplementary.

Angles A and B form a linear pair.

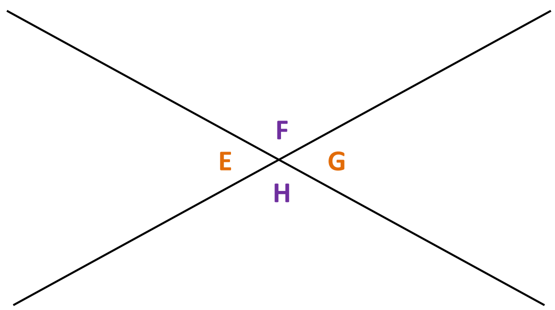
$m\angle A + m\angle B = 180^0$




Complementary Angles

Angles C and D are complementary.

$m\angle C + m\angle D = 90^0$




Vertical Angles

Angles which are opposite each other when two lines cross are vertical angles.

Angles E and G are vertical angles.
Angles F and H are vertical angles.

$m\angle E = m\angle G \quad and \quad m\angle F = m\angle H$

In addition, each angle is supplementary to the two angles adjacent to it. For example:

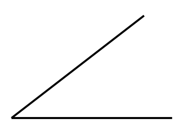
Angle E is supplementary to Angles F and H.

An acute angle is one that is less than $90^0$. In the illustration above, angles E and G are acute angles.

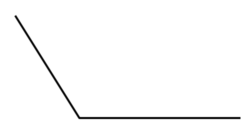
A right angle is one that is exactly $90^0$.

An obtuse angle is one that is greater than $90^0$. In the illustration above, angles F and H are obtuse angles.

A straight angle is one that is exactly $180^0$.



Acute



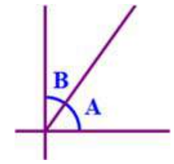
Obtuse

Right



Straight

**Example 1.9:** Two angles are complementary. The measure of one angle is 21° more than twice the measure of the other angle. Find the measures of the angles.




Drawing the situation described in the problem is often helpful.

Let the two angles be called angle **A** and angle **B**. Let's rewrite the problem in terms of these two angles.

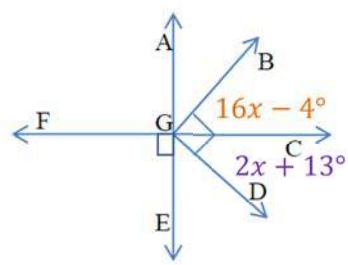
Angles A and B are complementary. $m\angle A = 21° + 2(m\angle B)$.

Let the measures of the angles be represented by the names of the angles. Then,

$$\left.\begin{aligned} A + B &= 90° \\ A &= 21° + 2B \end{aligned}\right\} \longrightarrow \begin{aligned} 2A + 2B &= 180° \\ +\ \ A - 2B &= \ \ 21° \\ \hline 3A \qquad &= 201° \\ \mathbf{A} \qquad &= \mathbf{67°} \end{aligned} \longrightarrow \begin{aligned} A + B &= 90° \\ 67° + B &= 90° \\ \hline \mathbf{B} &= \mathbf{23°} \end{aligned}$$

The measures of the two angles then, are, $\mathbf{67°}$ and $\mathbf{23°}$

**Example 1.10:** If $m\angle BGC = 16x - 4°$ and $m\angle CGD = 2x + 13°$, find the value of $x$ so that $\angle BGD$ is a right angle.




$\angle BGD$ is a right angle (i.e., $m\angle BGD = 90°$).

Then, $(16x - 4°) + (2x + 13°) = 90°$

$$\begin{aligned} 18x + 9° &= 90° \\ 18x \quad &= 81° \\ \boldsymbol{x} \quad &= \mathbf{4.5°} \end{aligned}$$

**Example 1.11:** Find $m\angle 1$ if $\angle 1$ is complementary to $\angle 2$, $\angle 2$ is supplementary to $\angle 3$, and $m\angle 3 = 126°$.

Let's turn this into equations because the English is confusing.

$m\angle 1 + m\angle 2 = 90°$ (complementary)

$m\angle 2 + m\angle 3 = 180°$ (supplementary)

$m\angle 3 = 126°$

Working with these equations from bottom to top, we get:

$m\angle 3 = 126°$

$m\angle 2 + m\angle 3 = m\angle 2 + 126° = 180°$, so $m\angle 2 = 54°$

$m\angle 1 + m\angle 2 = m\angle 1 + 54° = 90°$ so $\boldsymbol{m\angle 1 = 36°}$

# Geometry
## Conditional Statements

A conditional statement contains both a hypothesis and a conclusion in the following form:

If **hypothesis**, then **conclusion**.

For any conditional statement, it is possible to create three related conditional statements, as shown below. In the table, $p$ is the hypothesis of the original statement and $q$ is the conclusion of the original statement.

Statements linked below by red arrows must be either both true or both false.

| Type of Conditional Statement | Example Statement is: |
|---|---|
| **Original Statement:** If $p$, then $q$. $(p \rightarrow q)$<br>• Example: If a number is divisible by 6, then it is divisible by 3.<br>• The original statement may be either true or false. | TRUE |
| **Converse Statement:** If $q$, then $p$. $(q \rightarrow p)$<br>• Example: If a number is divisible by 3, then it is divisible by 6.<br>• The converse statement may be either true or false, and this does not depend on whether the original statement is true or false. | FALSE |
| **Inverse Statement:** If not $p$, then not $q$. $(\sim p \rightarrow \sim q)$<br>• Example: If a number is not divisible by 6, then it is not divisible by 3.<br>• The inverse statement is always true when the converse is true and false when the converse is false. | FALSE |
| **Contrapositive Statement:** If not $q$, then not $p$. $(\sim q \rightarrow \sim p)$<br>• Example: If a number is not divisible by 3, then it is not divisible by 6.<br>• The Contrapositive statement is always true when the original statement is true and false when the original statement is false. | TRUE |

**Note also that:**

- When two statements must be either both true or both false, they are called **equivalent statements**.
  - The original statement and the contrapositive are equivalent statements.
  - The converse and the inverse are equivalent statements.
- If both the original statement and the converse are true, the phrase **"if and only if"** (abbreviated **"iff"**) may be used. For example, **"A number is divisible by 3 iff the sum of its digits is divisible by 3."**

# Geometry
## Basic Properties of Algebra

### Properties of Equality and Congruence.

| Property | Definition for Equality | Definition for Congruence |
|---|---|---|
| | For any real numbers **a**, **b**, and **c**: | For any geometric elements **a**, **b** and **c**. (e.g., segment, angle, triangle) |
| Reflexive Property | $a = a$ | $a \cong a$ |
| Symmetric Property | $If\ a = b, then\ b = a$ | $If\ a \cong b, then\ b \cong a$ |
| Transitive Property | $If\ a = b\ and\ b = c, then\ a = c$ | $If\ a \cong b\ and\ b \cong c, then\ a \cong c$ |
| Substitution Property | If $a = b$, then either can be substituted for the other in any equation (or inequality). | If $a \cong b$, then either can be substituted for the other in any congruence expression. |

### More Properties of Equality. For any real numbers **a**, **b**, and **c**:

| Property | Definition for Equality |
|---|---|
| Addition Property | $If\ a = b, then\ a + c = b + c$ |
| Subtraction Property | $If\ a = b, then\ a - c = b - c$ |
| Multiplication Property | $If\ a = b, then\ a \cdot c = b \cdot c$ |
| Division Property | $If\ a = b\ and\ c \neq 0, then\ a \div c = b \div c$ |

### Properties of Addition and Multiplication. For any real numbers **a**, **b**, and **c**:

| Property | Definition for Addition | Definition for Multiplication |
|---|---|---|
| Commutative Property | $a + b = b + a$ | $a \cdot b = b \cdot a$ |
| Associative Property | $(a + b) + c = a + (b + c)$ | $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ |
| Distributive Property | $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$ | |

# Geometry
# Inductive vs. Deductive Reasoning

## Inductive Reasoning

Inductive reasoning uses observation to form a hypothesis or conjecture. The hypothesis can then be tested to see if it is true. The test must be performed in order to confirm the hypothesis.

**Example:** Observe that the sum of the numbers 1 to 4 is $(4 \cdot 5/2)$ and that the sum of the numbers 1 to 5 is $(5 \cdot 6/2)$. Hypothesis: the sum of the first n numbers is $(n * (n + 1)/2)$. Testing this hypothesis confirms that it is true.

## Deductive Reasoning

Deductive reasoning argues that if something is true about a broad category of things, it is true of an item in the category.

**Example:** All birds have beaks. A pigeon is a bird; therefore, it has a beak.

There are two key types of deductive reasoning of which the student should be aware:

- Law of Detachment. **Given that $p \rightarrow q$, if $p$ is true then $q$ is true.** In words, if one thing implies another, then whenever the first thing is true, the second must also be true.

  **Example 2.1:** Start with the statement: "If a living creature is human, then it has a brain." Then because you are human, we can conclude that you have a brain.

- Syllogism. **Given that $p \rightarrow q$ and $q \rightarrow r$, we can conclude that $p \rightarrow r$.** This is a kind of transitive property of logic. In words, if one thing implies a second and that second thing implies a third, then the first thing implies the third.

  **Example 2.2:** Start with the statements: "If my pencil breaks, I will not be able to write," and "if I am not able to write, I will not pass my test." Then I can conclude that "If my pencil breaks, I will not pass my test."

# Geometry
# An Approach to Proofs

Learning to develop a successful proof is one of the key skills students develop in geometry. The process is different from anything students have encountered in previous math classes, and may seem difficult at first. Diligence and practice in solving proofs will help students develop reasoning skills that will serve them well for the rest of their lives.

## Requirements in Performing Proofs

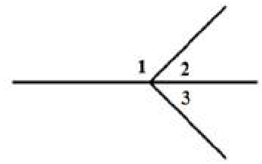
- Each proof starts with a set of "givens," statements that you are supplied and from which you must derive a "conclusion." Your mission is to start with the givens and to proceed logically to the conclusion, providing reasons for each step along the way.
- Each step in a proof builds on what has been developed before. Initially, you look at what you can conclude from the" givens." Then as you proceed through the steps in the proof, you are able to use additional things you have concluded based on earlier steps.
- Each step in a proof must have a valid reason associated with it. So, each statement in the proof must be furnished with an answer to the question: "Why is this step valid?"

## Tips for Successful Proof Development

- At each step, think about what you know and what you can conclude from that information. Do this initially without regard to what you are being asked to prove. Then look at each thing you can conclude and see which ones move you closer to what you are trying to prove.
- Go as far as you can into the proof from the beginning. If you get stuck, …
- Work backwards from the end of the proof. Ask yourself what the last step in the proof is likely to be. For example, if you are asked to prove that two triangles are congruent, try to see which of the several theorems about this is most likely to be useful based on what you were given and what you have been able to prove so far.
- Continue working backwards until you see steps that can be added to the front end of the proof. You may find yourself alternating between the front end and the back end until you finally bridge the gap between the two sections of the proof.
- Don't skip any steps. Some things appear obvious, but actually have a mathematical reason for being true. For example, $a = a$ might seem obvious, but "obvious" is not a valid reason in a geometry proof. The reason for $a = a$ is a property of algebra called the "reflexive property of equality." Use mathematical reasons for all your steps.

**Proof examples** (may use information presented later in this handbook)

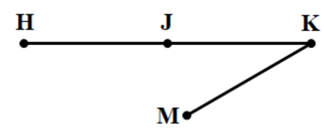
**Example 2.3:** Given: $m\angle 1 + m\angle 3 = 180°$. Prove: $\angle 2 \cong \angle 3$.




Recall that congruent angles have the same measure.

| Step | Statement | Reason |
|---|---|---|
| 1 | $m\angle 1 + m\angle 3 = 180°$ | Given. |
| 2 | $\angle 1$ and $\angle 3$ are supplementary. | If the sum of two angles is 180°, then the angles are supplementary. |
| 3 | $\angle 1$ and $\angle 2$ form a linear pair. | Diagram. |
| 4 | $\angle 1$ and $\angle 2$ are supplementary. | If two angles form a linear pair, then the angles are supplementary. |
| 5 | $\angle 2 \cong \angle 3$ | If two angles are supplementary to the same angle, then they are congruent. |

**Example 2.4:** Given: $\overline{KJ} \cong \overline{MK}$, J is the midpoint of $\overline{HK}$.
Prove: $\overline{HJ} \cong \overline{MK}$.




Recall that congruent segments have the same measure.

Thought process. Based on the givens, it appears that the three segments identified in the diagram are all congruent. That is, $\overline{HJ} \cong \overline{KJ} \cong \overline{MK}$. We need to work from the congruence we are given to the one we want to prove by considering how the segments relate to each other one pair at a time.

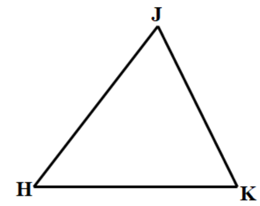
| Step | Statement | Reason |
|---|---|---|
| 1 | $\overline{KJ} \cong \overline{MK}$<br>$J$ is the midpoint of $\overline{HK}$ | Given |
| 2 | $\overline{KJ} \cong \overline{HJ}$ | A midpoint creates two congruent segments. |
| 3 | $\overline{HJ} \cong \overline{MK}$ | Transitive property of congruence (in this case, two segments that are each congruent to a third segment are congruent to each other). |

Note: purple text in the proof is explanatory and is not required to complete the proof.

**Example 2.5:** Given: $\angle H \ncong \angle K$.

Prove: $\Delta JHK$ is not isosceles with base $\overline{HK}$.

Note: the " $\ncong$ " symbol means "is not congruent to".

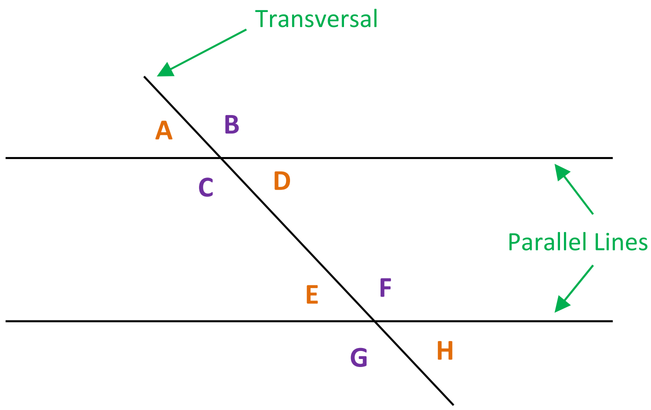



We will use proof by contradiction on this problem. In proof by contradiction, we assume that the opposite of the conclusion is true, then show that is impossible. This implies that the original assumption is false, so its opposite (what we want to prove) must be true.

| Step | Statement | Reason |
|---|---|---|
| 1 | $\angle H, \angle K$ not congruent | Given |
| 2 | Assume $\Delta JHK$ is isosceles with base $\overline{HK}$. | Assumption intended to lead to a contradiction. |
| 3 | $JK = JH$ | Euclid's definition of isosceles triangle. |
| 4 | $\overline{JK} \cong \overline{JH}$ | Definition of congruent segments. |
| 5 | $\angle H \cong \angle K$ | Angles opposite congruent sides in a triangle are congruent. |
| 6 | Contradiction | We are given $\angle H, \angle K$ are not congruent. |
| 7 | $\Delta JHK$ is not isosceles with base $\overline{HK}$. | Assumption in Step 2 must be false. |

Additional proofs are provided throughout this handbook.

# Geometry
# Parallel Lines and Transversals




**Alternate**: refers to angles that are on opposite sides of the transversal.

**Consecutive:** refers to angles that are on the same side of the transversal.

**Interior**: refers to angles that are between the parallel lines.

**Exterior**: refers to angles that are outside the parallel lines.

## Corresponding Angles

**Corresponding Angles** are angles in the same location relative to the parallel lines and the transversal. For example, the angles on top of the parallel lines and left of the transversal (i.e., top left) are corresponding angles.

Angles **A** and **E (top left)** are **Corresponding Angles**. So are angle pairs **B** and **F (top right)**, **C** and **G (bottom left)**, and **D** and **H (bottom right)**. Corresponding angles are congruent.

## Alternate Interior Angles

Angles **D** and **E** are **Alternate Interior Angles**. Angles **C** and **F** are also alternate interior angles. Alternate interior angles are congruent.

## Alternate Exterior Angles

Angles **A** and **H** are **Alternate Exterior Angles**. Angles **B** and **G** are also alternate exterior angles. Alternate exterior angles are congruent.

## Consecutive Interior Angles

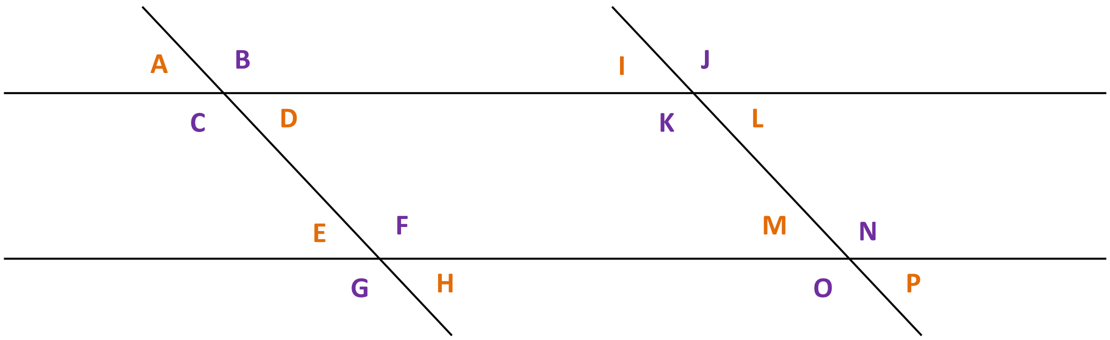
Angles **C** and **E** are **Consecutive Interior Angles**. Angles **D** and **F** are also consecutive interior angles. Consecutive interior angles are supplementary.

*Note that angles* ***A****,* ***D****,* ***E****, and* ***H*** *are congruent, and angles* ***B****,* ***C****,* ***F****, and* ***G*** *are congruent. In addition, each of the angles in the first group are supplementary to each of the angles in the second group.*

# Geometry
# Multiple Sets of Parallel Lines

## Two Transversals

Sometimes, the student is presented two sets of intersecting parallel lines, as shown above. Note that each pair of parallel lines is a set of transversals to the other set of parallel lines.




In this case, the following groups of angles are congruent:

- Group 1: Angles **A**, **D**, **E**, **H**, **I**, **L**, **M** and **P** are all congruent.
- Group 2: Angles **B**, **C**, **F**, **G**, **J**, **K**, **N**, and **O** are all congruent.
- Each angle in the Group 1 is supplementary to each angle in Group 2.

**Some Examples: In the diagram above (Two Transversals), with two pairs of parallel lines, what types of angles are identified and what is their relationship to each other?**

**Example 3.1:** $\angle D$ and $\angle I$.

These angles are alternate interior angles; they are congruent.

**Example 3.2:** $\angle C$ and $\angle J$.

These angles are alternate exterior angles; they are congruent.

**Example 3.3:** $\angle J$ and $\angle N$.

These angles are corresponding angles; they are congruent.

**Example 3.4:** $\angle F$ and $\angle M$.

These angles are consecutive interior angles; they are supplementary.

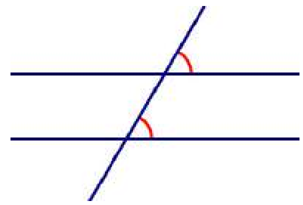
**Example 3.5:** $\angle G$ and $\angle L$.

These angles do not have a name, but they are supplementary.

# Geometry
# Proving Lines are Parallel

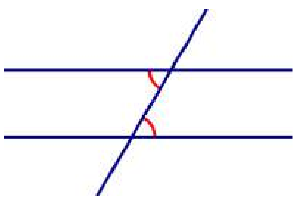
The properties of parallel lines cut by a transversal can be used to prove two lines are parallel.

## Corresponding Angles



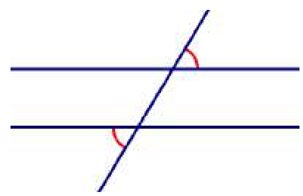
If two lines cut by a transversal have congruent corresponding angles, then the lines are parallel. Note that there are 4 sets of corresponding angles.

## Alternate Interior Angles



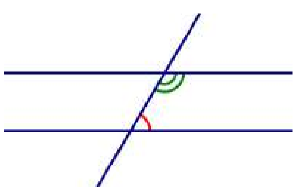
If two lines cut by a transversal have congruent alternate interior angles congruent, then the lines are parallel. Note that there are 2 sets of alternate interior angles.

## Alternate Exterior Angles



If two lines cut by a transversal have congruent alternate exterior angles, then the lines are parallel. Note that there are 2 sets of alternate exterior angles.

## Consecutive Interior Angles

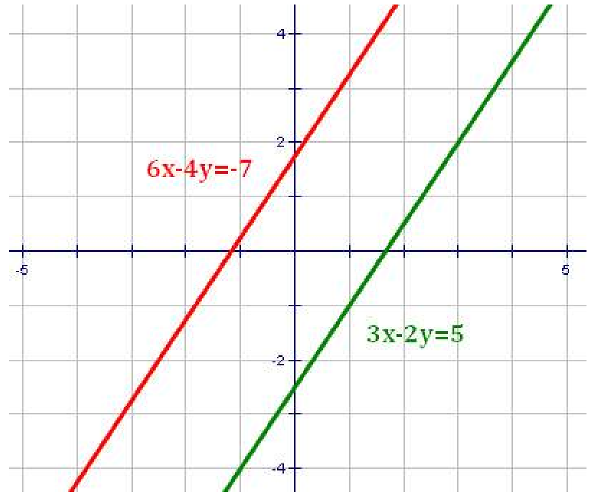


If two lines cut by a transversal have supplementary consecutive interior angles, then the lines are parallel. Note that there are 2 sets of consecutive interior angles.

# Geometry
# Parallel and Perpendicular Lines in the Coordinate Plane

## Parallel Lines

**Two lines are parallel if their slopes are equal.**




- In $y = mx + b$ form, if the values of $m$ are the same.

  **Example 3.6:** $y = 2x - 3$ and
  $y = 2x + 1$

- In Standard Form, if the coefficients of $x$ and $y$ are proportional between the equations.

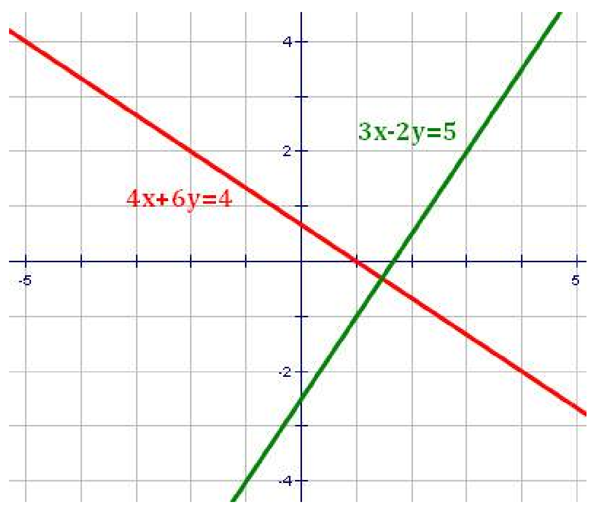
  **Example 3.7:** $3x - 2y = 5$ and
  $6x - 4y = -7$

- Also, if the lines are both vertical (i.e., their slopes are undefined).

  **Example 3.8:** $x = -3$ and
  $x = 2$

## Perpendicular Lines

**Two lines are perpendicular if the product of their slopes is $-1$.** That is, if the slopes have different signs and are multiplicative inverses.




- In $y = mx + b$ form, the values of $m$ multiply to get $-1$..

  **Example 3.9:** $y = 6x + 5$ and
  $y = -\frac{1}{6}x - 3$

- In Standard Form, if you add the product of the $x$-coefficients to the product of the $y$-coefficients and get zero.

  **Example 3.10:** $4x + 6y = 4$ and
  $3x - 2y = 5$ because $(4 \cdot 3) + \big(6 \cdot (-2)\big) = 0$

- Also, if one line is vertical (i.e., $m$ is undefined) and one line is horizontal (i.e., $m = 0$).

  **Example 3.11:** $x = 6$ and
  $y = 3$

**Example 3.12:** Write the equation of the perpendicular bisector of CD if $\mathrm{C}(-4, 3)$ and $\mathrm{D}(-8, -9)$.

Line containing $\overline{CD}$:

$$m = \frac{-9 - 3}{-8 - (47)} = \frac{-12}{-4} = 3$$

Midpoint of $(-4, 3)$ and $(-8, -9)$ is halfway between them: $(-6, -3)$

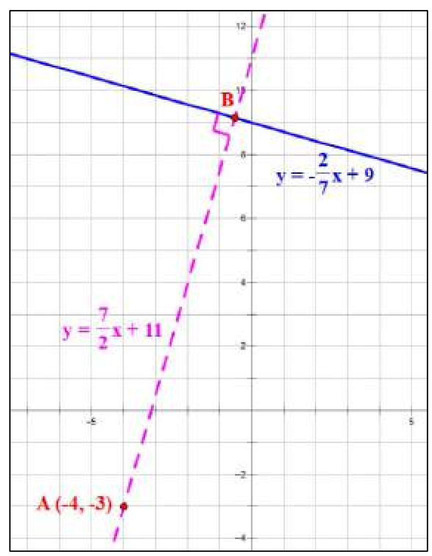
Perpendicular bisector: Slope is the "negative reciprocal" of the slope of $\overleftrightarrow{CD}$ because the lines are perpendicular. Also, $(-6, -3)$ is a point on the perpendicular bisector.

$$m = -\frac{1}{3}$$

**Equation:** $\boldsymbol{y + 3 = -\frac{1}{3}(x + 6)}$ **or** $\boldsymbol{y = -\frac{1}{3}(x + 6) - 3}$ **or** $\boldsymbol{y = -\frac{1}{3}x - 5}$

point-slope form — $h$-$k$ form — slope-intercept form

**Example 3.13:** Write an equation of the line that can be used to calculate the distance between $(-4, -3)$ and the line $y = -\frac{2}{7}x + 9$.




The distance between a point and a line is the length of the segment connecting the point to the line at a right angle. See the diagram to the right.

So, this question is asking for the equation of the line perpendicular to $y = -\frac{2}{7}x + 9$ that contains the point $(-4, -3)$, but is not asking us to calculate the distance.

The perpendicular line will have a slope that is the opposite reciprocal of the original line:

$$m = -\frac{1}{-\frac{2}{7}} = \frac{7}{2}$$

Then, the equation of the perpendicular line (in $h$-$k$ form) is:

$$\boldsymbol{y = \frac{7}{2}(x + 4) - 3}$$

Note: If we were asked to calculate the distance between Point A and the line $y = -\frac{2}{7}x + 9$, we would first need to find Point B at the intersection of the two lines shown, and then measure the distance between the two points using the distance formula.

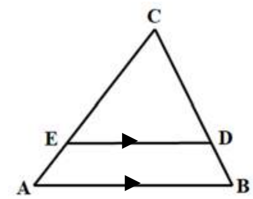
# Geometry
# Proportional Segments

## Parallel Line in a Triangle

**A line is parallel to one side of a triangle iff it divides the other two sides proportionately.**

This if-and-only-if statement breaks down into the following two statements:

- If a line (or ray or segment) is parallel to one side of a triangle, then it divides the other two sides proportionately.
- If a line (or ray or segment) divides two sides of a triangle proportionately, then it is parallel to the third side.

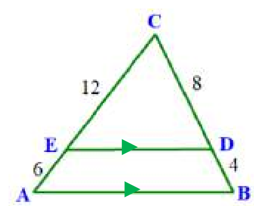



In the diagram to the right, we see that $\overline{AB} \parallel \overline{ED}$. We can conclude that:

$\frac{CE}{EA} = \frac{CD}{DB}$ and $\frac{CE}{CD} = \frac{EA}{DB}$ as well as a number of other equivalent proportion equalities.

Conversely, if we knew one of the proportions above, but were not given that the segments were parallel, we could conclude that $\overline{AB} \parallel \overline{ED}$ because of the equal proportions.

**Example 3.14:** Determine whether $\overline{AB} \parallel \overline{ED}$ in the diagram to the right.




Let's check the proportions. Is $\frac{CE}{CD} = \frac{EA}{DB}$ ?

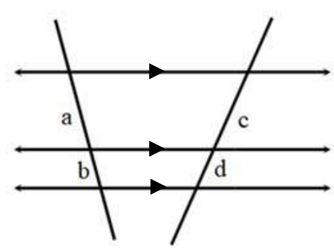
$\frac{CE}{CD} = \frac{12}{8} = \frac{3}{2}$ $\qquad$ $\frac{EA}{DB} = \frac{6}{4} = \frac{3}{2}$

Since the proportions of the two sides are equal, we can conclude that $\boldsymbol{\overline{AB} \parallel \overline{ED}}$**.**

## Three or More Parallel Lines

**Three or more parallel lines divide any transversals proportionately.**

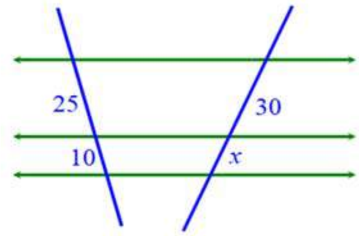
In the diagram to the right, we see that the three horizontal lines (or rays or segments) are parallel. We can conclude that:




$\frac{a}{b} = \frac{c}{d}$ and $\frac{a}{c} = \frac{b}{d}$.

The converse of this is not true. That is, if three or more lines divide transversals into proportionate parts, *it is not necessarily true* that the lines are parallel.

**Example 3.15:** Given that the three horizontal lines in the diagram to the right are parallel, what is the values of $x$?




The three parallel horizontal lines in the diagram divide the vertical lines into proportional segments.

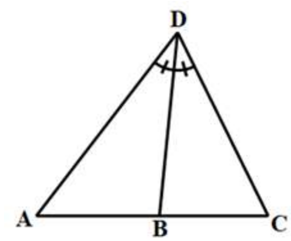
$$\frac{25}{10} = \frac{30}{x}$$

$$25x = 300$$

$$\boldsymbol{x = 12}$$

## Angle Bisector

**An angle bisector in a triangle divides the opposite sides into segments that are proportional to the adjacent sides.**

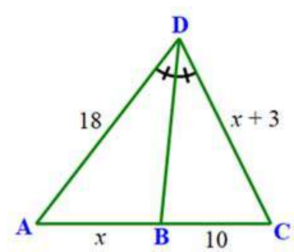



In the diagram to the right, we see that $\angle D$ is bisected, creating segments $\overline{AB}$ and $\overline{BC}$ opposite $\angle D$. We can conclude that:

$$\frac{AB}{AD} = \frac{BC}{DC} \text{ and } \frac{AB}{BC} = \frac{AD}{DC}.$$

The converse of this is also true. That is, if a line (or ray or segment) through a vertex of a triangle splits the opposite side into segments that are proportional to the adjacent sides, then, that line (or ray or segment) bisects the vertex angle. That is, if the above proportions are true, then $\overline{DB}$ bisects $\angle D$.

**Example 3.16:** Find the value of $x$ in the diagram.




An angle bisector in a triangle divides the opposite sides into segments that are proportional to the adjacent sides. So,

$$\frac{18}{x} = \frac{x+3}{10}$$

$$x(x+3) = 18 \cdot 10$$

$$x^2 + 3x = 180$$

$$x^2 + 3x - 180 = 0$$

$$(x-12)(x+15) = 0 \quad \rightarrow \quad x = 12, -15$$

If $x = -15$, we have negative side lengths, so we discard the solution $x = -15$.

If $x = 12$, the sides of $\Delta BAD$ would be $18, 15, 22$, which makes a valid triangle.

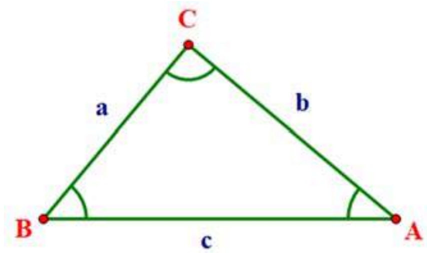
Conclude: $\boldsymbol{x = 12.}$

# Geometry
# What Makes a Triangle?

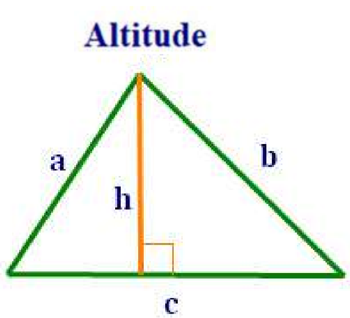
**Definition** – A triangle is a plane figure with three sides and three angles.

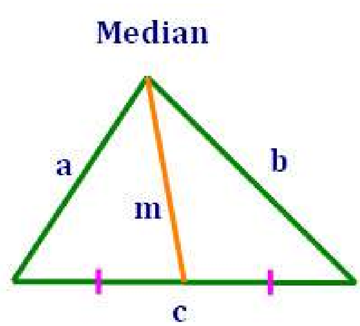
- Draw three points that are not on the same line, connect them, and you have a triangle. The three points you started with are called vertices.
- Three points determine a plane, so a triangle must have all of its parts on the same plane.

## Parts of a Triangle



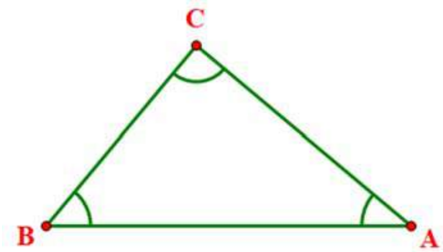

- Vertices – the points where the sides intersect. In the diagram to the right, the vertices are the red points. Vertices are typically labeled with capital letters.
- Legs – the sides of a triangle are also called the triangle's legs. In diagrams, the lengths of the legs are often represented by lower case letters corresponding to the angles opposite them.
- Angles (interior angles) – the angles formed at each vertex are the triangle's angles. In the diagram above, the triangle has interior angles $\angle A, \angle B, \angle C$ indicated by the green arcs at the vertices. These angles could be named in various ways, for example:
  - $\angle A = \angle BAC = \angle CAB$.
  - Naming the angle with a single vertex is acceptable if there is no ambiguity about which angle is being referenced, e.g., $\angle A$.
  - If any ambiguity exists as to which angle is being referenced, the angle must be named using three points: two of the points must be on the sides enclosing the angle and the vertex must be in the middle, e.g., $\angle BAC$ or $\angle CAB$.
  - Alternatively, an angle may be named with a letter or symbol next to its arc.
- Altitudes – line segments from each vertex to the opposite side of the triangle that are perpendicular to that opposite side. In the diagram below left, an altitude is labeled h.
- Medians – line segments from each vertex to the midpoint of the opposite side of the triangle. In the diagram below right, a median is labeled m.



### Sum of Interior Angles

The sum of the interior angles of a triangle is $180°$. If two of the interior angles in a triangle have known measures, the measure of the third can be easily calculated. For example, in the diagram to the right, if $m\angle A$ and $m\angle B$ are known, $m\angle C$ can be calculated as:



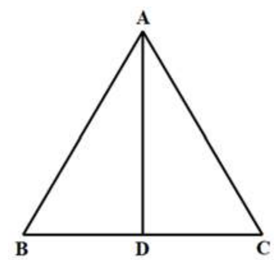

$$m\angle C = 180° - m\angle A - m\angle B.$$

**Third Angle Theorem:** If two interior angles in one triangle are congruent to two interior angles in another triangle, then the third interior angles in the two triangles are congruent.

This follows from the fact that the sum of the three interior angles in each triangle must be $180°$.

**Example 4.1:** Given $\overline{AD} \perp \overline{BC}$, $\overline{AD}$ bisects $\angle BAC$, prove $\angle ABD \cong \angle ACD$.

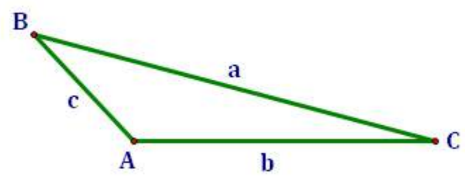
This can be proven in multiple ways. Let's prove it with the Third Angle Theorem.




| Step | Statement | Reason |
|---|---|---|
| 1 | $\overline{AD} \perp \overline{BC}$.<br>$\overline{AD}$ bisects $\angle BAC$. | Given. |
| 2 | $\angle ADB$ is a right angle.<br>$\angle ADC$ is a right angle. | $\overline{AD} \perp \overline{BC}$. Perpendicular lines form right angles. |
| 3 | $\angle ADB \cong \angle ADC$. | All right angles are congruent (they all measure $90°$). |
| 4 | $\angle BAD \cong \angle CAD$. | $\overline{AD}$ bisects $\angle BAC$. |
| 5 | $\angle ABD \cong \angle ACD$ | Third Angle Theorem (triangles are $\Delta ADB$ and $\Delta ADC$). |

# Geometry
# Inequalities in Triangles

Angles and their opposite sides in triangles are related. In fact, this is often reflected in the labeling of angles and sides in triangle illustrations.




Angles and their opposite sides are often labeled with the same letter. An upper case letter is used for the angle and a lower case letter is used for the side.

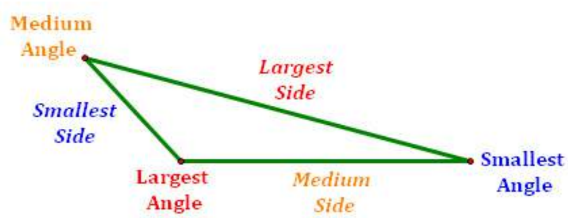
The relationship between angles and their opposite sides translates into the following triangle inequalities:

$$\text{If } \ m\angle C < m\angle B < m\angle A, \ \text{ then } \ c < b < a$$

$$\text{If } \ m\angle C \leq m\angle B \leq m\angle A, \ \text{ then } \ c \leq b \leq a$$

That is, in any triangle,

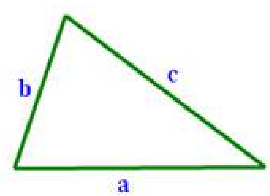
- The largest side is opposite the largest angle.
- The medium side is opposite the medium angle.
- The smallest side is opposite the smallest angle.




## Other Inequalities in Triangles

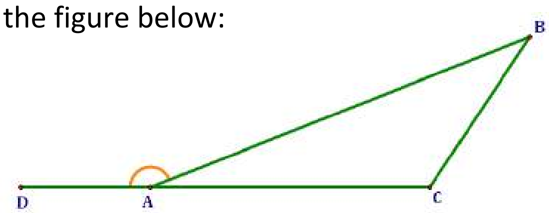
**Triangle Inequality:** The sum of the lengths of any two sides of a triangle is greater than the length of the third side. Also, the difference of the lengths of any two sides is smaller than the length of the third side. If $a > b$:

$a - b < c < a + b$ and similar for the other sides.




**Exterior Angle Inequality:** The measure of an external angle is greater than the measure of either of the two non-adjacent interior angles. That is, in the figure below:

$$m\angle DAB > m\angle B \quad \text{and} \quad m\angle DAB > m\angle C.$$




**Exterior Angle Equality:** The measure of an external angle is equal to the sum of the measures of the two non-adjacent interior angles. That is, in the figure to the right:

$$m\angle DAB = m\angle B + m\angle C.$$

**Note:** the Exterior Angle Equality is typically more useful than the Exterior Angle Inequality.

## Sides of a Triangle

The lengths of the sides of a triangle are limited: given the lengths of any two sides, the length of the third side must be greater than their difference and less than their sum. That is, if the sides of a triangle have lengths $a$, $b$, and $c$, and you know the values of, for example, $a$ and $b$ with $a$ the larger of the two, then:

$$a - b < c < a + b$$

**Example 4.2:** If a triangle has two sides with lengths 13 and 8, what are the possible lengths of the third side?

If we let $c$ represent the length of the third side of a triangle, with $a = 13, b = 8$, then:

- $c$ must be greater than the difference of $a$ and $b$: $c > 13 - 8 \quad \rightarrow \quad c > 5$.
- $c$ must be less than the sum of $a$ and $b$: $c < 13 + 8 \quad \rightarrow \quad c < 21$.

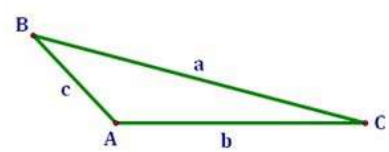
If we put all of this together in a single inequality, we get:

$$13 - 8 < c < 13 + 8$$

$$\mathbf{5 < c < 21}$$

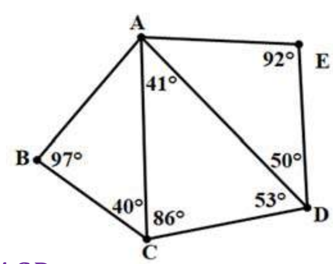
Also, as indicated above, there are limits to the lengths of sides if the measures of the interior angles of the triangle are known. In particular,

- The longest side of a triangle is opposite the largest interior angle.
- The shortest side of a triangle is opposite the smallest interior angle.




In general, if we know that $m\angle C < m\angle B < m\angle A$, then we know that $c < b < a$.

**Example 4.3:** Identify the longest segment in the diagram shown.




Let's see what we know in each of the triangles. Note that:

- The sum of the angles in each triangle must be 180° and
- Sides across from larger angles in the same triangle are larger.

In $\Delta ABC$:

- $m\angle BAC = 43°$
- $AB < BC < AC$

In $\Delta ADE$:

- $m\angle EAD = 38°$
- $DE < AE < AD$

In $\Delta ACD$:

- $CD < AC < AD$

Therefore, the two candidates for longest segment are $\overline{AC}$ and $\overline{AD}$. Looking closer at the above inequalities, we notice that in $\Delta ACD$, we have $AC < AD$. Therefore, the longest segment is: $\boldsymbol{\overline{AD}}$.

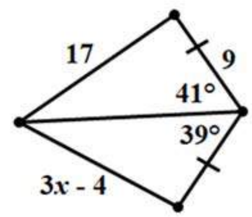
The discussion above addresses angles within a single triangle. There is another relationship that allows us to compare the lengths of sides in two different triangles. In particular,

> If two triangles have two pairs of congruent sides, consider the angles between the congruent sides. The triangle with the larger of these angles has the larger side opposite that angle. This is illustrated in the next example.

**Example 4.4:** Find the range of values for $x$.

**Note: never trust the relative sizes of angles and sides in a diagram.** For example, the two sides with length 9 in this diagram are drawn with different lengths!

We know two things involving $x$:




- The side labeled $3x - 4$ must be positive. So, $3x - 4 > 0$.
- The two angles shown (39°) and (41°) share two congruent sides (one side with length 9 and one side of unknown length that is shared by the two angles). Therefore, the side opposite the smaller angle must be smaller than the side opposite the larger angle. So, $3x - 4 < 17$.

Combining these two inequalities into a single compound inequality, and solving:

| | |
|---|---|
| Starting inequality: | $0 < 3x - 4 < 17$ |
| Add 4: | $4 < 3x < 21$ |
| Divide by 3: | $\boldsymbol{\frac{4}{3} < x < 7}$ |

**Example 4.5:** Given $\Delta\text{ABC}$ with $\text{A}(-3, 4), \text{B}(7, 1), \text{C}(2, -1)$, and median $\overline{\text{AD}}$, find the coordinates of point D.

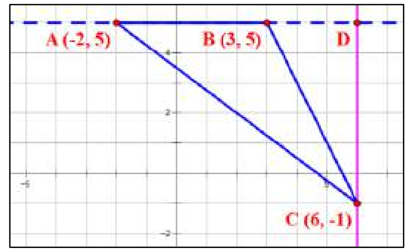
Many times, you need to draw the situation for a given problem. This is *not* one of those times.

Point D is the midpoint of the side of the triangle opposite the given vertex.

In this problem, Point A is the vertex in question (it is on the median $\overline{AD}$). So, Point D is the midpoint of the points $\text{B}(7, 1)$ and $\text{C}(2, -1)$.

So, the coordinates of Point D are: $[\,(7, 1) + (2, -1)] \div 2 = \mathbf{(4.5, 0)}$

**Example 4.6:** Given $\Delta ABC$ with $A(-2,5), B(3,5), C(6,-1)$, and altitude $\overline{CD}$, find the coordinates of point D.




An altitude of a triangle is a line segment drawn from a vertex to a point on the opposite side (extended, if necessary) that is perpendicular to that side.

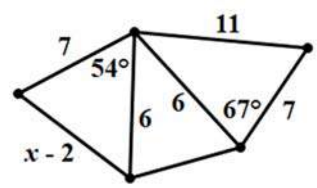
This problem is very straightforward once you graph it. To find the base point of the altitude, we can look at the intersection of the two lines on which Point D lies.

Line containing $\overline{BA}$: $y = 5$

Line containing $\overline{CD}$. $x = 6$ is perpendicular to $y = 5$ and contains $C(6, -1)$.

Therefore, **Point D** has coordinates: $(\mathbf{6}, \mathbf{5})$.

**Example 4.7:** Write and solve an inequality for $x$.




Each side must have a positive measure, so: $x - 2 > 0$

$x > 2$

Also, in the triangle on the left, we have:

$7 - 6 < x - 2 < 7 + 6$

$1 < x - 2 < 13$

$3 < x < 15$

Next, both outside triangles have sides of length 6 and 7 with angles between them.

Since the measure of the angle in the triangle on the left $(54°)$ is less than the one in the triangle on the right $(67°)$, the opposite side on the left must be less than the opposite side on the right. So, $x - 2 < 11$.

$x < 13$

Putting it all together, we have: $3 < x$, equivalent to $x > 3$, which is more restrictive than $x > 2$, so we use the more restrictive $3 < x$.

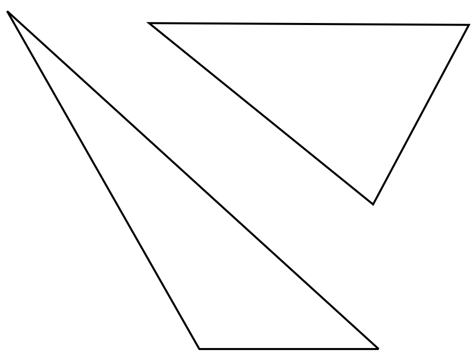
We also have: $x < 13$, which is more restrictive than $x < 15$, so we use the more restrictive $x < 13$.

Finally, since $3 < x$ and $x < 13$, we have $\mathbf{3 < x < 13}$

# Geometry
## Types of Triangles

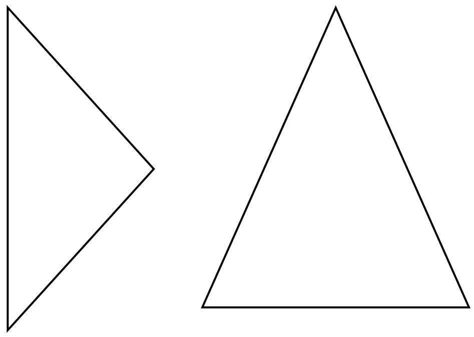
### Scalene

A Scalene Triangle has 3 sides of different lengths. Because the sides are of different lengths, the angles must also be of different measures.



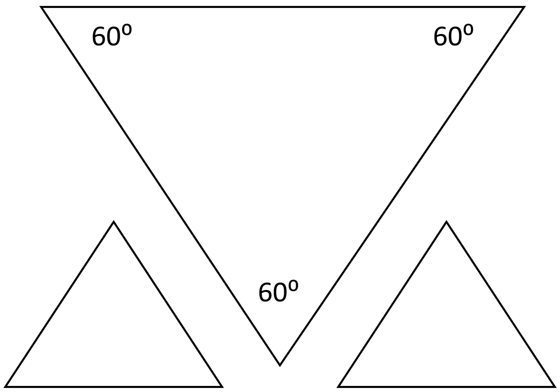
### Isosceles

An Isosceles Triangle has 2 sides the same length (i.e., congruent). Because two sides are congruent, two angles must also be congruent.



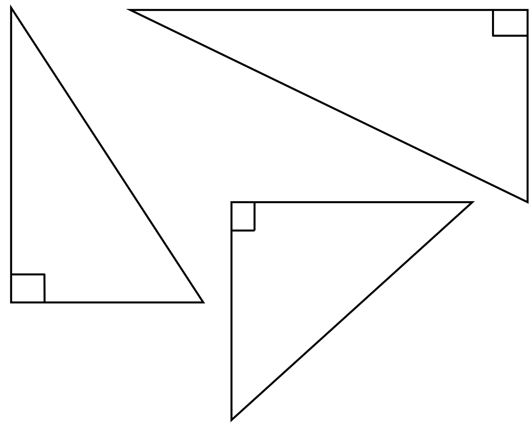
### Equilateral

An Equilateral Triangle has all 3 sides the same length (i.e., congruent). Because all 3 sides are congruent, all 3 angles must also be congruent. This requires each angle to be $60^o$.

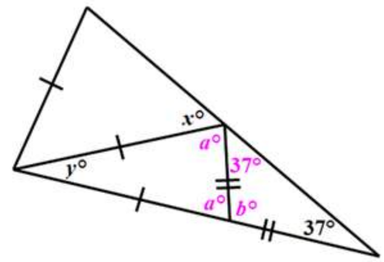



### Right

A Right Triangle is one that contains a $90^o$ angle. It may be scalene or isosceles, but cannot be equilateral. Right triangles have sides that meet the requirements of the Pythagorean Theorem.



**Example 4.8:** Find the values of $x$ and $y$ based on the diagram.




This problem becomes easier if we label a few more angles. See the diagram on the right.

Angles opposite congruent sides in isosceles triangles are congruent, which helps with our labeling.

In the triangle on the right, the sum of the interior angles must be 180°, so,

$b = 180 - 37 - 37 = 106.$

The adjacent angles marked $a°$ and $b°$ form a linear pair, so,

$a = 180 - 106 = 74.$

The center triangle has two angles of $a°$ and one angle of $y°$, which must add to 180°, so,

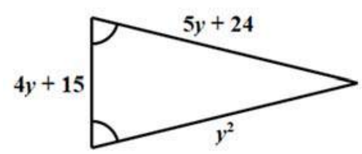
$\boldsymbol{y} = 180 - 74 - 74 = \mathbf{32}.$

Finally, along the top right, angles marked 37°, $a°$, and $x°$ must add to 180° in order to form a straight angle, so,

$\boldsymbol{x} = 180 - 37 - 74 = \mathbf{69}.$

**Example 4.9:** Find the value of $y$ and the perimeter of the triangle.

Legs opposite congruent angles in isosceles triangles are congruent.




$y^2 = 5y + 24$
$y^2 - 5y - 24 = 0$
$(y - 8)(y + 3) = 0$
$\boldsymbol{y} = \mathbf{8, -3}$ (2 possibilities)

If we plug each of these values into the lengths of the sides shown in the diagram, we always get positive numbers, so there are two cases. If we had gotten a length that was negative for either $y = 8$ or $y = -3$, we would have had to discard that solution.

The perimeter of the triangle is: $P = y^2 + (4y + 15) + (5y + 24) = y^2 + 9y + 39.$

Case 1 ($y = 8$): $P = y^2 + 9y + 39 = 8^2 + 9 \cdot 8 + 39 = \mathbf{175}.$ (we are not given units)

Sides of this triangle are 64, 64, 47, which gives a viable triangle.

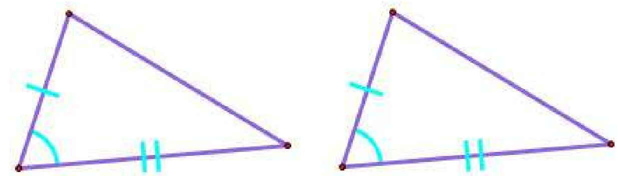
Case 2 ($y = -3$): $P = y^2 + 9y + 39 = (-3)^2 + 9 \cdot (-3) + 39 = \mathbf{21}.$

Sides of this triangle are 9, 9, 3, which gives a viable triangle.

# Geometry
# Congruent Triangles

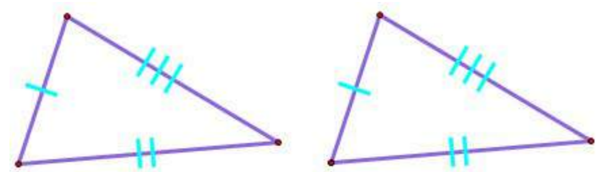
The following theorems present conditions under which triangles are congruent.

## Side-Angle-Side (SAS) Congruence



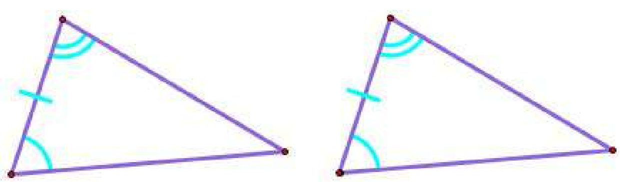
**SAS congruence** requires the congruence of two sides and the angle between those sides. Note that there is no such thing as SSA congruence; the congruent angle must be between the two congruent sides.

## ide-Side-Side (SSS) Congruence



**SSS congruence** requires the congruence of all three sides. If all of the sides are congruent then all of the angles must be congruent. The converse is not true; there is no such thing as AAA congruence.

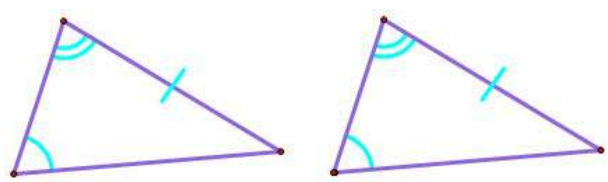
## Angle-Side-Angle (ASA) Congruence



**ASA congruence** requires the congruence of two angles and the side between those angles.

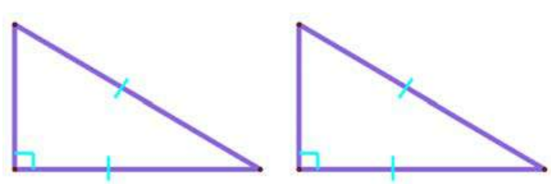
*Note: ASA and AAS combine to provide congruence of two triangles whenever any two angles and any one side of the triangles are congruent.*

## Angle-Angle-Side (AAS) Congruence



**AAS congruence** requires the congruence of two angles and a side which is not between those angles.

## Hypotenuse Leg (HL) Congruence

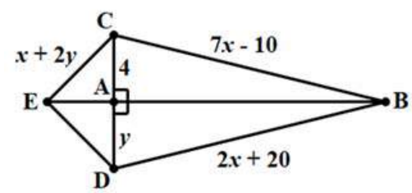


**HL** can be used if the triangles in question have right angles. It requires the congruence of the hypotenuse and *one* of the other legs.

## CPCTC

**CPCTC** means **"corresponding parts of congruent triangles are congruent."** It is a very powerful tool in geometry proofs and is often used shortly after a step in the proof where a pair of triangles is proved to be congruent.

**Example 4.10:** Given that $\overline{BE}$ is a perpendicular bisector of $\overline{CD}$, find $ED$.




In the diagram, $\overline{CA} \cong \overline{DA}$ because $\overline{BE}$ bisects $\overline{CD}$.

So, $\Delta CAB \cong \Delta DAB$ by SAS, and $\Delta CAE \cong \Delta DAE$ by SAS.

The two hypotenuses (yep, that's the plural form of hypotenuse) of the triangles on the right side of the diagram are congruent. So,

$7x - 10 = 2x + 20$

$5x = 30$

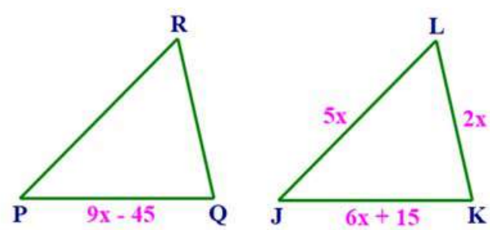
$x = 6$

$CA = DA$, so $y = 4$

Finally, $ED = EC = x + 2y$ (because $\Delta CAE \cong \Delta DAE$, and $\overline{ED}$ and $\overline{EC}$ are corresponding parts of those congruent triangles).

$ED = EC = 6 + 2(4) = \mathbf{14}$

**Example 4.11:** Given $\Delta PQR \cong \Delta JKL, PQ = 9x - 45, JK = 6x + 15, KL = 2x, JL = 5x$, what is the value of $x$?




It's helpful to draw a picture for this problem.

Notice that congruent segments $\overline{PQ}$ and $\overline{JK}$ have measures $9x - 45$ and $6x + 15$. Then:

$9x - 45 = 6x + 15$

$3x = 60$

$\boldsymbol{x = 20}$

We are not quite finished, even though we found a value for $x$. We need to check the sides of $\Delta JKL$ to make sure this results in a viable triangle:

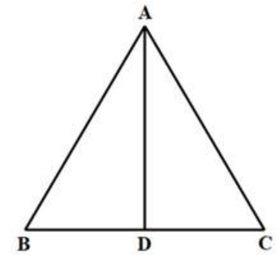
$2x = 40, \quad 5x = 100, \quad 6x + 15 = 135$

Sides of $40, 100, 135$ are viable in a triangle because $40 + 100 > 135$.

Note that if $PQ = 12x - 45$, we would have calculated $x = 10$. Then, the sides would have been $20, 50, 75$, which is not a viable triangle because $20 + 50 < 75$. If this were the case, this problem would have no solution.

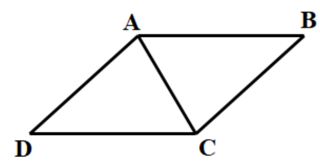
**Example 4.12:** Given $\overline{AD} \perp \overline{BC}$, $\overline{AD}$ bisects $\angle BAC$, prove $\angle B \cong \angle C$.

It looks like we want to head toward $\Delta ADB \cong \Delta ADC$, and use CPCTC.




| Step | Statement | Reason |
|---|---|---|
| 1 | $\overline{AD} \perp \overline{BC}$.<br>$\overline{AD}$ bisects $\angle BAC$. | Given. |
| 2 | $\angle ADB$ is a right angle.<br>$\angle ADC$ is a right angle. | $\overline{AD} \perp \overline{BC}$. Perpendicular lines form right angles. |
| 3 | $\angle ADB \cong \angle ADC$. | All right angles are congruent. |
| 4 | $\overline{AD} \cong \overline{AD}$. | Reflexive property of congruence. |
| 5 | $\angle BAD \cong \angle CAD$. | $\overline{AD}$ bisects $\angle BAC$. |
| 6 | $\Delta ADB \cong \Delta ADC$ | ASA congruence theorem. |
| 7 | $\angle B \cong \angle C$ | CPCTC. |

**Example 4.13:** Given $\overline{AD} \parallel \overline{CB}$, $\overline{AB} \parallel \overline{CD}$, prove $\angle B \cong \angle D$



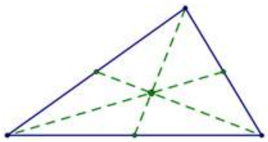

With parallel lines, we will typically look for alternate interior angles or corresponding angles to prove things. Also, this looks like a situation where we prove congruent triangles and can use CPCTC.

| Step | Statement | Reason |
|---|---|---|
| 1 | $\overline{AD} \parallel \overline{CB}$.<br>$\overline{AB} \parallel \overline{CD}$. | Given. |
| 2 | $\angle BAC \cong \angle DCA$. | Alternate interior angles of $\overline{AB} \parallel \overline{CD}$, with $\overline{AC}$ a transversal. |
| 3 | $\angle BCA \cong \angle DAC$. | Alternate interior angles of $\overline{AD} \parallel \overline{CB}$, with $\overline{AC}$ a transversal. |
| 4 | $\overline{AC} \cong \overline{AC}$. | Reflexive property of congruence. |
| 5 | $\Delta BAC \cong \Delta DCA$ | ASA congruence theorem. |
| 6 | $\angle B \cong \angle D$ | CPCTC. |

# Geometry
# Centers of Triangles

The following are all points which can be considered the center of a triangle.

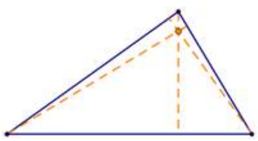
### Centroid (Medians)



The centroid is the intersection of the three medians of a triangle. A median is a line segment drawn from a vertex to the midpoint of the side of the triangle that is opposite the vertex.

- The centroid is located 2/3 of the way from a vertex to the opposite side. That is, the distance from a vertex to the centroid is double the length from the centroid to the midpoint of the opposite line.
- The medians of a triangle create 6 inner triangles of equal area.

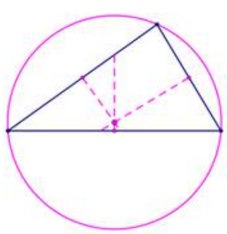
### Orthocenter (Altitudes)



The orthocenter is the intersection of the three altitudes of a triangle. An altitude is a line segment drawn from a vertex to a point on the opposite side (extended, if necessary) that is perpendicular to that side.

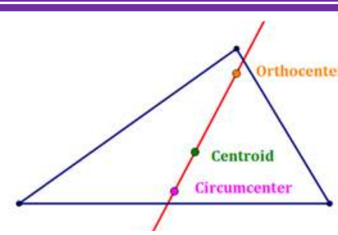
- In an acute triangle, the orthocenter is inside the triangle.
- In a right triangle, the orthocenter is the right angle vertex.
- In an obtuse triangle, the orthocenter is outside the triangle.

### Circumcenter (Perpendicular Bisectors)



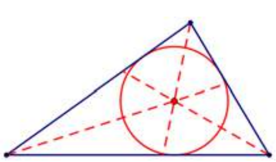
The circumcenter is the intersection of the perpendicular bisectors of the three sides of the triangle. A perpendicular bisector is a line which both bisects the side and is perpendicular to the side. The circumcenter is also the center of the circle circumscribed about the triangle.

- In an acute triangle, the circumcenter is inside the triangle.
- In a right triangle, the circumcenter is the midpoint of the hypotenuse.
- In an obtuse triangle, the circumcenter is outside the triangle.



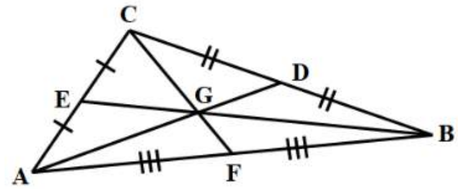

**Euler Line**: Interestingly, the centroid, orthocenter and circumcenter of a triangle are collinear (i.e., lie on the same line, which is called the Euler Line).

### Incenter (Angle Bisectors)



The incenter is the intersection of the angle bisectors of the three angles of the triangle. An angle bisector cuts an angle into two congruent angles, each of which is half the measure of the original angle. The incenter is also the center of the circle inscribed in the triangle.

**Example 4.14:** Given $\Delta CAB$, $CG = 3x - 2$, $GF = x + 3$, find $x$ and $CF$.




**Centroid**

- The centroid is the intersection of the three medians of a triangle.
- A median is a line segment drawn from a vertex to the midpoint of the side of the triangle that is opposite the vertex.
- The centroid is located 2/3 of the way from a vertex to the opposite side.
- The medians of a triangle create 6 inner triangles of equal area.

From the diagram, we can see that Points $D, E, F$ are midpoints of the sides of $\Delta ABC$. So, $\overline{AD}, \overline{BE}, \overline{CF}$ are medians of $\Delta ABC$.

Point $G$ is the centroid of $\Delta ABC$ because it is the intersection of the three medians of the triangle. Therefore,

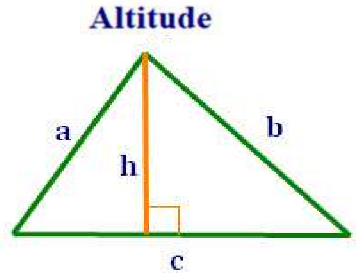
$$CG = 2(GF)$$
$$3x - 2 = 2(x + 3)$$
$$3x - 2 = 2x + 6$$
$$\boldsymbol{x = 8}$$

Then, $\mathbf{CF} = CG + GF = (3x - 2) + (x + 3) = 4x + 1$

$$= 4(8) + 1 = \mathbf{33}$$

# Geometry
## Length of Altitude, Median and Angle Bisector

### Altitude (Height)



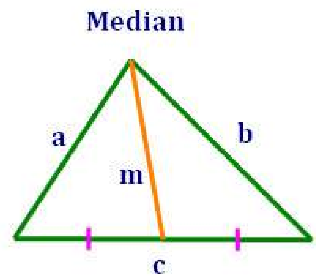

The formula for the length of a height of a triangle is derived from Heron's formula for the area of a triangle:

$$h = \frac{2\sqrt{s\,(s-a)\,(s-b)\,(s-c)}}{c}$$

where, $s = \frac{1}{2}(a+b+c)$, and

$a, b, c$ are the lengths of the sides of the triangle.

### Median

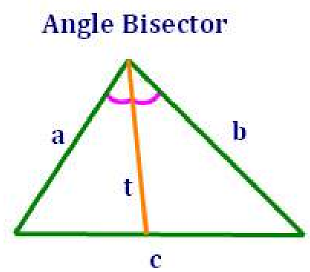



The formula for the length of a median of a triangle is:

$$m = \frac{1}{2}\sqrt{2a^2 + 2b^2 - c^2}$$

where, $a, b, c$ are the lengths of the sides of the triangle.

### Angle Bisector

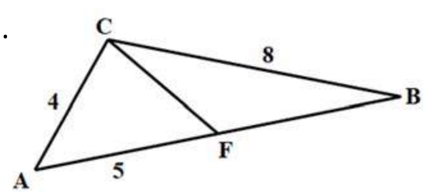



The formula for the length of an angle bisector of a triangle is:

$$t = \sqrt{ab\left(1 - \frac{c^2}{(a+b)^2}\right)}$$

where, $a, b, c$ are the lengths of the sides of the triangle.

**Example 4.15:** Find the length of $\overline{CF}$, if $\overline{CF}$ is a median of $\Delta ABC$.




Point F bisects $\overline{AB}$, so $AB = 2 \cdot 5 = 10$. From the formula above, we have:

$$CF = \frac{1}{2}\sqrt{2 \cdot AC^2 + 2 \cdot CB^2 - AB^2}$$

$$= \frac{1}{2}\sqrt{2 \cdot 4^2 + 2 \cdot 8^2 - 10^2} = \frac{1}{2}\sqrt{60} = \frac{1}{2} \cdot 2\sqrt{15} = \sqrt{15}$$

# Geometry
# Polygons - Basics

## Basic Definitions

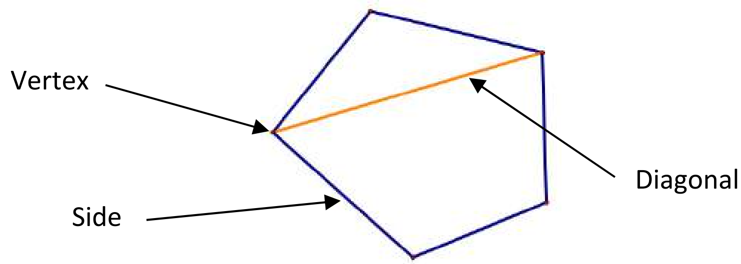
**Polygon:** a closed path of three or more line segments, where:

- no two sides with a common endpoint are collinear, and
- each segment is connected at its endpoints to exactly two other segments.

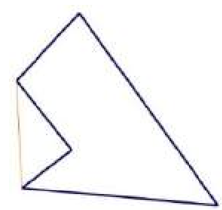
**Side:** a segment that is connected to other segments (which are also sides) to form a polygon.

**Vertex:** a point at the intersection of two sides of the polygon. (plural form: **vertices**)

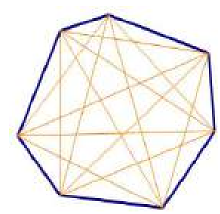
**Diagonal:** a segment, from one vertex to another, which is not a side.




**Concave:** A polygon in which it is possible to draw a diagonal "outside" the polygon. (Notice the orange diagonal drawn outside the polygon at right.) Concave polygons actually look like they have a "cave" in them.



**Convex:** A polygon in which it is not possible to draw a diagonal "outside" the polygon. (Notice that all of the orange diagonals are inside the polygon at right.) Convex polygons appear more "rounded" and do not contain "caves."



## Names of Some Common Polygons

| Number of Sides | Name of Polygon | Number of Sides | Name of Polygon |
|---|---|---|---|
| 3 | Triangle | 9 | Nonagon |
| 4 | Quadrilateral | 10 | Decagon |
| 5 | Pentagon | 11 | Undecagon |
| 6 | Hexagon | 12 | Dodecagon |
| 7 | Heptagon | 20 | Icosagon |
| 8 | Octagon | $n$ | $n$-gon |

Names of polygons are generally formed from the Greek language; however, some hybrid forms of Latin and Greek (e.g., undecagon) have crept into common usage.

# Geometry
# Polygons – More Definitions

## Definitions

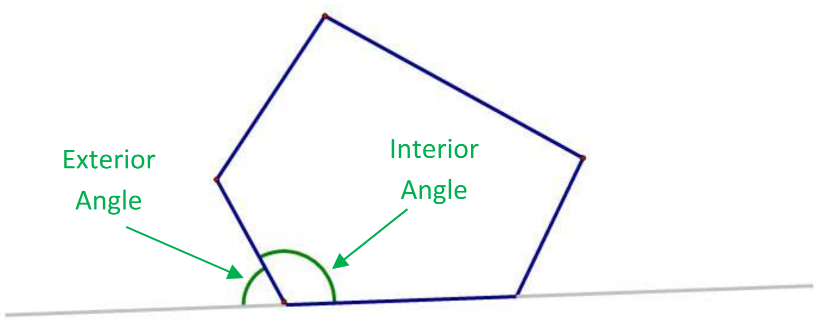
**Equilateral:** a polygon in which all of the sides are equal in length.

**Equiangular:** a polygon in which all of the angles have the same measure.

**Regular:** a polygon which is both equilateral and equiangular. That is, a ***regular polygon*** is one in which all of the sides have the same length and all of the angles have the same measure.

**Interior Angle:** An angle formed by two sides of a polygon. The angle is inside the polygon.

**Exterior Angle:** An angle formed by one side of a polygon and the line containing an adjacent side of the polygon. The angle is outside the polygon.




**"Advanced" Definitions:**

**Simple Polygon:** *a polygon whose sides do not intersect at any location other than its endpoints.* Simple polygons always divide a plane into two regions – one inside the polygon and one outside the polygon.

**Complex Polygon:** *a polygon with sides that intersect someplace other than their endpoints (i.e., not a simple polygon). Complex polygons do not always have well-defined insides and outsides.*

**Skew Polygon:** *a polygon for which not all of its vertices lie on the same plane.*

## How Many Diagonals Does a Convex Polygon Have?

Believe it or not, this is a common question with a simple solution. Consider a polygon with $\boldsymbol{n > 3}$ sides and, therefore, $\boldsymbol{n}$ vertices.

- Each of the $\boldsymbol{n}$ vertices of the polygon can be connected to $(\boldsymbol{n-3})$ other vertices with diagonals. That is, it can be connected to all other vertices except itself and the two to which it is connected by sides. So, there are $[\,\boldsymbol{n \cdot (n-3)}]$ lines to be drawn as diagonals.
- However, when we do this, we draw each diagonal twice because we draw it once from each of its two endpoints. So, the number of diagonals is actually half of the number we calculated above.
- Therefore, the number of diagonals in an $n$-sided polygon is:

$$\frac{n \cdot (n-3)}{2}$$

# Geometry
# Interior and Exterior Angles of a Polygon

## Interior Angles

The sum of the interior angles in an $n$-sided polygon is:

$$\sum = (n-2) \cdot 180°$$

If the polygon is regular, you can calculate the measure of each interior angle as:

$$\frac{(n-2) \cdot 180°}{n}$$

**Notation:** The Greek letter **"Σ"** is equivalent to the English letter **"S"** and is math short-hand for a summation (i.e., addition) of things.

| Interior Angles | | |
|---|---|---|
| **Sides** | **Sum of Interior Angles** | **Each Interior Angle** |
| 3 | $180^0$ | $60^0$ |
| 4 | $360^0$ | $90^0$ |
| 5 | $540^0$ | $108^0$ |
| 6 | $720^0$ | $120^0$ |
| 7 | $900^0$ | $129^0$ |
| 8 | $1{,}080^0$ | $135^0$ |
| 9 | $1{,}260^0$ | $140^0$ |
| 10 | $1{,}440^0$ | $144^0$ |

## Exterior Angles

No matter how many sides there are in a polygon, the sum of the exterior angles is:

$$\sum = 360^0$$

If the polygon is regular, you can calculate the measure of each exterior angle as:

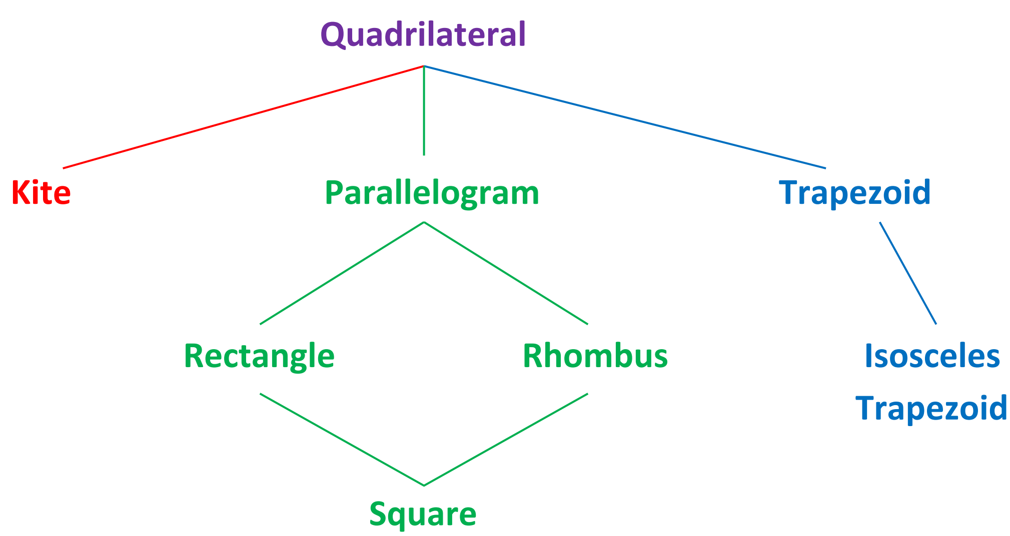
$$\frac{360^0}{n}$$

| Exterior Angles | | |
|---|---|---|
| **Sides** | **Sum of Exterior Angles** | **Each Exterior Angle** |
| 3 | $360^0$ | $120^0$ |
| 4 | $360^0$ | $90^0$ |
| 5 | $360^0$ | $72^0$ |
| 6 | $360^0$ | $60^0$ |
| 7 | $360^0$ | $51^0$ |
| 8 | $360^0$ | $45^0$ |
| 9 | $360^0$ | $40^0$ |
| 10 | $360^0$ | $36^0$ |

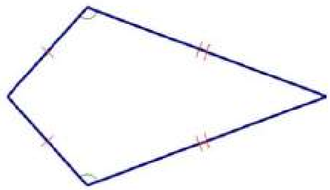
# Geometry
## Definitions of Quadrilaterals

| Name | Definition |
|---|---|
| **Quadrilateral** | A polygon with 4 sides. |
| **Kite** | A quadrilateral with two consecutive pairs of congruent sides, but with opposite sides not congruent. |
| **Trapezoid** | A quadrilateral with exactly one pair of parallel sides. |
| **Isosceles Trapezoid** | A trapezoid with congruent legs. |
| **Parallelogram** | A quadrilateral with both pairs of opposite sides parallel. |
| **Rectangle** | A parallelogram with all angles congruent (i.e., right angles). |
| **Rhombus** | A parallelogram with all sides congruent. |
| **Square** | A quadrilateral with all sides congruent and all angles congruent. |

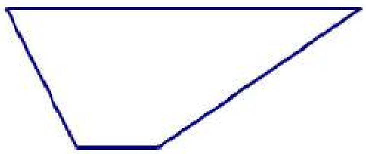
### Quadrilateral Tree:



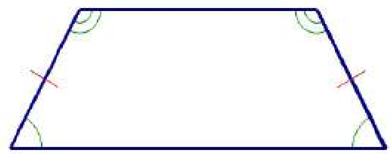
# Geometry
## Figures of Quadrilaterals



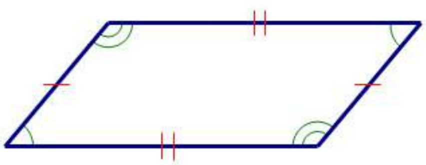
### Kite
- 2 consecutive pairs of congruent sides
- 1 pair of congruent opposite angles
- Diagonals perpendicular



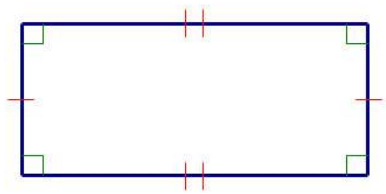
### Trapezoid
- 1 pair of parallel sides (called "bases")
- Angles on the same "side" of the bases are supplementary



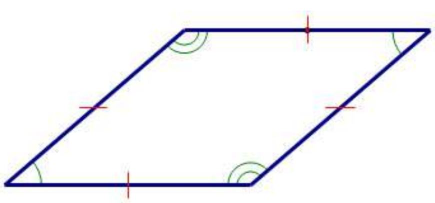
### Isosceles Trapezoid
- 1 pair of parallel sides
- Congruent legs
- 2 pair of congruent base angles
- Diagonals congruent



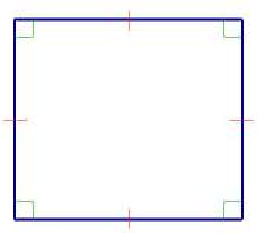
### Parallelogram
- Both pairs of opposite sides parallel
- Both pairs of opposite sides congruent
- Both pairs of opposite angles congruent
- Consecutive angles supplementary
- Diagonals bisect each other



### Rectangle
- Parallelogram with all angles congruent (i.e., right angles)
- Diagonals congruent

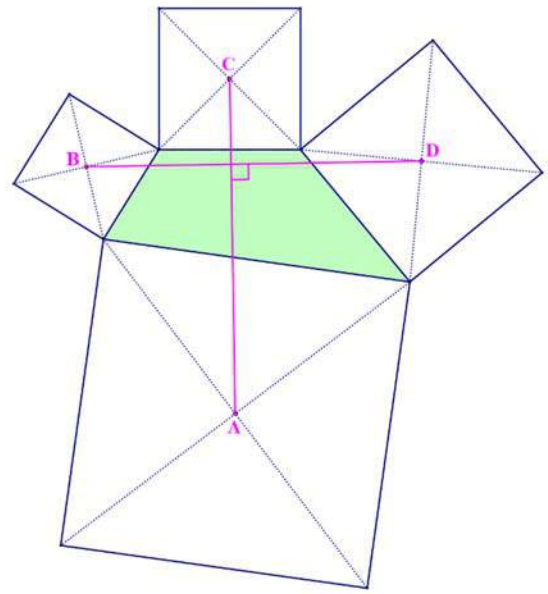


### Rhombus
- Parallelogram with all sides congruent
- Diagonals perpendicular
- Each diagonal bisects a pair of opposite angles



### Square
- Both a Rhombus and a Rectangle
- All angles congruent (i.e., right angles)
- All sides congruent

## Amazing Property of Quadrilaterals



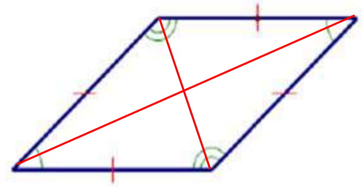

Steps:

1. Draw any quadrilateral (green in diagram).
2. Construct squares along each side of the quadrilateral.
3. Connect the midpoints of opposite squares with segments.

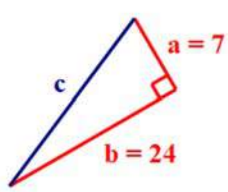
Result: The two segments connecting the midpoints of the squares are congruent and perpendicular.

Diagram: In the diagram to the right, segments $\overline{AC}$ and $\overline{BD}$ are both congruent and perpendicular. It may not look like the segments are the same length, but they are. On a printed page, each segment is 4.4 cm long. Amazing!

**Example 6.1:** Find the side length of the rhombus if its diagonals measure 14 inches and 48 inches.



Lets take one triangle from the inside of the rhombus shown to the right. See below.




We know that the diagonals are perpendicular, so we have a right triangle. The two red sides of the triangle are half of the length of the diagonals from which they come.

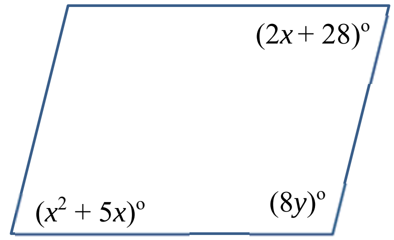
We have sides, then, of $a = 14 \div 2 = 7$ and $b = 48 \div 2 = 24$.

It remains for us to calculate the value of $c$. Let's use the Pythagorean Theorem:

$$c^2 = a^2 + b^2$$
$$c^2 = 7^2 + 24^2$$
$$c^2 = 49 + 576 = 625$$

$\boldsymbol{c = 25}$ **inches** (remember to use units in the answer because they are in the statement of the problem).

**Example 6.2:** Find the values of $x$ and $y$ in the parallelogram below.




In a parallelogram, opposite angles are congruent and consecutive angles are supplementary. This gives us:

$x^2 + 5x = 2x + 28$
$x^2 + 3x - 28 = 0$
$(x + 7)(x - 4) = 0$
$x = -7$ or $4$

If $x = -7$, then the angles involved are equal to $(2x + 28)° = (2(-7) + 28)° = 14°$.
If $x = 4$, then the angles involved are equal to $(2x + 28)° = (2(4) + 28)° = 36°$.

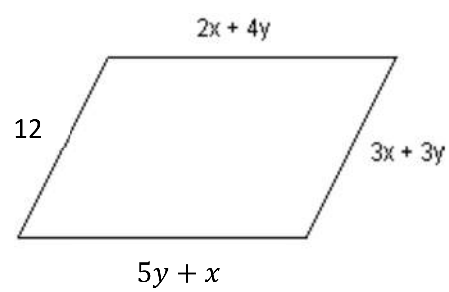
Both of these angle values are possible, so we have two cases:

If $x = -7$, then the angles involved are $14°$. Since consecutive angles are supplementary, $14 + 8y = 180$.
$8y = 166$
$y = 20.75$ and the solution is: $\boldsymbol{x = -7, y = 20.75}$

If $x = 4$, then the angles involved are $36°$. Since consecutive angles are supplementary, $36 + 8y = 180$.
$8y = 144$
$y = 18$ and the solution is: $\boldsymbol{x = 4, y = 18}$

Both solutions are valid because they both result in positive angle values.

**Example 6.3:** Find the values of $x$ and $y$ so that the figure shown is a parallelogram.




In order for the figure to be a parallelogram, opposite sides must be congruent. So,

Top and bottom sides:
$2x + 4y = 5y + x$
$x = y$

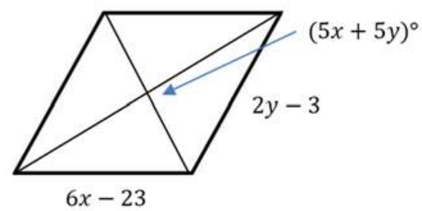
Left and right sides:
$3x + 3y = 12$
$x + y = 4$
$y = 4 - x$

Combine above results:
$x = 4 - x$
$2x = 4$
$x = 2$

$x = y$
$2 = y$

Solution: $\boldsymbol{x = 2}$, $\boldsymbol{y = 2}$

**Example 6.4:** Find the values of $x$ and $y$ from the rhombus below.




In a rhombus, the diagonals intersect at right angles, so:

$5x + 5y = 90$
$x + y = 18$
$y = 18 - x$

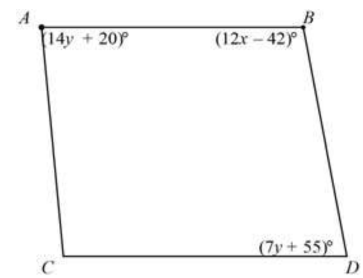
In a rhombus, the sides have the same length, so:

$6x - 23 = 2y - 3$
$6x - 20 = 2y$
$3x - 10 = y$

Combining the two equations:

$3x - 10 = 18 - x$ → $y = 18 - x$
$4x = 28$ → $y = 18 - 7$
$\mathbf{x = 7}$ → $\mathbf{y = 11}$

**Example 6.5:** Find the values of $x$ and $y$ if $\overline{AD} \cong \overline{BC}$ and ABCD is a parallelogram.

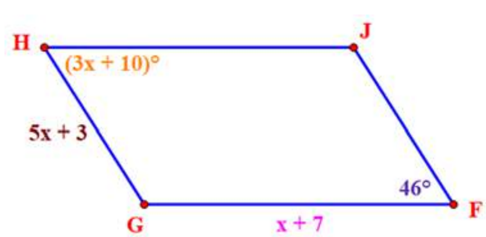



$\overline{AD}$ and $\overline{BC}$ are diagonals. Since the diagonals are congruent and ABCD is a parallelogram, we conclude that ABCD is a rectangle. Therefore, all four interior angles measure 90°. So,

| | | |
|---|---|---|
| $14y + 20 = 90$ | $12x - 42 = 90$ | $7y + 55 = 90$ |
| $14y = 70$ | $12x = 132$ | $7y = 35$ |
| $\mathbf{y = 5}$ | $\mathbf{x = 11}$ | $y = 5$ |

Note that the first and third column result in the same value for $y$. If this were not the case, we would say this problem is overdetermined, and there would be no solution for $y$.

**Example 6.6:** What is the measure of $\overline{HJ}$ in the parallelogram below.




First, opposite angles in a parallelogram have equal measures, so we can find $x$ as follows:

$3x + 10 = 46$
$3x = 36$
$x = 12$

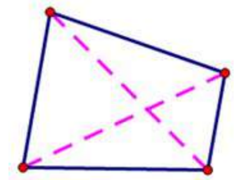
Then, opposite sides have the same length, so

$\mathbf{HJ} = \text{FG} = x + 7 = 12 + 7 = \mathbf{19}$

**Example 6.7:** If a quadrilateral has congruent diagonals, is it a rectangle: never, sometimes, or always?

For a problem like this, it is a good idea to draw the required shape, but to put as little structure in the shape as allowed by the question.

**Sometimes.** Rectangles have congruent diagonals, but it is possible to construct a quadrilateral with congruent diagonals that is not a rectangle. See the figure to the right, which has congruent diagonals.



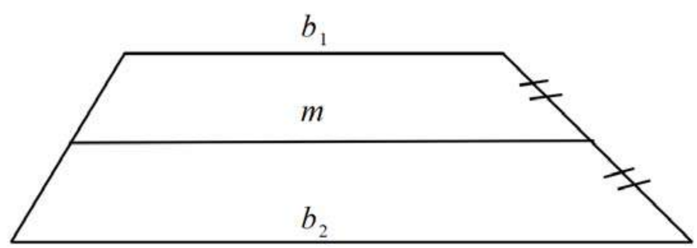
**Example 6.8:** If a quadrilateral is a rhombus, then it is a parallelogram: never, sometimes, or always?

**Always.** This can be seen in the quadrilateral tree at the beginning of this chapter. A rhombus is defined to be a parallelogram with four congruent sides.

**Example 6.9:** A triangle can be a kite: never, sometimes, or always?

**Never.** Triangles have three sides, but kites have four congruent sides.

**Example 6.10:** Given a trapezoid with bases of length $2x^2 - 14$ cm and $8x + 4$ cm, and a midline of length $m = 5x + 15$ cm. find the length of the midline.




$m$ is the average (mean) of $b_1$ and $b_2$. So,

$$m = \frac{b_1 + b_2}{2}$$

$$5x + 15 = \frac{(2x^2 - 14) + (8x + 4)}{2}$$

$$10x + 30 = (2x^2 - 14) + (8x + 4)$$

Next, collect terms, all on one side of the = sign.

$$0 = 2x^2 - 2x - 40$$
$$0 = (x^2 - x - 20)$$
$$0 = (x - 5)(x + 4)$$

$$x = 5, -4$$

$x$ cannot be $-4$ because that would make $m$ negative ( $m = 5(-4) + 15 = -5$ ), and negative lengths are not allowed.

Therefore, $x = 5$, so

$m = 5x + 15 = 5(5) + 15 =$ **40 cm**

# Geometry
## Characteristics of Parallelograms

| Characteristic | Square | Rhombus | Rectangle | Parallelogram |
|---|---|---|---|---|
| 2 pair of parallel sides | ✓ | ✓ | ✓ | ✓ |
| Opposite sides are congruent | ✓ | ✓ | ✓ | ✓ |
| Opposite angles are congruent | ✓ | ✓ | ✓ | ✓ |
| Consecutive angles are supplementary | ✓ | ✓ | ✓ | ✓ |
| Diagonals bisect each other | ✓ | ✓ | ✓ | ✓ |
| All 4 angles are congruent (i.e., right angles) | ✓ | | ✓ | |
| Diagonals are congruent | ✓ | | ✓ | |
| All 4 sides are congruent | ✓ | ✓ | | |
| Diagonals are perpendicular | ✓ | ✓ | | |
| Each diagonal bisects a pair of opposite angles | ✓ | ✓ | | |

Notes: Red ✓-marks are conditions sufficient to prove the quadrilateral is of the type specified.
Green ✓-marks are conditions sufficient to prove the quadrilateral is of the type specified if the quadrilateral is a parallelogram.

# Geometry
# Parallelogram Proofs

## Proving a Quadrilateral is a Parallelogram

To prove a quadrilateral is a parallelogram, prove any of the following conditions:

1. Both pairs of opposite sides are parallel. (note: this is the definition of a parallelogram)
2. Both pairs of opposite sides are congruent.
3. Both pairs of opposite angles are congruent.
4. An interior angle is supplementary to both of its consecutive angles.
5. Its diagonals bisect each other.
6. A pair of opposite sides is both parallel and congruent.

## Proving a Quadrilateral is a Rectangle

To prove a quadrilateral is a rectangle, prove any of the following conditions:

1. All 4 angles are congruent.
2. It is a parallelogram and its diagonals are congruent.

## Proving a Quadrilateral is a Rhombus

To prove a quadrilateral is a rhombus, prove any of the following conditions:

1. All 4 sides are congruent.
2. It is a parallelogram and Its diagonals are perpendicular.
3. It is a parallelogram and each diagonal bisects a pair of opposite angles.

## Proving a Quadrilateral is a Square

To prove a quadrilateral is a square, prove:

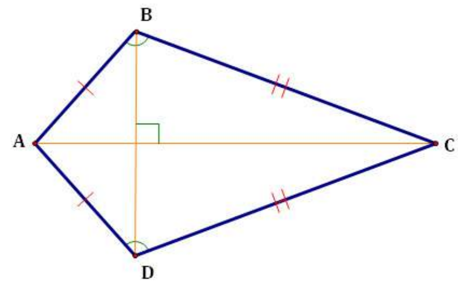
1. It is both a Rhombus and a Rectangle.

# Geometry
# Kites and Trapezoids

## Facts about a Kite

To prove a quadrilateral is a kite, prove:

- It has two pair of congruent sides.
- Opposite sides are not congruent.



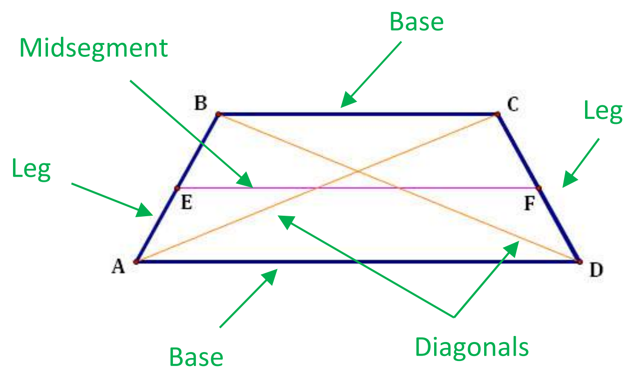

Also, if a quadrilateral is a kite, then:

- Its diagonals are perpendicular
- It has exactly one pair of congruent opposite angles.

## Parts of a Trapezoid

Trapezoid ABCD has the following parts:

- $\overline{AD}$ and $\overline{BC}$ are **bases**.
- $\overline{AB}$ and $\overline{CD}$ are **legs**.
- $\overline{EF}$ is the **midsegment**.
- $\overline{AC}$ and $\overline{BD}$ are **diagonals**.
- Angles $A$ and $D$ form a pair of **base angles**.
- Angles $B$ and $C$ form a pair of **base angles**.




## Trapezoid Midsegment Theorem

The **midsegment** of a trapezoid is parallel to each of its bases and: $EF = \frac{1}{2}(AD + BC)$.

## Proving a Quadrilateral is an Isosceles Trapezoid

To prove a quadrilateral is an isosceles trapezoid, prove any of the following conditions:

1. It is a trapezoid and has a pair of congruent legs. (definition of isosceles trapezoid)
2. It is a trapezoid and has a pair of congruent base angles.
3. It is a trapezoid and its diagonals are congruent.

# Geometry
# Introduction to Transformation

A **Transformation** is a mapping of the pre-image of a geometric figure onto an image that retains key characteristics of the pre-image.

## Definitions

The **Pre-Image** is the geometric figure before it has been transformed.

The **Image** is the geometric figure after it has been transformed.

A **mapping** is an association between objects. Transformations are types of mappings. In the figures below, we say *ABCD is mapped onto A'B'C'D'*, or $ABCD \longrightarrow A'B'C'D'$. The order of the vertices is critical to a properly named mapping.

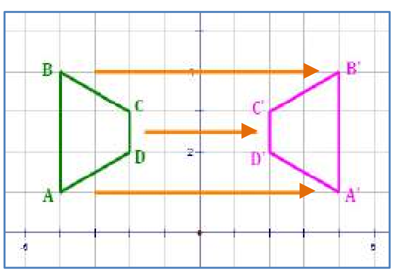
An **Isometry** is a one-to-one mapping that preserves lengths. Transformations that are isometries (i.e., preserve length) are called **rigid transformations**.

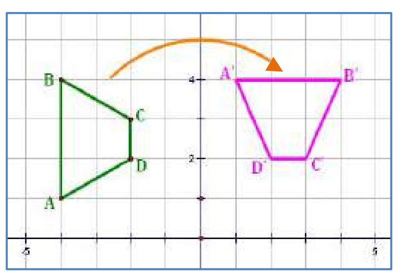
## Isometric Transformations

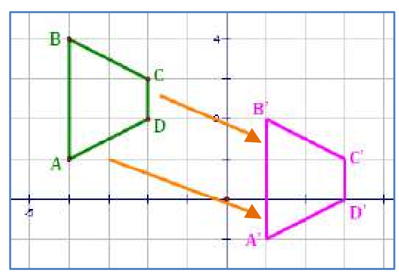
**Reflection** is flipping a figure across a line called a "mirror." The figure retains its size and shape, but appears "backwards" after the reflection.

**Rotation** is turning a figure around a point. Rotated figures retain their size and shape, but not their orientation.

**Translation** is sliding a figure in the plane so that it changes location but retains its shape, size and orientation.




## Table of Characteristics of Isometric Transformations

| Transformation | Reflection | Rotation | Translation |
|---|---|---|---|
| Isometry (Retains Lengths)? | Yes | Yes | Yes |
| Retains Angles? | Yes | Yes | Yes |
| Retains Orientation to Axes? | No | No | Yes |

# Geometry
# Introduction to Transformation (cont'd)

## Transformation of a Point

A point is the easiest object to transform. Simply reflect, rotate or translate it following the rules for the transformation selected. By transforming key points first, any transformation becomes much easier.

## Transformation of a Geometric Figure

To transform any geometric figure, it is only necessary to transform the items that define the figure, and then re-form it. For example:

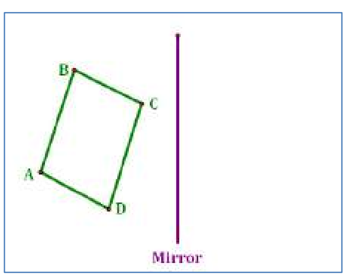
- To transform a **line segment**, transform its two endpoints, and then connect the resulting images with a line segment.
- To transform a **ray**, transform the initial point and any other point on the ray, and then construct a ray using the resulting images.
- To transform a **line**, transform any two points on the line, and then fit a line through the resulting images.
- To transform a **polygon**, transform each of its vertices, and then connect the resulting images with line segments.
- To transform a **circle**, transform its center and, if necessary, its radius. From the resulting images, construct the image circle.
- To transform **other conic sections (parabolas, ellipses and hyperbolas)**, transform the foci, vertices and/or directrix. From the resulting images, construct the image conic section.

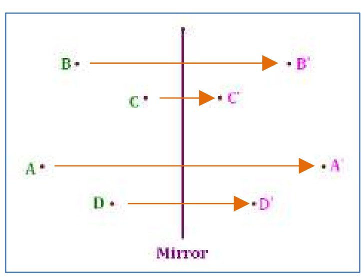
**Example 7.1:** Reflect Quadrilateral ABCD over the mirror shown.

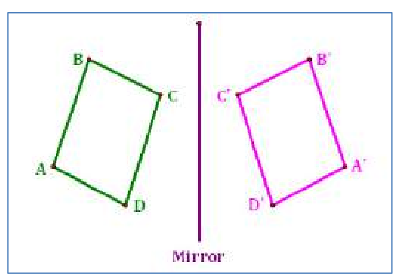
To reflect a point over a mirror:

- Connect the point to the mirror with a segment that is perpendicular to the mirror.
- Draw the segment again, in the same direction, beyond the mirror.
- Place the image point at the end of the second segment.

See the diagrams below.



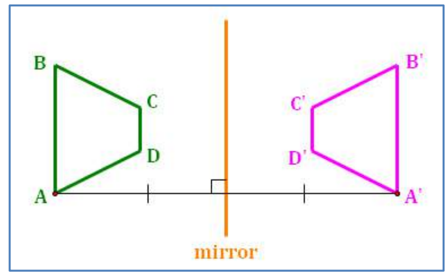



# Geometry
# Reflection

## Definitions

**Reflection** is flipping a figure across a mirror.

The **Line of Reflection** is the mirror through which the reflection takes place.




Note that:

- The line segment connecting corresponding points in the image and pre-image is bisected by the mirror.
- The line segment connecting corresponding points in the image and pre-image is perpendicular to the mirror.

## Reflection through an Axis or the Line $y = x$

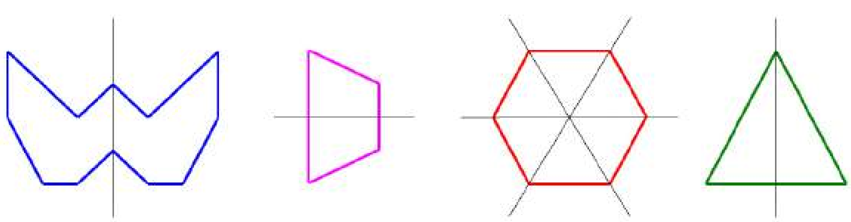
Reflection of the point (a, b) through the $x$- or $y$-*axis* or the line $y = x$ gives the following results:

| Pre-Image Point | Mirror Line | Image Point |
|---|---|---|
| (a, b) | x-axis | (a, -b) |
| (a, b) | y-axis | (-a, b) |
| (a, b) | the line: $y = x$ | (b, a) |

**If you forget the above table**, start with a point such as $(3, 2)$ on a set of coordinate axes. Reflect the point through the selected line and see which set of "a, b" coordinates works.

## Line of Symmetry

A **Line of Symmetry** is any line through which a figure can be mapped onto itself. The thin black lines in the following figures show their axes of symmetry:

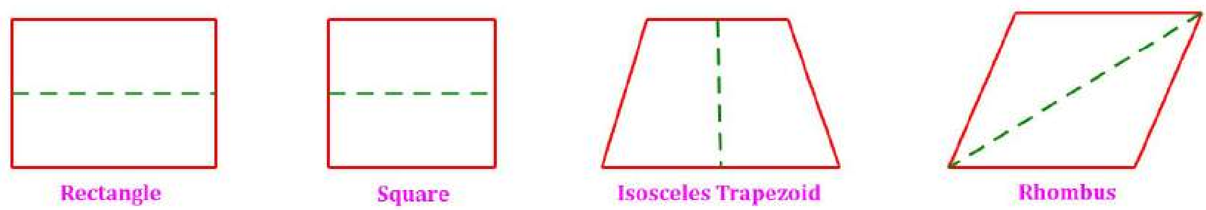


**Example 7.2:** Which of the following quadrilaterals has line symmetry? Square, rectangle, isosceles trapezoid, rhombus?

A figure has line symmetry if it is possible to draw a line so that the image looks the same when reflected over the line.

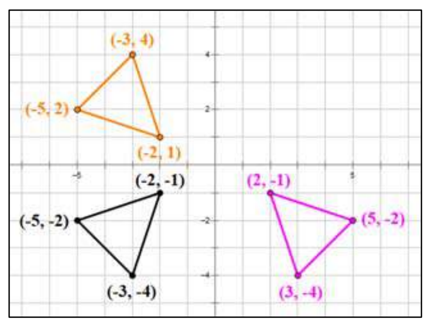
In drawing the figures to help answer this problem, it is important to draw them in their most general form. For example, when considering a rhombus, we would not want to draw a square (even though a square is a type of rhombus) to analyze because a rhombus is not required to have the right angles contained in a square. Doing so could lead us to the wrong conclusions.

In the figures below, lines of symmetry are drawn as dashed green segments.




**Answer: all of the quadrilaterals mentioned have line symmetry.**

**Example 7.3:** Reflect $\Delta ABC$ over the $x$-axis and over the $y$-axis. What are the $x$ and $y$ coordinates after reflection?




Starting coordinates (black in the diagram):

$(-2,-1), (-3,-4), (-5,-2)$

After reflection over the $x$-axis (orange in the diagram):

$x$-values are unchanged. $y$-values change sign.

$(-2,1), (-3,4), (-5,2)$

After reflection over the $y$-axis (magenta in the diagram):

$x$-values change sign. $y$-values are unchanged.

$(2,-1), (3,-4), (5,-2)$

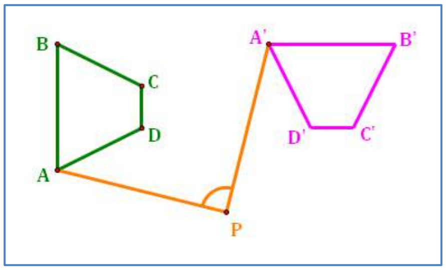
Notice the symmetry in the diagram. Symmetry is often noticed because it looks “pretty.”

# Geometry
# Rotation

## Definitions

**Rotation** is turning a figure by an angle about a fixed point.

The **Center of Rotation** is the point about which the figure is rotated. Point ***P***, at right, is the center of rotation.




The **Angle of Rotation** determines the extent of the rotation. The angle is formed by the rays that connect the center of rotation to the pre-image and the image of the rotation. Angle ***P***, at right, is the angle of rotation. Though shown only for Point $A$, the angle is the same for any of the figure's 4 vertices.

**Note:** In performing rotations, it is important to indicate the direction of the rotation – clockwise or counterclockwise.

## Rotation about the Origin

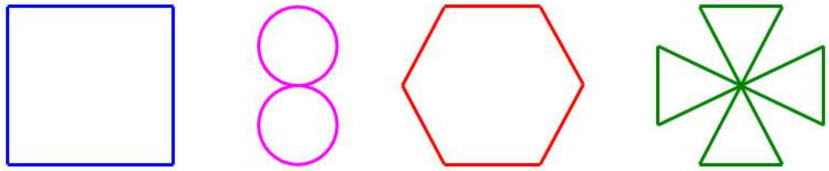
Rotation of the point (a, b) about the origin (0, 0) gives the following results:

| Pre-Image Point | Clockwise Rotation | Counterclockwise Rotation | Image Point |
|---|---|---|---|
| (a, b) | $90^0$ | $270^0$ | (b, -a) |
| (a, b) | $180^0$ | $180^0$ | (-a, -b) |
| (a, b) | $270^0$ | $90^0$ | (-b, a) |
| (a, b) | $360^0$ | $360^0$ | (a, b) |

**If you forget the above table**, start with the point (3, 2) on a set of coordinate axes. Rotate the point by the selected angle and see which set of "a, b" coordinates works.

## Rotational Symmetry

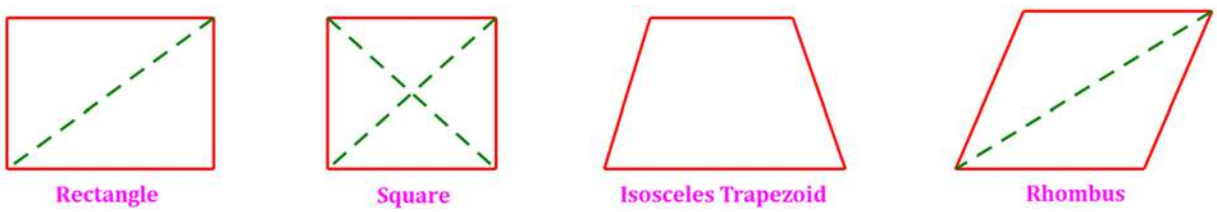
A figure in a plane has **Rotational Symmetry** if it can be mapped onto itself by a rotation of $180^0$ or less. Any regular polygon has rotational symmetry, as does a circle. Here are some examples of figures with rotational symmetry:



**Example 7.4:** Which of the following quadrilaterals has rotational symmetry? Square, rectangle, isosceles trapezoid, rhombus?

A figure has rotational symmetry if it is possible to rotate the image and get a result that looks the same. The order of a rotational symmetry is the number of positions the shape can take (within a 360˚ rotation) and look the same.

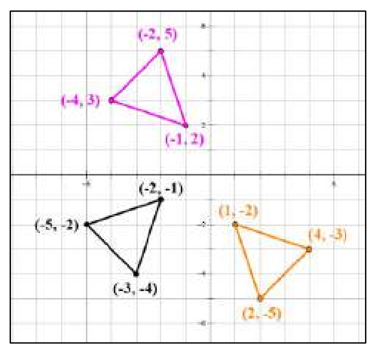
In the figures below, lines of symmetry are drawn as dashed green segments.




**Answer:** A rectangle has rotational symmetry of order 2 (0˚ and 180˚ rotations).
A square has rotational symmetry of order 4 (0˚, 90˚, 180˚ and 270˚ rotations).
An isosceles trapezoid does not have rotational symmetry.
A rhombus has rotational symmetry of order 2 (0˚ and 180˚ rotations).

**Example 7.5:** Rotate $\Delta ABC$ counterclockwise by 90° about the origin and, separately, clockwise by 90° about the origin. What are the $x$ and $y$ coordinates after rotation?

Rotating "about" a point means that the point is the center of rotation.




Starting coordinates (black in the diagram):

$(-2,-1), (-3,-4), (-5,-2)$

A rotation of 90° counterclockwise about the origin produces a mapping of $(a,b) \rightarrow (-b,a)$. That is, the $x$ and $y$ coordinates switch and the new $x$-value changes its sign.

After rotation about the origin, (orange in the diagram):

$(1,-2), (4,-3), (2,-5)$

A rotation of 90° clockwise about the origin produces a mapping of $(a,b) \rightarrow (b,-a)$. That is, the $x$ and $y$ coordinates switch and the new $y$-value changes its sign.

After reflection over the $y$-axis (magenta in the diagram):

$(-1,2), (-4,3), (-2,5)$

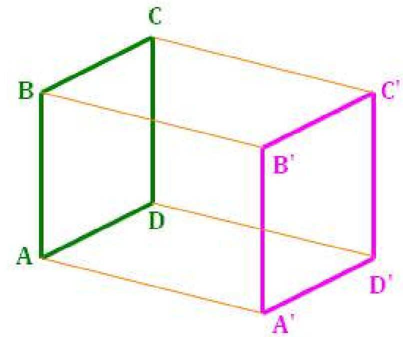
Notice that the two rotations produce coordinates that are a 180° rotation from each other. That is, rotating $(1,-2), (4,-3), (2,-5)$ by 180° gives $(-1,2), (-4,3), (-2,5)$, and rotating $(-1,2), (-4,3), (-2,5)$ by 180° gives $(1,-2), (4,-3), (2,-5)$. That's because a point rotated 90° counterclockwise the same point rotated 90° clockwise are 180° apart.

# Geometry
# Translation

## Definitions

**Translation** is sliding a figure in the plane. Each point in the figure is moved the same distance in the same direction. The result is an image that looks the same as the pre-image in every way, except it has been moved to a different location in the plane.

Each of the four orange line segments in the figure at right has the same length and direction.

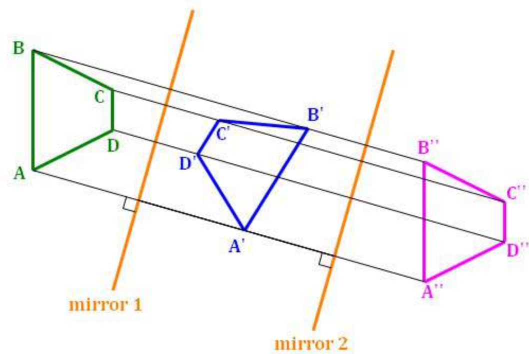



## When Two Reflections = One Translation

If two mirrors are parallel, then reflection through one of them, followed by a reflection through the second is a translation.

In the figure at right, the black lines show the paths of the two reflections; this is also the path of the resulting translation. Note the following:

- The distance of the resulting translation (e.g., from $A$ to $A''$) is double the distance between the mirrors.
- The black lines of movement are perpendicular to both mirrors.




## Defining Translations in the Coordinate Plane (Using Vectors)

A translation moves each point by the same distance in the same direction. In the coordinate plane, this is equivalent to moving each point the same amount in the *x-direction* and the same amount in the *y-direction*. This combination of *x-* and *y-direction* movement is described by a mathematical concept called a **vector**.

In the above figure, translation from $A$ to $A''$ moves $10$ in the *x-direction* and the $-3$ in the *y-direction*. In vector notation, this is: $\overrightarrow{AA''} = \langle 10, -3 \rangle$. Notice the "half-ray" symbol over the two points and the funny-looking brackets around the movement values.

So, the translation resulting from the two reflections in the above figure moves each point of the pre-image by the vector $\overrightarrow{AA''}$. Every translation can be defined by the vector representing its movement in the coordinate plane.

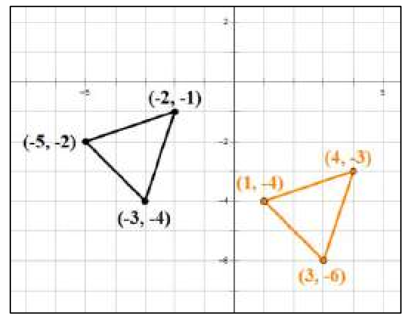
## Translation Coordinate Form

Translations are often shown as coordinates with an enclosed mapping, so a translations of $(x, y) \rightarrow (x - 2, y + 5)$ means decrease the $x$-values of translated points by 2 and increase the $y$-values of translated points by 5.

**Example 7.6:** Translate the triangle shown in the diagram according to the mapping: $(x, y) \rightarrow (x + 6, y - 2)$.

Starting coordinates (black in the diagram):

$(-2, -1), (-3, -4), (-5, -2)$




After translation (orange in the diagram):

$x$-values increase by 6. $y$-values decrease by 2.

$(4, -3), (3, -6), (1, -4)$

When you look at the result of a translation in a graph, it often looks like we just slid the figure from one place to another (which we did). The shape retains its shape and orientation.

**Example 7.7:** If a point $(3, 6)$ is translated so that its image is $(-1, 12)$, what is the translation coordinate form of the translation?

This question boils down to asking how far $x$ moved and how far $y$ moved, from preimage $(3, 6)$ to image $(-1, 12)$. The easiest way to answer this is to subtract the two points to obtain the movement vector, then convert that to the desired form.

| | |
|---|---|
| Image: | $(-1, 12)$ |
| Preimage: | $-(\ \ 3,\ \ 6)$ |
| Movement vector: | $\langle -4,\ \ 6 \rangle$ |

To obtain the translation coordinate form, add the movement vector $\langle -4,\ \ 6 \rangle$ to the general point $(x, y)$.

$\boldsymbol{(x, y) \rightarrow (x - 4, y + 6)}$

# Geometry
# Compositions

When multiple transformations are combined, the result is called a **Composition of the Transformations**. Two examples of this are:

- Combining two reflections through parallel mirrors to generate a translation (see the previous page).
- Combining a translation and a reflection to generate what is called a **glide reflection**. The glide part of the name refers to translation, which is a kind of gliding of a figure on the plane.

> **Note:** In a **glide reflection**, if the line of reflection is parallel to the direction of the translation, it does not matter whether the reflection or the translation is performed first.

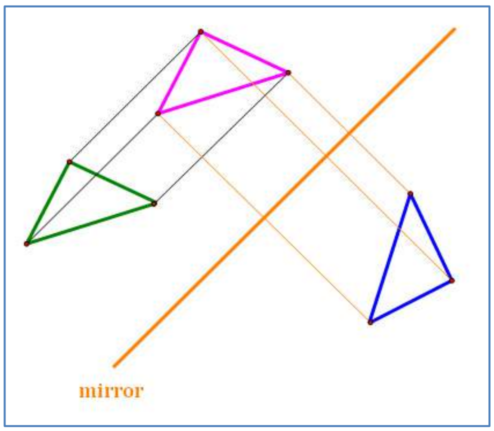



**Figure 1: Translation followed by Reflection.**

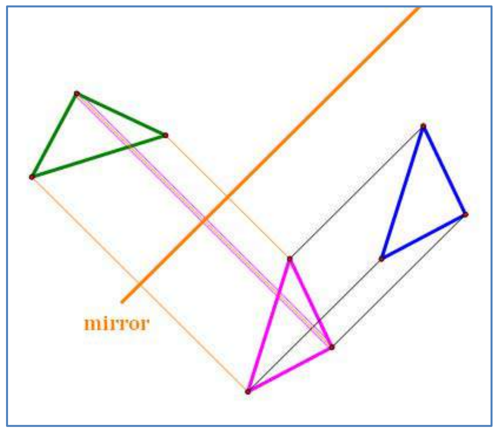



**Figure 2: Reflection followed by Translation.**

## Composition Theorem

The composition of multiple isometries is as Isometry. Put more simply, if transformations that preserve length are combined, the composition will preserve length. It is also true that if transformations that preserve angle measure are combined, the composition will preserve angle measure.

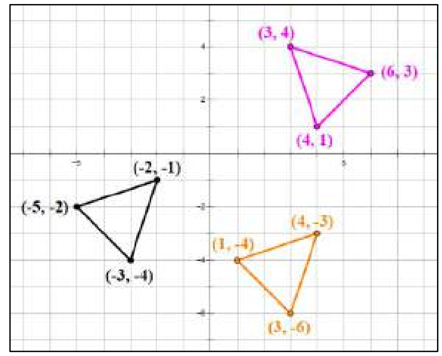
## Order of Composition

Order matters in most compositions that involve more than one class of transformation. If you apply multiple transformations of the same kind (e.g., reflection, rotation, or translation), order generally does not matter; however, applying transformations in more than one class may produce different final images if the order is switched.

**Example 7.8:** Translate the triangle shown in the diagram according to the mapping: $(x, y) \to (x+6, y-2)$, followed by a counterclockwise rotation of 90°.

Starting coordinates (black in the diagram):

$(-2,-1), (-3,-4), (-5,-2)$




After translation (orange in the diagram):

$x$-values increase by 6. $y$-values decrease by 2.

$(4,-3), (3,-6), (1,-4)$

A rotation of 90° counterclockwise about the origin produces a mapping of $(a, b) \to (-b, a)$. That is, the $x$ and $y$ coordinates switch and the new $x$ value changes its sign.

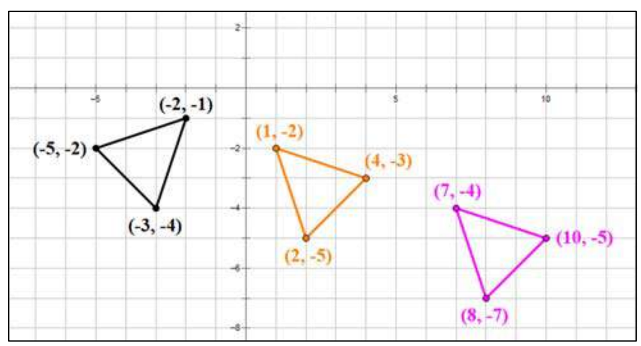
After a subsequent rotation about the origin, (magenta in the diagram):

$(3,4), (6,3), (4,1)$

**Example 7.9:** Reverse the order of the transformations in the previous example. That is, Rotate the triangle shown in the diagram counterclockwise by 90°, followed by translation according to the mapping: $(x, y) \to (x+6, y-2)$.

Starting coordinates (black in the diagram):

$(-2,-1), (-3,-4), (-5,-2)$




A rotation of 90° counterclockwise about the origin produces a mapping of $(a, b) \to (-b, a)$. That is, the $x$ and $y$ coordinates switch and the new $x$ value changes its sign.

After rotation about the origin, (orange in the diagram):

$(1,-2), (4,-3), (2,-5)$

After a subsequent translation (magenta in the diagram):

$x$-values increase by 6. $y$-values decrease by 2.

$(7,-4), (10,-5), (8,-7)$

Notice that the examples above involved performing the same transformations on the same starting triangle, but in a different order. The results are very different, illustrating that order matters in compositions that involve more than one class of transformation.

# Geometry
## Rotation About a Point Other than the Origin

Rotating an object about a point involves rotating each point of the object by the same angle about that point. For a polygon, this is accomplished by rotating each vertex and then connecting them with segments, so you mainly have to worry about the vertices, which are points. An example of the process of rotating a point about another point is described below. It is a good example of what can be accomplished with a composition of transformations.

Let's define the following points:

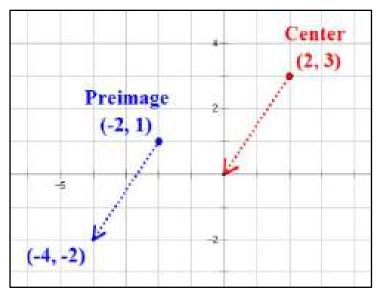
- The point about which the rotation will take place, i.e., the center of rotation: $(x_0, y_0)$.
- The initial point (before rotation), i.e., the preimage: $(x_1, y_1)$.
- The final point (after rotation), i.e. the image: $(x_2, y_2)$.

The problem is to determine $(x_2, y_2)$ if we are given $(x_0, y_0)$ and $(x_1, y_1)$. It involves 3 steps:

1. Convert the problem to one of rotating a point about the origin (a much easier problem).
2. Perform the rotation.
3. Reverse the translation in Step 1.

We consider each step separately, algebraically and geometrically, in the following example:

**Example 7.10:** Rotate a point by 90$^o$ about another point.



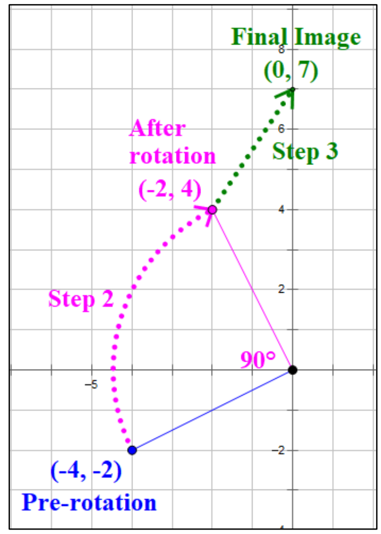

**Step 1: Convert the problem to one of rotating a point about the origin:**

First, we translate our point $(-2, 1)$ and the center of rotation $(2, 3)$ so that the center of rotation moves to $(0, 0)$. That involves subtracting (2, 3) from both the point and the center.

| General Situation | Example |
|---|---|
| Points in this step<br>• Rotation Center: $(x_0, y_0)$<br>• Initial point: $(x_1, y_1)$<br>• Image of translation | Points in this step<br>• Rotation Center: $(2, 3)$<br>• Initial point: $(-2, 1)$<br>• Image of translation |
| Translate our point by subtracting $(x_0, y_0)$ from $(x_1, y_1)$. The resulting image is:<br>$(x_1 - x_0, y_1 - y_0)$ | Translate our point by subtracting $(2, 3)$ from $(-2, 1)$. The resulting image is:<br>$(-4, -2)$ |

The next steps depend on whether we are making a clockwise or counter clockwise rotation.

## Example 7.10a: Clockwise Rotation:




### Step 2: Perform the rotation about the origin:

Rotating 90$^o$ clockwise about the origin $(0,0)$ is simply a process of switching the $x$- and $y$-values of a point and negating *the new* $y$*-term*. That is, $(x,y)$ becomes $(y,-x)$ after clockwise rotation by 90$^o$.

| General Situation | Example |
|---|---|
| Pre-rotated point (from Step 1): $(x_1-x_0, y_1-y_0)$ Point after rotation: $(y_1-y_0, -x_1+x_0)$ | Pre-rotated point (from Step 1): $(-4,-2)$ Point after rotation: $(-2,4)$ |

### Step 3: Reverse the translation performed in Step 1.

To do this, simply translate the image of rotation by the coordinates of the center of rotation (adding back what was subtracted in Step 1).

| General Situation | Example |
|---|---|
| Point after rotation (from Step 2): $(y_1-y_0, -x_1+x_0)$ Add back the point of rotation $(x_0, y_0)$ $(y_1-y_0+x_0, -x_1+x_0+y_0)$ which gives us the final image: $(\boldsymbol{x_2}, \boldsymbol{x_2})$ | Point after rotation (from Step 2): $(-2,4)$ Add back the center of rotation $(2,3)$: $(0,7)$ |

Finally, here are the formulas for $\boldsymbol{x_2}$ and $\boldsymbol{y_2}$:

**Clockwise Rotation**

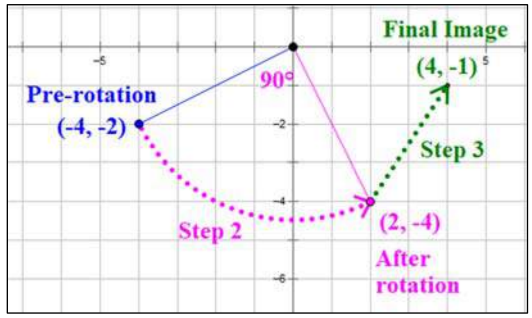
$$\boldsymbol{x_2 = \quad y_1 - y_0 + x_0}$$

$$\boldsymbol{y_2 = -x_1 + x_0 + y_0}$$

*Notice that the formulas for clockwise rotation (this page) and counter-clockwise rotation (next page) by 90$^o$ are the same except the terms in magenta are negated between the formulas.*

Interesting note: If you are asked to find the point about which a rotation occurred, you can substitute the values for the starting point $(x_1, y_1)$ and the ending point $(x_2, y_2)$ in the above equations and solve for $x_0$ and $y_0$.

### Example 7.10b: Counterclockwise Rotation:




**Step 2: Perform the rotation about the origin:**

Rotating 90º counterclockwise about the origin $(0, 0)$ is simply a process of switching the $x$- and $y$-values of a point and negating *the new $x$-term*. That is, $(x, y)$ becomes $(-y, x)$ after counterclockwise rotation by 90º.

| General Situation | Example |
|---|---|
| Pre-rotated point (from Step 1):<br>$(x_1 - x_0, y_1 - y_0)$<br>Point after rotation:<br>$(-y_1 + y_0, x_1 - x_0)$ | Pre-rotated point (from Step 1):<br>$(-4, -2)$<br>Point after rotation:<br>$(2, -4)$ |

**Step 3: Reverse the translation performed in Step 1.**

To do this, simply translate the image of rotation by the coordinates of the center of rotation (adding back what was subtracted in Step 1).

| General Situation | Example |
|---|---|
| Point after rotation (from Step 2):<br>$(-y_1 + y_0, x_1 - x_0)$<br>Add back the point of rotation $(x_0, y_0)$<br>$(-y_1 + y_0 + x_0, x_1 - x_0 + y_0)$<br>which gives us the final image: $(\boldsymbol{x_2}, \boldsymbol{x_2})$ | Point after rotation (from Step 2):<br>$(2, -4)$<br>Add back the point of rotation (2, 3):<br>$(4, -1)$ |

Finally, here are the formulas for $\boldsymbol{x_2}$ and $\boldsymbol{y_2}$:

**Counterclockwise Rotation**

$$\boldsymbol{x_2} = -\boldsymbol{y_1} + \boldsymbol{y_0} + \boldsymbol{x_0}$$

$$\boldsymbol{y_2} = \boldsymbol{x_1} - \boldsymbol{x_0} + \boldsymbol{y_0}$$

*Notice that the formulas for clockwise rotation (this page) and counter-clockwise rotation (next page) by 90º are the same except the terms in magenta are negated between the formulas.*

Interesting note: The point half-way between the clockwise and counterclockwise rotations of 90º is the center of rotation, $(x_0, y_0)$. In the example, halfway between $(0, 7)$ and $(4, -1)$ is $(2, 3)$.

# Geometry
# Ratios Involving Units

## Ratios Involving Units

When simplifying ratios containing the same units:

- Simplify the fraction.
- Notice that the units disappear. They behave just like factors; if the units exist in the numerator and denominator, the cancel and are not in the answer.

**Example 8.1:**

$$\frac{3 \text{ inches}}{12 \text{ inches}} = \mathbf{\frac{1}{4}}$$

**Note:** the unit "inches" cancels out, so the answer is $\frac{1}{4}$, not $\frac{1}{4}$ *inch*.

When simplifying ratios containing different units:

- Adjust the ratio so that the numerator and denominator have the same units.
- Simplify the fraction.
- Notice that the units disappear.

**Example 8.2:**

$$\frac{3 \text{ inches}}{2 \text{ feet}} = \frac{3 \text{ inches}}{(2 \text{ feet}) \cdot (12 \text{ inches/foot})} = \frac{3 \text{ inches}}{24 \text{ inches}} = \mathbf{\frac{1}{8}}$$

## Dealing with Units

Notice in the above example that units can be treated the same as factors; they can be used in fractions and they cancel when they divide. This fact can be used to figure out whether multiplication or division is needed in a problem. Consider the following:

**Example 8.3:** How long did it take for a car traveling at 48 miles per hour to go 32 miles?

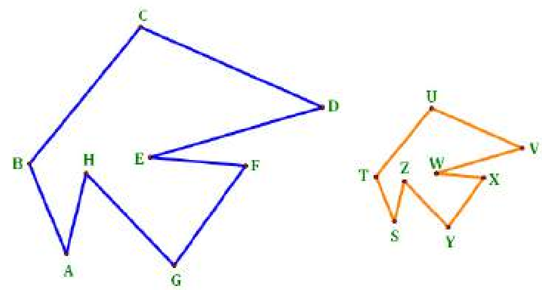
Consider the units of each item: $32 \text{ miles}$ $\quad 48 \,\frac{\text{miles}}{\text{hour}}$

- If you multiply, you get: $(32 \text{ miles}) \cdot \left(48 \frac{\text{miles}}{\text{hour}}\right) = 1{,}536 \frac{\text{miles}^2}{\text{hour}}$. This is clearly wrong!
- If you divide, you get: $(32 \text{ miles}) \div \left(48 \frac{\text{miles}}{\text{hour}}\right) = \frac{32}{48} \text{ miles} \cdot \left(\frac{\text{hour}}{\text{miles}}\right) = \frac{2}{3} \text{hour}$. Now, this looks reasonable. Notice how the "miles" unit cancel out in the final answer.

We could have solved this problem by remembering that $distance = rate \cdot time$, or $d = rt$. However, paying close attention to the units also generates the correct answer. In addition, **the "units" technique always works, no matter what the problem!**

# Geometry
# Similar Polygons

## In similar polygons,




- Corresponding angles are congruent, and
- Corresponding sides are proportional.

Both of these conditions are necessary for two polygons to be similar. Conversely, when two polygons are similar, all of the corresponding angles are congruent and all of the sides are proportional.

## Naming Similar Polygons

Similar polygons should be named such that corresponding angles are in the same location in the name, and the order of the points in the name should "follow the polygon around."

**Example 8.4:** The polygons above could be shown similar with the following names:

$$ABCDEFGHI \sim STUVWXYZ$$

It would also be acceptable to show the similarity as:

$$DEFGHIABC \sim VWXYZSTU$$

Any names that preserve the order of the points and keeps corresponding angles in corresponding locations in the names would be acceptable.

## Proportions

One common problem relating to similar polygons is to present three side lengths, where two of the sides correspond, and to ask for the length of the side corresponding to the third length.

**Example 8.5:** In the above similar polygons, if $BC = 20, EF = 12, and\ WX = 6, what\ is\ TU?$

This problem is solvable with proportions. To do so properly, it is important to relate corresponding items in the proportion:

$$\frac{BC}{TU} = \frac{EF}{WX} \quad \longrightarrow \quad \frac{20}{TU} = \frac{12}{6} \quad \longrightarrow \quad TU = 10$$

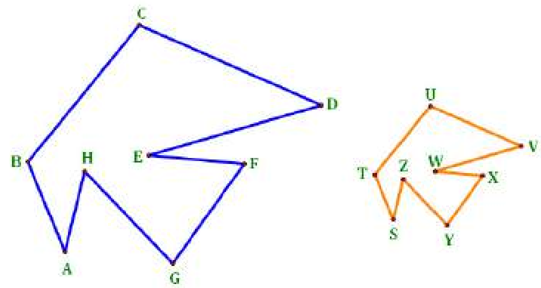
Notice that the left polygon is represented on the top of both proportions and that the left-most segments of the two polygons are in the left fraction.

# Geometry
## Scale Factors of Similar Polygons

From the similar polygons below, the following is known about the lengths of the sides:

$$\frac{AB}{ST}=\frac{BC}{TU}=\frac{CD}{UV}=\frac{DE}{VW}=\frac{EF}{WX}=\frac{FG}{XY}=\frac{GH}{YZ}=\frac{HA}{ZA}=k$$

That is, the ratios of corresponding sides in the two polygons are the same and they equal some constant $k$, called the **scale factor** of the two polygons. The value of $k$, then, is all you need to know to relate corresponding sides in the two polygons.




### Finding the Missing Length

Any time the student is asked to find the missing length in similar polygons:

- Look for two corresponding sides for which the values are known.
- Calculate the value of $k$.
- Use the value of $k$ to solve for the missing length.

$k$ is a measure of the relative size of the two polygons. Using this knowledge, it is possible to put into words an easily understandable relationship between the polygons.

- Let Polygon 1 be the one whose sides are in the numerators of the fractions.
- Let Polygon 2 be the one whose sides are in the denominators of the fractions.
- Then, it can be said that **Polygon 1 is $k$ times the size of the Polygon 2**.

**Example 8.6:** In the above similar polygons, if $BC = 20, EF = 12, and\ WX = 6, what\ is\ TU$?

Seeing that $EF$ and $WX$ relate, calculate:

$$\frac{EF}{WX}=\frac{12}{6}=2=k$$

Then solve for $TU$ based on the value of $k$:

$$\frac{BC}{TU}=k \quad \rightarrow \quad \frac{20}{TU}=2 \quad \rightarrow \quad TU=10$$

Also, since $k=2$, the length of every side in the blue polygon is double the length of its corresponding side in the orange polygon.

# Geometry
# Dilation of Polygons

A dilation is a special case of transformation involving similar polygons. It can be thought of as a transformation that creates a polygon of the same shape but a different size from the original. Key elements of a dilation are:

- Scale Factor – The scale factor of similar polygons is the constant $k$ which represents the relative sizes of the polygons.
- Center – The center is the point from which the dilation takes place.

Note that $k > 0$ and $k \neq 1$ in order to generate a second polygon. Then,

- If $k > 1$, the dilation is called an "enlargement."
- If $k < 1$, the dilation is called a "reduction."

## Dilations with Center (0, 0)

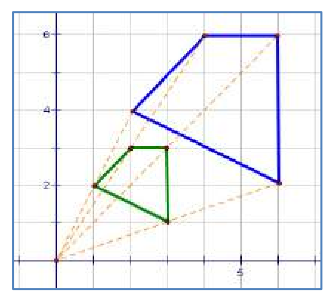
In coordinate geometry, dilations are often performed with the center being the origin $(0, 0)$. In that case, to obtain the dilation of a polygon:

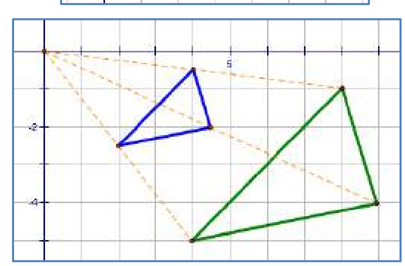
- Multiply the coordinates of each vertex by the scale factor $k$, and
- Connect the vertices of the dilation with line segments (i.e., connect the dots).

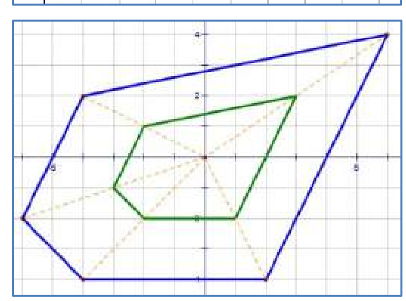
**Examples:**

In the following examples:

- The green polygon is the original.
- The blue polygon is the dilation.
- The dashed orange lines show the movement away from (enlargement) or toward (reduction) the center, which is the origin in all 3 examples.





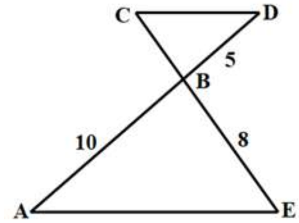


Notice that, in each example:

$$\begin{pmatrix}\text{distance from center}\\ \text{to a vertex of the}\\ \text{dilated polygon}\end{pmatrix} = k \cdot \begin{pmatrix}\text{distance from center}\\ \text{to a vertex of the}\\ \text{original polygon}\end{pmatrix}$$

This fact can be used to construct dilations when coordinate axes are not available. Alternatively, the student could draw a set of coordinate axes as an aid to performing the dilation.

**Example 8.7:** Given that $\frac{AB}{DB} = \frac{BE}{BC} = \frac{AE}{DC}$, what is the scale factor of $\Delta ABE$ to $\Delta DBC$?




There is only one set of corresponding sides to work with in this diagram, so there is only one ratio we can calculate directly from the diagram. Fortunately, we are given $\frac{AB}{DB} = \frac{BE}{BC} = \frac{AE}{DC}$, so that's all we need.

$$\frac{AB}{DB} = \frac{10}{5} = 2.$$

Therefore, the scale factor of $\Delta ABE$ to $\Delta DBC$ is **2**.

**Example 8.8:** Given a triangle with vertices at $(1, 4), (5, -2), (-2, 3)$, what are the coordinates of the vertices of the triangle after dilation $D: (x, y) \rightarrow (2x, 3y)$?

The coordinates of the preimage are $(1, 4), (5, -2), (-2, 3)$.

The dilation doubles all $x$- values and triples all $y$-values. So, the coordinates of the image are:

$$\mathbf{(2, 12), (10, -6), (-4, 9)}$$

**Example 8.9:** Given two similar cubes have a scale factor of $4: 3$, what is the ratio of their volumes?

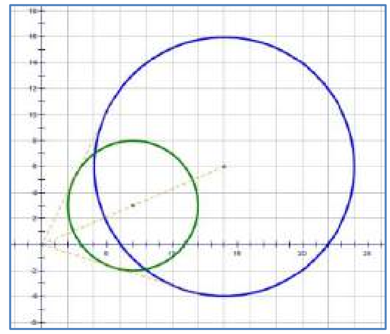
Volumes exist in three dimensions, so the ratio of their volumes would be the third power (i.e., the cube) of the scale factor. In fact, that's why the third power of a number is referred to as the "cube" of the number.

$$\textbf{Ratio} = \left(\frac{4}{3}\right)^3 = \mathbf{\frac{64}{27}}.$$

**ADVANCED**

# Geometry
# More on Dilation

## Dilations of Non-Polygons



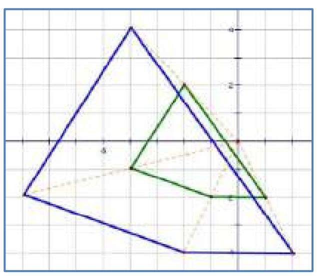
Any geometric figure can be dilated. In the dilation of the green circle at right, notice that:

- The dilation factor is 2.
- The original circle has center $(7, 3)$ and radius $= 5$.
- The dilated circle has center $(14, 6)$ and radius $= 10$.

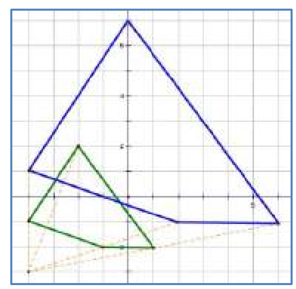
So, the center and radius are both increased by a factor of $k = 2$. This is true of any figure in a dilation with the center at the origin. All of the key elements that define the figure are increased by the scale factor $k$.

## Dilations with Center $(a, b)$

In the figures below, the green quadrilaterals are dilated to the blue ones with a scale factor of $k = 2$. Notice the following:

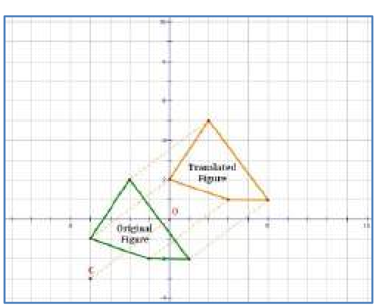


In the figure to the left, the dilation has center $(0, 0)$, whereas in the figure to the right, the dilation has center $(-4, -3)$. The size of the resulting figure is the same in both cases (because $k = 2$ in both figures), but the location is different.

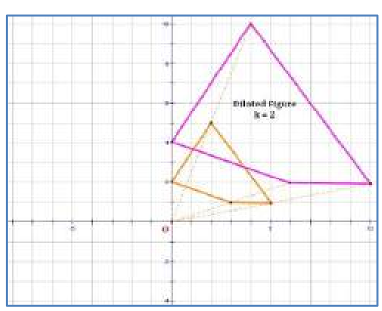


Graphically, the series of transformations that is equivalent to a dilation from a point $(a, b)$ other than the origin is shown below. Compare the final result to the figure above (right).

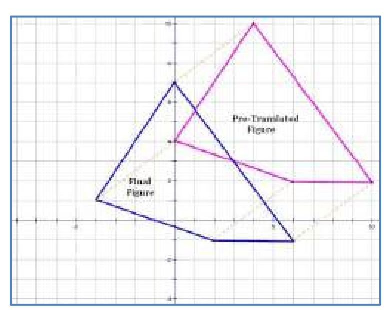
- Step 1: Translate the original figure by $(-a, -b)$ to reset the center at the origin.
- Step 2: Perform the dilation.
- Step 3: Translate the dilated figure by $(a, b)$. These steps are illustrated below.




Step 1




Step 2

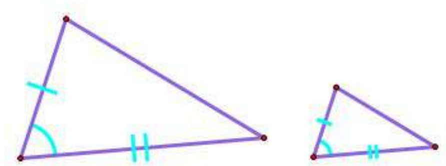



Step 3

# Geometry
# Similar Triangles

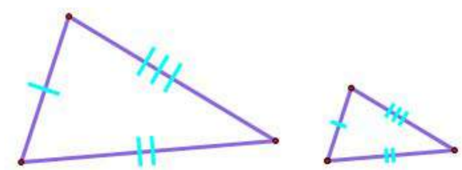
The following theorems present conditions under which triangles are similar.

## Side-Angle-Side (SAS) Similarity



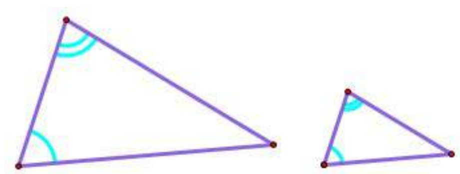
**SAS similarity** requires the proportionality of two sides and the congruence of the angle between those sides. Note that there is no such thing as SSA similarity; the congruent angle must be between the two proportional sides.

## Side-Side-Side (SSS) Similarity



**SSS similarity** requires the proportionality of all three sides. If all of the sides are proportional, then all of the angles must be congruent.

## Angle-Angle (AA) Similarity

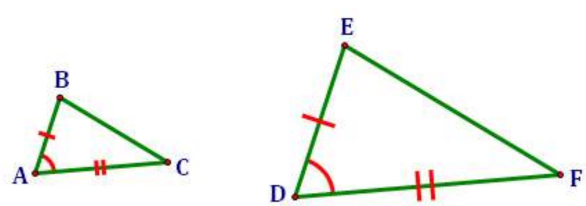


**AA similarity** requires the congruence of two angles and the side between those angles.

## Similar Triangle Parts

In similar triangles,

- Corresponding sides are proportional.
- Corresponding angles are congruent.




Establishing the proper names for similar triangles is crucial to line up corresponding vertices. In the picture above, we can say:

$$\Delta ABC \sim \Delta DEF \quad \text{or} \quad \Delta BCA \sim \Delta EFD \quad \text{or} \quad \Delta CAB \sim \Delta FDE \quad \text{or}$$
$$\Delta ACB \sim \Delta DFE \quad \text{or} \quad \Delta BAC \sim \Delta EDF \quad \text{or} \quad \Delta CBA \sim \Delta FED$$

All of these are correct because they match corresponding parts in the naming. Each of these similarities implies the following relationships between parts of the two triangles:

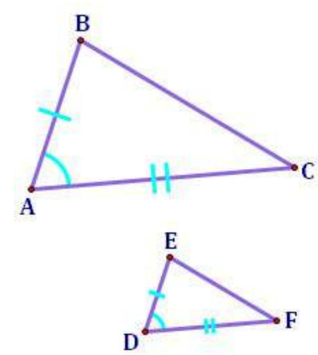
$$\angle A \cong \angle D \quad \text{and} \quad \angle B \cong \angle E \quad \text{and} \quad \angle C \cong \angle F$$

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$$

# Geometry
# Proportion Tables for Similar Triangles

## Setting Up a Table of Proportions

It is often useful to set up a table to identify the proper proportions in a similarity. Consider the figure to the right. The table might look something like this:




| Triangle | Left Side | Right Side | Bottom Side |
|---|---|---|---|
| **Top Δ** | $AB$ | $BC$ | $CA$ |
| **Bottom Δ** | $DE$ | $EF$ | $FD$ |

The purpose of a table like this is to organize the information you have about the similar triangles so that you can readily develop the proportions you need.

## Developing the Proportions

To develop proportions from the table:

- Extract the columns needed from the table:

| | |
|---|---|
| $AB$ | $BC$ |
| $DE$ | $EF$ |

- Eliminate the table lines.
- Replace the horizontal lines with "division lines."
- Put an equal sign between the two resulting fractions:

$$\frac{AB}{DE} = \frac{BC}{EF}$$

**Also from the above table,**

$$\frac{AB}{DE} = \frac{CA}{FD}$$

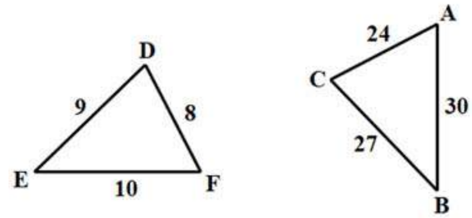
$$\frac{BC}{EF} = \frac{CA}{FD}$$

**Solving for the unknown length of a side:**

You can extract any two columns you like from the table. Usually, you will have information on lengths of three of the sides and will be asked to calculate a fourth.

Look in the table for the columns that contain the 4 sides in question, and then set up your proportion. Substitute known values into the proportion, and solve for the remaining variable.

**Example 8.10:** Are the triangles in the diagram similar? If so, write the similarity statement and state the theorem used to determine the similarity.




We only have the sides to work with, so we must check proportions. The easiest way to do this is by increasing the sizes of the sides of the triangles as you move from left to right in the proportions. So, we want to know if:

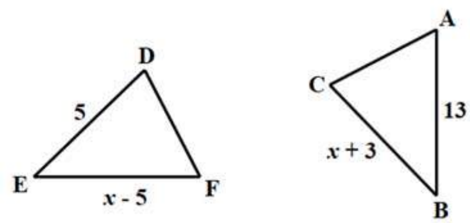
$$\frac{\text{smallest value}}{\text{smallest value}} = \frac{\text{middle value}}{\text{middle value}} = \frac{\text{largest value}}{\text{largest value}}$$

Side lengths for one triangle go in the numerators of the fractions and side lengths for the other triangle go in the denominators of the fractions. So, we want to know if:

$$\frac{8}{24} = \frac{9}{27} = \frac{10}{30} \ ?$$

Simplifying the fractions, we get: $\frac{1}{3} = \frac{1}{3} = \frac{1}{3}$. Then, **by the SSS Similarity Theorem, $\Delta FDE \sim \Delta ACB$.**

**Example 8.11:** If $\Delta EDF \sim \Delta BCA$, what is the value of $x$?




Let's be careful with letter order in setting up our proportion for this problem. In identifying proportions, refer to the names of the triangles that the lengths are coming from.

$$\frac{\text{first letter } (E), \text{last letter } (F)}{\text{first letter } (E), \text{second letter } (D)} = \frac{\text{first letter } (B), \text{last letter } (A)}{\text{first letter } (B), \text{second letter } (C)}$$

$$\frac{EF}{ED} = \frac{BA}{BC}$$

$$\frac{x-5}{5} = \frac{13}{x+3}$$

$$(x-5)(x+3) = 13 \cdot 5$$

$$x^2 - 2x - 15 = 65$$

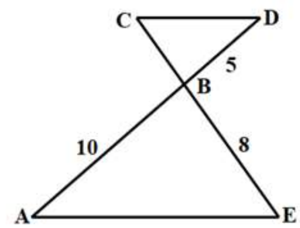
$$x^2 - 2x - 80 = 0$$

$$(x-10)(x+8) = 0$$

$$x = 10, -8.$$

Notice that $x = -8$ would give negative lengths in the diagram, so we discard that solution. So, $\boldsymbol{x = 10}$.

**Example 8.12:** In the figure to the right, $\overleftrightarrow{AE} \parallel \overleftrightarrow{CD}$ and $\overline{AD}$ intersects $\overline{CE}$ at point $B$. Find the length of $\overline{CE}$.




First, we need to find the similarity in the diagram, then find the appropriate proportion.

- $\angle ABE \cong \angle DBC$ because they are vertical angles.
- $\angle A \cong \angle D$ because they are alternate interior angles of parallel lines $\overleftrightarrow{AE}$ and $\overleftrightarrow{CD}$ with transversal $\overline{AD}$.
- $\Delta ABE \sim \Delta DBC$, then, by the **AA Similarity Theorem**.

The proportion we want must follow the lettering in the similarity.

$\frac{AB}{DB} = \frac{EB}{CB}$, with the large triangle in the numerator of the fractions and the small triangle in the denominator of the fractions in the proportion.

$\frac{10}{5} = \frac{8}{CB}$

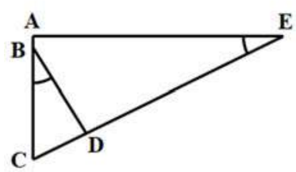
$10 \cdot CB = 40$

$CB = 4$

$\boldsymbol{CE} = EB + CB = 8 + 4 = \mathbf{12}$

**Example 8.13:** Given: $\angle CEA \cong \angle CBD$.

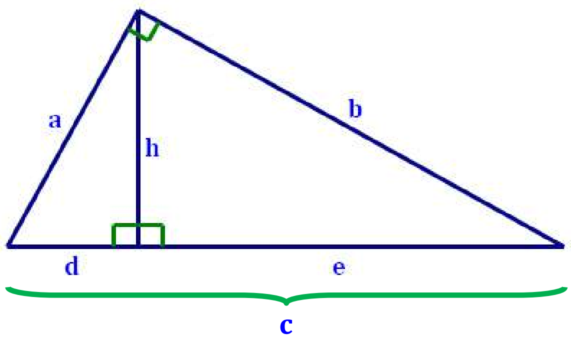
Prove: $AZ \cdot XY = AB \cdot ZY$




| Step | Statement | Reason |
|---|---|---|
| 1 | $\angle CEA \cong \angle CBD$ | Given. |
| 2 | $\angle C \cong \angle C$ | Reflexive property of congruence. |
| 3 | $\Delta CEA \cong \Delta CBD$ | AA Similarity Theorem.<br>Angles in Steps 1 and 2. |
| 4 | $\frac{AC}{AE} = \frac{DC}{DB}$ | Corresponding sides in similar triangles are proportional. |
| 5 | $\boldsymbol{AC \cdot DB = DC \cdot AE}$ | **Multiplicative property of equality (applied twice).** |

# Geometry
## Three Similar Triangles

A common problem in geometry is to find the missing value in proportions based on a set of three similar triangles, two of which are inside the third. The diagram often looks like this:




**Pythagorean Relationships**

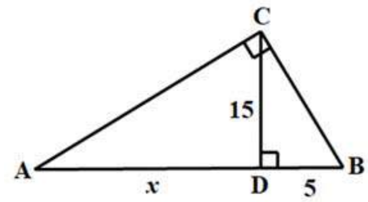
| | |
|---|---|
| Inside triangle on the left: | $d^2 + h^2 = a^2$ |
| Inside triangle on the right: | $h^2 + e^2 = b^2$ |
| Outside (large) triangle: | $a^2 + b^2 = c^2$ |

## Similar Triangle Relationships

Because all three triangles are similar, we have the relationships in the table below. These relationships are not obvious from the picture, but are very useful in solving problems based on the above diagram. Using similarities between the triangles, 2 at a time, we get:

| From the two inside triangles | From the inside triangle on the left and the outside triangle | From the inside triangle on the right and the outside triangle |
|---|---|---|
| $\frac{h}{d} = \frac{e}{h}$ or $h^2 = d \cdot e$ | $\frac{a}{d} = \frac{c}{a}$ or $a^2 = d \cdot c$ | $\frac{b}{e} = \frac{c}{b}$ or $b^2 = e \cdot c$ |
| **The height squared = the product of: the two parts of the base** | **The left side squared = the product of: the part of the base below it and the entire base** | **The right side squared = the product of: the part of the base below it and the entire base** |

**Example 8.14:** Solve for the value of $x$ in the diagram.



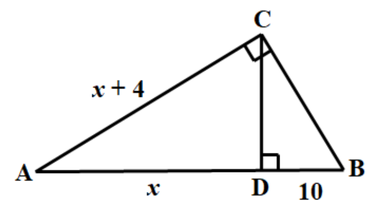

From the chart on the previous page:

The height squared = the product of the two parts of the base.

$15^2 = 5x$

$x = \frac{225}{5} = \mathbf{45}$

**Example 8.15:** Solve for the value of $x$ in the diagram.




From the chart on the previous page:

The left side squared = the product of the part of the base below it and the entire base.

$(x+4)^2 = x(x+10)$

$x^2 + 8x + 16 = x^2 + 10x$

$16 = 2x$

$x = \frac{16}{2} = \mathbf{8}$

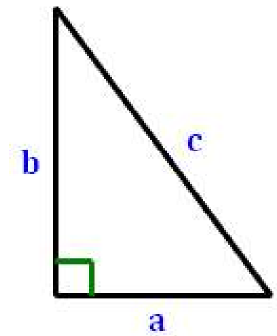
# Geometry
# Pythagorean Theorem

In a right triangle, the Pythagorean Theorem says:

$$\boldsymbol{a^2 + b^2 = c^2}$$

where,

- $\boldsymbol{a}$ and $\boldsymbol{b}$ are the lengths of the legs of a right triangle, and
- $\boldsymbol{c}$ is the length of the hypotenuse.

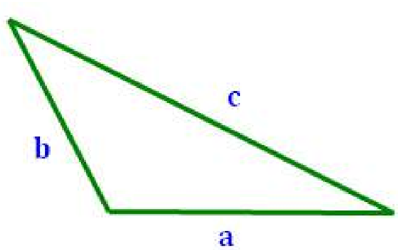



## Right, Acute, or Obtuse Triangle?

In addition to allowing the solution of right triangles, the Pythagorean Formula can be used to determine whether a triangle is a right triangle, an acute triangle, or an obtuse triangle.

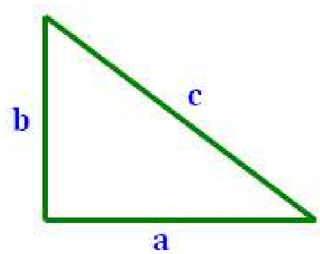
To determine whether a triangle is obtuse, right, or acute:

- Arrange the lengths of the sides from low to high; call them $a$, $b$, and $c$, in increasing order
- Calculate: $a^2, b^2,$ and $c^2$.
- Compare: $a^2 + b^2 \text{ vs. } c^2$
- Use the illustrations below to determine which type of triangle you have.



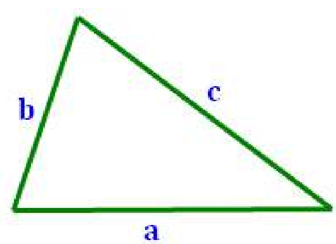

**Obtuse Triangle**

$\boldsymbol{a^2 + b^2 < c^2}$




**Right Triangle**

$\boldsymbol{a^2 + b^2 = c^2}$




**Acute Triangle**

$\boldsymbol{a^2 + b^2 > c^2}$

**Example 9.1:**

Triangle with sides: 7, 9, 12

$$7^2 + 9^2 \;\; vs. \;\; 12^2$$
$$49 + 81 < 144$$
$$\rightarrow \;\; \textit{Obtuse Triangle}$$

**Example 9.2:**

Triangle with sides: 6, 8, 10

$$6^2 + 8^2 \;\; vs. \;\; 10^2$$
$$36 + 64 = 100$$
$$\rightarrow \;\; \textit{Right Triangle}$$

**Example 9.3:**

Triangle with sides: 5, 8, 9

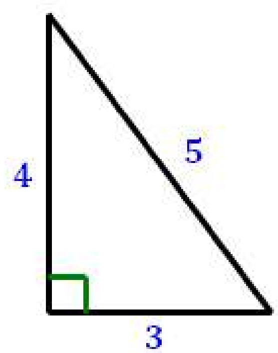
$$5^2 + 8^2 \;\; vs. \;\; 9^2$$
$$25 + 64 > 81$$
$$\rightarrow \;\; \textit{Acute Triangle}$$

# Geometry
# Pythagorean Triples

**Pythagorean Theorem:** $a^2 + b^2 = c^2$

Pythagorean triples are sets of 3 positive integers that meet the requirements of the Pythagorean Theorem. Because these sets of integers provide "pretty" solutions to geometry problems, they are a favorite of geometry books and teachers. Knowing what triples exist can help the student quickly identify solutions to problems that might otherwise take considerable time to solve.

## 3-4-5 Triangle Family

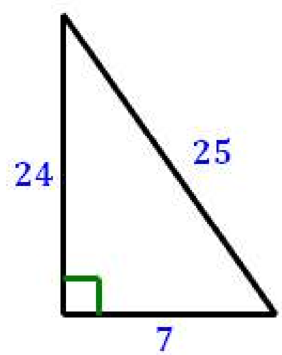



**Sample Triples**

3-4-5
6-8-10
9-12-15
12-16-20
30-40-50

$$3^2 + 4^2 = 5^2$$
$$9 + 16 = 25$$

## 7-24-25 Triangle Family

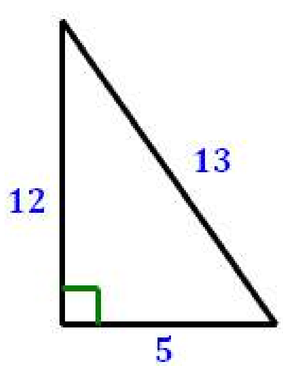



**Sample Triples**

7-24-25
14-48-50
21-72-75
. . .
70-240-250

$$7^2 + 24^2 = 25^2$$
$$49 + 576 = 625$$

## 5-12-13 Triangle Family

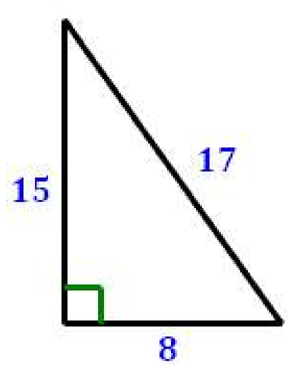



**Sample Triples**

5-12-13
10-24-26
15-36-39
. . .
50-120-130

$$5^2 + 12^2 = 13^2$$
$$25 + 144 = 169$$

## 8-15-17 Triangle Family

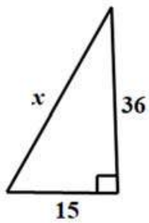



**Sample Triples**

8-15-17
16-30-34
24-45-51
. . .
80-150-170

$$8^2 + 15^2 = 17^2$$
$$64 + 225 = 289$$

**Example 9.4:** Find the value of $x$.



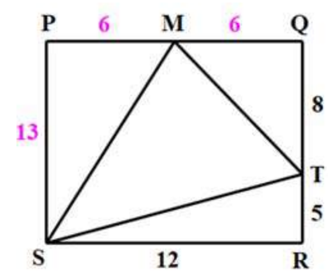

$x^2 = 15^2 + 36^2$

$x^2 = 225 + 1296$

$x^2 = 1521$

$\boldsymbol{x = 39}$

**Example 9.5:** M is the midpoint of $\overline{PQ}$ in rectangle $PQRS$. What is the perimeter of $\Delta MST$.




The measures in black in the diagram are given, so we add the ones in magenta. Then,

$ST = \sqrt{5^2 + 12^2} = 13$

$TM = \sqrt{6^2 + 8^2} = 10$

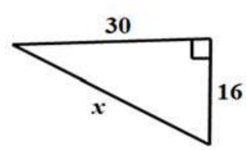
$MS = \sqrt{6^2 + 13^2} = \sqrt{205}$

$\boldsymbol{P(\Delta MST)} = ST + TM + MS$

$= 13 + 10 + \sqrt{205}$

$= \boldsymbol{23 + \sqrt{205}}$

**Example 9.6:** A treasure is buried 16 paces north and 30 paces west of a landmark. How many paces is the treasure from the landmark via a direct route?



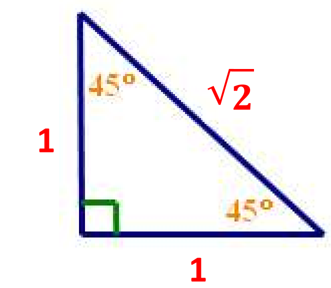

$x^2 = 16^2 + 30^2 = 1156$

$\boldsymbol{x} = \sqrt{1156} = \boldsymbol{34}$ **paces**

# Geometry
## Special Triangles

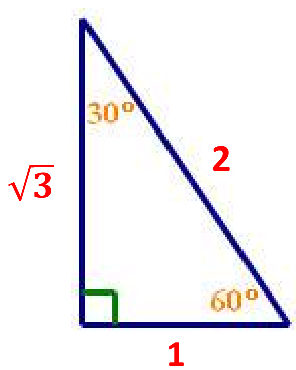
The relationship among the lengths of the sides of a triangle is dependent on the measures of the angles in the triangle. For a right triangle (i.e., one that contains a $90^o$ angle), two special cases are of particular interest. These are shown below:

### $45^o$-$45^o$-$90^o$ Triangle




In a **$45^o$-$45^o$-$90^o$ triangle**, the congruence of two angles guarantees the congruence of the two legs of the triangle. The proportions of the three sides are: $\mathbf{1 : 1 : \sqrt{2}}$. That is, the two legs have the same length and the hypotenuse is $\sqrt{2}$ times as long as either leg.

### $30^o$-$60^o$-$90^o$ Triangle




In a **$30^o$-$60^o$-$90^o$ triangle**, the proportions of the three sides are: $\mathbf{1 : \sqrt{3} : 2}$. That is, the long leg is $\sqrt{3}$ times as long as the short leg, and the hypotenuse is $\mathbf{2}$ times as long as the short leg.

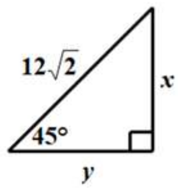
In a right triangle, we need to know the lengths of two sides to determine the length of the third. The power of the relationships in the special triangles lies in the fact that we need only know the length of one side of the triangle to determine the lengths of the other two sides.

**Example Side Lengths**

| $45^o$-$45^o$-$90^o$ Triangle | |
|---|---|
| $1 : 1 : \sqrt{2}$ | $2 : 2 : 2\sqrt{2}$ |
| $\sqrt{2} : \sqrt{2} : 2$ | $\sqrt{3} : \sqrt{3} : \sqrt{6}$ |
| $3\sqrt{2} : 3\sqrt{2} : 6$ | $25 : 25 : 25\sqrt{2}$ |

| $30^o$-$60^o$-$90^o$ Triangle | |
|---|---|
| $1 : \sqrt{3} : 2$ | $2 : 2\sqrt{3} : 4$ |
| $\sqrt{2} : \sqrt{6} : 2\sqrt{2}$ | $\sqrt{3} : 3 : 2\sqrt{3}$ |
| $3\sqrt{2} : 3\sqrt{6} : 6\sqrt{2}$ | $25 : 25\sqrt{3} : 50$ |

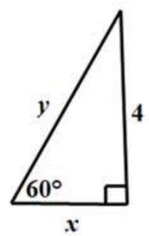
**Example 9.7:** Find the values of $x$ and $y$.




$$x = 12\sqrt{2} \div \sqrt{2} = \mathbf{12}$$

$$y = x = \mathbf{12}$$

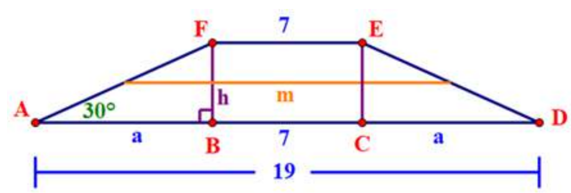
**Example 9.8:** Find the values of $x$ and $y$.




$$x = 4 \div \sqrt{3} = \frac{4}{\sqrt{3}} = \frac{4}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\mathbf{4\sqrt{3}}}{\mathbf{3}}$$

$$y = 2 \cdot \frac{4\sqrt{3}}{3} = \frac{\mathbf{8\sqrt{3}}}{\mathbf{3}}$$

**Example 9.9:** Find the area of the isosceles trapezoid shown. All measures are in meters (m).




$m$ is the midsegment of the trapezoid.

In the figure:

- BF and CE are drawn perpendicular to both BC and FE.
- $\Delta ABF \cong \Delta DCE$, both are right triangles.
- BCEF is a rectangle.

We want the total area of the trapezoid. The formula for this is:

$$Area = \frac{b_1 + b_2}{2} \cdot h = m \cdot h$$

$$m = \frac{7 + 19}{2} = 13$$

$h$ is determined using the proportions of a $30° - 60° - 90°$ $(1 : \sqrt{3} : 2)$ triangle: $\Delta ABF$.

a is the length of the long side of $\Delta ABF$.

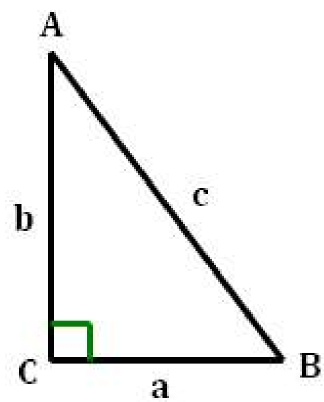
$$\mathrm{a} = \frac{19 - 7}{2} = 6$$

$$h = \frac{\mathrm{a}}{\sqrt{3}} = \frac{6}{\sqrt{3}} = 2\sqrt{3}$$

Finally, $\boldsymbol{Area} = 13 \cdot 2\sqrt{3} = \mathbf{26\sqrt{3}\ m^2}$

# Geometry
## Trig Functions and Special Angles

### Trigonometric Functions




**SOH-CAH-TOA**

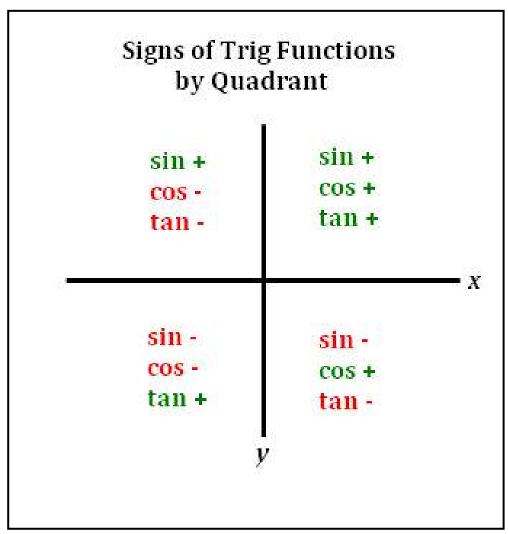
$\sin = \dfrac{opposite}{hypoteneuse}$ $\sin A = \dfrac{a}{c}$ $\sin B = \dfrac{b}{c}$

$\cos = \dfrac{adjacent}{hypoteneuse}$ $\cos A = \dfrac{b}{c}$ $\cos B = \dfrac{a}{c}$

$\tan = \dfrac{opposite}{adjacent}$ $\tan A = \dfrac{a}{b}$ $\tan B = \dfrac{b}{a}$

### Special Angles

| Trig Functions of Special Angles | | | | |
|---|---|---|---|---|
| **Radians** | **Degrees** | $\sin\theta$ | $\cos\theta$ | $\tan\theta$ |
| 0 | $0^o$ | $\frac{\sqrt{0}}{2} = 0$ | $\frac{\sqrt{4}}{2} = 1$ | $\frac{\sqrt{0}}{\sqrt{4}} = 0$ |
| $\pi/6$ | $30^o$ | $\frac{\sqrt{1}}{2} = \frac{1}{2}$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{1}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$ |
| $\pi/4$ | $45^o$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{2}$ | $\frac{\sqrt{2}}{\sqrt{2}} = 1$ |
| $\pi/3$ | $60^o$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{1}}{2} = \frac{1}{2}$ | $\frac{\sqrt{3}}{\sqrt{1}} = \sqrt{3}$ |
| $\pi/2$ | $90^o$ | $\frac{\sqrt{4}}{2} = 1$ | $\frac{\sqrt{0}}{2} = 0$ | *undefined* |

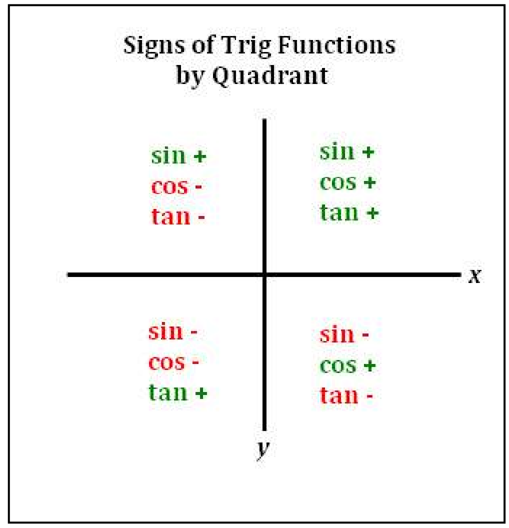


# Geometry

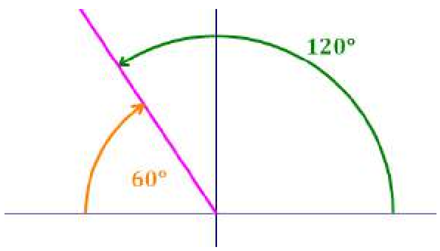
## Trigonometric Function Values in Quadrants II, III, and IV

In quadrants other than Quadrant I, trigonometric values for angles are calculated in the following manner:

- Draw the angle $\boldsymbol{\theta}$ on the Cartesian Plane.
- Calculate the measure of the angle from the x-axis to $\boldsymbol{\theta}$.
- Find the value of the trigonometric function of the angle in the previous step.
- Assign a "+" or "−" sign to the trigonometric value based on the function used and the quadrant $\boldsymbol{\theta}$ is in.




**Examples:**




**Example 9.10:** $\boldsymbol{\theta}$ **in Quadrant II** – Calculate: $(180^0 - m\angle\theta)$

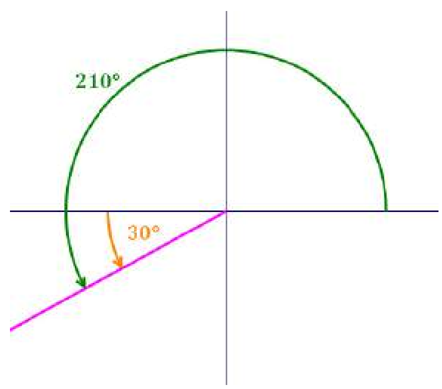
For $\theta = 120^0$, base your work on $180° - 120° = 60°$

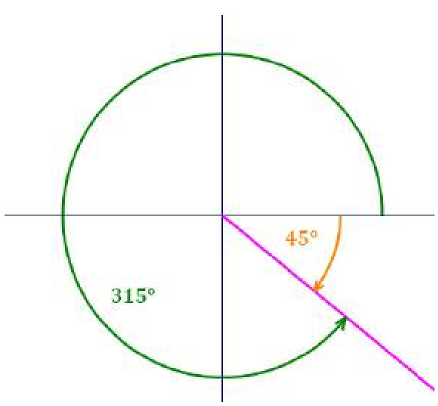
$\sin 60° = \frac{\sqrt{3}}{2}$, so: $\mathbf{\sin 120° = \frac{\sqrt{3}}{2}}$

**Example 9.11:** $\boldsymbol{\theta}$ **in Quadrant III** – Calculate: $(m\angle\theta - 180^0)$

For $\theta = 210^0$, base your work on $210° - 180° = 30°$

$\cos 30° = \frac{\sqrt{3}}{2}$, so: $\mathbf{\cos 210° = -\frac{\sqrt{3}}{2}}$



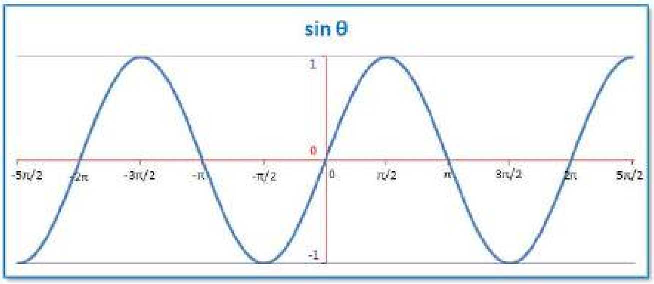



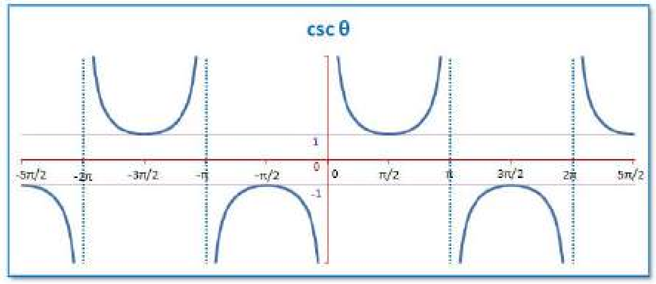

**Example 9.12:** $\boldsymbol{\theta}$ **in Quadrant IV** – Calculate: $(360^0 - m\angle\theta)$

For $\theta = 315^0$, base your work on $360° - 315° = 45°$

$\tan 45° = 1$, so: $\mathbf{\tan 315° = -1}$

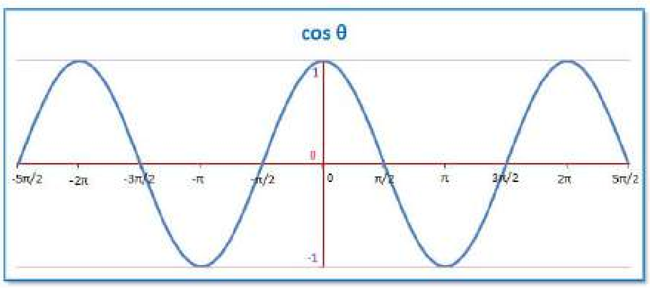
# Geometry
## Graphs of Trigonometric Functions

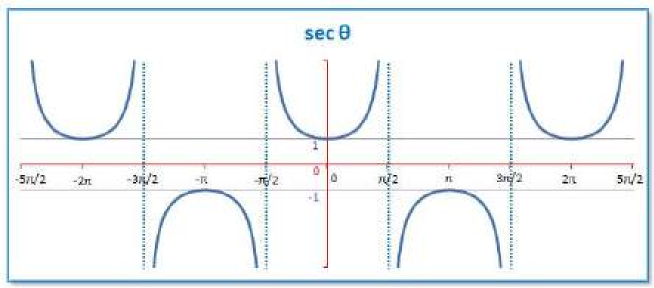



The sine and cosecant functions are inverses. So:

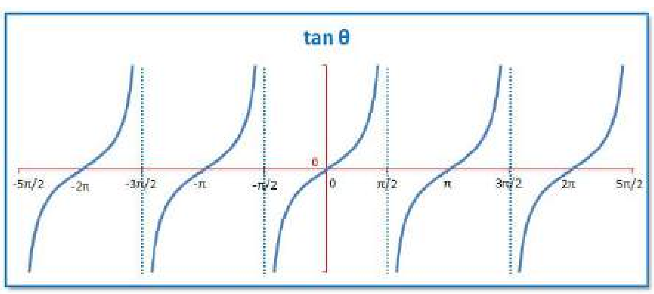
$$\sin\theta = \frac{1}{\csc\theta} \quad \text{and} \quad \csc\theta = \frac{1}{\sin\theta}$$

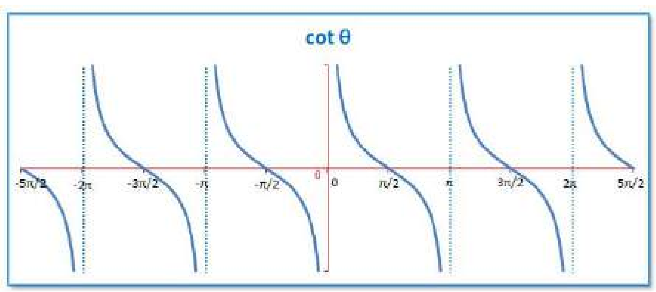



The cosine and secant functions are inverses. So:

$$\cos\theta = \frac{1}{\sec\theta} \quad \text{and} \quad \sec\theta = \frac{1}{\cos\theta}$$

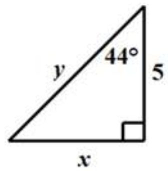



The tangent and cotangent functions are inverses. So:

$$\tan\theta = \frac{1}{\cot\theta} \quad \text{and} \quad \cot\theta = \frac{1}{\tan\theta}$$

**Example 9.13:** Find the values of $x$ and $y$. Round values to 2 decimal places.



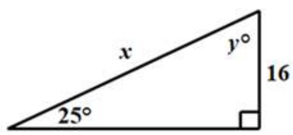

$\tan 44° = \frac{x}{5}$

$\boldsymbol{x} = 5 \cdot \tan 44° \approx \mathbf{4.83}$

$\cos 44° = \frac{5}{y}$

$\boldsymbol{y} = \frac{5}{\cos 44°} \approx \mathbf{6.95}$

**Example 9.14:** Find the values of $x$ and $y$. Round values to 2 decimal places.




$\sin 25° = \frac{16}{x}$

$\boldsymbol{x} = \frac{16}{\sin 25°} \approx \mathbf{37.86}$

$y + 25° = 90°$

$\boldsymbol{y} = 90° - 25° \approx \mathbf{65°}$

**Example 9.15:** $\cos x = 0.5$. What is $\sec x$? $\csc y = 4$. What is $\sin y$?

$\cos x = 0.5$

$\mathbf{sec}\,\boldsymbol{x} = \frac{1}{\cos x} = \frac{1}{0.5} = \mathbf{2}$

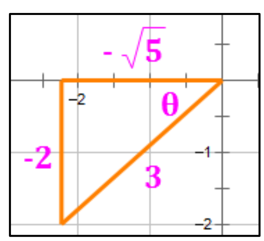
$\csc y = 4$

$\sin y = \frac{1}{\csc y} = \frac{1}{4} = \mathbf{0.25}$

**Example 9.16:** $\sin\theta = -\frac{2}{3}$, $\tan\theta > 0$. Find the values of $\sec\theta$ and $\cot\theta$.

Notice that $\sin\theta < 0$, $\tan\theta > 0$. Therefore, $\theta$ is in $Q3$, so we draw the angle in that quadrant.

In $Q3$, $y$ is negative; $r$ is always positive. Since $\sin\theta = \frac{y}{r} = -\frac{2}{3}$, we let $y = -2$, $r = 3$.




Using the Pythagorean Theorem, we calculate the length of the horizontal leg of the triangle: $\sqrt{3^2 - (-2)^2} = \sqrt{5}$. Since the angle is in $Q3$, $x$ is negative, so we must have $x = -\sqrt{5}$.

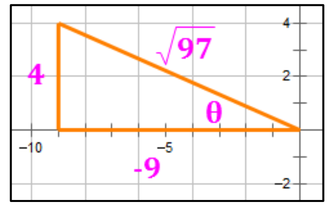
Then, $\sec\theta = \frac{1}{\cos\theta} = \frac{r}{x} = \frac{3}{-\sqrt{5}} = -\frac{3\sqrt{5}}{5}$

And, $\cot\theta = \frac{1}{\tan\theta} = \frac{x}{y} = \frac{-\sqrt{5}}{-2} = \frac{\sqrt{5}}{2}$

**Example 9.17:** $\cot\theta = -\frac{9}{4}$, $\cos\theta < 0$. Find the value of $\csc\theta$ and $\cos\theta$.

Notice that $\cot\theta < 0$, $\cos\theta < 0$. Therefore, $\theta$ is in $Q2$, so we draw the angle in that quadrant.

In $Q2$, $x$ is negative, and $y$ is positive. Since $\cot\theta = \frac{x}{y} = -\frac{9}{4}$, we let $x = -9$, $y = 4$.



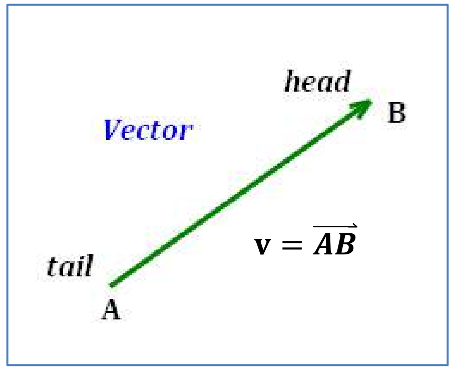

Using the Pythagorean Theorem, we can calculate the length of the hypotenuse of the triangle: $r = \sqrt{(-9)^2 + 4^2} = \sqrt{97}$.

Then, $\csc\theta = \frac{1}{\sin\theta} = \frac{r}{y} = \frac{\sqrt{97}}{4}$

And, $\cos\theta = \frac{x}{r} = \frac{-9}{\sqrt{97}} = \frac{-9\sqrt{97}}{97}$

# Geometry
# Vectors

## Definitions




- A **vector** is a geometric object that has both magnitude (length) and direction.
- The **Tail** of the vector is the end opposite the arrow. It represents where the vector is moving from.
- The **Head** of the vector is the end with the arrow. It represents where the vector is moving to.
- The **Zero Vector** is denoted $\mathbf{0}$. It has zero length and all the properties of zero.
- Two vectors are **equal** is they have both the same magnitude and the same direction.
- Two vectors are **parallel** if they have the same or opposite directions. That is, if the angles of the vectors are the same or 180° different.
- Two vectors are **perpendicular** if the difference of the angles of the vectors is 90° or 270°.

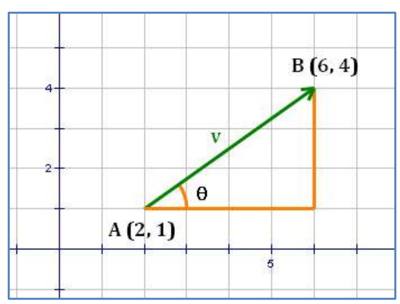
## Magnitude of a Vector

The distance formula gives the magnitude of a vector. If the head and tail of vector $\mathbf{v}$ are the points $A = (x_1, y_1)$ and $B = (x_2, y_2)$, then the magnitude of $\mathbf{v}$ is:

$$|\mathbf{v}| = |\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Note that $|\overrightarrow{AB}| = |\overrightarrow{BA}|$. The directions of the two vectors are opposite, but their magnitudes are the same.

## Direction of a Vector




The direction of a vector is determined by the angle it makes with a horizontal line. In the figure at right, the direction is the angle $\theta$. The value of $\theta$ can be calculated based on the lengths of the sides of the triangle the vector forms.

$$\tan\theta = \frac{3}{4} \qquad \text{or} \qquad \theta = \tan^{-1}\left(\frac{3}{4}\right)$$

where the function $\tan^{-1}$ is the inverse tangent function. The second equation in the line above reads "$\theta$ is the angle whose tangent is $\frac{3}{4}$."

# Geometry
# Operations with Vectors

It is possible to operate with vectors in some of the same ways we operate with numbers. In particular:

## Adding Vectors

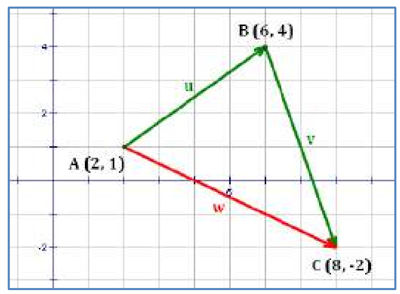
Vectors can be added in rectangular form by separately adding their *x*- and *y-components*. In general,

$$\mathbf{u} = \langle u_1, u_2 \rangle$$

$$\mathbf{v} = \langle v_1,\ v_2 \rangle$$

$$\mathbf{u} + \mathbf{v}\ =\ \langle u_1,\ u_2 \rangle + \langle v_1,\ v_2 \rangle\ =\ \langle u_1 + v_1,\ u_2 +\ v_2 \rangle$$

**Example 9.18:** In the figure at right,




$$\mathbf{u} = \langle 4, 3 \rangle$$

$$\mathbf{v} = \langle 2, -6 \rangle$$

$$\mathbf{w}\ =\ \mathbf{u} + \mathbf{v}\ =\ \langle 4, 3 \rangle + \langle 2, -6 \rangle\ =\ \langle 6, -3 \rangle$$

## Vector Algebra

| | | |
|---|---|---|
| $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ | $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ | $\mathrm{a} \cdot (\mathbf{u} + \mathbf{v}) = (\mathrm{a} \cdot \mathbf{u}) + (\mathrm{a} \cdot \mathbf{v})$ |
| $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{w} + \mathbf{v})$ | $\mathbf{0} \cdot \mathbf{u} = \mathbf{0}$ | $(\mathrm{a} + \mathrm{b}) \cdot \mathbf{u} = (\mathrm{a} \cdot \mathbf{u}) + (\mathrm{b} \cdot \mathbf{u})$ |
| $\mathbf{u} + \mathbf{0} = \mathbf{u}$ | $1 \cdot \mathbf{u} = \mathbf{u}$ | $(\mathrm{ab}) \cdot \mathbf{u} = \mathrm{a} \cdot (\mathrm{b} \cdot \mathbf{u}) = \mathrm{b} \cdot (\mathrm{a} \cdot \mathbf{u})$ |

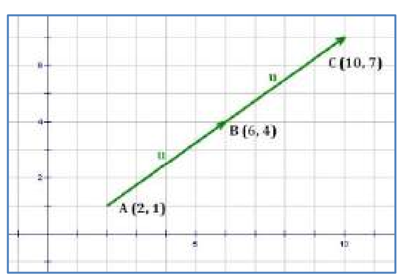
## Scalar Multiplication

Scalar multiplication changes the magnitude of a vector, but not the direction. In general,

$$\mathbf{u} = \langle u_1, u_2 \rangle$$

$$k \cdot \mathbf{u} = \langle k \cdot u_1, k \cdot u_2 \rangle$$

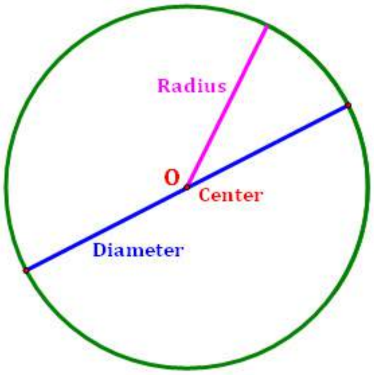
**Example 9.19:** In the figure at right,




$$\mathbf{u} = \langle 4, 3 \rangle$$

$$2 \cdot \mathbf{u}\ =\ 2 \cdot \langle 4, 3 \rangle\ =\ \langle 8, 6 \rangle$$

# Geometry
## Parts of Circles




**Center** – the middle of the circle. All points on the circle are the same distance from the center.

**Radius** – a line segment with one endpoint at the center and the other endpoint on the circle. The term "radius" is also used to refer to the distance from the center to the points on the circle.

**Diameter** – a line segment with endpoints on the circle that passes through the center.

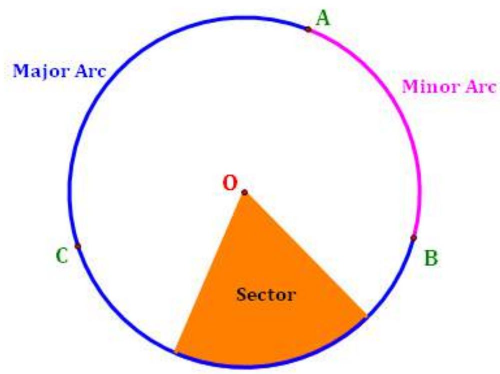
**Arc –** a path along a circle.

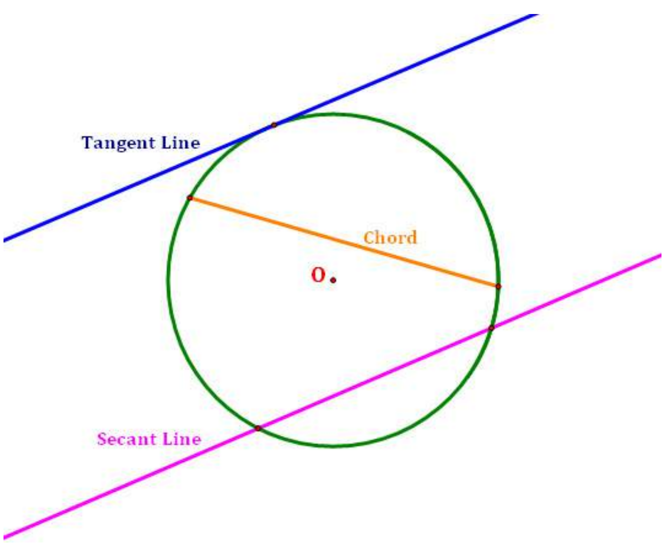
**Minor Arc** – a path along the circle that is less than $180^o$.

**Major Arc** – a path along the circle that is greater than $180^o$.

**Semicircle** – a path along a circle that equals $180^o$.

**Sector** – a region inside a circle that is bounded by two radii and an arc.



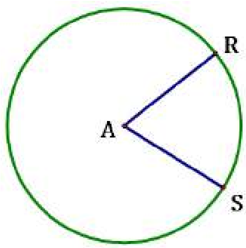

**Secant Line** – a line that intersects the circle in exactly two points.

**Tangent Line**– a line that intersects the circle in exactly one point.

**Chord** – a line segment with endpoints on the circle that does not pass through the center.

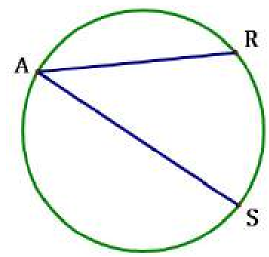
# Geometry
## Angles, Arcs, and Segments

### Central Angle



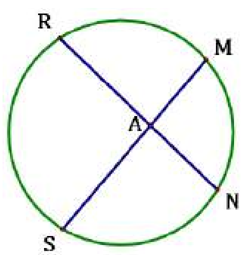

$$m\angle A = m\,\widehat{RS}$$

### Inscribed Angle



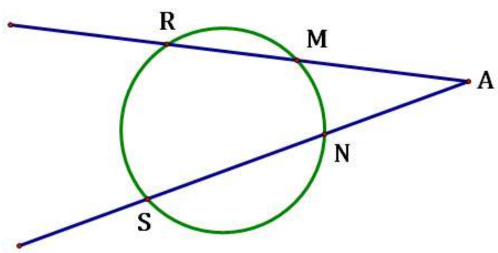

$$m\angle A = \frac{1}{2}\,m\,\widehat{RS}$$

### Vertex inside the circle



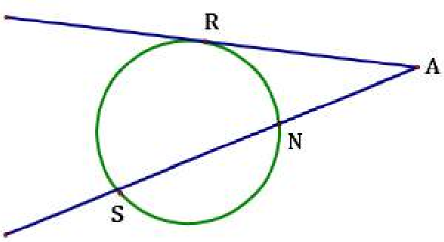

$$m\angle A = \frac{1}{2}\left(m\,\widehat{RS} + m\,\widehat{MN}\right)$$
$$RA \cdot AN = SA \cdot AM$$

### Vertex outside the circle



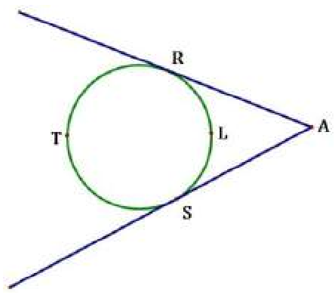

$$m\angle A = \frac{1}{2}\left(m\,\widehat{RS} - m\,\widehat{MN}\right)$$
$$AM \cdot AR = AN \cdot AS$$

### Tangent on one side



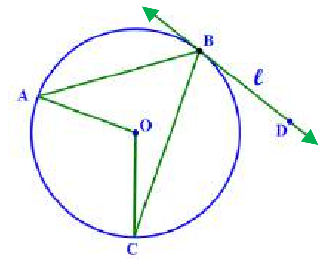

$$m\angle A = \frac{1}{2}\left(m\,\widehat{RS} - m\,\widehat{RN}\right)$$
$$AR^2 = AN \cdot AS$$

### Tangents on two sides




$$m\angle A = \frac{1}{2}\left(m\,\widehat{RTS} - m\,\widehat{RLS}\right)$$
$$AR = AS$$

**Circle Vocabulary:**



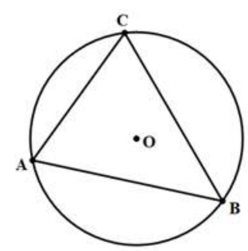

- Subtended angle: an angle whose two rays pass through the endpoints of a geometric object (e.g., an arc on a circle).
- An arc subtends an angle. An angle is subtended by an arc.
- In the diagram to the right, $\widehat{AC}$ subtends both $\angle ABC$ and $\angle AOC$. Both $\angle ABC$ and $\angle AOC$ are subtended by $\widehat{AC}$.

- Naming a circle: Circles are typically named by their centers. In the diagram above, we would refer to the circle as Circle O. Typically, the point at the center of a circle is named O or a letter close to O in the English alphabet.
- Interior point: a point whose distance from the center of the circle is less than the radius of the circle. That is, the point is inside the circle.
- Exterior point: a point whose distance from the center of the circle is more than the radius of the circle. That is, the point is outside the circle.

- Central angle: An angle with its vertex at the center of a circle. In the diagram above, $\angle AOC$ is a central angle.
- Inscribed angle: An angle with its vertex on a circle and its rays passing through the circle. In the diagram above, $\angle ABC$ is an inscribed angle.
- Tangent-chord angle: An angle with its vertex on a circle, one ray tangent to the circle, and one ray passing through the circle. In the diagram above, line $\ell$ is tangent to Circle O at Point B. $\angle ABD$ and $\angle CBD$ are tangent-chord angles.

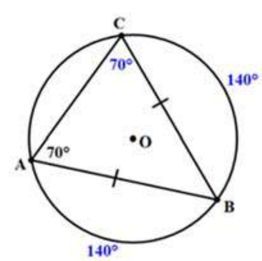
- Circumscribed polygon: A polygon outside a circle, with all of the sides of the polygon tangent to the circle. Circumscribed polygons are typically regular (i.e., they have equal angle measures and equal side lengths).
- Inscribed polygon: A polygon inside a circle, with all of its vertices on the circle.

**Example 10.1:** Given: $\overline{AB} \cong \overline{BC}$ and $m\angle A = 70°$, find $m\ \widehat{ABC}$.




The top diagram to the right is given with this problem. In order to solve the problem, we add a few things to the top diagram to get the bottom diagram.

In the bottom diagram, items in black are given, and items in blue are derived as follows:

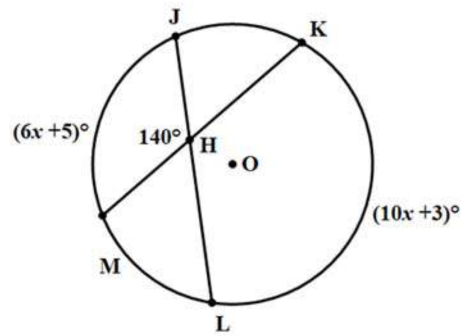



$m\angle C = m\angle A = 70°$ because they are opposite congruent sides in a triangle.

$m\ \widehat{BC} = m\ \widehat{AB} = 2 \cdot 70° = 140°$ because the arcs subtend angles of 70°.

$\boldsymbol{m\ \widehat{ABC}} = m\ \widehat{AB} + m\ \widehat{BC} = 140° + 140° = \mathbf{280°}$

**Example 10.2:** Solve for $x$ in the circle provided.




H is a vertex inside the circle, so we have the relationship:

$$140° = \frac{1}{2}(m\,\widehat{JM} + m\,\widehat{KL})$$

$$140 = \frac{1}{2}[(6x + 5) + (10x + 3)]$$

$$140 = 8x + 4$$

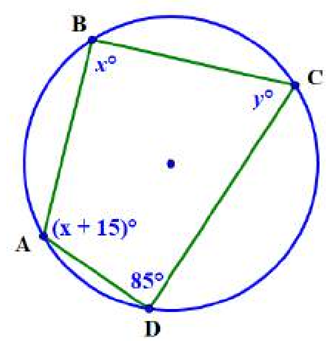
$$136 = 8x$$

$$\mathbf{17 = x}$$

### Facts about Circles

- The circumference of a circle is: $C = \pi d = 2\pi r$, where $d$ is the diameter of the circle and $r$ is the radius of the circle.
- The area of a circle is $C = \pi r^2$, where $r$ is the radius of the circle.
- A diameter spits a circle into two arcs, each of measure $180°$.
- All radii in a circle are congruent. Likewise, all radii in congruent circles are congruent.
- If a quadrilateral is inscribed in a circle, then opposite angles of the quadrilateral are supplementary.

**Example 10.3:** Solve for $x$ and $y$.




Opposite angles in a quadrilateral inscribed in a circle add to 180°.

$$x + 85 = 180 \quad \rightarrow \quad \boldsymbol{x = 105}$$

$$m\angle A = (x + 15)° = (105 + 10)° = 120°$$

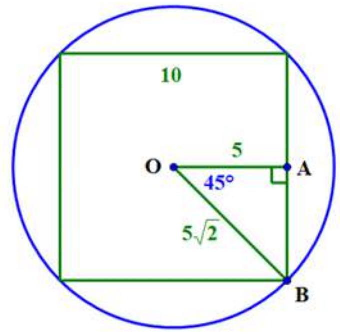
$$y + 120 = 180 \quad \rightarrow \quad \boldsymbol{y = 60}$$

### Facts about Chords

- The distance of a chord from the center of a circle is measured from the center of the circle to the midpoint of the chord. The radius extending through the midpoint of the chord is perpendicular to the chord.
- The perpendicular bisector of a chord passes through the center of the circle.
- Chords that are the same distance from the center of the same circle or congruent circles are congruent.

**Example 10.4:** A square with area $100\ \text{cm}^2$ is inscribed in a circle. Find the exact value of the area of the circle.

If a square has an area of 100, it must have a side length of:
$s = \sqrt{100} = 10$.




We create $\Delta\text{OAB}$ in the diagram to find the radius of Circle O. $\Delta\text{OAB}$ is a 45°-45°-90° triangle with sides of length 5, so the hypotenuse, $\text{OB} = 5\sqrt{2}$.

The radius of the circle is the length of $\overline{\text{OB}}$. $\text{OB} = 5\sqrt{2}$.

Finally, the area requested is:

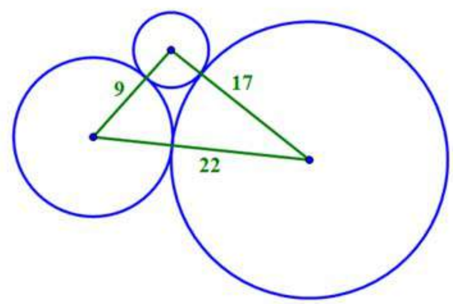
$$C = \pi r^2 = \pi \cdot \left(5\sqrt{2}\right)^2 = \mathbf{50\pi\ cm^2}.$$

**Example 10.5:** Given three tangent circles with distances between their radii of 9, 17, 22, find the radii of the circles.

Let's call the radii of the three circles $a, b, c$. Then,

$$a + b = 9, \quad b + c = 22, \quad a + c = 17$$

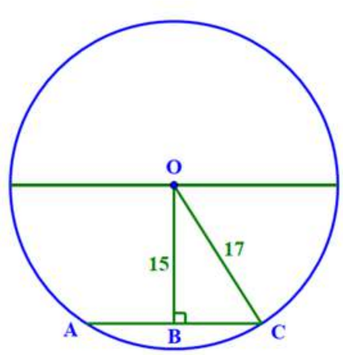
Solve.




$$\begin{array}{r} a + c = 17 \\ -a - b = -9 \\ \hline c - b = 8 \end{array} \quad \rightarrow \quad \begin{array}{r} -b + c = 8 \\ b + c = 22 \\ \hline 2c = 30 \\ \mathbf{c = 15} \end{array}$$

With $c = 15$, we get $\mathbf{b = 7}$, $\mathbf{a = 2}$ from the starting equations.

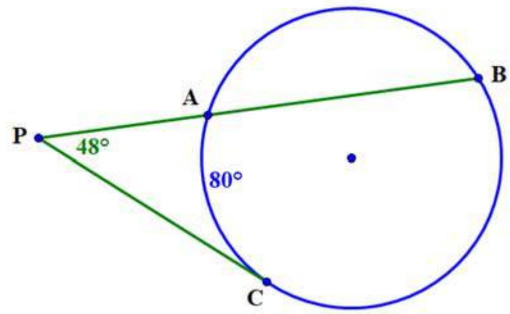
**Example 10.6:** Find the length of a chord that is 15 cm from the center of a circle with a diameter of 34 cm.




The figure to the left diagrams this problem. All radii of the circle are 17 cm in length. The distance from the center to the chord ($\overline{\text{AC}}$) is 15 cm, and $\overline{\text{AC}}$ is perpendicular to the segment drawn from the center to the chord, $\overline{\text{OB}}$.

$$\begin{aligned} \mathbf{AC} &= 2 \cdot \text{AB} \\ &= 2 \cdot \sqrt{17^2 - 15^2} \\ &= 2 \cdot 8 = \mathbf{16} \end{aligned}$$

**Example 10.7:** Given $m\angle P = 48°$ and $m\,\widehat{AC} = 80°$, what is $m\,\widehat{AB}$?




$$48° = \frac{1}{2}(m\,\widehat{BC} - m\,\widehat{AC})°$$

$$48° = \frac{1}{2}(m\,\widehat{BC} - 80)°$$

$$96° = (m\,\widehat{BC} - 80)°$$

$$m\,\widehat{BC} = 176°$$

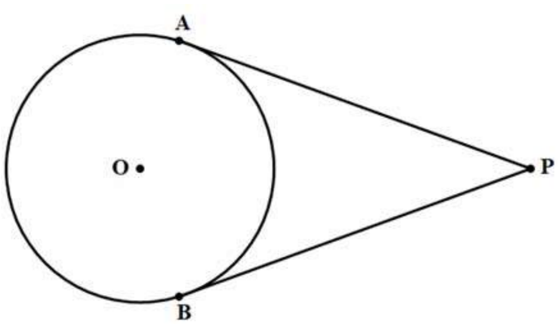
$$m\,\widehat{AB} = 360° - m\,\widehat{AC} - m\,\widehat{BC}$$

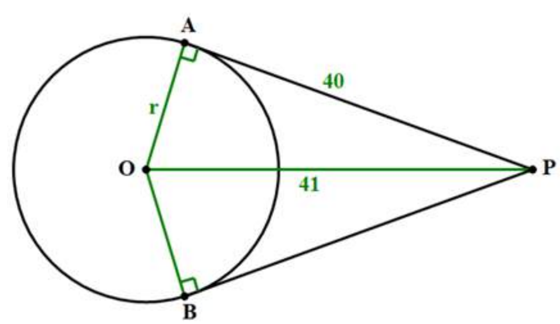
$$\boldsymbol{m\,\widehat{AB}} = 360° - 80° - 176° = \mathbf{104°}$$

**Facts about Tangents**

- Tangents to a circle from an external point are congruent.
- A tangent to a circle is perpendicular to the radius of the circle that intersects the tangent at the point of tangency.
- If two lines that are tangent to a circle intersect at an external point, then the line containing the point of intersection and the center of the circle bisects the angle formed by the two tangents.
- All of these facts about tangents are illustrated in the example below.

**Example 10.8:** $\overline{PB}$ and $\overline{PA}$ are tangent to Circle $O$, $PA = 40$ and $PO = 41$. Find PB and the radius of the circle.




The above left diagram is given with this problem. In order to solve the problem, we add a few things to get the above right diagram.

Tangents to a circle from an external point are congruent, so $\boldsymbol{PB} = PA = \mathbf{40.}$

There are right angles at the points of tangency. Pythagoras will help us get the radius.

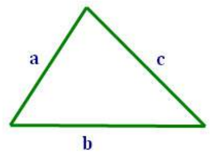
$$\boldsymbol{r} = \sqrt{41^2 - 40^2} = \mathbf{9}$$

# Geometry
# Perimeter and Area of a Triangle

## Perimeter of a Triangle

The perimeter of a triangle is simply the sum of the measures of the three sides of the triangle.

$$P = a + b + c$$

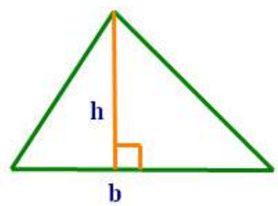



## Area of a Triangle

There are two formulas for the area of a triangle, depending on what information about the triangle is available.

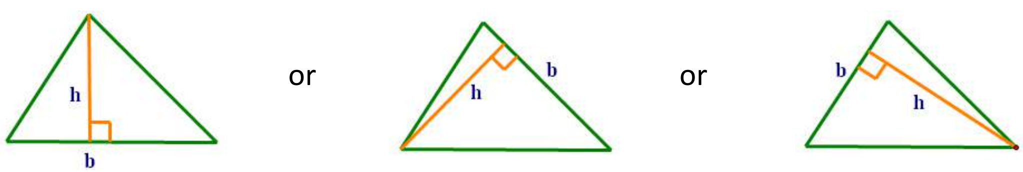
**Formula 1:** The formula most familiar to the student can be used when the base and height of the triangle are either known or can be determined.

$$A = \frac{1}{2}bh$$

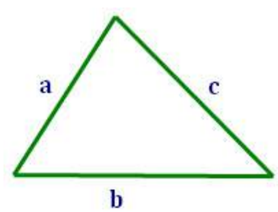



where, $b$ is the length of the base of the triangle.
$h$ is the height of the triangle.

**Note:** The base can be any side of the triangle. The height is the measure of the altitude of whichever side is selected as the base. So, you can use:




**Formula 2:** **Heron's formula** for the area of a triangle can be used when the lengths of all of the sides are known. Sometimes this formula, though less appealing, can be very useful.

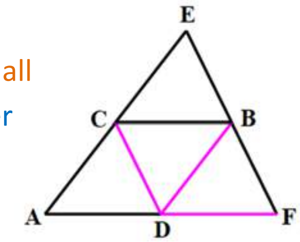



$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

where, $s = \frac{1}{2}P = \frac{1}{2}(a+b+c)$. **Note:** $s$ is sometimes called the semi-perimeter of the triangle.

$a, b, c$ are the lengths of the sides of the triangle.

**Example 11.1:** C, B, D are midpoints. BD = 12 cm, DF = 11 cm, CD = 10.4 cm. Find the perimeter of $\Delta AEF$.




The four small triangles formed by connecting midpoints C, B, D are all congruent. The perimeter of the $\Delta AEF$ will be double the perimeter of any of the four interior triangles.

We are given the three lengths shown in magenta in the diagram. Let's use the perimeter of $\Delta DBF$ as our basis to calculate the perimeter of $\Delta AEF$.

$$P(\Delta DBF) = BD + BF + DF$$

Of the three distances in the formula, we are missing $BF$, but fortunately we know that $BF = CD = 10.4$. Then,

$$P(\Delta DBF) = BD + BF + DF = 12 + 10.4 + 11 = 33.4.$$

$$\boldsymbol{P(\Delta AEF)} = 2 \cdot P(\Delta DBF) = 2 \cdot 33.4 = \mathbf{66.8\ cm.}$$

**Example 11.2:** Given a triangle with vertices at $(1, 4), (5, -2), (-2, 3)$, what is the perimeter of the triangle after dilation? Round to 2 decimals.

The coordinates of the preimage are $(1, 4), (5, -2), (-2, 3)$.

The dilation doubles all $x$- values and triples all $y$-values. So, the coordinates of the image are: $(2, 12), (10, -6), (-4, 9)$

**Distances:**

From $(2, 12)$ to $(10, -6)$, the distance is: $\sqrt{(10-2)^2 + (-6-12)^2} = \sqrt{388} \approx 19.70.$

From $(10, -6)$ to $(-4, 9)$, the distance is: $\sqrt{(-4-10)^2 + \big(9-(-6)\big)^2} = \sqrt{421} \approx 20.52.$

From $(-4, 9)$ to $(2, 12)$, the distance is: $\sqrt{(2-(-4))^2 + (12-9)^2} = \sqrt{45} \approx 6.71.$

**Perimeter** $\approx 19.70 + 20.52 + 6.71 = \mathbf{46.93}$

**Example 11.3:** If a triangle has lengths of 8, 9, and 15 m, what is its area? Round to 2 decimals.

Using Heron's formula,

$$s = \frac{8+9+15}{2} = 16$$

$$\boldsymbol{A} = \sqrt{16(16-8)(16-9)(16-15)} = \sqrt{16 \cdot 8 \cdot 7 \cdot 1} = \sqrt{896} \approx \mathbf{29.93\ m^2}$$

ADVANCED

# Geometry
# More on the Area of a Triangle

## Trigonometric Formulas

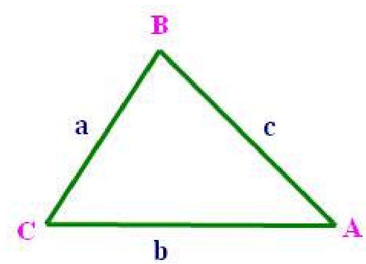
The following formulas for the area of a triangle come from trigonometry. Which one is used depends on the information available:

**Two angles and a side:**

$$A = \frac{1}{2} \cdot \frac{a^2 \cdot \sin B \cdot \sin C}{\sin A} = \frac{1}{2} \cdot \frac{b^2 \cdot \sin A \cdot \sin C}{\sin B} = \frac{1}{2} \cdot \frac{c^2 \cdot \sin A \cdot \sin B}{\sin C}$$

**Two sides and an angle:**

$$A = \frac{1}{2}\, ab\ \sin C = \frac{1}{2}\, ac\ \sin B = \frac{1}{2}\, bc\ \sin A$$

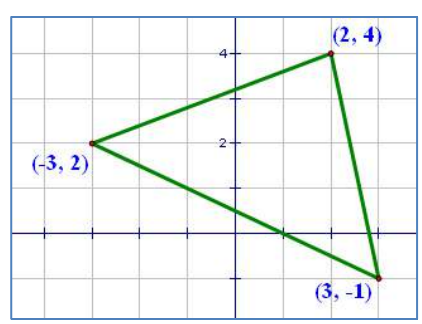



## Coordinate Geometry

If the three vertices of a triangle are displayed in a coordinate plane, the formula below, using a determinant, will give the area of a triangle.

Let the three points in the coordinate plane be: $(x_1, y_1),\ (x_2, y_2),\ (x_3, y_3)$. Then, the area of the triangle is one half of the absolute value of the determinant below:

$$A = \frac{1}{2} \cdot \left| \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right|$$

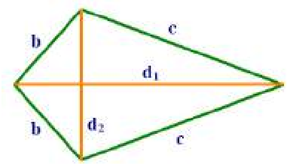
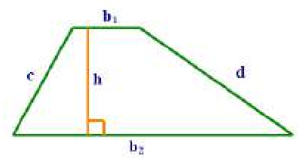
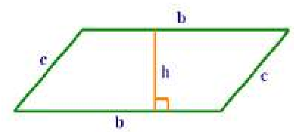
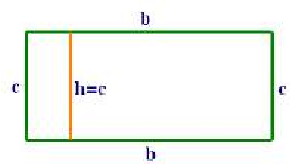
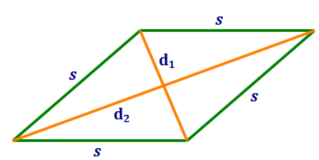
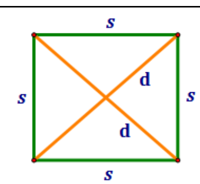



**Example:** For the triangle in the figure at right, the area is:

$$A = \frac{1}{2} \cdot \left| \begin{vmatrix} 2 & 4 & 1 \\ -3 & 2 & 1 \\ 3 & -1 & 1 \end{vmatrix} \right|$$

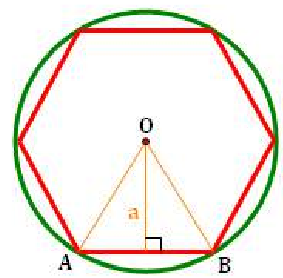
$$= \frac{1}{2} \cdot \left| \left( 2 \begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} - 4 \begin{vmatrix} -3 & 1 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} -3 & 2 \\ 3 & -1 \end{vmatrix} \right) \right| = \frac{1}{2} \cdot 27 = \frac{27}{2}$$

# Geometry
## Perimeter and Area of Quadrilaterals

| Name | Illustration | Perimeter | Area |
|---|---|---|---|
| **Kite** |  | $P = 2b + 2c$ | $A = \frac{1}{2}(d_1 d_2)$ |
| **Trapezoid** |  | $P = b_1 + b_2 + c + d$ | $A = \frac{1}{2}(b_1 + b_2)h$ |
| **Parallelogram** |  | $P = 2b + 2c$ | $A = bh$ |
| **Rectangle** |  | $P = 2b + 2c$ | $A = bh$ |
| **Rhombus** |  | $P = 4s$ | $A = bh = \frac{1}{2}(d_1 d_2)$ |
| **Square** |  | $P = 4s$ | $A = s^2 = \frac{1}{2}(d^2)$ |

# Geometry
# Perimeter and Area of Regular Polygons

## Definitions – Regular Polygons




- The **center** of a polygon is the center of its circumscribed circle. Point $\boldsymbol{O}$ is the center of the hexagon at right.
- The **radius** of the polygon is the radius of its circumscribed circle. $\overline{\boldsymbol{OA}}$ and $\overline{\boldsymbol{OB}}$ are both radii of the hexagon at right.
- The **apothem** of a polygon is the distance from the center to the midpoint of any of its sides. **a** is the apothem of the hexagon at right.
- The **central angle** of a polygon is an angle whose vertex is the center of the circle and whose sides pass through consecutive vertices of the polygon. In the figure above, $\angle \boldsymbol{AOB}$ is a central angle of the hexagon.

## Area of a Regular Polygon

$$A = \frac{1}{2}aP$$

where, $a$ is the apothem of the polygon

$P$ is the perimeter of the polygon

## Perimeter and Area of Similar Figures

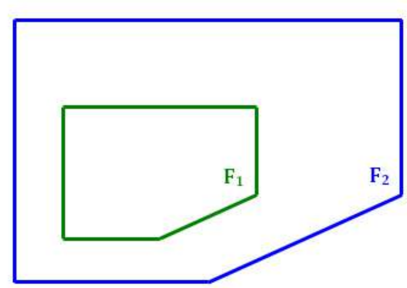
Let $\mathbf{k}$ be the scale factor relating two similar geometric figures $\mathbf{F_1}$ and $\mathbf{F_2}$ such that $\mathbf{F_2} = \mathbf{k} \cdot \mathbf{F_1}$.

Then,

$$\frac{\textbf{Perimeter of } \mathbf{F_2}}{\textbf{Perimeter of } \mathbf{F_1}} = \mathbf{k}$$

and

$$\frac{\textbf{Area of } \mathbf{F_2}}{\textbf{Area of } \mathbf{F_1}} = \mathbf{k^2}$$



**Example 11.4:** The scale factor of two similar polygons is $5:2$. The perimeter of the larger polygon is $40\text{ ft}$ and its area is $100\text{ ft}^2$. What are the perimeter and area of the smaller polygon?

Scale factors and perimeter are both linear measures.

For perimeter, we have the proportion:

$$\frac{5}{2} = \frac{40}{P}$$

$$P = \frac{40 \cdot 2}{5} = \mathbf{16\ ft}$$

For area, we have the proportion:

$$\left(\frac{5}{2}\right)^2 = \frac{100}{A}$$

$$\frac{25}{4} = \frac{100}{A}$$

$$A = \frac{100 \cdot 4}{25} = \mathbf{16\ ft^2}$$

**Example 11.5:** Two similar figures have areas of 80 and 180. Find the ratio of their perimeters:

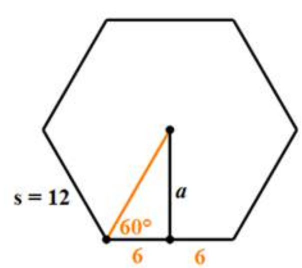
Area ratios are the squares of the corresponding linear ratios. Perimeters are linear measures. Therefore, we have the proportion:

Area ratio: $k^2 = \frac{80}{180} = \frac{4}{9}$

The small figure's area is in the numerator of the above fraction and the large figure's area is in the denominator of the above fraction. Then,

Perimeter ratio: $k = \sqrt{k^2} = \sqrt{\frac{4}{9}} = \frac{2}{3}$

**Example 11.6:** What is the length of the apothem of a regular hexagon with side length 12 cm?

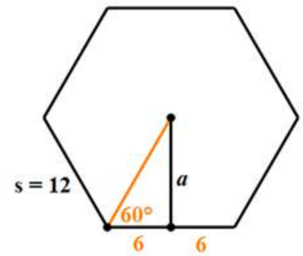



The apothem splits the bottom side of the hexagon in half, i.e., into two segments of length 6.

Each interior angle in a regular hexagon is 120°, so half of that is 60°. This gives us a 30°-60°-90° triangle, with one side of the triangle being the apothem. We can calculate, then:

$$a = 6 \cdot \sqrt{3} = \mathbf{6\sqrt{3}\ cm}$$

**Example 11.7:** What is the area of a regular hexagon with side length 12 cm?




The perimeter of this regular hexagon is:

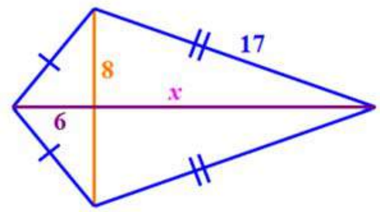
$$P = (6 \text{ sides}) \cdot (12 \text{ cm per side}) = 72 \text{ cm}$$

The length of the apothem is $6\sqrt{3}$ from the previous example.

The area of the regular hexagon in the diagram is:

$$A = \frac{1}{2}aP = \frac{1}{2}(6\sqrt{3}) \cdot 72 = \mathbf{216\sqrt{3}\ units^2}$$

**Example 11.8:** What is the area of the kite in the diagram? All measurements are in inches.




We need the lengths of the diagonals of the kite.

The vertical diagonal has length $d_1 = 8 + 8 = 16$.

To find the horizontal diagonal, we need the help of Pythagoras.

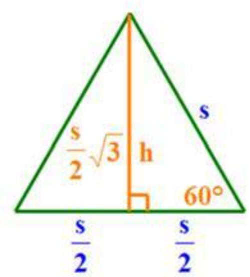
$$x^2 + 8^2 = 17^2 \quad \rightarrow \quad x = 15$$

$$d_2 = 15 + 6 = 21$$

Finally, we have:

$$A = \frac{1}{2}d_1 d_2 = \frac{1}{2}(16)(21) = \mathbf{168\ in^2}$$

**Example 11.9:** Derive a formula for the area of an equilateral triangle with side length $s$.




Let the height of the equilateral triangle be $h$. We need to find $b$.

We draw an altitude from the top of the triangle to the base, creating a pair of congruent interior triangles. This results in 30°-60°-90° triangles, each with base $\frac{s}{2}$. The length of the height, then, is $\frac{s}{2}\sqrt{3}$.

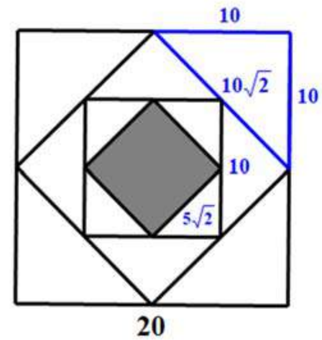
The length of the whole base is: $2 \cdot \frac{s}{2} = s$.

Finally,

$$A = \frac{1}{2}bh = \frac{1}{2}s \cdot \left(\frac{s}{2}\sqrt{3}\right) = \frac{\sqrt{3}}{4}s^2$$

**Example 11.10:** Successive squares are formed by joining the midpoints of each side. If the outermost square has a side length of 20 m, what is the area of the shaded square?

Notice that we are able to create a 45°-45°-90° triangle in the upper right corner of the diagram. Working in from the outer square to the next inner square, we see that the side lengths of the squares shrink by a factor of $\sqrt{2}$.




Since the side lengths shrink by a factor of $\sqrt{2}$, the areas of successive squares must shrink by a factor of $\left(\sqrt{2}\right)^2 = 2$.

The outer square has an area of: $A = 20^2 = 400 \text{ units}^2$.

The shaded square is three squares in from the outer square, so its area must be:

$$A = 400 \cdot \left(\frac{1}{2}\right)^3 = \mathbf{50\ m^2}$$

**Example 11.11:** If $\Delta ABC \sim \Delta DEF, AC = 22 \textit{ and } DF = 55$, what is the ratio of the area of $\Delta ABC$ to the area of $\Delta DEF$.

The ratio of the areas is the square of the ratio of the linear measures.

$$r = \frac{\Delta ABC \text{ area}}{\Delta DEF \text{ area}} = \left(\frac{22}{55}\right)^2 = \left(\frac{2}{5}\right)^2 = \frac{\mathbf{4}}{\mathbf{9}}$$

**Example 11.12:** If the ratios of the areas of two similar polygons is $\frac{121}{196}$, what is the ratio of their perimeters?

The ratio of the areas is the square of the ratio of the linear measures. So, the ratio of linear measures (e.g., perimeter) is the square root of the ratio of the areas.

$$r = \sqrt{\frac{121}{196}} = \frac{\mathbf{11}}{\mathbf{14}}$$

# Geometry
# Circle Lengths and Areas

## Circumference and Area

$C = 2\pi \cdot r$ is the **circumference** (i.e., the perimeter) of the circle.

$A = \pi r^2$ is the **area** of the circle.

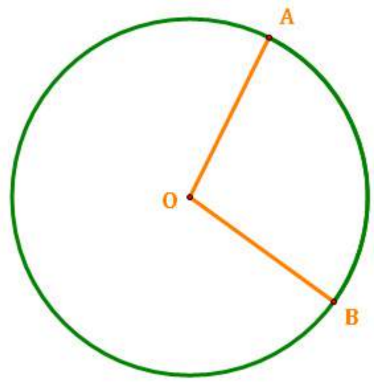
where: $r$ is the radius of the circle.

## Length of an Arc on a Circle

A common problem in the geometry of circles is to measure the length of an arc on a circle.

**Definition:** An arc is a segment along the circumference of a circle.

$$arc\ length = \frac{m\widehat{AB}}{360} \cdot C$$




where: $m\angle\widehat{AB}$ is the measure (in degrees) of the arc. Note that this is also the measure of the central angle $\angle AOB$.

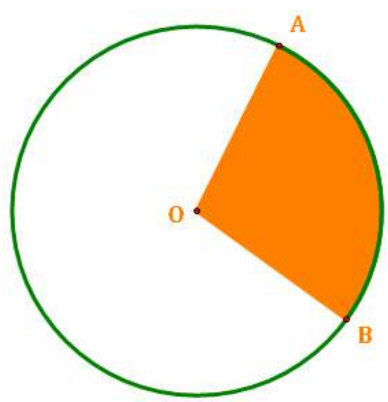
$C$ is the circumference of the circle.

## Area of a Sector of a Circle

Another common problem in the geometry of circles is to measure the area of a sector a circle.

**Definition:** A sector is a region in a circle that is bounded by two radii and an arc of the circle.

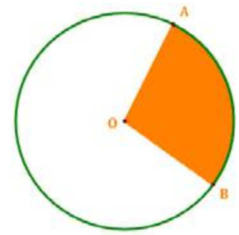
$$sector\ area = \frac{m\widehat{AB}}{360} \cdot A$$




where: $m\angle\widehat{AB}$ is the measure (in degrees) of the arc. Note that this is also the measure of the central angle $\angle AOB$.

$A$ is the area of the circle.

**Example 11.13:** What is the area of the shaded region if $m\angle \mathrm{AOC} = 95°$ and $m\,\widehat{\mathrm{AB}} = 53\pi$ m?

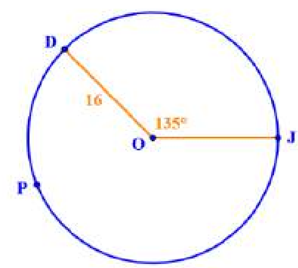



The length of the arc is $\dfrac{360-95}{360} = \dfrac{53}{72}$ of the circumference of the circle.

$$C = 53\pi \div \frac{53}{72} = 72\pi = 2\pi r \quad \rightarrow \quad r = 36$$

$$A_{region} = \frac{53}{72} \cdot A_{circle} = \frac{53}{72} \cdot \pi r^2 = \frac{53}{72} \cdot \pi \cdot 36^2 = \mathbf{954\pi\ m^2}$$

**Example 11.14:** What is the length of major arc $\widehat{\mathrm{DPJ}}$ if $m\angle \mathrm{DOJ} = 135°$ and the diameter of the circle is 16 meters.



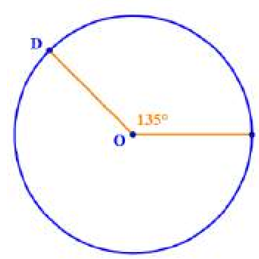

The circumference of the circle is: $C = \pi d = 16\pi$ m.

$\widehat{\mathrm{DPJ}}$ has the same measure as the central angle subtended by it.

So, $m\,\widehat{\mathrm{DPJ}} = 360° - 135° = 225°$. Then,

$$\textbf{length of } \widehat{\mathbf{DPJ}} = \frac{225}{360} \cdot 16\pi = \mathbf{10\ m.}$$

**Example 11.15:** Find the length of minor arc $\widehat{\mathrm{DJ}}$ if $m\angle \mathrm{DOJ} = 135°$ and the area of the circle is $25\pi\ \mathrm{cm}^2$.




$A = \pi r^2 = 25\pi \quad \rightarrow \quad r = 5$

$C = 2\pi r = 2 \cdot \pi \cdot 5 = 10\pi$ m.

$\widehat{\mathrm{DJ}}$ has the same measure as the central angle subtended by it.

So, $m\,\widehat{\mathrm{DJ}} = 135°$, or $\dfrac{135}{360} = \dfrac{3}{8}$ of the circumference of the circle.

The length of major arc $\widehat{DJ}$, then is:

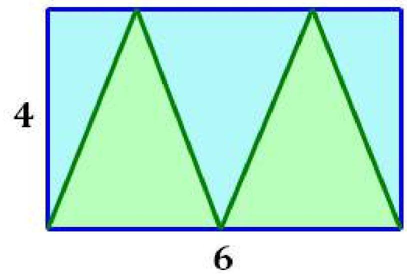
$$\textbf{length of } \widehat{\mathbf{DJ}} = \frac{3}{8} \cdot 10\pi = \mathbf{\frac{15}{4}\ cm.}$$

# Geometry
## Area of Composite Figures

To calculate the area of a figure that is a composite of shapes, consider each shape separately.

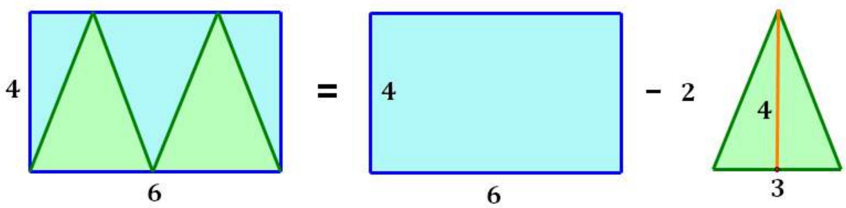
### Example 11.16:

Calculate the area of the blue region in the figure to the right.




To solve this:

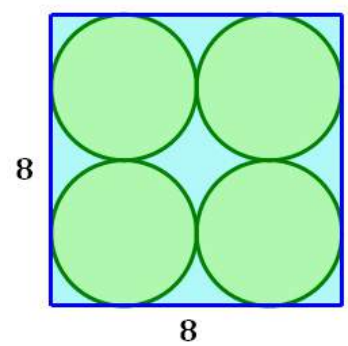
- Recognize that the figure is the composite of a rectangle and two triangles.
- Disassemble the composite figure into its components.
- Calculate the area of the components.
- Subtract to get the area of the composite figure.




$$\textit{Area of Region} = (4\cdot 6) - 2\left(\frac{1}{2}\cdot 4\cdot 3\right) = 24 - 12 = 12$$

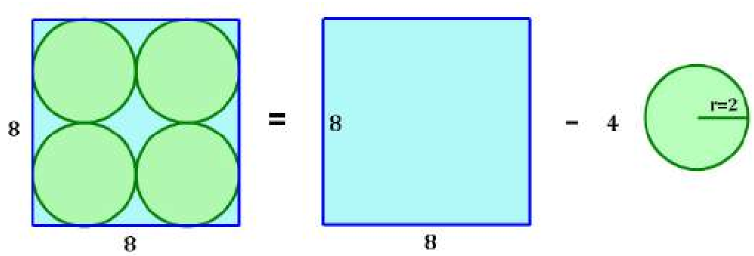
### Example 11.17:

Calculate the area of the blue region in the figure to the right.



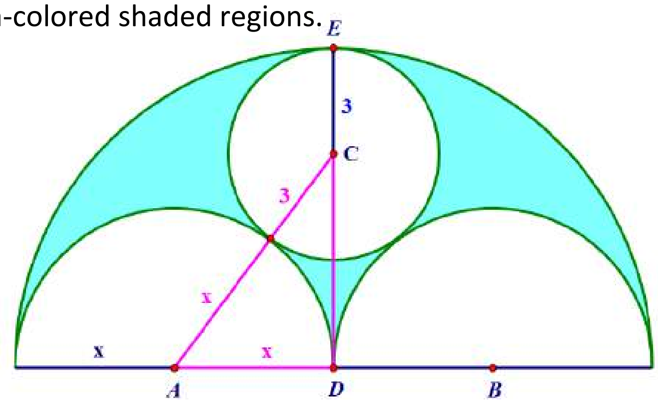

To solve this:

- Recognize that the figure is the composite of a square and a circle.
- Disassemble the composite figure into its components.
- Calculate the area of the components.
- Subtract to get the area of the composite figure.




$$\textit{Area of Region} = 8^2 - 4(\pi\cdot 2^2) = 64 - 16\pi \sim 13.73$$

**Example 11.18:** Two congruent semicircles and a full circle are arranged inside a large semicircle as shown in the diagram. The radius of the smaller semicircles is $x$. The radius of the full circle is 3. Find the total area of the aqua-colored shaded regions.




First, let's find $x$:

$\Delta ADC$ is a right triangle, so using the Pythagorean Theorem:

$AD^2 + CD^2 = AC^2$

$x^2 + CD^2 = (x+3)^2$

$x^2 + CD^2 = x^2 + 6x + 9$

$CD^2 = 6x + 9$

We also know that $2x$ is the radius of the large (outer) semicircle.

Then, on line segment $DE$ (also a radius of the large semicircle):

$2x = CD + 3$

$CD = 2x - 3$

$CD^2 = (2x-3)^2 = 4x^2 - 12x + 9$

Then, set the two expressions for $CD^2$ equal to each other:

$4x^2 - 12x + 9 = 6x + 9$

$4x^2 - 18x = 0$

$2x(2x - 9) = 0$

$x = 0, \dfrac{9}{2}$

The answer $x = 0$ makes no sense, so we must have: $x = \dfrac{9}{2}$

---

The shaded area of the diagram is developed as follows:

$$A_{\text{large semicircle}} = \frac{1}{2}\pi\left(\frac{9}{2}+\frac{9}{2}\right)^2 = \frac{81}{2}\pi$$

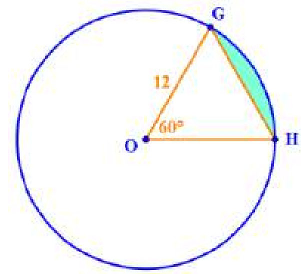
$$A_{\text{small semicircle}} = \frac{1}{2}\pi\left(\frac{9}{2}\right)^2 = \frac{81}{8}\pi$$

$$A_{\text{full circle}} = \pi(3)^2 = 9\pi$$

$$A_{\text{shaded}} = A_{\text{large semicircle}} - 2\cdot A_{\text{small semicircle}} - A_{\text{full circle}}$$

$$\boldsymbol{A_{\text{shaded}}} = \frac{81}{2}\pi - 2\cdot\frac{81}{8}\pi - 9\pi = \boldsymbol{\frac{45}{4}\pi}$$

**Example 11.19:** What is the area of the region shaded in the diagram? All measurements are in feet.

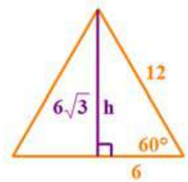



Shaded area = sector area – triangle area.

Sector area $= \frac{60}{360} \cdot \pi \cdot 12^2 = 24\pi$.

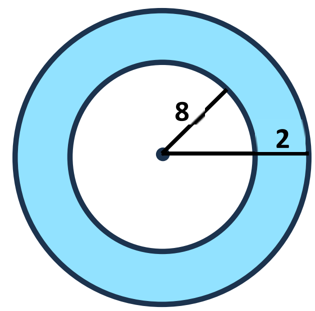
The orange triangle is equilateral with sides of length 12 ft. This allows us to complete the its measurements as shown below. Then,

Triangle area $= \frac{1}{2}\text{bh} = \frac{12 \cdot 6\sqrt{3}}{2} = 36\sqrt{3}$




**Shaded area** $= \mathbf{24\pi - 36\sqrt{3}}$ **ft²**

**Example 11.20:** What is the area of the annulus shaded in the diagram?




An annulus is the area between two circles, so its area is the difference of the areas of the two circles:

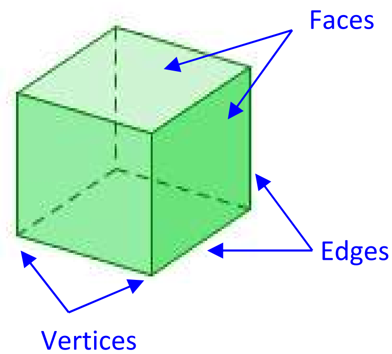
$A_{\text{large}} = \pi r_{\text{large}}{}^2 = \pi \cdot (8+2)^2 = 100\pi$

$A_{\text{small}} = \pi r_{\text{small}}{}^2 = \pi \cdot 8^2 = 64\pi$

$A_{\text{annulus}} = A_{\text{large}} - A_{\text{small}} = 100\pi - 64\pi = \mathbf{36\pi\ units^2}$

# Geometry
# Polyhedra

## Definitions




- A **Polyhedron** is a 3-dimensional solid bounded by a series of polygons.
- **Faces** are the polygons that bound the polyhedron.
- An **Edge** is the line segment at the intersection of two faces.
- A **Vertex** is a point at the intersection of two edges.
- A **Regular** polyhedron is one in which all of the faces are the same regular polygon.
- A **Convex** Polyhedron is one in which all diagonals are contained within the interior of the polyhedron. A **Concave** polyhedron is one that is not convex.
- A **Cross Section** is the intersection of a plane with the polyhedron.

## Euler's Theorem

Let: $F$ = the number of faces of a polyhedron.
$V$ = the number of vertices of a polyhedron.
$E$ = the number of edges of a polyhedron.

Then, for any polyhedron that does not intersect itself,

$$\boldsymbol{F + V = E + 2}$$

**Example 12.1: Euler's Theorem**

The cube above has …

- 6 faces
- 8 vertices
- 12 edges

$$\mathbf{6 + 8 = 12 + 2} \quad \checkmark$$

## Calculating the Number of Edges

The number of edges of a polyhedron is one-half the number of sides in the polygons it comprises. Each side that is counted in this way is shared by two polygons; simply adding all the sides of the polygons, therefore, double counts the number of edges on the polyhedron.

**Example 12.2:** Consider a soccer ball. It is polyhedron made up of 20 hexagons and 12 pentagons. Then the number of edges is:

$$\boldsymbol{E} = \frac{1}{2} \cdot [(20 \cdot 6) + (12 \cdot 5)] = \mathbf{90}$$

Note: use of the $\frac{1}{2}$ factor reflects each edge being counted twice in the sides of the polygons.

# Geometry
# A Hole in Euler's Theorem

**Topology** is a branch of mathematics that studies the properties of objects that are preserved through manipulation that does not include tearing. An object may be stretched, twisted and otherwise deformed, but not torn. In this branch of mathematics, a donut is equivalent to a coffee cup because both have one hole; you can deform either the cup or the donut and create the other, like you are playing with clay.

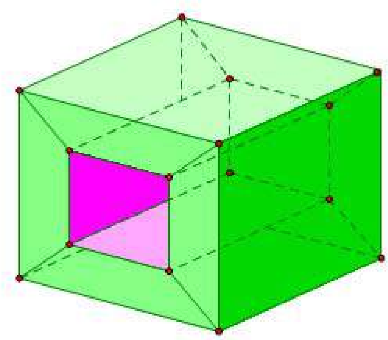
All of the usual polyhedra have no holes in them, so Euler's Equation holds. What happens if we allow the polyhedra to have holes in them? That is, what if we consider topological shapes different from the ones we normally consider?

## Euler's Characteristic

When Euler's Equation is rewritten as $\boldsymbol{F - E + V = 2}$, the left-hand side of the equation is called the **Euler Characteristic**.

*The Euler Characteristic of a shape is:* $\boldsymbol{F - E + V}$

## Generalized Euler's Theorem



Let: $F$ = the number of faces of a polyhedron.

$V$ = the number of vertices of a polyhedron.

$E$ = the number of edges of a polyhedron.

$g$ = the number of holes in the polyhedron. $g$ is called the **genus** of the shape.

Then, for any polyhedron that does not intersect itself,

$$\boldsymbol{F - E + V = 2 - 2g}$$

Note that the value of Euler's Characteristic can be negative if the shape has more than one hole in it (i.e., if $g \geq 2$)!

**Example 12.3:**

The cube with a tunnel in it has …

$F = 16, E = 32, V = 16$

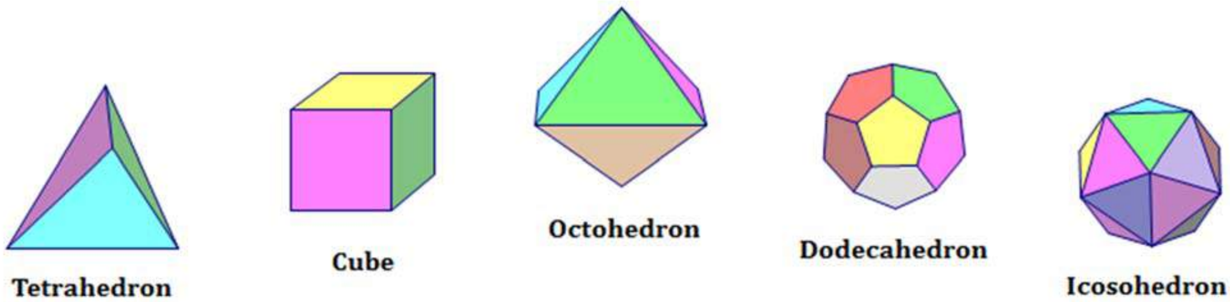
so, $\boldsymbol{F - E + V = 0}$

Then, $0 = 2 - 2g$

$\boldsymbol{g = 1}$ **hole** ✓

# Geometry
# Platonic Solids

A **Platonic Solid** is a convex regular polyhedron with faces composed of congruent convex regular polygons. There five of them:




## Key Properties of Platonic Solids

It is interesting to look at the key properties of these regular polyhedra.

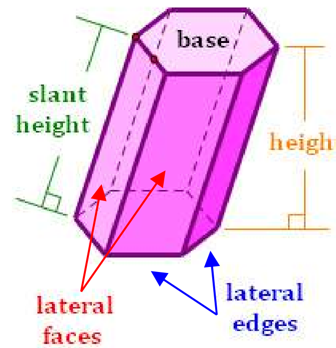
| Name | Faces | Vertices | Edges | Type of Face |
|---|---|---|---|---|
| **Tetrahedron** | 4 | 4 | 6 | Triangle |
| **Cube** | 6 | 8 | 12 | Square |
| **Octahedron** | 8 | 6 | 12 | Triangle |
| **Dodecahedron** | 12 | 20 | 30 | Pentagon |
| **Icosahedron** | 20 | 12 | 30 | Triangle |

Notice the following patterns in the table:

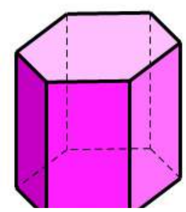
- All of the numbers of faces are even. Only the cube has a number of faces that is not a multiple of 4.
- All of the numbers of vertices are even. Only the octahedron has a number of faces that is not a multiple of 4.
- The number of faces and vertices seem to alternate (e.g., cube 6-8 vs. octahedron 8-6).
- All of the numbers of edges are multiples of 6.
- There are only three possibilities for the numbers of edges – 6, 12 and 30.
- The faces are one of: regular triangles, squares or regular pentagons.

# Geometry
# Prisms

## Definitions




- A **Prism** is a polyhedron with two congruent polygonal faces that lie in parallel planes.
- The **Bases** are the parallel polygonal faces.
- The **Lateral Faces** are the faces that are not bases.
- The **Lateral Edges** are the edges between the lateral faces.
- The **Slant Height** is the length of a lateral edge. Note that all lateral edges are the same length.
- The **Height** is the perpendicular length between the bases.
- A **Right Prism** is one in which the angles between the bases and the lateral edges are right angles. Note that in a right prism, the height and the slant height are the same.
- An **Oblique Prism** is one that is not a right prism.
- The **Surface Area** of a prism is the sum of the areas of all its faces.
- The **Lateral Area** of a prism is the sum of the areas of its lateral faces.



Right Hexagonal Prism

## Surface Area and Volume of a Right Prism

**Surface Area:** $SA = Ph + 2B$

**Lateral SA:** $SA = Ph$

**Volume:** $V = Bh$

where, $P = the\ perimeter\ of\ the\ base$
$h = the\ height\ of\ the\ prism$
$B = the\ area\ of\ the\ base$

## Cavalieri's Principle

If two solids have the same height and the same cross-sectional area at every level, then they have the same volume. This principle allows us to derive a formula for the volume of an oblique prism from the formula for the volume of a right prism.

## Surface Area and Volume of an Oblique Prism

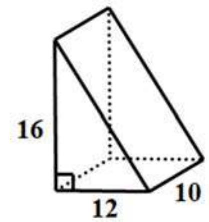
**Surface Area:** $SA = LSA + 2B$

**Volume:** $V = Bh$

where, $LSA = the\ lateral\ surface\ area$
$h = the\ height\ of\ the\ prism$
$B = the\ area\ of\ the\ base$

> The lateral surface area of an oblique prism is the sum of the areas of the faces, which must be calculated individually.

**Example 12.4:** Find the volume of the triangular prism.



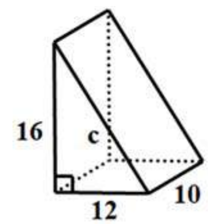

This is a right prism, with a triangle for a base. First, find $B$, the area of the triangular base.

$$B = \frac{1}{2}(12)(16) = 96$$

The height is the length perpendicular to the base. So, $h = 10$.

Finally, $\boldsymbol{V} = Bh = 96 \cdot 10 = \mathbf{960}$

**Example 12.5:** Find the lateral surface area and the total surface area of the triangular prism.




The formula for the surface area of a prism is:

$SA = \text{Ph} + 2\text{B}$, where P is the perimeter of the base, h is the height of the prism, and B is the area of one base. Ph is also called the lateral surface area of the prism.

The height is the length of a segment perpendicular to the base. So, $h = 10$.

The base is a triangle, so we need to calculate the length of its hypotenuse in order to calculate the perimeter, P. Pythagoras will help us with this; the hypotenuse has length:

$$c = \sqrt{12^2 + 16^2} = 20$$

We can now calculate: $\text{P} = 12 + 16 + 20 = 48$. Therefore,

$$\boldsymbol{LSA} = P \cdot h = 48 \cdot 10 = \mathbf{480}$$

The area of one triangular base of the prism is: $B = 96$ from the prior example.

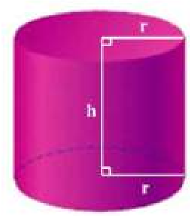
The total surface area of the triangular prism, then, is:

$$\boldsymbol{SA} = 480 + 2 \cdot 96 = \mathbf{672}.$$

# Geometry
# Cylinders

## Definitions

- A **Cylinder** is a figure with two congruent circular bases in parallel planes.
- The **Axis** of a cylinder is the line connecting the centers of the circular bases.
- A cylinder has only one **Lateral Surface**. When deconstructed, the lateral surface of a cylinder is a rectangle with length equal to the circumference of the base.
- There are no **Lateral Edges** in a cylinder.
- The **Slant Height** is the length of the lateral side between the bases. Note that all lateral distances are the same length. The slant height has applicability only if the cylinder is oblique.
- The **Height** is the perpendicular length between the bases.




- A **Right Cylinder** is one in which the angles between the bases and the lateral side are right angles. Note that in a right cylinder, the height and the slant height are the same.
- An **Oblique Cylinder** is one that is not a right cylinder.

- The **Surface Area** of a cylinder is the sum of the areas of its bases and its lateral surface.
- The **Lateral Area** of a cylinder is the areas of its lateral surface.

## Surface Area and Volume of a Right Cylinder

**Surface Area:** $SA = Ch + 2B$
$= 2\pi rh + 2\pi r^2$

**Lateral SA:** $SA = Ch = 2\pi rh$

**Volume:** $V = Bh = \pi r^2 h$

where, $C = \text{the circumference of the base}$
$h = \text{the height of the cylinder}$
$B = \text{the area of the base}$
$r = \text{the radius of the base}$

## Surface Area and Volume of an Oblique Cylinder

**Surface Area:** $SA = Pl + 2B$

**Volume:** $V = Bh = \pi r^2 h$

where, $P = \text{the perimeter of a}$ right section* $\text{of the cylinder}$
$l = \text{the slant height of the cylinder}$
$h = \text{the height of the cylinder}$
$B = \text{the area of the base}$
$r = \text{the radius of the base}$

* A **right section** of an oblique cylinder is a cross section perpendicular to the axis of the cylinder.

**Example 12.6:** Find the volume of a right cylinder that has a diameter of 6 cm and a height of 10 cm.


For a cylinder, $V = \pi r^2 h$. In this case, $r = 6 \div 2 = 3, h = 10$.

$V = \pi r^2 h = \pi \cdot 3^2 \cdot 10 = 90\pi \text{ cm}^3$

**Example 12.7:** Find the lateral surface area and the total surface area of a right cylinder that has a diameter of 6 cm and a height of 10 cm

The formula for the surface area of a cylinder is:

$SA = 2\pi rh + 2\pi r^2$, where $r$ is the radius of the base, $h$ is the height of the cylinder, and $\pi r^2$ is the area of one base. $2\pi rh$ is also called the lateral surface area of the right cylinder.

The radius is half the diameter: $r = 6 \div 2 = 3$

The height is the length of the side perpendicular to the base. So, $h = 10$.

Therefore,

$\boldsymbol{LSA} = 2\pi rh = 2\pi \cdot 3 \cdot 10 = \mathbf{60\pi}$

The area of one circular base of the cylinder is: $\pi r^2 = \pi(3)^2 = 9\pi$.

The total surface area of the right cylinder, then, is:

$\boldsymbol{SA} = 60\pi + 2 \cdot 9\pi = \mathbf{78\pi}$.

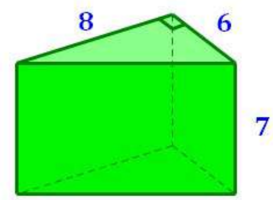
# Geometry
## Surface Area by Decomposition

Sometimes the student is asked to calculate the surface are of a prism that does not quite fit into one of the categories for which an easy formula exists. In this case, the answer may be to decompose the prism into its component shapes, and then calculate the areas of the components. Note: this process also works with cylinders and pyramids.

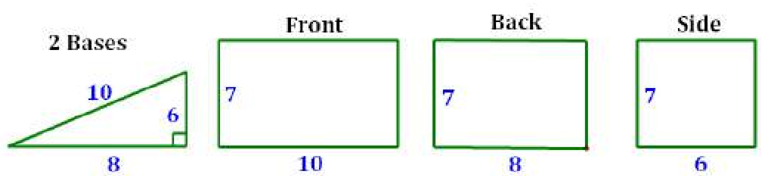
### Decomposition of a Prism

To calculate the surface area of a prism, decompose it and look at each of the prism's faces individually.

**Example 12.8:** Calculate the surface area of the triangular prism:




To do this, first notice that we need the value of the hypotenuse of the base. Use the Pythagorean Theorem or Pythagorean Triples to determine the missing value is $\mathbf{10}$. Then, decompose the figure into its various faces:




The surface area, then, is calculated as:

$$SA = (2\ Bases) + (Front) + (Back) + (Side)$$

$$\boldsymbol{SA} = 2 \cdot \left(\frac{1}{2} \cdot 6 \cdot 8\right) + (10 \cdot 7) + (8 \cdot 7) + (6 \cdot 7) = \mathbf{216}$$

### Decomposition of a Right Cylinder

**Example 12.9:** Calculate the surface area of the cylinder:

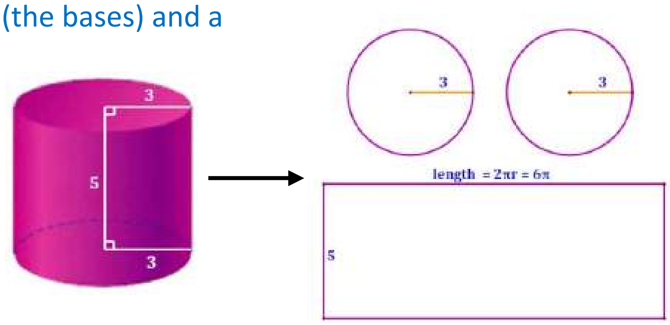
The cylinder is decomposed into two circles (the bases) and a rectangle (the lateral face).

The surface area, then, is calculated as:

$$SA = (2\ tops) + (lateral\ face)$$

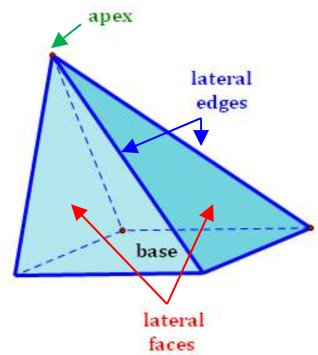
$$SA = 2 \cdot (\pi \cdot 3^2) + (6\pi \cdot 5)$$

$$\boldsymbol{SA = 48\pi \approx 150.80}$$



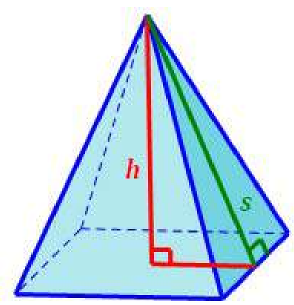
# Geometry
# Pyramids

## Pyramids




- A **Pyramid** is a polyhedron in which the base is a polygon and the lateral sides are triangles with a common vertex.
- The **Base** is a polygon of any size or shape.
- The **Lateral Faces** are the faces that are not the base.
- The **Lateral Edges** are the edges between the lateral faces.
- The **Apex** of the pyramid is the intersection of the lateral edges. It is the point at the top of the pyramid.
- The **Slant Height** of a regular pyramid is the altitude of one of the lateral faces.
- The **Height** is the perpendicular length between the base and the apex.

- A **Regular Pyramid** is one in which the lateral faces are congruent triangles. The height of a regular pyramid intersects the base at its center.
- An **Oblique Pyramid** is one that is not a right pyramid. That is, the apex is not aligned directly above the center of the base.




- The **Surface Area** of a pyramid is the sum of the areas of all its faces.
- The **Lateral Area** of a pyramid is the sum of the areas of its lateral faces.

## Surface Area and Volume of a Regular Pyramid

**Surface Area:** $SA = \frac{1}{2}Ps + B$

**Lateral SA:** $SA = \frac{1}{2}Ps$

**Volume:** $V = \frac{1}{3}Bh$

where, $P = the\ perimeter\ of\ the\ base$
$s = the\ slant\ height\ of\ the\ pyramid$
$h = the\ height\ of\ the\ pyramid$
$B = the\ area\ of\ the\ base$

## Surface Area and Volume of an Oblique Pyramid

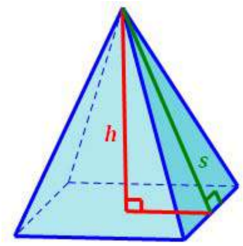
**Surface Area:** $SA = LSA + B$

**Volume:** $V = \frac{1}{3}Bh$

where, $LSA = the\ lateral\ surface\ area$
$h = the\ height\ of\ the\ pyramid$
$B = the\ area\ of\ the\ base$

The lateral surface area of an oblique pyramid is the sum of the areas of the faces, which must be calculated individually.

**Example 12.10:** Calculate the volume of the square pyramid shown if the perimeter of the base is 64 and the height is 15.



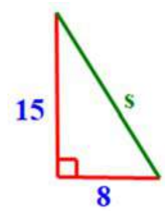

For a square pyramid, we introduce a factor of $\frac{1}{3}$ into the volume calculation, relative to a prism. The same factor is used in the calculation of a cone, relative to a cylinder. The origins of the $\frac{1}{3}$ factor come from Calculus and the fact that we are working in 3 dimensions.

If the perimeter of the base is 64, then the length of one base edge is: $64 \div 4 = 16.$

Our base is a square with area: $B = 16^2 = 256.$ $h = 15.$

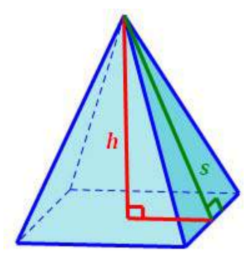
$$\boldsymbol{A} = \frac{1}{3}Bh = \frac{1}{3}(256)(15) = \mathbf{1280}$$

**Example 12.11:** Calculate the slant height of the face of the square pyramid in the previous example.




If we look inside the pyramid, we can see a triangle that has a height of length $h = 15$, a leg that is half the length of a base edge of the pyramid ($16 \div 2 = 8$) and a hypotenuse of the slant height ($s$). Use the Pythagorean Theorem, then, to determine: $\mathbf{s} = \sqrt{15^2 + 8^2} = \mathbf{17}$

**Example 12.12:** Calculate the lateral surface area and the total surface area of the square pyramid in the previous example.




The formula for the surface area of a square pyramid is:

$SA = \frac{1}{2}\text{Ps} + \text{B},$ where P is the perimeter of the base, s is the slant height of the pyramid, and B is the area of the base. $\frac{1}{2}Ps$ is also called the lateral surface area of the pyramid.

From the previous two examples, we know that $\text{P} = 64$ and $\text{s} = 17.$ Therefore,

$$\boldsymbol{LSA} = \frac{1}{2} \cdot P \cdot s = \frac{1}{2} \cdot 64 \cdot 17 = \mathbf{544}$$

The base length is 16, so the area of the base is: $B = 16^2 = 256.$

The total surface area of the square pyramid, then, is:

$$\boldsymbol{SA} = 544 + 256 = \mathbf{800}.$$

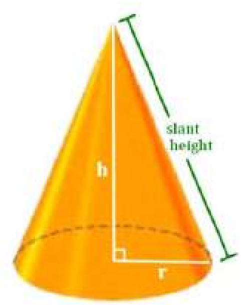
# Geometry
# Cones

## Definitions

- A **Circular Cone** is a 3-dimensional geometric figure with a circular base which tapers smoothly to a vertex (or apex). The apex and base are in different planes. Note: there is also an elliptical cone that has an ellipse as a base, but that will not be considered here.
- The **Base** is a circle.
- The **Lateral Surface** is area of the figure between the base and the apex.
- There are no **Lateral Edges** in a cone.
- The **Apex** of the cone is the point at the top of the cone.
- The **Slant Height** of a cone is the length along the lateral surface from the apex to the base.
- The **Height** is the perpendicular length between the base and the apex.

- A **Right Cone** is one in which the height of the cone intersects the base at its center.
- An **Oblique Cone** is one that is not a right cone. That is, the apex is not aligned directly above the center of the base.

- The **Surface Area** of a cone is the sum of the area of its lateral surface and its base.
- The **Lateral Area** of a cone is the area of its lateral surface.




## Surface Area and Volume of a Right Cone

**Surface Area:** $SA = \pi rs + \pi r^2$

**Lateral SA:** $SA = \pi rs$

**Volume:** $V = \frac{1}{3}Bh = \frac{1}{3}\pi r^2 h$

where, $r = \text{the radius of the base}$
$s = \text{the slant height of the cone}$
$h = \text{the height of the cone}$
$B = \text{the area of the base}$

## Surface Area and Volume of an Oblique Cone

**Surface Area:** $SA = LSA + \pi r^2$

**Volume:** $V = \frac{1}{3}Bh = \frac{1}{3}\pi r^2 h$

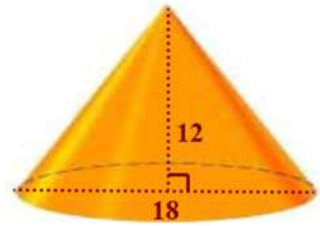
where, $LSA = \text{the lateral surface area}$
$r = \text{the radius of the base}$
$h = \text{the height of the cone}$

| |
|---|
| There is no easy formula for the lateral surface area of an oblique cone. |

**Example 12.13:** Calculate the exact volume of the right cone shown.

For a cone, $V = \frac{1}{3}\pi r^2 h$. In this case, $r = 18 \div 2 = 9, h = 12$.

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \cdot 9^2 \cdot 12 = 324\pi \text{ cm}^3$$




**Example 12.14:** Find the lateral surface area and the total surface area of a right cone shown.

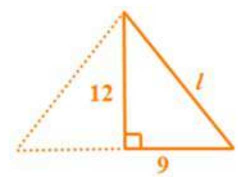
The formula for the surface area of a cone is:

$SA = \pi r l + \pi r^2$, where r is the radius of the base, $l$ is the slant height of the cone, and $\pi r^2$ is the area of the base. $\pi r l$ is also called the lateral surface area of the right cone.

The radius is half the diameter: $r = 18 \div 2 = 9$

The height is given in the diagram. $h = 12$.

A cross-sectional view of a cone is a triangle. We want to examine the right triangle in the cross-section to determine the slant height, $l$. Pythagoras will help us with this; the hypotenuse has length:




$$c = \sqrt{9^2 + 12^2} = 15$$

Therefore,

$$\boldsymbol{LSA} = \pi r l = \pi \cdot 9 \cdot 15 = \mathbf{135\pi}$$

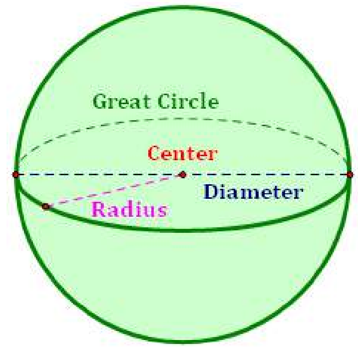
The area of the circular base of the cone is: $\pi r^2 = \pi(9)^2 = 81\pi$.

The total surface area of the right cone, then, is:

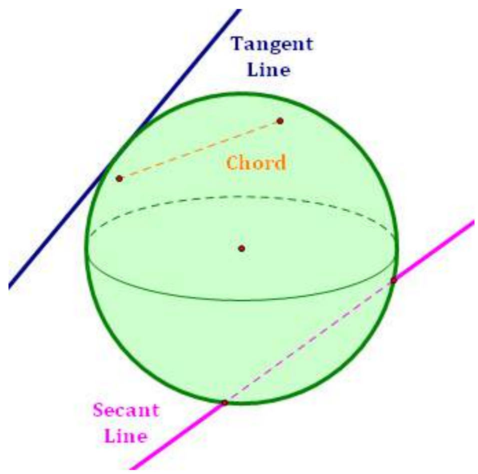
$$\boldsymbol{SA} = 135\pi + 81\pi = \mathbf{216\pi}.$$

# Geometry
# Spheres

## Definitions




- A **Sphere** is a 3-dimensional geometric figure in which all points are a fixed distance from a point. A good example of a sphere is a ball.
- **Center** – the middle of the sphere. All points on the sphere are the same distance from the center.
- **Radius** – a line segment with one endpoint at the center and the other endpoint on the sphere. The term "radius" is also used to refer to the distance from the center to the points on the sphere.
- **Diameter** – a line segment with endpoints on the sphere that passes through the center.
- **Great Circle** – the intersection of a plane and a sphere that passes through the center.
- **Hemisphere** – half of a sphere. A great circle separates a plane into two hemispheres.
- **Secant Line** – a line that intersects the sphere in exactly two points.
- **Tangent Line**– a line that intersects the sphere in exactly one point.
- **Chord** – a line segment with endpoints on the sphere that does not pass through the center.




## Surface Area and Volume of a Sphere

**Surface Area:** $SA = 4\pi r^2$

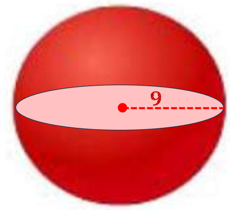
**Volume:** $V = \frac{4}{3}\pi r^3$

where, $r = the\ radius\ of\ the\ sphere$

**Example 12.15:** Find the volume of a sphere with radius 9.

The volume of a sphere is: $V = \frac{4}{3}\pi r^3$. In this case, $r = 9$.

$$\boldsymbol{V} = \frac{4}{3}\pi(9)^3 = \boldsymbol{972\pi}$$




**Example 12.16:** Find the surface area of a sphere with radius 9.

The surface area of a sphere is: $SA = 4\pi r^2$. In this case, $r = 9$.

$$\boldsymbol{SA} = 4\pi(9)^2 = \boldsymbol{324\pi}$$

Interestingly, in Calculus, you will learn that the formula for the surface area of a sphere is the derivative of the formula for the volume of a sphere. That is:

$$V = \frac{4}{3}\pi r^3 \qquad \frac{dV}{dr} = 4\pi r^2 = SA$$

This also occurs with the formulas for the area and circumference of a circle.

$$A = \pi r^2 \qquad \frac{dA}{dr} = 2\pi r = C$$

**Example 12.17:** The Earth has a volume is approximately 1.08 trillion km$^3$. Assuming that the Earth is a sphere, estimate its radius to the nearest kilometer and to the nearest mile.

The volume of a sphere is: $V = \frac{4}{3}\pi r^3$. In this case, $V = 1{,}080{,}000{,}000{,}000$.

Get your calculator ready.

$$1{,}080{,}000{,}000{,}000 = \frac{4}{3}\pi r^3$$

$$257{,}831{,}007{,}809 = r^3$$

$$\boldsymbol{r} = \sqrt[3]{257{,}831{,}007{,}809} = 6{,}364.7065 \approx \mathbf{6,365\ km}$$

$$\boldsymbol{r} = (6{,}364.7065\text{ km}) \cdot \left(0.62137119\frac{\text{km}}{\text{mile}}\right) = 3{,}954.8453 \approx \mathbf{3,955\ miles}$$

**Example 12.18:** Approximate the circumference of the Earth in kilometers and miles.

Using the radius estimates from the prior example:

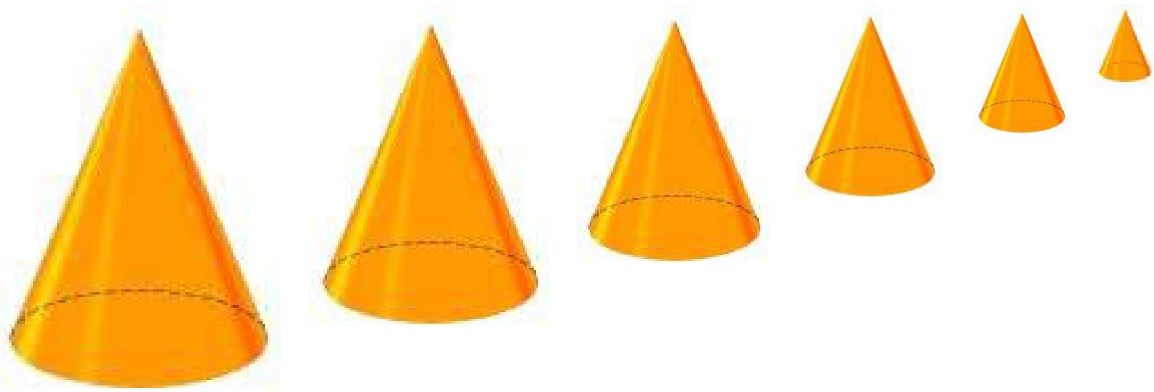
Kilometers: $\boldsymbol{C} = 2\pi r = 2\pi \cdot 6{,}364.7065\text{ km} \approx \mathbf{39,991\ km}$ or about $40{,}000$ km.

Miles: $\boldsymbol{C} = 2\pi r = 2\pi \cdot 3{,}954.8453\text{ miles} \approx \mathbf{24,849\ miles\ km}$ or about $25{,}000$ miles.

Given the accuracy of our starting values, two significant digits in our answers is about the best we can hope for. 40,000 km and 25,000 miles are real estimates of the circumference of the Earth to two significant digits.

# Geometry
## Similar Solids

**Similar Solids** have equal ratios of corresponding linear measurements (e.g., edges, radii). So, all of their key dimensions are proportional.



## Edges, Surface Area and Volume of Similar Figures

Let $\mathbf{k}$ be the scale factor relating two similar geometric solids $\mathbf{F_1}$ and $\mathbf{F_2}$ such that $\mathbf{F_2 = k \cdot F_1}$. Then, for corresponding parts of $\mathbf{F_1}$ and $\mathbf{F_2}$,

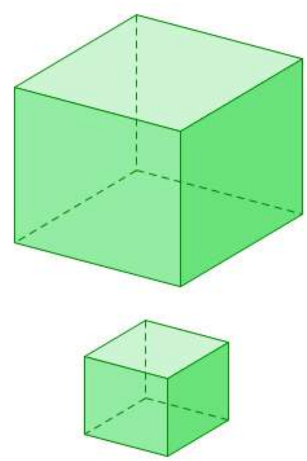
$$\frac{\textbf{Edge of } \mathbf{F_2}}{\textbf{Edge of } \mathbf{F_1}} = \mathbf{k}$$

and

$$\frac{\textbf{Surface Area of } \mathbf{F_2}}{\textbf{Surface Area of } \mathbf{F_1}} = \mathbf{k^2}$$

And

$$\frac{\textbf{Volume of } \mathbf{F_2}}{\textbf{Volume of } \mathbf{F_1}} = \mathbf{k^3}$$



These formulas hold true for any corresponding portion of the figures. So, for example:

$$\frac{\text{Total Edge Length of } F_2}{\text{Total Edge Length of } F_1} = k \qquad \frac{\text{Area of a Face of } F_2}{\text{Area ofa Face of } F_1} = k^2$$

**Example 12.19:** Two similar octahedrons have edges of lengths 4 and 12. Find the ratio of their volumes.

$$\text{Volume ratio} = (\text{Linear ratio})^3$$

$$\textbf{Volume ratio} = \left(\frac{4}{12}\right)^3 = \left(\frac{1}{3}\right)^3 = \mathbf{\frac{1}{27}}$$

**Example 12.20:** Two similar icosahedrons have volumes of 250 and 686. Find the ratio of their surface areas.

Call the linear ratio between similar objects $k$. Then:

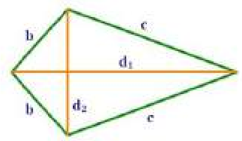
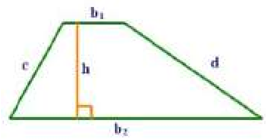
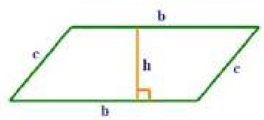
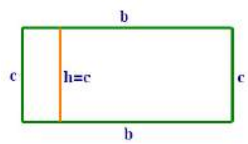
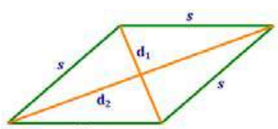
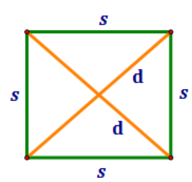
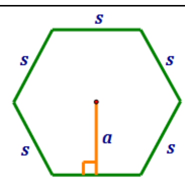
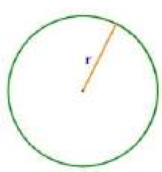
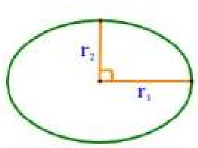
Linear measure : area : volume have relative ratios of $k : k^2 : k^3$. To get from a volume ratio to a surface area ratio, we need to take the cube root of the volume ratio (to get from volume to linear) and square the result (to get from linear to area). Alternatively, we could take the 2/3 power of the volume relativities to get the same answer.

$$\textbf{Area ratio} = \left(\sqrt[3]{\frac{250}{686}}\right)^2 = \left(\sqrt[3]{\frac{125}{343}}\right)^2 = \left(\frac{5}{7}\right)^2 = \mathbf{\frac{25}{49}}$$

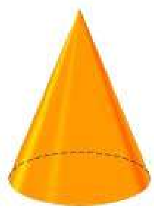
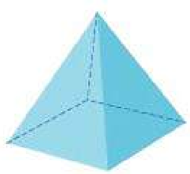
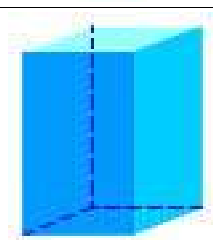
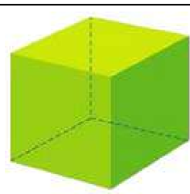
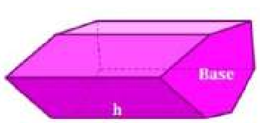
Alternative Method:

$$\textbf{Area ratio} = \left(\frac{250}{686}\right)^{\frac{2}{3}} = \mathbf{\frac{25}{49}}$$

## Geometry
## Summary of Perimeter and Area Formulas – 2D Shapes

| Shape | Figure | Perimeter | Area |
|---|---|---|---|
| **Kite** |  | $P = 2b + 2c$<br>$b, c = sides$ | $A = \frac{1}{2}(d_1 d_2)$<br>$d_1, d_2 = diagonals$ |
| **Trapezoid** |  | $P = b_1 + b_2 + c + d$<br>$b_1, b_2 = bases$<br>$c, d = sides$ | $\mathrm{A} = \frac{1}{2}(\mathrm{b}_1 + \mathrm{b}_2)\mathrm{h}$<br>$\mathrm{b}_1, \mathrm{b}_2 = \text{bases}$<br>$\mathrm{h} = \text{height}$ |
| **Parallelogram** |  | $P = 2b + 2c$<br>$b, c = sides$ | $\mathrm{A} = \mathrm{bh}$<br>$b = base$<br>$h = height$ |
| **Rectangle** |  | $P = 2b + 2c$<br>$b, c = sides$ | $\mathrm{A} = \mathrm{bh}$<br>$b = base$<br>$h = height$ |
| **Rhombus** |  | $P = 4s$<br>$s = side$ | $A = bh = \frac{1}{2}(d_1 d_2)$<br>$d_1, d_2 = diagonals$ |
| **Square** |  | $P = 4s$<br>$s = side$ | $A = s^2 = \frac{1}{2}(d_1 d_2)$<br>$d_1, d_2 = diagonals$ |
| **Regular Polygon** |  | $P = ns$<br>$n = number\ of\ sides$<br>$s = side$ | $A = \frac{1}{2}\, a \cdot P$<br>$a = apothem$<br>$P = perimeter$ |
| **Circle** |  | $C = 2\pi r = \pi d$<br>$r = radius$<br>$d = diameter$ | $A = \pi r^2$<br>$r = radius$ |
| **Ellipse** |  | $P \approx 2\pi\sqrt{\frac{1}{2}(r_1{}^2 + r_2{}^2)}$<br>$r_1 = major\ axis\ radius$<br>$r_2 = minor\ axis\ radius$ | $A = \pi r_1 r_2$<br>$r_1 = major\ axis\ radius$<br>$r_2 = minor\ axis\ radius$ |

# Geometry
## Summary of Surface Area and Volume Formulas – 3D Shapes

| Shape | Figure | Surface Area | Volume |
|---|---|---|---|
| **Sphere** |  | $SA = 4\pi r^2$<br>$r = radius$ | $V = \frac{4}{3}\pi r^3$<br>$r = radius$ |
| **Right Cylinder** |  | $SA = 2\pi rh + 2\pi r^2$<br>$h = height$<br>$r = radius\ of\ base$ | $V = \pi r^2 h$<br>$h = height$<br>$r = radius\ of\ base$ |
| **Cone** |  | $SA = \pi rl + \pi r^2$<br>$l = slant\ height$<br>$r = radius\ of\ base$ | $V = \frac{1}{3}\pi r^2 h$<br>$h = height$<br>$r = radius\ of\ base$ |
| **Square Pyramid** |  | $SA = 2sl + s^2$<br>$s = base\ side\ length$<br>$l = slant\ height$ | $V = \frac{1}{3}s^2 h$<br>$s = base\ side\ length$<br>$h = height$ |
| **Rectangular Prism** |  | $SA = 2 \cdot (lw + lh + wh)$<br>$l = length$<br>$w = width$<br>$h = height$ | $V = lwh$<br>$l = length$<br>$w = width$<br>$h = height$ |
| **Cube** |  | $SA = 6s^2$<br>$s = side\ length\ (all\ sides)$ | $V = s^3$<br>$s = side\ length\ (all\ sides)$ |
| **General Right Prism** |   | $SA = Ph + 2B$<br>$P = Perimeter\ of\ Base$<br>$h = height\ (or\ length)$<br>$B = area\ of\ Base$ | $V = Bh$<br>$B = area\ of\ Base$<br>$h = height$ |

# Trigonometry Reference

### Function Relationships

$$\sin\theta = \frac{1}{\csc\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

$$\cos\theta = \frac{1}{\sec\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{1}{\cot\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

### Opposite Angle Formulas

$$\sin(-\theta) = -\sin(\theta)$$

$$\cos(-\theta) = \cos(\theta)$$

$$\tan(-\theta) = -\tan(\theta)$$

$$\cot(-\theta) = -\cot(\theta)$$

$$\sec(-\theta) = \sec(\theta)$$

$$\csc(-\theta) = -\csc(\theta)$$

### Cofunction Formulas (in Quadrant I)

$$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right) \qquad \cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$$

$$\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right) \qquad \cot\theta = \tan\left(\frac{\pi}{2} - \theta\right)$$

$$\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right) \qquad \csc\theta = \sec\left(\frac{\pi}{2} - \theta\right)$$

### Angle Addition Formulas

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\,\tan B}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\,\tan B}$$

### Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1$$

$$\tan^2\theta + 1 = \sec^2\theta$$

$$\cot^2\theta + 1 = \csc^2\theta$$

### Half Angle Formulas

$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}} = \frac{1-\cos\theta}{\sin\theta} = \frac{\sin\theta}{1+\cos\theta}$$

### Double Angle Formulas

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$$

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$$

### Product-to-Sum Formulas

$$\sin A\cdot\sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$$

$$\cos A\cdot\cos B = \frac{1}{2}\left[\cos(A-B) + \cos(A+B)\right]$$

$$\sin A\cdot\cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$$

$$\cos A\cdot\sin B = \frac{1}{2}\left[\sin(A+B) - \sin(A-B)\right]$$

### Triple Angle Formulas

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

$$\tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

### Power Reducing Formulas

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta}$$

### Sum-to-Product Formulas

$$\sin A + \sin B = 2\cdot\sin\left(\frac{A+B}{2}\right)\cdot\cos\left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B = 2\cdot\sin\left(\frac{A-B}{2}\right)\cdot\cos\left(\frac{A+B}{2}\right)$$

$$\cos A + \cos B = 2\cdot\cos\left(\frac{A+B}{2}\right)\cdot\cos\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B = -2\cdot\sin\left(\frac{A+B}{2}\right)\cdot\sin\left(\frac{A-B}{2}\right)$$

### Arc Length

$S = r\theta$

### Law of Sines

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

### Law of Cosines

$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = a^2 + c^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

### Law of Tangents

$$\frac{a-b}{a+b} = \frac{\tan\left[\frac{1}{2}(A-B)\right]}{\tan\left[\frac{1}{2}(A+B)\right]}$$

**mathguy.us**

### Mollweide's Formulas

$$\frac{a+b}{c} = \frac{\cos\left[\frac{1}{2}(A-B)\right]}{\sin\left(\frac{1}{2}C\right)}$$

$$\frac{a-b}{c} = \frac{\sin\left[\frac{1}{2}(A-B)\right]}{\cos\left(\frac{1}{2}C\right)}$$

### Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta = \text{cis}\,\theta$$

### DeMoivre's Formula

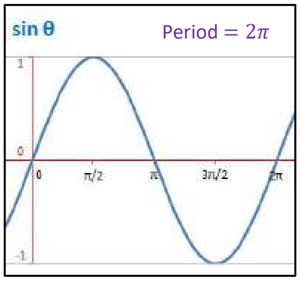
$$(r\,\text{cis}\,\theta)^n = r^n\,\text{cis}\,(n\theta)$$

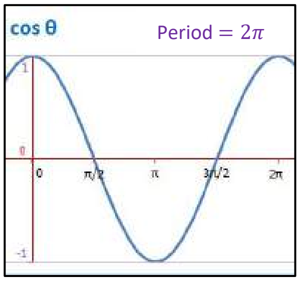
### Polar Multiplication and Division

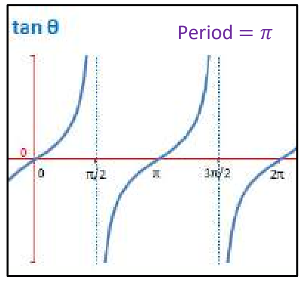
Let: $a = r_1\,\text{cis}\,\theta \qquad b = r_2\,\text{cis}\,\varphi$

$$a\cdot b = r_1 r_2\,\text{cis}\,(\theta+\varphi) \qquad \frac{a}{b} = \frac{r_1}{r_2}\,\text{cis}\,(\theta-\varphi)$$

# Trigonometry Reference



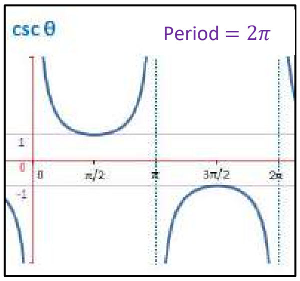

$y = A \cdot f(Bx - C) + D$

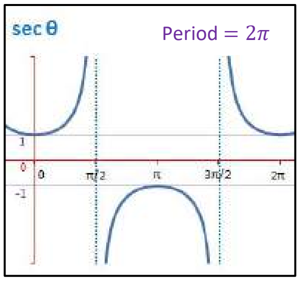
Amplitude: $|A|$

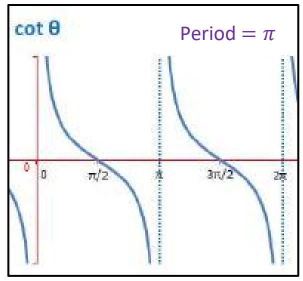
Period: $\dfrac{parent\ "f"\ period}{B}$

Phase Shift: $\dfrac{C}{B}$ →

Vertical Shift: $D$




## Harmonic Motion

$d = a\cos\omega t$ or

$d = a\sin\omega t$

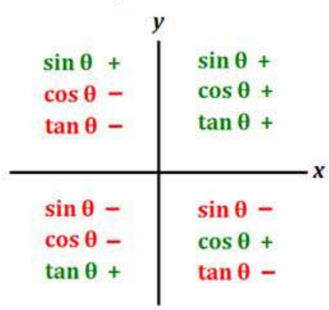
$f = \dfrac{1}{\text{period}} = \dfrac{\omega}{2\pi}$

$\omega = 2\pi f, \quad \omega > 0$

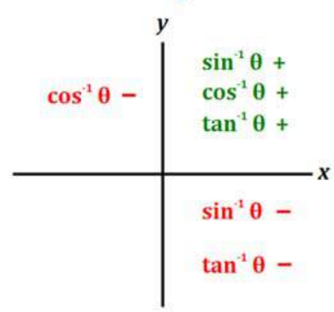
## Trig Functions of Special Angles (Unit Circle)

| $\theta$ Rad | $\theta°$ | $\sin\theta$ | $\cos\theta$ | $\tan\theta$ |
|---|---|---|---|---|
| 0 | 0° | 0 | 1 | 0 |
| $\pi/6$ | 30° | $1/2$ | $\sqrt{3}/2$ | $\sqrt{3}/3$ |
| $\pi/4$ | 45° | $\sqrt{2}/2$ | $\sqrt{2}/2$ | 1 |
| $\pi/3$ | 60° | $\sqrt{3}/2$ | $1/2$ | $\sqrt{3}$ |
| $\pi/2$ | 90° | 1 | 0 | *undefined* |

## Signs of Trig Functions by Quadrant




## Locations of Principal Values of Inverse Trig Functions




## Rectangular/Polar Conversion

| Rectangular | Polar |
|---|---|
| $(x, y)$ | $(r, \theta)$ |
| $x = r\cos\theta$<br>$y = r\sin\theta$ | $r = \sqrt{x^2 + y^2}$<br>$\theta = \tan^{-1}\left(\frac{y}{x}\right)$ |
| $a + bi$ | $r\,(\cos\theta + i\sin\theta)$<br>or $r\ cis\theta$ |
| $a = r\cos\theta$<br>$b = r\sin\theta$ | $r = \sqrt{a^2 + b^2}$<br>$\theta = \tan^{-1}\left(\frac{b}{a}\right)$ |
| $a\mathbf{i} + b\mathbf{j}$ | $\lVert\mathbf{v}\rVert \angle\theta$ |
| $a = \lVert\mathbf{v}\rVert\cos\theta$<br>$b = \lVert\mathbf{v}\rVert\sin\theta$ | $\lVert\mathbf{v}\rVert = \sqrt{a^2 + b^2}$<br>$\theta = \tan^{-1}\left(\frac{b}{a}\right)$ |

## Triangle Area

$$A = \frac{1}{2}bh$$

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

$$s = \frac{1}{2}P = \frac{1}{2}(a+b+c)$$

$$A = \frac{1}{2}\left(\frac{a^2\sin B\sin C}{\sin A}\right)$$

$$A = \frac{1}{2}ab\sin C$$

$$A = \frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right|$$

$$A = \frac{1}{2}\lVert\mathbf{u}\rVert\,\lVert\mathbf{v}\rVert\,\sin\theta$$

## Vector Properties

$0 + \mathbf{u} = \mathbf{u} + 0 = \mathbf{u}$

$\mathbf{u} + (-\mathbf{u}) = (-\mathbf{u}) + \mathbf{u} = 0$

$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$

$m(n\mathbf{u}) = (mn)\mathbf{u}$

$m(\mathbf{u} + \mathbf{v}) = m\mathbf{u} + m\mathbf{v}$

$(m + n)\mathbf{u} = m\mathbf{u} + n\mathbf{u}$

$1(\mathbf{v}) = \mathbf{v}$

$\lVert m\mathbf{v}\rVert = |m|\,\lVert\mathbf{v}\rVert$

Unit Vector: $\dfrac{\mathbf{v}}{\lVert\mathbf{v}\rVert}$

## Vector Dot Product

$\mathbf{u} \circ \mathbf{v} = (u_1 \cdot v_1) + (u_2 \cdot v_2)$

$\mathbf{u} \circ (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \circ \mathbf{v}) + (\mathbf{u} \circ \mathbf{w})$

## Vector Cross Product

$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} = u_1 v_2 - u_2 v_1$

$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$

## Angle between Vectors

$\cos\theta = \dfrac{\mathbf{u} \circ \mathbf{v}}{\lVert\mathbf{u}\rVert\,\lVert\mathbf{v}\rVert}$ $\quad$ $\sin\theta = \dfrac{\lVert\mathbf{u} \times \mathbf{v}\rVert}{\lVert\mathbf{u}\rVert\,\lVert\mathbf{v}\rVert}$

$\perp$ iff $\mathbf{u} \circ \mathbf{v} = 0$ $\quad$ $\parallel$ iff $\mathbf{u} \times \mathbf{v} = 0$

# Geometry Handbook
# Index

# Geometry Handbook
# Index

# Geometry Handbook
# Index

# Geometry Handbook
# Index

# Geometry Handbook
# Index

# Geometry Handbook
# Index

# Geometry Handbook
# Index