A COLLECTION OF MATH RIDDLES

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Contents

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This is a list of some of my favorite math riddles. I have not tried to cite sources for these riddles, though I can tell you where I first heard lots of them.

1. One Hundred Hats

A prison guard tells his 100 prisoners that they will be playing the following game. The guard will line up the prisoners single-file, all facing the front of the line. The guard will place a red or a black hat on each prisoner's head. There could be any combination of red and black hats (perhaps 50 and 50, perhaps 99 and 1, perhaps 100 and 0, etc.) and the hats could be in any order. Each prisoner can only see the hats of those in front of him in line, and not his own hat or those hats behind him. The guard will start with the prisoner at the end of the line (who can see all hats except his own), and ask "Is your hat red or black?". If the prisoner responds correctly, he will be set free. Otherwise he will remain in jail. The guard will then ask the second to last prisoner in line the same question, with the same consequences. The guard will keep moving forward, asking the same question, until all 100 prisoners have been asked. Before the guard begins the game, the prisoners are allowed to devise a strategy. Once the game begins, the prisoners cannot communicate, except by answering "red" or "black" in response to the guard. They can't use volume, intonation, or pauses in their response to communicate, as the guard will notice all these tricks (it's a math riddle). Your task is to devise a strategy so that at most one prisoner remains in jail, and so that at least 99 prisoners are set free. How do you do it?

2. Three Hats

There are 3 prisoners in a room. Two are wearing red hats, and one is wearing black hat. Each prisoner can see all hat colors except his own. The prison guard tells them, "Each of you is wearing a red or a black hat. Tell me your hat color, and you will be set free." Since none of the prisoners can see their own hat, nobody responds. Next the guard adds, "At least one person is wearing a red hat." Soon afterwards, a prisoner correctly identifies his hat color. How?

3. Tiling a Chess Board

(Part I) Cover the top right and bottom left squares of an 8×8 chessboard. Can you tile the remaining 62 squares with 31 tiles of size 2×1?

(Part II) Cover any 3 corner squares of a 30×30 board. Can you tile the remaining 897 squares with 299 tiles of size 3×1?

4. One Hundred Quarters

There are 100 quarters lying flat on a table. Twenty have heads facing up and the remaining eighty have tails up. You are blindfolded and gloved so that you cannot see or feel which are are heads and which are tails. You are allowed to pick up and move the quarters however you like, but at the end of your manipulations each of the 100 quarters must be showing either heads or tails. Your task is to split the 100 quarters into two groups that are guaranteed to have the same number of heads facing up. How do you do it?

5. Die Magic Trick

Learn how to perform the following magic trick. You place a six-sided die on a table. You pick a volunteer, and then turn the other way so that you cannot see the volunteer nor the die. You ask the volunteer to perform a $90°$ rotation of their choosing to the die. (Note: a cube has six 90-degree rotations: the first two are spinning the cube either 90-degrees clockwise or counterclockwise while keeping the same face on top. The last four are rotating the cube so one of the four side faces is now on top. Together, this makes six possible 90-degree rotations.) Ask the volunteer to perform eight more $90°$ rotations of their choosing, for a total of nine rotations. Tell the volunteer to secretly choose whether to perform a tenth $90°$ rotation or not. Turn back around, examine the die, and declare whether or not a tenth rotation was performed.

6. Card Magic Trick

Learn how to perform the following magic trick with a partner. A volunteer selects five random cards from a 52 card deck and gives the five cards to Partner A. Partner A removes and hides one card and orders the remaining four cards however he likes. Partner B examines the four ordered cards, and then names the hidden card.

7. Three Loops

Tie three pieces of string into three loops so that:

(1) The three loops are linked – you can't pull one loop arbitrarily far away from the others.
(2) After cutting ANY one of the three loops, the remaining two loops are not linked – you can pull them arbitrarily far apart.

8. Sink the Sub

You are trying to destroy an enemy submarine. The sub has an integer starting position p. The sub also has a fixed integer velocity v. The submarine moves as follows: at turn 0 it is at position p; at turn 1 it is at position $p + v$; at turn 2 it is at position $p + 2v$, at turn 3 it is at position $p + 3v$, etc. The values of the integers p and v are unknown to you. At each turn you get to fire a missile at one integer location on the numberline. Devise a strategy for firing your missiles so that you are guaranteed to eventually hit the enemy submarine.

9. Ten Hats

A prison guard tells his ten prisoners that they will be playing the following game. The guard will arrange the prisoners in a circle and place a hat on each prisoner's head. Each hat will be one of ten colors, and the prisoners are told the ten possible colors in advance. There could be any combination of hats: all ten hats could be of one color, the ten hats could be of ten different colors, or the ten hats could be of any combination in-between (there are 10^{10} possibilities). Each prisoner can see the hat colors of the nine other prisoners but not his own hat color. Simultaneously, the prisoners must guess their own hat colors. If at least one prisoner guesses his hat color correctly, then all prisoners are released. If no prisoners guess their hat color correctly, then all prisoners remain in jail. Before the game begins, the prisoners are allowed to devise a strategy. Your task is to devise a strategy so that at least prisoner guesses his hat color correctly, and hence all prisoners are set free. How do you do it?

10. Lockers and Wallets

Suppose I take the wallets from you and 99 of your friends. I randomly place them in a room with 100 lockers, one wallet per locker. I let you and each of your friends inside the room one at a time. Each of you gets to look inside 50 lockers of your choosing. You may inspect the wallets you find inside, even inspect the driver's license to see who it belongs to. You're hoping to open the locker with your wallet. Whether you succeed or not, you leave all the wallets where you found them and leave all the locker doors closed. You then exit the room and don't get to communicate to your friends in the waiting room. If everybody finds his wallet in the 50 lockers they open, then your team wins and you all get your wallets back. If a single person doesn't find her wallet, then your team loses and nobody gets their wallet back. You get to discuss a team strategy before anybody enters the room to open 50 lockers, but once the locker-opening process begins there's no more communication of any kind. The riddle is to find a strategy so that your team wins with probability at least 30% (in fact, with probability 31.18%). How do you do it?

The naive strategy is for each prisoner to open 50 boxes at random, but then the prisoners get freed with only probability $(1/2)^{100}$, which is way less than 30%. I think it's pretty counterintuitive that it's possible to succeed with probability 30%.

11. An Unfair Coin

You have a (possibly) unfair coin that lands heads with probability $0 < p < 1$ and that lands tails with probability $1 - p$. Unfortunately, you don't know the value of p. How do you simulate a fair coin? That is, how do you use your unfair coin to create a random event that occurs with probability exactly 0.5? You are allowed to toss your unfair coin as many times as you want.

12. The Devil's Chessboard

The Devil places one quarter on each of the 64 squares of a chessboard, randomly facing heads or tails up. He arbitrarily selects a square on the board of his choosing, called the Magic Square, which he reveals to you. You get to examine the chessboard and flip a single quarter of your choosing, from heads to tails or vice versa. Now, a friend of yours enters the room. Just by looking at the coins, she must tell the Devil the location of the Magic Square. What strategy do you and your friend devise before the game begins so that you can always win?

13. One Hundred Light Bulbs

100 light bulbs are in a line, and all are turned off. Perform the following steps.

(1) change the state of every bulb (IE turn them all on).
(2) change the state of every second bulb (IE turn every second bulb off).
(3) change the state of every third bulb.
(4) change the state of every fourth bulb.
...
(n) change the state of every n-th bulb.

After Step 100, which bulbs are turned on? There is a simple description of such numbers.

14. Twenty-Four

You must write a mathematical expression that equals 24, with the following restrictions. You must use the numbers $3, 3, 8,$ and 8 exactly once, and you may not use any other numbers. You may use any of the symbols $+, -,$ x, /, (, and) as many times as you need, but no other symbols. You're not allowed to "paste" 3 and 8 together to get 38 (for example), although as far as I know this doesn't help.

15. Twenty-Five Horses

You have 25 horses, all of different speeds. You have a five-lane race track and so can only race five horses at a time. After running a race, you know the five horses' relative speeds, but not their times. How many races does it take to determine the fastest, second fastest, and third fastest horse?

16. A Duel

There is a three-way duel between duelers A, B, and C. Dueler A's gun is accurate with probability $\frac{1}{3}$, B's gun is accurate with probability $\frac{2}{3}$, and C's gun is accurate every time. The duelers take turns shooting until only one dueler remains. Dueler A gets to shoot first, B second, and C third. With his first shot, where should participant A shoot?

17. Double Russian Roulette

Two bullets are put in adjacent chambers of a six chamber revolver. You spin the chamber, shoot, and luckily you fire a blank. Since this is double roulette, you must fire the gun one more time. However, you get to choose whether or not to spin the chamber before your second shot. Which choice gives you the best chance of survival?

18. Rope around the Earth

Imagine there is a circular rope around the equator of the earth and a circular rope around the equator of a basketball. Increase the length of each rope so that there is now a 1 meter gap between each circular rope and the corresponding sphere. To which rope did you have to add more length?

19. Meeting

There is a meeting of 30 people. Assume that if person A knows B, then B knows A. Is it possible that everybody at the meeting knows a different number of people?

20. Dinner Party

You are at a dinner party with 5 couples. That evening, no person shakes hands with their partner. At the end of the party, you ask all 9 other people how many different hands each shook, and get 9 different responses. How many hands did your wife shake?

21. Friends and Enemies

There are six people in a room. Each pair among them are either friends or enemies. Must there be a group of three people who are either all friends or all enemies?

22. Dividing Cake

From a large filled-in rectangle, a smaller rectangle has been removed. This smaller rectangle can be at any position or orientation inside the larger rectangle. You have only an unmarked straightedge and a pencil. Draw a straight line that divides what remains of the large rectangle into two regions of equal area.

(Now you know how to divide a cake, with an awkwardly removed rectangular slice, into two equal portions using only a single straight cut.)

23. Wine Cork

Learn the wine cork hand trick (you will need to see a demonstration).

24. NIM

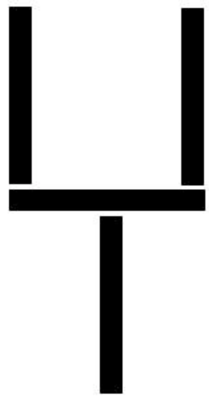
Learn how to win the game of NIM every time, so long as you get to pick whether to go first or second.

25. Cheese

You are given a block of cheese and a knife. After cutting the cheese, assume that the resulting pieces stay in place (they don't move at all). What is the maximum number of pieces you can produce with five straight cuts?

26. Wine Glass

Below is a wine glass made of four sticks. Moving only two of the sticks, turn the wine glass upside-down.



27. Airplane Loading

50 passengers are loading a 50-seater airplane. They are loading in a random order. There is one crazy passenger who lost his ticket and, when he enters the plane, will choose a seat at random. The rest of the passengers will choose their assigned seat; in the case that their assigned seat is already taken, they will choose a random seat. What is the probability that the last passenger to load sits in his assigned seat?

28. Squashed Bee

Two trains are a mile apart, headed towards each other on the same track at 30 mph. A bee starts at one train and zig-zags between the two trains at 60 mph, until it is squashed when the trains collide. How far does the bee travel before it is squashed?

29. Average Train Speed

A train travels 500 miles, completing the trip with an average speed of 50 miles per hour. However, it travels at different speeds along the way. It seems plausible that nowhere along the 500 miles of track is there a segment of 50 miles which the train traversed in exactly one hour. Prove that this is not the case.

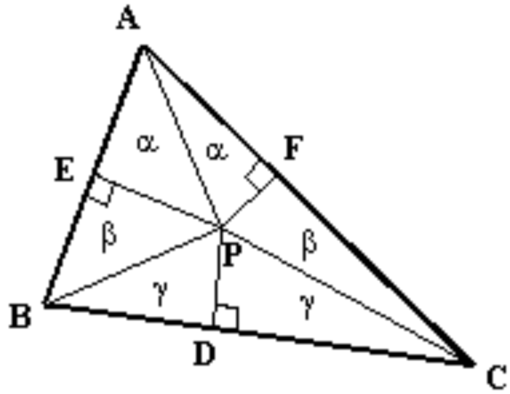
30. South by East by North

Find all spots on the earth where you can go a mile south, a mile east, and finally a mile north and end up in your starting location.

31. All Triangles are Isosceles

Below is a proof that all triangles are isosceles. Clearly not all triangles are isosceles, so this proof must be wrong. Where is the error in this proof?

Let ABC be an arbitrary triangle. Consider the angle bisector at point A and the perpendicular bisector of edge BC. Let P be their intersection point. We have $|BP| = |CP|$ since edge PD is the perpendicular bisector of edge BC. Drop a perpendicular altitude from P to edge AB (intersecting at E), and drop a perpendicular altitude from P to edge AC (intersecting at F). Since triangles AEP and AFP are right triangles with equal angles at A and with equal hypotenuses, they are congruent, and so $|AE| = |AF|$ and $|EP| = |FP|$. Since $|BP| = |CP|$ and $|EP| = |FP|$, we know that right triangles BEP and CFP are congruent. Hence $|BE| = |CF|$. Finally, we have $|AB| = |AE| + |BE| = |AF| + |CF| = |AC|$, and so triangle ABC is isosceles.



32. Bridge Crossing

Four friends must cross a bridge. They must do it in pairs, as they only have one flashlight, and the flashlight is only bright enough for two walkers to see at once. So, two will have to cross with the flashlight, one will have to bring it back, two more will have to cross with the flashlight, one will have to bring it back, etc. until all are on the other side. Two friends walk at the pace of the slower. Suppose the four friends take 1, 2, 5, and 10 minutes (respectively) to cross the bridge. What is the quickest way for all four of them to cross?

33. Hill Climb

A man climbs a hill. He starts at 6am and ends at 6pm. He sleeps that night at the top. The next day, he walks back down the same path, starting again at 6am and ending at 6pm. Show that there exists a time of day, between 6am and 6pm, when the man is at the same spot on the hill on both days (since the man may walk at an irregular pace, that time need not be noon).

34. Missing Dollar

Three customers at a restaurant split a \$30 bill, \$10 apiece. When the waiter brings the \$30 to the manager, the manager realizes there has been a mistake: the bill should only have been \$25. The crooked waiter returns \$1 to each customer and keeps the remaining \$2 for himself. So, each customer has paid \$9. And \$2 went to the waiter. This accounts for $3 \times 9 + 2 = 29$ of the dollars. But the original bill was \$30. What happened to the missing dollar?

35. Ants on a Log

There are nine ants at arbitrary positions on a circle (say, a log). All ants walk at the same speed. However, each ant could be walking either clockwise or counterclockwise. When two ants collide, they bounce (change directions) and continue. Will there ever be a time at which all nine of the original starting positions are again occupied by ants?

36. Knight's tour

(Part I) Starting from the top left square of a chessboard, is it possible for a knight to travel from square to square (according to legal chess moves) so that he visits every square exactly once, and so that he ends at the bottom right square?

(Part II) Can you place a knight on a 5×5 chessboard and have him travel from square to square (according to legal chess moves) so that he visits each square on the board exactly once, and so that from his ending location he is attacking his starting location?

37. Rubik's Cube Path

Consider a $3 \times 3 \times 3$ grid of cubes (IE the cubes are arranged like the small cubes in a Rubik's cube, with the center cube included). Two cubes in this grid are considered adjacent if they share a square face. Is there a path which starts in the center cube, travels from one cube to an adjacent cube, and visits each cube in the grid exactly once?

38. Rubik's Cube Cuts

Suppose you want to cut a Rubik's cube into its 27 individual smaller cubes. You are only allowed to make straight cuts, but you are allowed to restack the pieces however you like before a cut. Prove that at least 6 cuts are required. (It turns out that 6 cuts are also sufficient, even without restacking.)

39. Four Consecutive Integers

Show that the product of four consecutive positive integers is never a perfect square.

40. GEB Sequence

What is the next number in the sequence 1, 3, 7, 12, 18, 26, 35, 45, 56, 69, ... ?

41. LAS Sequence

What is the next number in the sequence 1, 11, 21, 1211, 111221, ... ?

42. Say Red

One by one, flip over the top card from a 52 card deck. At any point, you may decide that the next card is your selected card. If your selected card is red, you win, and if it is black, you lose. You must select your chosen card before the deck runs out. What strategy gives you the best chance of winning?

43. Map on a Torus

Divide the torus into seven regions so that seven colors are needed to color the resulting map (that is, each region touches all of the other six).

44. Circle's Center in Triangle

(Part I) Choose three random points on a circle. What is the probability that the defined triangle contains the center of the circle?
(Part II) What is the probability that a tetrahedron defined by four random points on a sphere contains the center of the sphere?

45. Three Nested Triangles

You have 3 nested triangles, each circumscribed inside the larger. The edges of the largest are parallel to those of the smallest. Show that

$$\text{(area of largest triangle)} \times \text{(area of smallest triangle)} = \text{(area of middle triangle)}^2.$$

46. 789-th Digit

Consider the number $(1+\sqrt{2})^{3456}$. In its decimal expansion, what is the 789-th digit after the decimal point? (Do not use a computer.)

47. Trucks

There are two roads between points A and B, and it is possible for two trucks to go from A to B, one on each road, so that the trucks are never more than 100 meters apart. Now, let one truck start at A and the other at B. Using these same two roads, is it possible that one truck can go from A to B, and the other from B to A, so that the two trucks are never within 101 meters of each other?

48. Cheating Husbands

In a village there exist some men who are having affairs with the wives of other men. There is a custom which requires a woman to kill her husband the morning after she discovers that he is having an affair with another woman. Also, every woman knows whether every man besides her husband is having an affair or not. So life in this village goes on peacefully since no woman knows for sure that her own husband is cheating on her. Unfortunately, an Oracle visits one day and proclaims that at least one man in the village is having an affair. What happens after this and why?

49. Two Switches

There are 100 prisoners in jail. They are allowed to devise a strategy. Afterwards, the warden will put them in separate cells. Periodically he will bring a single prisoner at a time to visit a room in which there are two binary switches. The initial state of these switches is unknown. All prisoners will be brought to the room many times, but not necessarily sequentially or in any fixed time frame. The prisoners must devise a strategy so that, eventually one of them can tell the warden for certain that all 100 prisoners have visited the room.

50. Nickel through Penny Hole

Cut a whole the size of a penny in a piece of paper. Without tearing the paper, can you pass a nickel through this hole? If so, do it. If not, prove it is physically impossible.

51. Geometric Series

For $|r| < 1$, explain why the formula

$$1 + r + r^2 + r^3 + ... = \frac{1}{1-r}$$

makes sense. In other words, if you didn't know that this formula were true, how might you dream up this formula so that you could try to prove it?

52. Divisible by Forty

Suppose both $2n+1$ and $3n+1$ are prefect squares, where n is a positive integer. Show that n is divisible by 40.

53. Envelopes Paradox

Warning: this is a paradox, not a riddle. In the paragraph below you will be mislead, and you must figure out exactly how you were mislead.

Consider the following game. I have two envelopes, one of which contains exactly twice as much money as the other. You do not know which envelope is which. You are allowed to select one envelope, open it, count how much money is inside, and then decide whether or not to switch envelopes. You keep the money inside your final selection. What should your strategy be in order to maximize your payoff? Randomness tells you that it doesn't matter whether you switch envelopes or not– each envelope is equally likely to be the more valuable one. However, an expected value calculation tells you that you should ALWAYS switch after your initial selecton: if your initial envelope contains x dollars, then the expected value of the other envelope is $\frac{1}{2}(2x) + \frac{1}{2}(\frac{x}{2}) = \frac{5}{4}x > x$. Explain this paradox.

54. Achilles and the Tortoise Paradox

Warning: this is a paradox, not a riddle.

Achilles and a tortoise are racing. Since Achilles is twice as fast as the tortoise, he gives the tortoise a head start. Say the tortoise starts at point A. By the time Achilles gets to point A, the tortoise will have moved ahead to point B. By the time Achilles gets to point B, the tortoise will have moved ahead to C. This continues forever, and so Achilles will never catch the tortoise. Why is this argument incorrect?

55. Bike Riddle

Warning: This is a physics riddle, not a math riddle.

Hold a bike up straight (or put training wheels on it). Orient the pedals so that the right pedal is at the bottom, and tie a string to the right pedal. If you pull backwards on the string (away from the front of the bike), will the bike move forwards or backwards?

56. Stable Pairings

Suppose we have $2n$ people where $n \geq 2$, and that each person has ranked all others in order of preference. An arrangement of the $2n$ people into n pairs of two is called a *pairing*. A pairing is called *unstable* if there exists two people A and B who are not paired together but A prefers B to A's partner, and B prefers A to B's partner. Therefore, a pairing is called *stable* if for any two people A and B who are not paired, either A prefers his partner to B's partner, or B prefers his partner to A's partner, or both.

Find an assignment of the preferences for all $2n$ people such that, given those preferences, no stable pairing exists.

57. Coffee and Tea

Suppose you have two mugs. Mug A contains coffee and mug B contains an equal amount of tea. You take a spoonful of the coffee in mug A and mix it thoroughly into mug B. Then, you take a spoonful of mug B and put it in mug A. Which quantity is greater: the amount of tea in mug A or the amount of coffee in mug B?

Part II: What if mugs A and B initially contain different amounts of coffee and tea? What if you don't mix thoroughly?

58. Integral Rectangles

Suppose a rectangle can be exactly covered by smaller rectangles, each which has at least one side length an integer value. Prove that the large rectangle has at least one side length an integer value.

59. One Hundred Hats, Parts II and III

This is an extension of Riddle 1: One Hundred Hats.

(Part II) Suppose that instead of 100 prisoners we have a countably infinite number of prisoners. Each prisoner can see the hat colors of the infinite number of prisoners in front of him, but not the hat colors of the finite number of prisoners behind him. Suppose now that all the prisoners have to say "red" or "black" simultaneously — IE, they can't listen for the guesses of the prisoners behind them in line. Your task is to devise a strategy so that at most a finite number of prisoners guess incorrectly and remain in jail. You can assume that each prisoner has an extremely large brain in order to construct and remember an "uncountable" strategy.

(Part III) Same as Part II, except now with an arbitrary number of hat colors instead of just two (red and black).

60. Expected Determinant

Let A be an $n \times n$ matrix with entry i, j equal to 0 or 1 with equal probability. If $n = 1$, then the expected value of the determinant $\det(A)$ is $1/2$. For $n > 1$, what is the expected value of $\det(A)$?

61. Five Coins

Given five identical coins, can you arrange them so that every coin touches each of the other four?

62. Last quarter on the table

I really enjoy the following riddle, available at `http://m.nsa.gov/news-features/puzzles-activities/puzzle-periodical/2016/puzzle-periodical-04.shtml`.

63. 400 coins

A rectangular table can be completely covered by 100 (overlapping) coins of radius 2. Prove that it can also be completely covered by 400 (overlapping) coins of radius 1.