7. Work and Kinetic Energy

A) Overview

This unit introduces two important new concepts: *kinetic energy* and *work*. These concepts are defined in terms of the fundamental concepts from dynamics (*force* and *mass*) and kinematics (*displacement* and *velocity*). We will find that integrating Newton's second law through a displacement will result in an equation that links these two concepts of kinetic energy and work. This equation (called the work-energy theorem or sometimes, the center of mass equation), allows us to easily answer many questions that would be very difficult using Newton's second law directly

B) Work and Kinetic Energy in One Dimension

We begin our introduction of work and energy by considering the simple one-dimensional situation shown in Figure 7.1. A force is applied to an object, causing it to accelerate. We say that the force acting over time causes the change in velocity. We can

Figure 7.1
Aconstant force $\boldsymbol{F_{net}}$ is applied to a an object over a time interval (from t_1 to t_2) that results in a change in its velocity and in its displacement.

quantify this statement by integrating the force over time to obtain the relationship between this integral of the force over time and the change in velocity.

$$F_{net} = m\frac{dv}{dt}$$

$$\int_{t_1}^{t_2} F_{net}\,dt = m\int_{v_1}^{v_2} dv$$

We could also describe this situation by saying that the force acting through a *distance* caused the change in velocity. How do we quantify this description? Well, consider the motion at time *t*: in the next instant of time *dt*, the velocity will change by an amount *dv* which is equal to the acceleration times *dt*.

$$dv = adt$$

In this same time *dt*, the position will change by an amount which is equal to the velocity

$$dx = vdt$$

times *dt*. We apply Newton's second law, to replace the acceleration by the net force divided by the mass and then combine the equations to eliminate *dt* and obtain the equation:

$$dv = \frac{F_{net}}{m}\frac{dx}{v}$$

If we now integrate this equation, we obtain the relationship between the integral of the net force over the displacement and the change in the square of the velocity.

$$m\int_{v_1}^{v_2} v\,dv = \int_{x_1}^{x_2} F_{net}\,dx$$

$$\frac{1}{2}m\Delta(v^2) = \int_{x_1}^{x_2} F_{net}\,dx$$

We define the integral of the force over the displacement to be the ***work*** done by the force

$$W \equiv \int_{x_1}^{x_2} F\,dx$$

and the quantity ½ the mass times the velocity squared to be the ***kinetic energy*** of the

$$K \equiv \frac{1}{2}mv^2$$

particle. Thus we see that in one dimension, the work done on the object by the net force is equal to the change in that object's kinetic energy. Work and kinetic energy both are measured in Joules, where 1 Joule is defined to be 1 N-m.

We will do a simple example in the next section that illustrates the use of these concepts.

C) Example

Figure 7.2 shows a box of mass 6 kg that is initially at rest. A horizontal force of magnitude 24 N is now applied and the box begins to move. We would like to determine the speed of the box when it is at a distance of 8 m from its initial position.

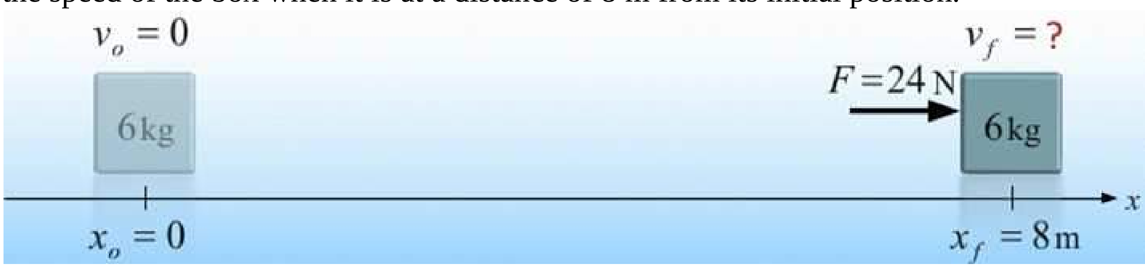

Figure 7.2
A constant force o 24 N is applied to a box of mass 6 kg that is initially at rest. What is the speed of the box when it reaches a distance of 8m from its starting point?

How do we go about making this calculation? We could first use Newton's second law to determine the acceleration, and then use this acceleration in one of our kinematics equations to determine the time it takes to travel 8 m, and then use another kinematics equation to determine the speed at this time.

There is, however, an easier way. Namely, we can simply equate the work done by the net force (which is just the applied force in this case) to the change in the kinetic energy of the box.

$$W = \Delta K$$

Now, since the applied force is constant, we can take it outside the integral, and then the work done is just equal to the product of the applied force and the distance, Since the box was initially at rest, the change in kinetic energy of the box is just equal to its final kinetic energy.

$$W = F \cdot (x_f - x_i) = 24N \cdot 8m$$

$$\Delta K = \tfrac{1}{2} m v_f^2 = \tfrac{1}{2} \cdot 6kg \cdot v_f^2$$

Consequently, we see that the final speed is equal to 8 m/s.

We've just seen how helpful this connection between the work done and the change in kinetic energy in one dimension can be. In order to generalize this connection to more than one dimension, we will need to introduce the concept of the dot product of two vectors, which we will do in the next section.

D) Dot Product

So far, all operations we have performed on vectors have produced another vector. When we add or subtract vectors, the result is another vector. When we multiply a scalar by a vector, the result is also a vector.

How do we multiply two vectors? There are actually two different products of two vectors. The cross product of two vectors produces another vector and we will discuss this operation later in the course. In this section we will introduce the *dot product* of two vectors which produces a scalar.

The dot product of any two vectors is defined to be the product of the magnitudes of the vectors and the cosine of the angle between them as shown in Figure 7.3..

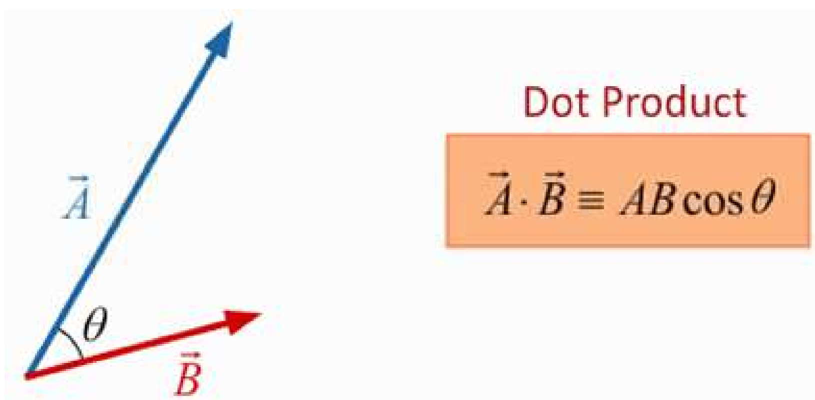

Figure 7.3
The dot product of two vectors is defined to be a scalar that measures the projection of one vector along the other..

Therefore, if the two vectors are parallel, the dot product is equal to the arithmetic product.

If the two vectors are perpendicular, the dot product is equal to zero. If the two vectors are anti-parallel, the dot product is equal to minus the arithmetic product. The dot product of two vectors is a measure of the projection of one vector along the other.

The dot product is used to define the components of vectors. For example, A_x and A_y are the dot products of ***A*** with the unit vectors in the x and y directions, respectively.

Figure 7.4 shows two vectors ***A*** and ***B***, defined in terms of their x and y components. If

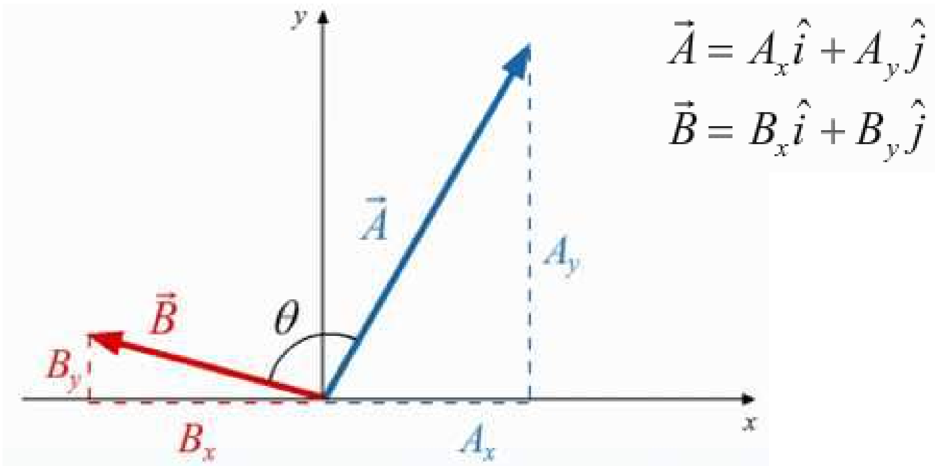

Figure 7.4
The representation of vector ***A*** and ***B*** in terms of their components (A_x, A_y) and (B_x,B_y) using the dot product and the unit vectors $(\hat{i}, \hat{j})$ along the (*x*,*y*) axes.

we now take the dot product of these two vectors, we see that the only terms that survive are the products of the same components

$$\vec{A} \bullet \vec{B} = A_x B_x + A_y B_y$$

In the next section, we will use the dot product to define the work done by a force

E) Work-Kinetic Energy Theorem

In an earlier section we determined that in one dimension, the integral of the net force over the displacement of an object is proportional to the change in the square of its velocity We will now make use of the dot product to generalize this result to more than one dimension.

We start by defining the work done by a force on an object as the integral of the dot product of the force and the displacement. We can see from this definition that only

$$W \equiv \int \vec{F} \bullet d\vec{\ell}$$

the part of the force that is parallel to the displacement contributes to the calculation of the work done by the force. For example if you pull a box across a horizontal frictionless floor, both the weight and the normal force are perpendicular to the displacement so that neither of these forces does any work on the box. Only the tension force has a component in the direction of the displacement, so only the tension force does any work on the box.

Figure 7.5 shows a box being pulled up a frictionless ramp so that we have displacements in two dimensions, x and y. We can now basically repeat the derivation of the one

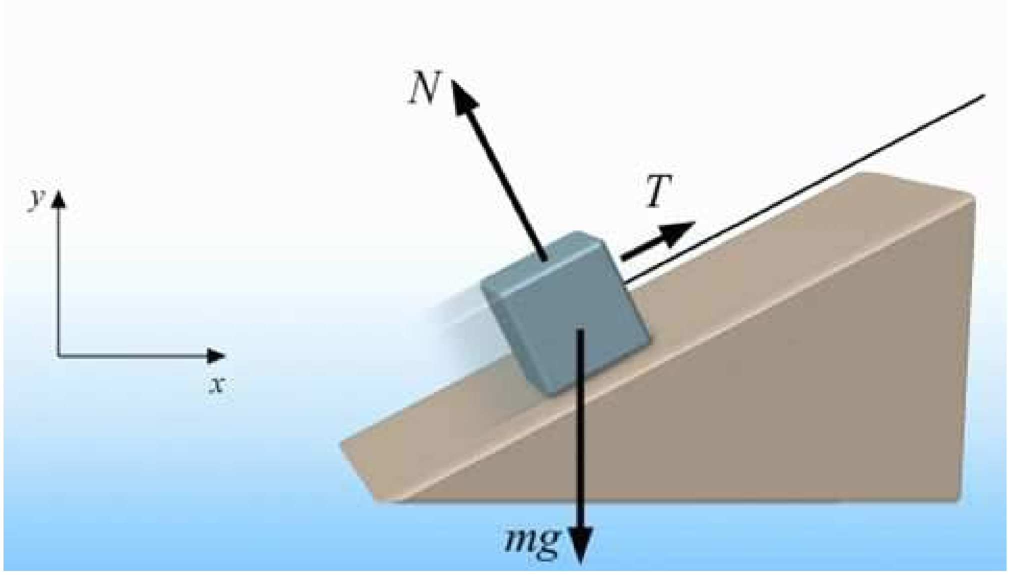

Figure 7.5
A free-body diagram for a box of mass m being pulled up a a ramp by a taut rope.

dimensional case for each component. That is, we start from the equations for the change in velocity and displacement during a time dt.

$$dv_x = a_x dt \qquad dx = v_x dt$$
$$dv_y = a_y dt \qquad dy = v_y dt$$

Once again we apply Newton's second law to replace the acceleration by the net force divided by the mass and combine the equations to eliminate dt and obtain two equations, one for each dimension.

$$mv_x dv_x = F_{net\,x} dx \qquad mv_y dv_y = F_{net\,y} dy$$

We now integrate each equation and add them to obtain our final result:

$$m\left[\int v_x dv_x + \int v_y dv_y\right] = \int (F_{net\,x} dx + F_{net\,y} dy)$$

The left hand side of the equation is equal to the change in kinetic energy of the box. The right hand side of the equation is the work done by the net force, since the dot product can be expanded as the sum of the products of the components.

$$\Delta K = W_{net}$$

We now see that defining the work in terms of the dot product of the force and the displacement allows us to generalize our one-dimensional result. *The change in the kinetic energy of an object is equal to the work done on that object by the net force.* We call this statement the *Work Kinetic Energy Theorem*. In the next few sections we will explore the work done by several forces.

F) Examples: Work Done by Gravity Near the Surface of the Earth

Figure 7.6a shows a mass *m* being moved along some arbitrary path connecting

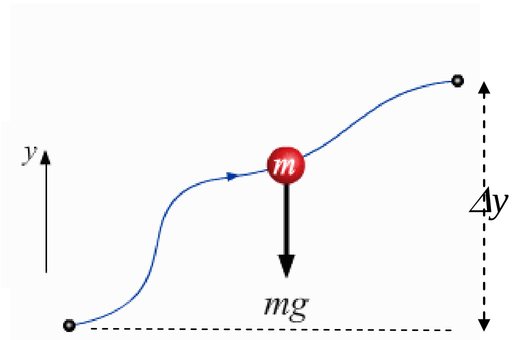

Figure 7.6a
An object of mass *m* is moved along an arbitrary path between two fixed points separated by a vertical distance *Δy*.

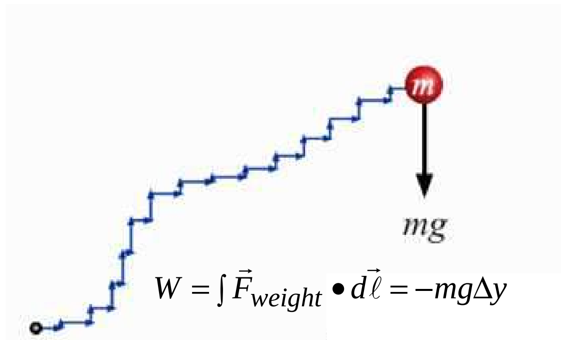

Figure 7.6b
The work done by the weight force is calculated by approximating the path by a series of infinitesimal horizontal and vertical segments. The only non-zero contributions come from the vertical segments.

two fixed points near the surface of Earth. We want to calculate the work done by gravity along this path. We will first approximate the path as a series of infinitesimal straight line segments along the horizontal and vertical directions as shown in Figure 7.6b.. Now the work done along all of the horizontal segments is zero since the force (the weight) is always perpendicular to the direction of the horizontal segment. Therefore, the total work done by gravity along this path is just the sum of the work done by gravity along the vertical segments. Since the force (the weight) points down and the vertical segment (dy) points up, the dot product of these two vectors is just equal to $-mg\ dy$. Therefore, the total work done by gravity along the path is just equal to minus the product of the weight and the total vertical displacement.

$$W = -mg\Delta y$$

Note that this formula for the work done by the weight does *not* depend of the path taken, but only on the difference in height of the initial and final points. When the work done by a force during some motion only depends on the endpoints of the motion, but not on the details of the path, we say that the force is ***conservative***. A consideration of conservative forces will lead us to the important concept of potential energy in the next unit.

We will now use the result for the work done by the weight in an example that illustrates the power of the work-kinetic energy theorem. Figure 7.7 shows two balls released at the same time from the same height.*h*. One ball simply falls with constant acceleration g. The other skids down a frictionless curved surface. The free fall problem is one we've solved many times; we know the velocity and position of the ball at any time. We use this information to determine how long the ball is in the air and with what velocity it hits the ground. The skidding ball looks to be a much more difficult problem.

The net force here is certainly not constant! Therefore, we cannot use our kinematic equations for constant acceleration. Indeed, finding the position and velocity

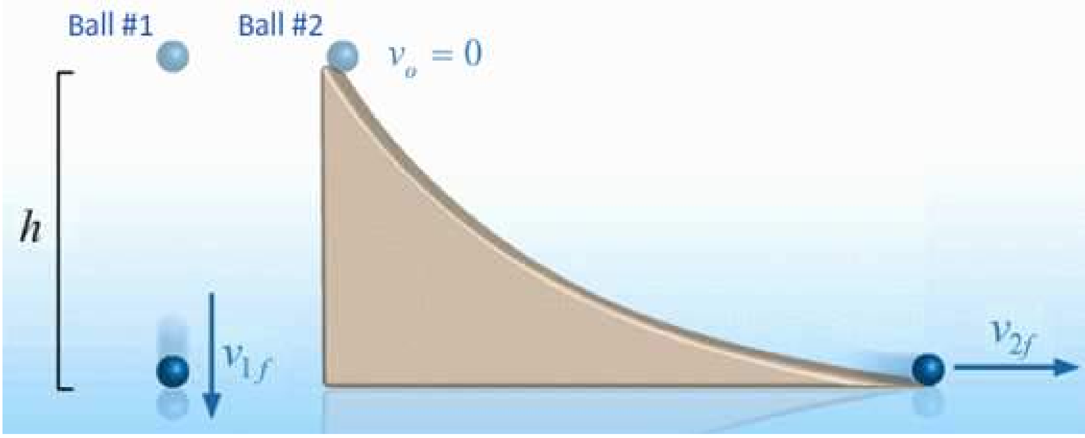

Figure 7,7
Two balls are dropped at the same tiem from the same height. Ball 1 falls freely while Ball 2 skids down a frictionless curved surface..

of the skidding ball at any time is difficult. However, we can use the work-kinetic energy theorem to easily find the final velocity.

Only two forces act on the skidding ball: the normal force provided by surface, and the weight of the ball. Now, the ball always moves parallel to the plane. The normal force is always perpendicular to the plane. Therefore, work done by the normal force, given by the dot product of the normal force and the displacement is zero! Consequently, the work done by the net force here is just equal to the work done by the weight:

$$W = -mg\Delta y$$

Applying the work-kinetic energy theorem, we see that the final speed of the skidding ball is exactly the same as the final speed for the dropped ball!

$$-mg\Delta y = \frac{1}{2}mv_{2f}^2$$

$$v_{2f} = \sqrt{2g\Delta y}$$

We will close this unit with a discussion of two more examples of the work done by conservative forces.

G) Work Done by a Variable Force: Spring

We have calculated the work done by the weight, a constant force, when the orientation of the path relative to the force was changing. We will now calculate the work done by a spring in one dimension. In this case, the orientation of the path relative to the force is simple, but the magnitude of the force changes as we move.

We define the origin or the coordinate system to correspond to the e relaxed length of the spring as shown in Figure 7.8. The force exerted by the spring on the object attached to its end as a function of position is proportional to the extension or compression of the spring. As we move the object between two positions x_1 and x_2, the force on the object clearly changes. Breaking the movement into tiny steps we see that

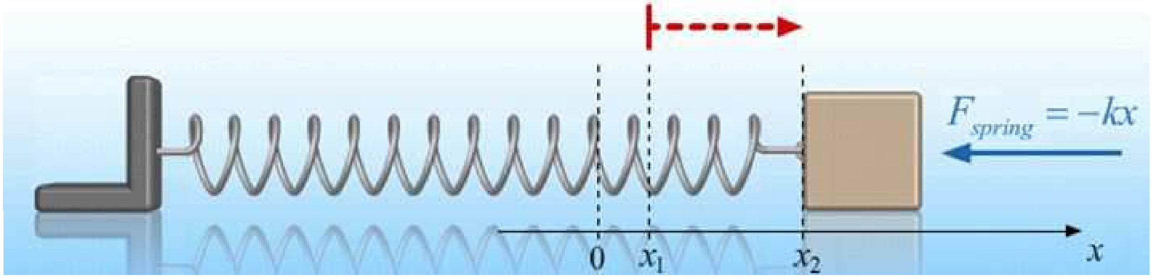

Figure 7.8
A spring with spring constant k exerts a restoring force on an object of mass *m*. The coordinate system is defined such that $x = 0$ corresponds to the relaxed length of the spring. To calculate the work done by the spring as the object moves between two positions, x_1 and x_2, we must integrate the varying restoring force from x_1 to x_2..

the work done by the spring along each step will depend on the position : To find the total work done we need to integrate this expression between x_1 and x_2.

$$W_{1\to 2} = -k\int_{x_1}^{x_2} x\,dx = -\frac{1}{2}k(x_2^2 - x_1^2)$$

Figure 7.9 shows a plot of the force as a function of displacement as the spring is moved from x_1 to x_2. The area under the curve represents the work done by the force.

Note that the formula for the work done by a spring that we just derived depends only on the endpoints of the motion, x_1 and x_2. In fact, if we first stretch the spring from its equilibrium position, we see that the spring force points in the opposite direction to the displacement and therefore does negative work. If we then compress the spring back to its equilibrium position, the spring force has the same direction, but the displacement is now in the opposite direction so that the spring force does positive work. In fact the magnitudes of these negative and positive works are equal, so that we see that when the spring is returned to its original position, the net work done is zero, the signature of a conservative force.

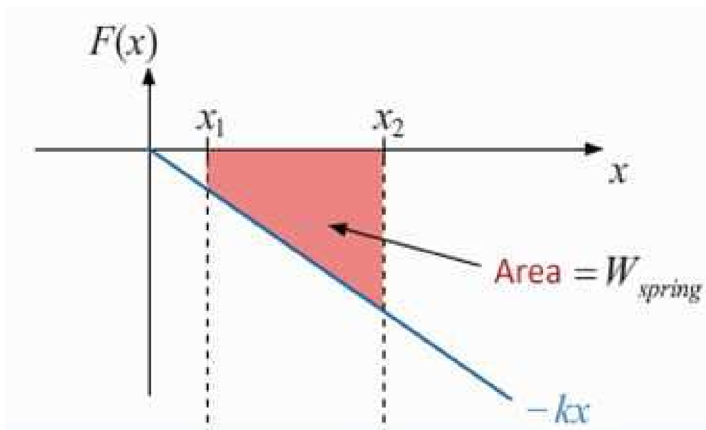

Figure 7.9
A plot of the force exered by the spring in Fig 7.8 as a function of the extension of the spring. The work done by the spring as the object is moved from x_1 to x_2 is given by the shaded area under the curve.

H) Work Done by Gravity Far From Earth

As a final example, we will consider the work done by gravity on an object of mass *m* that moves in three dimensions along some arbitrary path between two fixed points $\boldsymbol{r_1}$ and $\boldsymbol{r_2}$ that are far from the surface of Earth as shown in Figure 7.10. In this case both the direction of path relative to the direction of the force, as well as the magnitude of the force, can change along the path.

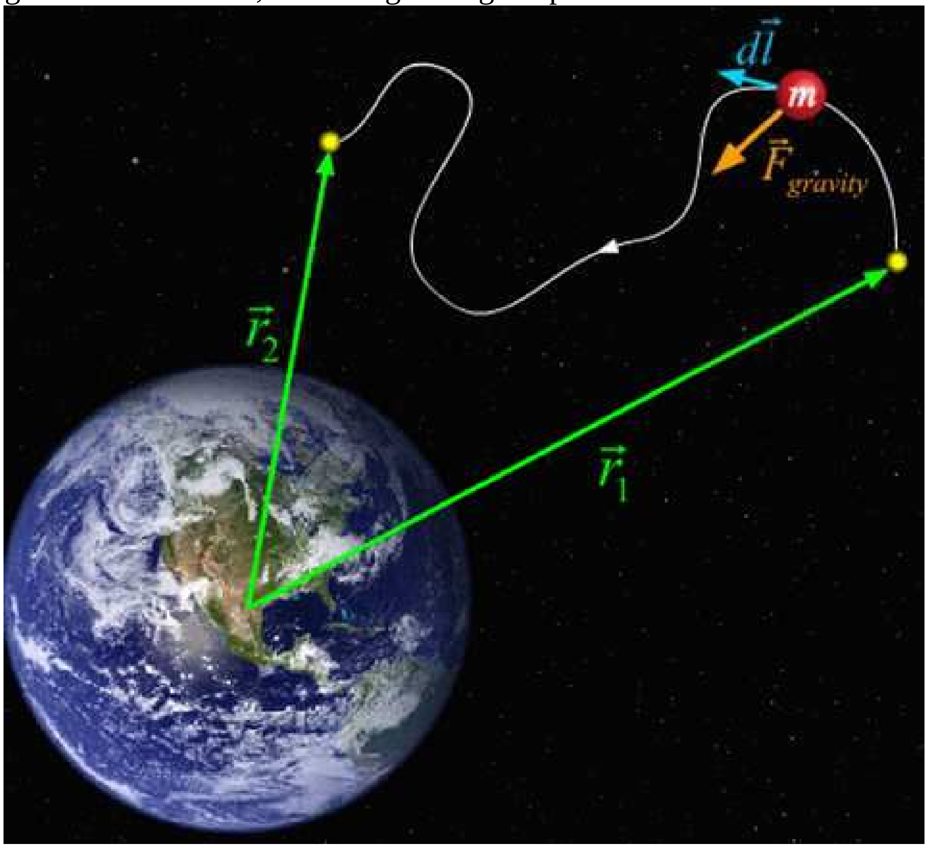

Figure 7.10
To calculate the work done by gravity as a mass *m* is moved along the curved path shown, we need to evaluate an integral in which the magnitude of the force $\boldsymbol{F_{gravity}}$ is changing as is the angle between the force and the path element.

The magnitude of the gravitational force that the earth exerts on this object is given by the universal gravitational force law; the direction of this force is always radially inward, towards the center of the Earth. Once again, we can break the path between $\mathbf{r_1}$ and $\mathbf{r_2}$ into tiny steps as shown in Figure 7.11. . One such step is shown magnified in the Figure. This step is in turn broken into two even smaller steps, one that is parallel to the radial direction and one which is perpendicular to the radial direction. Since the force is radial, the work done when moving along the perpendicular step is zero! The work done when moving along the radial step is just equal to the force at that point times the radial displacement.

$$dW = \vec{F}_{gravity} \bullet d\vec{\ell} = -G\frac{M_E m}{r^2}dr$$

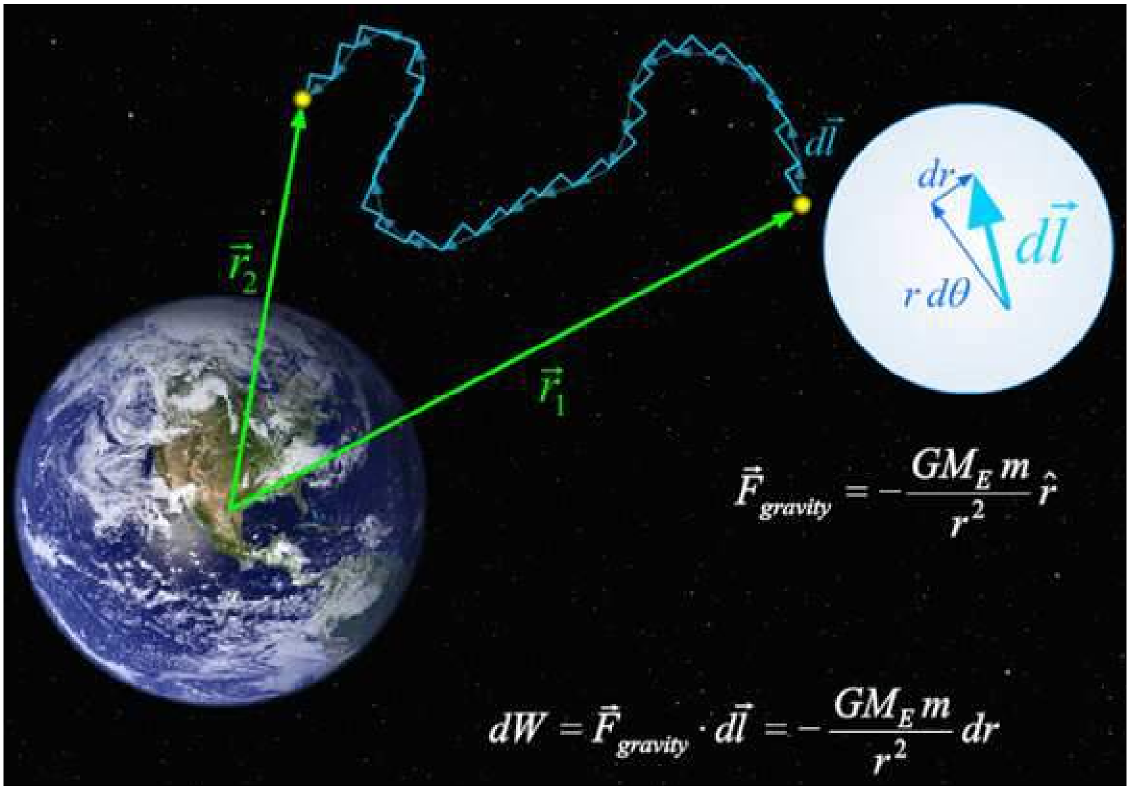

Figure 7.11
To evaluate the work done by gravity, we approximate the curved path as a series of infinitesimal elements each of which are in turn broken down into two elements, one radial and one tangential. The work done along each tangential element is zero since the force is radial. Consequently, the work done by gravity only depends on r_1 and r_2, the radial distances of the endpoints of the path.

The total work is therefore just the sum of the work done by all of the radial steps. To make this sum, we do the integral and obtain our result: that the work done by the

$$W_{1\to 2} = -\int_{r_1}^{r_2} \frac{GM_E m}{r^2}\,dr = GM_E m\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

gravitational force as an object is moved from $\boldsymbol{r_1}$ to $\boldsymbol{r_2}$ is proportional to $1/r_2 - 1/r_1$. Once again we see that the work depends only on the endpoints and not on the specific path, demonstrating that the gravitational force is indeed a conservative force.

Main Points

- **Work and Kinetic Energy Definitions**

The Kinetic Energy of an objecit is defined to be ½ the product of the mass and the square of its velocity.

$$K \equiv \frac{1}{2}mv^2$$

Thw Work done by a force as an object is moved between two points along some path is defined to eb the dot product of the force with the displacement along the path between those two points.

$$W_{1\to2} \equiv \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}\cdot d\vec{l}$$

- **Work-Kinetic Energy Theorem**

Integrating Newton's second law, we obain the work-kinetic energy theore: that the work done by the net force on an object as it moves between two ppoints is equal to the change in its kinetic energy.

$$W_{Net} = \Delta K$$

- **Conservative Forces**

Conservative forces aredefined to be those forces in which the work done by them does NOT depend on the path, only on the endpoints.

The gravitational force is a conserative force

Work done by the Gravitational Force

$$W = -mg\Delta y$$ (near Earth)

$$W_{1\to2} = GM_E m\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$ (general expression)

The spring force is a conserative force

Work done by the Spring Force

$$W_{1\to2} = -\frac{1}{2}k(x_2^2 - x_1^2)$$