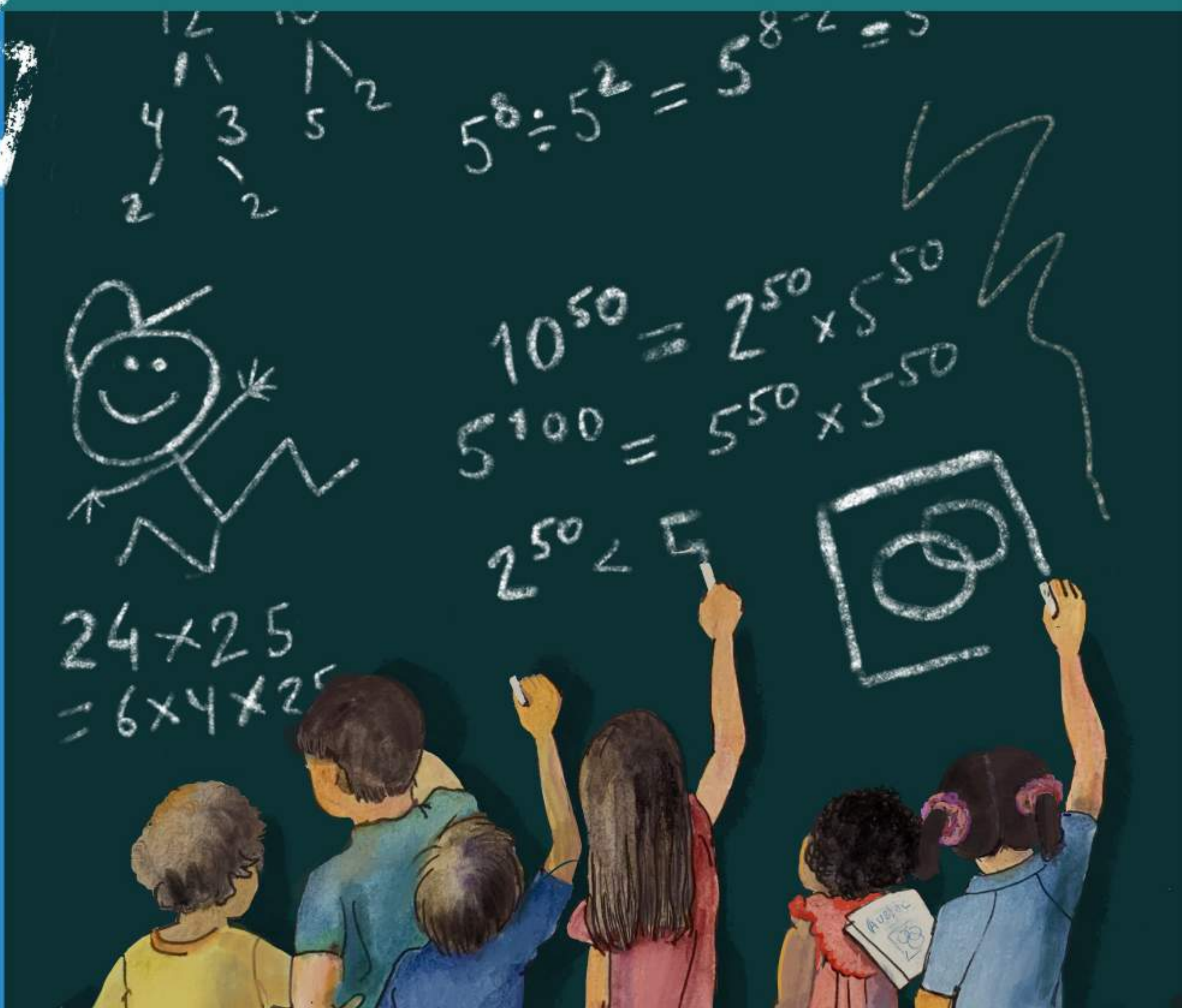
Natasha Rozhkovskaya

Blue Book of Mathematics for Elementary School Teachers



Edition 1.0

Cover design by Natasha Rozhkovskaya
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Contents

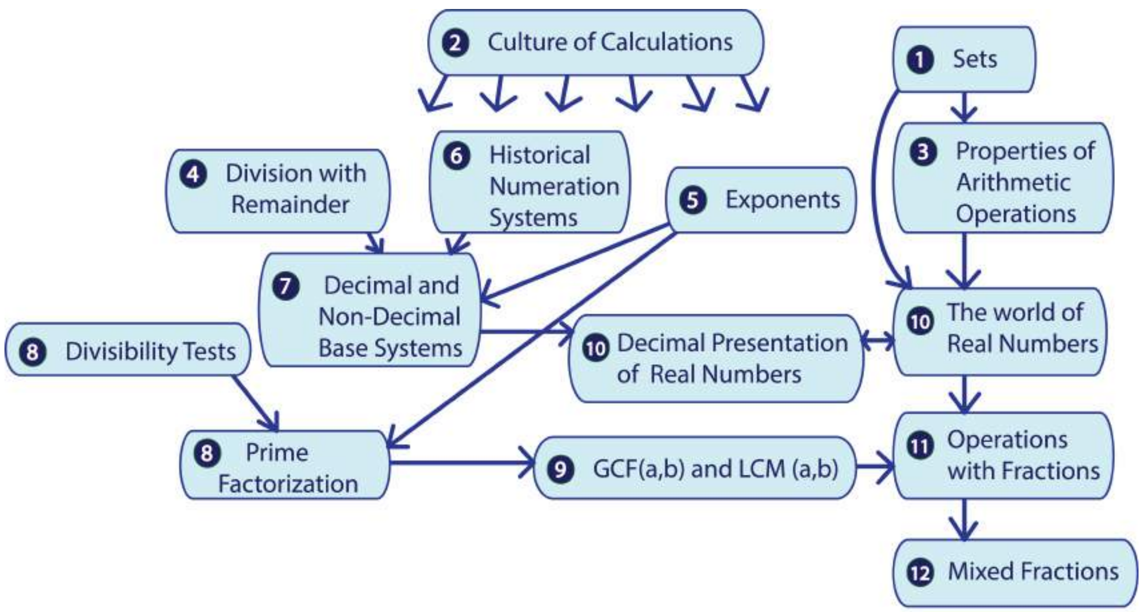
0. INTRODUCTION

0.1 About This Book

This book is a write-up of lectures given by the author at Kansas State University. The main audience of the course are students with prospective careers in elementary school education. Math courses like this determine the success of future generations in mathematics, sciences, and technology.

Note that this course is not about methodology of mathematics. The main subject of the course is mathematics itself. Our primary goal is to achieve a deeper understanding of notions that stand behind basic mathematics of elementary school. In these lectures, we review elementary mathematics within the larger picture of modern mathematics.

The following diagram describes the relations between chapters.



Each chapter includes the overview of definitions, methods, and solved examples. A list of exercises is provided at the end. It is planned to publish the list of answers to the exercises in a separate appendix.

0.2 About the Author

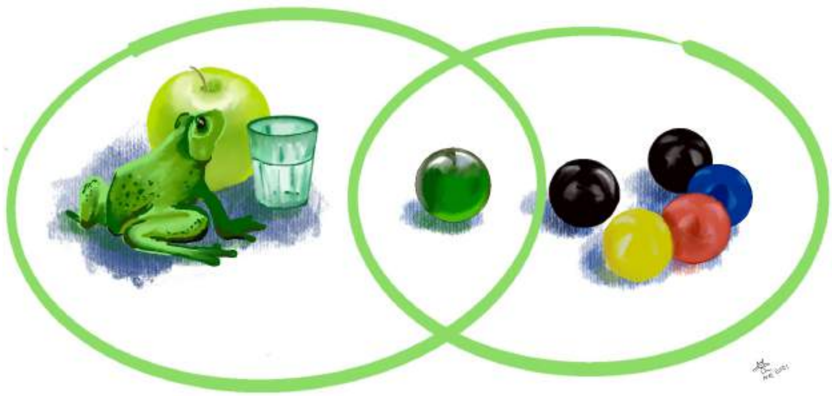
Natasha Rozhkovskaya is Professor of Mathematics at Kansas State University, USA. Her research interests are representation theory and quantum integrable systems. She is a co-author of one research monograph and the author of two books on popularization of mathematics. Her twenty years of teaching experience include a broad range of math classes, from math circles for seven-years old to advanced topic courses in representation theory for graduate students.

0.3 Acknowledgments

The author would like to thank all the students of Math 320 course for their active participation in the class. She is very grateful to her esteemed colleague Professor Gabriel Kerr for valuable comments and recommendations that improved the text significantly. She expresses her special gratitude to the editor Tamara Rozhkovskaya for her generous help in preparation of the lectures for publication. The project is supported by The Open/Alternative Textbook Initiative at Kansas State University.

1. SETS

1.1 The Language of Sets



Sets

No matter what kind of objects we count in our daily life (for example, apples, birds, money, days, people), we use the same mathematics. And it always works since an abstract notion of a number does not need to know the characteristics of the objects being counted. In this chapter, we go further and introduce an abstract language of sets. It describes mathematically *collections of objects*.

A *set* is a collection of objects. Objects in the collection are called *elements* of the set.

Example 1.1 We can talk about the set of three colors

$$S = \{\text{Red, Blue, Green}\}$$

This set is defined by *listing* its elements. In mathematical notation, the elements of a set in the list are separated by commas and the list is enclosed with the braces { and }. Sets are usually denoted by capital letters. We used the letter S to denote the set.

Example 1.2 Let K be the set of states that have common border with Kansas. We described this set by words, using the *defining property* of its elements. We can also list all elements of the set K:

$$K = \{\text{Nebraska, Missouri, Colorado, Oklahoma}\}$$

The set K in Example 1.2 contains four elements. In mathematical notation, we write this fact as $|K| = 4$. We say that A is a *finite set* if it consists of a finite number of elements. In this case, the elements can be listed. Sets which are not finite are called *infinite*.

Remark The notation $\#K$ and $n(K)$ for the number of elements of a set is also commonly used.

Example 1.3 The set of all counting numbers $\{1, 2, 3, \ldots\}$ is infinite. The set $\{1, 2, 3, 4, 5\}$ is finite.

In many cases, a set can be defined in one of the following ways:

- through *verbal description*
- *listing elements* of the set
- using *formal mathematical symbols* that describe the defining properties of elements of the set

Example 1.4 Let A be the set of all counting numbers less than 13. This set is described by words (verbal description). The list of its elements provides another description

$$A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$$

One can also convert the verbal description to mathematical formulas. The result may look like

$$A = \{a \text{ is a counting number} \,|\, a < 13\}$$

Here, "a is a counting number" specifies that the set consists of numbers of a certain type and $a < 13$ specifies that we want to consider the numbers possessing this particular property.

$a \in A$ means that *a is an element of a set A*, while $a \notin A$ means that *a is not an element of a set A*.

Example 1.5 $B = \{1, 2, 3, 4, 5\}$. Then $2 \in B$, but $7 \notin B$. Let $A = \{$☺, ✿, ☆$\}$. Then ☺ $\in A$, but ☹ $\notin A$.

The symbol $\varnothing$ denotes the *set containing no elements*. This set is called the *empty set*.

Example 1.6 The set of people who live on the planet Saturn has no elements. This set is empty.

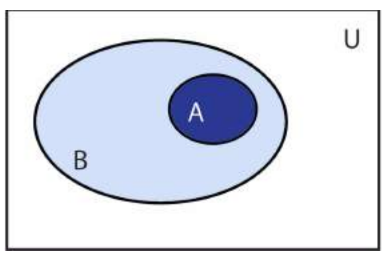
1.2 Examples of Sets of Numbers

We work with different sets of numbers that play an important role in mathematics.

- *Counting numbers* are $\{1,2,3,4,5,6,\ldots\}$.
- *Whole numbers* include zero and counting numbers $\{0,1,2,3,4,5,6,\ldots\}$.
- *Even counting numbers* include counting numbers that are multiples of two $\{2,4,6,8,\ldots\}$.
- *Odd counting numbers* include counting numbers that are not multiples of two $\{1,3,5,7\ldots\}$.

1.3 Subsets and Equal Sets

We introduce other notions of set theory with the help of diagrams. These diagrams are called *Venn diagrams*.



$A \subset B$

The rectangle represents the *universal set* U. The universal set is our "world." For example, if we solve a problem about people, U is the collection of all people; we do not consider chairs, cats, dogs, or any other objects. The ovals A and B represent sets of elements from the universal set U. We also see that A is contained in B, which represents the situation where A is a *subset* of B.

A set A is called a *subset* of a set B if every element of A is also an element of B. We write $A \subset B$.

Example 1.7 $A = \{a,b,c,d,e\}$ and $B = \{a,c,f,e,b,g,d\}$. Is it true that $A \subset B$?

Solution It is true since every element of A can be found in the list of elements of B.

Example 1.8 Are these statements true for any set A:

(a) $A \subset A$ (b) $\varnothing \subset A$

Solution Both statements are true.

(a) $A \subset A$ since every element of A belongs to A.

(b) The argument behind this statement is a little bit twisted. The empty set $\varnothing$ contains no elements. In particular, it contains no elements that do not belong to A. Hence we conclude that $\varnothing$ is a subset of any set.

A subset A of a set B is called *proper* if it is not the whole set and is not empty: $A \subset B$, $A \neq B$, and $A \neq \varnothing$.

Example 1.9 List all subsets of the set $S = \{1,2,3,4\}$.

Solution

- Subsets with four elements: $\{1,2,3,4\}$ (this is the set S itself).
- Subsets with three elements: $\{1,2,3\}$, $\{1,2,4\}$, $\{1,3,4\}$, and $\{2,3,4\}$.
- Subsets with two elements: $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{2,3\}$, $\{2,4\}$, and $\{3,4\}$.
- Subsets with one element: $\{1\}$, $\{2\}$, $\{3\}$, and $\{4\}$.
- Subsets with no elements: $\varnothing$.

There are 16 subsets of the set of four elements. The subsets S and $\varnothing$ are not proper, whereas the remaining subsets are proper.

We say that two sets A and B are *equal* and write $A = B$ if A and B are identical as collections of elements.

Note that when we compare sets with listed elements, the order of elements in the list is not important.

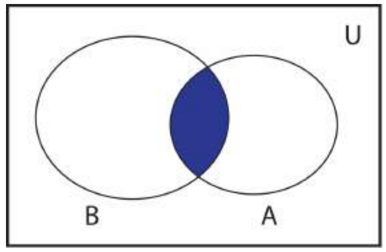
Example 1.10 $\{a,b,c,d\} = \{c,b,d,a\} = \{b,a,d,c\} = \ldots$

Also, listing an element two or more times does not change the set. Usually, we list each element only once.

Example 1.11 $\{a,d,d,d\} = \{a,d\}$.

1.4 Operations with Sets

The common part of two ovals in the diagram represents the *intersection* of sets A and B.



$A \cap B$

> $A \cap B$ denotes the set of all elements that belong to both A and B. This set is called the *intersection* of A and B.

Example 1.12

(a) $\{a, b\} \cap \{c, d, e, f\} = \varnothing$

(b) $\{a, b, c, f\} \cap \{c, d, e, f\} = \{c, f\}$

(c) $\{1, 2, \text{✿}, A\} \cap \{1, 2, \text{☆}, 3, A\} = \{1, 2, A\}$

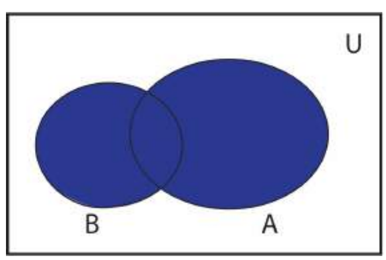
(d) $A \cap \varnothing = \varnothing$

Example 1.13 $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 4, 6, 8, 10\}$. List all elements of $A \cap B$.

Solution $A \cap B = \{2, 4\}$.

Remark To find all elements of the intersection, one can organize the work as follows. For each element of A check if it belongs to B. If yes, include this element in the intersection. If no, skip it and check the next one.

The shaded part of the diagram is called the *union* of A and B.



$A \cup B$

> $A \cup B$ denotes the set of all elements that belong to A or B (or both). This set is called the *union* of A and B.

Example 1.14

(a) $\{a,b\} \cup \{c,d,e,f\} = \{a,b,c,d,e,f\}$

(b) $\{a,b,c,f\} \cup \{c,d,e,f\} = \{a,b,c,d,e,f\}$

(c) $\{1, 2,$ ☆, ●$\} \cup \{1,$ ☆, $3, A\} = \{1,2,3,$ ☆, ●, $A\}$

(d) $A \cup \varnothing = A$

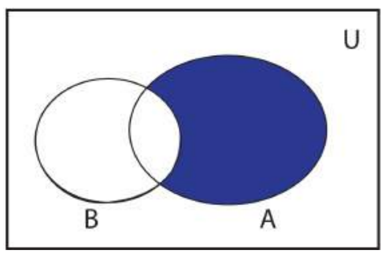
Example 1.15 $A = \{a,b,c\}$ and $B = \{b,c,d,e\}$. List all elements of $A \cup B$.

Solution $A \cup B = \{a,b,c,d,e\}$.

Remark To find all elements of the union, one can organize the work as follows. List all elements of A first, and then add all elements of B that are not yet included in the list.

Remark The symbols for the union and intersection are often confused. One can memorize that $\cup$ looks like the letter U and hence stands for *Union*.

The shaded part of the diagram is called the *difference* of the set B from the set A.



$A - B$

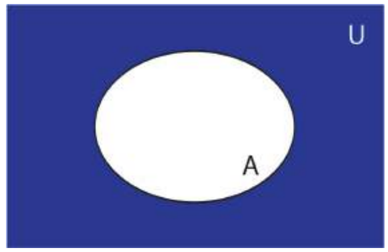
$A - B$ denotes the set of all elements of A that do not belong to B: $x \in A$, but $x \notin B$. This set is called the *difference* of the set B from the set A.

Example 1.16 $A = \{a,b,c,d,e\}$ and $B = \{b,c,d,e,f,g\}$. List all elements of $A - B$ and $B - A$.

Solution $A - B = \{a\}$ and $B - A = \{f,g\}$.

Remark $A - B$ and $B - A$ are different sets!

The area outside a set A is called the *complement* of A.



$\overline{A}$

$\overline{A}$ denotes the set of all elements in the universal set U that do not belong to A. This set is called the *complement* of A.

Remark In some books, the notation A^c is used for the complement of a set A.

The notion of $\overline{A}$ depends on the universal set, which can be seen from the following example.

Example 1.17

(a) Let U be the set of whole numbers, and let A be the set of odd counting numbers. Describe $\overline{A}$.

(b) Let U be the set of counting numbers, and let A be the set of odd counting numbers. Describe $\overline{A}$.

Solution

(a) $\overline{A} = \{0, 2, 4, 6, 8 \ldots\}$

(b) $\overline{A} = \{2, 4, 6, 8 \ldots\}$

Note that the answers to (a) and (b) are different because the universal sets are different.

Example 1.18 Find

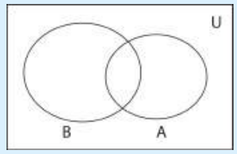
(a) $\varnothing \cap A$ (b) $\varnothing \cup A$ (c) $A - A$

Solution

(a) $\varnothing \cap A = \varnothing$ (b) $\varnothing \cup A = A$ (c) $A - A = \varnothing$

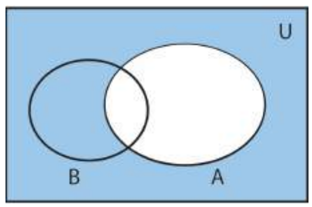
1.5 Operations with Sets by Venn Diagrams

Example 1.19 Shade the region representing $\overline{A} \cap \overline{B}$.

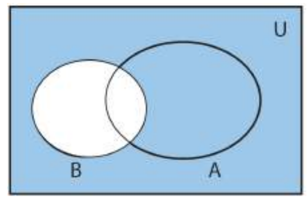


Solution Problems like this one become easier if we break it up into separate steps. First, let us shade $\overline{A}$ and $\overline{B}$.

$\overline{A}$

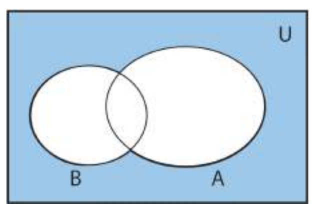


$\overline{B}$

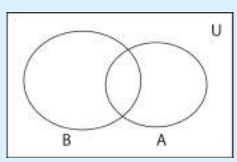


The intersection of $\overline{A}$ and $\overline{B}$ contains the parts of the diagram that are shaded in *both* intermediate sketches.

Answer: $\overline{A} \cap \overline{B}$

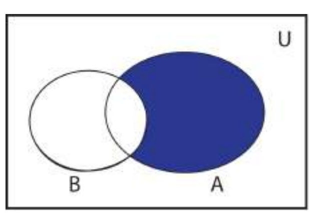


Example 1.20 Shade the region representing $(A - B) \cup (B - A)$.

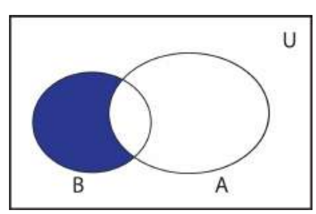


Solution First, we draw the diagrams of $(A - B)$ and $(B - A)$.

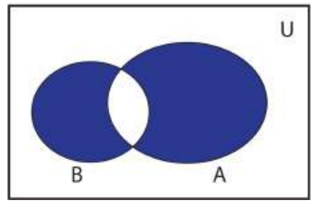
$A - B$



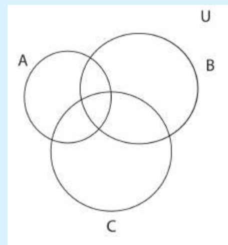
$B - A$



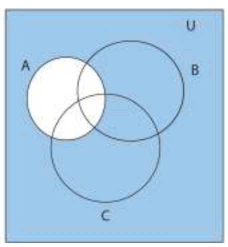
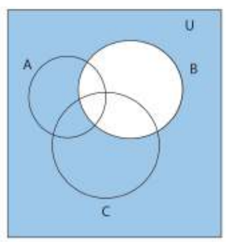
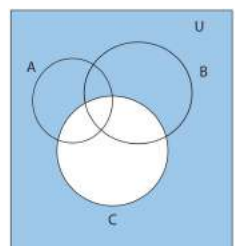
The union of these two sets contains the parts that are shaded in *at least one* intermediate sketch.

Answer: $\overline{A} \cap \overline{B}$ 

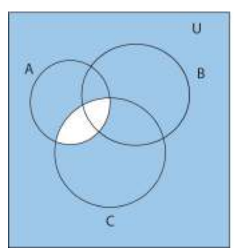
Example 1.21 Shade the region representing $(\overline{A} \cup \overline{C}) \cap \overline{B}$.



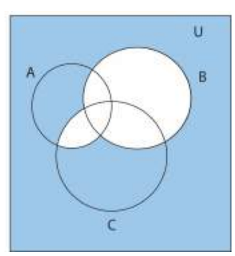
Solution Again, we do one step at a time. First, we shade in the intermediate diagrams of $\overline{A}$, $\overline{B}$, and $\overline{C}$.

$\overline{A}$  $\overline{B}$  $\overline{C}$ 

Then we shade $\overline{A} \cup \overline{C}$ which corresponds to at least one shading in the diagrams of $\overline{A}$ and $\overline{C}$.

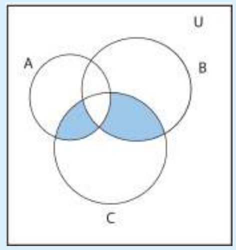
$\overline{A} \cup \overline{C}$ 

The final answer corresponds to the shading in both diagrams of $\overline{A} \cup \overline{C}$ and $\overline{B}$.

Answer: $(\overline{A} \cup \overline{C}) \cap \overline{B}$ 

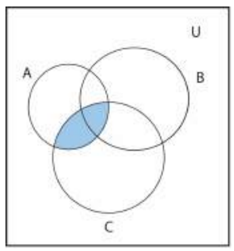
We learned to represent formulas with set operations by diagrams. Let us go in the opposite direction and reconstruct a formula from a shaded diagram.

Example 1.22 Represent the shaded region as a result of operations with sets.

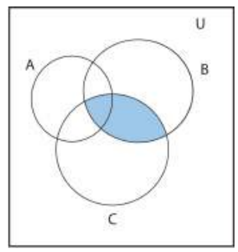


Solution Note that there can be more than one possible solution of this problem. The two parts of the shaded region remind us of intersections.

$A \cap C$

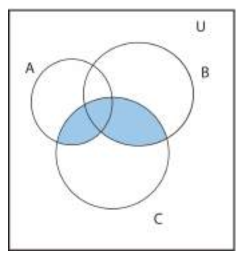


$B \cap C$



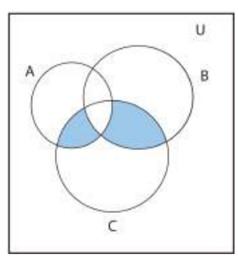
If we take the union of $(A \cap B) \cup (B \cap C)$, we almost get the desired picture, but with one extra shaded area.

$(A \cap C) \cup (B \cap C)$



This extra shaded area is $A \cap B \cap C$. We can remove it by taking the difference.

$(A \cap C) \cup (B \cap C) - A \cap B \cap C$



Answer: $(A \cap C) \cup (B \cap C) - A \cap B \cap C$.

1.6 Word Problems Solved by Venn Diagrams

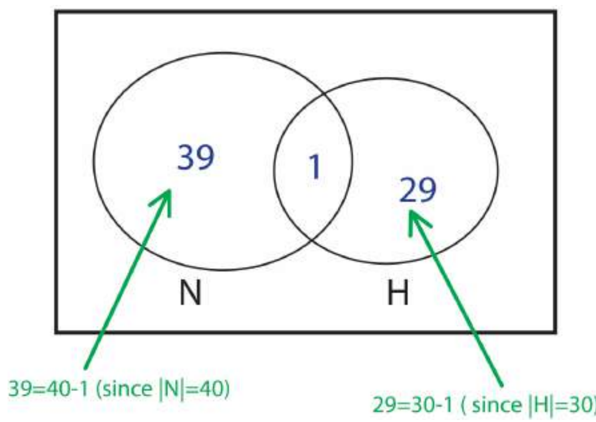
There are word problems of a certain type that can be solved effectively with the help of Venn diagrams.

Example 1.23 In a class, 40 students visited Nebraska, 30 students visited Ohio, one student visited both Nebraska and Ohio. How many students visited Nebraska or Ohio (at least one of these states)?

Solution In problems like this one, we want to avoid counting the same elements twice. We can write the problem in the language of sets. Let N be the set of students who visited Nebraska, and let H be the set of students who visited Ohio:

$$|N| = 40, \quad |H| = 30, \quad |N \cap H| = 1, \quad |N \cup H| = ?$$

In each part of the diagram, we write the number of elements, starting with $|N \cap H| = 1$. We write the number that corresponds to the bounded part. The whole oval N has 40 elements. Thus, the remaining part without the intersection has $40 - 1 = 39$ elements. Similarly, the remaining part of H has $30 - 1 = 29$ elements.



Thus, the number of elements of the union $N \cup H$ is just the sum of numbers in the parts:

$$|N \cup H| = 39 + 1 + 29 = 69$$

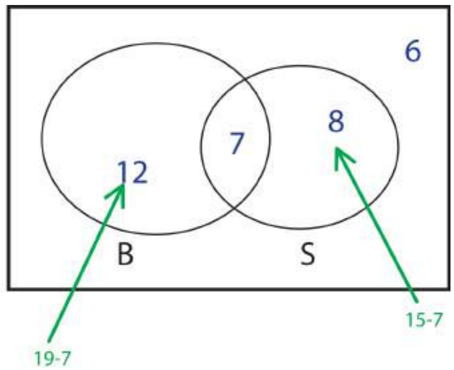
Answer: 69 students.

Example 1.24 In a science classroom, 19 students have at least one brother, 15 students have at least one sister, 7 students have at least one brother and at least one sister, and 6 students do not have any siblings at all. How many students are in the classroom?

Solution We formulate the problem in the language of sets. Let U be the set of all students in the class. It is the universal set. Let B be the set of students with at least one brother, and let S be the set of students with at least one sister. Then, in terms of sets, we have

$$|B| = 19, \quad |S| = 15, \quad |B \cap S| = 7, \quad |\overline{B \cup S}| = 6\,(\text{outside } B \text{ and } S), \quad |U| = ?$$

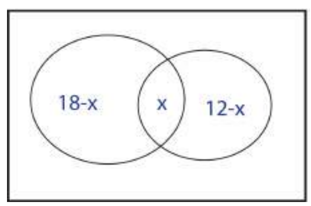
In each part of the diagram, we write the number of elements, starting from $|B \cap S| = 7$.



Then $|U| = 12 + 7 + 8 + 6 = 33$.
Answer: 33 students.

Example 1.25 Find $|S \cap T|$ if $|S| = 18$, $|T| = 12$, and $|S \cup T| = 23$.

Solution We need to find $|S \cap T|$, so we cannot fill in numbers in the intersection first as in the previous examples. Nevertheless, let us denote $x = |S \cap T|$ and express the values in the parts of the diagram through x.



If we add all the parts, we get the number of elements of $S \cup T$. This allows us to find x:

$$\begin{aligned} |S \cup T| &= 18 - x + x + 12 - x = 30 - x \\ 23 &= 30 - x \\ x &= 7 \end{aligned}$$

Answer: $|S \cap T| = 7$.

1.7 Exercises

1.7.1 Examples of sets

Exercise 1.1 Describe the following sets by the listing method or indicate that the set is empty.

(a) The whole numbers less than 8.

(b) The odd counting numbers between 4 and 20.

(c) The whole numbers less than 0.

(d) The odd counting numbers less than 10 and divisible by four.

(e) The whole numbers between 4 and 12 (inclusive).

(f) The whole numbers strictly less than 15, but not less than 5.

(g) The even counting numbers less than 15.

(h) The odd counting numbers less than 20, but greater than 18.

Exercise 1.2 Which of the following sets are equal to $\{0,1,2,3,4\}$?

(a) $\{4,3,2,1,0\}$.

(b) $\{0,4,2,6.1\}$.

(c) The whole numbers less than 5.

(d) The counting numbers less than 5.

Exercise 1.3 Which of the following sets are equal to $\{6,8,10\}$?

(a) $\{10,8,6\}$.

(b) $\{6,6,10,8\}$.

(c) The even numbers between 5 and 11.

(d) The intersection of the sets $\{1,2,3,4,5,6,7,8,9,10\}$ and $\{5,6,7,8,9,10,11\}$.

(e) The intersection of the sets $\{1,2,3,4,5,6,7,8,9,10\}$ and $\{6,8,10,12\}$.

1.7.2 Subsets

Exercise 1.4 List all subsets of the set {❀, ☺, A}. Which of them are proper?

Exercise 1.5 List all subsets of the set $\{1,2,3,4\}$. Which of them are proper?

Exercise 1.6 List all subsets of the set $\{A,B,C\}$. Which of them are proper?

Exercise 1.7 True or false?

(a) empty set is a subset of itself

(b) set of odd counting numbers is infinite

(c) intersection of two infinite sets can be finite

(d) union of two infinite sets can be finite

1.7.3 Operations with sets

Exercise 1.8 $A = \{1,2,3,4\}$, $B = \{4,5,6,7\}$, and $C = \{1,2,7,8\}$. List elements of the sets

(a) $A \cup B$ (b) $A \cap B$ (c) $(A \cup B) \cap C$ (d) $A \cup (B \cap C)$

Exercise 1.9 $A = \{a,b,c,d\}$, $B = \{b,c\}$, and $C = \{e\}$. List elements of the sets

(a) $A \cup B \cup C$ (c) $A - C$ (e) $B - A$

(b) $A \cap B$ (d) $A - B$

Exercise 1.10 $X = \{1,2,3\}$, $Y = \{3,4,5\}$, and $Z = \{4,5,1\}$. List elements of the sets

(a) $X \cup Y$ (c) $X \cup Z$ (e) $Y \cup Z$

(b) $X \cap Y$ (d) $X \cap Z$ (f) $Y \cap Z$

Exercise 1.11 $A = \{1,2,3\}$ and $B = \{2,3,4\}$. List elements of the sets

(a) $A \cup B$ (b) $A \cap B$ (c) $A - B$

Exercise 1.12 $A = \{1,3,5,7,9\}$ and $B = \{2,4,6,8,10\}$. List elements of the sets

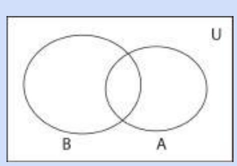
(a) $A \cup B$ (b) $A \cap B$ (c) $A - B$

Exercise 1.13 $A = \{1,2,3\}$, $B = \{2,3,4\}$, and $C = \{3,4,5\}$. List elements of the sets

(a) $A \cup B \cup C$ (b) $A \cap B \cap C$ (c) $(B - A) \cap C$

1.7.4 Representation of operations with sets by Venn diagrams

Exercise 1.14 Draw Venn diagrams of the sets A and B. Shade the area that represents the result of operations.



(a) $\overline{A} \cup B$

(b) $(A - B) \cup B$

(c) $A \cup (B - \overline{A})$

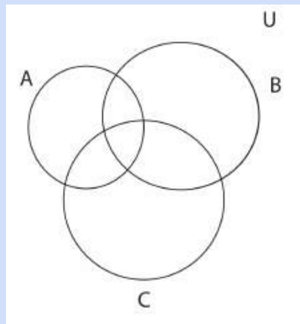
(d) $\overline{A} \cup \overline{B}$

(e) $\overline{A} \cap B$

(f) $(A - B) \cup (B - A)$

(g) $(A - B) \cap (B - A)$

Exercise 1.15 Draw Venn diagrams for the sets A, B, and C. Shade the area that represents the result of operations.



(a) $(A \cap B) \cap C$

(b) $(A \cup B) \cap C$

(c) $(A \cup C) \cap B$

(d) $(A \cup C) \cap \overline{B}$

(e) $(A \cup C) - (B \cap C)$

(f) $A \cup (C - B)$

(g) $(A \cup B) - (B \cap C)$

(h) $(\overline{A} \cup \overline{C}) \cap \overline{B}$

(i) $\overline{A} - (C \cap \overline{B})$

1.7.5 Word problems solved by Venn diagrams

Exercise 1.16 All the students in a class play soccer or football. How many students are there in the class if 18 students play soccer, 12 students play football, and 5 play both soccer and football?

Exercise 1.17 In a class, 50 students visited Nebraska, 40 students visited Ohio, 10 students visited both Nebraska and Ohio. How many students visited Nebraska or Ohio?

Exercise 1.18 At a breakfast buffet, 117 guests chose coffee, 133 guests chose juice, and 27 guests chose both coffee and juice. If each person choses at least one of these beverages, how many people visited the buffet?

Exercise 1.19

(a) Find $|S \cup T|$ if $|S| = 10$, $|T| = 8$, and $|S \cap T| = 2$

(b) Find $|S \cup T|$ if $|S| = 15$, $|T| = 6$, and $|S \cap T| = 5$

(c) Find $|S \cap T|$ if $|S| = 18$, $|T| = 7$, and $|S \cup T| = 23$

(d) Find $|S \cap T|$ if $|S| = 10$, $|T| = 9$, and $|S \cup T| = 17$

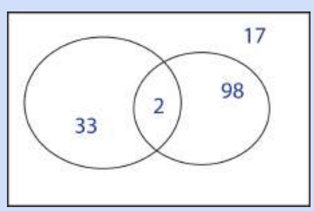
Exercise 1.20 In a science classroom, 20 students have at least one brother, 10 students have at least one sister, 5 students have both at least one brother and at least one sister, and 10 students do not have any siblings at all. How many students are in the classroom?

Exercise 1.21 $|A| = 8$ and $|A \cup B| = 8$. Is it true that $B \subset A$? Explain.

Exercise 1.22 $|A| = 10$ and $|A \cup B| = 10$. What can you say about $B - A$?

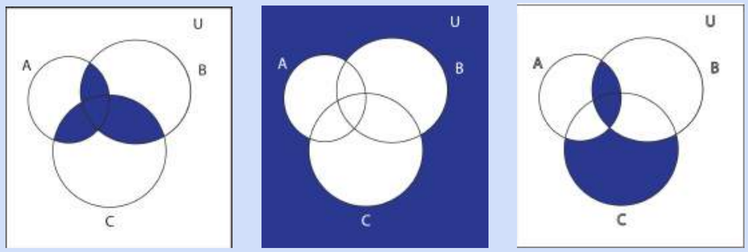
Exercise 1.23 Students were offered two selective courses: theater and ceramics. The council made four lists: students who selected only theater, students who selected exactly one course from these two, students who selected at least one course from these two, and students who selected both courses. Which of the lists is the longest?

Exercise 1.24 Create a problem that can be solved by the diagram



Exercise 1.25 Anita drew a diagram representing the sets of animals, animals with long tails, mammals, and cats. Which of the following pictures is Anita's diagram?

Exercise 1.26 Represent the following shaded regions as the results of applying the operations $\cap$, $\cup$, and $-$ to the sets A, B, and C.



Exercise 1.27

(a) Recall the definitions of geometric shapes: parallelogram, rhombus, rectangle, square. Draw Venn diagrams representing inclusions of sets of these types of quadrilaterals.

(b) On a piece of paper, Joseph drew 19 rectangles, 15 rhombuses, and 7 squares. How many parallelograms did Joseph draw?

2. CULTURE OF CALCULATIONS

Counting numbers.

2.1 Important Reasons to Review Calculation Methods

This may be one of the most important chapters of this course. We will solve many basic arithmetic exercises very similar to the ones that we used to do in school. You may wonder why we should go over all of these things again. The answer is that these basic exercises will put our discussion into the light of the *culture of mathematics*.

Basic exercises of this and further chapters have several objectives.

- We bring our computational skills to a higher, more professional level.
- We focus not only on answers, but also on elegant and efficient solutions.
- We discuss the reasons for performing some calculations in a certain way.
- We develop a stronger sense and intuition of mathematics.
- We learn about "personalities" of numbers and how to become better "friends" with them.

- Maybe, we will be able to get rid of some ineffective habits in calculations.
- Finally, with our new professional level we become true *ambassadors of mathematical culture* for future generations.

2.2 About Calculators

As in many math courses, in this chapter we specifically ask you to *avoid the use of calculators*.

Question Why do you think students are asked to work without calculators in many math classes?

Possible answer. Here is our version of an answer, you may certainly have a different opinion.

No doubt, calculators are very useful for tedious tasks. Yet, many easy problems can be (and in most cases should be) solved without the aid of technology, and here are some arguments for that. Solving even the most basic math problem gives more than just a review of arithmetic operations. It is even more than obvious training in logic and analytical thinking.

- Every math calculation is an exercise in planning, setting the key elements, analyzing the results, controlling possible mistakes.
- Working without a calculator fosters our self-confidence. We gain true independence from technology. We continue to believe in ourselves and our skills. It is crucial for solving advanced problems.
- Many teachers admit that very often difficulties in STEM courses are due not to the complexity of the new material, but to poor foundation in basic arithmetic skills.
- As we benefit from physical exercises (even though riding a car is easier), we benefit from mental activities unaided by technology: every calculation without a calculator is a step of intellectual self-improvement.
- It is not a secret that good counting skills have an impact on the quality of our daily life. Unnoticeably for ourselves, we quickly and confidently make small household calculations in our minds, often gaining benefits of a value while shopping, making a budget, and in professional activities.

Let us keep these arguments in mind and work on the exercises of this chapter without a calculator.

2.3 Incorrect Usage of Equal Sign

First, we would like to discuss not how to solve problems, but how to *write solutions*.

Consider the following problem and its "solution."

Example 2.1 Find the sum of the first six consecutive counting numbers.

"Solution." To find the sum of the first six consecutive numbers $1,2,3,4,5,6$, we add

$$1+2=3+3=6+4=10+5=15+6=21$$

Answer: 21.

Comments on the "solution." The idea of the solution is correct, and the answer is correct. Yet, the solution itself is written improperly: it contains *wrong statements* which read as false equalities

$$\begin{aligned}
&1+2=3=(!)6=3+3\\
&3+3=6=(!)10=6+4\\
&6+4=10=(!)15=10+5\\
&10+5=15=(!)21=15+6
\end{aligned}$$

Since each part of the equality has a different value, the whole argument looses its validity. This is an example of *incorrect usage of the equal sign*, which is, unfortunately, widespread in papers of students of all ages, in school, and at a college level.

Probably, this sloppiness comes from a desire to save time and effort. However, such "shortcuts" should be avoided since incorrectly placed equal signs produce a lot of confusion with serious consequences for students. Example 2.1 is very simple, and we have had no difficulties in understanding the steps, even though they were not connected properly, but, in more complicated situations, a carelessly placed equal sign may mess up the whole argument. Not only the reader (for example, a grader of the homework) would get confused, but the author of the solution would risk getting lost in their own invalid statements and make other mistakes.

Question If the solution of Example 2.1 is not written properly, what would be a *correct way* to write it?

Answer We can suggest two standard correct ways to write down solutions of this kind.

Solution 1. To find the sum of the first six consecutive numbers 1, 2, 3, 4, 5, 6, we add

$$\begin{aligned}
&1+2=3\\
&3+3=6\\
&6+4=10\\
&10+5=15\\
&15+6=21
\end{aligned}$$

Answer: 21.

In Solution 1, we write each step separately. Note that *each line is a correct statement*.

Solution 2. To find the sum of the first six consecutive numbers 1, 2, 3, 4, 5, 6, we add

$$1+2+3+4+5+6=3+3+4+5+6=6+4+5+6=10+5+6=15+6=21$$

Answer: 21.

In Solution 2, we write all arguments and intermediate steps in one line, but we are careful to *include all necessary information at each step* to keep equalities true.

2.4 Parentheses and Imposed Order

Example 2.2 One student wrote

$$50-30-20=40$$

and another student wrote

$$50-30-20=0.$$

Who has a correct answer? What could be a reason behind the mistake of the other student?

Solution It is clear that the second student has the correct answer to the problem. We may guess that the first student wanted to do the calculation in a different order, but forgot to put *parentheses* to indicate the change of the order of operations:

$$50-(30-20)=40$$

Let us look at more examples of calculations where the order is changed with the help of parentheses.

Example 2.3 Compute $100-2\times30-20$.

Solution $100-2\times30-20=100-60-20=20$.

Example 2.4 Put the parentheses in the statements to make them correct equalities:

(a) $100-2\times30-20=80$ (b) $100-2\times30-20=980$

Solution

(a) $100-2\times(30-20)=80$ (b) $(100-2)\times(30-20)=980$

We discussed the significance of proper placement of the equal sign and parentheses. More generally, formulas in mathematics are like sentences in a language: symbols and their placement are important for the meaning of the statement. We need to keep this in mind when we explain our solutions to others and teach our students these elements of mathematical culture.

2.5 Some Techniques for Better Calculations

We decided to work on easy problems without a calculator. Let us review some standard techniques that will broaden the collection of problems that we can call *easy*.

2.5.1 Regrouping of terms

Example 2.5 Compute $703+64+7$ without a calculator.

Solution If you can do this calculation in your head, that is really good! In general, this should not be a difficult mental calculation for anyone, especially since this is an example where *regrouping of terms* helps make everything very simple:

$$703+64+7 \quad = \underbrace{703+7}_{\text{add first}}+64 = 710+64 = 774$$

We used that $3+7=10$, and 10 is a "nice" number that is easy to add to other numbers. Sometimes, changing the order of terms leads to simpler and more elegant calculations. In some sense, regrouping is based on a search for terms that are "good friends with each other" meaning that, added together, they produce nice round numbers. Examples below illustrate the method.

Example 2.6 Find the sum $22+46+18$.

Solution We can group together 22 and 18 since they add up to a round number:

$$22+46+18 \quad = \underbrace{22+18}_{\text{add first}}+46 = 40+46 = 86$$

Example 2.7 Find the sum $134+408+166$.

Solution We change the order to use that 134 and 166 add up nicely to a round number:

$$134+408+166 \quad = \underbrace{134+166}_{\text{add first}}+408 = 300+408 = 708$$

Example 2.8 Find the sum $1+2+3+4+5+6+7+8+9$.

Solution A trick for this sum is also based on regrouping of terms in pairs "small + large," where every pair produces the same sum:

$$\begin{aligned} 1+2+3+4+5+6+7+8+9 &= (1+9)+(2+8)+(3+7)+(4+6)+5 \\ &= 10+10+10+10+5 = 4\times 10+5 = 45 \end{aligned}$$

Each pair adds up to 10, there are four pairs, and 5 in the middle is left without a pair, so we add it separately.
Answer: 45.

Example 2.9 Find the sum $1+2+3+4+\cdots+96+97+98+99$ of all the numbers from 1 to 99.

Solution The same trick works here. Even in this huge sum of 99 numbers, we can match them into pairs that add up to the same number:

$$1+2+3+4+\cdots+96+97+98+99$$

The sum in each pair is 100, so we need to figure out how many pairs are there. For this purpose let us look closer at the terms in the middle of this sum

$$\ldots 48+49+50+51+52\ldots$$

Note that there are 49 pairs that add up to 100 and 50 is left in the middle without a pair. We conclude that

$$1+2+3+4+\cdots+96+97+98+99 = 49\times 100+50 = 4950$$

Answer: 4950.

Example 2.10 Find the sum $11+12+13+14+15+16+17+18+19$.

Solution Using the same trick, we find

$$11+12+13+14+15+16+17+18+19 = 4\times 30+15 = 135$$

Answer: 135.

Example 2.11 A teacher graded an exam and put the student's scores in a table. Now, the scores in the table must be added. Even with a calculator it is faster to insert for each student 3 or 4 numbers than 10 numbers. In each example below, show how the teacher can combine scores or use other techniques to make calculations more efficient.

(a)

Problem	1	2	3	4	5	6	7	8	9	10
Score	10	10	10	5	5	4	3	10	10	10

(b)

Problem	1	2	3	4	5	6	7	8	9	10
Score	10	10	10	10	8	10	10	8	8	10

Solution (a) For the scores of the first student we can do the following steps.

Step 1. Count how many the 10's there are in the row. This gives 6×10.

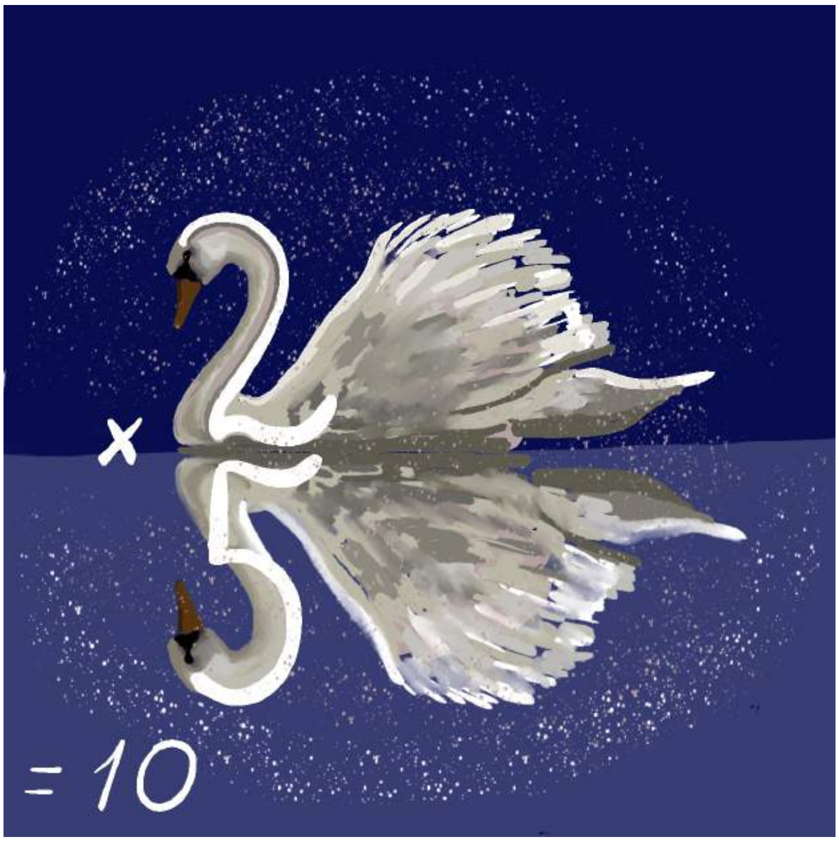
Step 2. Note that $5+5=10$.

Then the computation of the total score reduces to the calculation

$$6 \times 10 + (5+5) + 4 + 3 = 60 + 10 + 7 = 77$$

(b) The scores of the second student are short of the perfect scores two points in three problems, which can be used for the calculation

$$10 \times 10 - 2 \times 3 = 100 - 6 = 94$$



Two and five are very good friends.

2.5.2 Two and Five Are Friends Forever

Example 2.12 Compute 24×25 without a calculator.

Solution Recall that $25 \times 4 = 100$. It would be nice to use this fact since the multiplication by 100 is easy. Indeed, we note that 24 contains 4 as a factor:

$$24 = 6 \times 4$$

From these two observations we conclude that

$$24 \times 25 = 6 \times \underbrace{4 \times 25} = 6 \times 100 = 600.$$

Answer: 600.

In Example 2.12, we used a rule that informally can be stated as follows.

> Two and five are very good friends.

Indeed, recall that

$$
\begin{aligned}
&2 \times 5 = 10 \\
&2^2 \times 5^2 = 4 \times 25 = 100 \\
&2^3 \times 5^3 = 8 \times 125 = 1000 \\
&\dots\dots\dots\dots\dots\dots
\end{aligned}
$$

> When multiplying numbers, look for copies of 2 and 5 hidden in the factors that could be matched to create easy factors 10, 100, 1000,....

Examples below illustrate how to use this advice.

Example 2.13 Compute $4 \times 17 \times 5 \times 5$ without a calculator.

Solution We match two copies of 5 with two copies of 2 "hidden" in $4 = 2 \times 2$ to solve this problem easily:

$$4 \times 17 \times 5 \times 5 = 2 \times 2 \times 17 \times 5 \times 5 = 10 \times 17 \times 10 = 1700.$$

Example 2.14 Compute 32×125 without a calculator.

Solution This example illustrates that it is *useful to recognize powers of 2 and 5*. We note that $32 = 2^5$ and $125 = 5^3$, which helps us to make the very simple calculation

$$32 \times 25 = 2 \times 2 \times 2 \times 2 \times 2 \times 5 \times 5 \times 5 = 2 \times 2 \times 10 \times 10 \times 10 = 4000.$$

Example 2.15 Compute without a calculator:

(a) 8×25 (b) 28×25 (c) $13 \times 25 \times 2 \times 6$

Solution

(a) $8 \times 25 = 2 \times 4 \times 25 = 2 \times 100 = 200$

(b) $28 \times 25 = 7 \times 4 \times 25 = 7 \times 100 = 700$

(c) $13 \times 25 \times 2 \times 6 = 13 \times 25 \times 2 \times 2 \times 3 = 13 \times 100 \times 3 = 3900$

2.5.3 Take a Penny, Leave a Penny method for addition

Example 2.16 What is the value of $398 + 26$?

Solution We can use that one of the terms *is very close to a nice round number*. Observe that 398 is almost the nice round number 400. In particular, it would be easy to add

$$400 + 26 = 426$$

We can use this observation to solve the initial problem

$$398 + 26 = (400 - 2) + 26 = (400 + 26) - 2 = 426 - 2 = 424$$

Example 2.17 What is the value of $300 - 78$?

Solution We use the same trick, but in the opposite direction. The straightforward subtraction of 78 from the round number 300 forces us to borrow units, which is not very convenient. At the same time, it is easy to calculate

$$298 - 78 = 220$$

We use this observation to simplify the calculation

$$300 - 78 = (298 + 2) - 78 = (298 - 78) + 2 = 220 + 2 = 222$$

Example 2.18 What is the value of $198 + 234$?

Solution Observe that $198 = 200 - 2$. Then

$$198 + 234 = (200 - 2) + 234 = (200 + 234) - 2 = 434 - 2 = 432$$

2.5.4 Take a Penny, Leave a Penny method for multiplication

Similar ideas may be applied to multiplication problems.

Example 2.19 Compute 25×39 without a calculator.

Solution Using that 39 is almost 40, we get

$$25 \times 39 = 25 \times (40 - 1) = 25 \times 40 - 25 \times 1 = 25 \times 4 \times 10 - 25 = 1000 - 25 = 975$$

Example 2.20 Compute 35×101 without a calculator.

Solution Note that $101 = 100 + 1$, so

$$35 \times 101 = 35 \times (100 + 1) = 35 \times 100 + 35 \times 1 = 3500 + 35 = 3535$$

2.6 Distinctiveness of Numbers

We have discussed several methods that help us simplify common calculations. We make a couple of more general remarks.

- The methods above are based on the idea of reducing the required calculation to round numbers since the addition and multiplication of round numbers is not that difficult.
- While there are many tricks for faster computations, the key is not in memorizing them, but in establishing *good intuition* about numbers. Many people think that numbers are faceless digital abstractions, but this is absolutely not the case! Numbers, like people, have characters. Hopefully, this course helps us to become better friends with many useful numbers. Such an informal attitude with a personal touch towards numbers improves calculations in the same way as good knowledge of individual characteristics and capabilities of team members leads to efficient teamwork.

2.7 Exercises

A calculator should not be used in any exercise of this section.

2.7.1 Regrouping of terms

Exercise 2.1 Regroup the terms to simplify calculations. Compute the result.

(a) $505 + 19 + 45$

(b) $208 + 17 + 22$

(c) $7 + 456 + 93$

(d) $213 + 32 + 68$

(e) $22 + 205 + 595 + 78$

(f) $437 + 39 + 13$

(g) $122 + 73 + 58$

(h) $353 + 22 + 7$

(i) $133 + 101 - 13 + 149$

(j) $15 + 327 + 25$

(k) $256 + 1037 + 44$

(l) $46 + 22 + 18$

(m) $664 + 13 + 87$

(n) $119 + 56 + 11$

(o) $765 + 208 + 135$

(p) $143 + 96 + 57$

(q) $37 + 25 + 43$

(r) $78 - 15 + 89 + 22$

(s) $456 + 32 + 68$

Exercise 2.2 Regroup the terms to find the sum in a simple way.

(a) $2+7+5+27+14+18+15+6$

(b) $111+112+113+114+115+116+117+118+119$

Exercise 2.3 A teacher graded an exam and put the student's scores in a table. Now the scores in the table must be added. In each example below, show how the teacher can combine scores or use other techniques to make calculations more efficient.

(a)

Problem	1	2	3	4	5	6	7	8	9	10
Score	10	5	5	5	5	7	3	10	10	10

(b)

Problem	1	2	3	4	5	6	7	8	9	10
Score	10	10	10	10	8	9	10	10	9	10

(c)

Problem	1	2	3	4	5	6	7	8	9	10
Score	10	10	10	10	7	6	3	0	4	10

2.7.2 Take a Penny, Leave a Penny

Exercise 2.4 Show how to add or subtract 1 or 2 to make calculations simpler.

(a) $1819+1153$

(b) $348+19\,997$

(c) $999+2\,036$

(d) $1201-166$

(e) $2702-138$

(f) $56-29$

(g) $542+79$

(h) $359+596$

(i) $1301-168$

(j) $5,000-997$

Exercise 2.5 Show how to add or subtract 1 or 2 to make calculations simpler.

(a) 23×21

(b) 39×102

(c) 15×9

(d) 13×199

(e) 17×98

(f) 14×199

(g) 14×19

(h) 25×38

(i) 35×98

(j) 27×999

2.7.3 Two and Five Are Friends Forever

Exercise 2.6 Match powers of 2 and 5 to create powers of 10 to simplify calculations.

(a) $5 \times 15 \times 4$

(b) 24×25

(c) 16×25

(d) 64×25

(e) 5×22

(f) $25 \times 180 \times 4$

(g) $18 \times 12 \times 125$

(h) 36×25

(i) $5 \times 125 \times 64$

(j) $25 \times 25 \times 25 \times 4 \times 4 \times 4 \times 4$

(k) $16 \times 25 \times 7$

(l) $2 \times 4 \times 25 \times 3 \times 2$

(m) $8 \times 7 \times 25$

(n) 84×5

2.7.4 Parentheses and imposed order

Exercise 2.7 Calculate.

(a) $512 - 15 - 10$

(b) $512 - (15 - 10)$

(c) $512 + 15 - 10$

(d) $512 + (15 - 10)$

(e) $110 - 2 \times (50 - 30)$

(f) $(110 - 2 \times 50) - 30$

(g) $(110 - 2) \times 50 - 30$

Exercise 2.8 Put the parentheses to make the following equalities correct statements:

(a) $200 \times 16 + 14 = 6000$

(b) $70 - 40 - 20 = 50$

(c) $50 - 10 + 20 = 20$

(d) $50 - 20 \times 5 - 1 = 120$

3. ARITHMETIC OPERATIONS

Odd numbers.

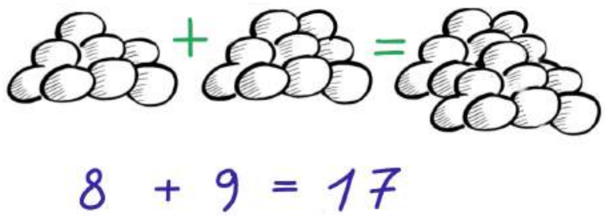
3.1 Introduction

The *arithmetic operations* are the operations of addition +, subtraction −, multiplication ×, and division ÷. The goal of this chapter is to discuss relations between the arithmetic operations and review their properties. We start with addition and will see that the introduction of other arithmetic operations is in some sense implied by the invention of addition.

3.2 Addition

In most cases, scientific innovations are introduced to make our daily life easier, and the addition operation is not an exception. Probably, addition was one of the earliest mathematical inventions of our civilization.

Imagine that you have two piles of stones with 8 stones in one pile and 9 in the other. If you put all the stones in one pile, you *do not need to recount* stones one-by-one again since you *memorized* from previous experience that $8+9=17$. One can say that the *basic addition* *consists of memorized results of counting*. In other words, a long time ago we started using addition to avoid the tedious task of recounting objects.



Moreover, this is probably how we invented counting numbers since *all counting numbers can be obtained by successive addition of one*. Indeed, we can write the sequence

$$
\begin{aligned}
&1\\
&2 = 1 + 1\\
&3 = 2 + 1\\
&4 = 3 + 1\\
&\ldots\ldots\ldots
\end{aligned}
$$

Every counting number will appear in this sequence.

3.3 The Number Zero

The set of whole numbers is the set consisting of all counting numbers and the *number zero*.

Question What is the number zero?

Answer While everyone knows what zero is, it is not easy to give a good *mathematical definition* of this very special number. Many people would say that zero is just nothing, but this does not explain the notion. The term *nothing* itself would need an explanation in mathematics. (For example, is the empty set also nothing? What is the difference between zero and the empty set?) It is convenient for us to introduce the number zero through its special property in relation to the addition operation.

> *Zero* is such a number that for any number a
>
> $$0 + a = a + 0 = a$$
>
> One says that zero has the *additive unit property*.

3.4 Multiplication

After introducing addition, multiplication does not come as a completely new operation. For counting numbers multiplication is nothing else but the *repetitive addition* of the same number. Indeed, the repetitive addition like

$$5+5+5+5+5+5+5+5+5+5+5+5+5+5$$

is very long to write, difficult to read, and it is easy to loose a summand. The well-known short notation was introduced for such expressions. For example, 5×14 is the short notation for the operation of addition of 5 to itself 14 times. The meaning of multiplication is reflected in the standard definition of this operation.

Let a and b be counting numbers. Then

$$a \times b = \underbrace{a + \cdots + a}_{b \quad \text{times}}$$

Multiplication also has a special number with the *unit property*.

The number 1 has the *multiplicative unit property*: For any number a

$$a \times 1 = 1 \times a = a$$

Note that zero also has a special multiplicative property.

The number 0 has the *multiplicative property*: For any number a

$$a \times 0 = 0 \times a = 0$$

3.5 Subtraction

Subtraction is the "backwards" operation for addition.

Example 3.1 There are 8 fish in a fish tank. Five fish are red, and the remaining ones are green. How many green fish are in the fish tank?

Solution Of course, the answer is 3 green fish, and the argument is based on the logic of addition: one has to add 3 to 5 to get 8 or

$$8 - 5 = 3 \quad \text{because we know that} \quad 5 + 3 = 8$$

This is *exactly* the idea of the definition of subtraction.

Let a and b be two numbers. Then $c = b - a$ means that c is a number such that $b = c + a$.

At first glance, this definition looks obvious, but it is not the case. We introduce the new symbol $-$ and explain its meaning through the addition operation $+$.

For whole numbers b and a the value of $b - a$ is not necessarily a whole number since we can get a negative value if $a > b$. We discuss this issue in Section 3.8 and Chapter 10.

3.6 Division

Similarly to subtraction, division is the "backwards" operation for multiplication.

Let a and b be two numbers with $a \neq 0$. Then $c = b \div a$ means that c is a number such that $b = c \times a$.

For whole numbers b and $a \neq 0$ the value of $b \div a$ is not necessarily a whole number. We discuss this issue in Section 3.8 and Chapter 10.

3.7 Properties of Operations

The arithmetic operations $+$, $\times$, $-$, and $\div$ have properties that we use in every day math, even though we rarely give them any thought.

3.7.1 Commutativity of addition and multiplication

Commutativity of addition and multiplication means that we can perform these operations with any order of terms.

Commutativity of addition and multiplication: For any two numbers a and b

$$a + b = b + a$$

$$a \times b = b \times a$$

This famous property significantly simplifies our daily life. Imagine that every time at a grocery store, we would have to think whether we should pay first for milk and then for bread or in the other way because the total will be different! Also we note that commutativity is not a guaranteed property of operations in mathematics.

Example 3.2 Is it true that for any two counting numbers a and b

$$a \div b = b \div a?$$

Solution This is certainly false. To show that the statement is false, it suffices to give any example of two numbers that fails the statement. For example, take $a = 20$ and $b = 10$. Then

$$20 \div 10 = 2 \quad \text{and} \quad 10 \div 20 = \frac{1}{2}, \quad \text{but} \quad 2 \neq \frac{1}{2}$$

This example shows that *division is not a commutative operation*.

When we want to prove that a statement is *false*, it is *enough to give just one example* of a number that fails the statement. When we believe that a statement is *true* and want to *prove* that it is true, *one example is not enough*. Rather, a more elaborate argument covering all possible cases is necessary.

Example 3.3 Subtraction is not a commutative operation. For example,

$$5 - 3 \neq 3 - 5$$

Example 3.4 Professional folklore of mathematicians contains a famous informal example of noncommutativity. If you put on socks first and then shoes, the result is not the same as if you do this in the other order.

3.7.2 Associativity of addition and multiplication

Associativity of addition and multiplication: For any three numbers a, b, and c

$$(a + b) + c = a + (b + c)$$
$$(a \times b) \times c = a \times (b \times c)$$

Example 3.5 A student formulated "associativity for division" as

$$(a \div b) \div c = a \div (b \div c) \quad \text{for any counting numbers } a,\ b,\ \text{and } c$$

Show that this statement is not true in general.

Solution Such "associativity" does not hold for *all* counting numbers. For example, take $a = 1$, $b = 2$, and $c = 4$. Then

$$(a \div b) \div c = (1 \div 2) \div 4 = \frac{1}{2} \div 4 = \frac{1}{8}$$
$$a \div (b \div c) = 1 \div (2 \div 4) = 1 \div \frac{1}{2} = 2$$

and, certainly, $\frac{1}{8} \neq 2$.

Remark In Example 3.5, we did some calculations with fractions. These kinds of calculations are reviewed in Chapter 11.

3.7.3 Distributivity

Distributivity properties describe interaction of the operations of multiplication (or division) with the operations of addition (or subtraction).

Distributivity: For any three numbers a, b, and c

$$\begin{aligned} &a \times (b + c) = a \times b + a \times c \\ &a \times (b - c) = a \times b - a \times c \\ &(b + c) \div a = b \div a + c \div a \quad (a \neq 0) \\ &(b - c) \div a = b \div a - c \div a \quad (a \neq 0) \end{aligned}$$

Example 3.6 We stated without proof that for any three counting numbers a, b, and c the distributivity property for division holds

$$(b + c) \div a = b \div a + c \div a$$

Is it true that for counting numbers a, b, and c

$$a \div (b + c) = a \div b + a \div c?$$

Solution No, this statement is wrong. For example, take $a = 1$, $b = 1$, and $c = 1$. Then

$$a \div (b + c) = 1 \div (1 + 1) = \frac{1}{2}$$
$$a \div b + a \div c = \frac{1}{1} + \frac{1}{1} = 1 + 1 = 2$$

and $\frac{1}{2} \neq 2$. Thus, $a \div (b + c) \neq a \div b + a \div c$.

3.7.4 Summary of properties of the arithmetic operations

Let us summarize the properties discussed above.

Commutativity of addition and multiplication: For any two numbers a and b

$$a + b = b + a$$
$$a \times b = b \times a$$

Associativity of addition and multiplication: For any three numbers a, b, and c

$$(a + b) + c = a + (b + c)$$
$$(a \times b) \times c = a \times (b \times c)$$

Distributivity: For any three numbers a, b, and c

$$a \times (b + c) = a \times b + a \times c$$
$$a \times (b - c) = a \times b - a \times c$$
$$(b + c) \div a = b \div a + c \div a \quad (a \neq 0)$$
$$(b - c) \div a = b \div a - c \div a \quad (a \neq 0)$$

Additive and multiplicative units
The number 0 is a unique whole number that has the *additive unit property*: For any whole number a

$$a + 0 = 0 + a = a$$

The number 1 is a unique whole number that has the *multiplicative unit property*: For any whole number a

$$a \times 1 = 1 \times a = a$$

Multiplicative property of zero: For any whole number a

$$a \times 0 = 0 \times a = 0$$

Example 3.7 What properties of the arithmetic operations are illustrated by the following examples.

(a) $2 + 7 + 8 = 2 + 8 + 7$

(b) $10 \times (100 - 1) = 10 \times 100 - 10 \times 1$

(c) $125 \times 3 \times 8 = 125 \times 8 \times 3$

(d) $134 + 0 = 0 + 134$

(e) $5 + 0 = 5$

(f) $1 + 0 = 1$

(g) $1 + 0 = 0 + 1$

(h) $0 \times (5 + 7) = 0$

(i) $5 \times 1 = 5$

(j) $5 \times (3 + 2) = (3 + 2) \times 5$

Solution

(a) We change the order of two terms and use the commutativity of addition:

$$2 + 7 + 8 = 2 + 8 + 7$$

(b) $10 \times (100 - 1) = 10 \times 100 - 10 \times 1$ is an example of distributivity.

(c) We use the commutativity of multiplication.

$$125 \times 3 \times 8 = 125 \times 8 \times 3$$

(d) We change the order of 0 and 134:

$$134 + 0 = 0 + 134$$

This is an example of the commutativity of addition (not the additive unit property of zero!)

(e) $5 + 0 = 5$ is an example of the additive unit property of zero.

(f) $1 + 0 = 1$ is an example of the additive unit property of zero.

(g) We change the order of 0 and 1:

$$1 + 0 = 0 + 1$$

This is an example of the commutativity of addition.

(h) $0 \times (5 + 7) = 0$ illustrates the multiplicative property of zero.

(i) $5 \times 1 = 5$ illustrates the multiplicative unit property of one.

(j) $5 \times (3 + 2) = (3 + 2) \times 5$ is based on the commutativity of multiplication.

3.8 Closure Property

Question What arithmetic operations on whole numbers always produce a whole number as their result?

Answer The result of the addition or multiplication of any two whole numbers is always a whole number. We can also subtract and divide whole numbers, but, in many cases, the result is not a whole number, for example, $3 - 5 = -2$ and $2/3$ are not whole numbers. We say that the set of whole numbers is *closed* under the operation of addition or multiplication and is *not closed* under the operations of subtraction and division. In other words, addition and multiplication do not take us outside the "world of whole numbers," but for subtraction and division this world is too small. When the world becomes too small, this leads to introduction of new numbers (negative integers and fractions). These expansions are discussed in Chapter 10.

The general notion of the *closure* of a set under an operation is formulated as follows.

A set S of numbers is *closed under addition* if for any two numbers in the set S their sum again belongs to S.

$$\text{if } a, b \in S, \text{ then } a + b \in S.$$

The closure of a set under other operations is introduced in a similar way.

Example 3.8 Determine whether the following sets are closed under the operations $+$ and $\times$.

(a) even numbers $\{0, 2, 4, \dots\}$

(b) odd numbers $\{1, 3, 5, \dots\}$

(c) $\{2, 5, 11\}$

(d) $\{0\}$

(e) $\{0, 1\}$

(f) $\{0, 2\}$

(g) $\{0, 1, -1\}$

Solution

(a) The set of even numbers $\{0, 2, 4, \dots\}$ is closed under both operations since

$$\text{even} + \text{even} = \text{even}$$
$$\text{even} \times \text{even} = \text{even}$$

(b) The set of odd numbers $\{1, 3, 5, \dots\}$ is not closed under addition. For example, $1 + 3 = 4$ and, more generally,

$$\text{odd} + \text{odd} = \text{even}$$

The set of odd numbers is closed under multiplication since

$$\text{odd} \times \text{odd} = \text{odd}$$

(c) The set $\{2, 5, 11\}$ is not closed under addition and is not closed under multiplication. For example, we can take $2 + 5 = 7$ and note that $7 \notin \{2, 5, 11\}$. Similarly, $5 \times 11 = 55$ and $55 \notin \{2, 5, 11\}$.

(d) The set $\{0\}$ is closed under both addition and multiplication since

$$0 + 0 = 0, \quad 0 \times 0 = 0$$

(e) The set $\{0, 1\}$ is not closed under addition, but is closed under multiplication. For the first statement we note that $1 + 1 = 2$ and $2 \notin \{0, 1\}$. To prove the second statement, we can make the table of multiplication for all the three elements of this set to check that the result of multiplication always belongs to this set:

$$0 \times 0 = 0, \quad 1 \times 0 = 0 \times 1 = 0, \quad 1 \times 1 = 1$$

(f) The set $\{0, 2\}$ is not closed under addition and is not closed under multiplication since we can take $2 + 2 = 4$ and $2 \times 2 = 4$, but $4 \notin \{0, 2\}$.

(g) The set $\{0, 1, -1\}$ is not closed under addition since $1+1 = 2$, but $2 \notin \{0, 1-1\}$. It is closed under multiplication, which again can be checked by using the multiplication table

$\times$	-1	0	1
-1	1	0	-1
0	0	0	0
1	-1	0	1

We see that all the results of multiplication belong to the set $\{0, 1, -1\}$.

3.9 Why Is Division by Zero Not Allowed?

We all know that it is forbidden to divide by zero, but do you know why? More precisely, why expressions like $\frac{a}{0}$ (for example, $\frac{15}{0}$) are *not well defined*? This is not a prohibition voluntarily invented by professors or math teachers, but a law *dictated by nature*. Mathematics itself cannot stand the division by zero, and an attempt to validate such an operation unavoidably leads to a *logical contradiction*.

Let us try to understand the obstruction. Assume for a moment that we <u>can</u> find a well defined number c that equals $\frac{a}{0}$ for a given counting number a. We see shortly that this leads us to a contradiction. As a result, we have to admit that our assumption is not valid, that is, there is no a well defined number $\frac{a}{0}$.

For example, take $a = 5$. Recall that, by the definition of the division operation, if c equals $\frac{5}{0}$, then

$$c \times 0 = 5$$

By the multiplicative property of zero, the left-hand side can be simplified as $c \times 0 = 0$, and we get the contradictory impossible equality

$$0=5 \quad !$$

This means that our assumption is wrong and it is not possible to find a number c that would be equal to $\frac{5}{0}$.

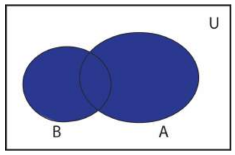
Remark For the sake of simplicity of exposition, we presented an argument for $a = 5$, but exactly the same argument works for any $a \neq 0$. For $a = 0$ the expression $\frac{0}{0}$ is not a well defined number because if $c = \frac{0}{0}$, then $c \times 0 = 0$, and this is true for *any* number c, not just one. So, there is ambiguity.

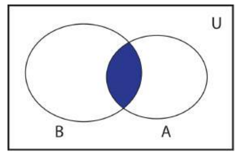
For any number a the expression $\frac{a}{0}$ cannot be well defined.

3.10 Properties of Operations with Sets

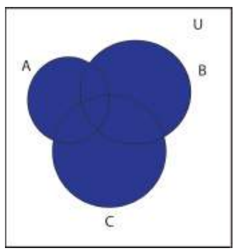
Recall that, in Section 1.4, we introduced the operations $A \cup B$, $A \cap B$, and $A - B$ with sets A and B. It is clear that they have some similarities with the arithmetic operations on numbers. Using Venn diagrams, one can investigate properties of these operations with sets to discover the following.

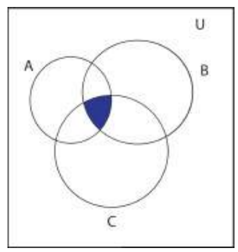
- *Commutativity* and *associativity* hold for the union and intersection.





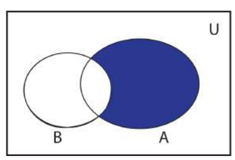
$A \cup B = B \cup A$ $\qquad$ $A \cap B = B \cap A$

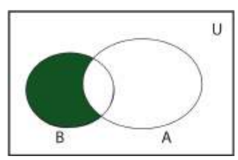




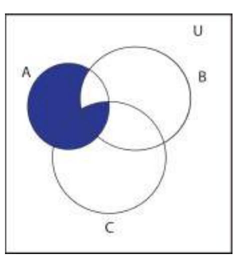
$A \cup (B \cup C) = (A \cup B) \cup C$ $\qquad$ $A \cap (B \cap C) = (A \cap B) \cap C$

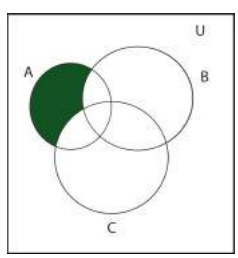
- *Commutativity* and *associativity* do not hold for the difference operation.





$A - B \neq B - A$





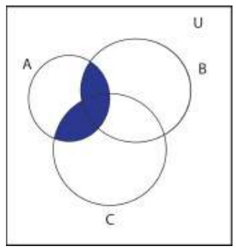
$A - (B - C) \neq (A - B) - C$

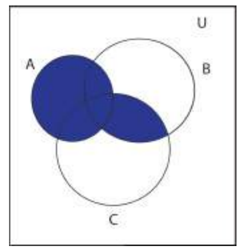
Example 3.9 Do the following analogs of distributivity hold?

(a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Solution We represent each side of the equality by a Venn diagram and see that they coincide in both cases, which illustrates that both statements (a) and (b) are true.





$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $\quad$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

3.11 Exercises

3.11.1 Properties of the arithmetic operations

Exercise 3.1 What properties of the arithmetic operations are illustrated by the following examples.

(a) $1279 + 3847$ is a whole number

(b) $7 + 5 = 5 + 7$

(c) $53 + 47 = 50 + 50$

(d) $1 + 0 = 1$

(e) $1 + 0 = 0 + 1$

(f) $7 \cdot (5 - 4) = 7 \cdot 5 - 7 \cdot 4$

(g) $(2 + 11) \cdot 0 = 0$

(h) $6 \cdot (5 \cdot 13) = (6 \cdot 5) \cdot 13$

(i) $(5 + 7) \cdot 7 = 5 \cdot 7 + 7 \cdot 7$

(j) $4 \cdot 6 = 6 \cdot 4$

(k) $6 \cdot (3 + 2) = (3 + 2) \cdot 6$

(l) $10 \cdot (3 + 7) = 10 \cdot 3 + 10 \cdot 7$

Exercise 3.2 Each of the following equalities illustrates one of the properties of addition. Substitute an appropriate number or a word for the question mark to complete the statement. Identify the properties illustrated by the examples.

(a) $4 + \ ? \ = 4$

(b) $3 + 8 = \ ? \ + 3$

(c) $(2 + 1) + 13 = 2 + (\ ? \ + 13)$

(d) $(3 + 2) + 10 = \ ? \ + (3 + 2)$

(e) $(5 + 11) + 8 = (11 + \ ? \ ) + 8$

(f) $(11 + 120)$ is a ? number

Exercise 3.3 Klaus claims that the following statements are correct, but he is wrong. For each of the statements give an example that would illustrate that the statement is wrong.

(a) If a and b are whole numbers, then a/b is also a whole number.

(b) $a/b = b/a$ for any two nonzero whole numbers a and b.

(c) $(a/b)/c = a/(b/c)$ for any three nonzero whole numbers a, b, and c.

(d) $a/(b+c) = a/b + a/c$ for any three nonzero whole numbers a, b, and c.

3.11.2 Closure of sets of numbers under arithmetic operations

Exercise 3.4 Which of the following sets are closed under (A) addition (B) multiplication? If a set is closed, give an argument. If not, give an example.

(a) The set $\{0, 10, 20, 30, \ldots\}$ of multiples of 10.

(b) The set $\{0, 3, 6, 9, \ldots\}$ of multiples of 3.

(c) The set $\{0, 2, 4, 6, \ldots\}$ of even numbers.

(d) The set $\{1, 3, 5, 7, \ldots\}$ of odd numbers.

(e) The set $\{1\}$.

(f) The set $\{0, 1, 2\}$.

(g) The set $\{0\}$.

(h) The set $\{2, 5, 8, 11, 14, \ldots\}$ that contains every third number and starts from 2.

(i) The set of whole numbers less than 19.

(j) The set of whole numbers greater than 19.

(k) The set of whole numbers with $\{3\}$ removed $\{0, 1, 2, \quad 4, 5, 6, 7, 8, 9, , \ldots\}$.

(l) The set of whole numbers with $\{6\}$ removed $\{0, 1, 2, 3, 4, 5, \quad 7, 8, 9, \ldots\}$.

(m) The set $\{2, 4\}$.

(n) The set $\{0, 1\}$.

(o) The set $\{0, 3\}$.

3.11.3 Definition of the number zero

Exercise 3.5 We asked a group of elementary school students to define zero. Here are their answers. Which of them are logically wrong? Which of them are ambiguous? What properties of zero are mentioned? Which of them are very close to our definition of zero?

(a) A number that is worth nothing.

(b) An oval that means nothing.

(c) Zero is nothing in a sentence. I have 0 mugs. A closed figure that is all round.

(d) Nothing, a number that indicates nothing.

(e) A number that represents nothing.

(f) A round number that is worth nothing.

(g) Nothing.

(h) Nothing as a number. It can be used as a space between digits.

(i) A number that contributes nothing when added to a number.

(j) So, when you add zero to a number, the solution is always that number.

(k) When you divide a number with zero,... well the solution is ... ZERO.

(l) The number that you get when you subtract a number from itself.

(m) It is what you get when you subtract any number from itself, like one minus one.

(n) Nothing, a number between negative and positive.

4. DIVISION WITH REMAINDER

Division with remainder.

4.1 Division That Stays Within the Set of Whole Numbers

As we discussed earlier, the set of whole numbers is not closed under division. For example, $5 \div 3 = \dfrac{5}{3}$ is not a whole number. At the same time, we know another operation which is very close to division and does not take us outside the set of whole numbers: *division with remainder*.

Example 4.1 15 can be divided by 7 with the remainder 1:

$$15 = 2 \times 7 + 1$$

More generally, let a and b be two counting numbers with $a \geq b > 0$. Then there exist whole numbers q and r such that $b > r \geq 0$ and

$$a = qb + r$$

We call q the *quotient*, b the *divisor*, and r the *remainder*. We say that *a divided by b gives the quotient q with the remainder r*. The commonly used notation for division with remainder looks like

$$a \div b = q\, R\, r$$

Example 4.2 Divide 49 by 8 with remainder.

Solution Since $49 = 6 \times 8 + 1$, we get $49 \div 8 = 6R1$.

Example 4.3 Divide with remainder

(a) 58 by 7

(b) 118 by 25

Solution

(a) Since $58 = 7 \times 8 + 2$, we get $58 \div 7 = 8R2$.

(b) Since $118 = 25 \times 4 + 18$, we get $118 \div 25 = 4R18$.

Remark The remainder is always less than the divisor, $r < b$.

For large values of a division with remainder is performed through long division or by using a calculator. It is acceptable in our course to use a calculator in the cases where calculations seem to be too lengthy. Sometimes, relations between numbers allow us to divide with remainder in a relatively easy way without a calculator or long division.

Example 4.4 Divide 569 by 57 with remainder.

Solution We can avoid long division or usage of a calculator if we find an "obvious" multiple of 57 close to 569. Note that 569 is almost 570 and $570 = 57 \times 10$. We can use this observation as follows:

$$569 = 570 - 1 = 10 \times 57 - 1$$

Note that the remainder cannot be negative, $r \neq -1$, so we adjust the last expression as

$$569 = 10 \times 57 - 1 = (9 + 1) \times 57 - 1 = 9 \times 57 + (57 - 1) = 9 \times 57 + 56$$

Answer: $569 \div 57 = 9R56$.

Example 4.5 Divide 278 by 25 with remainder.

Solution We find an "obvious" multiple of 25 close to 278, for example, 250, and then adjust it:

$$278 = 250 + 28 = 250 + 25 + 3 = 25 \times 10 + 25 + 3 = 25 \times (10 + 1) + 3 = 25 \times 11 + 3$$

This calculation gives $278 \div 25 = 11R3$.

Example 4.6 True or false?

(a) The remainder of division of 15 by 3 is zero.

(b) Since $63 = 5 \times 10 + 13$, we get $63 \div 10 = 5R13$.

(c) Since $64 = 5 \times 13 - 1$, we get $64 \div 5 = 13R(-1)$.

Solution

(a) Yes, this is true: $15 = 5 \times 3 + 0$ and $15 \div 3 = 5R0$.

(b) No, this is false since the remainder cannot be larger than the divisor and $13 > 10$. The correct statement is $63 \div 10 = 6R3$.

(c) No, this false since the remainder is always a nonnegative number: $r \geq 0$. The correct statement is $64 \div 5 = 12R4$.

In Chapter 7, we use the division by powers of 2, 3, ... with remainder. The next exercise prepares us for this work. Recall the values of the first few powers of 2:

$$2^0 = 1, \quad 2^1 = 2, \quad 2^2 = 4, \quad 2^3 = 8, \quad 2^4 = 16, \quad \ldots$$

Example 4.7 Complete division with remainder (substitute an appropriate quotient for the blue question mark and an appropriate remainder for the green question mark).

(a) $31 = ? \times 16 + ?$

(b) $15 = ? \times 8 + ?$

(c) $7 = ? \times 4 + ?$

(d) $3 = ? \times 2 + ?$

(e) $1 = ? \times 1 + ?$

Solution

(a) $31 = 1 \times 16 + 15$

(b) $15 = 1 \times 8 + 7$

(c) $7 = 1 \times 4 + 3$

(d) $3 = 1 \times 2 + 1$

(e) $1 = 1 \times 1 + 0$

Remark In Chapter 7, we learn that this particular calculation implies that the number 31 is represented in the binary system as

$$31 = 11111_{\text{two}}$$

This fact will be explained in detail later. Now, we consider the example.

Example 4.8 What are the remainders of the division of the number

$$a = 1 \times 2 \times 3 \times 4 \times 5 \times 6 + 1$$

by 3, 5, 2, 10, 12?

Solution All these questions have the same answer: the remainder is 1. Indeed, we see that a is a multiple of 3 plus 1.

$$a = (1 \times 2 \times 4 \times 5 \times 6) \times 3 + 1, \quad 3 > 1 > 0$$

So, 1 is the remainder of the division of a by 3. Similarly,

$$\begin{aligned}
a &= (1\times 2\times 3\times 4\times 6)\times 5+1, \quad 5>1>0\\
a &= (1\times 3\times 4\times 5\times 6)\times 2+1, \quad 2>1>0\\
a &= (1\times 3\times 4\times 6)\times 10+1, \quad 10>1>0\\
a &= (1\times 3\times 4\times 5)\times 12+1, \quad 12>1>
\end{aligned}$$

In each case, the remainder is 1.

4.2 Exercises

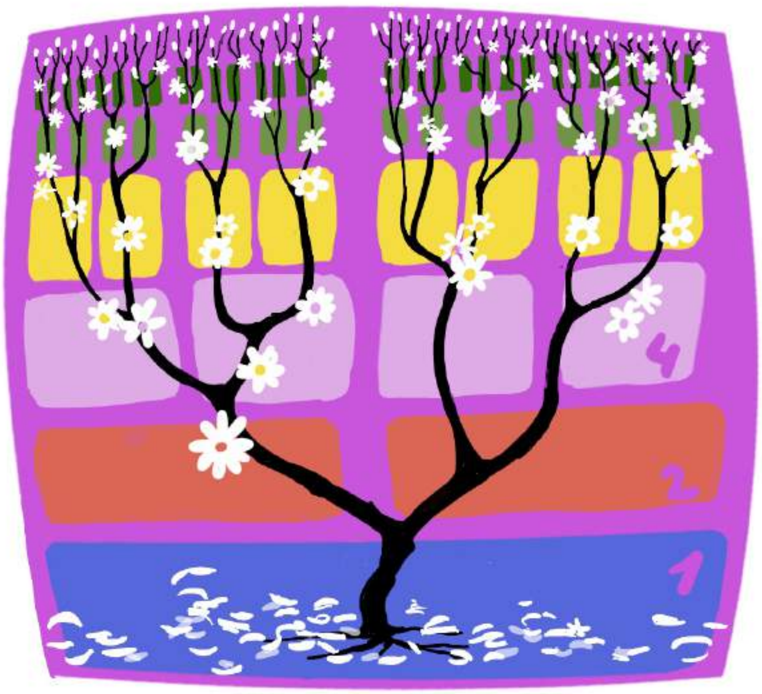
Exercise 4.1 Complete division with remainder.

(a) 12 by 5
(b) 128 by 13
(c) 2137 by 100
(d) 10 000 by 9
(e) 128 by 3
(f) 333333 by 33
(g) 49 by 8
(h) 73 by 8
(i) 58 by 7
(j) 400 by 57
(k) 487 by 17
(l) 456 by 6
(m) 118 by 23
(n) 683 by 5
(o) 5 by 638
(p) $(15\cdot A+2)$ by 3
(q) $(5\cdot 7\cdot B+2)$ by 5

Exercise 4.2 Find the remainder of the division of $n = 1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6+1$ by 20.

Exercise 4.3 Without calculating, determine whether the sum $1231+36785+513$ is even or odd.

5. EXPONENTS



The powers of two.

5.1 Definition of Powers of Numbers

Many students lack confidence in problems that involve powers of numbers. The unusual notation a^k and properties of exponential expressions may look rather puzzling. In reality, we should not be afraid of exponents since it is not a new operation, but the abbreviated notation for *repetitive multiplication*. Indeed, recall that we already saw the abbreviated notation $\times$ for the repetitive addition

$$5 \times 6 = 5 + 5 + 5 + 5 + 5 + 5$$

Similarly, we use the exponential notation for the repetitive multiplication

$$5^6 = 5 \times 5 \times 5 \times 5 \times 5 \times 5$$

Let k be a counting number, and let a be a whole number. Then a^k stands for the repetitive multiplication of a by itself k times

$$a^k = \underbrace{a \times \cdots \times a}_{k \quad \text{times}}$$

The number k is called the *exponent* or *power* of a. The number a is called the *base*. The expression a^k is read "a to the power k." For any $a \neq 0$ we define

$$a^0 = 1$$

Remark The expression 0^0 is not well defined (the assumption of the existence of such a number leads to a contradiction).

5.2 Properties of Exponents

All properties of exponents follow from the definition and properties of multiplication. Whenever you have doubts about a certain property of an exponential expression, write it out using multiplication and check whether the situation becomes more transparent.

Example 5.1 Simplify $5^3 \times 5^4$ and write the result in the form 5^k.

Solution It is easy to get confused here: should we multiply 3×4 or add $3 + 4$ to get the value of k? An answer becomes obvious if we go back to the definition of exponents:

$$5^3 \times 5^4 = \underbrace{5 \times 5 \times 5}_{3 \quad \text{times}} \times \underbrace{5 \times 5 \times 5 \times 5}_{4 \quad \text{times}} = \underbrace{5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5}_{7 \quad \text{times}}$$

Thus,

$$5^3 \times 5^4 = 5^{3+4} = 5^7$$

For any whole number a and any two counting numbers m and n

$$a^m a^n = a^{m+n}$$

This is proved by exactly the same argument:

$$a^m \times a^n = \underbrace{a \times \cdots \times a}_{m \quad \text{times}} \times \underbrace{a \times \cdots \times a}_{n \quad \text{times}} = a^{m+n}$$

Example 5.2 Simplify $5^8 \div 5^2$ and write the result in the form 5^k.

Solution We again go back to the definition. Then the answer becomes obvious:

$$5^8 \div 5^2 = \frac{5\times5\times5\times5\times5\times5\times\cancel{5\times5}}{\cancel{5\times5}} = \underbrace{5\times5\times5\times5\times5\times5}_{6\quad\text{times}}. = 5^6$$

Hence

$$5^8 \div 5^2 = 5^{8-2} = 5^6$$

> For any counting number a and any two counting numbers m and n
>
> $$a^m \div a^n = a^{m-n}$$

Further, let $a \neq 0$ and $m = 0$ in the formula $a^m \div a^n = a^{m-n}$:

$$a^0 \div a^n = a^{-n}$$

Since $a^0 = 1$, the left-hand side simplifies to $a^0 \div a^n = 1 \div a^n = \dfrac{1}{a^n}$, and we get another useful relation.

> For any counting number a and a counting number n
>
> $$\frac{1}{a^n} = a^{-n}$$

This formula helps us convert fractions to exponents.

Example 5.3 Simplify $5^{-3} \times \dfrac{1}{5^2}$ and write the result in the form 5^k.

Solution

$$5^{-3} \times \frac{1}{5^2} = 5^{-3} \times 5^{-2} = 5^{-3-2} = 5^{-5}$$

Example 5.4 Simplify $(5^3)^4$ and write the result in the form 5^k.

Solution

$$(5^3)^4 = 5^{3\times4} = 5^{12}$$

Why do we multiply exponents in this case? Again, this becomes clear when we write everything out using multiplication:

$$(5^3)^4 = \underbrace{5^3\times5^3\times5^3\times5^3}_{4\quad\text{times}}. = \underbrace{(5\times5\times5)\times(5\times5\times5)\times(5\times5\times5)\times(5\times5\times5)}_{3\times4\quad\text{times}} = 5^{12}$$

For any whole number a and any two counting numbers m and n

$$(a^m)^n = a^{m\times n}$$

By the same argument as in the example above, it is proved that

$$(a^m)^n = \underbrace{a^m \times \cdots \times a^m}_{n \quad \text{times}} = \underbrace{(a\times\cdots\times a)\times\cdots\times(a\times\cdots\times a)}_{m\times n \quad \text{times}} = a^{mn}$$

Example 5.5 Write $2^3 \times 5^3$ in the form a^k.

Solution Three copies of 5 and three copies of 2 can be matched to make 3 copies of 10:

$$2^3 \times 5^3 = 2 \times 2 \times 2 \times 5 \times 5 \times 5 = 10\times 10\times 10 = 10^3.$$

This example illustrates the general rule.

For any two whole numbers a and b and any counting number m

$$a^m \times b^m = (a\times b)^m$$

Example 5.6 Substitute an appropriate value for the question mark.

(a) $6^5 = 3^? \times 2^?$ (b) $8\times 3^3 = ?^3$ (c) $6^{10}\times 2^{10} = ?^{10}$

Solution

(a) $6^5 = (3\times 2)^5 = 3^5\times 2^5$.

(b) Note that $8 = 2^3$. Then $8\times 3^3 = 2^3\times 3^3 = (2\times 3)^3 = 6^3$.

(c) $6^{10}\times 2^{10} = (6\times 2)^{10} = 12^{10}$.

Example 5.7 True or false?

(a) $3^4\times 3^2 = 3^8$?

(b) $(3^2)^3 = 3^6$?

(c) $\dfrac{1}{3^{-2}} = 3^{-2}$?

(d) $3^7 \div 3^{-2} = 3^5$?

Solution

(a) $3^4\times 3^2 = 3^{4+2} = 3^6$, false

(b) $(3^2)^3 = 3^{2\times 3} = 3^6$, true

(c) $\dfrac{1}{3^{-2}} = 3^{-(-2)} = 3^2$, false

(d) $3^7\div 3^{-2} = 3^{7-(-2)} = 3^{7+2} = 3^9$, false

5.3 Summary of Properties of Exponents

For any whole numbers a and b and any counting numbers n and m

$$a^m = \underbrace{a \times \cdots \times a}_{m \quad \text{times}}$$

$$a^m a^n = a^{m+n}$$

$$a^m \div a^n = a^{m-n} \quad (a \neq 0)$$

$$a^{-n} = \frac{1}{a^n} \quad (a \neq 0)$$

$$(a^m)^n = a^{mn}$$

$$a^n b^n = (ab)^n$$

$$a^0 = 1$$

5.4 More Examples of Problems with Exponents

Let us apply properties of exponents to solve these examples.

Example 5.8 Simplify $2^2 \times 2^2 \times 2^2 \times 2^2 \times 2^2$ and write the result in the form 2^k.

Solution $2^2 \times 2^2 \times 2^2 \times 2^2 \times 2^2 = 2^{2+2+2+2+2} = 2^{10}$.

Example 5.9 Simplify $((((2^2)^2)^2)^2)^2$ and write the result the form 2^k.

Solution $((((2^2)^2)^2)^2)^2 = 2^{2\times2\times2\times2\times2} = 2^{(2^5)} = 2^{32}$.

Example 5.10 Simplify $2^{2^{2^{2^2}}}$ and write the result in the form 2^k.

Solution We start from the top exponent of the expression. First, we use $2^2 = 4$, then $2^4 = 16$, and, finally, $2^{16} = 65536$.

$$2^{2^{2^{2^2}}} = 2^{2^{2^4}} = 2^{2^{16}} = 2^{65536}$$

Example 5.11 Simplify $\dfrac{25^2}{5^4}$.

Solution Note that $25 = 5^2$.

$$\frac{25^2}{5^4} = \frac{(5^2)^2}{5^4} = \frac{5^4}{5^4} = 1$$

Remark Usually, we try to avoid large numbers at intermediate steps. Rather than evaluating $25^2 = 625$, we prefer to express everything in powers of 5 since these are small numbers that are easier to manipulate.

Example 5.12 Write $6^3 \div 6^{-7} \times 6^{11}$ in the form 6^k.

Solution

$$6^3 \div 6^{-7} \times 6^{11} = 6^3 \times 6^7 \times 6^{11} = 6^{3+7+11} = 6^{21}$$

Example 5.13 Write in the form a^k:

$$\frac{2^5 \times 4^6}{2^2} \times \frac{1}{8^2}$$

Solution We can express everything in powers of 2 since $4 = 2^2$ and $8 = 2^3$:

$$\frac{2^5 \times (2^2)^6}{2^2} \times \frac{1}{(2^3)^2} = \frac{2^5 \times 2^{12}}{2^2 \times 2^6} = \frac{2^{17}}{2^8} = 2^{17-8} = 2^9$$

Remark The answer 8^3 is also correct since $2^9 = (2^3)^3 = 8^3$.

Example 5.14 Write in the form a^k.

(a) $2^5 \times 8^2$ (b) $3^{20} \times 2^{20}$ (c) $25^{10} \times 2^{20}$ (d) $4^2 \times 9 \times 25$

Solution

(a) $2^5 \times 8^2 = 2^5 \times (2^3)^2 = 2^5 \times 2^6 = 2^{11}$

(b) $3^{20} \times 2^{20} = (3 \times 2)^{20} = 6^{20}$

(c) $25^{10} \times 2^{20} = (5^2)^{10} \times 2^{20} = 5^{20} \times 2^{20} = (5 \times 2)^{20} = 10^{20}$

(d) $4^2 \times 9 \times 25 = 4^2 \times 3^2 \times 5^2 = (4 \times 3 \times 5)^2 = 60^2$

Example 5.15 Solve for x.

(a) $(11^x)^4 = 11^{40}$ (b) $11^4 \times 11^x = 11^{40}$ (c) $11^4 \times 3^x = 33^4$

Solution

(a) $(11^x)^4 = 11^{4x} = 11^{40}$, so $4x = 40$ and $x = 10$

(b) $11^4 \times 11^x = 11^{4+x} = 11^{40}$, so $4 + x = 40$ and $x = 36$

(c) $33^4 = 11^4 \times 3^4$, so $x = 4$

Example 5.16 Find $5^2 - 4^2 + 2^5 \div 2^3$.

Solution

$$5^2 - 4^2 + 2^5 \div 2^3 = 25 - 16 + 2^2 = 25 - 16 + 4 = 13$$

Example 5.17 Find

$$4 \times 8 - \frac{6^2 \times 4}{2^3 \times 3^2}$$

Solution First, let us simplify the fraction

$$\frac{6^2 \times 4}{2^3 \times 3^2} = \frac{(2 \times 3)^2 \times 2^2}{2^3 \times 3^2} = \frac{2^2 \times 3^2 \times 2^2}{2^3 \times 3^2} = 2$$

Note that we avoid producing large numbers at intermediate steps. We break the factors into smaller ones and cancel as many as we can. Then the final step of calculation is

$$4 \times 8 - \frac{6^2 \times 4}{2^3 \times 3^2} = 32 - 2 = 30$$

Example 5.18 Simplify

$$\frac{2 \times 3^3}{15 - 2 \times 3}$$

Solution The denominator is simplified as $15 - 2 \times 3 = 15 - 6 = 9$. Then

$$\frac{2 \times 3^3}{15 - 2 \times 3} = \frac{2 \times 3^3}{9} = \frac{2 \times 3^3}{3^2} = 2 \times 3 = 6$$

5.5 Evaluation of Powers of Numbers

The ability to recognize powers of different numbers, for example,

$$25 = 5^2, \quad 8 = 2^3, \quad 16 = 4^2, \quad \ldots$$

may be of great assistance in different computations. Let us practice this skill.

Example 5.19 Represent in the form a^k.

(a) 4
(b) 9
(c) 16
(d) 25
(e) 49
(f) 81
(g) 100
(h) 125
(i) 128
(j) 1024
(k) 1 000 000
(l) 243

Solution

(a) $4 = 2^2$
(b) $9 = 3^2$
(c) $16 = 4^2 = 2^4$
(d) $25 = 5^2$
(e) $49 = 7^2$
(f) $81 = 9^2 = 3^4$
(g) $100 = 10^2$
(h) $125 = 5^3$
(i) $128 = 2^7$
(j) $1024 = 2^{10}$
(k) $1000000 = 1000^2 = 100^3 = 10^6$
(l) $243 = 3^5$

Example 5.20 Write down (a) the first five powers of 2 and (b) the first five powers of 3. (It is allowed to use a calculator for this exercise.)

Solution

(a) $2^1 = 2 \quad 2^2 = 4 \quad 2^3 = 8 \quad 2^4 = 16 \quad 2^5 = 32$

(b) $3^1 = 3 \quad 3^2 = 9 \quad 3^3 = 27 \quad 3^4 = 81 \quad 3^5 = 243$

5.6 Comparison of Powers of Numbers

One can compare two exponential numbers when they have the *same base* or the *same exponent*. Other cases usually require more work.

Example 5.21 Which of the numbers 10^{50} or 6^{50} is greater?

Solution $10^{50} > 6^{50}$ since $10 > 6$. By the definition of exponents, we have 50 copies of each multiplied

$$\underbrace{10 \times \cdots \times 10}_{50 \quad \text{times}} > \underbrace{6 \times \cdots \times 6}_{50 \quad \text{times}}$$

Let a and b be whole numbers, and let k be a counting number.

If $a > b$, then $a^k > b^k$.

Note that, in this rule, we have the same exponent k in a^k and b^k.

Example 5.22 Which of the numbers 10^{50} or 10^{100} is greater?

Solution $10^{50} < 10^{100}$ because $50 < 100 = 50 + 50$.

$$\underbrace{10 \times \cdots \times 10}_{50 \text{ times}} < \underbrace{10 \times \cdots \times 10}_{50 \text{ times}} \times \underbrace{10 \times \cdots \times 10}_{50 \text{ times}}$$

Let a be a whole number, and let k and l be counting numbers.

If $k > l$, then $a^k > a^l$.

In this rule, the base of a^k and a^l is the same.

Remark In this course, we compare integer powers of whole numbers. From our previous mathematical experience we know that the notion of an exponent can be extended to negative or rational numbers, for example, $\left(\frac{1}{2}\right)^3$, $(-5)^{\frac{1}{7}}$. In this case, comparison rules can be different. For example, take $k = 3$, $l = 2$, and $a = \frac{1}{2}$. Then $3 > 2$, but $\left(\frac{1}{2}\right)^3 < \left(\frac{1}{2}\right)^2$ since $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$ and $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$.

Example 5.23 Which of the numbers 10^{50} or 5^{100} is greater?

Solution 1 The bases 10 and 5 are different, so are the exponents 50 and 100. This means that some extra work is required to match exponential expressions. Note that $10 = 2 \times 5$, so $10^{50} = 2^{50} \times 5^{50}$, while $5^{100} = 5^{50} \times 5^{50}$. Then we need to compare $2^{50} \times 5^{50}$ and $5^{50} \times 5^{50}$. Since $2 < 5$, we have $2^{50} < 5^{50}$. Hence

$$10^{50} = 2^{50} \times 5^{50} < 5^{50} \times 5^{50} = 5^{100}$$

Answer: $10^{50} < 5^{100}$.

Solution 2 Note that $5^2 = 25$, so $5^{100} = 5^{2\times 50} = 25^{50}$. Since $10 < 25$, we can write

$$10^{50} < 25^{50} = 5^{100}$$

Answer: $10^{50} < 5^{100}$.

Example 5.24 Which of the numbers 3^{-10} or 1 is greater?

Solution

$$3^{-10} = \frac{1}{3^{10}} < 1$$

5.7 Exercises

5.7.1 Definition of a power of a number

Exercise 5.1 Using exponents, write the following in the form $a^k b^m$.

(a) $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$

(b) $6 \cdot 7 \cdot 7 \cdot 6 \cdot 6 \cdot 7$

(c) $a \cdot b \cdot a \cdot b \cdot a \cdot b$

(d) $5 \cdot 5 \cdot 5 \cdot 2 \cdot 5 \cdot 2 \cdot 5$

(e) $x \cdot y \cdot y \cdot y \cdot y \cdot x \cdot x$

Exercise 5.2 Write the following as powers of 10.

(a) $2^4 \cdot 5^4$

(b) $2 \cdot 2 \cdot 2 \cdot 5 \cdot 5 \cdot 5$

5.7.2 Properties of exponents

Exercise 5.3 True or false?

(a) $5^{20 \times} 7^{20} = 35^{20}$

(b) $3^5 \times 4^2 = 6^9$

(c) $9 \times 3^4 = 3^6$

(d) $((6^2)^2)^2) = 6^8$

(e) $4^{10} \times 3^{20} = 6^{20}$

(f) $(17^5)^2 = 17^7$

(g) $8^5 \times 3^5 = 11^5$

(h) $15^4 = 5^4 \times 3^4$

(i) $((2^3)^3)^3 = 2^9$

(j) $(((2^3)^2)^{-3})^{-2} = 1$

(k) $2^3 \times 2^{-2} \times 2^3 \times 2^{-2} = 1$

(l) $\dfrac{2^3}{3^2} \div \dfrac{2^3}{3^2} = 1$

(m) $\dfrac{2^3}{3^2} \times \left(\dfrac{2^3}{3^2}\right)^{-1} = 1$

(n) $2^3 \times 3^3 = 5^3$

(o) $2^3 \times 4^3 = 8^3$

(p) $6^4 = 4^2 \times 3^4$

Exercise 5.4 Rewrite the following expressions with a single exponent a^k (do not evaluate).

(a) $7^2 \cdot 7^5$

(b) $5^8 \cdot 3^8$

(c) $9^{10}/3^5$

(d) $4^{17}/4^5$

(e) $8 \cdot 2^8$

(f) $9^4 \cdot 16^6$

Exercise 5.5 True or false?

(a) $2^3 \cdot 3^2 = (2\cdot 3)^3$

(b) $7^{10}\cdot 8^{10} = (7\cdot 8)^{10}$

(c) $3^5 \cdot 2^4 = 6^9$

(d) $(2\cdot 7)^5 = 2^5\cdot 7^5$

(e) $6\cdot 2^3 = 12^3$

(f) $(2^3)^{10} = 2^{30}$

(g) $25\cdot 2^4 = 2^4\cdot 5^2$

(h) $3^4\cdot 3^5\cdot 3 = 3^{\cdot 10}$

(i) $9^{20}\cdot 2^{40} = 6^{40}$

(j) $((5^2)^2)^2 = 5^8$

Exercise 5.6 Simplify and write the result in the form 2^k.

(a) $2^3\times 2^3\times 2^3\times 2^3\times 2^3$ (b) $((((2^3)^3)^3)^3)^3$ (c) 2^{3^3}

Exercise 5.7 Find x.

(a) $5^8\cdot 5^x = 5^{12}$ (b) $(5^x)^4 = 5^{12}$ (c) $5^x\cdot 7^x = 35^{12}$

Exercise 5.8 Insert parentheses to obtain true statements.

(a) $5\times 4^2 - 2^2 + 3^2 = 69$ (b) $3^2 - 4\times 2^3 - 3 = 25$ (c) $2\times 3^2 - 2^2\times 10 = 140$

Exercise 5.9 Simplify

(a) $4^2 - \dfrac{2^3\times 3^2}{6}$ (b) $5^2 - 4^2 - 2^5 \div 2^3$ (c) $\dfrac{2\times(4-1)^4}{15-2\times 3}$

Exercise 5.10 Simplify

(a) $2^2 + 4\times(3^2-6)^2$

(b) $\dfrac{5+3\times(4^2-3^2)+2^2}{5^2-10}$

5.7.3 Evaluation of powers of numbers

Exercise 5.11 Evaluate without a calculator. Put in order from the smallest to the largest numbers.

$$3^3,\quad 2^4,\quad 8^2$$

Exercise 5.12 Find the values of the following exponential expressions.

(a) 2^3
(b) 5^2
(c) 3^4
(d) 4^3
(e) 2^4
(f) 99^1
(g) 11^2
(h) 13^0
(i) 10^4

Exercise 5.13 By what number should the following expressions be multiplied to obtain powers of 10?

(a) 2
(b) 5
(c) $2 \cdot 5^2$
(d) $2^4 \cdot 5^2$
(e) $2^3 \cdot 5^6$
(f) 80
(g) 16
(h) 50

Example: 25 must be multiplied by 4 to get $25 \times 4 = 100 = 10^2$.

Exercise 5.14 Which of the following numbers are powers a^k, $k > 1$, of some counting numbers.

(a) 4
(b) 7
(c) 9
(d) 16
(e) 25
(f) 49
(g) 56
(h) 81
(i) 100
(j) 125
(k) 128
(l) 243
(m) 1024
(n) 1000000

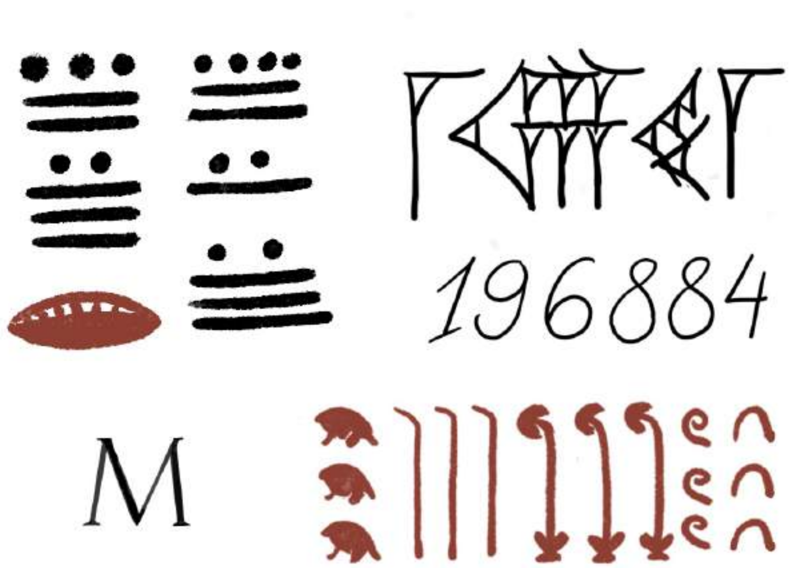
Example: The number 8 is a power of 2 since $8 = 2^3$. The number 6 cannot be written as a power a^k of some other number a and $k > 1$.

5.7.4 Comparison of powers of numbers

Exercise 5.15 Using properties of exponents, determine the larger number in the following pairs:

(a) 6^{10} or 3^{20}
(b) 5^{10} or 2^{10}
(c) $2^{10} \times 5^{10}$ or $2^{10} \times 2^{10}$
(d) 4^9 or 2^{20}
(e) 15^{10} or 5^{20}
(f) 10^{50} or 6^{50}
(g) 10^{50} or 10^{100}
(h) 10^{50} or 5^{100}
(i) 3^{-10} or 1

6. HISTORICAL NUMERATION SYSTEMS



Mysterious symbols.

6.1 Introduction

Question Can you guess what these pictures mean?

Answer These pictures represent numbers written in numeration systems of different civilizations, Mayan, Roman, Babilonian, Egyptian, and in the numeration system that we use today. To figure out the values of these numerals, one has to understand the meaning of the symbols and the rules of the numeration systems. The topic of this chapter is *how people write down numbers*. We want to compare some historical numeration systems with the system that we use in our daily life. This is the best way to understand (and to appreciate) our own way of writing numbers.

Our plan is as follows.

- Review the key features of the numeration system that we use today.
- Outline characteristics of some *historical numeration systems* which existed at different times in different civilizations. We will take a note of their advantages and disadvantages over our system.
- In Chapter 7, we will talk about *nondecimal numeration systems*, including the *binary system* which is used in computers.

Remark The modern version of the Hindu-Arabic numeration system is prevalent worldwide, but it is far from being the only numeration system used by people. More than one hundred numeration systems have existed over the past five thousand years. In this chapter we will outline some features of several historical numeration systems, but it is important to understand that the accurate description of the nuances and variations of mathematical language of any civilization is complicated. The summary in this chapter is based on the information from *Numerical Notation: A Comparative History* by Stephen Chrisomalis (Cambridge University Press, 2010). For the complete view of the topic it is recommended to read more on the academic studies of numerical notations by historians, archeologists, anthropologists, linguists, and mathematicians.

6.2 Western (Hindu–Arabic) Numeration System

The conventional term for the system that we use today is *Hindu-Arabic* or *Arabic* numerals, referring to their historical origins. Since symbols in those ancient scripts differ from the ones that we commonly use today, some scholars prefer the term *Western numerals*.

How do we write numbers today? How could we explain to aliens the meaning, for example, of 538? We would probably tell them that the mathematical meaning of this numeral is contained in its *expanded form*

$$538 = 5 \times 100 + 3 \times 10 + 8 \times 1$$

Note that our numeration system is a representation of every number as the sum of powers of 10

$$1,\ 10,\ 100,\ 1000,\ \ldots$$

multiplied by the values of digits

$$538 = 5 \times 10^2 + 3 \times 10^1 + 8 \times 10^0$$

Grouping into sets of powers of 10 is the fundamental principle of our numeration system. We say that our system is the *base ten* numeral system or the *decimal system*.

Remark In Chapter 7, we discuss systems that are based on powers of other numbers. For example, the *binary system* uses powers of 2

$$1,\ 2,\ 4,\ 8,\ 16,\ \ldots$$

instead of powers of 10.

We used digits 3, 5, and 8 to write down the number 538. More generally, the ten *digits*

$$0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9$$

are the symbols that we use in combination to represent all possible numbers.

Question What is the difference between a *digit* and a *number*?

Answer Digits are symbols. They are similar to letters of an alphabet. Numbers are notions that are written with the help of digits in the same manner as words are written with the help of letters. There are only ten digits, but infinitely many numbers.

Note that the order of digits in a numeral is important: 538 and 835 are different numbers. We say that our system is *positional* or that it uses the *place-value* principle.

6.3 Tally Numeration System

The tally marks maybe the oldest way to count things: each counted object is represented by a stroke, a tally mark. Each numeral is a line of several strokes.

Question What are advantages and disadvantages of this system?

Answer The obvious advantage of the tally numeration system is its simplicity. We still use tally marks to count scores in games, attendance, and some intermediate results.

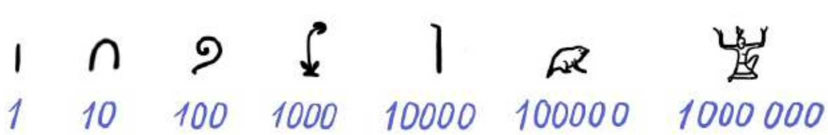
Among the main disadvantages is that the system is not practical for counting large numbers.

- Large numbers require many individual symbols.
- It is difficult to read such numerals, and it is easy to make a mistake when you write them.

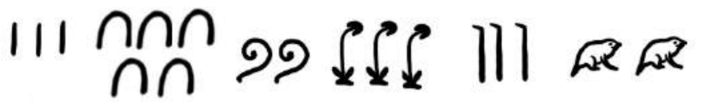
Very often, for the sake of legibility the tally marks are arranged in groups of five, where the last mark is placed across the other four in the group:

6.4 Egyptian Numeration System

One of the most well-studied hieroglyphic systems is the ancient Egyptian numeration system. It was in common use between about 3250 BCE and 400 CE in the Nile Valley area. The Egyptian numeration system has the *decimal structure*. As in our system, it is *based on powers of 10*. For every power of 10 a new symbol was introduced, which roughly looked like



The system has the simple *additive* principle: The values of individual numerals are just added together. For example,



would correspond to

$$3 + 50 + 200 + 3\,000 + 30\,000 + 200\,000 = 233\,253$$

Question What are advantages and disadvantages of the Egyptian numeration system?

Answer The system is intuitive and simple. It uses less symbols for large numbers than tally system. We also note that the order of symbols in this system is not very important. Indeed, there exist inscriptions written from left to right, right to left, and top to bottom. At the same time, still large numbers are represented by lengthy numerals. For example, 18 symbols are needed to write the numeral

which is just 567 written with three symbols in our system. Thus, it takes longer time to write these numerals. In addition, the symbols themselves are not that easy to reproduce.

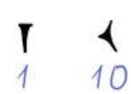
6.5 Babylonian Positional System

The cuneiform systems were commonly used in Mesopatamia for over 1,500 years, starting from around 2000 BCE. There were several types of cuneiform numeration systems. Many of them were additive and based on groups of ten. Very often numbers like 60, 600, 3600 have had the special notation. The origins of our 60 minutes in one hour and the 360° division of the circle can be traced to the influence of Babylonian astronomy and mathematics on Western mathematics through Ancient Greeks.

The most famous example of an old Mesopatamian numeration is the *Babylonian positional system*. It was not the main numeration system in the region for economic and cultural purposes, but for a short period has had a limited use in astronomy and higher mathematics. Yet, the elegance of this *base 60 positional* numeration system and resemblance of the decimal system continue to astonish many historians of mathematics.

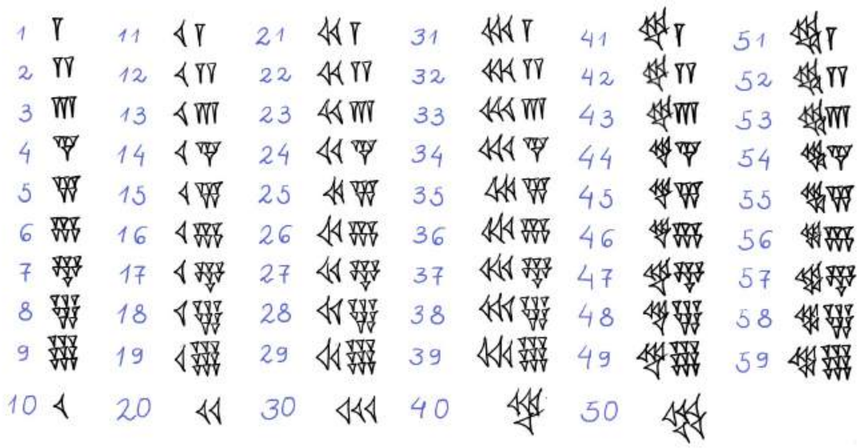
Here are the main principles of the Babylonian numeration system.

- The system uses only *two symbols*



Note that the shape of signs is dictated by the shape of writing tools. These symbols are convenient to reproduce by impressing a sharp stylus onto wet clay.

- Up to the number 59, they are represented in an additive way:



- For a larger number, the positions of combination of basic symbols encode the expansion of numerals in *powers of* 60

$$1,\ 60,\ 3600,\ \ldots$$

For example, according to the rules of the system, the numeral

consists of 44 in the leftmost position (corresponding to $60^2 = 3600$), 20 in the middle position (corresponding to $60^1 = 60$), and 11 in the last position (corresponding to $60^0 = 1$). All together, this gives the value

$$44 \times 60^2 + 20 \times 60 + 11 = 159\,611$$

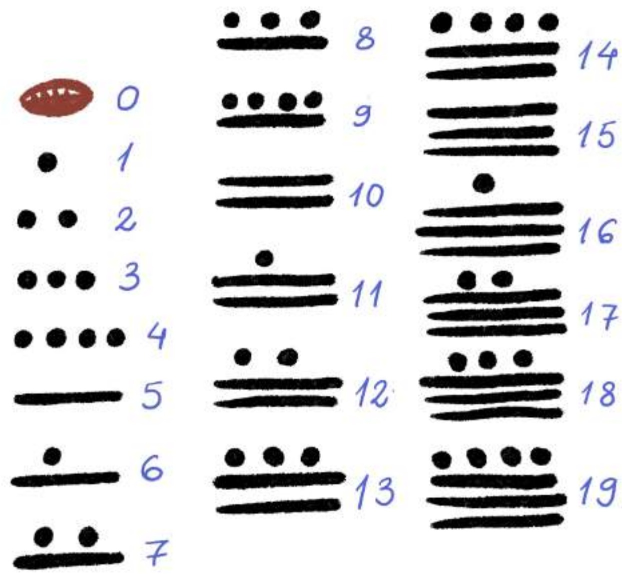
Thus, the system is based on the place-value principle and the *order of symbols* is important.

- The system is very advanced, but it may produce confusing situations. There is *no zero symbol* that would serve as a place holder, and for this reason without context it may be difficult to distinguish the symbol  as 1 from 60 or  as 11 from $10 \times 60 + 1 = 701$. To resolve this ambiguity, some texts used a bigger symbol for 60 or a large empty space to indicate the empty position. Sometimes, numerals were arranged by placing positional values in columns.

6.6 Maya Numeration System

The Mesoamerican numeration systems started to develop as early as 400 BCE and were in use until the Spanish conquests of the sixteenth century. Most of the information on the Maya numeration system comes from carved inscriptions and a few surviving codices. The main application of these numerals is calendarical, enumerating periods of time. This may explain some peculiarities of this numeration system.

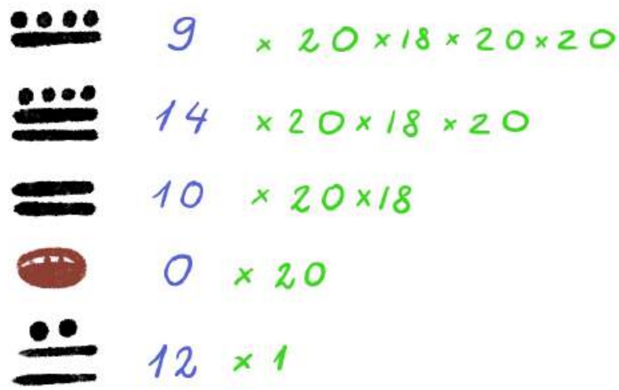
- Small numbers from 1 to 19 were written in bar-and-dot symbols by *additive principles*, where the dot stands for 1 and the bar stands for 5.



- The system had a *zero* which served as a *place-holder* within the numeral. Different symbols were used for zero, in codices the most common sign looked like a "shell."

- Twenty was a distinguished number that occasionally was denoted by a special glyph with the meaning of the *moon* or *lunar month*.

- Recall that when we write a date like (03 – 28 – 2002), each position indicates certain time units: the 28th day, the third month, the two thousand second year since the starting point of the calendar. In the context of a calendar, a Mayan numeral could be written in columns, where each level would also correspond to a time period: a day, a lunar month of twenty days, a year of eighteen lunar months (or 360 days), twenty years, four hundred years, with each successive period after that to be twenty times the previous one. (Here, we again stress that real variations of historical numeration systems are complicated, and we only outline some simplified ideas.) When a numeral phrase represents a date, its expanded form calculates the amount of time between the starting point of the Maya calendar and the date. For example, using these rules, we can write



which means 12 days, 0 lunar months, $(10 + 14 \times 20 + 9 \times 20^2 =)3890$ years or

$$12 + 0 \times 20 + 10 \times 18 \times 20 + 14 \times 18 \times 20^2 + 9 \times 18 \times 20^3$$
$$= 12 + 0 \times 20 + 10 \times 360 + 14 \times 7200 + 9 \times 144000$$
$$= 12 + 3600 + 122400 + 1296000$$
$$= 1\,422\,012 \text{ days from a start of the Mayan calendar}$$

- Observe that Mayan numerals are based on groups of

$$1, \quad 20, \quad 18 \times 20, \quad 18 \times 20^2, \quad 18 \times 20^3 \ldots$$

On the other hand, we almost never convert our dates to numbers of days, so, even with the importance of this exercise, it is quite possible that Mayan numerals were not applied this way by their users.

6.7 Roman Numeration System

Occasionally, we see numbers written in the Roman numerals in our daily life (for example, dates, clock faces). Due to the Roman conquests it was the *main numeral system* in all of Europe for nearly 1800 years, until only seven centuries ago it was replaced by the more effective Hindu-Arabic system. Let us review the main principles of the system.

The basic Roman numerals are given by the table

Basic Roman numeral	value
I	1
V	5
X	10
L	50
C	100
D	500
M	1000

Numbers are made out of combinations of basic numerals by using addition and, sometimes, subtraction.

Example 6.1 What are the values of the Roman numerals?

(a) VII (b) XXV (c) LVII (d) CCLXXXII

Solution

(a) VII$= 5 + 1 + 1 = 7$

(b) XXV$= 10 + 10 + 5 = 25$

(c) LVII$=50 + 5 + 1 + 1 = 57$

(d) CCLXXXII $= 100 + 100 + 50 + 10 + 10 + 10 + 2 = 282$

In some cases, to prevent numbers from being too long, the *subtraction notation* was used.

Example 6.2

(a) Instead of writing 4 as IIII, one interprets $4 = 5 - 1$, which is represented as IV$= 5 - 1 = 4$ (the subtracted I is placed to the left of V).

(b) Instead of writing 9 as VIIII, one interprets $9 = 10 - 1$, which is represented as IX$= 10 - 1 = 9$ (the subtracted I is placed to the left of X).

(c) Instead of writing 40 as XXXX, one interprets $40 = 50 - 10$, which is represented as XL$= 50 - 10 = 40$ (the subtracted X is placed to the left of L).

The subtraction rule is a little bit tricky due to some restrictions.

Example 6.3 How is 49 written as a Roman numeral?

Solution It is tempting to write 49 = IL which represents $49 = 50 - 1$. But, according to the existing standard rules for Roman numerals, this turns out to be not correct. These rules state that the subtraction notation is used only for the following pairs of Roman symbols:

Subtraction principle is used	value
IV	4 =5-1
IX	9 =10-1
XL	40=50-10
XC	90 =100-10
CD	400=500-100
CM	900=1000-100

For example, according to this principle, we do not write 99 $=100-1=$IC. Instead, we use 99= 90+ 9=XC+IX= XCIX.

Example 6.4 Convert the following numbers to Roman numerals:

(a) 21	(c) 39	(e) 499	(g) 1999
(b) 19	(d) 501	(f) 543	(h) 2150

Solution

(a) 21=XXI

(b) 19=X+ IX= XIX

(c) 39= 30+9= XXXIX

(d) 501=CI

(e) 499= 400+90+9=CD+XC+IX= CDXCIC

(f) 543=500+40+3= D+XL+III= DXLIII

(g) 1999=1000+900+90+9 = M+ CM+XC+IX= MCMXCIX

(h) 2150= 2000+100+50=MM+C+L= MMCL

Example 6.5 Find the values of the following Roman numerals:

(a) XVII (b) CI (c) DXIX (d) MCMLXXIV

Solution

(a) XVII= X+VII= 10+7=17

(b) CI=C+I= 100+1=101

(c) DXIX= D+X+IX= 500+10+9=519

(d) MCMLXXIV= M+CM+LXX+ IV=1000+900+70+4=1974

Question What are advantages and disadvantages of the Roman numeration system?

Answer One can quickly note that the Roman numerals are not difficult to use for addition and subtraction of whole numbers, but multiplication and division are complicated. The system does not have zero and is not well designed for operations with fractions. The more progressive Hindu-Arabic numeration system accelerated the development of mathematical sciences in Europe.

6.8 Exercises

Exercise 6.1 List advantages and disadvantages of the historical systems discussed in the class.

Exercise 6.2 What is the difference between a digit and a number?

Abstract manipulations with numerals written in historical numeration systems can represent exercises that are mathematically correct, but have no historical meaning. For this reason we consider here only problems on Roman numerals.

Exercise 6.3 Write the numbers from 1 to 20 in the Roman numerals.

Exercise 6.4 Write the following Roman numerals as decimals:

(a) IV
(b) VI
(c) LXX
(d) XI
(e) XC
(f) CMX
(g) CCLXXI
(h) MCCCXLIV
(i) MMMCXLIX

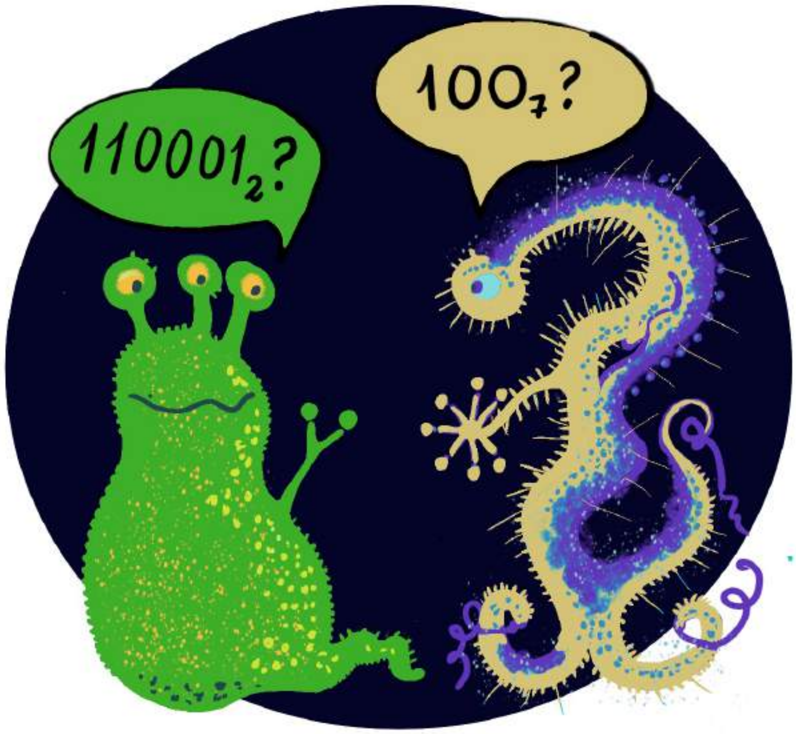
Exercise 6.5 Explain why these collections of symbols do not represent correct Roman numerals. Suggest what number was incorrectly written and give a correct Roman numeral.

(a) VIIII
(b) XVX
(c) ID

Exercise 6.6

(a) Write today's date in Roman numerals.
(b) Write the year of your birth in Roman numerals
(c) Write the year of the birth of one of your family members in Roman numerals.

7. NONDECIMAL BASE SYSTEMS



Base two numerals and base seven numerals.

7.1 Decimal Numeration System

In Chapter 6, we looked at examples of how different civilizations recorded numerals. Let us review the main features of our numeration system.

- Our system is called *decimal* since it uses groups of powers of ten

$$10^0 = 1, \quad 10^1 = 10, \quad 10^2 = 100, \quad 10^3 = 1000, \quad \ldots$$

- All numbers are written with the help of ten symbols, called *digits*,

$$0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9$$

- Our system is *positional*. A number is represented by a sequence of digits placed in one line *in order*. For example, 123 and 321 represent different numbers since the digits are placed in different order.

- *Positions* of digits carry information about the *powers of ten* that should be used to recover the expanded form of the number.

Example 7.1 The number 2364 has the expanded form

$$\begin{aligned}2364 &= 2\times 1000 + 3\times 100 + 6\times 10 + 4\times 1\\ &= 2\times 10^3 + 3\times 10^2 + 6\times 10^1 + 4\times 10^0\end{aligned}$$

Let us mark the positions of the digits of the number 2364 with labels $0, 1, 2, 3, \ldots$ from right to left:

2	3	6	4
3	2	1	0

Then the labels exactly match the powers of ten in expanded form. For example, 4 is placed in position 0, so it should be multiplied by 10^0.

This way of encoding numbers is very efficient and allows us to write very large numbers in a compact nonambiguous form.

7.2 Is Ten a Special Number?

We use powers of ten in our numeration system. We saw that many civilizations used ten in their numeration systems too. It may look like that ten is a special number that happens to be the easiest for calculations. This is not exactly the case. From the point of view of mathematics, the abstract number ten and its powers are not "easier" than many other numbers. It is commonly suggested that the choice of ten for the base of our numeration system is historically due to the possession of ten fingers as a convenient counting tool. Quite possible that if we would use 60 or 20 as the base of our system (as some civilizations did!), these numbers and their multiples would be the nicest round numbers for us. Hence the number ten is special for our numeration system more for *historical* or anthropological reasons than for its *mathematical* properties.

In this chapter, we discuss numeration systems that have the same principle as our system, but use a nondecimal base. We may not see these systems often in our daily life, but they do have applications to modern science and technology. Discussion of general systems will again contribute to a better understanding of our numeration system.

7.3 Conversion from Binary to Decimal System

Imagine that, one day, we meet aliens from another planet, where a similar numeration system is used, but with powers of *two* instead of ten:

$$2^0 = 1, \quad 2^1 = 2, \quad 2^2 = 4, \quad 2^3 = 8, \quad \ldots$$

We can compare the main features of our *decimal* (base ten) system and their *binary* (base two) system.

Our (base ten) system	Aliens' (base two) system
We use powers of ten $10^0 = 1$, $10^1 = 10$, $10^2 = 100$, $10^3 = 1000, \ldots$	They use powers of two $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, $2^3 = 8, \ldots$
We use ten symbols as digits $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ which allows us to write any number	They use two symbols as digits $\{0, 1\}$ which allows them to write any number

To distinguish numbers written in the binary system from those written in our decimal system, we indicate the base at the end of the number. For example, the number 111_{two} is written in the language of aliens and 111 is written in our system.

We would like to translate some numbers of aliens into our language. How are we going to do this?

Example 7.2 What number is represented by the base two numeral 111_{two}?

Solution To decode this number, we need to find the expanded form of the numeral. It is done in the same way as in our system, but with powers of two. First, mark the positions of digits

1	1	1	*two*
2	1	0	

The positions tell us what power of 2 is matched with each digit in expanded form. We simplify the expanded form to get the final answer

$$111_{\text{two}} = 1 \times 2^{2} + 1 \times 2^{1} + 1 \times 2^{0} = 4 + 2 + 1 = 7$$

Answer: 111_{two} represents the number 7.

Example 7.3 What number is represented by the base two numeral 1101_{two}?

Solution Label the positions of digits, reconstruct the expanded form, and simplify.

1	1	0	1	*two*
3	2	1	0	

$$1101_{\text{two}} = 1 \times 2^{3} + 1 \times 2^{2} + 0 \times 2^{1} + 1 \times 2^{0} = 8 + 4 + 0 + 1 = 13$$

Answer: $111_{\text{two}} = 13$.

Example 7.4 What number is represented by the base two numeral 101_{two}?

Solution

1	0	1 *two*
2	1	0

The positions tell us how to match powers of 2 with digits in expanded form

$$101_{\text{two}} = 1 \times 2^{2} + 0 \times 2^{1} + 1 \times 2^{0} = 4 + 0 + 1 = 5$$

Answer: $101_{\text{two}} = 5$.

Remark While we explain the binary system with the help of imaginary communication with aliens, the system itself is not imaginary at all. It is widely used at our planet in computer related technologies, where the digit 1 represents the situation "a signal is coming" and the digit 0 means "no signal."

7.4 Conversion from Other Base Systems to Decimal System

We discussed the base ten and base two numeration systems. Any counting number greater than one can serve as the base of a positional numeration system, and it all works in a similar way. Let us look at examples.

Suppose that we meet aliens from a planet where the *base six* numeration system is used. This means the following.

- The expanded form of their numerals uses powers of six

$$6^0 = 1, \quad 6^1 = 6, \quad 6^2 = 36, \quad 6^3 = 216, \quad \ldots$$

- There are six symbols that serve as digits

$$\{0, 1, 2, 3, 4, 5\}$$

- The position of each digit in a numeral indicates the corresponding power of 6 in expanded form.

- To distinguish base six numerals, we put the label "six."

Example 7.5 Convert the base six numeral 10_{six} to a decimal number.

Solution Label positions of digits to reconstruct the expanded form

1	0	*six*
1	0	

$$10_{\text{six}} = 1 \times 6^1 + 0 \times 6^0 = 6 + 0 = 6$$

Answer: $10_{\text{six}} = 6$.

Example 7.6 Convert the base six numeral 231_{six} to a decimal number.

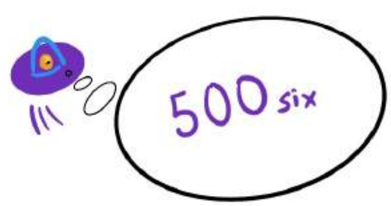
Solution

2	3	1	*six*
2	1	0	

$$231_{\text{six}} = 2 \times 6^2 + 3 \times 6^1 + 1 \times 6^0 = 2 \times 36 + 3 \times 6 + 1 \times 1 = 72 + 18 + 1 = 91$$

Answer: $231_{\text{six}} = 91$.

Example 7.7 Convert the base six numeral 500_{six} to a decimal number.



Solution

5	0	0	*six*
2	1	0	

$$500_{\text{six}} = 5 \times 6^2 + 0 \times 6^1 + 0 \times 6^0 = 5 \times 36 + 0 \times 6 + 0 \times 1 = 180$$

Answer: $500_{\text{six}} = 180$.

The above examples for the bases two and six give a clear idea of how to handle other base systems.

Example 7.8 Convert the base seven numeral 21_{seven} to a decimal number.

Solution

2	1	*seven*
1	0	

$$21_{\text{seven}} = 2 \times 7^1 + 1 \times 7^0 = 2 \times 7 + 1 \times 1 = 14 + 1 = 15$$

Answer: $21_{\text{seven}} = 15$.

7.5 Systems with a Base Greater Than Ten

Note that the base ten system uses ten digits, the base two system uses two digits, the base six system uses six digits, and so on. More generally, the base N system uses N digits. We need to make a comment on systems that have a base greater than ten. For example, consider the base twelve system. According to the general rule, we use twelve symbols as digits. It is tempting to choose ordinary numbers as such symbols

$$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$$

The idea would be correct, but there would be a problem with the last two "digits" 10 and 11 because they can produce ambiguity. Indeed, 10_{twelve} could be understood in two different ways:

1	0	*twelve*
1	0	

$$10_{\text{twelve}} = 1 \times 12^1 + 0 \times 12^0 = 12$$

or

	10	*twelve*
	0	

$$10_{\text{twelve}} = 10 \times 12^0 = 10$$

To avoid ambiguity, it is better to introduce other symbols for the "digits" 10 and 11, for example, $A = 10$ and $B = 11$. Then our collection of digits is

$$\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B\}$$

Example 7.9

$$A_{\text{twelve}} = 10 \times 12^0 = 10$$

$$10_{\text{twelve}} = 1 \times 12^1 + 0 \times 12^0 = 12$$

$$11_{\text{twelve}} = 1 \times 12^1 + 1 \times 12^0 = 13$$

$$AA_{\text{twelve}} = A \times 12^1 + A \times 12^0 = 10 \times 12 + 10 \times 1 = 130$$

$$1B_{\text{twelve}} = 1 \times 12^1 + B \times 12^0 = 1 \times 12 + 11 \times 1 = 23$$

$$5A_{\text{twelve}} = 5 \times 12^1 + A \times 12^0 = 60 + 10 = 70$$

7.6 Conversion from Decimal System to Other Base Systems

We learned how to "read a foreign language," that is, how to translate numerals of other base systems into our decimal system. Now, we would like to learn to "speak the foreign language," that is, to translate given decimal numerals into the other base language of aliens. This process involves *division with remainder* by powers of numbers.

Example 7.10 How is the number 395 converted to a base eight numeral?

$$395 = ???_{\text{eight}}$$

Solution Let us establish some facts about the base eight system.

- The system uses powers of eight. Let us write the first few of them

$$8^0 = 1, \quad 8^1 = 8, \quad 8^2 = 64, \quad 8^3 = 512, \quad \ldots$$

- The digits of the base eight system are $\{0,1,2,3,4,5,6,7\}$. Our final answer will be a numeral written with digits from this collection.

The transition "from one language to another" goes through the expanded form, so we want to write 395 as a combination of powers of 8

$$395 = ?8^{?} + ?8^{?} + \cdots + ?8^{1} + ?8^{0}$$

The coefficients of powers of eight (the green question mark ?) are digits of the base eight numeral, and powers of eight (the orange question mark ?) provide their positions. To find the digits, we divide 395 by the *largest power of 8 that can be fit into 395*

$$395 \quad \longrightarrow \quad \begin{array}{c} 8^0 = 1 \\ 8^1 = 8 \\ 8^2 = \underline{64} \\ \\ 8^3 = 512 \\ \cdots \end{array}$$

Since

$$8^2 = \underline{64} < 395 < 512 = 8^3$$

we have $8^2 = 64$ to be the largest power of 8 that fits into 395, so we *divide 395 by 64 with remainder*:

$$395 = 6 \times 64 + 11 = \boxed{6 \times 8^{2}} + 11$$

Next, we repeat the same step with the remainder 11:

$$11 \quad \longrightarrow \quad \begin{array}{c} 8^0 = 1 \\ 8^1 = \underline{8} \\ \\ 8^2 = 64 \\ 8^3 = 512 \\ \cdots \end{array}$$

We have

$$8^1 = \underline{8} < 11 < 64 = 8^2$$

The largest power of 8 that fits into 11 is 8^1, so we divide 11 by 8 with remainder:

$$11 = 1 \times 8 + 3 = \boxed{1 \times 8^{1}} + 3$$

We proceed with the new remainder 3:

$$3 \quad \longrightarrow \quad \begin{array}{c} 8^0 = \underline{1} \\ \\ 8^1 = 8 \\ 8^2 = 64 \\ 8^3 = 512 \\ \cdots \end{array}$$

We have

$$8^0 = \underline{1} < 3 < 8 = 8^1$$

and divide 3 by 1 with remainder:

$$3 = 3 \times 1 + 0 = \boxed{3 \times 8^{0}} + 0 (\leftarrow STOP)$$

The zero remainder tells us that we have to stop and the calculation is complete. Let us summarize the preformed steps of the successive division with remainder:

$$395 = \boxed{6 \times 8^{2}} + 11$$

$$11 = \boxed{1 \times 8^{1}} + 3$$

$$3 = \boxed{3 \times 8^{0}}$$

The boxed parts provide the answer: The green digits (quotients) obtained at each step are exactly the digits of the corresponding base eight numeral, and the orange powers of eight indicate their positions. Our calculation can be interpreted as the expanded base eight form

$$395 = 6 \times 8^{2} + 1 \times 8^{1} + 3 \times 8^{0}$$

We write the final answer

$$395 = 613_{\text{eight}}$$

Let us look at other examples.

Example 7.11 Convert the number 70 to a base eight numeral.

$$70 = ??_{\text{eight}}$$

Solution We again use powers of eight

$$8^{0} = 1, \quad 8^{1} = 8, \quad 8^{2} = 64, \quad 8^{3} = 512, \quad \ldots$$

and the digits $\{0, 1, 2, 3, 4, 5, 6, 7\}$. We perform the division with remainder by the largest power of eight that can be fit in the number at each step

$$8^{2} = 64 < 70 < 512, \quad \text{so} \quad 70 = 1 \times 64 + 6 = \boxed{1 \times 8^{2}} + 6$$

$$8^{0} = 1 < 6 < 8 = 8^{1}, \quad \text{so} \quad 6 = 6 \times 1 + 0 = \boxed{6 \times 8^{0}} + 0 (\leftarrow STOP)$$

The boxed parts

$$70 = \boxed{1 \times 8^{2}} + 6$$

$$6 = \boxed{6 \times 8^{0}} + 0$$

tell us that

$$70 = 1 \times 8^{2} + 0 \times 8^{1} + 6 \times 8^{0}$$

and we place the digits

1	0	6 *eight*
2	1	0

Answer: $70 = 106_{\text{eight}}$.

Example 7.12 Convert the number 74 to a base six numeral.

$$74 = ??_{\text{six}}$$

Solution We use powers of six

$$6^0 = 1, \quad 6^1 = 6, \quad 6^2 = 36, \quad 6^3 = 216, \quad \ldots$$

and the digits $\{0, 1, 2, 3, 4, 5\}$. Division with remainder yields

$$36 < 74 < 216, \quad \text{so} \quad 74 = 2 \times 36 + 2 = \boxed{2 \times 6^2} + 2$$

$$6^0 = 1 < 2 < 6 = 6^1, \quad \text{so} \quad 2 = 2 \times 1 + 0 = \boxed{2 \times 6^0} + 0 (\leftarrow STOP)$$

The boxed parts tell us the digits and their positions

$$74 = 2 \times 6^2 + 0 \times 6^1 + 2 \times 6^0$$

and we place the digits

2	0	2 *six*
2	1	0

Answer: $74 = 202_{\text{six}}$.

Example 7.13 Convert the number 100 to a base six numeral.

$$100 = ???_{\text{six}}$$

Solution We use powers of six

$$6^0 = 1, \quad 6^1 = 6, \quad 6^2 = 36, \quad 6^3 = 216, \quad \ldots$$

and the digits $\{0, 1, 2, 3, 4, 5\}$. Division with remainder yields

$$36 < 100 < 216, \quad \text{so} \quad 100 = 2 \times 36 + 28 = \boxed{2 \times 6^2} + 28$$

$$6 < 28 < 36 = 6^2, \quad \text{so} \quad 28 = 4 \times 6 + 4 = \boxed{4 \times 6^1} + 4$$

$$1 < 4 < 6, \quad \text{so} \quad 4 = 4 \times 6^0 + 0 = \boxed{4 \times 6^0} + 0 (\leftarrow STOP)$$

The boxed parts tell us the digits and their positions

$$100 = 2 \times 6^{2} + 4 \times 6^{1} + 4 \times 6^{0}$$

and we place the digits

$$\begin{array}{c|c|cl} 2 & 4 & 4 & six \\ \hline 2 & 1 & 0 & \end{array}$$

Answer: $100 = 244_{\text{six}}$.

7.7 Exercises

7.7.1 The expanded form of a numeral in decimal system

Exercise 7.1 Write the following numbers in expanded form.

(a) 70 (b) 923 (c) 48 200 123

Exercise 7.2 Write the following in standard place-value form.

(a) $1 \cdot 1000 + 6 \cdot 100 + 2 \cdot 1$

(b) $4 \cdot 10^5 + 3 \cdot 10^2 + 1 \cdot 10^0$

(c) $8 \cdot 10^4 + 2 \cdot 10$

(d) $1 \cdot 10^1 + 2 \cdot 10^0$

(e) $5 \cdot 10^6 + 3 \cdot 10^3 + 3 \cdot 10^0$

(f) $1 \cdot 10^5 + 5 \cdot 10^1$

7.7.2 Conversion from other base systems to decimal system

Exercise 7.3 How many digits (symbols) are used in the base seven system?

Exercise 7.4 How many digits (symbols) are used in the base fifty system?

Exercise 7.5 Why do we need to introduce extra symbols, besides the digits $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$, in the base thirteen system?

Exercise 7.6 What is wrong with the numeral 67_{seven}?

Exercise 7.7 What is wrong with the numeral 533_{four}?

Exercise 7.8 Write the following base six numerals in expanded base six form.

(a) 10_{six} (b) 125_{six} (c) 4510_{six}

Exercise 7.9 Convert the following numerals from the indicated base system to the base ten system.

(a) 110_{two} (d) 101_{two} (g) 123_{five}

(b) 1000_{two} (e) 100_{three} (h) 2032_{four}

(c) 100_{two} (f) 100_{five} (i) 102_{eight}

Exercise 7.10 True or false?

(a) $5_{\text{nine}} = 5$ (b) $30_{\text{four}} = 20_{\text{six}}$ (c) $200_{\text{three}} = 100_{\text{nine}}$

Exercise 7.11 The base thirteen numeration system has thirteen symbols $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C\}$, where $A = 10$, $B = 11$, and $C = 12$. Convert the following base thirteen numerals to base ten numerals.

(a) 133_{thirteen} (d) CA_{thirteen} (g) $10B_{\text{thirteen}}$

(b) C_{thirteen} (e) AC_{thirteen}

(c) $B1_{\text{thirteen}}$ (f) $1AC_{\text{thirteen}}$

Exercise 7.12 Write the numbers from 1 to 10 in the base five.

Exercise 7.13 Write the numbers from 1 to 10 in the base nine.

Exercise 7.14 What is the largest three-digit base two number?

Exercise 7.15 What is the largest three-digit base four number? What is the smallest four-digit base four number?

Exercise 7.16 What numeral follows 234_{five} in the base five system?

7.7.3 Conversion from decimal system to other base systems

Exercise 7.17 Convert the following base ten numerals to numerals in the indicated base.

(a) $395 = ?_{\text{eight}}$

(b) $748 = ?_{\text{four}}$

(c) $54 = ?_{\text{two}}$

(d) $74 = ?_{\text{six}}$

(e) $74 = ?_{\text{eleven}}$

Exercise 7.18 Find the missing base.

(a) $35 = 21_?$

(b) $43_{\text{six}} = 27_?$

(c) $51_{\text{seven}} = 36_?$

(d) $32 = 44_?$

(e) $31_{\text{four}} = 11_?$

(f) $50_{\text{eight}} = 10_?$

8. PRIME NUMBERS AND DIVISIBILITY TESTS

2,3,5,7,11,13... .

8.1 Prime Numbers

In this chapter, we talk about the *structure* of counting numbers. All of them are made of building blocks, *prime numbers*.

A *prime number* is a counting number that has exactly two divisors 1 and itself.

Remark The number *1 is not considered to be a prime number* since it has only one divisor.

Question Write down all prime numbers less than 50.

Answer $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47$.

Remark It is recommended to memorize all the prime numbers less than 50. This may be useful in many problems.

8.2 Prime Factorization

Prime numbers serve as building blocks for all counting numbers due to a very important statement.

Fundamental theorem of arithmetic. Each counting number greater than one can be expressed as the product of primes in a unique way up to the order of factors.

The *prime factorization* of a number is a representation of the number as the product of primes.

Example 8.1 Here are examples of some prime factorizations.

(a) $6 = 2 \times 3$

(b) $17 = 17$

(c) $60 = 2^2 \times 3 \times 5$

(d) $39 = 3 \times 13$

(e) $250 = 2 \times 5^3$

Note that $60 = 2 \times 6 \times 5$ and $250 = 10 \times 25$ are *not prime factorizations* since not all factors are prime. The order of factors in the prime factorization is not important. For example, the factorizations

$$60 = 2 \times 2 \times 3 \times 5 = 3 \times 2 \times 5 \times 2$$

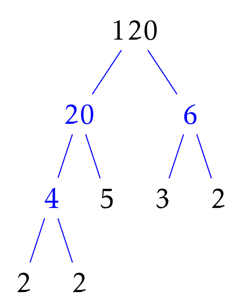
are considered to be equivalent.

Question How does one find the prime factorization of a number?

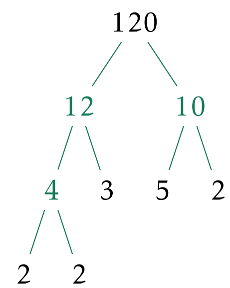
Answer One can do it step-by-step, as in this example:

$$120 = 20 \times 6 = 4 \times 5 \times 2 \times 3 = 2 \times 2 \times 5 \times 2 \times 3 = 2^3 \times 3 \times 5$$

Equivalently, one can use a factorization tree



Note that there may be different factorization trees, but *the resulting prime factorization is always the same up to the order of prime factors*.



This tree also gives $120 = 2^3 \times 3 \times 5$.

Example 8.2 Find the prime factorizations of the following numbers:

(a) 13 (b) 72 (c) 108 (d) 121 (e) 380 (f) 1000 (g) 129 (h) 179

Solution Using factorization trees, we find

(a) $13 = 13$

(b) $72 = 2^3 \times 3^2$

(c) $108 = 2^2 \times 3^3$

(d) $121 = 11^2$

(e) $380 = 2^2 \times 5 \times 19$

(f) $1000 = 2^3 \times 5^3$

(g) $129 = 3 \times 43$

(h) $179 = 179$

The last example, the prime factorization of 179, appears to be difficult since we have to figure out that this number is already prime. This leads to the important questions.

1. How does one start a factorization tree?
2. How does one determine whether a given number is prime?
3. More generally, how to determine what numbers divide a given number?

In many cases, *tests for divisibility* may help us answer these questions. For example, using such a test, one can, without performing division, quickly determine that the number 612311 is not divisible by 3. We learn *tests for divisibility by* 2, 3, 4, 5, 6, 8, 9, 10, 11, 12 (we skip 7). These tests are very practical for various computations. In particular, they save time, make computations more efficient, help to start factorization trees and simplify fractions.

8.3 Notation for Divisibility

The following notation is commonly used when divisibility is discussed.

Let a and b be counting numbers. We write $a \mid b$ and say that *a divides b* if there exists a counting number x such that $ax = b$. In this case, we can also say that b is a *multiple* of a and a is a *divisor* of b. We write $a \nmid b$ if *a does not divide b*.

For example, $4 \mid 8$ and $3 \mid 12$, but $3 \nmid 50$ and $2 \nmid 11$. We also need the definition of relatively prime numbers.

Two counting numbers are called *relatively prime* if their only common divisor is 1.

Example 8.3 The following pairs of numbers are relatively prime:

(a) 2 and 3

(b) $14 = 2 \times 7$ and $15 = 3 \times 5$

(c) $6 = 2 \times 3$ and $121 = 11 \times 11$

The numbers 21 and 28 are *not relatively prime* since $21 = 7 \times 3$, $28 = 7 \times 4$, and 7 is a nontrivial common divisor of both numbers 21 and 28.

8.4 Divisibility Tests

8.4.1 Test for divisibility by two

Example 8.4 Is the number 1234567898345 divisible by two? How did you get the answer?

Solution To determine whether a number is divisible by two, we need to look only at the last digit. Since the last digit 5 is odd, the whole number is odd, not divisible by two. This is exactly how the test for divisibility by 2 is formulated.

A number is *divisible by two* if and only if its ones digit is 0, 2, 4, 6, or 8.

8.4.2 Test for divisibility by three

Example 8.5 Is the number 121452 divisible by 3?

Solution Yes, it is. This follows from the divisibility test.

A number is *divisible by three* if and only if the sum of its digits is divisible by 3.

Usually, the sum of digits is smaller than the original number, so it is easier to check whether it is a multiple of 3. In our example,

$$\text{the sum of digits} = 1+2+1+4+5+2 = 15$$

Since $3 \mid 15$, we have $3 \mid 121452$ by the test.

Remark If a number has many digits, we do not need to add all of them to find the total sum. It is enough to show that the digits can be arranged in groups in such a way that, *in each group*, the sum is a multiple of three.

Example 8.6 Is the number 101422272 divisible by 3?

Solution By the test for divisibility by three, we check whether the sum of digits is divisible by three.

$$\underbrace{1+0+1+4}_{6}+\underbrace{2+2+2}_{6}+\underbrace{7+2}_{9}$$

Each group gives a sum that is a multiple of three. Hence the total sum is a multiple of three. Answer: Yes, the number 101422272 is divisible by three.

Example 8.7 Is the number 111222555 divisible by 3?

Solution By the test for divisibility by three, we check whether the sum of digits is divisible by three.

$$\underbrace{1+1+1}_{3}+\underbrace{2+2+2}_{6}+\underbrace{5+5+5}_{15}$$

Each group gives a sum that is a multiple of three. Hence the total sum is a multiple of three. Answer: Yes, the number 111222555 is divisible by three.

8.4.3 Test for divisibility by four

A number is divisible by four if and only if the number represented by its last two digits is divisible by 4.

Example 8.8 Is the number 12378943216 divisible by 4?

Solution Yes, this number is divisible by 4 since the last two digits form the number 16 and $4 \mid 16$.

Example 8.9 True or false? $4|356$

Solution True since $4|56$ (recall that $56 = 7 \times 8 = 7 \times 2 \times 4$).

Example 8.10 Does 4 divide 1022?

Solution No, it does not since 4 does not divide 22.

Example 8.11 List all digits that can replace the question mark of the number

$$123?0$$

to make this five-digit number divisible by

(a) 3 (b) 4 (c) 12

Solution Let x be a missing digit.

(a) By the test for divisibility by three, a number is divisible by 3 if the sum of digits $1 + 2 + 3 + x$ is divisible by 3. This means that x can be 0, 3, 6, or 9. These cases correspond to the numbers 12300, 12330, 12360, or 12390.

(b) By the test for divisibility by four, the last two digits form a number $x0$ which must be divisible by 4. This means that x can be 0, 2, 4, 6, or 8. These cases correspond to the numbers 12300, 12320, 12340, 12360, or 12380.

(c) Note that $12 = 3 \times 4$ and the numbers 3 and 4 are relatively prime. This means that a number is divisible by 12 if and only if it is divisible by both 3 and 4. Thus, x is among the common answers for (a) and (b), which are $x = 0$ or $x = 6$. The numbers 12300 and 12360 are divisible by 12.

8.4.4 Test for divisibility by five

The test for divisibility by five is easy and well known.

A number is divisible by five if and only if its ones digit is 0 or 5.

Example 8.12 $5 \nmid 17$, $5 \nmid 234$, and $5 \nmid 1001$, but $5 \mid 100000005$.

8.4.5 Test for divisibility by six

The test for divisibility by six is based on the two facts.

1. $6 = 2 \times 3$.
2. 2 and 3 are *relatively prime numbers*. They have no nontrivial common divisors.

A number is divisible by six if and only if it is divisible by both 2 and 3.

Example 8.13 Does 6 divide 222?

Solution Yes, since $2|222$ and $3|222$. Hence $6|222$.

Example 8.14 Does 6 divide 842?

Solution No. We have $2|842$, but $3 \nmid 842$ since the sum of digits is $8+4+2=14$ and $3 \nmid 14$.

8.4.6 Test for divisibility by eight

The test for *divisibility by seven* is not simple and for this reason is rarely applied. We skip it and go to the test for *divisibility by eight*. Observe that $8 = 2 \times 4$, but 2 and 4 are *not relatively prime*. This means that we cannot follow the same logic as in the test for divisibility by six. If a number is divisible by 2 and 4, this does not guarantee that it is divisible by 8.

Question Give an example of a number that is divisible by both 4 and 2, but is not divisible by $8 = 4 \times 2$.

Answer Many different examples can be suggested, for example, $4, 20, 36, \ldots$.

The true test for divisibility by eight has the following form.

A number is *divisible by eight* if and only if the number represented by its last three digits is divisible by 8.

Example 8.15 Does 8 divide 12800?

Solution Yes, since the last three digits form 800 and $8|800$.

Example 8.16 Is it true that $8|13056$?

Solution Yes, since the last three digits form $056 = 56$ and $8|56$.

Example 8.17 Does 8 divide 150008?

Solution Yes, since the last three digits form $008 = 8$ and $8|8$.

Example 8.18 Does 8 divide 406162?

Solution No, since the last three digits form $162 = 160 + 2$ and it is clear that $8 \nmid 162$.

8.4.7 Test for divisibility by nine

The test for divisibility by nine is similar to the test for divisibility by three.

> A number is *divisible by nine* if and only if the sum of its digits is divisible by 9.

Example 8.19 Does 9 divide 124567281?

Solution The sum of digits can be arranged in groups that are multiples of nine:

$$1+2+4+5+6+7+2+8+1 = \underbrace{1+2+6}_{9} + \underbrace{4+5}_{9} + \underbrace{7+2}_{9} + \underbrace{8+1}_{9}$$

We see that the sum of digits is divisible by 9, so $9 \mid 124567281$.

8.4.8 Test for divisibility by ten

> A number is *divisible by ten* if and only if its ones digit is zero.

8.4.9 Test for divisibility by eleven

This is a test of a new kind. After formulating the rule, we explain how to use it with the help of an example.

> To find out whether a number is *divisible by eleven*, complete the following steps.
>
> *Step* 1. Break the digits of the number in two groups: the digits corresponding to odd positions and the digits corresponding to even positions in the number.
>
> *Step* 2. Find the sums A_1 and A_2 of all digits in each group.
>
> *Step* 3. Find the absolute value of the difference $A_1 - A_2$. If it is zero or divisible by 11, then the number is divisible by 11. Otherwise, it is not divisible by 11.

Example 8.20 Let us show that the number 1224675 is not divisible by 11.

Step 1. We break the digits in two groups.

$$1 \ 2 \ 2 \ 4 \ 6 \ 7 \ 5$$

Step 2. The sum of digits in the first group is $A_1 = 1 + 2 + 6 + 5 = 14$. The sum of digits in the second group is $A_2 = 2 + 4 + 7 = 13$.

Step 3. The absolute value of the difference of two sums is $|A_1 - A_2| = 14 - 13 = 1$. It is not zero and is not divisible by 11. This implies that the number 1224675 is not divisible by 11.

Example 8.21 Is the number 183909 divisible by 11?

Solution

Step 1. We break the digits in two groups

$$1\quad 8\quad 3\quad 9\quad 0\quad 9$$

Step 2. The sum of digits in the first group is $A_1 = 1 + 3 + 0 = 4$. The sum of digits in the second group is $A_2 = 8 + 9 + 9 = 26$.

Step 3. The absolute value of the difference of two sums is $|A_2 - A_1| = 26 - 4 = 22$. We know that $11|22$, so $11|183909$.

Remark The difference $A_2 - A_1$ may be a positive or negative number, but only the absolute value of this expression matters. We can consider $A_2 - A_1$ if $A_2 \geq A_1$ and $A_1 - A_2$ if $A_1 \geq A_2$.

8.4.10 Test for divisibility by twelve

The test for divisibility by twelve is the last test that we discuss in these lectures.

> A number is divisible by twelve if and only if it is divisible by both 4 and 3.

This test is based on the facts that $12 = 4 \times 3$ and that 4 and 3 are relatively prime.

Example 8.22 Does 12 divide 888?

Solution Yes, since $3|888$ (check that $8 + 8 + 8 = 24$ and $3|24$) and $4|888$ (since $4|88$). Both 3 and 4 divide 888, so $12|888$.

Example 8.23 Take any eight-digit number and determine whether it is divisible by 2,3,4,5,6,8,9,10,11, 12.

Solution We solve this problem for the number 47928135.

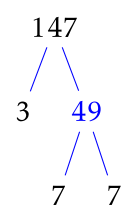
Number	Yes/No	Reason
2	no	the last digit is odd
3	yes	the sum of digits $= 4+7+9+2+8+1+3+5 = 39$ and $3 \mid 39$
4	no	$4 \nmid 35$ (another possible argument is that the number is not even)
5	yes	the last digit is 5
6	no	not divisible by 2
8	no	$8 \nmid 135$ (another possible argument is that the number is not even)
9	no	sum of digits is 39, and $9 \nmid 39$
10	no	does not end with zero
11	no	$A_1 = 4+9+8+3 = 24$. $A_2 = 7+2+1+5 = 15$. $\lvert A_1 - A_2 \rvert = \lvert 24 - 15 \rvert = 9$, and $11 \nmid 9$
12	no	not divisible by 4

8.5 Prime Factorization Using Divisibility Tests

We discussed that all counting numbers are made of building blocks, prime numbers. Recall that the prime factorization can be found by constructing *factorization trees*. Tests for divisibility may be very helpful to construct these trees since they help us *find divisors*. Let us look at examples.

Example 8.24 Find the prime factorization of 147.

Solution Using divisibility test, we note that $3 \mid 147$. This allows us to start the factorization tree with the division of 147 by 3.



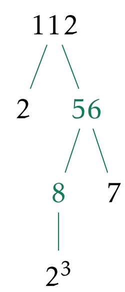
Answer: $147 = 3 \times 7^2$.

Example 8.25 Find the prime factorization of 429.

Solution By the test for divisibility by three, we note that $3 \mid 429$, so $429 = 3 \times 143$. By the test for divisibility by eleven, we may determine that $11 \mid 143$ and get $143 = 11 \times 13$.
Answer: $429 = 3 \times 11 \times 13$

Example 8.26 Find the prime factorization of 112.

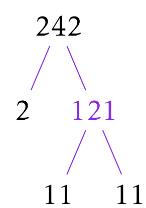
Solution It is an even number, so we start the factorization tree by dividing it by 2.



Answer: $112 = 2^4 \times 7$.

Example 8.27 Find the prime factorization of 242.

Solution



Answer: $242 = 2 \times 11^2$.

Example 8.28 Find the prime factorization of 148.

Solution We observe that $4|148$ and get $148 = 4 \times 37$. Here, we have to recall that 37 is a prime number. This example illustrates that it is good to know the first few prime numbers. Answer: $148 = 2^2 \times 37$.

8.6 More Properties of Divisibility

We learned several divisibility tests. Combining them, we can determine the divisibility by some other numbers.

- If 2 and 3 divide a number, then $6 = 2 \times 3$ also divides this number.
- If 4 and 3 divide a number, then $12 = 4 \times 3$ also divides this number.
- At the same time, if 2 and 6 divide a number, this does not imply, in general, that $12 = 2 \times 6$ also divides this number. The simplest example is 6 itself: $2|6$ and $6|6$, but $12 \nmid 6$.

Question For what kind of numbers, r and k, does the following test for divisibility hold: "If r and k divide a number, then $r \times k$ also divides this number?"

Answer We formulate the general rule.

Let r and k be *relatively prime*. If r and k divide a number, then $r \times k$ also divides the number.

Example 8.29 The numbers 4 and 7 are relatively prime. Any number that is divisible by both of them is a multiple of 28.

Example 8.30 The numbers 4 and 10 are not relatively prime. It may happen that 4 and 10 divide a number, but $40 = 4 \times 10$ does not. For example, take 20.

Example 8.31 Is the number 545436 divisible by 36?

Solution Note that $36 = 4 \times 9$ and 4 and 9 are relatively prime. If 4 and 9 divide a number, then 36 also divides this number. We have $4|545436$ (since $4|36$) and $9|545436$ (since $9|(5+4+5+4+3+6)$).
Answer: Yes, 545436 is divisible by 36.

Example 8.32 Is the number 1210 divisible by 110?

Solution Note that $110 = 11 \times 10$. The numbers 11 and 10 are relatively prime. If 11 and 10 divide a number, then 110 also divides this number. We have $10|1210$ and $11|1210$ (actually, $11^2 = 121$).
Answer: Yes, $110|1210$.

As a final remark, we note that the other way of writing our statement is always true.

If $n = r \times k$ and n divides a number, then both r and k divide the number too.
(We do not need r and k to be relatively prime for this property.)

Example 8.33 Since $12|144$, we have $2|144$ and $6|144$.

Example 8.34 We know that 24 divides a number N. What else divides N?

Solution All the divisors of 24 divide N, that is, $1, 2, 3, 4, 6, 8, 12, 24$.

8.7 Summary of Tests for Divisibility

- A number is *divisible by 2* if and only if its ones digit is 0, 2, 4, 6, or 8.
- A number is *divisible by 3* if and only if the sum of its digits is divisible by 3.
- A number is *divisible by 4* if and only if the number represented by its last two digits is divisible by 4.
- A number is *divisible by 5* if and only if its ones digit is 0 or 5.
- A number is *divisible by 6* if and only if it is divisible both by 2 and 3.
- A number is *divisible by 8* if and only if the number represented by its last three digits is divisible by 8.
- A number is *divisible by 9* if and only if the sum of its digits is divisible by 9.
- A number is *divisible by 10* if and only if its ones digit is zero.
- To find out whether a number is *divisible by 11*, complete the following steps.

Step 1. Break the digits of the number in two groups: the digits corresponding to odd positions and the digits corresponding to even positions in the number.

Step 2. Find the sums A_1 and A_2 of all digits in each group.

Step 3. Find the absolute value of the difference $A_1 - A_2$. If it is zero or divisible by 11, then the number is divisible by 11. Otherwise, it is not divisible by 11.

- A number is *divisible by 12* if and only if it is divisible by both 4 and 3.

8.8 Exercises

8.8.1 The prime factorization

Exercise 8.1 Find the factorization trees.

(a) 36

(b) 64

(c) 102

(d) 1000

(e) 243

(f) 380

Exercise 8.2 Find the prime factorizations.

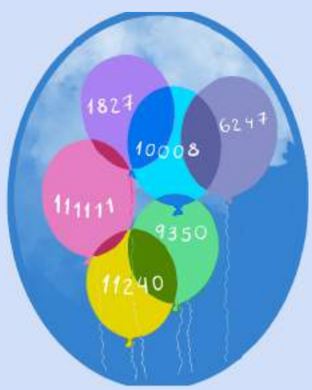
(a) 26
(b) 30
(c) 12
(d) 28
(e) 55
(f) 45
(g) 49
(h) 81
(i) 120
(j) 150
(k) 39
(l) 65
(m) 64
(n) 90
(o) 935

8.8.2 Tests for divisibility

Exercise 8.3 Which of the following numbers are divisible by 11?

(a) 111
(b) 2222
(c) 33333
(d) 333333
(e) 88888888
(f) 1010
(g) 1001
(h) 110011
(i) 101010
(j) 100001
(k) 11011011

Exercise 8.4 Pick any three balloons. Are the numbers on them divisible by 2, 3, 4, 5, 6, 8, 9, 10, 11, or 12?



Exercise 8.5 What digit could replace ■ so that the number 12345■6 would be divisible by 11?

Exercise 8.6 What digit could replace ■ so that the number 89567■2 would be divisible by 4?

Exercise 8.7 True or false? Explain your answer if an argument is needed.

(a) $3|9$

(b) 3 is a divisor of 12

(c) 4 is a factor of 16

(d) 6 is a factor of 3

(e) 16 is a multiple of 4

(f) $12|6$

(g) $11|11$

(h) $1|11$

(i) $11|1$

(j) $6|0$

(k) $0|5$

(l) 9 is a prime factor of 18

(m) If $2|a$, $6|a$, then $12|a$.

(n) If $6|ab$, then $6|a$ or $6|b$.

(o) If a counting number is divisible by 6 and 8, it must be divisible by 48.

(p) If a counting number is divisible by 8, it must be divisible by 4.

(q) If a counting number is divisible by 4, it must be divisible by 8.

(r) If a counting number is divisible by 9, it must be divisible by 3.

(s) If a counting number is divisible by 3 and 11, it must be divisible by 33.

(t) If a counting number is divisible by 2 and 4, it must be divisible by 8.

Exercise 8.8

(a) If 12 divides a, what else divides a?

(b) If 21 divides a, what else divides a?

Exercise 8.9 Prove without long computation that the following numbers are not prime.

(a) 333333 (b) 4561234 (c) 2341110

Exercise 8.10 Find the numbers that are divisible by 3, but not by 9.

(a) 321

(b) 3456

(c) 234567

(d) 123456789

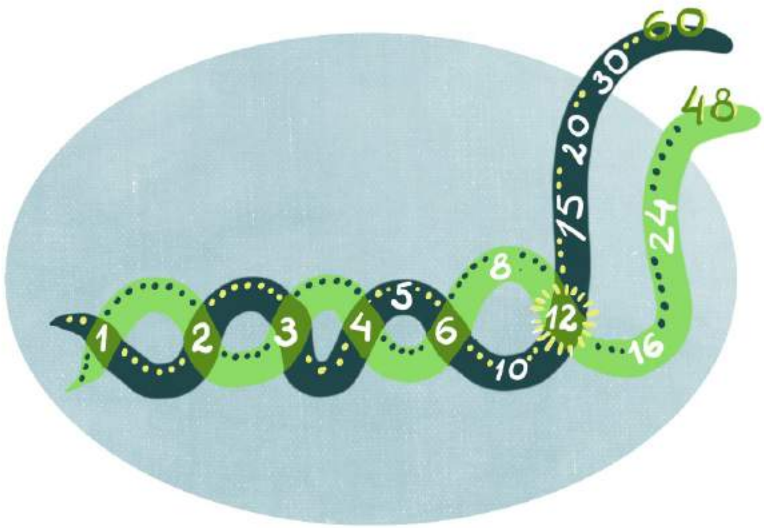
(e) 54321

(f) 456

Exercise 8.11 Find the numbers that are divisible by 4, but not by 8.

(a) 1740
(b) 32
(c) 240
(d) 3560
(e) 5112
(f) 34012

9. THE NUMBER OF DIVISORS, GCF(a, b), AND LCM(a, b)



The greatest common factor of 60 and 48 is 12.

9.1 The Number of Divisors

Question How many different numbers divide a given counting number?

Answer If a number is *small*, this question can be answered by *listing all divisors*. Let us look at examples.

Number	List of divisors	Number of divisors
5	$\{1, 5\}$	2
6	$\{1, 2, 3, 6\}$	4
8	$\{1, 2, 4, 8\}$	4
24	$\{1, 2, 3, 4, 6, 8, 12, 24\}$	8
37	$\{1, 37\}$	2
100	$\{1, 2, 4, 5, 10, 20, 25, 50, 100\}$	9
2400	???	???

The last two examples show the *disadvantages* of the listing method.

- The list may be quite long.
- It is easy to miss a divisor.
- It is not clear how to check the completeness of the list.

Luckily, there is a simple formula for the number of divisors of a given counting number. It does not require listing all the divisors. All that we need to know is the prime factorization of the counting number.

If the prime factorization of a number N is

$$N = p_1^{m_1} \cdot p_2^{m_2} \dots p_k^{m_k}$$

then the number of divisors of N is

$$(m_1 + 1)(m_2 + 1)\dots(m_k + 1)$$

The formula states that to find the number of divisors of N, we have to do the following steps.

Step 1. Find the prime factorization of N.

Step 2. Take the exponents of primes in the prime factorization. Add 1 to each exponent.

Step 3. Multiply out the resulting numbers. This gives the answer.

Example 9.1 How many divisors does 12 have?

Solution The prime factorization of 12 is $12 = 2^2 \times 3^1$. The exponents of primes in the prime factorization are 2 and 1. By the formula for the number of divisors, there are $(2+1)\times(1+1) = 3\times 2 = 6$ divisors of 12.

Example 9.2 How many divisors does the number 36 have?

Solution The prime factorization of 36 is $36 = 2^2 \times 3^2$. The exponents of primes in the prime factorization are 2 and 2. By the formula for the number of divisors, there are $(2+1)\times(2+1) = 9$ divisors of 36.

Example 9.3 We can re-establish the number of divisors of 100 found by the listing method. The prime factorization of 100 is $100 = 2^2\times 5^2$. By the formula for the number of divisors, there are $(2+1)\times(2+1) = 9$ divisors of 100.

Example 9.4 How many divisors does the number 18 have?

Solution The prime factorization $18 = 2^1 \times 3^2$ provides that there are $(1+1)\times(2+1) = 6$ divisors of 18.

Example 9.5 Find the number of divisors of the number 49.

Solution Since $49 = 7^2$, the number 49 has $2 + 1 = 3$ divisors.

Example 9.6 Let $N = 11^9 \times 2^5 \times 19^1$. How many divisors does the number N have?

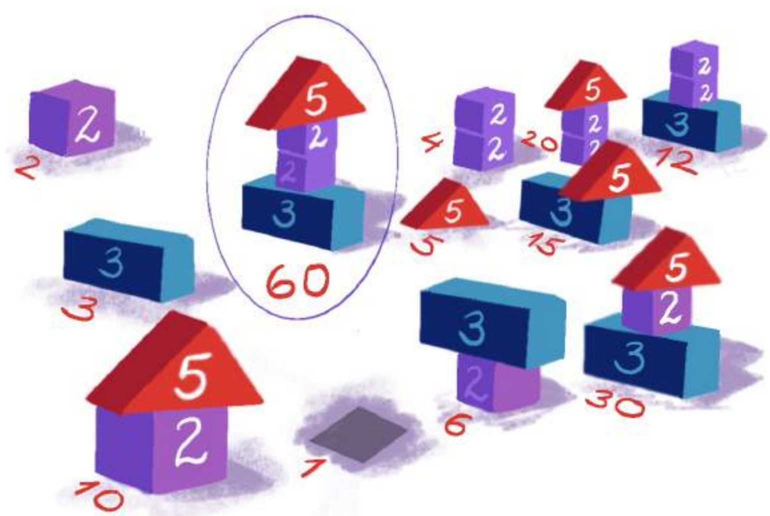
Solution The prime factorization of N is already provided and the exponents of the prime factors are 9, 5, and 1. By the formula for the number of divisors, there are $(9+1)\times(5+1)\times(1+1) = 120$ divisors of N.

Example 9.7 Let $N = 4^5$. How many divisors does the number N have?

Solution Note that 4 is *not prime*, so 4^5 is not the prime factorization. We first have to rewrite $N = 4^5 = (2^2)^5 = 2^{10}$. Hence there are $10 + 1 = 11$ divisors of N.

9.2 Explanation of Formula for the Number of Divisors

There is an unspoken rule in mathematics: The best way to remember a formula and learn its applications is to understand how the formula was deduced. In this section, we explain the formula for the number of divisors. In particular, we will see why we add 1 to each exponent in this formula.



All the divisors of 60.

The illustration shows all the divisors of the number 60. The prime factorization $60 = 3 \times 2^2 \times 5$ is interpreted as a tower built of blocks corresponding to the prime numbers 3, 2, 2, and 5. Any divisor of 60 can be thought of as a tower built of several blocks from the collection of two blocks 2, one block 3, and one block 5.

For example,

- The divisor $6 = 2^1 3^1 5^0$ is built of one block of type 2, one block of type 3, and no blocks of type 5.

- The divisor $20 = 2^2 3^0 5^1$ is built of two blocks of type 2, no blocks of type 3, and one block of type 5.

- The divisor $1 = 2^0 3^0 5^0$ is built of zero blocks of type 2, zero blocks of type 3, and zero blocks of type 5.

Suppose that we decided to build a divisor (tower). We have to choose how many blocks of each type to use. First, we decide how many blocks of type 2 will be used. We have *two blocks* of that type available, so we can use either both blocks (two), just one block (one), or no block at all (zero), that is, we have *three options*.

Next, we have *one block* of type 3. We can either use this block (one) or not (zero), that is, we have *two options*. The same is true for the block 5. We can use one or zero blocks of this type, which gives us *two options*.

Finally, we multiply (*three options* for blocks of type 2) × (*two options* for blocks of type 3) × (*two options* for blocks of type 5) $= 3 \times 2 \times 5 = 12$ ways to build a tower, which is the same as 12 ways to build a divisor of 60.

This argument can be generalized to explain the formula for the number of divisors. If we have m blocks of type p, we have $(p + 1)$ options to pick some of them, including the option of picking up no blocks of this type.

9.3 The Greatest Common Factor of Two Numbers

The greatest common factor and the least common multiple of two numbers describe relations between a and b, their divisors, and multiples. The commonly used abbreviations are GCF(a, b) for the greatest common factor and LCM(a, b) for the least common multiple of a and b.

We may not be even aware of that, but GCF(a, b) and LCM(a, b) are used in operations with fractions. In particular, GCF(a, b) and LCM(a, b) are used to simplify and perform the addition and multiplication of fractions in an efficient way. Operations with fractions will be discussed in more detail later.

We will review the notion of the *greatest common factor* of two numbers a and b and discuss the three ways to compute it.

1. The *listing method* based on the definition of GCF(a, b).

2. The *prime factorization method*.

3. The *Euclidean algorithm*.

After that we will review the notion of the *least common multiple* of two numbers a and b and discuss the two ways to compute it.

1. The *listing method* based on the definition of LCM(a, b).

2. The *prime factorization method*.

Let us start with GCF(a,b).

The *greatest common factor* of two counting numbers a and b is the largest counting number that divides both a and b. It is denoted by *GCF(a,b)*.

Remark Another commonly used name for the greatest common factor of two numbers is the *greatest common divisor* and the notation $gcd(a,b)$.

9.4 The Listing Method and Basic Properties of GCF(a,b)

Calculation of GCF(a,b) using the definition involves *listing divisors* of a and b and searching for the *largest common divisor* in these two lists. This method works well in *simple* cases.

Example 9.8 Find GCF$(10,6)$.

Solution The divisors of 10 are $\{1,2,5,10\}$, and the divisors of 6 are $\{1,2,3,6\}$. We see that 2 is the largest common number of these two sets.

Answer: GCF$(10,6) = 2$.

Example 9.9 Find GCF$(18,24)$ by using the definition of the greatest common factor of two numbers.

Solution The divisors of 18 are 1, 2, 3, 6, 9, 18, and the divisors of 24 are 1, 2, 3, 4, 6, 8, 12, 24. We see that 6 is the largest common element.

Answer: GCF$(18,24) = 6$.

Example 9.10 Find the greatest common factors.

(a) GCF$(3,6)$

(b) GCF$(100,100)$

(c) GCF$(2,15)$

(d) GCF$(6,60)$

(e) GCF$(9,12)$

(f) GCF$(13,26)$

(g) GCF$(8,27)$

(h) GCF$(125,3)$

(i) GCF$(1205,1)$

Solution

(a) GCF$(3,6) = 3$

(b) GCF$(100,100) = 100$

(c) GCF$(2,15) = 1$

(d) GCF$(6,60) = 6$

(e) GCF$(9,12) = 3$

(f) GCF$(13,26) = 13$

(g) $\text{GCF}(8,27)=1$

(h) $\text{GCF}(125,3)=1$

(i) $\text{GCF}(1205,1)=1$

These examples lead to important and very useful observations.

If $a|b$, then $\text{GCF}(a,b)=a$.

For example, $\text{GCF}(3,6)=3$, $\text{GCF}(12,60)=12$, and $\text{GCF}(13,26)=13$.

$\text{GCF}(a,a)=a$.

For example, $\text{GCF}(100,100)=100$.

$\text{GCF}(a,1)=1$ for any a.

For example, $\text{GCF}(123406,1)=1$.

Note that $\text{GCF}(2,15)=1$, $\text{GCF}(8,27)=1$, and $\text{GCF}(125,3)=1$ correspond to pairs of numbers a and b that have *no common divisors other than 1*. We already mentioned these pairs of counting numbers and a special name for them.

Two numbers a and b are *relatively prime* if and only if $\text{GCF}(a,b)=1$.

Example 9.11 Find $\text{GCF}(7,13)$, $\text{GCF}(12,12000)$, and $\text{GCF}(12345,1)$.

Solution 7 and 13 are prime numbers, so they are relatively prime. Hence $\text{GCF}(7,13)=1$. Further, $\text{GCF}(12,12000)=12$ since $12|12000$, and $\text{GCF}(12345,1)=1$.

9.5 The Prime Factorization Method for GCF(a,b)

In simple cases, we can find $\text{GCF}(a,b)$ by the definition, basic properties, or by listing divisors of a and b. However, it may not be an efficient way for large numbers. For example, how do we find $\text{GCF}(144,268)$ or $\text{GCF}(756,210)$? The lists of divisors may be quite long; it is easy to make a mistake and miss some factors. For this reason it is important to know, in addition to the definition, other methods for finding $\text{GCF}(a,b)$.

In this section, we discuss the *prime factorization method*. The idea is to build $\text{GCF}(a,b)$ from prime factors that are common for a and b.

Example 9.12 Find GCF$(48, 360)$.

Solution Using factorization trees, we can find the prime factorizations of the numbers 48 and 360:

$$48 = 2^4 \times 3^1, \quad 360 = 2^3 \times 3^2 \times 5^1$$

As any common divisor, GCF$(48, 360)$ is built from blocks, the prime factors that appear in the prime factorization of *both* numbers 48 and 360. We want to use the maximal possible number of common prime factors if we want to build the *largest* divisor. This means that we will use 2^3 and 3^1 since they enter both decompositions. We will not use 5 to construct GCF$(48, 360)$ since 5 does not enter the decomposition of 48. Thus,

$$\mathrm{GCF}(48, 360) = 2^3 \times 3 = 24$$

Example 9.13 Find GCF$(72, 264)$.

Solution Using factorization trees, we can find the prime factorization of the numbers 72 and 264:

$$72 = 2^3 \times 3^2, \quad 264 = 2^3 \times 3 \times 11$$

The common part of these two factorizations contains 2^3 and 3^1. Hence

$$\mathrm{GCF}(72, 264) = 2^3 \times 3 = 24$$

Example 9.14 Find GCF$(756, 210)$.

Solution Using factorization trees, we can find the prime factorization of the numbers 756 and 210:

$$756 = 2^2 \times 3^3 \times 7, \quad 210 = 2 \times 3 \times 5 \times 7$$

The common part of these two prime factorizations is as follows:

$$\mathrm{GCF}(756, 210) = 2^1 \times 3^1 \times 7^1 = 42$$

Example 9.15 Find the prime factorization of GCF(a, b) if

$$a = 2^{10} \times 3^{500} \times 5^7 \times 11^{203} \times 13^6$$

$$b = 2^2 \times 3^2 \times 5^8 \times 11^{204} \times 17^{10}$$

Solution We construct GCF(a, b) as the common part of the prime factorizations of a and b:

$$\mathrm{GCF}(a, b) = 2^2 \times 3^2 \times 5^7 \times 11^{203}$$

More precisely, using that $17^0 = 13^0 = 1$, we write a and b in the uniform way

$$a = 2^{10} \times 3^{500} \times 5^{7} \times 11^{203} \times 13^{6} \times 17^{0}$$

$$b = 2^{2} \times 3^{2} \times 5^{8} \times 11^{204} \times 13^{0} \times 17^{10}$$

and compare the exponents of prime factors for a and b. For each prime factor we choose the *smaller exponent* (highlighted in blue).

Remark In Example 9.15, the numbers, including the answer GCF(a,b), are too large to be calculated explicitly. We make the following convention: When the question of a problem is "find GCF(a,b)" or "find the explicit value of GCF(a,b)," it is expected that the answer is an explicit number. When the question of a problem is "find the prime factorization of GCF(a,b)," it suffices to write the answer as a product of prime factors since, probably, it is too large to be calculated explicitly.

Example 9.16 Find GCF(a,b) if

$$a = 2^7 \times 3^{15} \times 7^2 \times 11$$
$$b = 2 \times 5^{10} \times 7 \times 13$$

Solution We pick up a smaller exponent for each common prime factor in the decompositions of a and b to get

$$\mathrm{GCF}(a,b) = 2^1 \times 3^0 \times 5^0 \times 7^1 \times 11^0 \times 13^0 = 2 \times 7 = 14$$

Example 9.17 Find GCF$(15,100)$ by using the prime factorization.

Solution Since $15 = 3 \times 5$ and $100 = 2^2 \times 5^2$, we get GCF$(15,100) = 2^0 \times 3^0 \times 5^1 = 5$.

9.6 The Euclidean Algorithm for GCF(a,b)

Another powerful method for calculating GCF(a,b) uses *division with remainder*. The idea is to replace a calculation of GCF(a,b) with large numbers a and b with a calculation of GCF(b,r) with a "smaller" pair b, r. A sequence of such replacements reduces calculations to a simple case that is easy to compute. This method of subsequent substitutions is called the *Euclidean algorithm*. We first consider some examples illustrating the results produced at each step of the algorithm. After that we explain how the results are obtained.

Example 9.18 We will see shortly that the Euclidean algorithm provides the following sequence of equalities:

$$\mathrm{GCF}(1827,36) = \mathrm{GCF}(36,27) = \mathrm{GCF}(27,9) = 9$$

which implies GCF$(1827,36) = 9$. In the sequence of equalities, within each subsequent GCF(a,b) are smaller and simpler numbers, until we get the very simple case

GCF$(27, 9) = 9$.

Example 9.19 We will apply the Euclidean algorithm to compute

$$\text{GCF}(97, 89) = \text{GCF}(89, 8) = \text{GCF}(8, 1) = 1$$

and conclude that GCF$(97, 89) = 1$.

The Euclidean algorithm is a straightforward method for finding GCF(a, b) based on division with remainder. We state the general formula and illustrate its applications by examples.

Euclidean algorithm. Let $a > b$ be counting numbers. Then

$$GCF(a, b) = GCF(b, r)$$

where r is the remainder of the division of a by b

$$a = bq + r, \quad r < b < a$$

Remark Since $a > b$ and $b > r$, the new pair (b, r) is "smaller" than the original one (a, b).

Example 9.20 Find GCF$(1827, 36)$ by using the Euclidean algorithm.

Solution We divide 1827 by 36 with remainder

$$1827 = 50 \times 36 + 27$$

You can use a calculator for this calculation if you do not feel confident enough to obtain an answer by long division. Then GCF$(1827, 36)$= GCF$(36, 27)$, and we replace $(1827, 36)$ with $(36, 27)$. Next, we perform division with remainder for the new pair

$$36 = 1 \times 27 + 9$$

We write GCF$(36, 27)$= GCF$(27, 9)$. At this point, we either note that GCF$(27, 9) = 9$, and this is the answer, or continue the division until we get zero as a remainder, which means the end of the algorithm. The boxed divisor is the final answer:

$$27 = 3 \times \boxed{9} + 0 (\leftarrow STOP),$$
$$\text{GCF}(1827, 36) = 9$$

Here is the summary of all the steps of calculation. We suggest to organize solutions of other problems in this form.

	GCF$(1827, 36)$
$1827 = 50 \times 36 + 27$	=GCF$(36, 27)$,
$36 = 1 \times 27 + 9$	=GCF$(27, 9)$,
$27 = 3 \times \boxed{9} + 0 (\leftarrow STOP)$	=9

Answer: GCF $(1827, 36) = 9$.

Example 9.21 Find GCF$(97, 89)$.

Solution

	GCF$(97,89)$
$97 = 1 \times 89 + 8$	=GCF$(89,8)$,
$89 = 11 \times 8 + 1$	=GCF$(8,1)$,
$8 = 8 \times \boxed{1} + 0 (\leftarrow STOP)$	=1

Answer: GCF $(97, 89) = 1$.

Example 9.22 Find GCF$(246, 30)$.

Solution

	GCF$(246,30)$
$246 = 8 \times 30 + 6$	=GCF$(30,6)$,
$30 = 5 \times \boxed{6} + 0 (\leftarrow STOP)$	=6

Answer: GCF $(246, 30) = 6$.

Example 9.23 Find GCF$(546, 390)$.

Solution

	GCF$(546,390)$
$546 = 1 \times 390 + 156$	=GCF$(390,156)$,
$390 = 2 \times 156 + 78$	=GCF$(156,78)$,
$156 = 2 \times \boxed{78} + 0 (\leftarrow STOP)$	=78

Answer: GCF $(546, 390) = 78$.

Example 9.24 Find GCF$(271, 101)$.

Solution

	GCF$(271,101)$
$271 = 2 \times 101 + 69$	=GCF$(101,69)$
$101 = 1 \times 69 + 32$	=GCF$(69,32)$
$69 = 2 \times 32 + 5$	=GCF$(32,5)$
$32 = 6 \times 5 + 2$	=GCF$(5,2)$
$5 = 2 \times 2 + 1$	=GCF$(2,1)$
$2 = 2 \times \boxed{1} + 0 (\leftarrow STOP)$	=1

Answer: GCF $(271, 101) = 1$.

In this example, we went through all the steps of the Euclidean algorithm until we got zero as a remainder. Certainly, if the value of GCF(a,b) is obvious at an intermediate step (for example, it is clear that GCF$(32,5) = 1$), one can stop there and use this value as an answer. In such cases, there is no need to proceed further until the zero remainder.

9.7 The Least Common Multiple of Two Numbers

The notion of the least common multiple of two numbers is similar to the notion of the greatest common factor. The similarity of the notions often leads to confusion.

The *least common multiple* of two counting numbers a and b is the smallest number that is a multiple of both a and b. It is denoted by $\mathrm{LCM}(a,b)$.

Example 9.25 From the definition it is clear that $\mathrm{LCM}(3,6) = 6$, $\mathrm{LCM}(6,9) = 18$, and $\mathrm{LCM}(10,15) = 30$.

Let us compare the notions of $\mathrm{GCF}(a,b)$ and $\mathrm{LCM}(a,b)$ for $a = 24$ and $b = 36$. For this purpose we make the lists of *all divisors* and *all multiples* of each number.

List 1(a) Divisors of $a = 24$: $\{1,2,3,4,6,8,12,24\}$. This set is finite.

List 2(a) Multiples of $a = 24$: $\{24, 48, 72, 96, \ldots\}$. This set is infinite.

List 1(b) Divisors of $b = 36$: $\{1,2,3,4,6,9,12,18,36\}$. This set is finite.

List 2(b) Multiples of $b = 36$: $\{36, 72, 108, 144, \ldots\}$. This set is infinite.

Recall that List 1(a) and List 1(b) are used to find $\mathrm{GCF}(24,36)$. By definition, $\mathrm{GCF}(24,36)$ is the *largest* common number in List 1(a) and List 1(b), that is, 12.

Accordingly, List 2(a) and List 2(b) are used to find $\mathrm{LCM}(24,36)$. By definition, $\mathrm{LCM}(24,36)$. is the *smallest* common number in List 2(a) and List 2(b), that is, 72. Also note that the names of $\mathrm{GCF}(a,b)$ and $\mathrm{LCM}(a,b)$ prompt us to take the greatest element or the least element and infinite sets of multiples *have no "greatest" elements*.

9.8 The Listing Method and Basic Properties of LCM(a,b)

Example 9.26 Find $\mathrm{LCM}(15,25)$.

Solution We list the first few multiples of each number and mark the first common element of both sets.

Multiples of 15: $\{15, 30, 45, 60, \boxed{75}, 90, 105, 120, 135, 150, \ldots\}$

Multiples of 25: $\{25, 50, \boxed{75}, 100, 125, 150, \ldots\}$

Answer: $\mathrm{LCM}(15,25) = 75$.

Remark 150 is also a common multiple, but it is *not* the smallest common multiple.

Example 9.27 Find $\mathrm{LCM}(12,20)$.

Solution We list multiples of 12 and 20.

Multiples of 12: 12, 24, 36, 48, $\boxed{60}$, 72, ...

Multiples of 20: 20, 40, $\boxed{60}$, 80, ...

Answer: LCM$(12, 20) = 60$.

Let us look at the basic properties of the least common multiple of two numbers.

Example 9.28 It is easy to see that

$$\text{LCM}(3,6) = 6, \text{LCM}(25,100) = 100, \text{and LCM}(4,8) = 8.$$

If b is a multiple of a, then LCM$(a, b) = b$.

Example 9.29 It is clear that LCM$(1, 15) = 15$.

LCM$(1, a) = a$.

Example 9.30 Using the listing method, we find that

$$\text{LCM}(3,5) = 15, \text{LCM}(4,7) = 28, \text{and LCM}(4,9) = 36.$$

Note that $3 \times 5 = 15$, $4 \times 7 = 28$, and $4 \times 9 = 36$.
The following statement generalizes the last example.

If a and b are *relatively prime*, then LCM$(a, b) = a \times b$.

Remark It can be proved that the other way is also true: if a and b are not relatively prime, then LCM$(a, b) \neq a \times b$. We will see examples in the next section.

9.9 The Prime Factorization Method for LCM(a, b)

While no reasonable analogue of the Euclidean algorithm exists for LCM(a, b), we do have a prime factorization method similar to that for GCF(a, b).

Example 9.31 Find LCM$(12, 39)$.

Solution The numbers 12 and 39 are *not relatively prime*, so the product 12×39 cannot be the answer. The listing method does not seem very efficient here.

Multiples of 12: 12, 24, 36, 48, 60, 72, 84, 96, ...

Multiples of 39: 39, 78, 117, 156, ...

We have a rather long list of multiplies, but we have not yet reached a common multiple!

Luckily for us, there exists another way to find LCM(12, 39). Using the *prime factorizations* of the numbers 12 and 39, we find

$$12 = 2^2 \times 3, \quad 39 = 3 \times 13$$

A common multiple should contain both numbers 12 and 39, so it should be constructed from the prime factors of 12 and 39. These factors should be used in sufficient, but least necessary quantities. Hence we construct LCM(12, 39) as the product of several copies of 2, 3, and 13 taking the *largest available exponent* (highlighted in blue) for each prime factor

$$12 = 2^2 \times 3^1 \times 13^0$$
$$39 = 2^0 \times 3^1 \times 13^1$$

Answer: $\text{LCM}(12, 39) = 2^2 \times 3^1 \times 13^1 = 156$.

Example 9.32 Find LCM(36, 24).

Solution Using the prime factorizations

$$36 = 2^2 \times 3^2$$
$$24 = 2^3 \times 3^1$$

we find $\text{LCM}(36, 24) = 2^3 \times 3^2 = 8 \times 9 = 72$.

Example 9.33 Find LCM(15, 40).

Solution Using the prime factorizations

$$15 = 3 \times 5 = 2^0 \times 3^1 \times 5^1$$
$$40 = 2^3 \times 5 = 2^3 \times 3^0 \times 5^1$$

we find $\text{LCM}(15, 40) = 2^3 \times 3^1 \times 5^1 = 120$.

Example 9.34 Find the prime factorization of LCM(a, b) if

$$a = 3^2 \times 7^5 \times 11^{10} \times 13^2 \times 19^{100}$$
$$b = 2^3 \times 3^4 \times 7^2 \times 11^{12} \times 13$$

Solution We construct LCM(a, b) from prime factors, using for each of them the largest exponent of two available:

$$\text{LCM}(a, b) = 2^3 \times 3^4 \times 7^5 \times 11^{12} \times 13^2 \times 19^{100}$$

(The number is too large to be calculated explicitly.)

9.10 Summary

We can construct GCF(a,b) and LCM(a,b) based on the *prime factorizations* of a and b. For GCF(a,b), we take the *smaller* exponent for each factor that appears in the prime factorizations of of a and b. For LCM(a,b), we take the *larger* exponent for each factor in these two prime factorizations. GCF(a,b) can be calculated from the definition (by the listing method), the prime factorization, or the Euclidean algorithm. For LCM(a,b), we use the listing method that follows from the definition and the prime factorization method.

9.11 Product of GCF(a,b) and LCM(a,b)

We conclude this chapter with a nice fact concerning GCF(a,b) and LCM(a,b).

Example 9.35 Let $a = 8$ and $b = 6$. Find

(a) $a \times b$

(b) GCF(a,b)×LCM(a,b)

Solution

(a) $a \times b = 8 \times 6 = 48$

(b) GCF(a,b)×LCM(a,b) $= 2 \times 24 = 48$.

We got the same answer to both questions, and this is no coincidence.

GCF(a,b)×LCM(a,b) $= a \times b$ for any counting numbers a and b.

This statement is justified by the prime factorization method. We illustrate the idea of the proof by the example.

Example 9.36 Let

$$a = 2^5 \times 3^2 \times 5$$
$$b = 2^3 \times 3^4 \times 7^2$$

Show that GCF(a,b)×LCM(a,b) $= a \times b$.

Solution We have

$$a \times b = 2^5 \times 3^2 \times 5 \times 2^3 \times 3^4 \times 7^2 = 2^8 \times 3^6 \times 5 \times 7^2$$

To construct LCM(a,b), we pick up the larger exponents (highlighted in blue):

$$a = 2^5 \times 3^2 \times 5^1$$
$$b = 2^3 \times 3^4 \times 7^2$$

Then

$$\text{LCM}(a,b) = 2^5 \times 3^4 \times 5^1 \times 7^2$$

The exponents that have not been picked up are exactly the "smaller" ones. They are used in GCF(a,b):

$$\text{GCF}(a,b) = 2^3 \times 3^2$$

Then it is clear that

$$\text{LCM}(a,b) \times \text{GCF}(a,b) = 2^5 \times 3^4 \times 5^1 \times 7^2 \times 2^3 \times 3^2 = 2^8 \times 3^6 \times 5 \times 7^2 = a \times b$$

9.12 Exercises

9.12.1 The number of divisors

Exercise 9.1 Find the prime factorizations. List all the prime divisors. Find the number of different divisors.

(a) 630 (b) 144 (c) 28

Exercise 9.2 Find the number of different divisors.

(a) $2^5 \cdot 5^7 \cdot 13^9$ (b) $11 \cdot 3^{14} \cdot 2^{19}$

9.12.2 Properties of GCF(a,b)

Exercise 9.3 Explain the meaning of GCF(a,b).

Exercise 9.4 Find the greatest common factors using the definition or basic properties of GCF(a,b).

(a) GCF(3,9)
(b) GCF(2,10)
(c) GCF(2,15)
(d) GCF(200,200)
(e) GCF(13,17)
(f) GCF(6,60)
(g) GCF(10,15)
(h) GCF(1,13)
(i) GCF(7,49)
(j) GCF(6,12)
(k) GCF(4,6)
(l) GCF(2,13)
(m) GCF(2,4)
(n) GCF(5,12)
(o) GCF(9,12)
(p) GCF(3,6)
(q) GCF(7,70)
(r) GCF(1,213)
(s) GCF(2,3)
(t) GCF(7,11)
(u) GCF(4,7)
(v) GCF(7,25)
(w) GCF(8,27)
(x) GCF(12,18)
(y) GCF(10,30)

Exercise 9.5 Find the greatest common factors using the Euclidean algorithm.

(a) GCF$(25,3)$
(b) GCF$(7,29)$
(c) GCF$(999,30)$
(d) GCF$(13,49)$
(e) GCF$(27,18)$
(f) GCF$(125,15)$
(g) GCF$(27,25)$
(h) GCF$(60,45)$
(i) GCF$(100,15)$
(j) GCF$(24,54)$
(k) GCF$(39,91)$
(l) GCF$(72,160)$

Exercise 9.6 Find the exact values of the greatest common factors using the prime factorization method.

(a) GCF$(8,18)$
(b) GCF$(24,66)$
(c) GCF$(30,45)$
(d) GCF$(12,6)$
(e) GCF$(12,81)$
(f) GCF$(45,120)$

Exercise 9.7 Find the exact values of the greatest common factors GCF(a,b) for $a = 2^5 \cdot 5^7 \cdot 13^9$ and $b = 11 \cdot 3^{14} \cdot 2^{19}$.

Exercise 9.8 Find the greatest common factors using any method.

(a) GCF$(51,85)$
(b) GCF$(385,42)$
(c) GCF$(117,195)$

9.12.3 Properties of LCM(a,b)

Exercise 9.9 Explain the meaning of LCM(a,b).

Exercise 9.10 Find the least common multiples using the definition or basic properties of LCM(a,b).

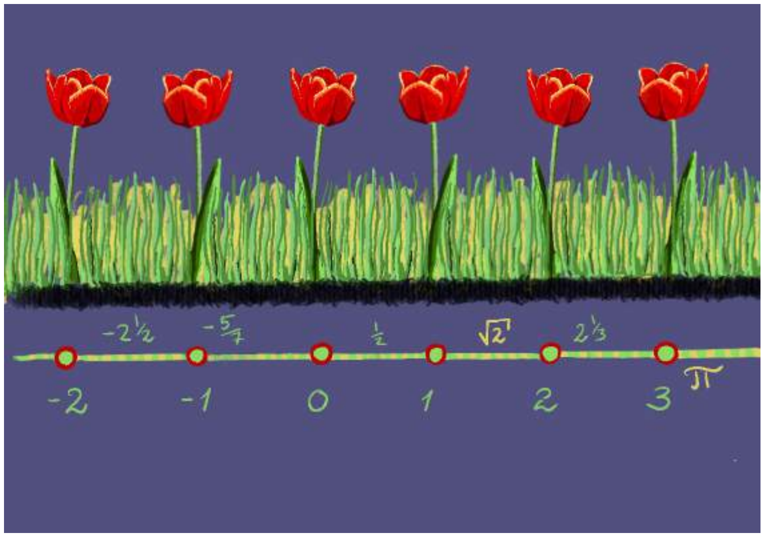
(a) LCM$(4,8)$
(b) LCM$(3,6)$
(c) LCM$(2,5)$
(d) LCM$(9,6)$
(e) LCM$(4,6)$
(f) LCM$(1,13)$
(g) LCM$(5,7)$
(h) LCM$(2,139)$

Exercise 9.11 Find the exact values of the following least common multiples using the prime factorization method.

(a) LCM$(99,2100)$
(b) LCM$(30,45)$
(c) LCM$(12,39)$
(d) LCM$(12,81)$
(e) LCM$(45,120)$

Exercise 9.12 Find the prime factorization of the least common multiple of the numbers $a = 2^5 \cdot 5^7 \cdot 13^9$ and $b = 11 \cdot 3^{14} \cdot 2^{19}$.

10. WORLD OF REAL NUMBERS



Integer, rational, and irrational numbers.

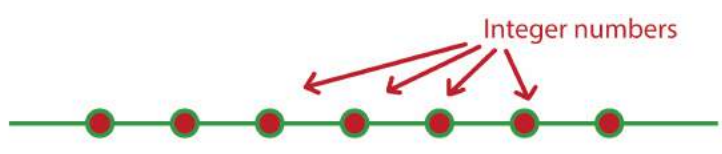
10.1 Introduction

In this chapter, we are going to expand our world of numbers. Until now, most of our discussions were related to the set of *whole numbers* $\{0,1,2,3,\dots\}$. On some occasions, we talked about *negative numbers* (such as -7, -1) or *fractions* (such as $1/3$, $2/5$). In this chapter, we review numbers of different types and discuss how they belong to the world of *real numbers*.

10.2 Counting, Whole, Negative, and Integer Numbers

Recall that we started with the set of *counting numbers* $\{1,2,3,4,\dots\}$. We observed that this set is *closed under the operations of addition and multiplication*. If we include *zero* in the list, we get the set $\{0,1,2,3,\dots\}$ of *whole numbers* that has the same closure property.

We also discussed that the subtraction operation takes us out of the world of whole numbers. For example, the result of the subtraction $3-5$ is not in the set of whole numbers. This motivates us to introduce *negative numbers* $\{-1,-2,-3,\dots\}$. Together with whole numbers, they create the set of *integers* $\{\cdots-3,-2,-1,0,1,2,3,\dots\}$. The integers are often represented by a discrete set of points on a line.



10.3 Rational Numbers

Note that the set of integers is *closed under the operations of addition, multiplication, and subtraction*. Yet, we know that it is *not closed under division*, for example, $1 \div 3$ is not integer. This leads to the further expansion of our number world to the set of *rational numbers*

$$\frac{1}{5}, \quad \frac{7}{20}, \quad \frac{-17}{55}, \quad \frac{8}{3}, \quad \ldots$$

> A *rational number* is a pair of two integers a and b with $b \neq 0$ written in the form $\frac{a}{b}$.

We know how to add, subtract, multiply, and divide rational numbers. We will review the arithmetic operations with fractions in Chapter 11. For the moment, we just take a note that the set of rational numbers is *closed under the operations of addition, subtraction, multiplication, and division by nonzero rational numbers*.

The same rational number can be represented by many different fractions.

10.4 Simplest Form of Fractions

The tricky part about rational numbers is that the same rational number can be represented by different pairs of integers. For example,

$$\frac{1}{3} = \frac{2}{6} = \frac{10}{30} = \frac{5}{15} = \ldots$$

Usually, we work with the *simplest form of a fraction*.

> Let a and b be two counting numbers. We say that the fraction $\frac{a}{b}$ is in *simplest form* if a and b are relatively prime.

Remark

1. If we allow a or b to be a negative integer, not just a counting number, we have to adjust slightly the notion of relatively prime numbers since we introduced it only for counting numbers. It is done in the simplest way. We say that two integers are *relatively prime* if their absolute values are relatively prime. Then a negative rational number a/b is in the simplest form if $a < 0$, $b > 0$, and a and b are relatively prime.

2. In some textbooks, fractions in simplest form are called *fractions in lowest terms*, *fractions in reduced form* or *reduced fractions*.

Example 10.1 The fractions $\frac{1}{3}$, $\frac{2}{5}$, and $\frac{4}{9}$ are in simplest forms. The fractions $\frac{3}{15}$, $\frac{4}{18}$, and $-\frac{2}{6}$ are not in simplest form.

Example 10.2 Is the fraction $\frac{42}{48}$ in simplest form?

Solution No, since 42 and 48 are not relatively prime. To put a fraction $\frac{a}{b}$ in simplest form, we have to divide the numerator and the denominator of the fraction by their greatest common factor GCF(a, b)

$$42 = 2 \times 3 \times 7, \quad 48 = 2^4 \times 3, \quad \text{GCF}(42, 48) = 2 \times 3 = 6$$
$$\frac{42}{48} = \frac{7 \times 6}{8 \times 6} = \frac{7}{8}$$

Since 7 and 8 are relatively prime, $\frac{7}{8}$ is the simplest form of $\frac{42}{48}$.

The simplest form can be used to establish equality of two fractions.

Two fractions $\frac{a}{b}$ and $\frac{c}{d}$ are *equal* if and only if they have the same simplest form.

Recall another way to check whether two fractions represent the same rational number.

Two fractions $\frac{a}{b}$ and $\frac{c}{d}$ are *equal* if and only if $ad = bc$.

10.5 Inclusions of Sets of Numbers

Any integer is a rational number because an integer a can be written in the form $\frac{a}{1}$:

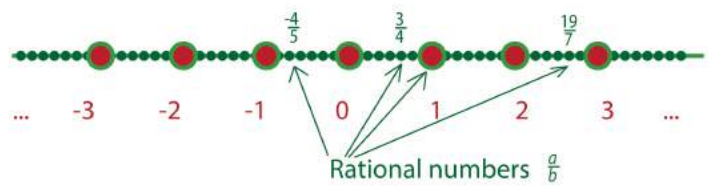
$$3 = \frac{3}{1}, \quad -5 = \frac{-5}{1} = \frac{-10}{2}, \quad \ldots$$

Using the symbol $\subset$ for inclusion of sets, we can write the following sequence of inclusions of number sets.

counting numbers $\subset$ whole numbers $\subset$ integers $\subset$ rational numbers

10.6 Rational Numbers on Line

If we mark all rational numbers on the same line, where we already placed integers, the rational numbers present an infinite set of points spread everywhere on the line.



The word *everywhere* literally means that, in every tiny segment of the line, there are (infinitely many) rational numbers.

Between any two rational numbers, one can find *another* rational number.

For example, the average of two rational numbers is again a rational number and stands exactly in the middle between them.

Example 10.3 Between $\frac{1}{4}$ and $\frac{3}{4}$ there is the rational number $\frac{1}{2}$

$$\frac{1}{4} < \frac{1}{2} < \frac{3}{4}$$

This follows from the fact that $\frac{1}{2}$ is the average of these two numbers.

$$\frac{1}{2} \times \left(\frac{1}{4} + \frac{3}{4}\right) = \frac{1}{2} \times \frac{4}{4} = \frac{1}{2} \times 1 = \frac{1}{2}$$

Between the fractions $\frac{3}{8}$ and $\frac{3}{4}$, we can find their average, the rational number $\frac{9}{16}$.

$$\frac{1}{2} \times \left(\frac{3}{8} + \frac{3}{4}\right) = \frac{1}{2} \times \left(\frac{3}{8} + \frac{6}{8}\right) = \frac{1}{2} \times \frac{9}{8} = \frac{9}{16}$$

$$\frac{3}{8} < \frac{9}{16} < \frac{3}{4}$$

Note that we can continue inserting more and more rational numbers between the new averages. This shows us:

Between any two rational numbers, one can find *infinitely many* rational numbers.

In other words, *rational numbers are spread densely everywhere* on the number line.

10.7 Irrational Numbers

Since rational numbers are spread everywhere on the line, it may appear that every point of the line corresponds to some rational number. People thought in this way for a while. They thought that rational numbers were the only numbers. Eventually, mathematicians of Ancient Greece realized that there are numbers that cannot be rational. For example, there exists a short and elegant proof that $\sqrt{2}$ is not rational, it cannot be written as $\frac{a}{b}$ with integers a and b. (Note that we do encounter $\sqrt{2}$ in real life, for example, as the length of the diagonal of a unit square, so it is a very concrete and useful number.)

Thus, to some extent contrary to our intuition, rational numbers are dense everywhere, but there are gaps between them. There are points on the line that *do not correspond to any rational numbers*. These points are called *irrational numbers*. A careful and consistent introduction to all kinds of numbers is a topic of very advanced math courses. In our course, we just take an irrational number to be a number that is not rational.

An irrational number *cannot be represented as* $\frac{a}{b}$ with integers a and b, $b \neq 0$.

Example 10.4 From previous experience we know examples of irrational numbers

$$\sqrt{2}, \quad \pi, \quad \sqrt{3}, \quad \sqrt[3]{25}$$

Remark The presence of the radical symbol $\sqrt{}$ in an expression does not necessarily mean that the expression is an irrational number. Sometimes, it can be simplified and turns out to be a rational number.

Example 10.5 The number $\sqrt{4}$ is rational despite the symbol $\sqrt{}$.

$$\sqrt{4} = \sqrt{2^2} = 2$$

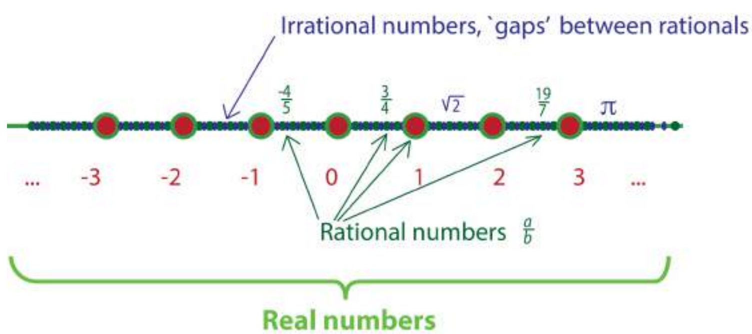
It is an integer (we already mentioned that integers are rational numbers).

10.8 Real Numbers

In summary, we have numbers of two types on the number line.

- Numbers that can be written in the form $\frac{a}{b}$ with integers a and $b \neq 0$. We call them *rational numbers*.
- Numbers that cannot be written in the form $\frac{a}{b}$ with integers a and $b \neq 0$. We call them *irrational numbers*.

All together, they create the set of all numbers that correspond to all points on the line, called the set of *real numbers*.



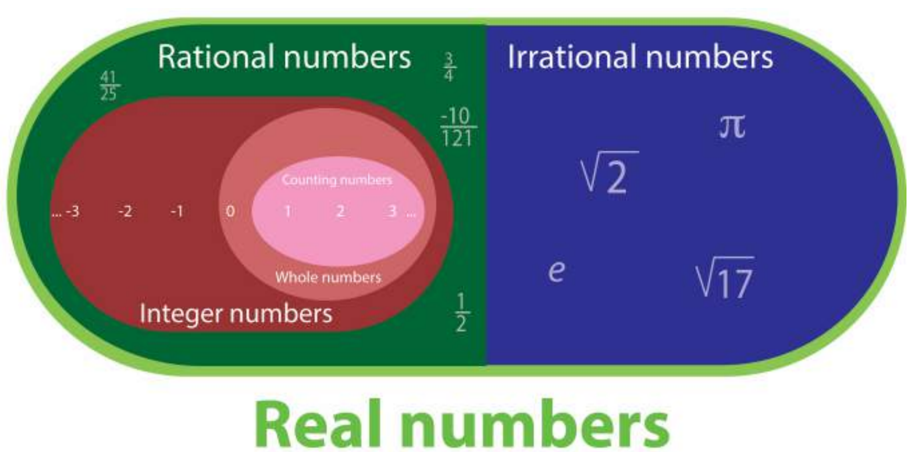
The inclusions of the above-mentioned sets look like

$$\boxed{\text{counting numbers} \subset \text{whole numbers} \subset \text{integers} \subset \text{rational numbers} \subset \text{real numbers}}$$

and, separately,

$$\boxed{\text{irrational numbers} \subset \text{real numbers}}$$

Another way to represent these relations is to use the Venn diagram



We note that the set of real numbers is *closed under addition, subtraction, multiplication, and division by nonzero numbers*.

Example 10.6 Which of the following numbers are whole, integer, rational, or irrational?

(a) 5
(b) 0
(c) -7
(d) $\frac{1}{2}$
(e) 0.5
(f) 12
(g) $\sqrt{2}$
(h) 1.2
(i) $-7\sqrt{3}$
(j) $\frac{6}{2}$
(k) π
(l) $\sqrt{9}$
(m) $-\sqrt{4}$

Solution Note that $\frac{6}{2} = 3$, $\sqrt{9} = 3$, and $-\sqrt{4} = -2$. Then

- whole numbers: $5, \quad 0, \quad 12, \quad \frac{6}{2} = 3, \quad \sqrt{9} = 3$
- integer numbers: all of the above and $-7, \quad -\sqrt{4}$
- rational numbers: all of the above and $\frac{1}{2}, \quad 0.5 = \frac{1}{2}, \quad 1.2 = \frac{12}{10}$
- irrational numbers: only $-7\sqrt{3}, \quad \pi, \quad \sqrt{2}$

10.9 Decimals

It turns out that the distinction between rational and irrational numbers is related to the *decimal representation of real numbers*.

Any real number can be written in decimal form. In particular, any rational number can be written in two ways: as a fraction and as a decimal.

Example 10.7 Here are examples of the decimal representation of rational numbers:

$$\frac{1}{10} = 0.1, \qquad \frac{1}{4} = 0.25$$

We often use decimals (such as 7.35, −567.145, 0.25, 800.01) in our daily life, especially since standard calculators work with the decimal representation of numbers. Let us take a closer look at the meaning of decimals.

Question What does the expression 132.54 mean?

Answer The answer becomes immediately clear when we recall our discussion on the decimal system in Chapter 7. Recall that 132 encodes the expanded form of the number 132 in powers of 10.

$$132 = 1 \times 10^2 + 3 \times 10^1 + 2 \times 10^0$$

Similarly, the fractional part of 132 encodes the expanded form of this part in *negative powers* of 10.

$$\begin{aligned} 132.54 &= 1 \times 100 + 3 \times 10 + 2 \times 1 + 5 \times \frac{1}{10} + 4 \times \frac{1}{100} \\ &= 1 \times 10^2 + 3 \times 10^1 + 2 \times 10^0 + 5 \times 10^{-1} + 4 \times 10^{-2} \end{aligned}$$

In summary, the decimal representation of a real number encodes the expanded form of the number in powers of 10 (including negative powers, if necessary).

$$\ldots 1000, \quad 100, \quad 10, \quad 1, \quad \frac{1}{10}, \quad \frac{1}{100}, \quad \frac{1}{1000}, \quad \ldots$$

At the end of this chapter, we discuss advantages and disadvantages of the decimal representation of rational numbers over the representation in the form of a fraction.

10.10 Conversion from Fractions to Decimals

Example 10.8 What decimals correspond to the fractions $\frac{1}{2}$, $\frac{1}{5}$, and $\frac{3}{20}$?

Solution $\frac{1}{2} = 0.5$, $\frac{1}{5} = 0.2$, and $\frac{3}{20} = 0.15$.

This example is not difficult, but we can ask the more general question.

Question How does one convert a fraction to decimal form?

Answer In practice, we use a calculator (probably, it is the most common computation performed on calculators). Without a calculator, this would be a result of long division.

Example 10.9 What is the decimal form of $\frac{1}{8}$?

Solution Using long division, we find

$$\begin{array}{r} 0.125 \\ 8\,\overline{)\,1.000} \\ \underline{8} \\ 20 \\ \underline{16} \\ 40 \\ \underline{40} \\ 0 \end{array}$$

Answer: $\dfrac{1}{8} = 0.125$.

10.11 Nonterminating Decimal Representation

Example 10.10 What is the decimal form of $\dfrac{1}{3}$?

Solution Using long division, we find

$$
\begin{array}{r|llllll}
 & 0. & 3 & 3 & 3 & 3 & \cdots \\ \hline
3 & 1. & 0 & 0 & 0 & 0 & \cdots \\
 & 0. & 9 & & & & \\ \cline{2-7}
 & & 1 & 0 & & & \\
 & & 0 & 9 & & & \\ \cline{3-7}
 & & & 1 & 0 & & \\
 & & & 0 & 9 & & \\ \cline{4-7}
 & & & & 1 & 0 & \\
 & & & & 0 & 9 & \\ \cline{5-7}
 & & & & & 1 & \\
 & & & & & & \ddots
\end{array}
$$

We see that $\dfrac{1}{3}$ has the *nonterminating decimal representation*

$$\frac{1}{3} = 0.33333\ldots$$

The long division process shows that division gets into a loop and the true value of the decimal representation of $\dfrac{1}{3}$ consists of infinitely many digits 3 placed after the decimal point. Note that no matter how many digits for the division 1/3 are displayed on a calculator, there still be only finitely many of digits and the result on a calculator is an *approximation* of the real value of $\dfrac{1}{3}$:

$$\frac{1}{3} \neq \underbrace{0.33333\ldots3}_{\text{if finitely many digits}}$$

There is a convention to write infinite decimals by using the bar over the repeated periodic pattern.

$$\frac{1}{3} = 0.\overline{3} = \underbrace{0.33333\ldots}_{\text{infinitely many digits}}$$

Example 10.11 Using a calculator, we can find

$$\frac{7}{15} = 0.46666\ldots = 0.4\overline{6}$$

$$\frac{1}{14} = 0.07142857142\ldots = 0.0\overline{714285}$$

$$\frac{3}{128} = 0.234375 \quad \text{(terminating, finite number of digits)}$$

It is a good time to discuss the results of division on a calculator. Any calculator has only a *finite* number of spaces for digits to display the result, so it always gives only the first few digits of the decimal representation, even when the true representation is infinite. In this case, how do we tell whether *all* the digits of the result are displayed on the calculator or it is just an approximation to the true value? For example, if we see

$$\frac{2}{7} = 0.28571428571$$

on the display, is this the true value of $\frac{2}{7}$ or just the first few digits of an infinite decimal? And, if there are infinitely many digits, do they go in a repetitive pattern?

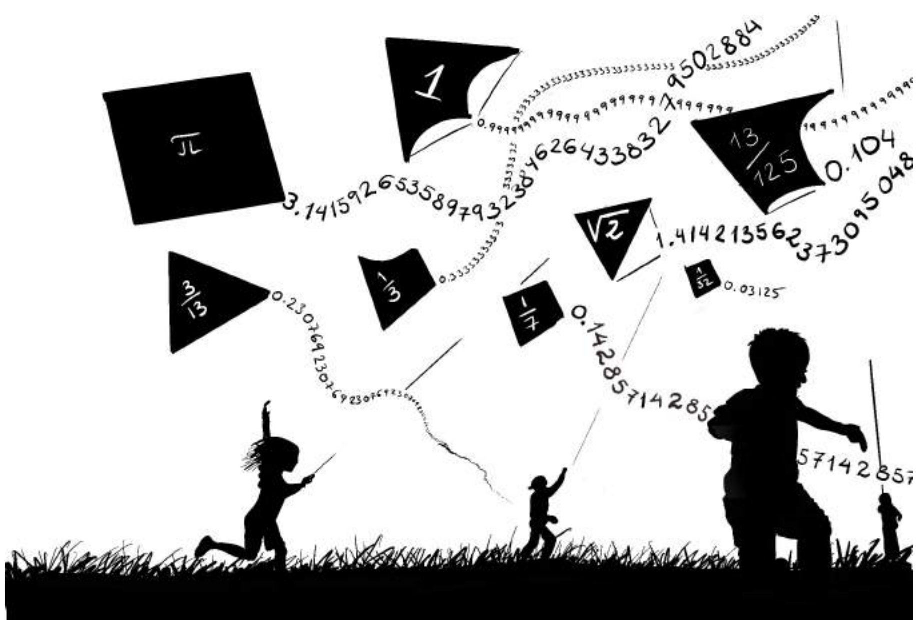
It turns out that the decimal representation of $\frac{2}{7}$ is *infinite periodic*:

$$\frac{2}{7} = 0.\overline{285714}$$

On the contrary, the decimal

$$\frac{31}{1024} = 0.0302734375$$

obtained by a calculator is the *exact value*. In this case, the decimal representation of $\frac{31}{1024}$ is *terminating*. It has exactly these finitely many digits.



Decimal representations of real numbers.

10.12 Types of Decimal Presentations of Real Numbers

We stated above that every real number has a decimal form. We can divide all real numbers into three types according to their decimal representation.

(A) *Terminating decimals*. These decimals have a finite number of nonzero digits after the decimal point.

(B) *Repeating nonterminating decimals*. These decimals have infinitely many nonzero digits after the decimal point that repeat in periodic pattern.

(C) *Nonrepeating nonterminating decimals*. These decimals have infinitely many nonzero digits after the decimal point and the digits do not create a periodic repetitive pattern.

Example 10.12

(A) The decimals 0.57, 12.456, and -310.48 are terminating.

(B) The decimals $0.\overline{3} = 0.333\ldots$, $12.45\overline{6} = 12.456666\ldots$, and $0.\overline{123} = 0.123123123\ldots$ are repeating nonterminating.

(C) The decimal whose digits form a row of all consecutive counting numbers

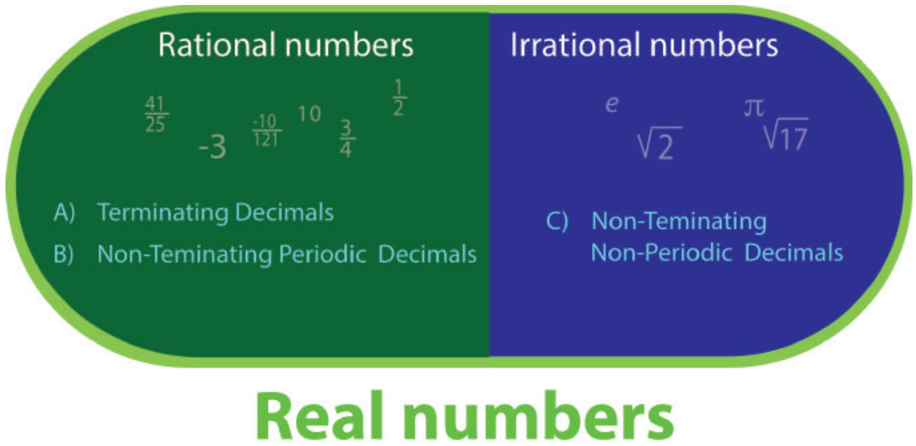
$$0.1234567891011\ldots$$

is a nonrepeating nonterminating decimal; its digits do not create a repetitive period. Also, it is not easy to prove, but is well known that the decimal representations of the irrational numbers $\sqrt{2}$ and π are of this type.

Question Is it possible to figure out the type of a decimal representation just by looking at the number, without actually converting it to a decimal?

Answer Yes, it is possible, and here is the rule. First of all, it is known that irrational numbers are exactly the real numbers that have type (C) decimal representation.

Irrational numbers have decimal representations of type (C). Rational numbers have decimal representations of type (A) or (B).



Thus, all rational numbers (fractions whose numerators and denominators are integers) have a terminating or repeating nonterminating representation. Next, we would like to ask about distinction between types (A) and (B).

Question How does one distinguish fractions with decimal representations of type (A) from fractions with decimal representations of type (B)?

Answer Let us try to guess a rule from the examples in the table below. The left column contains examples of rational numbers admitting terminating decimal representations (type (A)). The right column contains examples of rational numbers admitting (nonterminating periodic) decimal representations (type (B)).

Type (A) terminating decimal form	Type (B) repeating nonterminating decimal form
$\frac{1}{2^4}$ $\frac{7}{2^3\times 5}$ $\frac{17}{100}$	$\frac{1}{3}$ $\frac{3}{7}$ $\frac{17}{99}$
$\frac{3}{20}$ $\frac{13}{2^4\times 5^6}$ $\frac{27}{400}$	$\frac{7}{12}$ $\frac{3}{28}$ $\frac{1}{13\times 2^2}$
$\frac{121}{2^7\times 5^{64}}$ $\frac{9}{5^{43}}$ $\frac{137}{2^{18}}$	$\frac{137}{2^{18}\times 7}$ $\frac{121}{2^4\times 5^6\times 3^5}$ $\frac{9}{5^{43}\times 3^{12}}$

Can you guess the rule that distinguishes these two groups of numbers? The answer may be not evident, but here is a clue: look at the *prime factorizations* of the denominators.

Type (A) terminating decimal form	Type (B) repeating nonterminating decimal form
$\frac{1}{2^4}$ $\frac{7}{2^3\times 5}$ $\frac{17}{100}$	$\frac{1}{3}$ $\frac{3}{7}$ $\frac{17}{99}$
$\frac{3}{20}$ $\frac{13}{2^4\times 5^6}$ $\frac{27}{400}$	$\frac{7}{12}$ $\frac{3}{28}$ $\frac{1}{13\times 2^2}$
$\frac{121}{2^7\times 5^{64}}$ $\frac{9}{5^{43}}$ $\frac{137}{2^{18}}$	$\frac{137}{2^{18}\times 7}$ $\frac{121}{2^4\times 5^6\times 3^5}$ $\frac{9}{5^{43}\times 3^{12}}$

Note that the denominators of the fractions in the left column (type (A)) have only 2 and 5 in their prime factorizations, whereas for each fraction in the right column (type (B)) we have another factor, different from 2 and 5. The rule can be formulated as follows.

If the denominator of the simplest form of a fraction representing a rational number *contains only 2 and 5 in its prime factorization*, then the rational number has a *terminating decimal form*. Otherwise, it has a *repeating nonterminating decimal form*.

Thus, to determine whether a fraction has a decimal representation of type (A) or (B), we complete the following steps.

Step 1. Find the simplest form $\frac{a}{b}$ of a given fraction.

Step 2. Find the prime factorization of the denominator b.

Case 1. The prime factorization of b contains only powers of 2 and 5. Then the rational number is of type (A) (a terminating decimal).

Case 2. The prime factorization of b contains factors other than 2 and 5. Then the rational number is of type (B) (a nonrepeating and nonterminating decimal).

Example 10.13 Which of the following rational numbers have terminating decimal representations?

(a) $\frac{4}{3}$

(b) $\frac{7}{8}$

(c) $\frac{7}{24}$

(d) $\frac{11}{2^{100}\times 5^{100}}$

(e) $\frac{121}{13^{20}\times 2^2\times 5^3}$

(f) $\frac{3}{15}$

(g) $\frac{39}{60}$

Solution The rational numbers

(a) $\frac{4}{3}$

(c) $\frac{7}{24} = \frac{7}{2^3 \times 3}$

(e) $\frac{121}{13^{20} \times 2^2 \times 5^3}$

have the repeating nonterminating decimal representation of type (B) since they are written in simplest form and the denominators have factors other than 2 or 5.

The rational numbers

(b) $\frac{7}{8} = \frac{7}{2^3}$

(d) $\frac{11}{2^{100} \times 5^{100}}$

are in simplest form with denominators containing only powers of 2 and 5, so they have the terminating decimal representation of type (A).

The fractions $\frac{3}{15}$ and $\frac{39}{60}$ are not in simplest form, so we must simplify them first.

(f) $\frac{3}{15} = \frac{1}{5}$

(g) $\frac{39}{60} = \frac{13}{20} = \frac{13}{2^2 \times 5}$

Both fractions have the terminating decimal form.

Example 10.14 The result of a computation on a calculator is displayed as

$$\frac{1}{12} = 0.833333$$

Is this the exact value of $\frac{1}{12}$?

Solution No, it is not an exact value, but only an approximation by the first few digits of the true decimal representation. The rational number $\frac{1}{12}$ has the repeating nonterminating decimal representation, so its true decimal representation is infinite. One can find that

$$\frac{1}{12} = 0.8\overline{3}$$

Example 10.15 The result of a computation on a calculator is displayed as

$$\frac{1}{11} = 0.090909$$

What is the true decimal representation of $\frac{1}{11}$?

Solution The decimal representation of $\dfrac{1}{11}$ is infinite periodic. From the result of a computation on a calculator it is easy to guess the period: $\dfrac{1}{11} = 0.\overline{09}$.

Example 10.16 The result of a computation on a calculator is displayed as

$$\frac{3}{7} = 0.42857142857$$

What is the true decimal representation of $\dfrac{3}{7}$?

Solution The decimal representation of $\dfrac{3}{7}$ is infinite periodic. From the first few digits provided by the result of calculation we can guess the repeating pattern $\dfrac{3}{7} = 0\overline{428571}$.

Example 10.17 Let

$$N = \frac{1}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7}$$

A calculator helps us to find the value

$$\frac{1}{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7} = 0.000198412698$$

(a) Is N a rational number?

(b) Is the value provided by the calculator exact of approximate?

Solution

(a) Yes, N has the form $\dfrac{a}{b}$, where both $a = 1$ and $b = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7$ are integers.

(b) The result of a computation on a calculator is not an exact value of N in the decimal form. The prime decomposition of the denominator of N would contain other factors besides 2 and 5, so the decimal representation of N is repeating nonterminating. On calculator's display, we got only the first few of infinitely many digits.

10.13 Conversion from Decimals to Fractions

We discussed the conversion from rational numbers to decimals in detail. Now, we address the question.

Question How does one recover a fraction from a decimal?

In some cases, it is not difficult since we remember from our experience the decimal values of some fractions.

Example 10.18 Recall that

$$0.5 = \frac{1}{2}, \quad 0.\overline{3} = \frac{1}{3}, \quad 0.25 = \frac{1}{4}, \quad 0.1 = \frac{1}{10}, \quad 0.75 = \frac{3}{4}$$

There are *two basic rules* of conversion from decimals into fractions. First, we will discuss these rules and apply them in simple cases. After that we will look at more complicated situations.

10.13.1 Conversion from terminating decimals to fractions

Let us write a decimal of type (A) in the form

$$0.a_1a_2a_3\ldots a_k, \quad a_k \neq 0$$

Here, we mean that there are k digits $a_1, a_2, \ldots, a_k$ after the decimal point and the last digit a_k is nonzero. Then we write

$$0.\underbrace{a_1a_2a_3\ldots a_k}_{k\,\text{digits}} = \frac{a_1a_2a_3\ldots a_k}{10^k}$$

where the numerator is an integer, a k-digit number with digits $a_1, a_2, \ldots, a_k$. We simplify the fraction to get a final result. The rule becomes clear when we consider examples.

Example 10.19

(a) $0.\underbrace{18}_{2\text{digits}} = \dfrac{18}{\underbrace{100}_{10^2}} = \dfrac{9}{50}$

(b) $0.\underbrace{5}_{1\text{digit}} = \dfrac{5}{\underbrace{10}_{10^1}} = \dfrac{1}{2}$

(c) $0.\underbrace{018}_{3\text{digits}} = \dfrac{18}{\underbrace{1000}_{10^3}} = \dfrac{9}{500}$

(d) $0.\underbrace{25}_{2\text{digits}} = \dfrac{25}{\underbrace{100}_{10^2}} = \dfrac{1}{4}$

(e) $0.\underbrace{121}_{3\text{digits}} = \dfrac{121}{\underbrace{1000}_{10^2}}$

10.13.2 Conversion from repeating nonterminating decimals to fractions

Let us consider the simplest case of a decimal of type (B)

$$0.\overline{a_1a_2a_3\ldots a_k}$$

where the periodic pattern starts right after the decimal point and consists of k digits. Then we write

$$0.\underbrace{\overline{a_1a_2a_3\ldots a_k}}_{k\,\text{digits}} = \frac{a_1a_2a_3\ldots a_k}{\underbrace{99\ldots 9}_{k\text{–digit number}}}$$

where the numerator is an integer, a k-digit number with digits $a_1, a_2, \ldots, a_k$, and the denominator is the integer made out of k digits 9. We simplify the fraction to get a final result. Here are some examples.

Example 10.20

(a) $0.\underbrace{\overline{3}}_{1\text{digit}} = \frac{3}{9} = \frac{1}{3}$

(b) $0.\underbrace{\overline{6}}_{1\text{digit}} = \frac{6}{9} = 2/3$

(c) $0.\underbrace{\overline{36}}_{2\text{digits}} = \frac{36}{99} = \frac{4}{11}$

(d) $0.\underbrace{\overline{003}}_{3\text{digits}} = \frac{3}{999} = \frac{1}{333}$

(e) $0.\underbrace{\overline{120}}_{3\text{digits}} = \frac{120}{999} = \frac{40}{333}$

(f) $0.\underbrace{\overline{17}}_{2\text{digits}} = \frac{17}{99}$

10.13.3 Combination of two rules

Two basic rules are combined to convert more complicated decimals to fractions.

Example 10.21 Convert $0.0\overline{3}$ to a fraction.

Solution Note that this example is different from the examples in the previous section since the periodic pattern *starts from the second place* after the decimal point. Yet, we can reduce this case to the previous problem because

$$0.0\overline{3} = 0.033333\cdots = \frac{1}{10} \times 0.33333 = \frac{1}{10} \times 0.\overline{3}$$

Observe that $0.\overline{3}$ is the case considered in the previous section (the periodic pattern starts right after the decimal point) and we know that $0.\overline{3} = \frac{1}{3}$. Hence

$$0.0\overline{3} = \frac{1}{10} \times 0.\overline{3} = \frac{1}{10} \times \frac{1}{3} = \frac{1}{30}$$

Answer: $0.0\overline{3} = \frac{1}{30}$.

Example 10.22 Convert $0.00\overline{2}$ to a fraction.

Solution Note that

$$0.00\overline{2} = 0.002222\cdots = \frac{1}{100} \times 0.222\cdots = \frac{1}{100} \times 0.\overline{2}$$

Then $0.\overline{2}$ is the case considered in the previous section (a periodic pattern starts right after the decimal point) and we find $0.\overline{2} = \frac{2}{9}$. Hence

$$0.00\overline{2} = \frac{1}{100} \times 0.\overline{2} = \frac{1}{100} \times \frac{2}{9} = \frac{2}{900} = \frac{1}{450}$$

Answer: $0.00\overline{2} = \dfrac{1}{450}$.

Example 10.23 Convert $0.0\overline{17}$ to a fraction.

Solution We know that $0.\overline{17} = \dfrac{17}{99}$. Then

$$0.0\overline{17} = \frac{1}{10} \times 0.\overline{17} = \frac{1}{10} \times \frac{17}{99} = \frac{17}{990}$$

Answer: $0.0\overline{17} = \dfrac{17}{990}$.

Next example illustrates how to handle even more complicated cases.

Example 10.24 Convert $0.11\overline{6}$ to a fraction.

We show two ways to solve this problem.

Solution 1 We split the number into two parts (finite and infinite periodic), convert each part to a fraction, add the results, and simplify. More precisely,

$$0.11\overline{6} = 0.116666\cdots = 0.11 + 0.00\overline{6}$$

For the finite part we have

$$0.11 = \frac{11}{100}$$

For the infinite part we have

$$0.00\overline{6} = \frac{1}{100} \times 0.\overline{6} = \frac{1}{100} \times \frac{6}{9} = \frac{1}{150}$$

Finally,

$$0.11\overline{6} = 0.11 + 0.00\overline{6} = \frac{11}{100} + \frac{1}{150} = \frac{33}{300} + \frac{2}{300} = \frac{35}{300} = \frac{7}{60}$$

Answer: $0.11\overline{6} = \dfrac{7}{60}$.

Solution 2 Let x be the fraction with the decimal representation $0.11\overline{6}$. Then we can write

$$\begin{aligned} x &= 0.11666\ldots \\ 10x &= 1.16666\ldots \\ 10x - x &= 1.16666\cdots - 0.11666\cdots = 1.0500\cdots = 1.05 \\ 9x &= \frac{105}{100} \\ x &= \frac{105}{900} = \frac{7}{60} \end{aligned}$$

Answer: $0.11\overline{6} = \dfrac{7}{60}$.

Example 10.25 Convert 0.18, $0.\overline{18}$, and $0.1\overline{8}$ to fractions.

Solution

$$0.18 = \frac{18}{100} = \frac{9}{50},$$

$$0.\overline{18} = 0.181818... = \frac{18}{99} = \frac{2}{11},$$

$$0.1\overline{8} = 0.1 + 0.0\overline{8} = \frac{1}{10} + \frac{1}{10} \times 0.\overline{8} = \frac{1}{10} + \frac{1}{10} \times \frac{8}{9} = \frac{9+8}{90} = \frac{17}{90}$$

10.14 Decimal Representation of a Fraction is Not Unique

Note that there is a freedom in the choice of the period of a nonterminating periodic decimal. For example,

$$0.\overline{3} = 0.\overline{33} = 0.\overline{333} = 0.3\overline{333} \quad \ldots$$

We would like to mention one more interesting fact that reveals the sophisticated nature of decimals.

Example 10.26 Convert $0.\overline{9}$ to a fraction.

Solution Using the standard rule, we get

$$0.\overline{9} = \frac{9}{9} = 1$$

Thus,

$$1 = 0.9999999\ldots \quad !$$

This may look strange, since we know that $1 > 0.9$, and $1 > 0.99$, and $1 > 0.999$, and so on. The same is true for any decimal 0.99...9 made of any *finite* number digits 9. But the decimal 0.999... with an *infinite* number of digits 9 is not smaller, but equal to 1! This amazing fact is true, even if it seems to go against our intuition. We should consider this example as a warning that things may get quite unusual when we work with infinity. The explanation of this nontrivial observation is based on the notion of the *sum of a geometric series*, a topic considered in more advanced math courses.

10.15 Advantages and Disadvantages of Decimals over Fractions

We would like to talk about advantages and disadvantages of working with the decimal form of rational numbers in comparison with fractions. The list below was created with the participation of students.

Advantages of decimals over fractions

1. One of the main advantages of decimals is *historical*: traditionally, calculators display answers in decimals. Since today we use calculators more often than a piece of paper, we

have acquired a *habit* of working with decimals. Many modern calculators are capable of working with fractions as well, but still most of us use basic built-in features that operate with the decimal form of numbers.

2. Usually, *addition is easier* to perform in decimal form. For example, compare the calculation

$$\frac{7}{25}+\frac{13}{40}=\frac{7\times 8}{200}+\frac{13\times 5}{200}=\frac{56+65}{200}=\frac{121}{200}$$

with the same calculation in the decimal form

$$0.28+0.325=0.605$$

3. Very often the *magnitude* of a number is easier to read in decimal form. For example, it may be easier to see that 0.605 is slightly larger than 0.5 than $\dfrac{121}{200}>\frac{1}{2}$, even though 0.605 and $\dfrac{121}{200}$ are the same number.

Disadvantages of decimals over fractions

1. Many numbers have *infinite* decimal representations, which means that, in practice, we are forced to work with their *approximate values*. This may be one of the main disadvantages of decimal representations, which is an important point in science and engineering. In applications, one may want to be very careful about round up errors since they may accumulate and produce a lot of problems.

2. It turns out that *multiplication is often easier* for fractions than for decimals. For example, compare the calculation

$$\frac{1}{25}\times\frac{5}{13}=\frac{1\times\cancel{5}}{\cancel{5}\times 5\times 13}=\frac{1}{5\times 13}=\frac{1}{65}$$

with the same calculation in decimal form

$$0.04\cdot 0.\overline{384615}=?$$

Summary. Both fractional and decimal representations of rational numbers have their own pros and cons.

The choice of the form is dictated by the nature of computation.

10.16 Exercises

10.16.1 Recognition of rational and irrational numbers

Exercise 10.1 Which of the following numbers are whole?

(a) 5

(b) 0

(c) -7

(d) $\dfrac{1}{2}$

(e) 0.5

(f) $\dfrac{6}{3}$

(g) $-\dfrac{10}{5}$

Exercise 10.2 Which of the following numbers are integer?

(a) 5
(b) 0
(c) -7
(d) $\frac{1}{2}$
(e) 0.5
(f) $\frac{6}{3}$
(g) $-\frac{10}{5}$

Exercise 10.3 Which of the following numbers are rational, and which are irrational?

(a) -5
(b) 0
(c) $-7\sqrt{3}$
(d) $\frac{1}{2}$
(e) 0.3
(f) 12
(g) $\frac{8}{13}$
(h) π
(i) $\sqrt{2}$
(j) $\sqrt{9}$
(k) $-\sqrt{4}$

Exercise 10.4 Which of the following numbers are irrational?

(a) 0. 25
(b) $0.2\overline{5}$
(c) $0.\overline{25}$
(d) $0.\overline{3}$
(e) 28
(f) 7π
(g) $\sqrt{4}$
(h) $\sqrt{13}$
(i) $\sqrt{\frac{10}{40}}$
(j) $\sqrt{3}$
(k) $4\sqrt{3}$
(l) $\sqrt{100}$

10.16.2 Conversion from fractions to decimals

Exercise 10.5 What are advantages and disadvantages of using fractions over decimal representations in calculations?

Exercise 10.6 Using long division, convert each fraction to decimal form. Which of the fractions have terminating, and which have nonterminating periodic decimal representation?

(a) $\frac{7}{10}$
(b) $\frac{15}{4}$
(c) $\frac{1}{3}$
(d) $\frac{5}{6}$

Exercise 10.7 Using a calculator, convert each fraction to decimal form. For nonterminating periodic decimal representations write an answer by using the notation $\bar{c}$ for the repeating periodic part.

(a) $\frac{2}{125}$ (d) $\frac{4}{11}$ (g) $\frac{13}{9}$

(b) $\frac{18}{25}$ (e) $\frac{1}{7}$ (h) $\frac{1}{12}$

(c) $\frac{5}{6}$ (f) $\frac{1}{9}$ (i) $\frac{5}{13}$

Exercise 10.8 Convert the following fractions to decimals by completing the denominator to the full power of 10, if necessary. Do not use a calculator.

(a) $\frac{1}{1000}$ (d) $\frac{1}{4}$ (g) $\frac{7}{20}$ (j) $\frac{7}{25}$

(b) $\frac{8}{100}$ (e) $\frac{2}{5}$ (h) $\frac{1}{125}$

(c) $\frac{324}{10}$ (f) $\frac{5}{8}$ (i) $\frac{13}{200}$

Example.

$$\frac{2}{10} = 0.2, \qquad \frac{3}{4} = \frac{3 \times 25}{4 \times 25} = \frac{75}{100} = 0.75$$

10.16.3 Recognition of the type of decimal representation of a fraction

Exercise 10.9 Without converting to decimal form, determine whether the following numbers have (A) terminating decimal, (B) repeating nonterminating, or (C) nonrepeating nonterminating decimal representation.

(a) $\frac{34}{100}$ (h) $\frac{7}{100}$ (o) $\frac{13}{2^{13} \cdot 5^{24}}$

(b) $\frac{1}{4}$ (i) $\frac{3}{128}$ (p) $\frac{10}{111111}$

(c) $\frac{1}{1024}$ (j) $\frac{123}{347}$ (q) $\frac{8}{17}$

(d) $\frac{11}{366}$ (k) $\frac{1}{1250}$ (r) $\frac{13}{70}$

(e) $\frac{1}{\sqrt{2}}$ (l) $\frac{2}{3}$ (s) $\frac{16}{25}$

(f) $\frac{30}{60}$ (m) $\frac{3}{6}$ (t) $\frac{213}{555}$

(g) $\frac{13}{25}$ (n) $\frac{2}{13}$ (u) $\frac{333}{555}$

10.16.4 Conversion from decimals to fractions

Exercise 10.10 Convert the following decimals to fractions. Simplify

0.9	$0.\overline{003}$	0.16	$0.\overline{4}$	$0.1\overline{2}$
0.0025	$0.\overline{0003}$	0.125	$0.\overline{36}$	$2.1\overline{2}$
0.04	0.37	0.34	$0.\overline{9}$	$0.5\overline{1}$
1.25	$0.\overline{037}$	1.1	$0.\overline{12}$	$0.05\overline{1}$
0.3	$0.\overline{37}$	$0.\overline{4}$	$0.\overline{40}$	
$0.\overline{3}$	5.2	$0.\overline{312}$	$0.0\overline{2}$	
$0.\overline{03}$	0.02	$0.\overline{34}$	$0.00\overline{2}$	

10.16.5 Additional questions on repeating nonterminating decimals

Exercise 10.11

(a) Is the number $\dfrac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}$ rational or irrational?

(b) Computation on a standard calculator provides the decimal representation with 11 digits after the decimal point

$$\frac{1}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} = 0.00019841269$$

Is it the exact value or an approximate value of the number? Explain.

Exercise 10.12 Are the following numbers rational or irrational?

(a) $0.\overline{10} = 0.101010101010\ldots$

(b) $0.101001000100001000001\ldots$ (the number of zeros grows between ones)

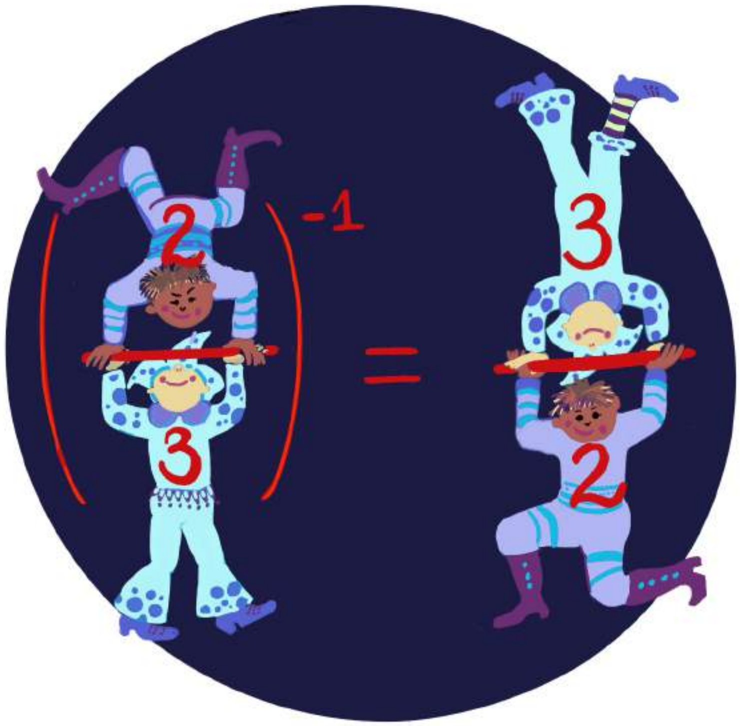
(c) $0.123456789101112131415\ldots$ (digits of all counting numbers written in the row)

Exercise 10.13 Convert the decimals $0.\overline{9}$, $0.\overline{99}$, and $0.\overline{9999}$ to fractions.

Exercise 10.14 Which of the numbers in the following pairs is larger?

(a) $0.\overline{345}$ or 0.345 (b) $0.\overline{345}$ or 0.3451 (c) $0.\overline{345}$ or 0.3455

11. OPERATIONS WITH FRACTIONS



The reciprocal of a fraction.

11.1 Arithmetic Operations with Fractions

Rational numbers can be represented as fractions with integer numerator and denominator. In this chapter, we review the arithmetic operations with fractions and discuss some techniques to perform these operations in an efficient and elegant way. In some sense, we again refer to the *culture of calculations*, as we did in Chapter 2.

Many people admit that, for integers, addition and subtraction seem to be easier operations than multiplication and division. Surprisingly, the situation is the opposite for fractions: the addition and subtraction of fractions are not very popular operations since they require using a *common denominator*.

11.2 Multiplication of Fractions

Here is important advice for the multiplication of fractions: *Simplify before multiplying*! The advantages of this idea are illustrated by the following examples.

Example 11.1 Multiply $\frac{5}{12} \times \frac{4}{15}$.

Solution A straightforward way is to multiply 5×4 in the numerator and 12×15 in the denominator first and then simplify. This way is not efficient, and there is a better way. We first cancel common factors in the numerators and denominators:

$$\frac{\cancel{5}^{\,1}}{\cancel{12}_{\,3}} \times \frac{\cancel{4}^{\,1}}{\cancel{15}_{\,3}} = \frac{1 \times 1}{3 \times 3} = \frac{1}{9}$$

The multiplication becomes very simple and we avoid large numbers in our calculation.

Example 11.2 Multiply $\frac{3}{20} \times \frac{2}{15} \times \frac{55}{7}$.

Solution

$$\frac{3}{\cancel{20}_{\,10}} \times \frac{\cancel{2}^{\,1}}{\cancel{15}_{\,3}} \times \frac{\cancel{55}^{\,11}}{7} = \frac{\cancel{3} \times 1 \times 11}{10 \times \cancel{3} \times 7} = \frac{11}{70}$$

Example 11.3 Multiply $\frac{20}{121} \times \frac{33}{40}$.

Solution

$$\frac{\cancel{20}^{\,1}}{\cancel{121}_{\,11}} \times \frac{\cancel{33}^{\,3}}{\cancel{40}_{\,2}} = \frac{1 \times 3}{11 \times 2} = \frac{3}{22}$$

11.3 Division of Fractions

Recall that the division by a fraction $\frac{c}{d}$ is equivalent to the multiplication by a flipped fraction $\frac{d}{c}$, called the *reciprocal fraction*.

Let a, b, c, and d be counting numbers. Then

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$$

For example,

$$\frac{1}{4} \div \frac{4}{5} = \frac{1}{4} \times \frac{5}{4} = \frac{5}{16}$$

It is recommended to perform division step-by-step.

Step 1. Convert the division problem into a multiplication problem.

Step 2. Simplify (cancel as many common factors as possible).

Step 3. Multiply fractions.

Warning. Do not simplify before reversing fractions. This may lead to incorrect results. For example,

$$\frac{1}{\cancel{4}} \div \frac{\cancel{4}}{5} \quad \text{“ = ”} \quad \frac{1}{5}$$

is not correct ! The correct way is shown above.

Example 11.4 Calculate $\frac{1}{3} \div \frac{7}{3}$.

Solution

$$\frac{1}{3} \div \frac{7}{3} = \frac{1}{\cancel{3}} \times \frac{\cancel{3}}{7} = \frac{1}{7}$$

Example 11.5 Calculate $\frac{17}{11} \div \frac{4}{11}$.

Solution

$$\frac{17}{11} \div \frac{4}{11} = \frac{17}{\cancel{11}} \times \frac{\cancel{11}}{4} = \frac{17}{4}$$

Example 11.6 Calculate $\frac{3}{5} \div \frac{5}{7}$.

Solution

$$\frac{3}{5} \div \frac{5}{7} = \frac{3}{5} \times \frac{7}{5} = \frac{21}{25}$$

Example 11.7 Calculate $\frac{6}{25} \div \frac{2}{5}$.

Solution

$$\frac{6}{25} \div \frac{2}{5} = \frac{\cancelto{3}{6}}{\cancelto{5}{25}} \times \frac{\cancelto{1}{5}}{\cancelto{1}{2}} = \frac{3 \times 1}{5 \times 1} = \frac{3}{5}$$

Example 11.8 Calculate $3 \div \left(\frac{1}{\frac{1}{3}}\right)$.

Solution

$$3 \div \left(\frac{1}{\frac{1}{3}}\right) = 3 \times \frac{\frac{1}{3}}{1} = 3 \times \frac{1}{3} = 1$$

Example 11.9 Solve for x.

(a) $x \times \frac{1}{4} = 2$

(b) $x \div \frac{1}{4} = 2$

(c) $\frac{1}{4} \div x = 2$

(d) $\frac{1}{4} \div \left(\frac{1}{x}\right) = 2$

(e) $\frac{1}{4} = 2 \div x$

(f) $\frac{1}{4} = 2 \times x$

Solution

(a) $x \times \frac{1}{4} \times 4 = 2 \times 4, \quad x = 2 \times 4 = 8$

(b) $x \div \frac{1}{4} = 2, \quad x \times 4 = 2, \quad x = \frac{2}{4} = \frac{1}{2}$

(c) $\frac{1}{4} \div x \times x = 2 \times x, \quad \frac{1}{4} = 2x, \; x = 8$

(d) $\frac{1}{4} \div \left(\frac{1}{x}\right) = 2, \quad \frac{1}{4} \times x = 2, \quad x = 8$

(e) $\frac{1}{4} \times x = 2, \quad x = \frac{1}{8}$

(f) $\frac{1}{4} = 2 \times x, \quad x = \frac{1}{8}$

11.4 Addition of Fractions

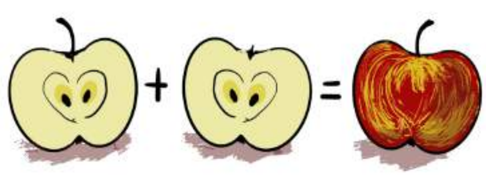
The addition of fractions is rarely a favorite operation of students since it is based on a complicated rule that involves a common denominator. Students may even wonder why they have to perform addition in this way rather than using some some simpler rule such as

$$\frac{a}{b} + \frac{c}{d} \text{ “ = ” } \frac{a+c}{b+d}$$

Question How would you answer the question why this “naive” rule is not used for adding fractions.

Answer Here is one of the possible explanations. The rules in mathematics are not created by a voluntary choice of scientists or math teachers, but are dictated by nature. Such rules exist because this is the *only possible way* for all things to work together without a contradiction.

We could find many examples when the “naive” addition rule does not work, but leads us to nonsense. Suppose that we have an apple that we cut into two halves.



so that

$$\frac{1}{2} + \frac{1}{2} = 1$$

Even in this simple situation, the easy "naive" rule gives us the wrong answer

$$\frac{1}{2}+\frac{1}{2} \quad \text{“}=\text{”} \quad \frac{1+1}{2+2}=\frac{2}{4}=\frac{1}{2} \quad !$$

This answer certainly contradicts our real life experience. Hence the "complicated" addition rule for fractions is dictated by the real world around us.

Let us review formulas for the addition of fractions. If both fractions have the *same denominator*, the rule is very simple.

Let a, b, and c be whole numbers with $b \neq 0$. Then

$$\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}$$

Example 11.10

$$\frac{1}{8}+\frac{5}{8}=\frac{1+5}{8}=\frac{6}{8}=\frac{3}{4}$$

$$\frac{1}{3}+\frac{2}{3}=\frac{1+2}{3}=\frac{3}{3}=1$$

The general case is reduced to this simple case by *creating a common denominator*.

Let a, b, c, and d be whole numbers with $d, b \neq 0$. Then

$$\frac{a}{b}+\frac{c}{d}= \underbrace{\frac{a\times d}{b\times d}+\frac{c\times b}{d\times b}}_{\text{the same denominator}} =\frac{ad+bc}{bd}$$

Example 11.11

$$\frac{1}{3}+\frac{1}{2}=\frac{1\times 2}{3\times 2}+\frac{1\times 3}{2\times 3}=\frac{2+3}{6}=\frac{5}{6}$$

We use the product $b \times d$ of denominators in these formulas as a common denominator. However, in practice, there is often a more efficient way.

Example 11.12 Calculate $\frac{8}{13}+\frac{41}{26}$.

Solution It would be a correct, but *not efficient* way to add fractions as

$$\frac{8}{13}+\frac{41}{26}=\frac{8\times 26}{13\times 26}+\frac{41\times 13}{26\times 13}=\frac{8\times 26+41\times 13}{26\times 13}=\ldots$$

Note that $26 = 2 \times 13$. So, we can use a *smaller common denominator*

$$\frac{8}{13}+\frac{41}{26}=\frac{8\times 2}{13\times 2}+\frac{41}{26}=\frac{16}{26}+\frac{41}{26}=\frac{57}{26}$$

This is the final answer since 57 and 26 are relatively prime, so the fraction $\frac{57}{26}$ is in simplest form.

Example 11.13 Calculate $\frac{61}{1000}+\frac{7}{10}$.

Solution Note that $1000 = 100 \times 10$. Then

$$\frac{61}{1000}+\frac{7}{10}=\frac{61}{1000}+\frac{7\times 100}{10\times 100}=\frac{61}{1000}+\frac{700}{1000}=\frac{761}{1000}$$

Example 11.14 Calculate $\frac{5}{12}+\frac{7}{18}$.

We solve this problem in two ways: using a straightforward general formula with 12×18 as a common denominator and using a smaller common denominator.

Solution 1.

$$\frac{5}{12}+\frac{7}{18}=\frac{5\times 18+7\times 12}{12\cdot 18}=\frac{90+84}{12\cdot 18}=\frac{174}{216}=\frac{29}{36}$$

Solution 2. Note that $12 = 6\times 2$ and $18 = 6 \times 3$. Hence 6 is already the common part of both denominators. Then

$$12\times 3 = 6\times 2\times 3 = 18\times 2$$

which can be used as a common denominator:

$$\frac{5}{12}+\frac{7}{18}=\underbrace{\frac{5\times 3}{2\times 6\times 3}+\frac{7\times 2}{3\times 6\times 2}}_{\text{the same denominator}}=\frac{5\cdot 3+7\cdot 2}{6\cdot 2\cdot 3}=\frac{15+14}{36}=\frac{29}{36}$$

There are obvious advantages of Solution 2. We avoided large numbers (operations with large numbers often produce many mistakes), and we did simpler calculations.

In summary, we can often add $\frac{a}{b}+\frac{c}{d}$ with a smaller common denominator than just $b\times d$. Actually, this smaller common denominator is $\mathrm{LCM}(b,d)$. Indeed, in the above examples, for common denominators we used the least common multiples of the denominators of given fractions

$$\mathrm{LCM}(26,13)=26,\quad \mathrm{LCM}(1000,10)=1000,\quad \mathrm{LCM}(12,18)=36$$

Example 11.15 Calculate $\frac{1}{8}+\frac{5}{6}$.

Solution We have $\mathrm{LCM}(8,6) = 24$ and

$$8\times 3 = 24 = 4\times 6$$

Then

$$\frac{1}{8}+\frac{5}{6}=\frac{1\times 3}{8\times 3}+\frac{5\times 4}{6\times 4}=\frac{3+20}{24}=\frac{23}{24}$$

11.5 Subtraction of Fractions

The same observations apply to the subtraction of fractions. When both fractions have the same denominator, we use the simple rule.

> Let a, b, and c be whole numbers with $b \neq 0$. Then
> $$\frac{a}{b} - \frac{c}{b} = \frac{a-c}{b}$$

When the denominators are different, we first try to make them the same.

> Let a, b, c, and d be whole numbers with $d, b \neq 0$. Then
> $$\frac{a}{b} - \frac{c}{d} = \underbrace{\frac{a \times d}{b \times d} - \frac{c \times b}{d \times b}}_{\text{the same denominator}} = \frac{ad - bc}{bd}$$

In practice, we often prefer to use LCM(b, d) as a common denominator rather than $b \times d$.

Example 11.16 Find $\frac{19}{77} - \frac{2}{11}$.

Solution

$$\frac{19}{77} - \frac{2}{11} = \frac{19}{77} - \frac{2 \times 7}{11 \times 7} = \frac{19 - 14}{77} = \frac{5}{77}$$

Example 11.17 Find $\frac{5}{12} - \frac{1}{18}$.

Solution

$$\frac{5}{12} - \frac{1}{18} = \frac{5 \times 3}{6 \times 2 \times 3} - \frac{1 \times 2}{6 \times 3 \times 2} = \frac{15 - 2}{36} = \frac{13}{36}$$

> Using LCM(b, d) instead of $b \times d$ as a common denominator in $\frac{a}{b} + \frac{c}{d}$ or $\frac{a}{b} - \frac{c}{d}$ simplifies calculations and reduces the risk of computational mistakes.

11.6 Comparison of Fractions

Sometimes, we may want to compare two rational numbers written as fractions. The case where both fractions have the same denominator is simple.

Let b be a counting number, and let a and c be integers. Then $\frac{a}{b} < \frac{c}{b}$ if and only if $a < c$.

Example 11.18 $\frac{5}{17} < \frac{8}{17}$ since $5 < 8$, and $\frac{-3}{5} < \frac{1}{5}$ since $-3 < 1$.

In the case of fractions with different denominators, we use the cross-product.

Let b and d be counting numbers, and let a and c be integers. Then $\frac{a}{b} < \frac{c}{d}$ if and only if $ad < bc$.

Example 11.19 $\frac{3}{4} < \frac{4}{5}$ since $3 \times 5 < 4 \times 4$.

11.7 Exercises

11.7.1 The simplest form of fractions

Exercise 11.1 Find the simplest form of the following fractions:

(a) $\frac{25}{100}$ (d) $\frac{75}{25}$ (g) $\frac{27}{840}$ (j) $\frac{32}{60}$

(b) $\frac{48}{63}$ (e) $\frac{35}{42}$ (h) $\frac{42}{49}$ (k) $\frac{216}{72}$

(c) $\frac{50}{72}$ (f) $\frac{36}{32}$ (i) $\frac{72}{24}$ (l) $\frac{600}{720}$

11.7.2 Addition of fractions using the least common denominator

Exercise 11.2 Find the smallest common denominator of given fractions. Add fractions without a calculator. Express the result in simplest form.

(a) $\frac{1}{8}+\frac{5}{8}$

(b) $\frac{1}{4}+\frac{1}{2}$

(c) $\frac{3}{7}+\frac{1}{3}$

(d) $\frac{8}{13}+\frac{41}{26}$

(e) $\frac{7}{22}+\frac{5}{121}$

(f) $\frac{61}{1000}+\frac{7}{10}$

(g) $\frac{5}{6}-\frac{8}{15}$

(h) $\frac{25}{8}-\frac{4}{5}$

(i) $\frac{3}{5}-\frac{4}{7}$

(j) $\frac{5}{6}-\frac{8}{15}$

(k) $\frac{1}{18}+\frac{1}{5}$

(l) $\frac{10}{11}-\frac{5}{12}$

(m) $\frac{8}{18}+\frac{2}{9}$

(n) $\frac{9}{28}-\frac{11}{35}$

(o) $\frac{7}{9}-\frac{3}{5}$

(p) $\frac{8}{9}+\frac{1}{12}+\frac{3}{8}$

Exercise 11.3 Explain how this calculation could be improved.

$$\frac{10}{12}+\frac{2}{36}=\frac{10\cdot 36+2\cdot 12}{12\cdot 36}=\frac{360+24}{12\cdot 36}=\frac{384}{432}=\frac{8}{9}$$

11.7.3 Multiplication of fractions

Exercise 11.4 Simplify before performing multiplication. Multiply without a calculator.

(a) $\frac{1}{2}\cdot\frac{4}{5}\cdot\frac{5}{6}$

(b) $\frac{1}{2}\cdot\frac{2}{3}$

(c) $\frac{9}{8}\cdot\frac{5}{36}\cdot\frac{7}{10}$

(d) $\frac{23}{17}\cdot\frac{51}{46}$

(e) $\frac{5}{16}\cdot\frac{56}{42}\cdot\frac{2}{15}$

(f) $\frac{11}{12}\cdot\frac{24}{77}$

(g) $\frac{5}{7}\cdot\frac{2}{9}\cdot\frac{7}{5}$

(h) $\frac{6}{25}\cdot\frac{5}{9}\cdot\frac{3}{2}$

Exercise 11.5 Find the powers.

(a) $\left(\frac{1}{2}\right)^3$

(b) $\left(\frac{4}{3}\right)^2$

(c) $\left(\frac{2}{3}\right)^4$

(d) $\left(\frac{10}{3}\right)^3$

Exercise 11.6 Explain how this calculation could be improved.

$$\frac{7}{9}\cdot\frac{3}{14}=\frac{63}{126}=\frac{1}{6}$$

11.7.4 Division of fractions

Exercise 11.7 Replace division with multiplication by the reciprocal of the divisor. Simplify and multiply without a calculator.

(a) $\frac{1}{2} \div \frac{1}{3}$

(b) $\frac{1}{2} \div \frac{1}{4}$

(c) $\frac{1}{4} \div \frac{4}{5}$

(d) $\frac{8}{9} \div \frac{4}{21}$

(e) $\frac{8}{5} \div \frac{16}{45}$

(f) $\frac{7}{27} \div \frac{49}{81}$

Exercise 11.8 Explain what is wrong with this calculation and correct it.

$$\frac{2}{5} \div \frac{10}{6} = \frac{1}{5} \div \frac{10}{3} = \frac{1}{1} \div \frac{2}{3} = \frac{3}{2}$$

Exercise 11.9 Calculate and reduce to the simplest form.

(a) $\frac{1}{2^2 \cdot 3^2} + \frac{1}{2 \cdot 3^2}$

(b) $\frac{1}{17 \cdot 5} + \frac{1}{5^2 \cdot 2}$

(c) $\frac{7}{8} - \left(\frac{2}{3} - \frac{1}{6}\right)$

(d) $\left(\frac{7}{8} - \frac{2}{3}\right) - \frac{1}{6}$

11.7.5 Simple equations with fractions

Exercise 11.10 Solve for x.

(a) $x \cdot \frac{3}{5} = \frac{4}{15}$

(b) $x \div \frac{2}{7} = \frac{7}{12}$

(c) $\frac{2}{3} \cdot x = \frac{4}{7}$

(d) $\frac{7}{9} \div x = \frac{7}{18}$

(e) $x \div \frac{1}{2} = \frac{1}{2}$

(f) $x \div \frac{12}{25} = \frac{3}{5}$

(g) $x \cdot \frac{12}{25} = \frac{3}{5}$

(h) $\frac{12}{25} \div x = \frac{3}{5}$

(i) $\frac{1}{\frac{1}{x}} = 5$

(j) $\frac{1}{\left(\frac{1}{\left(\frac{1}{x}\right)}\right)} = 5$

Exercise 11.11 Solve for x.

(a) $x \div 3 = 4$

(b) $3 \div x = 4$

(c) $3 = x \div 4$

(d) $3 \cdot x = 4$

(e) $3 = x \cdot 4$

(f) $3 = 4 \div x$

11.7.6 Comparison of fractions

Exercise 11.12 Which of the numbers in the following pairs is larger?

(a) $\frac{1}{2}$ and $\frac{1}{3}$

(b) $\frac{2}{5}$ and $\frac{1}{2}$

(c) $\frac{2}{3}$ and $\frac{3}{4}$

(d) $\frac{7}{8}$ and $\frac{8}{9}$

(e) $\frac{5}{6}$ and $\frac{7}{8}$

(f) $\frac{43}{50}$ and $\frac{1}{2}$

(g) $\frac{1041}{3040}$ and $\frac{3040}{1041}$

(h) $\frac{39}{40}$ and $\frac{38}{39}$

(i) $\frac{98}{99}$ and $\frac{97}{98}$

12. MIXED FRACTIONS

Mixed fractions show the magnitude of a rational number.

12.1 Different Representations of Rational Numbers

We know that

$$3.5 \qquad \frac{7}{2} \qquad 3\frac{1}{2}$$

represent three forms of the same number: *decimal representation*, (*improper*) *fraction form*, and *mixed fraction form*. Recall that to find the mixed form of an improper fraction $\frac{a}{b}$, where $a > b$ are counting numbers, we have to divide the numerator by the denominator with remainder. Each of these representations has advantages and disadvantages, but as any notion in mathematics, each form was invented for a purpose.

Question What are the advantages of writing rational numbers in the mixed fraction form?

Possible answer

- It seems that the main advantage of the mixed fraction form is that it gives us a better idea of the magnitude of a number. For example, it is hard to judge how large $\frac{55}{9}$ is unless we convert it to mixed form $\frac{55}{9} = 6\frac{1}{9}$ to see that it is "six and a little bit."

- Mixed fractions are commonly used in cooking recipes, vehicle manufacturing, construction, and many other types of production.

- Another advantage of mixed fractions is that huge numerators can be reduced to smaller ones by extracting integer parts.

12.2 Addition and Subtraction of Fractions in Mixed Form

Sometimes, converting fractions to mixed form simplifies the addition since the numerators become smaller. Yet, the mixed form of fractions is not well designed for the arithmetic operations. Extracting the integer part helps us to scale down the numerator; however, addition and subtraction still involve work with a common denominator and contain an inconvenient nuance of occasional borrowing of the units from the integer part. As for multiplication and division, these operations are tricky to perform in mixed form. In most cases, it is strongly recommended to convert fractions in mixed form to their improper form before multiplying or dividing them.

The important point about mixed fractions is that they are actually the *sum of two parts*: an integer and a fraction. For example,

$$6\frac{1}{9} = 6 + \frac{1}{9}$$

In this sense, a mixed fraction is not one number, but a *pair of numbers*. All difficulties of arithmetic operations with mixed fractions originate from this fact.

Example 12.1 The addition $6\frac{1}{9} + 5\frac{1}{18}$ is worked out separately on the integer and fractional parts, and then the results are collected in one mixed fraction

$$6\frac{1}{9} + 5\frac{1}{18} = 6 + \frac{1}{9} + 5 + \frac{1}{18} = (6+5) + \left(\frac{1}{9} + \frac{1}{18}\right) = 11 + \frac{1}{6} = 11\frac{1}{6}$$

Example 12.2 Find $2\frac{3}{5} + 3\frac{1}{2}$.

Solution

$$2\frac{3}{5} + 3\frac{1}{2} = (2+3) + \left(\frac{3}{5} + \frac{1}{2}\right) = 5 + \frac{11}{10} = 5 + 1\frac{1}{10} = 6\frac{1}{10}$$

Note that we have to extract the integer part from the result of addition of fractional parts.

Example 12.3 Find $6\frac{9}{10} + 1\frac{7}{15}$.

Solution

$$\begin{aligned} 6\frac{9}{10}+1\frac{7}{15} &= (6+1)+\left(\frac{9}{10}+\frac{7}{15}\right) \\ &= 7+\left(\frac{27}{30}+\frac{14}{30}\right) \quad \text{(we use that LCM(10,15) = 30)} \\ &= 7+\frac{41}{30} = 7+1\frac{11}{30} = 8\frac{11}{30} \end{aligned}$$

Example 12.4 Find $7\frac{1}{6}-4\frac{9}{16}$.

Solution

$$\begin{aligned} 7\frac{1}{6}-4\frac{9}{16} &= (7-4)+\left(\frac{1}{6}-\frac{9}{16}\right) \\ &= 3+\left(\frac{8}{48}-\frac{27}{48}\right) \quad \text{(we use that LCM(6,16) = 48)} \\ &= 3-\frac{19}{48} = \quad \text{(need to borrow)} \quad 2+\left(1-\frac{19}{48}\right) \\ &= 2+\left(\frac{48}{48}-\frac{19}{48}\right) = 2+\frac{29}{48} = 2\frac{29}{48} \end{aligned}$$

12.3 Multiplication and Division of Fractions in Mixed Form

Multiplication of mixed fractions is a common source of mistakes for students. If you ask students in a class to multiply, for example, $2\frac{1}{3}\times 3\frac{1}{2}$, it is very likely that someone will do it as

$$2\frac{1}{3}\times 3\frac{1}{2} \text{``}=\text{''} (2\times 3)+\left(\frac{1}{3}\times\frac{1}{2}\right) = 6+\frac{1}{6} = 6\frac{1}{6}$$

Of course, it is wrong. The flaw of this computation becomes clear if we again recall that mixed fractions are the sums of two parts

$$2\frac{1}{3} = 2+\frac{1}{3} \qquad 3\frac{1}{2} = 3+\frac{1}{2}$$

Then

$$2\frac{1}{3}\times 3\frac{1}{2} = \left(2+\frac{1}{3}\right)\times\left(3+\frac{1}{2}\right)$$

This reminds us how we multiply such expressions in algebra

$$(a+c)\times(b+d) = a\times c+b\times c+a\times d+b\times d$$

(Maybe this is what makes multiplying mixed fractions challenging: it contains elements of more advanced math.) Nevertheless, we need to perform multiplication according to the general rule

$$\left(2+\frac{1}{3}\right)\times\left(3+\frac{1}{2}\right) = 2\times 3+\frac{1}{3}\times 3+2\times\frac{1}{2}+\frac{1}{3}\times\frac{1}{2} = 6+1+1+\frac{1}{6} = 8\frac{1}{6}$$

We see that the multiplication of fractions in mixed form is quite a complicated procedure. At the same time, we remember that the multiplication of regular fractions (that could be in improper form) is not a difficult operation. This implies the general recommendation.

When multiplying or dividing fractions in mixed form, *first convert them to improper form*.

Example 12.5 Following this recommendation, we can obtain the much simpler calculation for the last example:

$$2\frac{1}{3}\times 3\frac{1}{2}=\frac{7}{3}\times\frac{7}{2}=\frac{49}{6}=8\frac{1}{6}$$

Example 12.6 Find $2\frac{2}{5}\div 3$.

Solution

$$2\frac{2}{5}\div 3=\frac{12}{5}\div 3=\frac{12}{5\times 3}=\frac{4}{5}$$

Example 12.7 Find $9\div 2\frac{1}{4}$.

Solution

$$9\div 2\frac{1}{4}=9\div\frac{9}{4}=9\times\frac{4}{9}=4$$

Example 12.8 Find $7\frac{1}{2}\div 2\frac{1}{2}$.

Solution

$$7\frac{1}{2}\div 2\frac{1}{2}=\frac{15}{2}\div\frac{5}{2}=\frac{15}{2}\times\frac{2}{5}=3$$

12.4 Exercises

12.4.1 Conversion of fractions to mixed form

Exercise 12.1 Convert the following fractions to mixed form:

(a) $\frac{3}{2}$ (b) $\frac{4}{3}$ (c) $\frac{28}{3}$ (d) $\frac{38}{13}$

12.4.2 Addition of fractions in mixed form

Exercise 12.2 Complete the addition. Write your final answer in mixed form.

(a) $2\frac{5}{7}+3\frac{2}{7}$

(b) $1\frac{1}{7}+5\frac{5}{7}$

(c) $3\frac{2}{7}+2\frac{6}{7}$

(d) $6\frac{9}{10}+1\frac{7}{15}$

(e) $4\frac{3}{5}+1\frac{2}{5}$

(f) $4\frac{13}{27}+8\frac{17}{27}$

(g) $4\frac{13}{25}+\frac{3}{50}$

(h) $6\frac{1}{12}+\frac{1}{18}$

(i) $5\frac{3}{5}+1\frac{3}{5}$

(j) $3\frac{2}{7}+2\frac{6}{7}$

(k) $3\frac{1}{5}+5\frac{2}{5}$

(l) $2\frac{1}{2}+\frac{1}{6}$

(m) $2\frac{3}{10}+6\frac{1}{9}$

(n) $4\frac{2}{5}+7\frac{4}{5}$

12.4.3 Subtraction of fractions in mixed form

Exercise 12.3 Subtract. Write your final answer in mixed form.

(a) $8\frac{3}{4}-4$

(b) $6\frac{2}{5}-3$

(c) $9\frac{3}{8}-3\frac{1}{16}$

(d) $9\frac{4}{9}-\frac{1}{3}$

(e) $8\frac{3}{16}-\frac{5}{8}$

(f) $1\frac{1}{5}-\frac{2}{5}$

(g) $1\frac{1}{9}-\frac{2}{9}$

(h) $1\frac{1}{8}-\frac{5}{8}$

12.4.4 Multiplication of fractions in mixed form

Exercise 12.4 Multiply. Write your final answer in mixed form.

(a) $1\frac{1}{5}\cdot 2$

(b) $2\frac{1}{5}\cdot 3$

(c) $8\cdot 3\frac{1}{4}$

(d) $2\cdot 5\frac{3}{4}$

(e) $2\frac{1}{5}\cdot 9$

(f) $1\frac{1}{5}\cdot\frac{1}{6}$

(g) $2\frac{2}{5}\cdot\frac{2}{3}$

(h) $7\frac{1}{3}\cdot 2\frac{1}{4}$

(i) $1\frac{1}{5}\cdot\frac{1}{6}$

(j) $\frac{3}{7}\cdot 3\frac{1}{3}$

12.4.5 Division of fractions in mixed form

Exercise 12.5 Divide. Write your final answer in mixed form.

(a) $1\dfrac{1}{5} \div 2$

(b) $2\dfrac{2}{5} \div 3$

(c) $9 \div 2\dfrac{1}{4}$

(d) $3\dfrac{1}{7} \div 2$

(e) $2 \div 5\dfrac{1}{3}$

(f) $2\dfrac{4}{5} \div 2$

(g) $\dfrac{5}{9} \div 4\dfrac{1}{6}$

(h) $\dfrac{5}{6} \div 3\dfrac{1}{3}$

Exercise 12.6 Simplify

(a) $2\dfrac{8}{15} - \left(1\dfrac{3}{10} + \dfrac{2}{5}\right)$

(b) $1\dfrac{4}{7} \cdot \dfrac{20 \div \frac{2}{15} + 25\frac{5}{7} \div 1\frac{1}{35}}{21\frac{7}{9} \div 4\frac{2}{3} - 1}$

12.4.6 Comparison of fractions in mixed form

Exercise 12.7 Compare the following fractions in mixed form.

(a) $1\dfrac{1}{2}$ and $1\dfrac{1}{3}$

(b) $3\dfrac{2}{5}$ and $2\dfrac{3}{5}$