Measurement and Geometry in Upper Primary School

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Koeno Gravemeijer
Eindhoven University of Technology

Nisa Figueiredo
Thomas more

F.H.J. van Galen
Utrecht University

Ronald Keijzer
Hogeschool iPabo

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Koeno Gravemeijer
Nisa Figueiredo
Els Feijs
Frans van Galen
Ronald Keijzer
Fokke Munk

TAL-project
Freudenthal Institute for Science and Mathematics
Education, Utrecht University

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Ada Molkenboer, Nathalie Kuijpers

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www.fi.uu.nl/publicaties/subsets/measurementgeometry/
This webiste gives direct links to the applets that are discussed in this book and offers the possibility to download the poster that is discussed in chapter 2

Preface

This book is the fourth – and final – publication in the TAL project series (*Tussendoelen Annex Leerlijnen - Interim goals and learning-teaching trajectories*). This TAL project was initiated by the Dutch Ministry of Education, Culture and Science. Its aim is to improve the quality of mathematics education by providing a perspective on didactic goals and learning-teaching trajectories, and the relationship between them.

The central focus of this book is on measurement and geometry in the upper grades of primary education. These are important topics which perhaps do not get the emphasis they deserve in education and the methods. You could even say that measurement and geometry, in a manner of speaking, build a bridge between everyday reality and mathematics. Measurement concerns the quantification of phenomena; consequently, it makes these phenomena accessible for mathematics. Geometry establishes the basis for understanding the spatial aspects of reality.

Measurement and geometry are also difficult subjects. The periodic performance assessment (PPON) of Cito (the Dutch testing and assessment organization) has shown that many pupils perform poorly on the measurement component. One pitfall for teachers is that the metric system of measurement seems so clear: we can move from one measurement to another by using a limited set of conversion rules. However, the clarity of the metric system is the product of centuries of development and the era of inches and yards was not that long ago. For pupils as well, it takes some time before they have completely figured out the system. This process of figuring out the system should be at the forefront because simply memorizing calculation rules leads to feigned results which have little long-term benefit for a child.

Perhaps most of all, geometry is a difficult subject for the teacher. There is a great diversity of subtopics, which make it infeasible to structure the entire subject in a linear fashion. At any rate, this is our contention in this book: the arguments in favor of one sequence of subtopics can often be applied just as effectively to another sequence. This makes it difficult for teachers to acquire an overview, and this in turn makes it difficult for them to help their students develop insight. Although a linear structure that comprises the entire domain of geometry education is impossible, it is certainly possible to structure specific areas. In this book, we illustrate this approach with several examples. In addition, we attempt to support teachers by describing which insights this concerns within the domain of geometry.

The specification ‘upper primary school’ in the title must be interpreted broadly. This book is a sequel to ‘*Young Children Learn Measurement and Geometry’* (Marja van den Heuvel-Panhuizen et al.,2004), which described the education in these subjects for the lower grades of primary school.

The publication of this book was preceded by a process of experimentation and discussion. We would like to express our appreciation especially for the teachers who contributed to this process. Specifically, we would like to thank Anneke van Vliet, Bente Schroot, Carlijn Bergmans, Christel Swen , Claar Rekers, Cleo Blenk, Corinne Huiskamp, Els Jasperse , Francis Hermsen, Gerda Veldhuizen, Janny Maas-Kamphof, Karin Kruidenier, Lia Oosterwaal, Mieke Brandenburg and Wouter Sluitman. In addition, we would like to thank everyone who provided recommendations and comments. Finally, we would like to thank Jan de Lange for his contribution in several draft versions.

We hope that this description of measurement and geometry not only provides support, but is also a source of inspiration for everyone who is involved in mathematics education in primary education.

Koeno Gravemeijer
Project leader TAL team

1 Introduction and outline

Arithmetic, measurement and geometry are closely related. You could even say that measurement and geometry build a bridge between everyday reality on one side and mathematics on the other. Measurement is what we do when we quantify reality, i.e. when we allocate numbers in order to acquire a grip on reality. With these numbers, we can calculate and make comparisons and predictions. For example, we can determine how much of something we need, how long something will last, or how much something will cost. Geometry establishes the basis for understanding the spatial aspects of reality. We use geometric knowledge even without being aware of it, for instance when we plan a route, furnish a room or interpret a plan. In geometry education, we try to expand this informal knowledge.

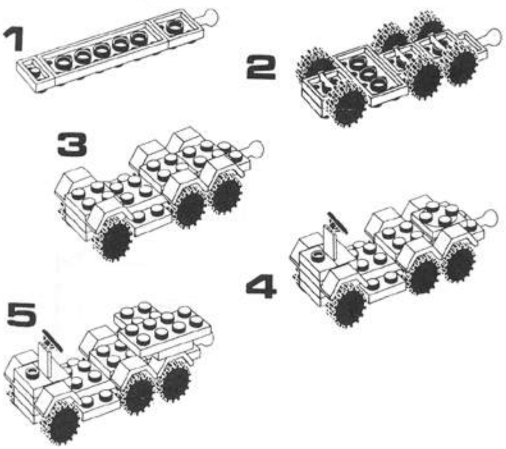


Figure 1.1.

It is crucial that pupils develop internal coherence in their knowledge. Nevertheless, measurement and geometry are often described as separate subtopics in arithmetic and mathematics. This book therefore first addresses measurement education in the upper grades of primary school, followed by geometry education. In addition, we discuss graphs in a separate chapter. This is because graphs are important for both measurement and geometry.

Graphs have a clear geometric character, but at the same time they are important tools for displaying measurement results. We see graphs primarily as a supplement to the components of measurement and geometry, which are subjects of this book. We therefore do not describe any explicit learning-teaching trajectories for this component.

Measurement and geometry are important topics which do not always get the emphasis they deserve in education. For example, one could question whether sufficient attention is given to measurement and whether this attention is on the correct aspects. Data from the periodic performance assessment (PPON, by Cito) show that pupils in the Netherlands often perform poorly in measurement and geometry, despite the fact that everyone is convinced of the practical importance of measurement. There are virtually no professions in which measurement, or interpreting data by means of measurement, do not play an important role. This also applies increasingly to the daily life of the average citizen.

Geometry suffers from the fact that teachers often view it as a topic that is not one of the core objectives. This is the case even though the concepts of 'mathematical literacy' and 'numeracy' indicate that geometry is a full-fledged and important component of the subject matter. Nevertheless, many teachers tend to believe that arithmetic is more important. In this book, we are therefore going to extensively address the importance of geometry, while making a distinction between its practical and educational value, preparatory value and intrinsic value.

Measurement

In this book, we characterize measurement as getting a grip on reality. We work this out in Chapter 2. Measurement is related to our need to capture reality in numbers and the resulting possibilities. In this context, we refer to the 'mathematical organization of reality'. One component of this approach is the development of mathematical tools. These tools include our system of measurements and formulas. The system is based upon various mathematical discoveries. We believe it is essential to make these discoveries explicit and to enable the pupils to experience their importance. We hope that this will allow pupils to acquire better understanding of the whole

system, so that they can ultimately reconstruct the relationships within the system themselves. Instead of memorizing rules, we want the pupils to understand how they can reason out the relationships between various measurements. To this end, they must understand the meaning of prefixes such as 'centi' and 'kilo', and be able to depict the various measurements themselves. They should also be able to refer to the measurement knowledge that is based on reference points in reality. A well-known example in this context is the realization that a door is generally 2 m tall.

These basic principles are worked out in greater detail in Chapter 3, which also provides a sketch of measurement education. The idea of seeing measurement and the development of the metric system as mathematical organization and the development of mathematical tools are worked out sequentially for the concepts of length, area and volume. Before this, attention is focused on the prefixes with which any metric unit can be expanded to become a series of measurements with a virtually unlimited range. The discussions on length, area and volume end with an explanation of their meaning and relationships Attention is also paid to the intended coherence and nature of the knowledge that the pupils should develop. After this, we focus on weight, time, speed and other quantities. This part of the book actually sketches out the rationale and structure behind the learning-teaching trajectories for measurement. In Chapter 3, we provide a brief summary of these learning-teaching trajectories and add a description of the intermediate attainment targets.

Geometry

We begin the description of the domain of geometry with a brief characterization in Chapter 4. In this chapter, we define geometry as getting a grip on space and as the mathematization of space. Here as well, we can refer to mathematical organization. This brief characterization is followed by a description of the basic principles, their legitimization and their characterization. This chapter addresses the above-named practical and educational value, preparatory value and intrinsic value. In the characterization we discuss the inquiry-based character, the differences in level and its consequences for the structure of the curriculum. After this, the content of the geometry taught in the upper grades of primary school is discussed. In the lower grades, the emphasis is still primarily on the activities of the pupils, while in the upper grades it shifts to explanation and reasoning. Consequently, we have chosen for a different structure than in the lower grades. Instead of structuring the material according to types of activities, we have chosen a structure that focuses more on the benefits of geometry education. In this way, we arrive at three terrains: orientation in space, two-dimensional and three-dimensional figures, visualization and representation. The first area concerns determining a position (localization), reasoning out what you can see from a specific viewpoint, and moving in space and describing this motion (navigation). The process of determining a position quickly leads to the exploration of maps and all related topics. The process of reasoning out what you can see from a specific viewpoint is the beginning of 'viewpoint geometry. This method uses lines of sight as a powerful instrument for reasoning out what you can or cannot see, or to determine the size of shadows. Navigation, or moving in space, requires geometric description aids that concern direction and changes in direction. An important aspect here is the distinction between a relative frame of reference, which is based on a vehicle or a person where the nose of the vehicle or person always points forward, and an absolute frame of reference such as the wind direction, which is independent of the person or the vehicle in question.

Geometry is a broad and complex area, where learning is based primarily on acquiring experiences and reflecting upon them. During this process, as can be expected, differences between pupils manifest themselves. Because reflecting on experiences delivers better results than focused instruction, the answer to these differences must lie in offering repeated possibilities to acquire insight. As a result, geometry does not lend itself very well to a priori planning and alignment. However, local alignment is possible; based on a rich activity we can indicate which geometric insights can be developed during this activity, what these insights are based on and what the next stage can be. For geometry, the descriptions of the learning-teaching teaching trajectories therefore have a different character than those from measurement. For geometry, we have chosen descriptions of the learning-teaching teaching trajectories which link up with three example activities based on the three terrains: orientation in space, two-dimensional and three-dimensional figures, and visualization and representation (Chapter 5). After describing the activity itself, we show how this activity can be placed

in a learning-teaching trajectory and what this learning-teaching trajectory looks like. Or more accurately, what it can look like. After all, differences in accents and sequences are possible, specifically because geometry is such a rich area. The most important aspects are the substantive analysis and the coherence between the various concepts, insights and techniques. In geometry education, we believe it is essential to carefully consider the bigger picture that surrounds a specific activity: What are the implications of this activity? What precedes this activity? What can follow this activity? The answers to these questions provide support for implementing the activity, provide insight into the support that you can offer the pupils and indicate the topics on which the pupils can reflect.
The attainment goals for geometry are summarized in Chapter 6, where we again use the division into three terrains: orientation in space, two-dimensional and three-dimensional figures, and visualization and representation.

Graphs

The book ends with a chapter on graphs (Chapter 7). Interpreting graphs is becoming an increasingly important component of mathematical literacy. Due to the increasing informatization of society, we must process more and more quantitative information. Graphs are a compact way to represent quantitative information. In Chapter 7, we address the various types of graphs such as bar graphs, circle graphs and line graphs. We also discuss important characteristics, conventions and possible pitfalls. The didactic principle here is that having students make graphs themselves can help them understand conventions and pitfalls. Consequently, we again return to the principles that form the theme of the education described in this book: investigating, reason, construct and understand.

Applets and poster

Direct links to the applets that are discussed in this book can be found at:
www.fi.uu.nl/publicaties/subsets/measurementgeometry/
This website also offers the possibility to download the poster that is discussed in chapter 2

2 A description of the domain of measurement

Gaining control of reality

Measurement is a way to gain control of reality. For example, we talk about how big something is or how heavy it is. Or we wonder how far away something is, how much it costs, how sweet it is, how hot it is, or how long something lasts. Measurement is a specific mathematical approach to reality. If we want pupils to learn to look at reality in a similar fashion, we must encourage them to structure and quantify situations in reality. For many children, this appears to be difficult. A periodic assessment conducted in the Netherlands (Cito 2005, *Balans (32) van het reken-wiskundeonderwijs aan het einde van de basisschool 4*) showed that children at the end of primary school often have difficulty, especially when working with area measurement. Pupils who have weak mathematics skills at the end of primary school do not understand area. Volume also causes problems. For pupils with weak mathematics skills, relating various liter measurements to each other is also difficult. Pupils with good mathematics skills are able to do this, but they also tend to become stymied when they start to work with cubic measurements.

It is important that pupils develop mathematical tools to measure things and to interpret the measurement results. This means that they must gain control of measurement-related concepts and procedures. For example, take the concept of area, which is linked to knowledge about procedures to determine the area of something. The most basic procedure is to measure in steps using a unit of area – 12 sheets of paper fit onto the table. This can ultimately lead to the formula 'length times width'. If the pupils acquire insight into this formula, then it becomes a mathematical tool for them.

The metric system belongs to our set of mathematical tools. The metric system offers a systematic structure for specific types of measurement. We have many different measurements for length – nanometer, meter, kilometer, to name only a few – but they can all be converted into each other via steps of 10. Moreover, the metric system links various types of measurement to each other; for example it links area and volume to length. We can read cm^2 as 'square centimeter', while thinking about small squares, but this obscures the fact that it was a great discovery to select a square with a standard unit of length on each side as a unit of area. This is not the case with older units of measurement, such as 'acre' for area and 'gallon' for volume.

What does this mean for education? When we look at primary education, we see that many pupils have difficulty with topics such as area and volume. This is understandable. The formulas appear to be similar, the conversion steps appear to be similar, and what's more, the names of many measurements are very similar. It is probably even more difficult to keep these aspects separate from each other because the pupils learn a 'ready-to-use' system, even though this system is actually the result of a long process of developing suitable tools such as useful units of measurement, a measurement system and suitable formulas. And because the system becomes relatively formal where area and volume measurements are concerned, we see that many pupils become stymied at that point.
We choose instead to have the pupils construct and discover as much as possible by themselves. This lays the foundation for insightful understanding. Moreover, by doing so we want the pupils to be able to reconstruct all kinds of connections later on.
For example, if you know
- that you can conceive of 1 m^2 as a 1 m x 1 m square;
- and that 1 cm^2 is a 1 cm x 1 cm square;
- and you also know that 1 m is equal to 100 cm;

then you can imagine that 100 rows of 100 cm^2 squares can fit onto 1 m^2.

As an alternative for memorizing the orders of magnitude and formulas, we therefore choose constructing, reconstructing and establishing relationships. To this end, in education we focus on the clever discoveries on which our system of units and formulas is based.

Of course, measurement is more than simply determining length, area and volume. In daily life, the children come into contact with many more 'measuring numbers'. For example:
- the car is travelling at 100 km/h, the speedometer points to '100';
- the packaging for children's medicine says that each tablet contains

e.g. 240 mg, and sometimes 120 mg;
- the sign on the elevator reads: 'maximum of 9 people or 700 kg';
- a sign on an old bridge reads: 'max. 2 tonnes';
- on the front page of the newspaper: 'today's weather: 18 degrees, wind southwest 4'.

Measurement extends to expressing virtually everything conceivable in numbers, such as measurements of 'uncertainty', 'consumer confidence' and the 'price increase percentage' of ice cream in the summer. Of course, during primary education we cannot address all possible quantities. But it is important for pupils to learn that measurement is a way to quantify the world. In the upper grades of primary school, pupils in any case learn to work with length, area, volume, weight, time, temperature and composite units such as speed and specific gravity.
In addition to the above measurement numbers, there are other composite units that concern 'so much per so much'. These units are sometimes difficult to perceive because they usually involve averages. For example, consider the number of calories per 100 g of fruit yogurt. Although in primary education we do not aim to have children understand such concepts completely, it is important that the pupils become acquainted with them.

Primary education addresses not only a broad spectrum of units, but there is also attention for measurement precision and using measurement instruments. In this context, it should be noted that measurement instruments increasingly have the character of a black box. Besides the digital scale and the digital thermometer, digital distance measuring devices are also available. With such types of apparatus, the repeated measurement in steps using a standard unit is no longer apparent. Teachers must therefore ensure that pupils do establish the relationship with measuring in steps.

Measurement education in the upper grades of primary school builds upon the education in the lower grades, when pupils became acquainted with many aspects of measurement. The quantity of length is central to education in the lower grades of primary school, but there is also emphatic attention for the quantities of volume and weight. Acquiring measuring skills begins with comparing, moves on to measuring in steps and then to reading measurement results. In length measurement, the pupils initially work with a natural unit of measurement. After this, a standard is introduced – the centimeter – and the pupils learn to work with a ruler and a tape measure. In a comparable fashion, volume and weight are introduced as standard units. For an extensive description of learning measurement in the lower grades of primary school, see the publication *'Young children learn measurement and geometry'* (2004).

Overview

In this chapter, we will show the possibilities for measurement education during the upper grades of primary school. We will begin with an additional characterization of this education and discuss why specific topics should be addressed in education. In the previous chapter, we indicated that measurement – in any case in the primary school – is largely related to knowledge of the units in the metric system. It is important that pupils grasp the underlying structure of this system; in this regard, the prefixes such as 'milli', 'centi', 'deci', 'deca', 'hecto' and 'kilo' are crucial. We propose that these prefixes be explicitly addressed in education; we also explain why this is important and how attention can be focused on this topic. The prefixes are closely linked to the system underlying the units of length. Some of these units of length are already known to the children when they enter the upper grades of primary school. Expanding the system of the units of length and becoming acquainted with the prefixes therefore go hand-in-hand. However, we still pay separate attention to the units of length. There are two reasons for this. Firstly, it is important to become acquainted with the units of length in order to function in everyday life. Secondly, knowledge of units of length is an important means to construct units of area and units of volume. We pay attention to acquiring units of length, area and volume, in that order. During this process, we show how other accents in education are often shifted from the current situation. The meagre performance of current education, especially regarding the aspects of area and volume, justify this attention. Especially when learning area and volume, we therefore emphasize that it is important for pupils to construct the units and the measurement procedures themselves. For example, this leads to the pupils spontaneously acquiring units of length, units of area and units of volume. We emphatically address such relationships between units. We close this chapter with a reflection on other units.

Chapter 3 is also concerned with measurement. In this chapter, we provide a brief summary of the learning-teaching trajectories, and we describe the intermediate attainment targets.

Basic principles, justification and characterization

Thorough attention to measurement in primary education is generally accepted. As a result, we seldom consider the actual importance of measurement. However, we will take this step here. We therefore begin with the basic principles of measurement education and we close with a characterization of this education.

The foundation: quantification and organization

We previously described measurement as 'gaining control of reality'. This process of gaining control of reality is realized by quantifying many aspects of reality. This is actually the power of measurement. You can capture reality in numbers (to a certain extent) and you can calculate with these numbers. A numerical approach to reality has become almost second nature for many people. This quantitative approach is not limited to elementary aspects such as measuring length, area, volume and weight. We also apply a quantitative approach to a wide range of physical and social phenomena. For example, aspects of reality are often quantified in the media.

Measurement originated from the need to compare objects or situations regarding some characteristic. This can lead to statements such as 'Jack is taller than Pete', or 'the level of criminality in New York is higher than in Amsterdam'. However, you can make much more exact statements using numbers. You could even say that measurement actually begins when you quantify the differences.

The basic idea of measurement is measuring in steps using a unit of measure. The number that you obtain in this way, the number of times that the unit of measure is stepped off, indicates the relation between the unit of measure and total measurement. The wonderful aspect of the numbers obtained in this way is that you can not only compare them with each other, but that you can also use them for calculation. The relation between the numbers also indicates the relation between the objects. For example, if you know that one sack of potting soil weighs 6 kg, then you can assume that two sacks weigh 12 kg, and that three sacks weigh 18 kg. And if you know the load capacity of your car, you also know how many sacks of potting soil you can purchase.

This proportional aspect is less self-explanatory than it appears; not every comparison that can be expressed in numbers concerns proportion numbers. For example, the points that the jury awards to figure skaters are not proportion numbers. The wind force measurements on the Beaufort Scale are not proportion numbers either; wind force 8 is not twice as hard as wind force 4. These numbers only indicate a ranking, where wind force 8 is harder than wind force 7.
Measuring in steps with a standard unit is a powerful aid in mathematically organizing situations in reality. During this process we preferably choose a unit of measurement that is suitable for the situation. For example, we measure travel distances in kilometers, but we measure living rooms in meters. Although for the latter situation, centimeters or millimeters are sometimes used as well. The convenient aspect of the system of units of length is that it is not such a problem to convert millimeters or centimeters into meters. Or to convert meters into kilometers, if this is required. After all, the conversion factors are all powers of 10. We are so accustomed to this, that we hardly realize that these units have actually been chosen in a very clever fashion.
The fact that this is not self-evident is shown by historical units of length, such as inches, yards and miles. All these units are useful for specific applications – such as the length of a nail, a piece of cloth or the distance from one town to another – but they are difficult to convert into each other. Viewed in this way, the development of a system of easily convertible units can also be understood as a way to organize things mathematically.

We emphasize this aspect of mathematical organization because the history of its development provides us with clues for measurement education. This concerns the entirety of useful discoveries that are hidden in the metric system. It also concerns the idea that we must make pupils aware of these discoveries if we want them to understand what they are doing. Because the pupils experience this mathematical organization themselves, we can

make them aware of these discoveries. In this way, we ultimately want the pupils to always be able to reconstruct the relationships within the system of units. With this objective in mind, we will now analyze the mathematical discoveries that established the basis for the metric system.

The metric system

Compared with the units that were used in earlier times, the metric system is well-organized:
- it makes optimal use of the structure of the *decimal system*;
- it uses the same *prefixes* wherever possible to link larger and smaller units to the standard unit;
- units of length, area and volume are linked together in a very convenient fashion.

Decimal structure and prefixes

Originally, all kinds of 'natural units' developed in practical situations. As a result, the 'old units' are usually situation-specific. We previously referred to the units of length inch, yard and mile and the awkward numbers that result when you try to convert one such unit into another. In that respect, the units of the metric system are much easier to use. Due to the consistent use of a factor of 10 for every subsequent unit, all units of length can be easily converted into each other. After all, multiplying or dividing by ten is very simple in the decimal system.

Thanks to prefixes such as kilo, hecto, deca, deci, centi and milli, a single standard unit (the meter) is sufficient to construct suitable units of length for all kinds of situations. The latter is the crux of the matter: *constructed units*, which are deliberately constructed in such a way that you can calculate with them easily. For education, this means that it is important for the pupils to fathom the usefulness of this structure and be able to master the prefixes so they can reconstruct the links between the standard units. This means that they can use the knowledge that 'kilo is 1000' to discover the fact that 1 km = 1000 m. With the aid of 'hecto is 100', for example, the pupils can reason that there are 10 hectometers in one kilometer.

The beauty of the system of prefixes is that it can also be used with many other quantities. This leads to a frequently used principle: 'If you know the basic unit, you can also construct the other units'. For example, this also applies to the basic unit 'gram' for weights and 'Joule' for energy. Used in reverse, this insight into the system can be a powerful aid for interpreting units. If you understand the system of prefixes – if you understand facts such as kilo = 1000 or centi = 1/100 – and you know what a gram is, then you can interpret all other units of weight.

The connection between units of length, area and volume

Another element of the organization of the metric system concerns the coherence between units of length, area and volume. Here as well, making a comparison with other units can clarify matters.

In the USA, people still use the *gallon* as a measure of volume, for example to buy fuel. Originally, the gallon was linked to the volume taken up by eight pounds of wine. Such a definition must have made it very difficult to link this unit of volume to units of length. However, at a certain time in history, the gallon was defined as the volume of a cylinder 6 inches tall and 7 inches in diameter; by using the formula $6 \times 3\frac{1}{2}^2 \times \pi$, the gallon could then be defined as 230.90706 cubic inches. This is a difficult value to calculate with, and therefore in 1706 it was rounded off to 231 inch^3 , and was ultimately defined as: 1 gallon = 224 inch^3. In the USA, this is still the official definition of a gallon.

Inches and gallons are typical examples of units that have been developed in a specific context; inches in the context of measuring lengths and gallons in the context of measuring the volume of liquids. However, in the metric system the units of area and units of volume were constructed using units of length. In this way, a square centimeter is equivalent to the area of a 1 x 1 cm square, and a cubic meter is equivalent to the volume of a cube which is 1 m on a side. As a result, without much difficulty you can calculate areas using formulas such as 'area = l x w', if you know the length and the width. And you can calculate volumes using the formula 'volume = l x w x h', when you know the length, width and height.

This is certainly a marvelous convenience, but for the pupils who do not know the history and do not perceive the correlation, this soon becomes juggling with numbers. For example, there are pupils who understand at a certain point that calculating volume has something to do with length, width and height, but who become completely confused when the volume

has a different shape than a rectangular block.

Conversely, this elegant system offers pupils interesting possibilities to construct or reconstruct correlations. When you know the prefixes and perceive the structure of the cubic units, you realize that that you can imagine m^3 as a cube 100 cm x 100 cm x 100 cm. On this basis you can reason out how many cm^3 go into 1 m^3.

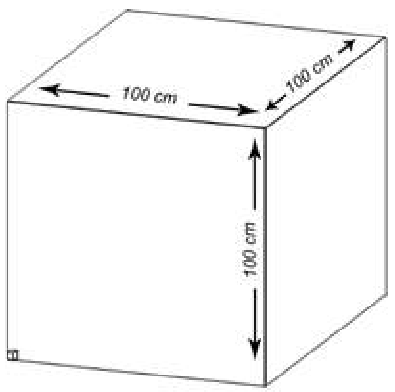


Figure 2.1

Basic principles

The above section shows that using units from the metric system and relating the units to each other is truly difficult, and that the highly polished metric system camouflages the conceptual difficulties. In primary education, formulas and procedures are generally taught in an insightful fashion, but 'seeing' the underlying structure only once is not enough. The insight disappears quickly when it is not continuously maintained. The result is that the pupils can no longer link the formulas and procedures they have learned to situations in practice.

The following example illustrates this situation. Pupils from the sixth grade were asked to determine how many cubic centimeters will fit into a number of large boxes.

The pupils were given access to all sorts of materials, such as the boxes themselves, rulers and blocks of one cubic decimeter. The children worked together in groups, and it turned out that all groups did use the cubic decimeter to measure the box in steps, but none of the pupils had come up with the idea of using the length, width and height of the box to calculate the volume. From the fact that the pupils did not use the volume formula l x w x h, which they certainly knew, we can ascertain that this formula did not have any actual meaning for these pupils. We suspect that this is more widespread problem and is not limited to this class alone.

The alternative approach, which we will work out in this book to deal with these problems, is based on the following basic principles:

1. Give the pupils a firmer foundation by making them aware of the general structure of the prefixes.
2. Ensure that the pupils become familiar with the metric units of length,

so that these units can serve as a basis for perceiving and remembering the relationships between the prefixes.
3. Limit the number of units of area and volume with which the pupils have to work with regularly.
4. Provide meaning to the area and volume formulas by investing a great deal in repeatedly reasoning out the underlying structure of flexible counting procedures.
5. Enable the pupils to perceive how units of length, area and volume are linked to each other in the metric system. And then, using this as a foundation, provide meaning to the conversion process within the metric system by having the pupils investigate the correlations between units instead of having them memorize the sequences of units.
6. Ensure the development of 'expertise in measurement'; for example knowing that the volume of a container of milk is 1 liter and that the height of a standard door is 2 meters.

Other quantities

The basic principles listed above specifically concern education involving length, area and volume. But we certainly want to expand the basic principles to the other quantities that are addressed in primary school.

The system that we describe for length, width and volume can be applied with little modification to the standard units for quantities such as weight, time, temperature and energy. For many of the units, the same – or partly the same – prefixes are used. For example, consider a millisecond or a kilojoule. It is also a beautiful fact that the kilogram is linked to the weight of 1 liter of water (under specific conditions of temperature, pressure, etc.). This can offer support when developing expertise in measurement: 'a one-liter container of milk weighs approximately 1 kg'.

> Strictly speaking, the statement, 'This weighs 1 kg', is not correct. Over the years, the metric system has evolved, and in the 1960s the SI system of units was implemented. In this system, 'weight' has been replaced by the concepts of 'mass' and 'force'. This distinction was made because the same object does not weigh the same under all conditions. On the moon, the same object would be lighter than on the earth. On the earth itself, there are also very small differences. However, the mass always stays the same. The weight of an object is therefore the force of gravity on that object. In this system, mass is expressed in kilograms and force is expressed in newtons. In daily practice, however, the kilogram is still used as a unit of weight, and in primary school we believe that there is no problem with continuing this practice.

Most of the well-known prefixes are not used with the quantity of time. For times shorter than one second, we use millisecond, nanosecond and so forth. But we also use units such as minute, hour, day and week. In this context it is especially important that the pupils become accustomed to this language, for example the language for telling time (digital or otherwise) and the language of dates. In addition, the pupils must learn to deal with the linkage between points in time and time duration.
The units used to measure temperature have also escaped from the system of prefixes. In this book, we will limit ourselves to reading and comparing temperatures. For that matter, a special aspect of temperature concerns the negative values, which require an adequate interpretation of the situation when making comparisons.
Finally, the composite quantities, such as speed and specific gravity, are especially difficult. These units require calculating with proportions of different quantities: time and distance for speed, and weight and volume for specific gravity.
However, the previously described general principles apply to all the above cases:
- measurement emerges as a method for organizing reality;
- measurement entails the development of mathematical tools, which enable you to tackle measurement problems.

Legitimization

The importance of measurement as a component of mathematics education in the primary school is not subject to discussion. However, in the light of disappointing pupil performance, especially for measuring area and volume, it is useful to look at the importance of measurement more specifically. During this process we can make a distinction between the general practical value, the educational value, the preparatory value and the intrinsic value.
Of these aspects, the *general practical value* and *educational value* are perhaps the most obvious. This concerns not only measuring itself, but also dealing with the measurement results. We take measurements in many day-to-day situations, for example while we are cooking and when we are furnishing a house or planning a garden. This gives us an indication of which units and which measurement activities should be given attention in education. This can be extended to include measuring in many vocational situations. For example, the frequent use of millimeters in engineering is a reason to pay attention to this unit in education.
However, at least as important as making measurements is being able to use measuring numbers. Consider aspects such as interpreting weights in grams and kilograms, lengths and distances in meters, centimeters or kilometers, and volume in liters, deciliters or cubic centimeters. The practical value of measurement also includes an understanding of the order of magnitude (expertise about measurements or benchmarks), considering the aspect of precision and working with measurement instruments.

The practical value and the *preparatory value* are very close to each other in many respects. For example, in secondary education pupils must frequently make measurements and deal with measuring numbers. Of course, this concerns not only mathematics, but also physics, chemistry, economics and vocational courses. In addition, measurement plays an indirect role: measurement provides a solid foundation for learning to calculate with whole numbers, decimals and fractions; the other way around, calculating with whole numbers and decimals is useful in measurement and learning how to measure.

The *intrinsic value* of measurement is perhaps less obvious. Nevertheless, pupils may enjoy measurement activities with a strong constructive or problem-solving character. In addition, there is the elegance of the metric

system, which will appeal to some pupils. However, most of the opportunities here appear to lie in experiencing measurement as the mathematical organization of reality. To us, this appears to be the primary way to do justice to the intrinsic value of measurement as a mathematical activity.

Characterization

Measurement in primary school concerns gaining control of reality. In the upper grades of primary school, the education is characterized by the pupils doing investigations, solving problems and reasoning.

Gaining control of reality
From the above, it should be clear that learning measurement is primarily characterized by solving problems and establishing relationships. To this end, the pupils develop a model of reality that is suitable for them. Children come into contact with measurement when they want to compare situations. They experience that you can approach this comparison more precisely if you have access to one or more units of measurement. Before the metric system was developed, these units were strongly linked to the context in which they were used. Following the introduction of the metric system, the situation has changed. We now measure with units that are based on a mathematical system. Moreover, we no longer actually measure in steps with a specific unit. Instead, we use measurement instruments. These instruments are an important aid for gaining control of the world around us. Measurement instruments, such as the ruler and measuring cup, show that measuring involves measuring in steps with a specific unit. With modern digital measurement instruments, pupils must learn that these also concern measuring in steps, but that this is invisible for the user.

Doing investigations, solving problems and reasoning
In the upper grades of primary school, measuring is presented to the children as an area of investigation. This area provides various problems that the pupils can tackle. By imagining the situation and then using this imaginary image to reason about the situation, the problems that have been posed can be tackled step-by-step.
Gaining control of the connections between units is such an area of investigation for children. To investigate units, the children must internalize the basic ingredients they need to build such connections. During this process,

the search for such relationships becomes a real problem for them. For example, they reason how measuring in steps with a unit actually works. The number of times that the unit is used results in a number of units. During this process, the size of the measurement unit determines the precision. When we measure a room in meters, the result is less precise than when we use a millimeter as a measurement unit. The choice of measurement unit depends on the situation. And if the measurement must be very precise, then we must explicitly take account of inaccuracies in the measurement process. These inaccuracies are an integral part of measurement. We can make measurement more accurate by measuring repeatedly and then averaging the results.
Measurement begins with choosing a unit. Measuring length leads to exploring units of length. During this exploration, the decimal structure emerges, and this is expanded by tackling problems involving area and volume. Knowledge of the decimal structure that is acquired in this way is an aid to gaining control of quantities besides length, area and volume. During this process, children use their knowledge of prefixes. However, this does not apply to composite units. These emerge as a proportion between two different quantities. Working with these units requires children to learn to reason proportionally. With the composite quantity of speed, this means that children learn that this concerns the proportion between distance travelled and time. The speed that is subsequently calculated is an average, where it is assumed that the same speed is maintained. If we want to determine the speed of an athlete who runs 100 meters in 10 seconds, then we assume that this sprinter has run the entire distance at the same speed; to calculate the speed in kilometers per hour, we may assume that he maintains this speed for an hour. A ratio table is a suitable mathematical tool that offers children the possibility of gaining control of situations in which speed or other composite units play a role. The use of this calculation algorithm also assumes that children recognize the two quantities in the composite unit, such as 'distance' and 'time' in the composite unit of speed. For pupils, this is not always a simple task.

In measurement education, the pupils continuously go back and forth between gaining control of reality and developing a mathematical tool for measuring. The further investigation of all kinds of recognizable situations leads to gaining further control of the situations, but also to the development and refinement of mathematical tools. Children can then use these new tools to tackle new problems.

Prefixes

Units can be 'customized' by linking them to a prefix. When you go for a long walk and want to indicate how far you have walked, you use kilometers. You use the prefix 'kilo' to make the unit manageable in this situation. Prefixes concern words and meanings, such as 'kilo', which pupils must have as ready knowledge at a certain point. The meanings of prefixes are therefore practiced. This is generally done using units of length, but other quantities can also be used to explore the prefixes.

tera	trillion	1 000 000 000 000
giga	billion	1 000 000 000
mega	million	1 000 000
kilo	thousand	1000
hecto	hundred	100
deca	ten	10
[no prefix]	one	1
deci	one tenth	0.1
centi	one hundredth	0.01
milli	one thousandth	0.001
micro	one millionth	0.000 001
nano	one billionth	0.000 000 001
pico	one trillionth	0.000 000 000 001

Figure 2.2

By regularly paying attention to the meaning of the prefixes, during the upper grades of primary school the pupils learn to understand many of these prefixes in various situations. We have listed the prefixes in figure 2.2, but by doing so we do not mean to suggest that you must also do this in education. The prefixes in this case should emerge gradually, where a complete overview is established only after some time has passed. In order to support such activities, we have made a poster, which can be used with the accompanying lessons (see figure 2.3).

Lesson on prefixes and points of reference
At the front of the class is a table with empty bottles of different sizes. The teacher asks the pupils what volume unit would be used for the various bottles. The units suggested by the pupils are compared with the units on the label. This is done, for example, with a bottle of shampoo. According to the label, the volume is 250 ml. The teacher then discusses with the children why the milliliter has been chosen and which other units could have been used. The pupils also know that the units cl, dl and liter could have been used. For example, 25 cl could have been used, or 0.25 liter.
The teacher takes the discussion in a somewhat different direction. She remarks on how easy it is to switch between units. This ease of conversion has to do with the prefixes 'milli', 'centi' and 'deci' that we put in front of 'liter'. The pupils report that they also encounter these prefixes with other units, such as the centimeter. The teacher uses the poster with prefixes to show the meanings of the prefixes.

In education, the prefixes can be used to construct new units and to determine if it would be convenient to use different units than the ones that are given. This applies, for example, to the units of volume that are derived from the liter (deciliters, centiliters and milliliters). Inquiry situations where these units of volume are compared with cubic units that are calculated with units of length offer many possibilities to acquire points of reference for units of volume and the relationships between them. And by taking measurements of objects using various units, the children also explore the relationships between the various units of volume that are derived from the meter.

The prefixes that are outlined in the middle of figure 2.2 (from 'kilo' through 'milli'), are the prefixes that are addressed in education; the pupils must become well acquainted with them. The prefixes that are used most frequently – especially in combination with units of length – are given the most attention. This concerns units such as the kilometer, meter, centimeter and millimeter. The latter unit is also given a great deal of attention due to its frequent use in engineering and in floor plans.

Figure 2.3 – This poster can be downloaded from www.fi.uu.nl/publicaties/subsets/measurementgeometry/

However, the less commonly used prefixes such as 'mega', 'giga' and 'terra' are also discussed with the pupils, especially in the context in which they are usually encountered: the computer. Until recently the capacity of a computer hard drive was shown in gigabytes, but this unit is already inadequate for indicating the capacity. People now refer to a hard drive with a storage capacity expressed in terabytes, and a new prefix is introduced: the ‘tera’, which stands for trillion. There is some confusion here, however, as these terms originally referred to powers of 2. A megabyte is still used sometimes for 1024 x 1024, or 1,048,576 bytes, but it is now officially defined as a million bytes of computer data. The SI-term for 1024^2 is a ‘mebibyte’. In the same way we now have ‘gigabyte’ and ‘gibibyte’, ‘terabyte’ and ‘tebibyte’.

The less frequently used prefixes, as stated above, will only be briefly touched upon. This also applies to the prefixes ‘micro’, ‘nano’ and ‘pico’. Conveniently, these prefixes also appear in the context of the computer. The dimensions of the individual components on computer chips (transistors) are generally specified in nanometers. For example, such a transistor is 90 nanometers in size. The chip manufacturers invest a great deal in making transistors smaller and smaller, for example so that it is possible to store more information on a chip.
Searching for examples of situations in which the prefixes ‘micro’, ‘nano’ and ‘pico’ are used can help the pupils to create an image of the situation. With the prefixes ‘milli’, ‘centi’, …, ‘kilo’, in time the pupils are expected to know the meaning.

Length as the basis

During the lower grades of primary schools, the pupils become familiar with the centimeter and meter as units of length. Measurement results are often formulated in expressions such as ‘1 meter 67’. In this way, the pupils become acquainted with the conversion rule: ‘100 centimeters is equal to 1 meter’. Later on, they learn that 1 km is the same as 1000 m. During this process, the necessary attention is paid to developing expertise in measurement. For example, the kilometer is linked to the length of a specific street in the immediate surroundings. A meter can be related to the height of a door, since most doors are approximately 2 m tall. Useful examples for centimeters can also be found.
Millimeters are introduced as a solution to a problem: what can you do if a centimeter is not precise enough? In the 15th century, the problem of making smaller units of length was tackled by Stevin. He proposed making units smaller by a factor of 10, and in the process actually invented the

decimal fractions. In the TAL publication *‘Fractions, Percentages, Decimals and Proportions’* we describe a number of lessons in which we make the children aware of the importance of this discovery. We essentially have them reinvent the process of decimal refinement. To compensate for the fact that most pupils are already somewhat familiar with the processes for refining units that are available in the metric system, we put these lessons into an historical context. It then turns out that choosing a suitable smaller unit is experienced as a real problem by the pupils. We therefore link the process of making units smaller to the fractions tenths, hundredths and thousandths, to establish an insightful basis for the decimal fractions. In this way, we can also provide meaning to the system of metric units of length.
If the pupils are already reasonably familiar with the various units of length, we can still bring up the underlying system for discussion. Why did people decide to make the centimeter smaller by tenths of a centimeter? It is possible to think of various reasons for this:

- You do not have to calculate with fractions, and you calculate in almost the same way as with whole numbers.
- You can use the same prefixes for many different magnitudes.
- In principle, you can continue to make the unit smaller and smaller, and in this way choose a suitable level of precision.

At the same time that they explored the usefulness of the millimeter, the pupils have also become thoroughly familiar with the relationship between centimeters and millimeters. In a similar fashion, the relationships between other units can also be explored. For example, if you drive past ten plus one road markers that are spaced one hectometer apart, you have travelled one kilometer, which is also the same as 1000 meters.
Gradually, the less commonly used units are addressed and the gaps in the system are filled. Although the hectometer, decameter and decimeter are not units that the pupils will use very frequently, the education focuses on learning to recognize the relationships between units of length in the system. This is because the units of length can serve as a handhold when making generalizations about the prefixes, ‘kilo’, ‘hecto’, ‘deca’, ‘deci’, ‘centi’ and ‘milli’. Therefore it is important that pupils become familiar with the various units of length and can easily deal with the relationships between them. This knowledge and skill can in turn be used to derive relationships between units of area and relationships between units of volume.

Area

The PPON survey in the Netherlands (Periodical survey of educational level) that we referred to in the introduction, has shown that measurement education is currently inadequate, especially regarding area and volume. Pupils are expected to master an extensive system of units of area and volume, but they appear to have serious difficulty with applying this knowledge. As we stated previously, we therefore believe it is preferable to make the pupils familiar with a number of units and applications that are relevant to daily life, and to place the emphasis on developing concepts, rules and procedures. The point of departure of our approach is to explicitly address the clever discoveries that are hidden within the system of units of area and the procedures for calculating area.

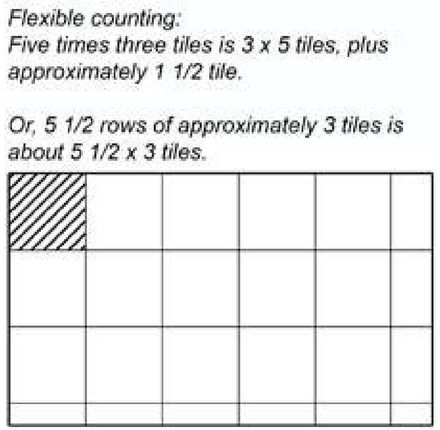


Figure 2.4

One of these primary discoveries is the idea that you can compare areas by measuring in steps with a suitable unit. For this purpose, it is preferable to choose a unit that you can count easily. For example, these units could be triangles, squares or rectangles. Squares work very well to measure rectangular areas. And then you can use flexible counting by taking advantage of the regularity: each row contains the same number of quares. You can then determine the total number of squares by multiplying the number of rows by the number of squares in each row.

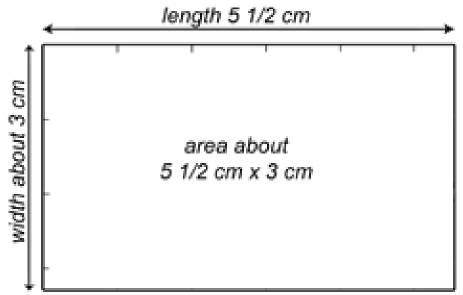


Figure 2.5

By introducing the square centimeter as a unit of area, the multiplication can be linked directly to the length and width. This is because the number

of 1 cm^2 squares that fit on one side of a rectangle is exactly equal to the length of the side in cm. Consequently, we can directly multiply the number of centimeters of the length and the number of centimeters of the width in order to obtain the total number of square centimeters (see figure 2.6).
In this way, the process of determining the area of a rectangle boils down to using the formula 'area is length times width'.

In education, these elements are generally addressed, but often much too quickly. In many cases, the pupils remember only the formula. If pupils can only reproduce the calculation rule, it becomes difficult for them to remember what this rule is based on. In that case, one must essentially 'reason in reverse'. The centimeters that are measured in steps on the sides of the rectangle must be seen as the sides of squares. You can draw these squares in your imagination (see figure 2.6).

You can easily count the number of 1 x 1 centimeter squares by seeing them as 'l' rows of 'w' squares. Therefore l x w squares of 1 x 1 centimeter, or l x w cm^2.
This is difficult reasoning, and most pupils do not arrive at this method by themselves. In addition, they quickly forget how the formula was devised. As a result, area calculations quickly become a trick which they do not understand. To prevent this, we must ensure that the area formula is thoroughly embedded in inquiry activities and reasoning, and is not supported only by a brief experience where the pupils 'see how it is'.

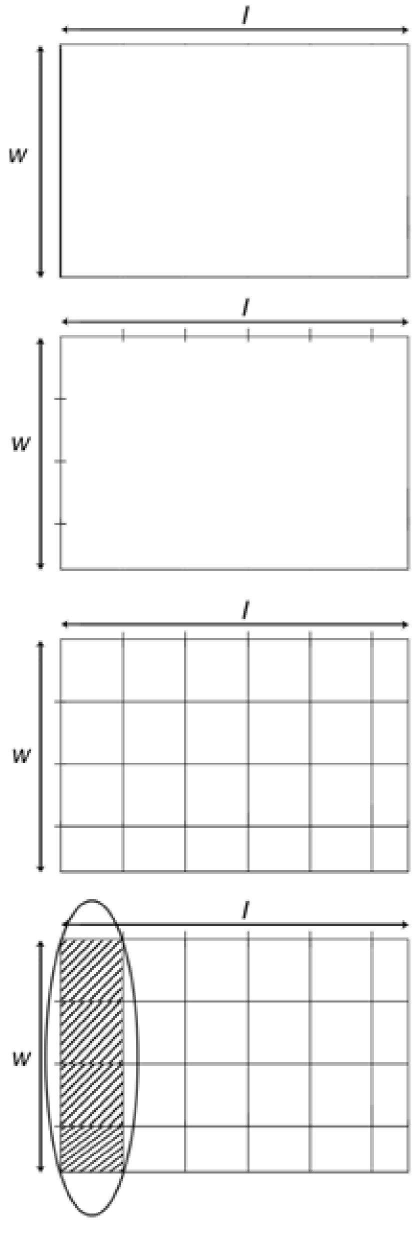


Figure 2.6

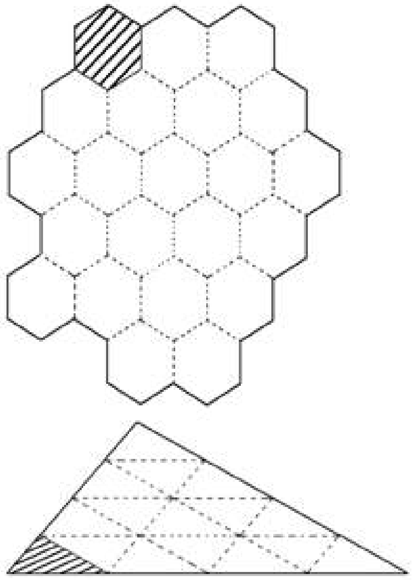


Figure 2.7

However, the problem with finding suitable inquiry activities is that the multiplication rule is very simple, and that counting squares inside rectangles provides little variation. In other words, this offers little challenge. The alternative is to pay more attention to figures besides rectangles which must be measured with other shapes. In this context, think of figures with angled or crooked sides, where triangular tiles, hexagons or rectangles are used as the measurement unit (see figure 2.7). As a result, the assignments have more the character of solving a problem. Because you can multiply in some cases and in other cases you cannot, the attention is focused on the formula itself: why do we use multiplication at all? This is important because multiplication in the rectangular model is not entirely self-evident for many pupils. The lesson report in the box 'Deca is ten' illustrates this situation.

When the pupils do not understand really the rectangle model, it also becomes difficult for them to understand the relationships between various units of area. Most of the pupils do not realize that inserting (or removing) zeros is based on flexibly counting smaller squares within a larger square. As a result, they lack the means to correct their misconceptions or errors. Here as well, a problem-oriented approach can offer a solution. For example, questions such as 'How many cm^2 go into a m^2?' can be presented as a problem several times during the year. And even if several pupils immediately call out that it is 100 times 100, the teacher may continue to ask questions, going back to the level of measuring in steps and flexible counting. Over time, the pupils will certainly memorize relationships such as 1 m^2 is equal to 10000 cm^2. Some of the pupils may realize that there is a system with 'repeated steps of 100 times' in the units of area. This is a wonderful realization, but we cannot expect this of all the pupils. Moreover, we must constantly keep an eye out for knowledge that is based on overly superficial generalizations.

Deca is ten

With their teacher, the pupils look at the TAL poster on prefixes. The prefixes 'milli', 'centi' and 'deci' have already been discussed. The less familiar prefix 'deca' has not yet come up for discussion. The pupils read the meaning of the prefixes on the poster. 'Deca' is 'ten' and therefore a decameter is a length of 10 meters, and a decaliter is a volume of 10 liters. Deca even gives meaning to a decasecond: this is 10 seconds. The meaning of deca helps to determine how long a decameter is.

Using the meaning of 'deca' as 'ten', a square decameter is considered. The pupils know that you can imagine this as an area which is 10 m by 10 m. This leads to the question: how many square meters fit into such a square decameter? This question briefly puzzles the pupils. Drawing and systematically counting units in the square decameter provides support while investigating this situation. The picture ultimately makes it clear that the answer is 100 m²

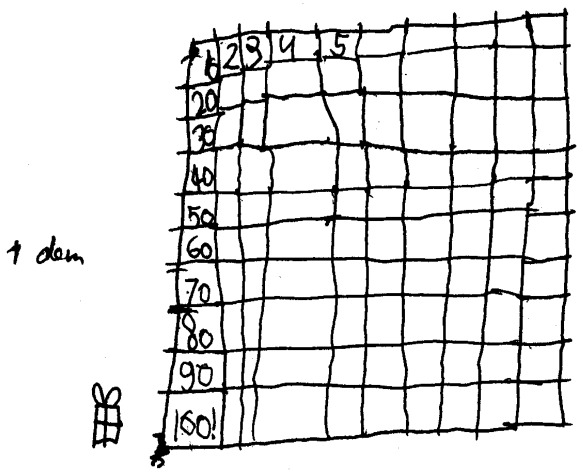


Figure 2.8

Despite this emphasis on meaning and the repeated derivation of flexible, abbreviated approaches, some of the pupils will start performing tricks with formulas and using them as meaningless calculation rules. Generally speaking, the teacher will then have to work with them so they can understand how to determine the area by flexible counting, so they can figure out why a known formula is or is not compatible with the given situation.
However, there are other ways to bring up these types of issues for discussion. This can be done by creating a conflict, for example by explaining that one pupil has calculated the area of a table by multiplying the length in meters by the width in centimeters. You can then ask the question, what does this result actually represent? The aim of this activity is to show the pupils that this result in fact indicates how often a rectangle measuring 1 meter x 1 centimeter fits onto the table. By transforming its shape, the pupils can also see this unit as a square decimeter.

Mary's mistake

'Mary measures her table. The length of the table is 2 m and the width is 52 cm. She concludes that the area of her table is 104.'

When this problem is addressed in fifth grade, one of the pupils immediately calls out that it must be square decimeters. Other pupils primarily see that a different approach has been chosen than the one they are familiar with.

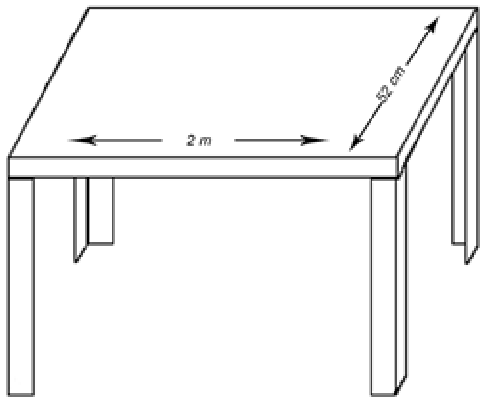


Figure 2.9

Translation: "I think she didn't understand what to do, just 2 x 52.
You have to do it like cm-cm, this is meter-cm."

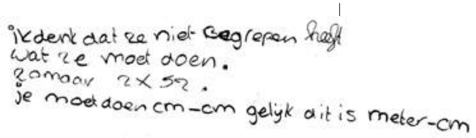


Figure 2.10

Other pupils try a different way to determine what Mary's approach could have been. This motivates the pupils to jointly explore the problem.

Translation:
"She added. She doubled 52 cm. It's all wrong."

"The table should be like this."

"She didn't do the length. She just added two lines of 52 cm."

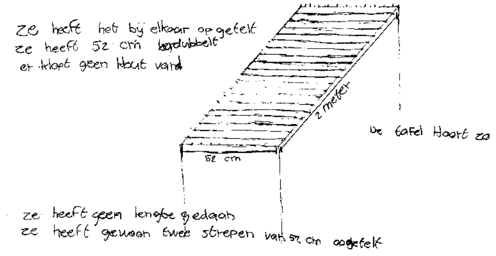


Figure 2.11

The teacher asks the pupils to explain what they think Mary has done. At the same time, he reveals that what she did was perhaps not entirely incorrect, but is simply different than what we are accustomed to.

This puts the pupils on a different track, where they – encouraged by the teacher – work out what the process of measuring the area actually involves. This concerns measuring an object in steps (using flexible shortcuts, if possible) with a chosen unit. Based on this approach, the teacher reformulates the question: 'Which unit could Mary have had in mind?' As an intermediate question, the teacher asks the pupils to think up various contexts where 2 x 52 would apply. The following contexts emerge:

- a parcel of land measuring 2 meters by 52 meters;
- two bags with 52 marbles in each one;
- …..

The teacher looks at these examples of contexts with the pupils. This always involves taking or visualizing two sets of 52 pieces. Which two sets could Mary have used in her approach?
This turns out to be a difficult question. One of the pupils thinks that she used the two-meter pieces. Another pupil adds: 'She actually used pieces of one meter by one centimeter.' The teacher draws a piece this size on the blackboard and then works out with the pupils whether this piece can be stepped off exactly 2 x 52 times. Everyone quickly agrees that this is the case. Mary had calculated that the area of her table was equal to 104 of those 1 m x 1 cm pieces.

Circumference and area

Circumference is essentially a simple concept, but when circumference and area are linked too quickly to the formulas that apply to the rectangle, children often go down the wrong track. In that case, the formulas 'circumference is length plus width, times 2' and 'area is length times width' soon become confused with each other. This problem can be prevented fairly easily by postponing the introduction of the formulas.
However, it is more difficult to deal with the misconception that there is a fixed relationship between the circumference and area of a figure. Pupils often tend to think that two figures with the same area also have the same circumference. It becomes even more interesting when we enlarge figures; in that case it is not at all self-evident that the area increases much more quickly than the circumference. If we look very carefully, we even see a relationship between the growth of the circumference and that of the area. When all of the sides of the figure are doubled in length, then the circumference also doubles, and the area becomes four times as large. And if all sides are tripled in length, then the circumference also becomes three times as large, but the area becomes nine times larger. We refer to this effect as quadratic enlargement, because the enlargement factor of the area is the square of the enlargement factor of the circumference (see figure 2.12).

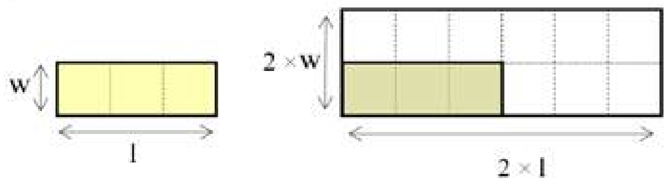


Figure 2.12

When determining circumference, children observe that this generally does not involve flexible counting or a formula. The circumference can be determined by counting or by using a ruler to measure all lines on the edge of the figure. Figure 2.13 demonstrates that circumference is not at all dependent on the area. The two grey figures both have an area of 10 tiles, while the circumference of the figure on the left is 22 linear units and that of the figure is only 14 linear units.

When the emphasis in education is placed on formulas for the area and circumference of a rectangle, pupils will perhaps start to think that a specific area has only one circumference. However, we want children to experience the fact that area and circumference are very different things. To aid in this process, assignments can be used where the children are asked to make a figure with a given circumference and area. This can also be done by asking them to think of figures which have the same circumference, but different areas.

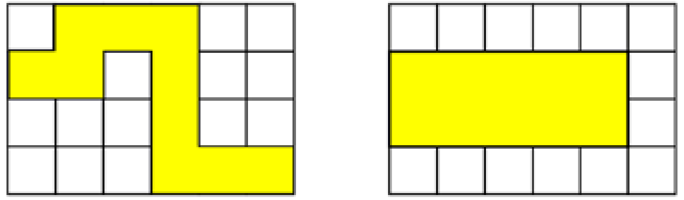


Figure 2.13

However, children should also be given the opportunity to investigate and learn standard approaches for determining area. Children start working with area in the lower grades of primary school. As early as second grade, the rectangle model is introduced when teaching the multiplication tables. This is the first time that determining area is addressed, albeit implicitly, as flexibly counting tiles. This process continues during the upper grades of primary school. We previously showed how pupils discover that it is agood idea to choose a suitable unit of area for measuring in steps. In many cases, a square tile is a suitable choice. This leads, for example, to flexible counting of the tiles in a rectangle, which in turn leads to the area formula for a rectangle. We previously indicated that a rapid introduction of this formula can lead to problems, and that working with other figures increases the children's understanding of how the formula operates. Here as well, there is often a reason to use flexible counting. Reflecting on flexible counting with a unit of area also helps the children to determine the area of other well-known figures, such as a parallelogram, a triangle and a trapezoid (see figure 2.16).

Computer programs for enlarging figures

If you make a square four times as tall and four times as wide, you make the area 16 times larger. With the computer program *Vergroten* (www.rekenweb.nl) pupils can discover that this rule also applies to rectangles with unequal sides (see figure 2.14). One of the tasks in the program is to make correct enlargements of 'carnival mirror photographs'. Measuring in steps with small rectangles that are exactly the same size as the original photograph can help pupils accomplish this task.

Figure 2.14

In the Gulliver computer assignments ('Gullivers travels', see figure 2.15), everything is based on the fact that Gulliver is 12 times larger than the Lilliputters. When the tailor wants to make a towel for him, he needs to sew together 12 x 12 Lilliput towels, or 144 towels. But to the astonishment of many pupils, this rule also applies to triangular towels.

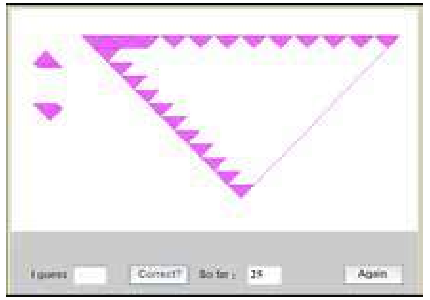
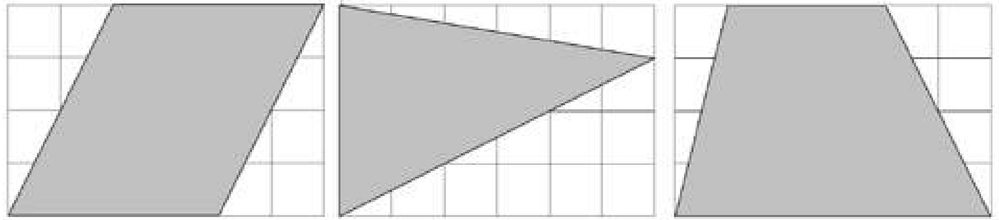


Figure 2.15

Figure 2.16



For example, the parallelogram allows itself to be easily transformed into a figure that contains only whole tiles. By shifting one-fourth of a tile four times, 16 whole tiles are created (see figure 2.17). We can determine this by counting. However, we can also quickly see this by shifting the tile at the lower left and that at the upper right to the adjacent columns. At once you see a rectangle of four tiles by four tiles.

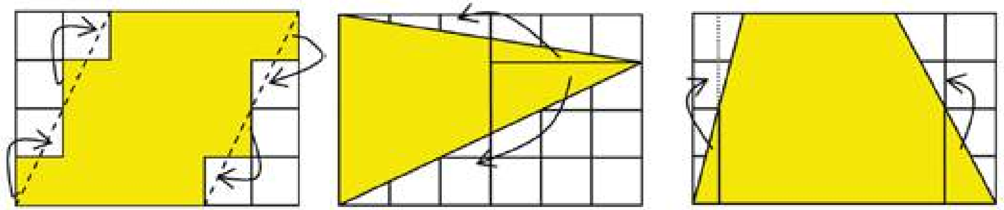


Figure 2.17

The parallelogram can also be transformed into a rectangle in another way. By drawing a vertical line from the right lower corner a triangle is created which fits exactly into the notch on the left side.
The other figures also allow themselves to be transformed into rectangles in a relatively simple fashion. We can cut up the triangle into a rectangle of 3 x 4 units and we can make the trapezoid into a rectangle measuring 4½ x 4 units. Obviously, transformation is not the only productive approach. For example, many children quickly see that the triangle is exactly half the area of the underlying rectangle. The parallelogram is created by removing eight tiles from the underlying rectangle, and with the trapezoid six tiles must be left out. Figure 2.17 shows how.

Flexible counting approaches to determine area also work with irregular figures. The strategy is then to repeatedly cut up the figure or its complement in such a way that whole tiles or other sub-figures are created for which the area can be easily determined. For example, this can be done when a structure of tiles lies over a map, where the dimensions of the tiles are known. In the next box, we show how this can be done when determining the area of the island of Texel.

The area of Texel

When determining the area of Texel we can use the map on the left. The tiles on this map grid are 5 x 5 km. The area of one tile is therefore 25 km². We combine parts of tiles to form whole tiles, and in this way we arrive at approximately 6 whole tiles for the entire island, or approximately 150 km².

Figure 2.18

If we have to use a map without a grid, then we can more or less design a grid ourselves. With a computer program like *Google Earth*, you can determine the distance from the southernmost point of the island to its northernmost point. This is 22 km. If we cross the island at its widest point, we travel 8 km. However, this is not enough to approximate the area of Texel. To accomplish this, we must therefore imagine that we can transform the island into a rectangle. We choose a rectangle that is 22 km x 7 km; the area that we remove on the side can be used to fill in the missing parts above and below. The area is therefore approximately 150 km².

Volume

As with units of area, we also focus on constructing units of volume. We therefore use the knowledge of connections between units of length to construct and derive relationships. With these units, we do not drill relationships, but focus much more on mastering the construction process. This construction (or reconstruction) must therefore be periodically practiced, and in different situations.

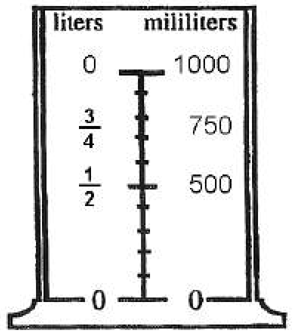


Figure 2.19

The results of the periodic assessment of education have shown that volume is a very difficult topic. This is perhaps because the volume in cubic units such as cm^3 and dm^3 is usually determined by making a calculation with the formula length x width x height. The actual measurement then concerns linear units of length, width and height. When we want to discuss the concept of volume, then we should obviously begin the exploration of volume with units that are derived from the liter. With units of liquid measure (or liquid capacity) such as the liter and centiliter, measuring cups and similar tools are used. In this situation, pupils must develop the skill of reading a scale (see figure 2.19).

An interesting aspect here is that there are often two units next to a scale line, such as liters and milliliters. Reading scale lines therefore offers an excellent context for relating various units to each other and for developing number relationships. On the measuring cup, for example, you can see that ¾ liter is the same as 750 milliliter. Moreover, reading scale lines is a skill that is used in many other contexts. For example, consider gauge glasses, petrol gauges, weighing scales and speedometers. For that matter, it should be noted that scale lines with pointers are being increasingly replaced by digital displays.

The gas and water meters that we encounter at home are a special case. The measured quantity of gas or water in this case is expressed in m^3. An interesting aspect here is that a cubic meter in this context manifests itself as the volume of a round pipe. The water meter actually determines how far the water has been pushed through the water line and uses cubic meters to show the corresponding quantity of water that has moved. This is an excellent context to make the pupils aware of the fact that a cubic meter does not necessarily have the shape of a cube.

However, we often deliberately choose a cube shape, for example when we want to have pupils investigate how many cubic centimeters fit into a cubic meter. They can construct this connection themselves by beginning with the meaning of 'cubic meter' and 'cubic centimeter'. With a cubic meter, you can assume there is a cube that is one meter in length, width and height, which expresses the concept of a 'cubic meter'. Analogously, a cubic centimeter can be seen as a cube which is one centimeter in length, width and height. Pupils can use these concepts to determine how many cubes of 1 x 1 x 1 centimeter fit into a cube of 1 x 1 x 1 meter. To this end, the pupil has to see that the bottom of the cubic meter contains 100 rows of 100 blocks of one cubic centimeter each, or 10,000 cubic centimeters. In the same way as with area, the lengths of the sides are used for flexible counting – in this case counting the blocks. Finally, the pupils must discover that 100 of such layers fit into a cube of one cubic meter. This results in 1,000,000 cm3 in 1 m^3.

In time, some of the pupils will develop ready knowledge of such relationships, as they did with the units of length. Other pupils – based on the units of length – will have to (and should be able to) reconstruct these relationships. In other words, they can derive the relationship again when they need it. To do this, the pupils must have acquired experience with investigating connections. Constructing such connections forms the basis for reconstructing them. After all, what we ultimately want to achieve is for the pupils to not be empty-handed when they can no longer remember what the relationship is between two units of volume.

The important thing is for the pupils to realize that finding the relation between two units is a matter of measuring in steps with one unit on top of the other. They should also know that the prefixes provide the necessary information, and that there are flexible ways to determine how many times one unit fits on top of another. After some time passes, many pupils will perceive that multiplying the length, width and height of all bars and blocks provides the volume. Investigating the relationships between units is in fact a component of calculating the volume of rectangular figures. In both cases

it is important that the pupils do not make the step to a formula or rule too quickly, but that they reason out the situation over and over.
Here we encounter the same problem as we did with area. Not much variation is possible if you limit yourself to determining the volume of bars and cubes within the metric system. Consequently, we again resort to non-standard units; learning to derive the relationships is benefited by also doing this in situations where there are no standard units or shapes that fit exactly. We provide several examples of this strategy below.

Soft drink

A large waste container in the shape of a soft drink can is shown (see figure 2.20). The question that accompanies this image can is: how many small soft drink cans can you fit into the large container? During this process you can use the small cans as a measurement unit, for example by determining how many times the underside of the small can fits into the lower surface (or upper surface) of the waste container. This tells you how many small cans you need to fill the large can to the height of one small can. You can then determine how many of these layers you need to fill the waste container by counting how many times the height of the small can fits onto the side of the large container. One alternative is, of course, to use the enlargement factor. We will return to this later.

Figure 2.20

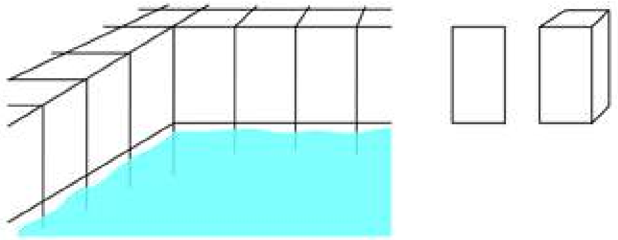


Figure 2.21

Swimming pools

When we want to compare the volume of two swimming pools, without having access to standard units, we can use the tiles along the side of the pool. We can use the tiles to design a unit. For example, a rectangular tile leads to the measurement unit that is shown in the illustration. This unit can be used to compare the volume of two swimming pools, if both swimming pools have the same tiles.

Volume formula

Developing the formula for calculating the volume of a block-shaped figure, volume = length x width x height, goes hand-in-hand with investigating the connections between different units of volume. Ultimately, you can reason out the volume formula by constructing a measurement unit in your imagination.

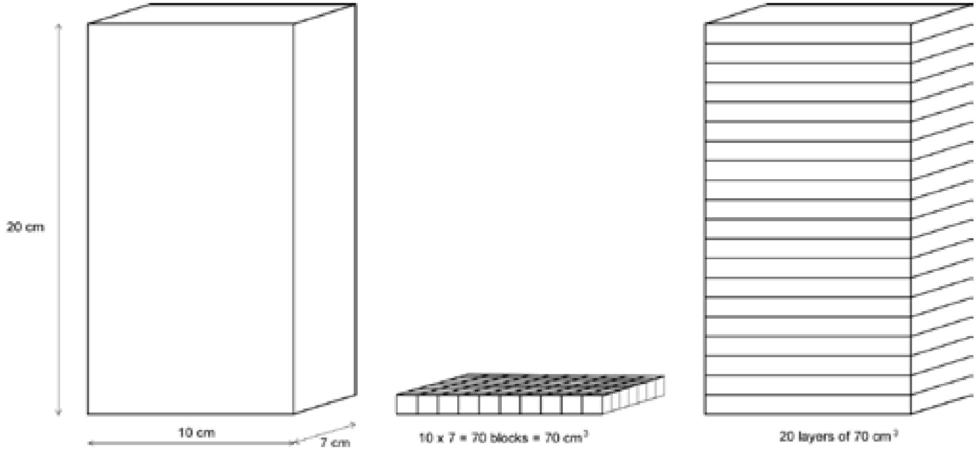


Figure 2.22

For example, if a box is 20 centimeters high, 10 centimeters long and seven centimeters wide, then you can ask yourself: how many cubic centimeter blocks will fit into this box? And then you can reason out that 10 x 7 x 20 of the blocks will fit into the box. To this end, you first determine how many blocks fit onto on the bottom, where you create a rectangular layer which is 10 cm long, 7 cm wide and 1 cm high; this layer therefore contains 10 x 7 blocks (see figure 2.22).
You can then use this layer of 70 blocks as a new unit, and you can determine how many of these layers will fit into the box. Because the box is 27 cm high, 20 layers of 70 cm³ each will fit into the box, which is a total of

1400 cm^3.
Here as well, a single experience with explanation or reasoning is generally insufficient for the pupils to remember and understand the volume formula. This is why we recommend using other examples besides blocks and cubes.

Liters and cubic units

In addition to the system of units of volume described extensively above, which is based on cubic units, there is a 'competitive' measurement unit that is derived from the liter. We introduced this system previously. In this system pupils encounter units such as liter, deciliter, centiliter and milliliter, especially on packaging for drinks and other foods. In this context, a distinction is also made between the volume of solids and the liquid capacity, but as we stated previously, we do not see any principle distinction between these systems and will therefore use the terms in parallel.
Pitfalls are encountered especially when relating these two systems. These pitfalls become apparent by means of the previously described prefixes. These prefixes are used for cubic units, because they are derived from the meter, and they are also used with liter-based units. The following example shows where the pitfalls lie. One centimeter is 100th of a meter. But this same proportion does not apply to the cubic centimeter. You could think that a cubic centimeter is 100th of a cubic meter. But with a cubic centimeter, we must conceive of a 1 centimeter linear unit that is placed in three dimensions to create unit of volume. One-hundredth of a cubic meter could therefore be a centi-cubic meter.
One aspect that plays a role here is that the prefixes used with liters have a different function than those used with cubic units. A centiliter is one-hundredth of a liter, but a cubic centimeter is one millionth of a cubic meter. The pupils must learn to keep these systems separate.
With the units of volume that are derived from the liter, the prefixes behave in the same way as with the units of length: one centiliter is indeed on-hundredth of a liter. This also explains why the liter is still being used in addition to the cubic decimeter and the cubic centimeter. The cubic decimeter and the cubic centimeter are separated by a factor of 1000; the liter units neatly fill this gap with the deciliter and the centiliter, which are flanked by the kiloliter, equal to a cubic meter, on one side, and the milliliter, equal to a cubic centimeter, on the other.

By working with the meanings of prefixes while investigating units of volume, the children can gain control of these concepts. However, it is important to repeatedly return to this stage.

For that matter, the liter harks back to an historical distinction between the 'volume' of solid substances and the liquid 'capacity'. Even today, it is still customary to use liters in the context of liquids and then to think about capacity. And to use units such as cubic centimeters for calculating the volume of a solid object like a timber beam. Strictly speaking, we do not have to make this distinction. Nevertheless, the deciliter and centiliter are useful intermediate units to supplement the system of cubic units.

A puddle of milk on the ground
The teacher explains what happened to her. She dropped a carton of milk, which leaked and made a big puddle on the ground. There was so much milk on the ground that she thought not much could be left in the one-liter carton. She links this situation to the central question: how can you see whether the puddle on the ground contains a whole liter of milk or less than this?
The children begin investigating the problem with the teacher. A liter of liquid is poured on a smooth floor in the school. The children measure the puddle of liquid. They use a ruler to measure the height of the puddle. They lay A4 sheets of paper on top of the puddle to measure the area.
Back in class, they examine the measurement results. The puddle is 3 mm in height, and approximately 5 A4 sheets of paper fit on top of the puddle. But, the teacher asks, is a puddle which is 3 mm high and 5 A4 sheets in area actually a liter?
The pupils previously learned that a liter fits exactly into a cubic decimeter. This provides a way to reformulate the question. Can we 'fold up' the puddle in such a way that it fits into the cubic decimeter?
This is not a simple question, and therefore the teacher proposes to see how many squares of 1 decimeter x 1 decimeter fit onto an A4 sheet. Approximately six squares fit onto the A4. Several pupils realize that you can therefore see the puddle as 30 pieces measuring 10 cm x 10 cm x 3 mm. One of the pupils stacks up these blocks in his imagination and then calls out: 'Then you would have a stack that is 9 cm tall, which is not a liter.' Using the cubic decimeter cube, the pupil shows what he means, and

then the other pupils also see that the puddle would fit approximately into the liter container.

Enlarging and reducing

We previously showed that when the sides of a rectangle are doubled in length, the area of the rectangle becomes four times larger. When the lengths are enlarged by a certain factor, the area is increased by the square of this enlargement factor. We refer to this as quadratic enlargement. We see also examples of quadratic enlargement in the relationships between units of area. If we enlarge a square centimeter to become a square decimeter, then we increase the lengths by a factor of 10. The result is that the area increases by a factor of 100, as we show in figure 2.23. The same illustration also shows that this principle applies in reverse. If we reduce all lengths by a factor of 10, the area is reduced by a factor of 100.

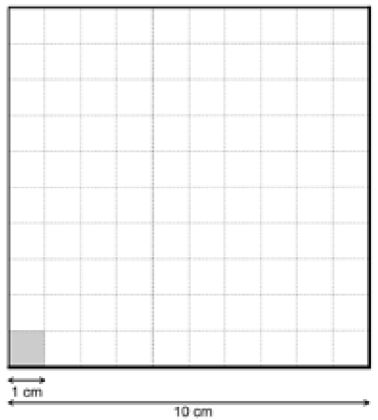


Figure 2.23

This property is easy to see with rectangular figures. However, quadratic enlargement or reduction also applies to more irregular figures. You can imagine these figures as being constructed of small squares, the area of which is quadratic enlarged or reduced.

Also, if the lengths are enlarged or reduced on an object, and the volume is determined, something special takes place with respect to the enlargement or reduction of the volume. This can be clearly seen when we compare a cube of one cubic centimeter with a cube of one cubic decimeter. The lengths are enlarged by a factor of 10, and the volume becomes 1000 times greater. We refer to this as cubic enlargement. Here as well, this concerns a phenomenon that applies both to rectangular figures and figures with a non-rectangular structure. For example, does this also apply to the situation with the small can and large can in figure 2.20 ? If the lengths of the large can are 4 times as large has those of the small can, then the volume of the large can is 4 x 4 x 4 = 64 times as large.

Meaning and relationships

We previously discussed the didactic consequences of the idea of rediscovering the inventions that are hidden in our system of units of length, area and volume. In this section, we will discuss the nature of the knowledge that we want to develop in this way. Besides tackling application problems and reasoning out relationships between units of measurement, attention must also be focused on the development of expertise in measurement. This means that the pupils can link the standard units to situations in reality. We previously referred to an example of such familiarity with benchmarks: ‘knowing that a standard door is approximately 2 m tall’. We can add other examples, such as knowing that a 1 liter carton of milk weighs approximately 1 kg (and also knowing that this does not apply to every liquid!), or knowing that a football field has an area of one-half hectare. This measurement expertise is important not only for the meaning and applicability of measurement, but it also offers additional anchor points to perceive and remember relationships. Ultimately, it is the aim for every pupil to develop a network of knowledge, relationships and insights that provides a handhold when reasoning about reality in quantitative terms.

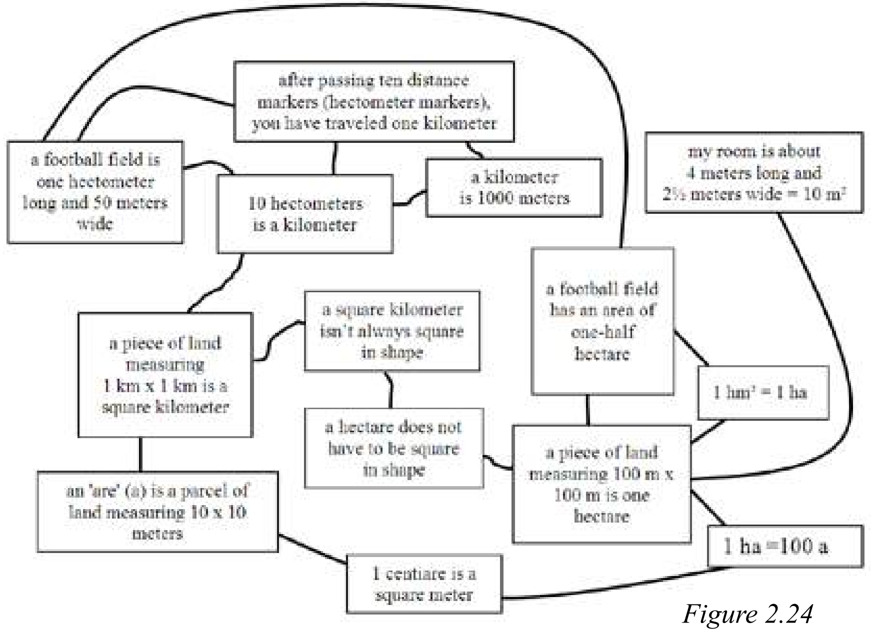


Figure 2.24

By means of various inquiry activities, class discussions and other experiences, every pupil gradually completes his or her network of meanings, benchmarks and relationships between units. One piece of a network of relationships concerning length and area could be similar to that shown in figure 2.24. However, we must emphasize that this concerns only a very small part of the network. In this example, we see among other things, relationships between the kilometer, hectometer, square kilometer and hectare (square hectometer), and insight into the meaning of the prefixes and the accompanying equalities such as 1 hm^2 = 1 ha. A square hectometer – a parcel of land measuring 100 x 100 meter – is called a hectare. The 'useful fact' that 1 hectare is equal to a 1 x 1 hectometer square then provides a good basis to determine the size of an 'are' and a 'centi-are'. A hectare is 10000 square meters, and an 'are' is 100 times smaller, so it is equal to 100 square meters, therefore it is a parcel of land measuring 10 x 10 m. The centi-are is one-hundredth of an 'are'. A centi-are is therefore one-hundredth of 100 square meters, or 1 square meter.
For volume, we can also sketch out a small part of the network (see figure 2.25). In this network, we not only see benchmarks that are linked to units derived from the liter, but also to the units of volume that are derived from the meter, such as the cubic centimeter and cubic decimeter.

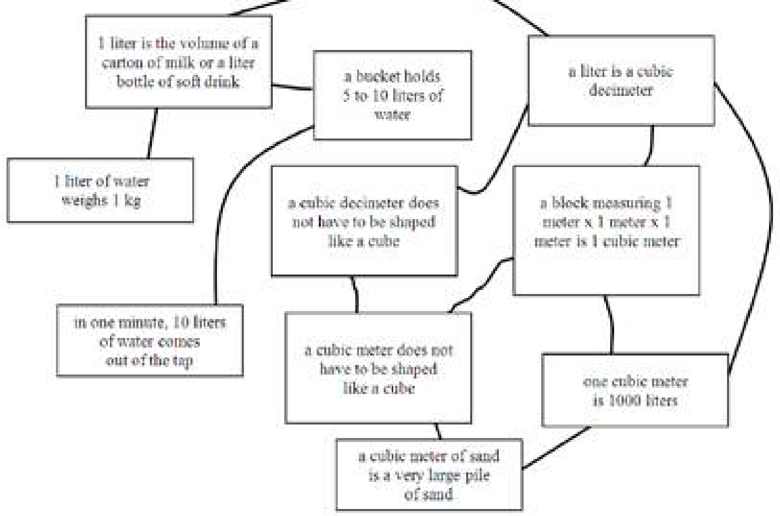


Figure 2.25

Weight, time, speed and other quantities

In the above sections we paid a great deal of attention to length, area and volume. Of course, there are more quantities that are addressed during the upper grades of primary school. The most important of these are weight, time, temperature and speed.

Weight

The quantity of weight fits beautifully into the metric system. The standard units are entirely proportional, and the prefixes are identical to those for units of length. The units for weight are created by combining the familiar prefixes with 'gram'. In this way, the pupils in the upper grades of primary school can reason that hectogram is another word for 100 grams, because 'hecto' stands for '100'. Let us briefly review what went before.
In the lower grades of primary school, children became acquainted with weight as a quantity by comparing the sensation of various weights. During first and second grade, a unit for weight was introduced. This was done, for example, by means of a spring scale that the pupils made themselves. The children experienced a relationship between the length (extending the spring on the scale) and the weight. In addition, the first standard weights were introduced, where the children came into contact with the standard unit of the 'gram'.
During these first measurements, children experienced the fact that weighing can yield different results. In the upper grades of primary school, the accuracy of measurement is discussed with the children. By weighing repeatedly, where the measurement results are then averaged, the children experience that the measurement can be made more accurate.
Measuring with units of weight, where a weighing scale is used, leads to exploring the relationship between the units 'gram' and 'kilogram', because both kilograms and grams are shown on the scale. In addition, the children acquire benchmarks for the units of weight. For example, virtually all children in third grade understand that you can think of a package of sugar when you want to understand how much a kilogram is. Moreover, outside of school children often become acquainted with older units such as the pound, and they become familiar with the kilogram via the abbreviation 'kilo'.
During the upper grades of primary school, children learn that the familiar

prefixes can also be used for the units of weight. By calculating and reasoning in this way with relationships between the units of weight, the children expand their benchmarks to include weights.

However, weight actually concerns the force that the earth exerts on a certain mass. Because this force – under normal conditions – is always the same, mass can be replaced by weight in almost all situations. This changes when we weigh very precisely. It then turns out that the weight of an object is related to the location on the earth where the measurement is made. The weight also changes completely if we move an object to the moon. During this process, the mass stays the same, but the weight becomes less. The smaller moon exerts less force on the object than the earth does. This distinction between mass and weight is discussed with children when events provide a reason for this, for example when a pupil describes being on a roller coaster, where you are virtually weightless for a few seconds. At that point, pupils may ask questions about what weight actually is and in which situations your weight becomes greater or smaller. The children can experiment with this, for example by standing on a scale while riding on a lift. Of course, they can think beforehand about what happens with weight.

There is yet another strange aspect about weight. In contrast to what many children think as a result of their experiences with water (and other liquids which are primarily composed of water), volume is something totally different than weight. Children can experience this directly by comparing the weight of a liter of water with the weight of a cubic decimeter of brick. The brick is much heavier.

Such experiences with comparing objects with differing weights, calls for quantification. This can be done by using a new quantity: specific gravity. A brick has a specific gravity of about 1500 grams per liter. This means that a brick of one cubic decimeter weighs 1500 grams. The specific gravity of Styrofoam, for example, is much lower, approximately 35 grams per liter. Specific gravity is a so-called composite unit. It is a proportion number that indicates the proportion between weight and volume. This quantity is also discussed occasionally in education, when events give a reason to do so.

Time

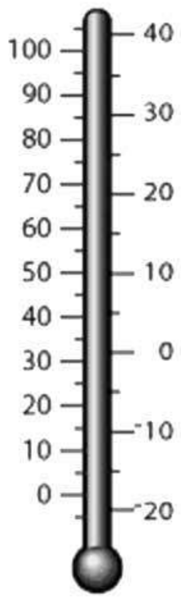
As a quantity, time often has a cyclical character. Pupils learn this during the lower grades of primary school. The same hours on the clock return every day. In the upper grades of primary school, the special character of time becomes a topic of discussion with the pupils.
In the lower grades of primary school, the children already experienced that some things take a long time and other things go past very quickly. They became increasingly aware of time. In first, second and third grades, these aspects were quantified. Children became acquainted with analogue time references in hours, quarter hours and minutes. Around the end of third grade, most children were able to read time on an analogue clock. In third and fourth grade there is also systematic attention for digital time references, with which most children are already familiar due to the many digital time devices they have at home.
In the upper grades of primary school, the knowledge about various time references is a reason to start calculating and reasoning with time. During this process there is emphatic attention for the further analysis of the cyclical character of time; for example the children talk with the teacher about the relationship between the rotation of the earth on its axis and the 24 hours in a day. In the upper grades of primary school, calculating and reasoning with time involve working on the following topics:

- day and night and the position of the sun;
- the phases of the moon;
- weeks, months, years;
- tenths and hundredths of a second;
- time differences on the earth.

Of course, there is also attention for very practical matters such as reading and interpreting a TV schedule, train and bus schedules and relating time duration and points in time to each other.

Temperature

Children come into contact with the quantity of temperature at a relatively young age. In the lower grades of primary school, this took place in the context of weather and other situations. For example, they experienced that today is colder than yesterday and they learned to associate this with the idea that ‘the temperature is lower’. As a rule, they also had experience



with body temperature. If you have a fever, your body temperature is high. During first and second grades, the children became acquainted with the thermometer. This device shows that measuring temperature involves measuring in steps using a unit. After all, if it is 3 degrees warmer today than it was yesterday, the liquid in the thermometer rises by three units of one degree.

This process of measuring in steps along the thermometer is analogous in many respects to measuring in steps using a ruler. But something strange happens with temperature. Unlike the ruler, the measurement scale on the thermometer continues below 0°. In grades 3 or 4, this leads to an initial exploration of negative numbers.
In fact, this scale is not fully proportional. If it was 15°C yesterday and 30°C today, then you cannot say that is twice as hot today. But we don't have to make a problem of this because in the upper grades of primary school, dealing with temperature is limited to reading temperatures and interpreting the information in terms of rising or falling.

Figure 2.26

Speed

Speed or velocity is one of the most difficult quantities with which pupils are confronted in the upper grades of primary school. This is because it is a composite quantity, a combination of time and distance, with the most customary units being kilometers per hour and meters per second.
Before the standard units are addressed, the children have certainly become acquainted with speed, for example, in the context of a foot race: the winner ran the fastest. In first, second and third grade, the children become informally acquainted with quantifying speed, for example, when they look closely at the speedometer of a car. Almost all children are familiar with road signs showing the speed limit. The road sign in the photograph shows that speed limit is 30 kph, which is not very fast. The road sign also shows why you should not drive very fast. There is road construction ahead.

Up to this point, reasoning about speed is not very difficult, but this changes when we start to implement speed measurements. The strange part about

speed measurements is that they refer to a type of imaginary situation. A speed of 55 km per hour means that you would travel 55 kilometers if you drove exactly this speed for exactly 1 hour. However, you do not have to drive for a whole hour to have a speed of 55 kph. If the speedometer points to 55, that is your speed at that instant.
In schoolbooks, this problem is often avoided by referring to 'average speed': "A car drives from Point A to point B with an average speed of 55 km/hour." You can use average speed for calculating, but this concept is far away from the actual experiences of the pupils and provides an incorrect picture. In middle school, the following question is often asked: "A car is driving at 60 km per hour, how long does it take it to go 500 meters?" The pupils respond with, "That is impossible to say, sometimes you have to put on the brakes, sometimes you have to accelerate." The pupils are thinking about average speed, which says nothing about speed on that short piece of road. With this question, you have to imagine that someone is theoretically travelling 500 meters with a constant speed of 60 km per hour.

Figure 2.27

The complex character of speed can be discussed in the context of 'speed traps', for example. The implementation of 'fixed average speed checks' (see box) is a good occasion to hold a discussion. Examples of questions for the pupils are:

- What does a speed camera measure?
- Why are the authorities implementing fixed average speed checks?

Speed cameras and fixed average speed checks

The police are using various methods to determine whether a car is driving too fast. For example, this is done by using a 'speed camera'. If you drive past the camera, it takes a snapshot if you are driving too fast. Because many drivers know where the speed cameras are located, they often reduce their speed just before they get to the camera.
In order to catch these speeders, fixed average speed checks are also being used on some sections of motorways.

Figure 2.28

There is a discussion about what a 'fixed average speed check' actually measures, and the operation of a 'fixed speed camera'. Based on this information, various methods for preventing speeding are compared.

The complexity of the concept of speed is also caused by the fact that you have to simultaneously pay attention to two quantities. Research has shown that pupils have the tendency to look primarily at the distance travelled, and lose sight of the time that is required to travel this distance. Viewed historically, it has also taken a long time before people accepted that the ratio between two different quantities can be used as a unit. We can therefore accept that there is a conceptual problem here. However, we believe that the ratio table can play an important role in making this concept accessible. We can then fall back on reasoning such as: 'in twice as much time, you drive twice as far'. If we drive 16 km in one hour, then we drive 32 km in two hours and 64 km in four hours. We ensure that the proportion between the distances is always equal to the proportion between the times. Conceptually speaking, this is less troublesome than conceiving of a unit that is composed of the proportion between the distance travelled and the required

time. The ratio table can also be used to convert speeds expressed in m/s into km/hour speeds.

Distance travelled	10 m	600	6 km	36 km
Time	1 sec	1 minute	10 minutes	1 hour

We see that 10 meters per second stands for a speed of 36 kilometers per hour. This is a speed that is almost impossible to maintain on a bicycle. You can also appreciate this when you realize that you travel from one wall of the gymnasium to the other in only one second when you are travelling at 10 meters per second.

In the same way, we can work from 60 km/hour to the speed over a distance of 500 meters.

Distance travelled	60 km	1 km	500 meters
Time	1 hour	1 minute	30 seconds

Speed is a quantity that is emphatically addressed during primary education. While investigating this unit, children learn that calculating a speed always concerns an average speed. Because they can measure the two underlying quantities themselves, children can also determine the speed themselves – for example with a ratio table. They then observe that taking repeated measurements and averaging the speeds that are found yields a more accurate answer.

Speed is a model for other composite units with which pupils come into contact. Generally speaking, the other composite units also involve finding an average, and an answer can also be made more accurate in these cases by measuring several times.

In conclusion

Many numbers that we encounter in daily life are measuring numbers. We want children in primary school to acquire a measurement repertoire that enables them to interpret these numbers in a meaningful fashion. The units that are used generally originate from an orderly mathematical system: the metric system. This necessitates that the children learn to see through this system. This has always been an important aim of measurement education and has led perhaps too frequently in the past to a one-sided focus in primary school on practicing calculation rules for converting units. This approach has been inadequate. In this situation, for example, the pupils become stymied when working with units of area and units of volume.

In this book, we propose a shift in accent in measurement education, where drills about relationships are replaced by repeatedly constructing these relationships. We have given suggestions on how to do this, which we can summarize as two standards for measurement education: ‘gaining control of the world’ and ‘developing mathematical tools’. In this chapter, we have shown how these core concepts are interwoven and how they can lead to measurement education being structured in such a way that all pupils work meaningfully with measurement.

3 Concise learning-teaching trajectories and intermediate attainment targets for measurement

In this chapter we will provide a concise summary of the learning-teaching trajectories and targets for measurement. In very broad terms, during the upper grades of primary school, measurement education focuses on flexible manipulation of units and reasoning with units. Activities therefore do not focus on imprinting relationships and calculation rules, but on searching for such relationships based on known unit meanings. In this chapter we will work out this approach for ‘benchmarks and measurement instruments’, ‘prefixes and units of length, ‘units of area’, ‘units of volume’, ‘weight’, ‘time’, ‘temperature’ and ‘speed and other composite quantities’. For the purpose of clarity, we will isolate these learning-teaching trajectories. Of course, this does not mean that these are isolated learning processes. Quite the contrary. Virtually all measurement situations can be placed in two or more of the above-mentioned clusters.

Benchmarks and measurement instruments

Learning-teaching trajectory

As time passes, children learn measurement facts. For example, they know that their thumb is approximately 1 centimeter wide and that a football field is approximately 100 meters long. We refer to these measurement facts as benchmarks. When learning length and weight, the pupils in the lower grades of primary school have already developed several of these benchmarks. In the upper grades, these benchmarks are expanded in two ways. Besides benchmarks for the centimeter and the meter, the pupils develop benchmarks for the kilometer and millimeter; besides benchmarks for the kilogram, the pupils develop benchmarks for the gram. Benchmarks are also developed for quantities besides length and weight. This concerns benchmarks for units of area such as the square meter, and later on for the hectare and square kilometer as well. For units of volume this concerns the milliliter, centiliter, deciliter and liter; for the so-called cubic units this concerns the cubic decimeter and cubic meter.

For quantities such as time, temperature and speed, there is also attention for the development of several benchmarks such as the approximate duration of a minute and an hour, the fact that water freezes below a temperature of 0° C, that water boils at 100° C, that your body temperature is approximately 37° C and that an outdoor temperature of around 22° C feels pleasant. Benchmarks for velocities are developed by referring to everyday situations, such as bicycling, walking or travelling in a train or a car. If you bicycle vigorously, you travel at more than 20 km/h, and if you walk vigorously, you have a speed of 5 or 6 km per hour. An express train has a top speed of 140 km/h, while a high-speed train can reach speeds of 300 km per hour. In Dutch cities, cars must often travel at less than 50 km/h, and on the motorway they can travel at 100 km/h.

The units of area and the cubic units of volume are constructed by the pupils. After constructing these units, the most familiar units will 'come alive' for the pupils. They can imagine the size of a cubic meter and they know that a hectare is the size of two football fields.
The pupils gain control of the units of length and units of volume by learning to use a suitable measurement instrument. This also applies to the benchmarks for weight, time, temperature and speed. The pupils become acquainted with many of these instruments. Initially, this concerns measurement instruments where the measurement action can be clearly seen. A balance is such an instrument. Later on, measurement instruments are used where the process of measuring in steps is less apparent, which is the case with digital measurement instruments. In fact, these instruments become a topic of investigation for the pupils. The central questions here are the following: how is the scale of the measurement instrument structured, which measurement results does the instrument provide in specific situations, and how can you explain that?

Targets

- Pupils develop an extensive arsenal of benchmarks for units that are used regularly in everyday practice.
- Pupils are capable of using simple everyday measurement instruments, even if the concrete measurement action of the instrument is no longer visible. They can read and interpret the measurement results that are provided by the instruments.

Prefixes and length

Learning-teaching trajectory

Units, including units of length, are given meaning by their prefixes. The pupils first learn the prefixes in the context of units of length. The graduated ruler, which the pupils know as a measurement instrument for length, gives meaning to the prefixes 'centi' and 'milli'. The graduated ruler can then help to make them aware of the meanings of these prefixes. There are 100 centimeters in one meter. One centimeter is therefore one-hundredth of a meter.
Soon there is attention for units of length, such as kilometer, that do not fall within the units on the graduated ruler. On the one hand, the definition of the kilometer as '1000 meters' provides an image of this unit. For example, it is the distance that you would walk if you took 1000 large steps. On the other hand, this definition also gives meaning to the prefix 'kilo'. This stands for 1000.
Other units of length, such as the decimeter of decameter, acquire their meaning directly from the prefixes. For example, 'deci' means one-tenth; therefore a decimeter is one-tenth of a meter. The graduated ruler then makes this visible.
After the prefixes 'milli', 'centi', 'deci', 'deca', hecto' and 'kilo' have been explicitly addressed, there is also attention for prefixes such as 'micro', 'nano', 'mega' and 'giga'.

Targets

Grades 3 and 4:
- Pupils can use a graduated ruler and a straight edge to measure lengths.
- Pupils know the prefixes 'milli', 'centi', …, 'kilo' and can use them to give meaning to units of length.

Grades 5 and 6:
- Pupils identify patterns of regularity in the prefixes and can explain this based on known units of length.
- Pupils identify 'micro', 'nano', 'mega' and 'giga' as prefixes.

Constructing units of area

Learning-teaching trajectory

Pupils in the lower grades of primary school learn how you can compare areas by placing them on top of each other. They also learn that you can determine the area by measuring in steps with a unit on the figure. This process of measuring in steps is expanded in the upper grades of primary school. At this stage, measuring in steps leads to flexible counting of the units. Generally speaking, this will involve using the structure of tiles or squares that are already present in the situation. To prevent premature generalizing, during this process the units being counted are varied. For example, they can be triangular, hexagonal, square or rectangular. The process of flexibly counting squares inside a rectangle then emerges as one of the discoveries.

At a certain point, the transition is made to situations where no structure with tiles or squares is present, and the pupils have to think up a unit themselves. In these situations, the pupils can build upon their insight that a square is a handy unit for covering or filling up rectangular figures. Moreover, because the pupils are familiar with the centimeter as a unit of length, they can reason that it is advantageous to use squares measuring one by one centimeter.
In this way, the cm² as a unit of area is constructed from the centimeter as a unit of length.

In addition to the rectangle, other regular figures (polygons) are certainly addressed, such as the parallelogram and the triangle. The pupils can use many strategies to determine the area of these regular polygons. Some of the pupils will perceive that general formulas also apply to such polygons. The choice for a standard unit sometimes results in this unit being less suitable. This is especially the case when irregular polygons are addressed. Then approximation must be used.
Following the introduction of the cm² as a standard unit, the standard units dm², m², hm² and km² are also introduced. Here as well, knowledge of the unit of length establishes the basis for the unit of area. For the unit of area hm², the alternative name hectare is provided. In addition, the pupils experience in different situations that these units do not necessarily have to form a square. As a rule, a shopping district with an area of one square kilometer does not have a square shape.
Finally, experience with measuring in steps and counting squares leads to a way to determine relationships between metric units of area.

Targets

Grades 3 and 4:
- With simple regular and irregular polygons, the pupils can determine the area by measuring in steps with a given unit.
- The pupils can determine the area of a figure by using a structure of tiles or squares that is already present.

Grades 5 and 6:
- The pupils know the standard units of area – cm², dm², m², hm² (ha) and km² – and perceive the relationships with the corresponding units of length.
- The pupils can calculate the area of simple regular and irregular polygons in standard units.
- The pupils can reason out the relationships between cm², dm², m², hm² (ha) and km².

Constructing units of volume

Learning-teaching trajectory

Learning about volume also begins in the upper grades of primary school with measuring in steps (or scooping out) with a unit. During their exploration of units of volume that are based on the liter, the pupils go back to the prefixes that were explored in the context of the units of length. In this way, the milliliter emerges as one-thousandth of a liter. Benchmarks for units of volume offer support to this process. During the exploration of these units of volume, the measuring cup takes an important role. The pupils are often familiar with the measuring cup from home and from measurement activities in the lower grades of primary school. Now the focus is on the unit or units on the measuring cup. At that point, for example, the pupils are asked how high the liquid would rise if a 33 cl beverage can was emptied into a one-liter unit.

With the cubic units, the emphasis is on measuring in steps in such a way that the result can be determined by flexible counting. The latter is important especially when determining how many small cubes fit into a regular solid such as a cube or a rectangular prism. In this situation, flexible counting ultimately leads to the insight that counting the small cubes can be reduced to a multiplication problem. However, before the pupils get to this point, many other situations are explored where a volume can be approximated by means of flexible counting.

Finally, the obvious question is asked: what is a suitable unit of volume which is compatible with the units of length and area that are known to the pupils? At this point, the pupils reason that it is advantageous to use small cubes measuring 1 cm x 1 cm x 1 cm. In this way, the cm^3 as a unit of volume is constructed from the centimeter as a unit of length.

Besides the unit of the cubic centimeter, the units of the cubic decimeter and cubic meter are introduced. Due to the names of these units, the pupils quickly see what they could look like. By discussing the fact that a liter is actually another name for a cubic decimeter, the pupils understand that a cubic decimeter does not necessarily have the shape of a cube (and that this also applies to the other cubic units).

When the pupils become familiar with the cubic centimeter and cubic decimeter, the relationships between these two units are also explored. This

once again involves measuring in steps (imaginarily or using tangible materials) and counting flexibly to determine how often the unit fits. In this way, the pupils discover that 1000 cm^3 fit into a cubic decimeter. Instead of having the pupils memorize this knowledge, the teacher makes sure that they repeatedly practice this process of measuring in steps and flexible counting. When this measuring in steps and flexible counting is discussed with the pupils, relationships between liter units and cubic units also emerge. The pupils discover that a cubic centimeter is a milliliter, because it fits 1000 times into a liter.

Targets

Grades 3 and 4:

- Using simple block structures, the pupils can determine the volume by thinking how many blocks fit inside.
- The pupils know the units of volume that are derived from the liter and can relate them to each other by using the meanings of the prefixes.

Grades 5 and 6:

- The pupils know cm^3, dm^3 and m^3 as cubic units of volume and can relate these to the corresponding units of length.
- The pupils can reason out the relationship between various units of volume, also between units of volume that are derived from the liter and cubic units of volume.
- The pupils can determine the volume of regular polygons such as cubes and rectangular prisms if the length, width and height are known.

Weight

In the lower grades of primary school, the children become acquainted with the kilogram and gram as units of weight. They come into contact with these units in daily life as well, for example during a visit to the market. There the pupils notice that other units are used, such as pound and ounce, and that kilogram is abbreviated as 'kilo'. In the upper grades of primary school, the units of the kilogram and gram are linked with the meaning given by the prefix 'kilo'. Kilogram means 1000 grams. The pupils learn that these two units of weight are the ones most commonly used in practice. Even so, their knowledge of the prefixes makes it possible for them to experiment with naming other units of weight. For example, the pupils can reason out that a decagram is the same as 10 grams, because 'deca' stands for 10.
The pupils acquire a wide range of benchmarks by weighing various weights with a set of scales. Digital scales provide a measurement result with a decimal fraction. The pupils also encounter such decimal fractions in daily life, for example on the price stickers of fresh products. These decimal fractions become a topic for discussion in the class, where they learn how the decimal fractions can be effectively rounded off and how you can interpret decimals as regular fractions. See also the Tal book *Fractions, Percentages, Decimals and Proportions - A Learning-Teaching Trajectory for Grade 4, 5 and 6.*
The careful examination of packaging provides another topic for discussion about measurement. When the packaging states both the weight and volume of the product, this provides a reason to talk about specific gravity.

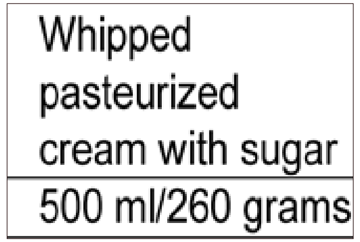


Figure 3.1

Targets

Grades 3 and 4:
- The pupils have access to benchmarks for the kilogram and gram.
- By using a set of scales, the students can determine the weight of an object in units of kilograms or grams, also when the measurement result is a decimal fraction.

Grades 5 and 6:
- By using prefixes, the pupils can reason out the meaning of various units of weight and convert one unit of weight into another.
- The pupils have a general idea of what specific gravity is.

Time

In the lower grades of primary school, the pupils become acquainted with the measurement of time. They become familiar with the time designations of the minute, quarter hour, half hour and hour. They begin to tell time. Many children in third grade can already tell time fairly well and they know the names of the days of the week and months of the year. In the upper grades of primary school, these skills continue to be practiced and are expanded by learning to read digital clocks. There is also attention for calculating with clock times. For example, this takes place in the context of using a television program schedule. If you want to watch a specific program, you must determine how long it will be before it begins. Another context is working with train and bus timetables.
There is also attention in education for calculating with the calendar. The pupils learn to determine how many days there will be from one event to the subsequent event. When learning to tell time and read timetables, the pupils generally work with the 'units' of the hour and minute. In other situations, they work with seconds, milliseconds, days, weeks, months, years and centuries.
Finally, there is attention for time zones. If there is a situation involving time differences, the teacher works with the pupils to determine how you can reason out whether the clock time at a specific location on the earth will be later or earlier than 'our' time. For example, you could ask: if you live in Amsterdam, when is the best time to phone someone who lives in New

York? Another topic that is addressed in this context is the beginning or end of daylight savings time. Do you turn the clock forward or back, and how can you reason this out?

Targets

Grades 3 and 4:
- The pupils know the following time designations: minutes, hours, seconds, milliseconds, days, weeks, months, years and centuries.
- The pupils can tell time with both analogue and digital clocks.
- The pupils can use a calendar and a diary.

Grades 5 and 6:
- The pupils can use timetables to look up information and to determine how long it will be before something happens in the future or how long ago something happened in the past, regardless of the time unit in which the information is given.
- The pupils can calculate with time.
- The pupils can reason out time differences in simple situations.

Temperature

The pupils become acquainted with temperature in the context of weather. In this way, they also acquire several benchmarks concerning temperature. In addition, they learn what the outdoor temperature is influenced by, and they learn to describe changes in temperature with a temperature graph. Ideally, the pupils make such a graph based on their own measurements. To this end, they learn to use a thermometer. In any case, they become acquainted with the traditional analogue thermometer. The temperature scale on this thermometer supports their reasoning because this scale is similar to the familiar number line, but has been expanded to include negative numbers. On many thermometers, the temperature is shown in both Celsius and Fahrenheit. The pupils become acquainted with these two systems of units. There is special attention in education for 0º C and 100º C, the freezing point and boiling point of water. In addition, the pupils learn that their own body temperature is approximately 37º C.

Targets

Grades 3 and 4:
- The pupils can read the temperature on a thermometer.
- The pupils can make a graph of the changes in temperature.

Grades 5 and 6:
- The pupils know that there are two systems for measuring temperature in degrees: Celsius and Fahrenheit.
- The pupils know that water boils at 100º C and freezes at 0º C.
- They know that their own body temperature is approximately 37º C.

Speed and other composite quantities

Speed is the most important composite quantity for pupils in the upper grades of primary school. Almost all pupils have sufficient informal experience with speed; they have travelled at high speed in a car, train or aeroplane or have gone very fast on a bicycle. During the upper grades of primary school, such experiences are quantified. The pupils learn to make a distinction between speed at a specific moment (instantaneous speed) and average speed. The initial ideas about speed are about instantaneous speed. But to describe speed, you must indicate how long it takes to travel a specific distance, or how long this would take if you travelled at a constant rate. The pupils learn that it does not matter if you express speed in meters per second or in kilometers per hour. A speed of 72 kilometers per hour is the same as 1.2 kilometers per minute, or 20 meters per second.
The pupils learn to distinguish this indication of speed from average speed. If a hiker walks 6 kilometers in one hour, this does not mean that he walked the same speed for the entire distance. Perhaps he started walking very quickly and then slowed down. In that case, 6 kilometers per hour is his average speed.
For calculating with velocities, the ratio table is used. In this way, the complex concept of speed becomes manageable, because time and distance each have their own place on the table.
In addition to speed, other composite quantities are addressed, such as population density, price per kilogram, precipitation rate, medicine dosage and

per capita income. The pupils reason out the information provided by these composite quantities and how they can calculate with them if necessary.

Targets

Grades 5 and 6:

- The pupils understand speed as a proportion number and can calculate with speed, for example by using a ratio table.
- The pupils understand the difference between average speed and instantaneous speed.
- The pupils can interpret unknown composite quantities and reason out the information that these units provide.

4 The domain of geometry

Introduction

Babies come into contact with geometry while they are still in the cradle. This happens sooner than with any other aspect of mathematics. Freudenthal, among other authors, has written beautifully about this topic in his book ‘Mathematics as an Educational Task’. He is one of the founders of Realistic Mathematics Education and writes ‘about the truncated cone that a drinking cup actually is’, ‘about the ball that the still incapable fingers try to hold’ and ‘growing up with geometry’. During this phase, the space for the child is still limited to the cradle, the playpen, the changing table, the baby bath and the baby’s room. But as the child grows older, this space continues to expand. The room turns out to be part of a house. The house is located on a street. There are other houses on the street. Grandma lives far away, and there are various ways to go there.

However, in addition to that continually expanding universe for children, they also discover that form and function are often related. The ball rolls, a tall cup tips over easily, blocks only slide, and building with blocks can be rather difficult. In short, besides gaining control of space, Freudenthal called this process ‘grasping space’ (‘Mathematics as an Educational Task’ – page 403), there is also the challenge of understanding various shapes and figures and how they are related.

Children often learn to grasp their surroundings quickly. They soon have little trouble finding their way to school. New surroundings offer new challenges. At the campground, there is a sign with a map showing where you are standing now. Your mother explains the map and shows you where your family’s tent is located. A map can be very useful, and children can learn to use maps when they are very young.

In a natural fashion, children thus come into contact with many aspects of geometry: they acquire a spatial sense, they recognize shapes and figures and they understand that an aspect of reality can be made visible in a picture. This summarizes the three aspects of geometry that are at the core of this book, which has the topic of geometry in the upper grades of primary school. Geometry in the lower grades of primary school is addressed in the TAL book (2004) *‘Young Children Learn Measurement and Geometry’.*

Because we come into contact with geometry on a daily basis, it is sometimes difficult to recognize it as part of mathematics education. However, it is indeed part of mathematics education and therefore deserves attention. After all, regardless of how natural geometry seems to be, learning and using geometry does not happen by itself; many skills are required. Observing, experiencing, taking action, rendering, analysing, comparing, explaining, seeing and making connections, classifying and predicting are skills that almost always play an important role.
Seen from this perspective, geometry deserves a prominent place in the primary school curriculum. Of course, it is self-evident that arithmetic itself is seen as the heart of all mathematics-related education in primary school. But there are no 'logical' reasons that could indicate the percentage of education that should be spent on arithmetic in relation to the time that is allocated to geometry.

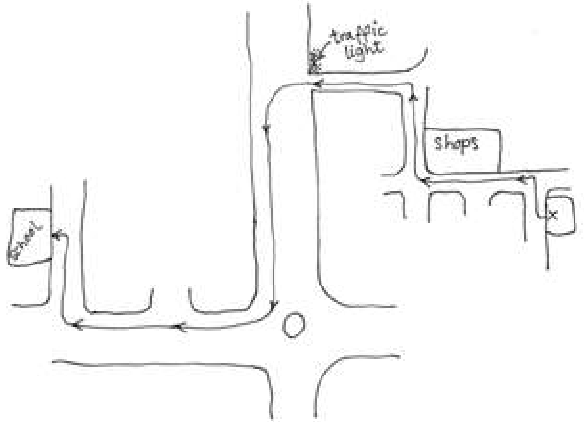


Figure 4.1

Learning geometry, other than exclusively by means of experiences in daily life, is valuable from various perspectives. First of all, there are arguments that emphasize the practical value of geometry. You frequently encounter geometry, implicitly and explicitly, and it is useful to be aware of the practical possibilities. For example, if you are giving directions it is convenient if you can draw a simple map (see figure 4.1). For building a sturdy structure, triangles are the strongest shape. In order to know what time of day an area will fall into shadow, you need to know about the effect of the position of the sun on the length and direction of the shadow. In short, geometry has a practical value, and this value should be made explicit in primary education.

But like almost every school subject, geometry also has a preparatory value. We refer here to the role of geometry in secondary education, assuming that geometry has a legitimate place in secondary education. But geometry

is also encountered in other disciplines. Maps are used in virtually all disciplines, but other types of 'representations' also appear (see figures 4.2, 4.3 and 4.4).

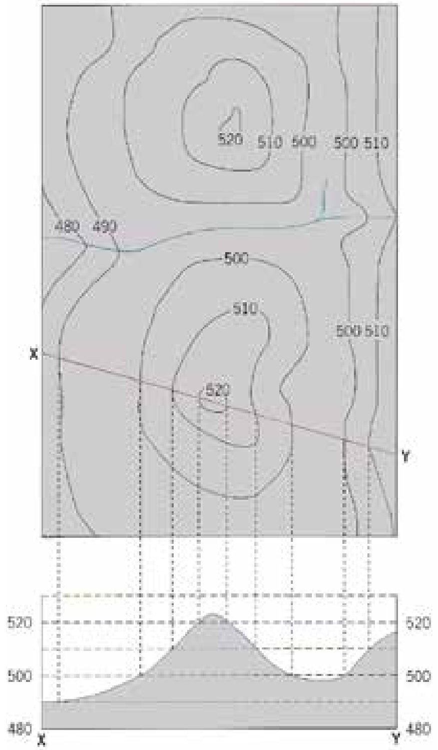


Figure 4.2

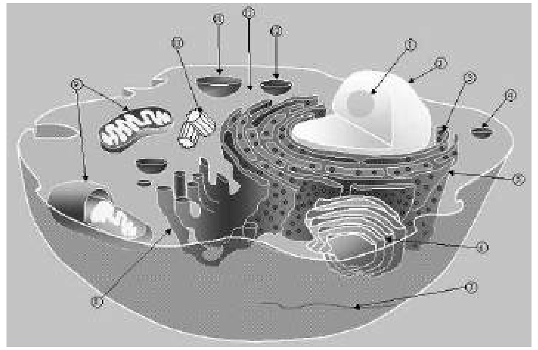


Figure 4.3

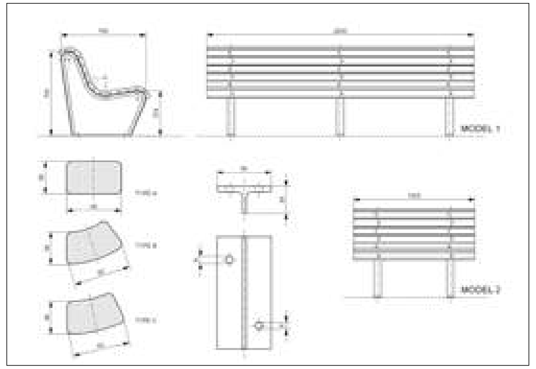


Figure 4.4

Geometry is not only an important and relatively independent discipline within mathematics, but also has an intrinsic value as a subject. Pupils must be brought into contact with its beauty. For example, you can see this beauty in the artistic patterns of the Moorish architects (see figure 4.5), or in the relationship between number patterns and their geometric representations.

Figure 4.5

An additional argument for more geometry in education is provided by children who have difficulty doing mathematics with numbers; geometric representations often give them support and motivation for continuing to learn about mathematics. In this example, we are not thinking about abstract geometry, but about everyday geometry that can be experienced. We ask questions such as: why does something appear to be smaller when it is further away? What is a blind angle?

Geometry is often involved with recognizing the geometric aspects in situations in the world around you. There are many situations with geometric aspects. In these situations, various aspects of geometry often become visible. But in order to recognize the integrated nature and coherence of various aspects of geometry, it is necessary to apply some structure. We make a distinction here between three terrains of geometry in the upper grades of primary school: 'spatial sense', 'plane and solid figures', and 'visualization and representation'.

Before we go further into these three aspects of geometry, we will focus on the basic principles, legitimization and characterization of the geometry that we have in mind.

Basic principles, legitimization and characterization

Basic principles

We discuss the following basic principles:
- building upon the geometry from grades K-2;
- mathematizing space.

Building upon the geometry from grades K-2

The geometry in the lower grades primary school is primarily concerned with exploring, observing and experiencing the surroundings in which children live and move. In conversations, children learn the language to give names to locations in three dimensions and on the plane. Words such as under, above, left, right, in front, behind, here, there and beside acquire meaning in many situations. Children are increasingly required to use their sense of direction, both indoors and outdoors. Orientation in space is a crucial concept. But children also see and play with all kinds of plane and solid figures, and they are always constructing things: with blocks, matches, folded paper and discarded materials such as packaging. Finally, they also begin moving figures around, they begin to see symmetries, they rotate figures and they become fascinated with shadows or projections. Due to having these tangible experiences and talking about them, the children improve their spatial sense, and their spatial insight grows.

In grades 3 and 4, aspects of analysis and explanation increasingly become part of the picture. In this way children are asked to imagine themselves at a different standpoint, analyze building block structures from a view, analyze a figure based on a number of fixed shapes (such as the Tangram) and use a mirror to ascertain symmetry and add the missing parts of figures based on their symmetry. At the same time, the geometry activities become more complicated. The language also becomes more formal. In grades K-2, language developed spontaneously as part of playful activities. Gradually more direction was provided to this process. Moreover, concepts have been increasingly developed by using examples that are 'on paper' instead of tangible situations. Seen from a distance, in grades 2 and 3, the structure has become more systematic. This also results in more coherence. During the class discussion about an activity, more emphasis will be placed on using ‘geometry language’. In this way the children not only learn the meaning of terms such as triangle, quadrangle, rectangle, square and cube, but also of concepts such as top view, side view, floor plan, mirror image or symmetrical, route and left front.

Mathematizing space

In the upper grades of primary school, the pupils are increasingly encouraged to make connections, create images, explain and predict. By doing so, they develop geometric concepts and instruments, such as plane, line, angle or view. In the upper grades of primary school the focus is increasingly on mathematizing space, in both a structural and formal sense. The pupils learn to model space, and based on these models they are increasingly capable of reasoning with, understanding and predicting geometric phenomena. A good example is a description of a long journey. The route of the journey can be tracked on a map (a projection of a sphere) or on a globe, and can be described in terms of coordinates, direction and/or duration. Questions such as the shortest distance between two locations on a sphere, the absolute and relative direction and the projection of the route on the map lead to activities such as experimenting, modeling and explaining.

It remains very important for the pupils to experience things, look at things and do things themselves. In the upper grades of primary school, the teacher reflects on these experiences with the pupils in order to describe the observations with increasing precision, to determine which representation is the most suitable for further analysis, to explain things and to make connections with other observations. The result should be that the children discover relationships and patterns by means of shared models and that they develop language to discover well-reasoned answers to questions posed by the teacher and those that emerge from their own sense of wonder. Curiosity and a sense of wonder do not always happen by themselves; the education focuses on teaching children to ask questions and on generating a sense of wonder. “What does a cross-section of an apple look like (see figure 4.6)?” is an example of such a question.

Figure 4.6

From there, the pupils can start looking for all kinds of cross-sections of three-dimensional figures, such as cross-sections of a cube. The teacher encourages the children to take an investigative approach and elicits all kinds of wonderful and valuable discoveries by asking good questions. In the follow-up discussion, the discoveries are formulated in words and shared.

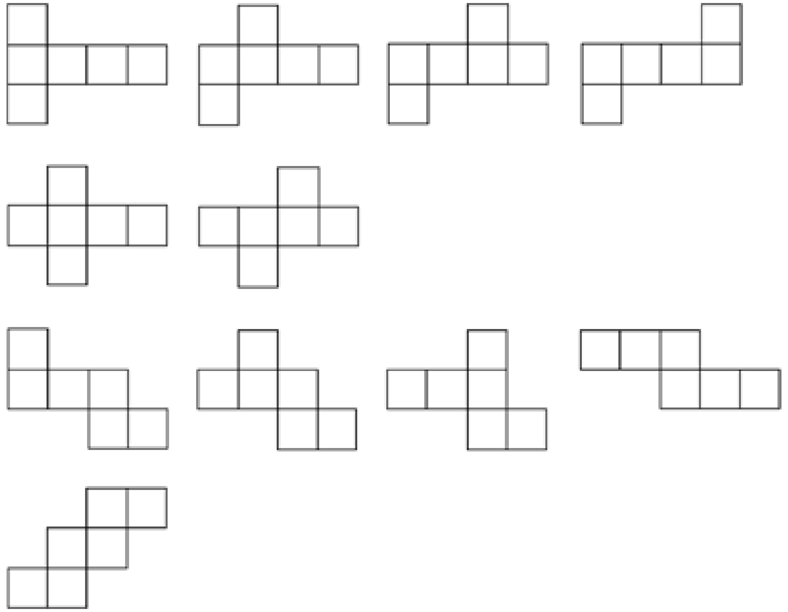


Figure 4.7

Acting, experiencing and looking therefore have more than the exploratory character of these activities in grades K-2; they now serve as an experiential basis for reasoning, constructing models and for reflection. At the same time, there is also a shift to dealing with mental images. Children learn to set up small inquiry projects to test their assumptions. They learn to deal with questions in a scientific fashion. They do this, for example, by systematically seeking proof that they have found all possible foldout shapes for the cube (see figure 4.7), or that they have found all possible structures that belong to the given views.

In addition, in the upper grades of primary schools there is a shift to more exact systematic description and explanation of geometric phenomena. For example, at a certain point pupils begin to use views and vision lines to explain what you can or cannot see from a specific position. They develop insight into the concept of scale, which they use to make drawings, such as floor plans. This means that many of the activities that took place in the lower grades of primary school are again addressed in the higher grades, such as drawing floor plans, working with mirrors, working with mosaics and symmetrical figures or foldout shapes, and building with blocks using floor plans with heights or views (see fig. 4.8).

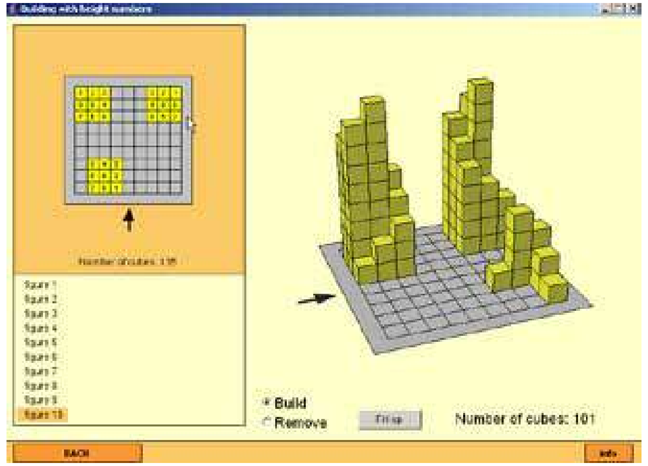


Figure 4.8

In the upper grades of primary school, however, more difficult problems are also tackled, which gives these activities greater depth. Higher demands are placed on the type of descriptions and explanations that the children come up with. The activities require more reflection and reasoning. During this process, the power of visual descriptions of a situation becomes emphatically apparent.

At the same time, pupils develop geometric language, which makes reflection and reasoning possible; the children can start using terms such as parallelogram or rhombus. Pupils can investigate the similarities and differences between quadrangles that you can cut out of a folded piece of paper, for example, or those that you can draw on graph paper (see figure 4.9 and 4.10).

After this, there is a need to give names to these various new figures. In this way, the pupils internalize more and more terms that they can use when discussing the geometry problems.

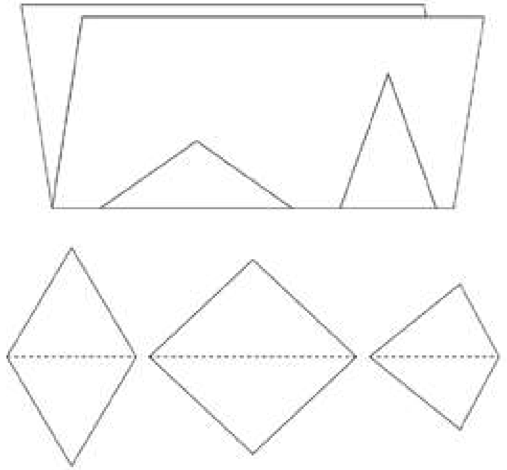


Figure 4.9

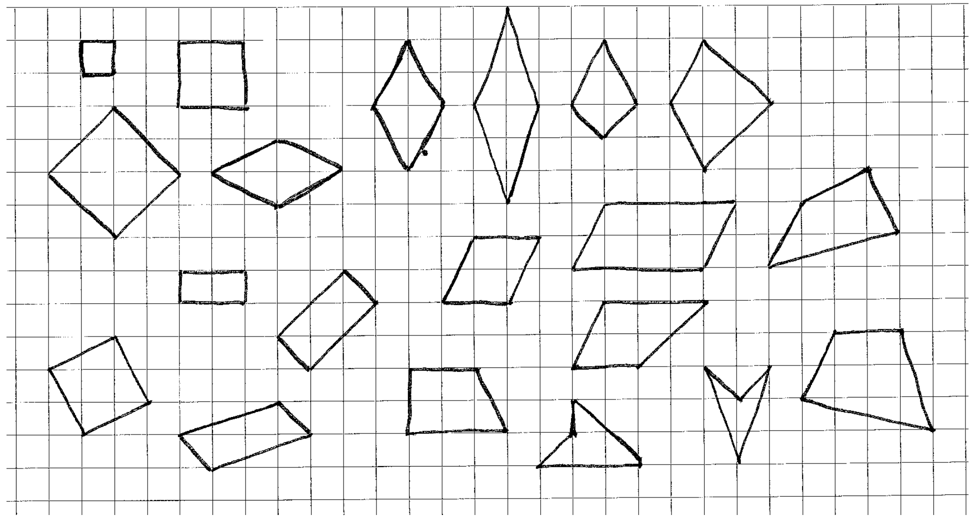


Figure 4.10

In summary

In retrospect, it can be stated that geometry education in the upper grades of primary school links up with the geometry in the lower grades. The geometry development of pupils in primary school begins very close to them with their own experiences, and is primarily intuitive, observational, qualitative and global in nature. Gradually, the perspective becomes more objective, quantitative and precise. An important aspect of this process is that the pupils continue to explore, although this takes place at a more abstract level. At the same time, the learning process focuses on description, i.e. the development of mathematical language and the skills of modeling; based on reflection and mental images, the pupils work on explaining phenomena with the aid of drawings, constructions and mathematical concepts. The pupils gradually learn how to make connections, generalize and make predictions, and these aspects become increasingly important. They learn to mathematize space, to take a mathematical view of space.

Legitimization

Generally speaking, three main values are attributed to geometry: its general practical and educational value, its preparatory value and its intrinsic value. We can recognize these three values in virtually all components of geometry. For instance, you learn to interpret a floor plan not only because it is useful in everyday life, but also because you need a foundation of this type of knowledge so you can use coordinates, directions and scale in secondary education. In addition, it is also good for learning concepts and appreciating geometry as a discipline, and therefore has an intrinsic value. With maps, we can think about the various projection methods that cartographers have developed through the centuries, which are a fascinating aspect of geometry. We will briefly discuss these three values of geometry below.

The practical and educational value of geometry

In primary education, geometry has a general practical and educational value. Everyone (this is the reason for the 'general') needs geometric knowledge in order to function as a citizen in this society. Everyone needs practical geometric knowledge, which is perhaps not even recognized as geometry. Such knowledge is used for locating something on a floor plan, planning a route on a map, giving directions to someone, using a sewing

pattern or putting together a cabinet with a set of instructions.

In addition, geometry activities have a personal educational value. This is because geometry promotes reasoning, a problem-solving capacity and the discerning attitude that you need in order to function well in society. For example, consider the process of classifying and organizing spatial situations when making models, graphs and diagrams.

The preparatory value of geometry

We define preparatory value not only as preparing for geometry in secondary education, but also as preparing for other topics in mathematics education and the material that is addressed in other subjects. In primary school, the foundation is created for the geometry in secondary education. In this way pupils in secondary education work at a formal level with concepts such as angle, line, line symmetry and rotational symmetry, coordinates, perpendicular and parallel lines, and with plane and solid figures such as the rectangular prism, cube, pyramid, sphere and cone. Primary education lays the foundation for secondary education by exploring these concepts intuitively and informally.

But primary school geometry not only lays the foundation for geometry in secondary school, it also has a preparatory value for other mathematical topics that are addressed after primary school, such as measuring, numeracy and mathematical operations. For example, measuring requires spatial insight and geometric models such as floor plans, maps and technical drawings, and schematic representations and concepts such as angles, lines, parallelism, shapes and figures. In addition, many geometric models, such as the rectangle model and bar model, are also used to develop numeracy and insight into mathematical operations.

Finally, when learning geometric models, visual representations are developed that serve an important role as an application in other fields such as geography (maps) or physics and engineering (technical drawings).

The intrinsic value of geometry

The last value is the intrinsic value of geometry. Depending on their talents, interests or needs, pupils can also develop a fascination for geometry and begin to see it as a wonderful subject. They discover that geometry is fun and that they want to develop additional geometry skills. This may concern the aesthetic aspect of geometry that you encounter in art and architecture,

where geometric transformations such as mirror images, as well as plane and solid figures, are used in many patterns.
One example is the wondrous world created by the Dutch artist and mathematician Escher. Exploring and analyzing such a world is a type of an activity that can keep children on the edge of their seats with excitement (see figure 4.11).
In addition, geometric puzzles such as those developed by the mathematician Rubik (renowned for his cube puzzle) can contribute to a sense of wonder and fascination for geometry.

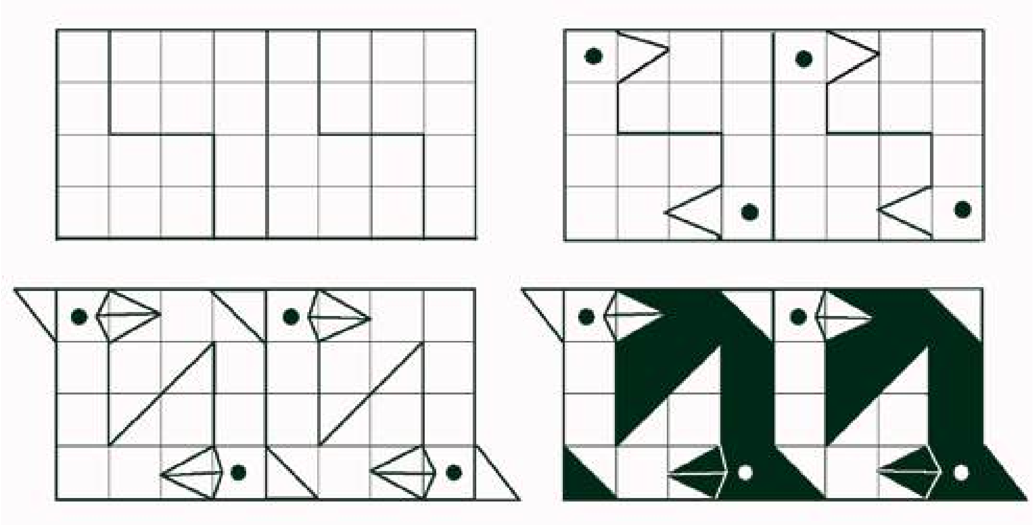


Figure 4.11

Finally, with the help of geometry you can also solve interesting problems in a beautiful, elegant and/or contemplative manner.
One example of such a problem is the sum of the first hundred odd numbers:
1 + 3 + 5 + .. + 197 + 199 = ? Combining them one by one leads to a time consuming result and isn't an elegant strategy. Combining the numbers in a contemplative manner can solve the problem faster and more elegant:
1 + 199 + 3 + 197 + … = 50 x 200 = 10000.

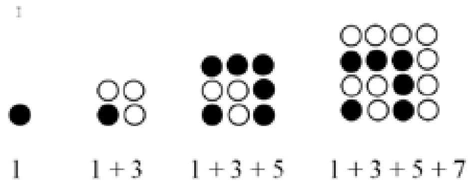


Figure 4.12

To see a certain geometric structure in this addition, may supply an even more elegant solution. Can you discover the addition in the growing geometrical pattern of figure 4.12?
If you see that n x n is the sum of the first n odd numbers, the solution is easy to find: the sum of the first odd numbers is 100 x 100 = 10000.

Finally, besides the above-mentioned legitimizations, there is also the element of motivation. Geometric problems can provide a positive stimulus to pupils to do their own investigations. Involvement in inquiry or constructive activities in geometry is especially beneficial for pupils with weak mathematics skills. The more gifted pupils can also enjoy themselves with these activities.

Characterization

Inquiry-based
Geometry education in the lower grades of primary school is characterized by a high proportion of inquiry activities. In the upper grades of primary school, the greater emphasis in geometry education on reasoning, comparing, rendering, understanding and explaining requires even richer situations. These situations not only compel children to do their own investigating, but they will also have greater complexity. The foundation that has been laid down in the lower grades makes it possible to increase the complexity and richness of inquiry tasks. With these richer, more complex tasks, the first phase of investigation continues to focus on exploration; this is the so-called exploring phase. However, this clearly demands a lot from the teacher. After all, by definition these rich and more complex inquiry activities often have an open character, and this provides opportunities for the children to go astray in undesired areas. Consequently, the teacher must not only have knowledge of and skill in geometry, but must also respond to the inquiry activities of the children with good timing.

A rich inquiry activity offers pupils the possibility to investigate a problem at more or less their own level, or to solve a problem in the same fashion. The pupils must work together interactively: they can share the work, they can reflect together, they can develop strategies together, and so on. Both the horizontal interaction (between the pupils) and the vertical interaction (between the pupils and teacher) support the inquiry work. When we speak of 'rich' inquiry activities, we refer not only to the mathematical content, but also to the form of these activities and the potential learning results.

Differences in level

In the geometry education in grades 3-6, there is a shift from informal to formal, with a growing emphasis on the development of mathematical models. The teacher must take account of the fact that the pupils provide 'solutions' at differing formal or mathematical levels. In fact, these differing levels are very desirable with respect to the learning process. In this way, pupils can come up with mathematics that may not be part of the lesson plan, but which they probably know from a practical situation at home or elsewhere. For example, they may use degrees when expressing directions. In the context of specific activities, most pupils can deal with the four cardinal directions north, east, south, and west, as well as the interim directions northwest, northeast, southwest and southeast, but the teacher should also be prepared for pupils to use degrees on their own initiative. Under certain conditions this can actually be very desirable: it meets a real need of the pupils to specify directions more precisely. As a teacher, you can also ask pupils who are ready for this challenge how they can indicate a direction on a map even more precisely. In this case, providing the pupils with a compass rose (see figure 4.12) leads to an almost automatic transition to using degrees. Here as well, during the discussion with the pupils, discoveries and strategies must be exchanged with the aim of developing a common language and bringing everyone to a higher level.

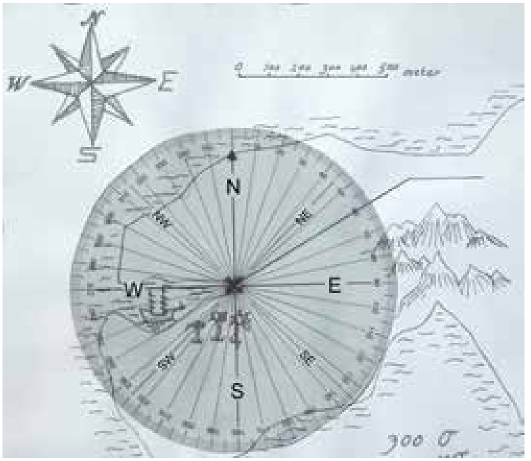


Figure 4.2

Structure of the curriculum

The pupils are initially encouraged to make a broad exploration of a specific concept. This takes place by experiencing or investigating tangible situations. Observation has an important role in this process. Based on these concrete experiences, a mathematical language is developed, and models are created that provide a mathematical rendering of these situations. At the same time, visual representations are frequently used (such as drawings). These representations are increasingly used to support explanations. Pupils are encouraged to make predictions based on reflection. Ultimately, a higher level of abstraction is achieved when the pupils begin to generalize and make connections.

The possibility of a method-based approach does not imply that only a single curriculum structure is possible for every component from the geometry programme. The complexity of the space around us means that the concepts are extremely interwoven, and this leads to many possibilities to relate these concepts together as part of a learning-teaching trajectory. Examples of a method-based approach to geometry in the upper grades of primary school will be sketched out in Chapter 5.

Finally, we note an increasing degree of precision of the instruments that are being used, such as instruments that indicate a position; this aspect is interwoven with the learning-teaching trajectories in geometry in the upper grades at primary school. Therefore, a certain degree of structure in the curriculum can also be provided in this regard.

The geometry taught in the upper grades of primary school

Introduction

The TAL publication on geometry for grades K-2 (‘*Young Children Learn Measurement and Geometry’*, Marja van den Heuvel-Panhuizen et al.,2004) describes three aspects of geometry, orientation, construction and operation. *Orientation* is defined as determining one’s own position and that of objects in space and being able to describe motion in space. During this process, elementary orientation concepts are used such as direction, angle, distance and coordinates; pupils must also be able to interpret a visual model of a spatial situation that is observed from a specific viewpoint. *Construction* is defined as activities where pupils make shapes and figures themselves, while *operation* is concerned more with thinking and working with these shapes and figures; this involves shifting, mirroring, rotating and projecting. This classification does not mean that a geometric activity always takes place entirely within one of these three aspects. During an activity, multiple aspects are often addressed, depending on the nature of the activity at that time. The three-part classification – orientation, construction and operation – which was chosen in the above-named TAL publication is concerned with what pupils do as part of a geometry activity. It also helps to structure education in the lower grades. After all, the geometry education during these school years is primarily concerned with acquiring tangible experiences and exploring space as part of rich activities. Aspects such as reflection, comparing, explaining, making mental images and predicting begin cautiously in grades 1 and 2, but these aspects are given more emphasis only in grade 3 and afterwards.

For geometry in the lower grades, the three-part classification – orientation, construction and operation – indicates what the pupils have *to do*. For geometry in the upper grades of primary school, we have chosen a three-part classification that indicates what the pupils have *to be capable* of doing. We can divide the geometric phenomena with which pupils in primary school come into contact into two global areas: ‘spatial sense’ and ‘plane and solid figures’. The first, which links up nicely with one of the three

areas from the lower grades, concerns gaining control of the space around you. The second concerns the classification, description and discovery of properties of geometric shapes and figures in two or three dimensions.
The third area is ‘visualization and representation’. During the study of space and figures in space, we need techniques which make it possible to visualize the stubborn three-dimensional reality in two dimensions, and to be able to interpret two-dimensional representations of that reality, i.e. to convert them back to three-dimensional reality. First of all, however, we must model reality in general in order to acquire some sort of overview. For example, a map is a model of part of reality. This part of reality is represented by a sketch or a very detailed (sometimes even three-dimensional) map; however, it remains a representation. The skills of ‘visualization’ and ‘representation’ are developed during the entire learning-teaching trajectory and are an important learning-teaching result in the long term.
In summary, in the upper grades of primary school we have chosen for a division into three terrains: ‘spatial sense’, ‘plane and solid figures’ and ‘visualization and representation’.

This does not mean that totally different geometry material is addressed in the upper grades of primary school than in the lower grades, but that there is a transition in the nature of the geometry activity. It is still important that children create all kinds of geometric constructions and that they can also operate tangibly in these activities, but the emphasis in these activities shifts from construction and operation, which primarily had an exploratory character in the lower grades, to establishing connections, comparing and reflecting on all sorts of questions, and explaining and predicting. For each of the three terrains, we will explain this shift in emphasis in more detail and elaborate upon it.

The three terrains

The first terrain in geometry for the upper grades of primary school concerns what we call ‘spatial sense’. Activities in this terrain focus on gaining control of the space around us. Gaining control of space is a skill that is important enough to become an attainment target. This concerns aspects such as: finding your way in the city, imagining the world as a sphere and being able to derive all kinds of relevant conclusions from this, or using a floor plan to find a good seat in the cinema or theatre so you have a good

view of the screen or stage.
The attainment targets can be described with action-oriented verbs such as ‘to localize’, ‘to take a standpoint’ and ‘to navigate in space’. These are verbs which can be used for describing processes and skills such as localizing or navigating in space by using maps, floor plans and the wind rose, and for using elements from vision geometry, such as views and vision lines, to reason out whether things can or cannot be seen. With this ‘seeing’ we can reason from various standpoints. In this first terrain, the practical value of the acquired knowledge and skills is paramount.

We refer to the second terrain as ‘plane and solid figures’. On this journey of discovery through space, pupils are confronted with many shapes and figures, first very informally, but later on there is a need to apply some structure and classification. There are pointed objects on which you can injure yourself, there are round objects that can roll and which you can move, there are block-shaped objects that remain in a stable position, there are flat objects and three-dimensional objects. Identifying all kinds of different shapes while acknowledging their corresponding properties, reasoning about the relationships between these objects – such as the relationship between a square and a rectangle – and reasoning about their properties: these are mental activities and processes that fall under the inquiry terrain of ‘plane and solid figures’. Pupils learn to think logically about the properties of figures and about how these figures can be combined, transformed or divided to make new figures and patterns.

We refer to the third terrain within geometry for the upper grades of primary school as ‘visualization and representation’. As part of activities in the first two terrains, it is often necessary or practical to be able to visualize and represent situations. This certainly applies to many aspects of mathematics, but it is especially important for geometry. Some techniques are not just instruments, but have evolved to become relatively independent components of geometry.

Spatial sense

Introduction

Spatial sense is concerned with activities and problem situations that involve gaining control of the space around you. In this context, geometry as a human activity can be described as mathematical world orientation: we want to explain why things appear as they do; explain how your observation depends on your location; being able to discover what another person can see; find your location, often with respect to another location; and determine the best way to give directions to someone else. Based on the observations, this also concerns seeking an explanation model for the strange behavior of a shadow that sometimes appears to go in front of you and sometimes follows behind you, and being able to explain why a tower in the distance appears to rise above everything, but if you are only a street away from the tower you may not be able to see it all.

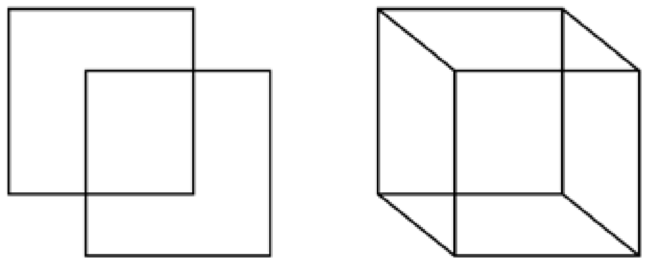
From the lower grades of primary school to the upper grades

The geometry education in grades K-2 centers on two aspects of orientation: localizing and taking a standpoint. In grades 3/4, a third aspect is added: turns and directions.
Localizing involves determining where something or someone is located. In addition, the localization data can be used to find something or someone. In grades K-2, the school itself is the most suitable context.

At the beginning of the upper grades of primary school, the pupils are directed towards a higher level of localization, and the children therefore go outside the school. The pupils are asked to make a scale model and to design a map of their neighborhood. They will also be confronted with schematic maps, such as maps that show the routes of busses and trains.
Later on, the pupils are steered towards working with ‘higher’ map skills. These higher skills include using coordinates, using directions that are more specific than the four cardinal directions (north, east, south and west), working with scale and understanding the differences between different maps of the same area, for example by determining what the various projections are, and making connections between these kinds of maps. Of

course, these projections are also part of studying spatial objects such as cubes and rectangular prisms, because here as well the children must be able to draw various representations. In these cases, a parallel projection is regularly chosen because the parallel lines then remain parallel. This projection technique is frequently used by designers and architects. However, this is not how we see a cube or rectangular prism in reality. We see a projection from a point.

In parallel projection, all lines that are parallel in reality are also drawn parallel. For making an image of a cube, this means that the front surface and back surface are drawn the same size. In this way, the pupils learn to make a simple drawing of a cube by drawing two slightly offset squares and connecting the corners.



In practice, we observe that sightlines are not parallel lines, but converge in the distance. Consider a railroad track as an example. This also applies to the cube; as seen by the eye, the sides converge. This is in accordance with the fact that we see the front of the cube as being larger than the back. Pupils can experience this by studying photographs or by drawing what they see on a sheet of glass while they are looking at a cube.

With taking a standpoint, one can think about looking at the same object from different positions. In grades K-2, pupils look at a self-made construction from multiple positions in order to make a construction drawing of the object. In grades 3-6, the experience that some objects can look differently from another standpoint leads the pupils to independently make various views of the objects. On Rekenweb.nl, two computer programs are available for reasoning with views: ‘Building with blocks’ and ‘Treasure island’.

At this stage, many children are also developing skills to mentally ‘read’ from different views and create a mental image of an object. In grades 3 and 4, the distinction between the various views (front, top and side) can be effectively addressed. Later on, inquiry is conducted at a higher level about the effect of viewing an object from various positions. It is seen as self-explanatory that a ball remains a circle regardless of their viewpoint, but that in a two-dimensional representation, a rectangular prism can appear to be a cube, and a circle can appear to be a line segment. The knowledge of views can be expanded upon by offering more complex problems and making more precise analyses. We will return to this as part of the ‘cube sawing’ activity that is a central focus of the learning-teaching trajectory ‘plane and solid figures’.

Vision lines play an important role in geometry. You can use vision lines to visualize what you can see and what you cannot see. In grades 3-6, pupils learn to use vision lines when drawing a view of an object (see figure 4.14). In their investigations into shadow images, pupils discover that light beams are closely related to vision lines; shadows and ‘blind spots’ (the areas that you cannot see) are very closely related in conceptual terms. When we are looking somewhere, we may not see a specific area because something is in the way. If we move to a different standpoint, this blind spot also moves. We can see something that we did not see previously. Something similar happens when we move a light source and determine what effect this has on the movement of the shadow. In both cases, the phenomenon can be explained by discovering that the vision lines and light beams are straight lines. And with these straight lines – for example in shadows and blind spots – we can perform

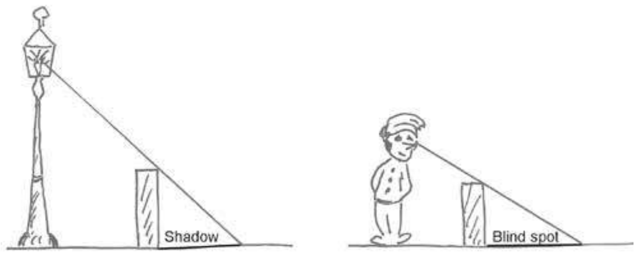


Figure 4.2

geometry, especially if we take into account the relationships that appear almost automatically.

With *turns and directions*, we are concerned with the role of turns during movements in space and the resulting changes in direction. Both of these aspects are addressed in grades K-2 in the context of walking along a route. The pupils experience the description of such a route, which uses concepts such as left turn and quarter turn, with their own bodies. These experiences are the basis for constructing mental images.
In grades 3-6, the angle concept is addressed from the context of the turn (see figure 4.15). The turn that you make, the direction in which you move and the bend in the path that results: all are expressed in angles.

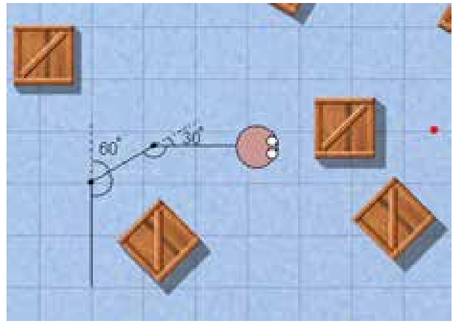


Figure 4.15

> *Computer programs for navigation*
> Directions and turns can be indicated in two ways: relatively and absolutely. The two computer programs 'Robot' and 'Koers' ('Navigation course') show the difference clearly (see figure 4.16 and 4.17). In the first computer program, a robot must be steered by indicating how it must turn on its own axis. A turn of 90° means a quarter turn to the right, regardless of the initial position of the robot.
>
> In 'Koers' a boat is steered according to the directions from the wind rose: N, NE, E, and so forth. In 'Koers' the turn is not specified with respect to the direction as seen from the boat (relative), but according to an independent system of directions (absolute).

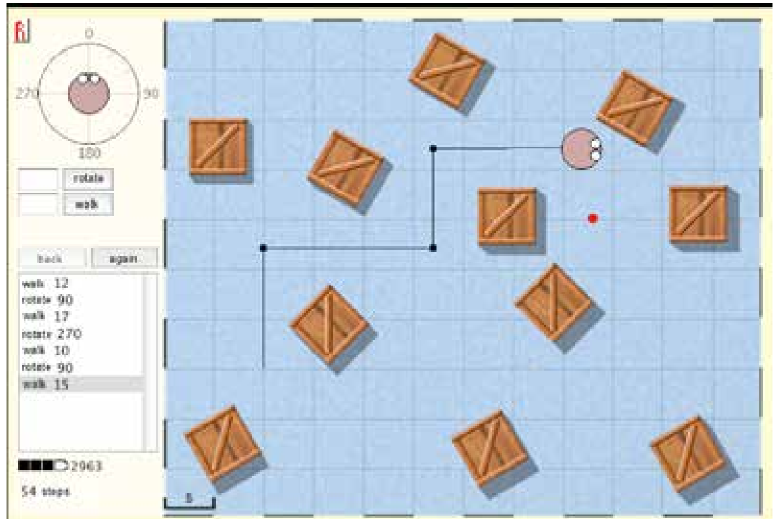


Figure 4.16

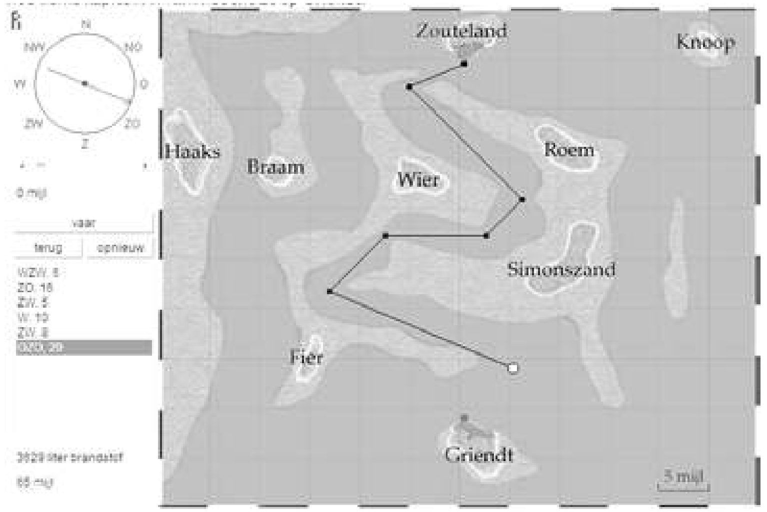


Figure 4.17

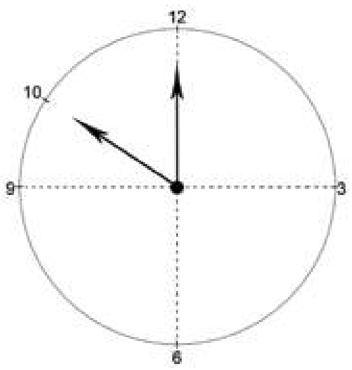


Figure 4.18

In grades 1 and 2, additional expressions are discussed such as quarter turn, half turn and whole turn. The relationship with 90° angles (right angles) can be addressed in grades 3-6.

The difference between a relative direction and an 'absolute' direction is also discussed. With respect to relative direction, from the reference point of your own position you can say: go straight ahead and then turn left at the second intersection, or 'the enemy is at 10 o'clock'. These formulations are based on the position of the speaker or the direction in which the vehicle or aircraft is moving. When we say 'go straight ahead' with reference to a vehicle, this refers to the direction of 12 o'clock; 10 o'clock means that we go forward and to the left (see figure 4.18).

If the direction is derived from an external system, then we refer to absolute direction. For example, the cardinal directions north, east, south, and west are absolute directions. North could be straight ahead, but it could just as easily be to the left. But there is also a problem with this system. If you continue driving north, you always arrive at the same point: the North Pole. That if we continuously travel west, we never get to the same point. Finally, turns also plays a role when investigating symmetries, specifically rotational symmetry.

Core insights

To link up with the distinctions as formulated for grades K-2, we propose to make a very analogous distinction for spatial sense in grades 3-6:

- localizing,
- taking a standpoint,
- moving in space (navigating).

One difference with grades K-2 is that the turns and directions are placed in the somewhat broader context of navigating.

Localizing

With localizing, we think of something that is static, something that stands still or has a fixed location somewhere. This location must be found or described. For example, you can think of where you are located in space: describe your own position with respect to the surroundings or indicate where you are located on a map. It also concerns localizing other people and objects in space. There are three core insights that are involved with localizing: insight into maps, insight into directions and insight into distances. With respect to maps, we encounter all kinds of variations. The map in an atlas is a completely different type than the map with train routes. A hiking map is not particularly suitable for automobiles. Google Earth provides satellite views of areas, but this is not a map for navigation purposes. We often see maps that are not drawn to scale, where the designers have allowed themselves all kinds of artistic and other kinds of freedom. Does a map have to have the same scale across the entire image? Or can the scale change? If it can change, how do you do that?

Figure 4.19

Directions (such as wind direction) are important, also regarding the static aspect of directions as part of localizing. Distances are also part of localizing, even if we are not moving ourselves. But distances are present on a map, and these distances are different; maps can specify distances with formulations like 'as the crow flies', or 'along the road'.

Taking a standpoint in space
When taking a standpoint in space, aspects can be specified that we can classify as 'vision geometry': what can someone see from their position and how do they view the surroundings? What is the difference between the view from point A and the view from point B? The questions that are addressed here are initially general and are considered to be qualitative, but they increasingly become quantitative, with more precise instruments and drawings. The skill that must be acquired is to be able to determine what you see and how you see from a specific position. This skill requires insight; vision lines help you understand what you can or cannot see from a specific position. We consider this insight to be at the core of vision geometry.

Moving in space (navigation)
Moving in space (navigation) is concerned with the dynamic aspect of spatial sense. An object or person moves from one location to another. This is a topic that greatly appeals to children. After all, travel means adventure, and there is a great deal to discover on the earth, but outer space also comes into the picture. In science fiction stories, new planets are discovered and journeys are made to the depths of the sea. Many computer games also have a journey as a theme. Remote-control vehicles are always popular among children.
In grades 3-6, children learn that motion can be specified on maps, in route descriptions and with navigation systems (see figure 4.20). They are introduced to the world of instruments that can be used to indicate a change in position. During this process, they encounter concepts such as angle, direction (wind directions, compass rose, degrees), distance, scale, coordinates, longitude and latitude and route (different routes and shortest route).

Figure 4.20

The most important core concepts in navigation are direction and change in direction, expressed as a turn or an angle, together with the distance to be travelled. You can observe navigation as a child, for example when you are sitting in the backseat of the car, or if you get lost at a mall. You can navigate as a child, for example when you ride your bicycle to school or when you play a computer game and you control the movement of a vehicle on the screen. You can describe navigation, for example by giving directions to a friend or by using a secret language on a treasure map. You can explain navigation, for example if you have to show the route of an aeroplane travelling from Amsterdam to New York on a globe. You can predict navigation, for example by designing various routes. Aspects of visual geometry are also related to navigation: what happens while you are 'en route' with what you see and how you see it? In short, the dynamics of motion offer many possibilities for addressing geometric concepts and thinking about them in a meaningful fashion.

Plane and solid figures

Introduction

The topic of plane and solid figures concerns the description of relationships between and properties of these figures. By reasoning about and with figures in a range of inquiry activities, the pupils become increasingly aware of the properties of figures and their mutual relationships. They learn to recognize figures and are gradually able to name them based on their properties, and not only based on their appearance. During this process, the pupils learn to express themselves with increasing precision by developing and using geometric 'language'.

From the lower grades of primary school to the upper grades

In the lower grades of primary school, the pupils acquire a fair amount of experience with plane and solid figures, especially concerning the aspects of 'constructing' and 'operating with shapes and figures', as referred to in the TAL publication *'Young Children Learn Measurement and Geometry'*.
In *construction*, three-dimensional (solid) and two-dimensional (plane) objects are constructed from blocks, waste materials (such as packaging and toilet paper tubes) and paper. Children acquire all kinds of experiences, such as how objects are put together and how they can be built. They create simple block structures based on views. During the playful learning process, words for geometric concepts appear naturally, not only words such as round and square, but also block, cube and cylinder. In kindergarten and first grade, elementary characteristics are addressed, such as rolling a cylinder and being able to stack blocks.
In *operation*, children acquire experiences with plane and solid figures. By means of activities with mirrors and shadows, pupils come into contact with symmetries and images of figures. During activities with mosaics, they create beautiful patterns. Pupils learn to determine the mirror image of simple figures. At a certain level, they can indicate how mirroring works. What happens on one side of the mirror line or mirror surface is shown in exactly the same way on the other side, but then as a mirror image. The explanations that pupils can provide about this phenomenon, and therefore the language that they use, will generally be simple and will be still be

strongly bound to the tangible contexts and figures of the activity itself. As part of this explanation, lines play an important role, and that is also the case when explaining how the direction, length and shape of shadows are related to the position of the sun.
During a wide range of construction activities, pupils can experience the properties of and relationships between polygons – primarily triangles and squares, but also a wide range of plane figures. Together, these experiences establish a rich basis for pupils in the upper grades of primary school to make these properties and relationships explicit and to work out their details.

At the beginning of the upper grades of primary school, the learning-teaching trajectory is primarily characterized by two developments: a shift from 'tangible' to 'mental' and an expansion of the constructions to include geometric constructions.
During the shift from tangible to mental, spatial cognition comes into the picture. The pupils think about objects (including three-dimensional ones) that can be constructed in the real world, and they think about how they will appear. By continuing to give 'action' a clear role in this process, the children develop insight into how the world is put together in terms of geometry.
For example, consider the geometric construction that is created in response to the question: how do you make a square? The following shapes and figures are addressed: square, rectangle, triangle, circle, cube, rectangular prism, cylinder and cone.
When operating with these shapes and figures, the topics addressed in grades K-2 – mirror images, mosaics and shadows – are discussed in greater depth with the aim of having the pupils expand their knowledge of plane figures.

In grades 3-6, the emphasis comes to lie on the relationships between and the properties of plane and solid figures. During this process, the descriptions become increasingly formal. In the case of a square, for example, we can no longer be satisfied with only identifying the figure and knowing how to draw it. The differences between a square and other quadrangles must not only be investigated and named, but the properties of the square will also become apparent by investigating the square itself. For example, the

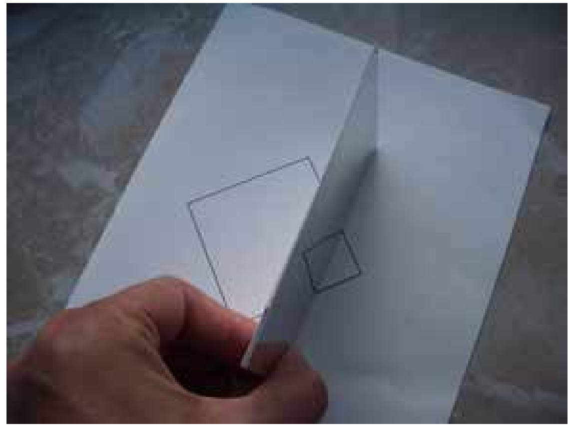


Figure 4.21

teacher can ask the pupils investigate a square with a mirror (see figure 4.21). Which quadrangles can you create in this way? How do you know that it is a square?
Figure 4.21

Pupils are encouraged to develop geometric insight into the classification of solid figures based on comparison, reasoning and making connections. This does not involve learning mathematical terms and definitions of concepts such as equilateral triangle, parallelogram or symmetry axis. That is a task for secondary education. But it does concern the development of the informal knowledge, language and insights that enable children to understand and formalize these concepts after they enter secondary education. Geometric terminology will be used, but on a limited scale and in a natural way. If a specific geometric term is introduced, this emerges directly from the need to name such a concept and to facilitate communication. For example, a term such as parallelogram can emerge because pupils discover that this figure is different from a rectangle, and they have a need to give it a name. More important, however, is that pupils explore polygons and understand how to recognize and name a number of differences. After this, they can use several properties of simple figures such as rectangles to determine when they have cut out or drawn a rectangle.

Core insights

By linking up with and expanding upon the aspects from the lower grades of primary school, we arrive at the following structure for the upper grades:
- properties of and relationships between figures;
- operations, transformations and constructions.

Properties of and relationships between figures
Gaining control – and wanting to gain control – of two and three-dimensional figures begins early and in an informal fashion. This is often based on three-dimensional reality, occasionally on the plane. During this process, the solid figures of the sphere, cylinder, cone and cube play the leading role. There is less emphasis on the plane figures of the circle, triangle, square and rectangle. The concept of the angle plays a very important role with these plane and solid figures, especially the right angle. Using this as a point of departure, the concepts of 'perpendicular' and 'parallel' can be addressed.

It goes without saying that pupils must know more than the properties of figures. For example, in order to reason logically about these concepts, they must know what a square is and what a rectangle is. This reasoning emerges when relationships between three-dimensional figures are addressed, for example in a lesson where the pupils investigate which figures are created when a cube is sliced in different ways. The discussion in the class then concerns questions such as: can a cube be sliced so that a pyramid is created (see fig. 4.22)?

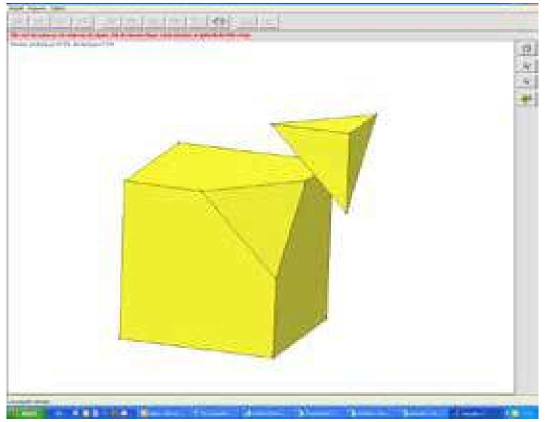


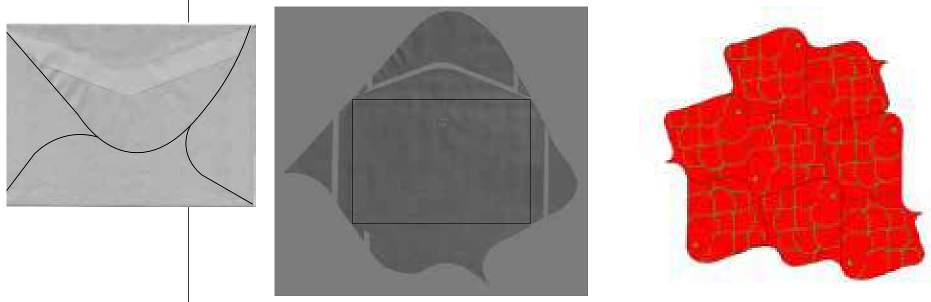
Figure 4.22

It is especially because the pupils encounter plane figures as part of space that it is vitally important to establish a relationship between solid and plane figures. For example, this concerns the question of what the folded out shape of a cube can look like, or the folded out shape of a cylindrical tin. The emphasis is on understanding and spatial reasoning. During this process, the angles of a polygon are important. Geometry education must establish an insightful relationship between the angles of a polygon and the angle as it is used when making turns and giving directions in the terrain of 'spatial sense'.

Operations, Transformations and Constructions

As in the lower grades of primary school, operations involve mirror images, mosaics and shadows. The pupils expand their knowledge of shapes by determining mirror images, making mosaics and playing new forms of Tangram. The latter two activities will emphatically involve working on the computer. The informal knowledge of various geometric operations and transformations can be expanded based on small-scale inquiry activities with plane and solid figures. Patterns can be analyzed with the aid of shifts, mirror images and rotations.

By using the 'envelope technique', you can always create a figure that you can use to tile a floor. Take a sealed envelope and cut it open as shown in the drawing. Make sure that you begin at a corner and end at a corner. Fold the envelope open. Trace the pattern several times onto another sheet of paper and cut out the figures. Try to tile a floor with the figures.



The following question can then be asked: do you always have to use an envelope, or can you also use a regular sheet of paper? If you can use a regular sheet of paper, how should you do this? Can you also use a square envelope? Or a triangular envelope? Can you design an interesting figure such as a fish, dog, or maybe a whale?

In addition, the previous activity of enlarging and reducing figures will continued, where the shape stays the same, but the size changes. Constructions can include 'sawing' pieces off of three-dimensional figures. The 'operation' of sawing ensures that new figures and shapes are 'constructed' (see figure 4.24). Of course, this can take place at various levels of abstraction and formality. Here as well, the computer can play a useful role and can help provide insight.

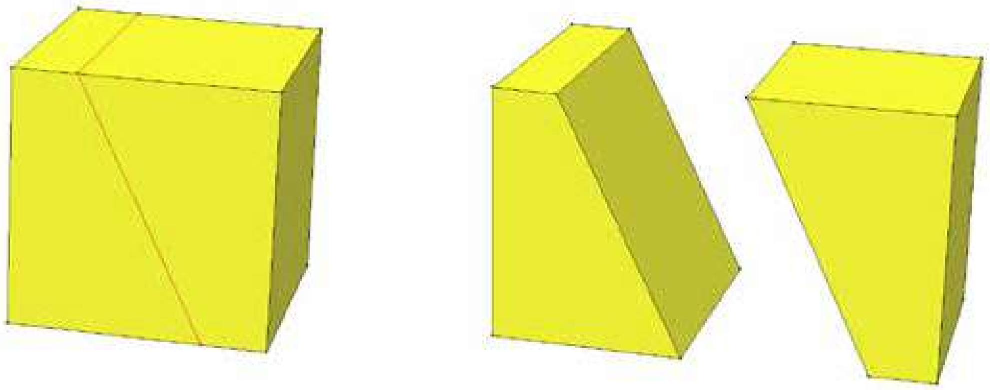


Figure 4.24

But this construction can also take place in two dimensions: how can you draw an equilateral triangle? For example, with the following question: can you 'construct' an equilateral triangle by sawing off a 'point' from a cube? This leads to experimentation, where children think up approaches to determine whether or not a triangle is equilateral. In addition, these experiments can lead to discovering a way to actually make the equilateral triangle. For that matter, it goes without saying that informal language is used with such experimentation. In the terrain of plane and solid figures, gaining control of space is also apparent in spatial reasoning about patterns and shapes that can be transformed into each other.

Visualization and representation

Introduction

The third terrain within geometry for the upper grades of primary school concerns the representation of two-dimensional and three-dimensional reality. This involves making a suitable drawing (in two-dimensions) representing a problem situation in order to analyze it and reason logically about it. Sometimes this means drawing a suitable cross-section of a solid figure, other times this means making a suitable projection drawing of a solid figure. Within this third terrain of geometry, pupils learn a number of aspects of two-dimensional representations of reality, such as the following. How do you make a two-dimensional representation? What information do they contain? How should you interpret them? Here are some examples: learning to read a map or floor plan, making a map of the surroundings or a foldout figure of a cube, drawing a cube or a rectangular prism, understanding how the solid figure is related to the three views, and determining how many views you need as a minimum to be certain about the shape of a figure.

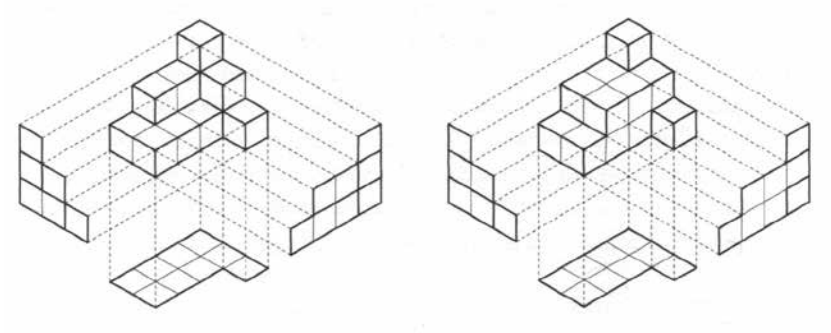


Figure 4.25

Of course, the terrain of 'visualization and representation' is strongly connected to the other two terrains of 'spatial sense' and 'plane and solid figures'. Geometric representations of reality play an important role in spatial sense. For example, you must be able to interpret a map in order to plan a route from one location to another. However, the emphasis now lies primarily on the representation itself: the process of interpreting and independently making representations, and developing the geometric instruments that you need to solve issues in the terrain of spatial orientation.

There are also many connections with the terrain of solid figures. For example, if you build a solid figure based on views of the object, you are not only involved with a geometric figure, but also with the knowledge and insight concerning the information that views contain about the solid figure. You could say that within the terrain of visualization and representation, pupils develop geometric instruments that enable them to geometrically represent both the capricious reality of the world around them and the mathematical reality of plane and solid figures.

From the lower grades of primary school to the upper grades

Visualization and representation are addressed in the geometry learning-teaching trajectory for the lower grades of primary school, but are not described separately (see: Young Children Learn Measurement and Geometry). However, in the upper grades of primary school, aspects of geometric representation are given increasing emphasis. Building upon the previous exploratory activities, such as drawing a castle that the pupils designed and built themselves, in the upper grades of primary school they develop instruments more systematically, and these instruments enable them to create representations. There is increasing attention to mathematical language and to models that enable the pupils to make improved explanations, predictions and generalizations.

Core insights for the upper grades of primary school

Compared to the other two terrains, 'visualization and representation' does not link up as seamlessly with the lower grades of primary school. This is primarily due to the instrumental character of learning to make representations and visualizations. This means that aspects of visualization and representation are also addressed in the other two terrains. Visual representations are an integral component of geometry, but they also play a role in other components of mathematics, such as number structures. It is therefore obvious that the pupils must be confronted with the many types of representations, especially those in geometry. Of these representations, projections are almost a science by themselves. Finally, an important aspect of representations and visualizations is their scale preservation. Therefore, we have proposed the following structure:

- types of visualizations and representations;
- projections;
- scale preservation.

Types of representations

A visualization or representation is a schematic representation of a specific part of reality. The level of detail can vary: on a photograph you see a many details, while a scale drawing provides more information than a sketch. Pupils come into contact with many types of geometric representations of reality that they learn to interpret by converting two dimensions to three dimensions. They also learn various ways of representing reality on paper by converting three dimensions to two dimensions. Ultimately, they must become skilled in using all kinds of visualizations and representations of reality.

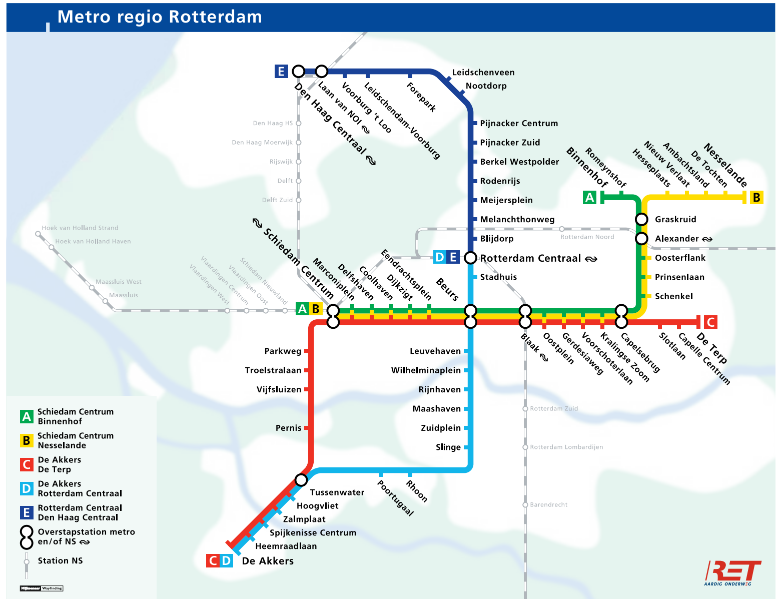


Figure 4.26

Pupils must acquire many skills in order to deal effectively with all sorts of representations. The following types of visualizations and representations are addressed in the upper grades of primary school: views (top view, side view and front view), perspective drawings, photographs (distortions), floor plans, various types of maps, schematic representations (such as a metro network, see figure 4.26), construction drawings, cut-outs, the same cube from different perspectives, graphs and foldout shapes of figures.

Projections

In geometry, projections are actually a separate science. Of course, they are addressed at an earlier stage in the component 'types of representations' because projections are a way to capture three-dimensional reality in two dimensions. In reality, you see parallel train tracks converging at the horizon, which is a source of questions for younger children. You can see this on a photograph, and it can also be drawn this way in perspective projection, complete with a vanishing point. But engineers prefer to draw the tracks so that they remain parallel to each other. The name of this method, parallel projection, is self-explanatory, although the drawings made with this projection are sometimes not very pretty (see fig. 4.27).

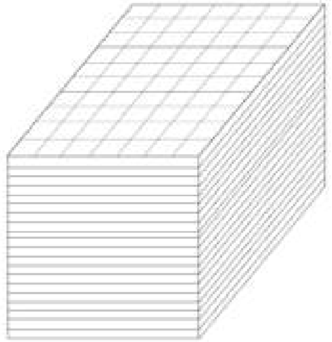


Figure 4.27

Even though the result appears less realistic, this representation makes it more suitable for measurements and determining proportions. Therefore, this projection is more frequently used in technical and mathematical drawings.

We encounter projections again in the context of shadows (see fig. 4.28). This now concerns the difference between shadow images caused by the sun and shadows thrown by the light of the lamp.

Figure 4.28

We then refer to the shadows thrown by a lamp as central projection (or point projection), and those caused by the sun as parallel projection. Based on their investigations into shadow images, pupils can understand the differences between both concepts and use them to solve problems. To this end, they will have to become skilled at using several elementary drawing techniques.
A third category of projections with which pupils become acquainted in the upper grades of primary school comes from the world of cartography. The earth is shaped like a sphere, and by doing simple experiments the pupils can perceive that it is difficult to arrive at a two-dimensional representation of the earth's surface. For hundreds of years, all kinds of techniques have been developed to offer a solution to this problem (see fig. 4.29, 4.30 and 4.31).

Figure 4.29 Cylindrical (equidistant) projection

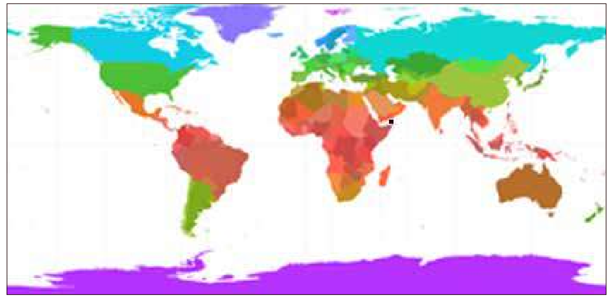


Figure 4.30 Mercator (conform = angle-preserving) projection

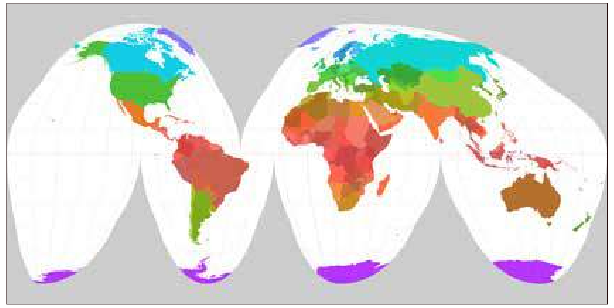


Figure 4.31 Goode (equal area) projection

Although projections clearly offer a wide range of possibilities, it will become obvious that we must be restrained about the formal aspects of this component in the upper grades of primary school. At this stage, we should primarily work on seeing the different types of projection, comparing them, citing their advantages and disadvantages, and connecting them to other aspects where possible. This topic will be explored in greater depth in secondary education.

Scale preservation

Finally, we will discuss the insight into scale preservation of representations and visualizations. Part of reality can be represented by sketches without taking account of proportions. In some cases, after all, this is sufficient. However, if it concerns examples such as a map or floor plan that you want to use for measuring purposes, then the proportions must be correct. If the distance AB is twice as long in reality as the distance CD, then this must also be the case on the map (internal proportions). Moreover, if you have to draw conclusions about distances in reality based on distances on a map, then you must also know the scale of the map. After all, the scale describes the proportion between distances on the map and distances in reality (external proportions). In other words: what reduction factor was used to make the map?

5 Learning-teaching trajectories in geometry

Introduction

In this chapter we describe the learning-teaching trajectories in geometry for each of the three terrains 'Spatial sense', 'Plane and solid figures', and 'Visualization and representation'. Every learning-teaching trajectory description is connected to an example activity that is described, analyzed and discussed based on the core insights. The following example activities are addressed:

- Navigation: treasure hunt on Moony Island;
- Construction: cube sawing
- Representation: bringing shadows into the picture.

The building blocks for a learning-teaching trajectory emerge during the description of the activity. The most important elements of such a learning-teaching trajectory are shown in a schematic representation, including the topics that are related to the activity. Finally, in order to picture the learning-teaching directory – or make it more concrete – we discuss a series of possible activities. These descriptions are examples, but they can indicate valuable content for the learning-teaching trajectory.

The description of a learning-teaching trajectory, or curriculum structure, has an inherent danger that it will be understood as the only structure that can lead to understanding the concepts to be learned. However, in the case of geometry, there are many possible routes to acquire insight into specific concepts and to learn to use them.

Navigation: Treasure hunt on Moony Island

Introduction

As part of the first geometry terrain for the upper grades of primary school, Spatial Sense, an example of activity has been developed as part of the

TAL project for pupils at the end of grade 5 or beginning of grade 6 with the title ‘Treasure hunt on Moony Island’. This activity is placed at the center of a learning-teaching trajectory description. However, you should realize that this is an example description: this learning-teaching trajectory does not cover all topics addressed within the terrain of ‘Spatial Sense’ in the upper grades of primary school. Moreover, it concerns one possible learning-teaching trajectory; it is possible that other connections between topics will be made, resulting in a different trajectory.

Before we look at the building blocks for the learning-teaching trajectory presented here as it emerged from the analysis of the activity ‘Treasure hunt on Moony Island’, we will describe the activity itself.

Target of the activities

The activities concerning Moony Island have been developed with the intention of enabling pupils to acquire experience with navigating from a distant position, i.e. drawing out or following a route, in this case on a map. They discover during this activity that two core insights are important: direction and distance .After all, if you want to indicate on a map how you can travel from point A to point B, you must know which direction to go and how long you must continue in this direction before you have reached the destination.

Besides acquiring insight into the concepts of direction and distance, it is the intention that the pupils fathom how the picture of the wind rose can be used as a measurement instrument for direction (the compass rose) and how the scale line of a map can be converted into an instrument (the measuring strip) for measuring distances. They learn to use these instruments to draw a described route on a map, as well as to adequately describe a route themselves. During this process the pupils experience that transferring directions to the plane corresponds with shifting lines in parallel. The pupils must realize the consequences of the degree of precision with which they are working.

The assignments

The context of the pupils’ activities concerns an island shaped like a certain tropical fish: Moony Island. The text for the pupils sketches out a number

of problems concerning a pirate treasure that was buried on the island a long time ago.

> Long ago, Redbeard the Pirate and his ship the Brigantine arrived at the bay of Moony Island, located in the South Pacific. He had an enormous treasure on his ship which he decided to bury on this island. And so he did. When he was finished, he made a map of the island on which he wrote how you can get to the treasure from the bay. Redbeard himself never returned to the island to reclaim his treasure, but the map of Moony Island was found …..
>
> On the map, you can see where Redbeard anchored his ship: X marks the spot.

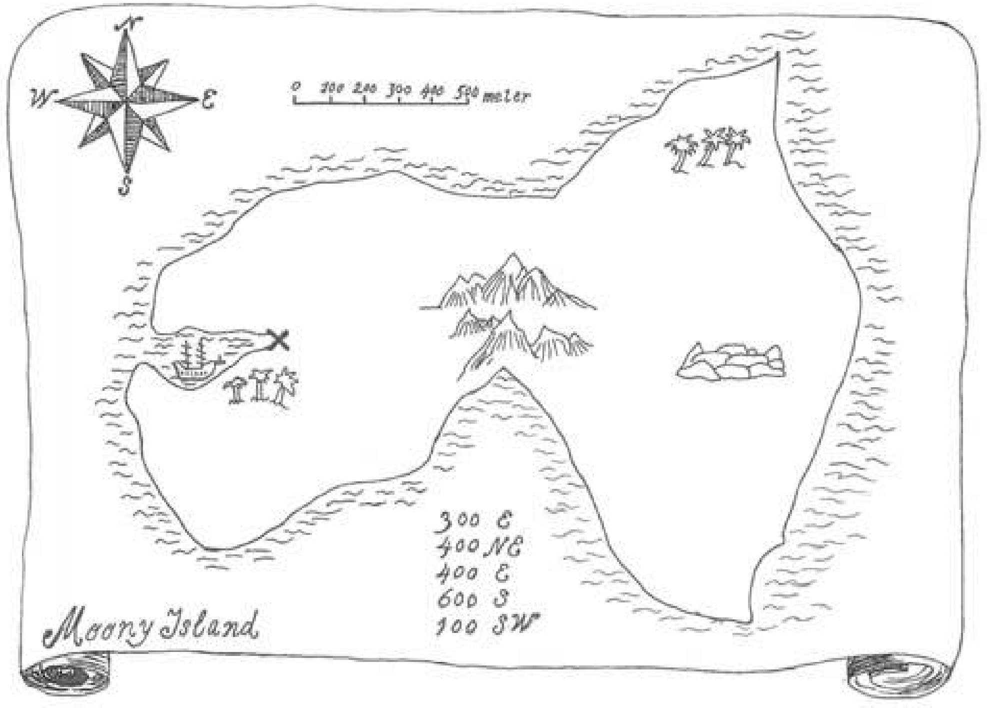


Figure 5.1

1. On the map, draw how you should walk according to Redbeard's directions to get to the treasure, and mark where the treasure is buried. Use the compass rose and the scale line.

However, during the many centuries that have passed in the meantime, the area between the mountains and the rocks on Moony Island has become an impenetrable swamp, which forces you to go north of the mountains and east of the rocks.

2. Draw another route to the treasure and write the instructions in the same way as Redbeard did. Use the compass rose and the scale line.

3. How much longer or shorter is the new route compared to that of Redbeard?

The pupils are given the treasure map, on which a wind rose is pictured with the four cardinal directions, and a scale line, which allows them to read distances up to 100 m with precision; in addition, they are given a transparent compass rose and a measurement strip to use for completing the assignments.

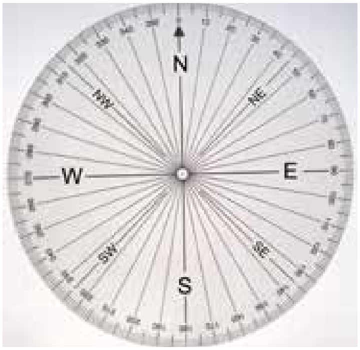


Figure 5.2

Figure 5.3

Follow-up to the assignments

The context of a treasure hunt using a treasure map is very suitable for bringing pupils into contact with navigation. All ingredients for a rich inquiry situation are present. The context is supported by providing an historical background to the origin of the treasure (piracy). An interesting introduction in the class, for example with an exciting story, would obviously be a good way to begin the activities.

Treasure maps are often shown with an X on the map to show where the treasure is buried, and the treasure hunter simply has to dig at that spot. However, Redbeard has made it more difficult: the X indicates the starting point for the treasure hunt, which takes place according to more or less cryptic instructions, either in reality (walking on the island) or in this case on the map. The process of decoding the route information and being able to draw the route on the map are the main ingredients of these activities.

Moony Island in the class

The teacher introduces the context of digging for a pirate treasure, and looks at the map of Moony Island with the pupils.

It was the time of the great explorers and traders who were searching for new destinations and who made maps of their journeys. At that time, there was still another group we did the same thing: the pirates. The teacher introduces Redbeard as a real villain and explains why he, like other pirates, always had to choose a location to hide their loot. This is followed by a story about how Redbeard came up with a cunning scheme to secretly hide his treasures at a location on Moony Island that only he knew about.

The children begin to realize that something involving treasure maps is about to happen. The teacher confirms this, shows them the map of Moony Island and asks them what they can see on the map. While she hangs up the map on the wall, the children talk about the name Moony Island and discover that the island has the shape of a Moonyfish. All the images that are drawn on the island are discussed.

Finally, the X is examined. Some of the children think that this shows where the treasure is buried, but the teacher explains that in that case the X would immediately show where the treasure is buried, and Redbeard was very secretive. The children philosophize some more about where the treasure could be located until the teacher asks the children to describe what else is on the map. The children discover the compass rose and scale line, talk about them and link them to the secret language on the map. During this process, some of the children use their arms to indicate what would happen if you go 400 E.
The teacher than explains what happened to Redbeard. After this, the children are all given a copy of the map with the assignment: find out where Redbeard buried the treasure.

The discussion of the map of Moony Island shows that the historical context of the problem situation is exciting for the students. During the discussion about the treasure map, it is striking that 'reading' the map is not very difficult for them. The pupils identify the images on the map and can give them a meaning. Most of the children have seen the wind rose before. The X is initially seen as the location of the treasure, but the pupils quickly realize that this is unlikely. The coding of the route is compared with other information on the map, and is consequently given a meaning. The pupils recognize the instructions with the wind directions, and they link the numbers with the directions to the numbers on the scale line. Before long, the children are already drawing the route. They want to find the treasure!

The children go to work in groups of two or three. The teacher passes out measurement strips, which they use to transfer the scale. They can use these strips on the drawing as a ruler.
In one of the groups, a boy is finished almost immediately. He says proudly, 'this is where the treasure is buried'. The teacher sits down with the group and asks why the boy is so certain that the treasure is located exactly there. In her question, she emphasizes the direction NE. The teacher gives the boy a wind rose so he can measure the direction again. He measures again and confirms his previous drawing.

The teacher tells the children that when they are done, they should compare their location for the treasure with that of a classmate. She explains that you can do this by placing the maps on top of each other and then looking to see whether the treasure is located at the same point on the map. The teacher encourages the children to work as precisely as possible. As a result, the children re-examine their drawings to adjust the location of the treasure if needed.

For the children who have quickly found the treasure, the teacher asks them to describe the return route of Redbeard when he walks back to the ship after burying the treasure. She asks another pupil to measure precisely how far it is along the route from the ship to the treasure.

The teacher gets into a discussion with a number of other children about why their maps are slightly different from each other. The pupils think that it has something to do with the direction of the drawn lines. They point to the angled lines. The teacher makes a step towards a refinement by asking what NNE Could mean. One of the boys explains that this is north of NE.

The teacher gives the class a new assignment (Assignment 2). Now the pupils have to draw and describe a new route related to the impassable swamp that now exists north of the mountains and east of the rocks. One of the boys wants to know what a direction is called if it is between NW and N. The teacher refers to the previously named NNE, and the pupils figure out NNW. After this she points to the direction between NW and W, and asks what this is called. The pupils do not come up with the answer WNW. The children are working at different rates, and some of them are unable to proceed any further by themselves. At the same time, some of the children have already drawn and measured the route along the swamp. They are allowed to make their own treasure map with a route that leads to the treasure.
Between these two groups, a clear difference in level and learning rate becomes apparent. The pupils help each other to refine the compass rose, and they check each other's work to determine whether they now make their own treasure map

The teacher wants everyone to measure the difference between the drawn routes (Assignment 3). With a few of the students, she holds a discussion about the degrees that are shown as numbers on the compass rose. During this discussion she asks: how many degrees are there in a full circle? And how many degrees are there in a half circle? Due to these questions, some of the pupils are challenged to define the direction even more precisely with degrees instead of using the cardinal points on the wind rose.

Drawing the described route is tackled at various levels. Some pupils focus strongly on the result and want to see the route on paper as quickly as possible. But this approach usually has a negative effect on the required precision. When comparing the locations of the treasure that have been found, the teacher takes the lead in indicating how this comparison can be made: by placing the pages on top of each other. After all, the clues are very clear, so all the children should end up at the same location. While drawing the route, the emphasis is primarily on transferring the direction: on the wind rose you can see which direction is east, but how do you draw a line to the east from the X? In geometric terms, this concerns shifting or transferring lines in parallel.

The follow-up assignment that the teacher presented to the fast pupils – describing the return trip of Redbeard – enables the pupils to acquire insight into the fact that every direction has an opposite direction. N becomes S and NW becomes SE. During the lesson, it turns out that the pupils have a rapidly increasing need to use more directions when describing a route. The limitation of the cardinal directions becomes immediately apparent in this lesson, and taking the next step towards degrees appears obvious. This is also one of the key points of the follow-up discussion that the teacher holds with the class.

The teacher first wants to know if the children have all ended up at the same treasure location. Initially, the pupils answer yes, but then start to hedge their answers with ‘not exactly’. The teacher wants to know more. What do you mean, ‘not exactly’? The pupils explain that they did come out at the same point because they exchanged routes

with each other. The teacher asks, what would have happened if you had not exchanged routes? One of the children thinks that in that case they might have come out one centimeter above. One centimeter, how many meters is that? The children reply with answers such as 100 m and 100 km. From 100 m it becomes 99.5 meters. The teacher confirms that it will not be exactly 100 m.

‘If you start to dig there, will you find the treasure?’ Most of the children don’t think so. The teacher asks how much 100 m is in reality. The answers include ‘as big as the classroom’, ‘the classroom and the playground’ and ‘a football field’. The teacher uses the example of the classroom plus the playground and confirms that this could be possible. Yet another pupil answers, ‘10 times a 10-metre ruler’. The teacher smiles and asks jokingly, how long is a 10-metre ruler? ‘Ten meters!’ responds the class in chorus.
The teacher asks again why the children did not all come out at exactly the same location. One of the children explains this is because the compass (pointing to the compass rose) is a little bit crooked. One pupil wants to explain how you can tell which direction is east. Her story concerns placing the transparent compass rose on the wind rose and then sliding it carefully. Another pupil wants to explain why she used numbers instead of directions such as N and NW. She was the only who did this. She explains that the numbers are degrees. You only have to write the numbers down to draw a direction. Is this useful? The pupil doesn’t know. Another pupil thinks that it is useful, but cannot explain why. All the pupils are asked to take their compass rose and find which number is next to NE. They quickly come up with the number 45. After determining several other directions and the accompanying degrees, one pupil remarks that 22.5 is not on the compass rose. Words such as NNE are discussed in that context. The teacher focuses on what one pupil said, that you can also look between the degree lines, for example between 45 and 46. One of the children realizes fairly quickly that this is 45.5, but he refers to this as 45.5.5. The teacher briefly refers to 42.25 and says that the children are on the right track.

During the follow-up discussion, experiences are shared and written down. For some pupils, this means that they are required to move to a higher level. This is especially the case when the discussion shifts to using degrees. When discussing the meaning of a deviation, most children start to think for the first time about the role of precision when drawing the route and the consequences of not following the route description precisely. Elements of the problem formulation include drawing on a floor plan while using the scale line, the meaning of scale in reality, and the consequences of imprecision. In order for the pupils to be motivated to work on the problem, it is important that they truly experience the problem. This approach can be successful, as shown in the above lesson report.
The problem of transferring a direction emerges naturally during the discussion because the teacher keeps asking about the differing treasure locations.

The activities in the perspective of a learning-teaching trajectory

An analysis of the assignments concerning Moony Island and activities in the class provides a view of the building blocks that can be used for a 'navigation' learning-teaching trajectory, as a component of Spatial Sense. The building blocks can be found in the schematic diagram of figure 5.12, where they have all been collected as a summary. We will explain each of the building blocks and provide suggestions for their possible application in education. During this process we consider the 'Treasure hunt on Moony Island' as the center of the learning-teaching trajectory.

In terms of geometry, the activities surrounding Moony Island concern *navigation* from a distant position on a *map*. With respect to navigation, there are two important aspects: *direction*, with the central question 'which direction do you take?' and *distance*, with the central question 'how far do you go in that direction?'. We will first examine the concept of direction.

Direction (turn and angle)

In the lower grades of primary school, pupils have acquired experience with moving in space based on their *own position*. For example, they can follow directions that concern a distance (number of *steps*) and a *turn*. A turn is in fact a change of direction: you turn left or right, therefore your new direction is related to your previous direction. Dividing a whole turn into a quarter turn, a half turn and a three-quarter turn links up with the image of the hands of the clock. After this, the pupils steered a *robot* with the aid of words such as go left, go right, go two steps ahead and make a quarter turn. The pupils have experienced that steering commands can direct the movement in space. During both navigation and steering a robot, we deal with *relative directions*. A steering activity, using turning and distance, is described in the box below.

As a game, pupils steer a 'robot' (a blindfolded pupil) through the class or another space. The process of steering a person in three-dimensional space naturally focuses on the movement of the feet. The route it is developed in this way can be defined as navigating in two-dimensional space (the floor as a plane). The 'robot' is steered according to the direction in which he or she is facing, and interprets the steering directions in the same way. This direction is the same as the direction in which the robot's feet are pointing. Based on this direction of view, another pupil steers the 'robot' to a predetermined target, for example to the door of the classroom. The language that is used is characterized by terms such as go straight ahead, go back, take a number of steps, make a quarter turn to the left, a quarter turn to the right, and so forth. With these terms, the ingredients of navigation – distance (in steps) and direction (in fractions of a turn) – become visible. The route that has been taken by the robot is recorded by drawing it on a floor plan of the room.

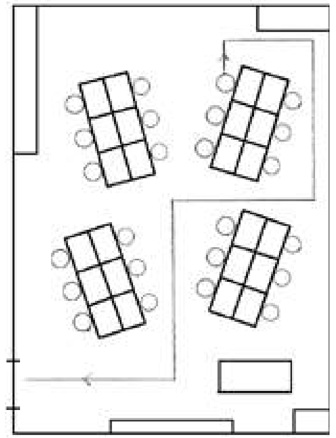


Figure 5.4

A follow-up activity addresses the visualization of the room (at scale) and the route that was walked as a line with angles. These are right angles and are not drawn by the pupils with an instrument such as a protractor. While working with the children it is possible to compare the destinations, to ascertain that there are differences and arrive at an explanation of how these differences occurred. Due to the size of the drawing, small differences in

Figure 5.5

drawing the angle will have few important consequences. The pupils also do not see these differences as being important enough to talk about. The children experience that it is necessary to agree about the exact meaning of changes in direction, such as a half turn, primarily in situations where the subsequent distance is greater (and therefore the deviation increases) or where the target is so small that you can easily miss it. The context of an aeroplane or a ship that can move freely in space is accessible in this situation. In the role of a pilot or steersman, the children can experience that small deviations in direction can have big consequences (see figure 5.5).

Besides changing directions when of making a turn, the pupils in the lower grades of primary school come into contact with aiming at a *target object*. For example, this concerns walking in the *direction* of the church tower. In this context, the navigation is done by the pupil who is moving. Based on his field of view, his knowledge of the surroundings and his sense of direction, he tries to steer towards the target object. A good example of the limitation of this method is the experience of walking towards the church tower, but once you get to the streets surrounding the church, you lose sight of the target and have to continue on your sense of direction only. A possible activity which gives more meaning to the distinction between direction and target object is described in the next box.

The teacher asks the children to all point with their right hands at the doorknob. What do you notice if you look at the direction all those arms are pointing? The children are given the assignment of drawing a top view of all their arms. Then the teacher asks all the children to point to the north (or to a familiar building or object at a relatively large distance from the school, such as a tower or a statue). The children are also asked to draw the top view of this situation.

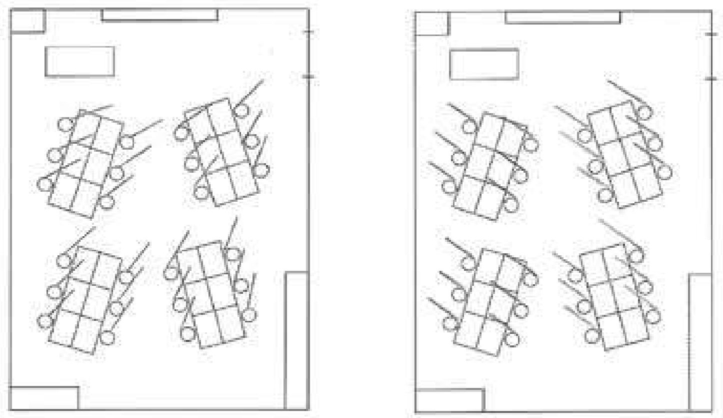


Figure 5.6

Based on the question 'What do you notice when you compare these two top views?', a discussion takes place about the difference between pointing at a specific point and pointing in a direction. Even better would be if the teacher went to the playground with the children and asked them do both of the 'pointing' assignments by drawing colored arrows on the ground. Examining the ' arrow field' leads to the discussion about the differences between direction and target object.

Direction becomes something that goes beyond the immediate surroundings. You no longer point at something, but in a specific direction.
It is important for pupils in the upper grades of primary school to realize that there are many ways to indicate direction. An example of a discussion about the concept of direction is shown in the next box.

The teacher talks with the children about the concept of direction by referring to phenomena such as wind direction, driving and walking direction, direction of view and signaling a direction. The fact that people use all kinds of indicators for direction is addressed during the discussion. For example, the discussion concerns a landmark; you must walk towards the tower. It could also concern another indicator based on the position of the sun, such as: keep the sun to your back. Direction can also be indicated by using the wind, such as: walk against the wind. The teacher summarizes the strengths and weaknesses of each indicator.

During the upper grades of primary school, the concept of direction is given additional content based on a *distant position*: looking at motion as shown on a map. The direction indicators that are used are related to the north as an *absolute direction*. With the activities involving Moony Island, the pupils are given tools to describe and determine direction, in terms other than quarter turn, half turn and three-quarter turn. Besides the above-mentioned methods to indicate direction, we can use the cardinal directions of north, east, south and west, but further refinements are also possible (such as north-northeast) and are sometimes essential. One possibility to bring up the cardinal directions for discussion is described in the box with figure 5.7.

The teacher asks the children about their knowledge of the four cardinal directions. Which way is north? And how can you determine which way is north? Among other aspects, this concerns deriving north from the position of the sun. The teacher asks the children who is familiar with the compass as an instrument for determining direction. The teacher may decide to that all the children should learn how to work with a compass. The wind rose is introduced as an instrument based on the compass. Besides the cardinal directions of N, E, S and W, the composite directions of NE, ZE, SW and NW are discussed.

In the near future, the pupils will be using the protractor, but in the meantime an interim phase is desirable and necessary. The *wind rose* and the

compass rose (with degrees), which is derived from the wind rose, can be used for this interim phase.

Finally, we will address the difference between *absolute and relative directions*. With relative direction, the starting point is your own position. An absolute direction exists outside yourself. When following routes that are shown on a map, absolute directions can be used, such as north and south, as well as relative directions, such as ‘towards Amsterdam’ or ‘towards New York’. We call the latter relative because the direction is based on the position from which you are travelling to Amsterdam or New York. This is an engaging topic for class discussion; an example of such a discussion is shown in the next box with figure 5.7.

In order to travel to a specific city, you can depart from various directions depending on where you are located with respect to the city. To get to Western Europe from the United States, you can go east, but you can also get there by going west. If you go west, you will travel through the Far East. The sun goes down in the west; in the Netherlands, we can see the sun ‘sinking into the sea’. On the East Coast of the United States, however, people cannot see the sun sinking into the sea. The North Pole is a specific location on the earth, but there is no West Pole. Travelling to the north ultimately means travelling to the North Pole. There is no final destination when you travel to the west. Travelling to the west can be replaced by travelling to the east. There are no final destinations in either direction.
The conclusion is that you will encounter directions such as west and north in both an absolute sense (such as Western Europe) and in a relative sense (for example, west of Amsterdam).

Figure 5.4

In terms of geometry, absolute direction means the parallel transfer of an ascertained direction, while the relative direction is obtained by connecting the point of departure and the ‘final destination’. The parallel shift requires

the pupils to fathom an idea which will not be completed until they are well into secondary education.
A separate problem is determining routes on the earth as a sphere. This means that if you travel in the absolute direction of north, you will always come out at the North Pole. If you travel east or west without stopping, you ultimately return to the place from which you departed.
In the context of navigation, as described above, we therefore have to deal with change in direction. This change in direction can be described, but the possibilities of *measuring* the change are also essential. During the Moony Island activities, both aspects are addressed. In this context, measuring involves comparing directions by placing the drawing and the wind rose on top of each other, and placing the drawings on top of each other. Description takes place by using the cardinal directions. Defining the *angle* that describes these changes in direction is an obvious step. We focus attention on the concept of the angle because it is so strongly affiliated with the concept of direction.

Angle

We need the mathematical concept of the angle to describe navigation activities on a map and to measure the result of this activity. The concept of the angle emerges from dynamic activities, in this case the change of direction. There are plenty of contexts in which we encounter this: the wind vane on the church tower, the windmills that turn towards the wind, the ship that changes course, the compass and the receiver dish that seeks a distant signal. In all cases, the change in direction can be expressed as a 'turn'. The static angle concept then comes into the picture when you look at the result of that change of direction: the tracks that remain behind. In fact, an outside angle changes to an inside angle (see figure 5.8).

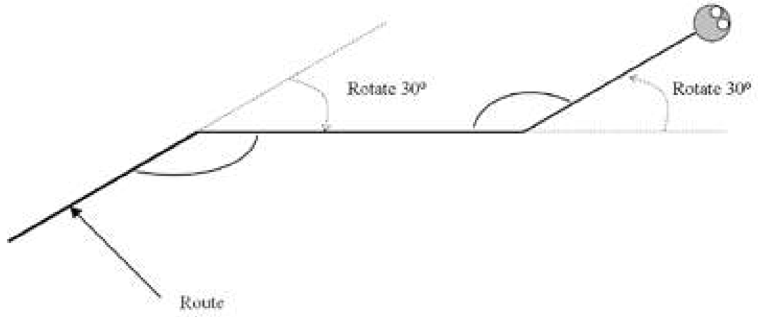


Figure 5.8

Within the terrain of 'plane and solid figures', concepts such as of triangle and quadrangle can be further analyzed concerning the word angle.
Measuring an angle is generally considered to be the domain of secondary education. Once the compass rose has been introduced, for example with the Moony Island activities, measuring angles can also be a useful expansion for some pupils in the upper grades of primary school. Investigating the sum of the angles in a triangle and a quadrangle is in store for secondary education.
A good context for the introduction of the static concept of the angle is a drawing robot or a sewing machine that embroiders figures. The appealing aspect is that a dynamic robot makes a static angle on the drawing. For a computer simulation of steering a robot, refer to the Robot program on rekenweb (www.rekenweb.nl). The static angle concept also comes emphatically into the picture when studying plane and solid figures. We refer here to the following example activity that is at the center of a learning-teaching trajectory description for 'plane and solid figures'. In addition, we encounter angles when studying the relationship between the position of the sun and the length of the shadows; after all, the angle of the rays of the sun to the Earth's surface is important for the length of the shadows. We are referring here to the final example activity in the learning-teaching trajectory description for 'Visualization and representation'.

Distance and scale

As stated previously, two concepts are essential with respect to navigation: direction and distance. We will now focus on the second concept.
As with direction, we can also indicate distances in different ways, for example with steps, in meters or in kilometers. As an indication of distance with the Moony Island activities, a *scale line* was used, which can be used to accurately read the distance up to 100 m. For drawing the route on the map, pupils can transfer this scale line to their *measuring strip*, which they can then use to step off distances on the map. As a result, we are of course now in the area of measurement, and will therefore not go any further. However, we will pay attention to the related concept of *scale*.
Being able to navigate on drawn-to-scale representations such as a map or a floor plan elicits a number of questions. Important aspects include not only the level of detail of the representation, such as the relevant elevation differences in the landscape and the relevant objects that are needed for navigation, but also the scale data and the question of what scale preservation actually means. Of course, the detail questions are addressed in geography. In the context of geometry, the aspect of obtaining scale data and the question of scale preservation are addressed.

Figure 5.9

In the first place, the true-to-scale representation of the earth (as a sphere) on the plane is, of course, an impossible task. This is easy to fathom when you try to paste a piece of paper onto a balloon. The various projections

that are used all have their own limitations. It would be going too far to discuss the types of projections and their limitations. However, for some pupils this can be an interesting challenge.

In terms of geometry, true-to-scale imaging entails making a representation where proportionally identical lengths are enlarged or reduced. The ratio between the lengths of images and reality can be described with a scale line, a description such as one centimeter on the map is equal to 5 km in reality and a proportion indicator such as 1: 500,000 (see figure 5.9).
Most modern maps use a scale line. The proportion indicator is the most abstract form. To give the pupils insight into the concept of scale, drawing an enlargement and a reduction of a simple object such as a square on graph paper is an important activity. On rekenweb, the computer program 'Vergroten' gives pupils the possibility to improve their understanding of the concept of 'proportional' (see figure 5.10).

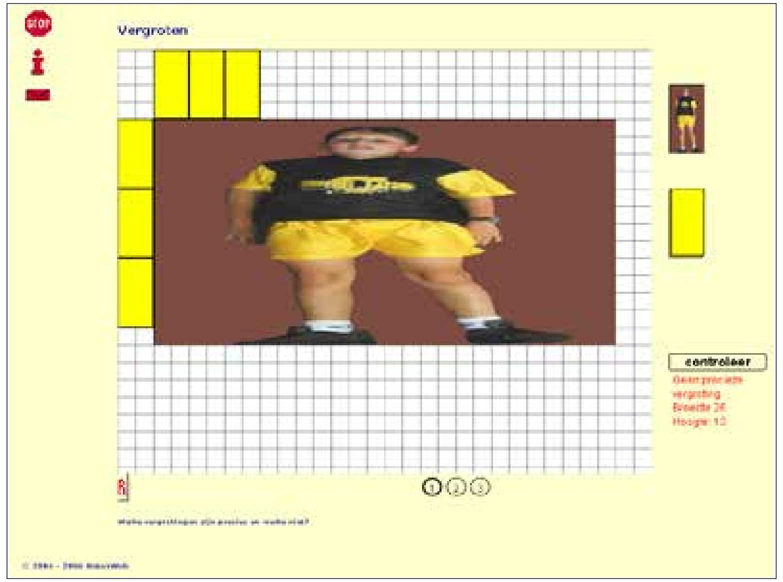


Figure 5.10

As we have seen above, you can navigate through space yourself and describe your journey, and you can steer an object from a distance from point A to point B. In the latter case, you need a representation of the space in

order to steer. In the upper grades of primary school, this representation is available in the form of a map or floor plan.
In the context of the Moony Island activities, we have assumed that the children at the end of grade 5 can imagine themselves in a constructed situation. The hidden treasure of a pirate comes into the picture by means of a treasure map that has been found. In this situation children can make use of a two-dimensional representation of space (a floor plan or *map*). The fact that pupils can imagine themselves in a situation also means that they can imagine how it is to go searching for treasure, that the treasure map can be a representation of reality, that you can represent a *route* by means of a drawn line, and that you - as a treasure hunter with such a treasure map and a compass rose - can actually walk the route from a starting point.
Before the pupils get started with the Moony Island activities, a class discussion during a previous lesson can be devoted to various ways to describe routes.

> The teacher begins the discussion with the question: can you tell a friend how to walk from school to your house? The pupils use descriptive language for their route description, for example:
> 'Go to the hardware shop, turn onto the street near a large building, near an apartment house you turn onto a street where there is a red car, and then go to number 21.'
>
> The following question focuses on the knowledge the children have about the school building. Can you tell someone how to walk from the classroom to the WC without going with them? Or how to get to the teachers WC? The pupils then use terms such as go straight ahead, turn left and take the second door on the left.
> During the class discussion, it clearly emerges that the pupils have different experiences with routes and descriptions of routes. For example, one pupil can describe the route from school to his own house in terms of changing directions at familiar landmarks that he encounters underway. Descriptions such as 'turn right at the tower, go across the bridge turn to the left, go into the alley at the red house, and after the viaduct turn the corner' indicate how the pupil navigates in his mind through his own environment. The fact that he does not realize that the landmarks he refers to, do not have any meaning for someone else, does not play a role. Pupils also do not consider using a drawing with a description. The description in language is very situational and is full of references to target objects. The same applies to route descriptions within the school building.
> As a third situation, the pupils are asked the route to a holiday address in Spain. Examples of answers are: 'Look on the map.' 'Use the Atlas!' One pupil is sure he knows the right answer: 'Get a TomTom' (a navigation system). The teacher wants to know what a navigation system is. 'You type in where you want to go and then it says, for example, that you have to turn right after 200 meters.'
> As the distance becomes greater, the discussion about the route must take place at an increasingly abstract level. The expectation is that the pupils will make a connection with a roadmap, but due to technical advances, the role of the map in route descriptions is being replaced – also in the world experienced by the pupils – by navigation systems.

The discussion about route descriptions and how you can give directions to someone has two benefits: the pupils take the important step of using a map (a representation) of the surroundings, and they can navigate better by using this representation.

We end this learning teaching trajectory description with two suggestions for sequels to the navigation activities (see box at figure 5.1). With both of these follow-up activities, pupils can acquire more in-depth knowledge concerning direction, distance and routes on maps.

> The pupils are given the assignment to draw the route from school to home on a self-made street map. During this activity, they pay attention to the scale of their street map. They also make a description of the route in terms of direction (using the compass rose) and distance.

On Rekenweb, the children play the game ‘Koers’, which requires them to use direction and distance for making a journey on a ship(see figure 5.11). In this game, the children can determine the direction by turning a needle on the compass rose.
A corresponding version shows the course in degrees (from the north). This new way to indicate direction gives the children the possibility to begin using the degree scale in subsequent lessons.

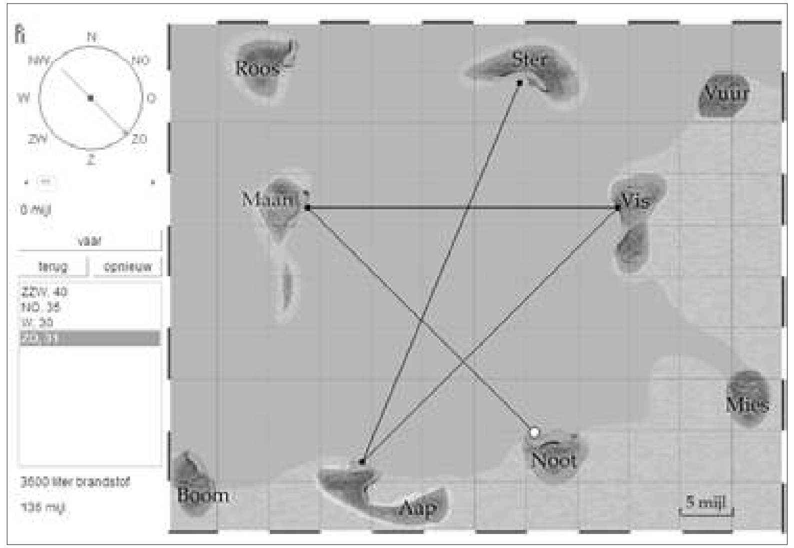


Figure 5.11

Building blocks for a learning-teaching trajectory in navigation

We will now summarize the above-named building blocks for a possible learning-teaching trajectory (see figure 5.12):

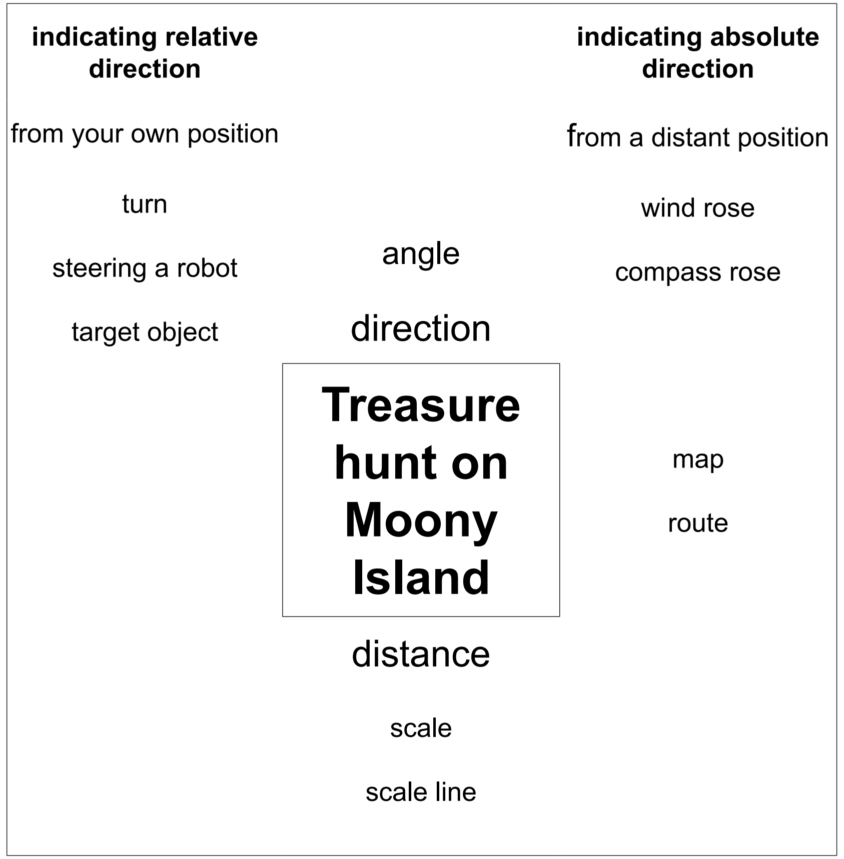


Figure 5.12. Schematic: Moony Island in the learning-teaching trajectory

Constructing: sawing cubes

Introduction

For the second terrain in geometry, 'plane and solid figures', an activity was developed in the TAL project which we call 'Sawing cubes'. It is advisable to spread this activity out across a number of lessons. A representation of the assignments that the pupils work on is followed by a description of the lesson. An analysis of the activities provides the building blocks for a learning-teaching trajectory description.

Cube sawing is at the center of a learning-teaching trajectory description for acquiring insight into and knowledge about plane and solid figures. Spatial cognition is the 'center of gravity' of this learning-teaching trajectory. In the same way as with the previous learning-teaching trajectory description, we will sketch out a possible learning-teaching trajectory and support it. Other choices, and consequently other accents, are possible for the structure of education within the targets we have presented concerning plane and solid figures.

Target of the activities

By literally placing solid figures on the cutting board and by sinking the figures in water, pupils investigate the properties of and relationships between plane and solid figures. The section that is created by cutting through a solid figure is called a 'cross-sectional plane'. You can use the actual cross-sectional plane to make a print to show that this is a plane figure with specific dimensions and a specific shape. The targets of the cube-sawing activity are: predicting the shape of the cross-sectional plane, seeking the relationship between the location where cross section is made and the resulting cross-sectional plane and reasoning about the properties of the figure that is created by cube sawing, based on knowledge about the solid figure.

The assignments

This activity can be introduced in class by cutting an apple in half; this provides a reason to discuss what making a cross-section means and what a 'cross-sectional plane' is. After this, the pupils begin investigating various cross-sections of a cube. After thinking about how they have to saw the cube in order to create a section with a specific shape, the pupils start sawing cubes, using a small hacksaw and cubes of oasis (florist foam). In this way, the process of mentally making a cross-section of the cube is confirmed by actually sawing the cube. After this, the pupils go to work on other assignments, where the cubes of oasis and saws remain available for pupils who want to use them.

For these assignments, four assignment sheets are used, where a workstation with specific materials is set up for each assignment sheet. The pupils can work on the assignments in groups, circulating between the workstations

For Assignment 1, a workstation is set up with a wire model of a cube which can be immersed in a bucket of water.

1. Assignments:
- Can you hold the wire cube in the water, so that exactly the same amount of the cube is above water and underwater?
- First think about how you would do this, and then try it.
- Can you do this in more than one way?
- On the work sheet, for each solution, draw the lines on a cube that show the water surface.
- Beside each cube which has been 'bisected' by the water surface, draw the corresponding cross-sectional plane.

Every assignment sheet includes a worksheet on which the pupils can draw the intersecting lines on the cube illustration and draw the corresponding cross-sectional plane (see figure 5.13).

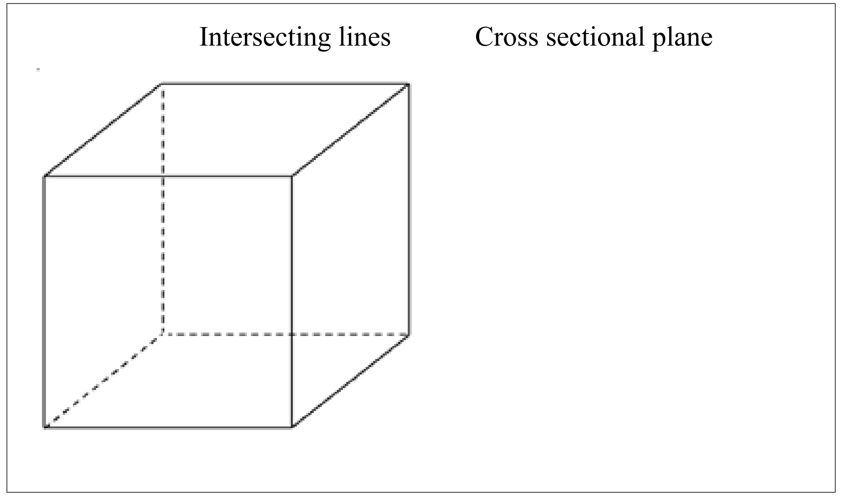


Figure 5.13

For assignment 2 pupils can use an open, transparent cube and a pitcher with colored water or soft drink. Moreover, they can draw on the cubes with non-permanent markers.

2. Assignments:
Pour a shallow layer of colored water into the cube.
- Think about how you can tip the cube so that the 'cross-sectional plane' (the water surface) is a square, a rectangle and a triangle. First think about how you would do this, and then try it.
- Can you also make a pentagon (a 5-sided figure)?
- Can you also make a hexagon (a 6-sided figure)?
- On the work sheet, for each solution, draw the lines on a cube that show the water surface.
- Beside each cube which has been 'bisected' by the water surface, draw the corresponding cross-sectional plane.

Assignment 3 is done at a workstation which is equipped with a closed, transparent cube on which three lines are drawn (see figure 5.14), a bucket of water, and non-permanent markers with which the pupils can draw on the cubes.

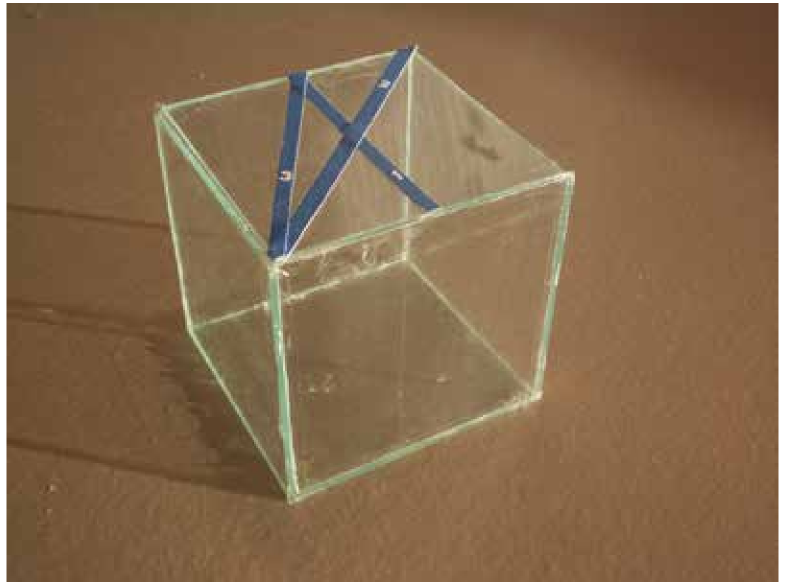


Figure 5.14

3. Assignments:
- There are three lines on the cube. If you place the cube in the water in such a way that Line 1 matches the water surface, what kind of shape does the water form around the cube?
- Do the same for Line 2.
- Do the same for Line 3. First think about the answer, and then perhaps try it out. If you are unable to think of the answer at first, you can try it out.
- On the worksheet, draw the lines on a cube that indicate the water surface: one cube for Line 1, one cube for Line 2 and one cube for Line 3.
- Next to each cube, draw the shape that the water forms around the cube.

A closed, opaque cube, a bucket of water and non-permanent markers are the materials that are available for assignment 4 (see box).

4. Assignments:

- How can you hold the cube in the water so that a three-sided pyramid sticks out above the water? How do you get the biggest pyramid?

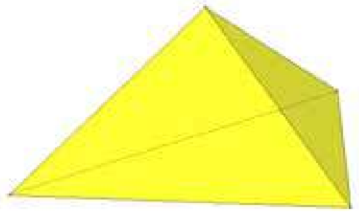


Figure 5.15

First think about the answer, and then try it out.

- How can you hold the cube in the water so that a 'Toblerone bar' shape (a triangular prism) sticks out above the water? How do you get the biggest prism?

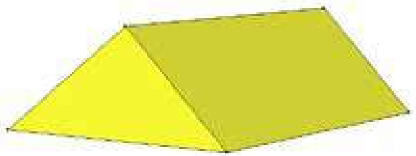


Figure 5.16

Once again, first think about the answer, and then try it out.

- For each solution, draw lines on a cube that indicate the water surface; do this on the worksheet for both the pyramid and the prism.
- Next to each cube, draw what the cross-sectional plane of the cube would look if the cube was cut along these lines.

Besides the workstations with the assignment sheets described above, the pupils could work on the same assignments by using the computer program 'Doorzien' (see www.wisweb.nl).

Sawing cubes in the class

To make the concept of a cross-sectional plane manageable, an apple is placed on the cutting board, and the teacher asks the pupils to think about the possible shape of the cross-sectional plane. This question is repeated for a number of different cross-sections of the apple.

The teacher slices through the apple, keeping the cut pieces together, and asks the pupils to make a drawing of what the cut apple would look like. After the pupils have finished drawing, the teacher pulls the apple apart and the pupils look to see if their drawings are correct.
Several more cross-sections are explored. In the drawings, many differences in detail are visible, for example when the pupils also draw the core or the stem.
When the teacher cuts off a small piece of the apple, this puts some children on the wrong track. They do not draw a view, but a picture on which both pieces can be seen (see figure 5.17).
During the discussion, the teacher holds the two pieces of apple together on the cut surface to show that the two cross-sectional surfaces always have the same size and shape.

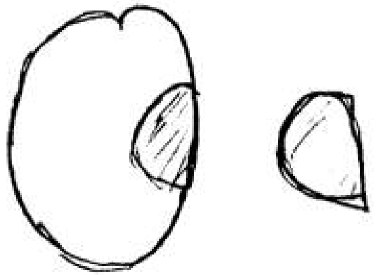


Figure 5.17

The discussion about cutting apples is completed by observing that both parts of a cross section have the same surface. This surface is called the 'cross-sectional plane'. Based on their experiences with activities such as potato stamps, the children can imagine that a section is a plane figure. Drawing the cross-sectional plane is easier due to the idea of stamping; after all, the drawing of the cross-sectional plan looks like a print from the potato stamp.
The teacher makes the transition from fruit to the cube. He shows the children a cube of Oasis foam and demonstrates how the cube will be cut with a small saw. The teacher asks the pupils to imagine what the cross-sectional plane will look like. Two of the pupils suggest a square, another suggests a rectangle.

However, the questions about the largest possible rectangle, a triangle, the largest possible triangle and other polygons are not at all easy for the pupils. The pupils are all given cubes of Oasis foam and small saws and get to work in pairs on the above questions.

The small cubes are viewed from all sides and rotated. The pupils show each other how they think the cube should be sawn, they try to persuade each other, and only then do they actually start sawing the cube. When they draw the cross-sectional plane, very few of the pupils use the sawn section itself as a stamp. Usually the pupils make a sketch of the shape of the sawn surface.

During the discussion, the pupils explain that their reasoning about finding the rectangle was initially based on a horizontal cross-section. As soon as the saw went in at an oblique angle, the pupils saw a rectangle appear. One pupil came up with the idea that the diagonal cross-section would provide the largest rectangle (see figure 5.18).

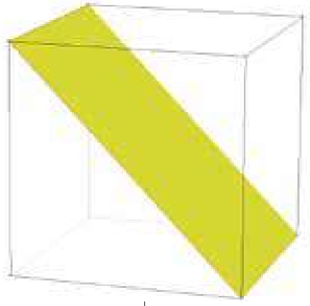


Figure 5.18

One pupil explains that it is very easy to find the triangle: you simply saw off a corner of the cube (see figure 5.19).

For many of the children, finding a pentagon and a hexagon as a cross-section is too difficult. They are unable to think of a sawing plane that will generate these figures. Consequently, the children bend the saw blades slightly and saw in a somewhat zigzag fashion through the foam.

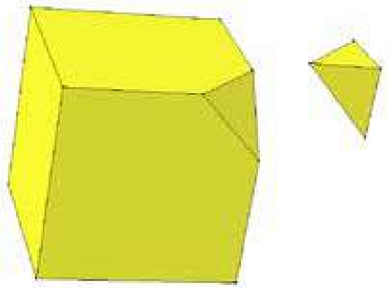


Figure 5.19

The pupils show their discoveries to each other. The teacher discusses the findings with the pupils.

In the classroom, a number of workstations are ready for the following round of activities. At each workstation, the assignment sheets are ready with a number of worksheets. There is a laboratory atmosphere in the class, where young scientists are busy with all kinds of cubes: wire cubes, open and closed cubes, transparent and opaque cubes, and cubes made of oasis foam for children who still unable to do without sawing.

Each inquiry activity has the following phases: predicting with the aid of spatial cognition, doing the tangible activity and then making two drawings: the cross-sectional plane as a figure and the cross-section in the pictured cube. Due especially to the many types of cubes, the pupils acquire experience with possible strategies for finding a cross-sectional plane. The teaching-learning process works toward a situation where the pupils can mentally create a cross-sectional plane while looking at the cube.

By using a layer of colored liquid in a *transparent cube*, the pupils investigate the possible figures of cross-sections. Here the pupils see the cross-sectional plane as the surface that lies inside the cube and is bordered by the sides of the cube. The pupils can view the surface from above and consequently acquire a good image of the plane figure. Drawing the surface on the cube pictured on the worksheet requires spatial reasoning about the lines on the sides of the cube. These lines, which are seen on the inside of the cube, are transposed into lines that are made visible on the outside of the cube.

They do the same investigation with a *solid cube* that is submerged in water. The lines intersecting with the water surface on the outside of the cube are drawn and then transferred to the illustration of the cube on the worksheet. The cross-sectional plane is made visible with a few lines on the sides of the cube. Based on the drawn lines, the pupils must imagine the cross-section as a plane.

The *wire model of the cube* is again submerged in water, which provides a different image of the cross-sectional plane (see figure 5.20). The pupils reason out how they should submerge the wire cube to obtain a specific shape on the water surface. The points where the wires intersect with the water surface show the contours of the cross-sectional plane. By transferring the intersection points to the cube illustrated on the worksheet and connecting the points with lines on the sides of the cube, the pupils see how to obtain the cross-sectional plane that is pictured. Here as well, the cross-sectional plane is drawn as lines on the outside of the cube shown on the worksheet.

Figure 5.20

Finally, lines are drawn on a *transparent cube* where it intersects the water surface (see figure 5.21). The pupils must think about how the cube should be tipped in order for these lines to become lines on a cross-sectional plane. The pupil sees the other intersecting lines of the cross-sectional plane return as parallel lines on the parallel sides of the cube.

The pupils continually discuss how a cross-sectional plane can be given a specific shape. During this process, they also ask themselves questions such as: why is it impossible to make a hexagon with an open cube (filled with water), but you can do this with all the other cubes? The pupils turn out to be very capable about noting their findings on the worksheet, on which wire models of cubes are shown. During this process as well, the pupils consult emphatically with each other and give each other directions..

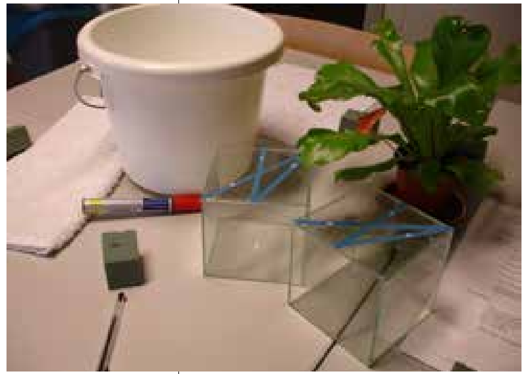


Figure 5.21

Drawing the cross-section as a plane figure provides an image of what the figure looks like, where the pupils must especially pay attention to the form characteristics. Drawing the cross-section in the pictured cube as a plane or as lines on the outside of the cube surface requires the pupils to convert a solid to a plane. As a result, the essence of how the plane is part of the cube comes into view.

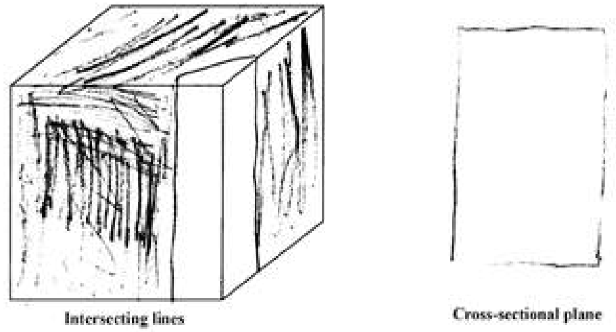


Figure 5.22

On the work sheet, a cube is shown in parallel projection. This type of projection has not been explicitly addressed, and its characteristics have not been discussed. Most of the pupils appear to be able to deal with it intuitively without being told explicitly that lines that are parallel in reality are also parallel on the drawing. This lesson is primarily concerned with how a cross-sectional plane in such a three-dimensional drawing can be made visible.

The pupil’s inquiry activities at the workstations are discussed afterwards by the teacher. It is especially the open, transparent cube with a layer of colored liquid inside that provides material for additional thought.

The follow-up discussion addresses all the figures that can be made with the open cube (with a layer of colored liquid), beginning with the square. According to one of the pupils, this is easy; if you just set the cube down, the liquid automatically goes into the shape of a square. It is also easy to make a rectangle by tipping the cube slightly. One pupil is able to explain how a triangle can be obtained: you set the cube on one of its corners. This links up with the previously acquired image of sawing off a corner of a cube. The question of how you can find the largest possible triangle is a difficult one, which many children cannot really answer.
The teacher brings up the hexagon after the pentagon. The pupils think that the hexagon is impossible because the cube is open. The water runs out when you try to make a hexagon. So you could never make a hexagon. Drawing the hexagon inside the cube is also too difficult for the children.

A number of pupils have succeeded in making a hexagon with the computer program ‘Doorzien’ (see www.wisweb.nl), which didn’t work with the open cube. The follow-up discussion shows the importance of the three steps of predicting, doing and visualizing for every assignment in this lesson. Pupils who started too quickly with tangible activities have difficulty with the mental step of reasoning about the cross-sectional plane.

The activities in the perspective of a learning-teaching trajectory

Based on an analysis of the assignments concerning ‘Cube sawing’ and the investigations that pupils conducted with cubes in various contexts and the class, we arrive at a possible learning-teaching trajectory description for the terrain of Plane and Solid Figures. We position Cube sawing at the center of the learning-teaching trajectory, and we look at what could precede this in education during the lower and upper grades of primary school, and what could follow. The building blocks for the curriculum are shown schematically in figure 5.23.

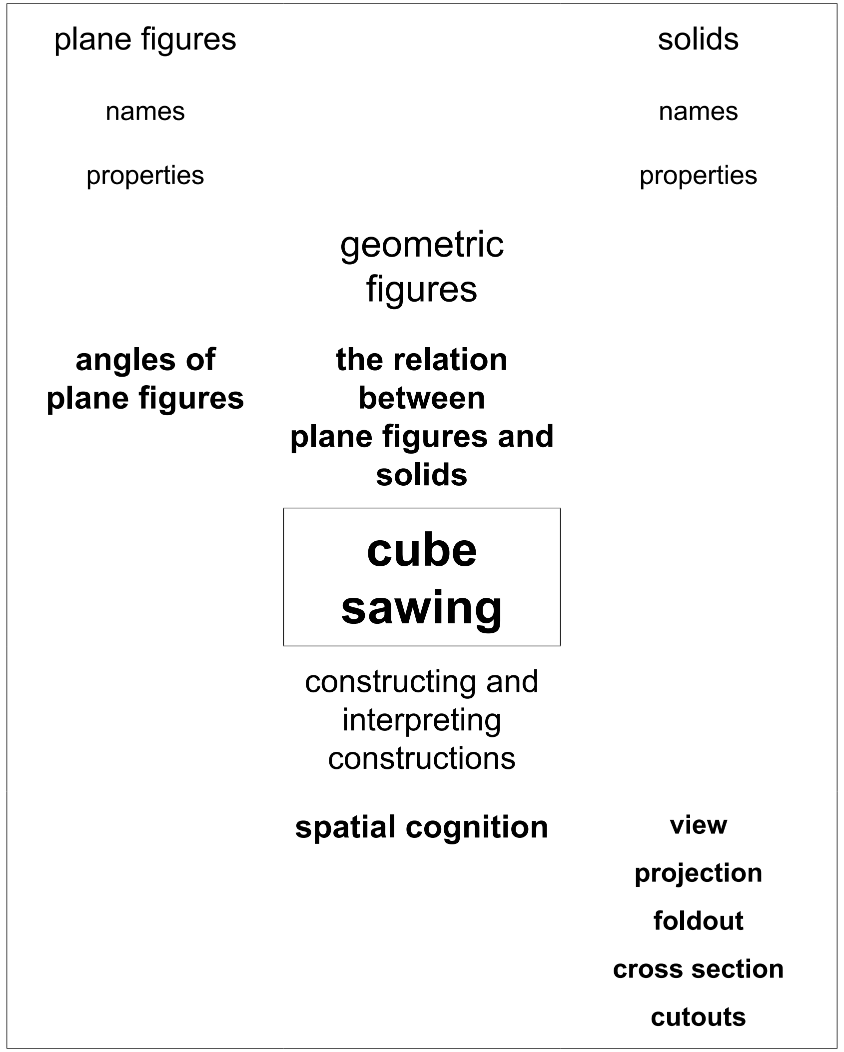


Figure 5.23 Schematic: Cube sawing in the learning-teaching trajectory

Cube sawing centers on make cross sections of cubes, so in the schematic we encounter two important geometric topics: *figures* and *cross-sections*. The location they have been given in the schematic does not mean that something precedes them in time. Besides Cube Sawing, we see important skills in the schematic that are required when sawing cubes: *spatial cognition* and *constructing*. We are now going to discuss figures and cross-sections, in that order. During this discussion we will indicate where pupils require spatial cognition or where they are involved with constructing.

Figures

For the Cube lesson, the pupils require some basic knowledge in order to participate usefully. This concerns knowledge of *plane figures* such as the triangle, quadrangle and polygon. In addition, the pupils should have thoroughly explored *solids* such as the cube, cylinder and rectangular prism. During the lower grades of primary school, these figures are linked as a form (or perhaps better as a shape), to their names, but without paying further attention to the properties that could be linked to the figures.
For pupils in the lower grades of primary school, the *plane figure* is often a thing with a name, a form and an area. For example, in some cases you can tile floors with them.

By using a number of plane figures, you can sometimes make a solid.

> Pupils investigate regular figures (polygons). With an equilateral triangle, a square, a regular pentagon and a regular hexagon as basic elements, pupils investigate which solid figures can be made by only using surfaces of triangles, only surfaces of squares, only surfaces of pentagons and only surfaces of hexagons. In the class, materials are available to actually construct the solids. This concerns materials with which you can make wire models as well as closed solids.

In the computer program *Tegels Leggen*, the assignment is to design tiles which can be used to make patterns that join together. The tiles can be rotated in four positions. Figure 5.24 shows a pattern that can theoretically be continued forever.

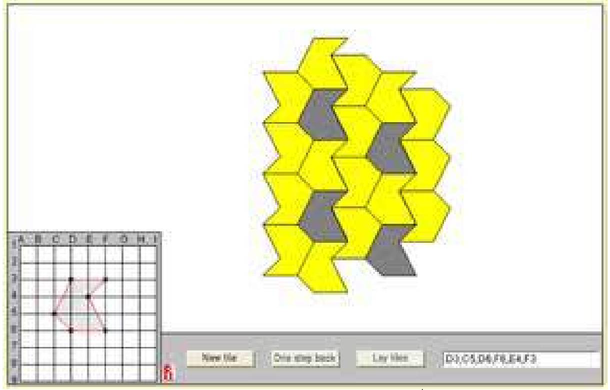


Figure 5.24

In the program *Mozaiek* patterns can be made with a special type of tile (see figure 5.25). These are polygons with sides all of the same length.
While experimenting, children will discover how the corners fit together. You can fill a corner of the hexagon with three triangles, and a corner of the dodecagon (regular 12-sided polygon) with one triangle plus one square. The outer edges can also be tiled in specific ways.

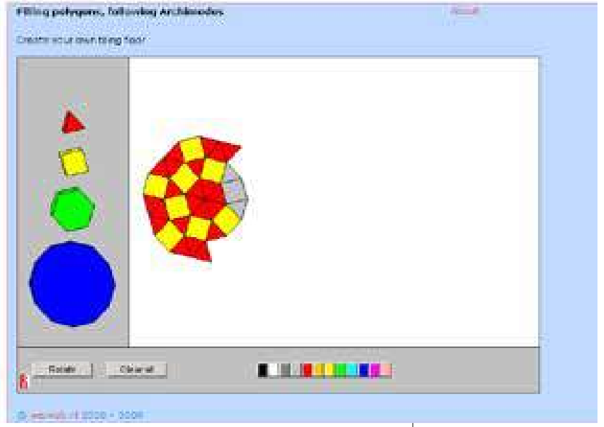


Figure 5.25

The programs can be found at www.rekenweb.nl.

You can place a mirror on some figures in such a way that the other half of the figure becomes visible in the mirror. This concerns informal knowledge about symmetrical figures.
Pupils can recognize simple reflection-symmetric figures and they know how you can find or check the symmetries of figures by folding or using a mirror. They can indicate reflection-symmetric points and lines in figures and develop ideas about what this has to do with the distance of these points to the axis of symmetry. The informal knowledge that they develop therefore has to do with (1) knowing that you can fold reflection-symmetric figures along specific lines so that the halves match each other; (2) knowing that a figure may not have such a symmetric folding line or multiple folding lines; (3) knowing that if you place a mirror along with the folding line, you will be able to see the entire figure

Sometimes the figure is not actual material, but is only a shaded or colored shape, such as a shadow.

> The pupils investigate the shadows cast by a rectangular prism, cube, cylinder, pyramid and cone, and they draw the possible forms of the shadow area. During this process, they investigate the distinction between lamplight (point source) and sunlight (parallel rays).

The essence of the figure emerges only when drawing the *plane figure*. It has become a geometric investigation object with specific characteristics. The process of drawing the plane figure progresses from a sketch to a precise shape with specific dimensions. During this process, tools come into the picture such as a ruler, a pair of compasses, and later on a compass rose or other angle measuring device.
During some activities in the lower grades of primary school, *characteristics* of plane figures are addressed implicitly, for example while playing Tangram and when folding a square from an A4 sheet of paper. In the upper grades of primary school, the attention gradually shifts from figures as shapes to figures as mathematical investigation objects. The pupils discover that a triangle appears in many forms depending on the size of the angles and the lengths of the sides. They also learn that names are given to a group of triangles based on their properties, such as the isosceles triangles, the right triangles and the equilateral triangles. Some triangular forms are symmetrical, others are not.
With *plane figures*, the idea of the angle deserves separate attention. This concerns the development of informal knowledge about the idea of the angle with plane figures. With a plane figure, you refer to an angle as two lines that join together. The length of the lines is unimportant; the angle concerns how much you have to rotate one line on the vertex until it coincides with the other line.

Ultimately, some pupils are able to distinguish the *angles* of figures and compare them in a simple fashion: they can recognize a right angle and compare it with other angles. They see that other angles are larger or smaller than a right angle. The pupils can develop the right angle concept at various levels. One child will be able to relate a right angle to a 90 degree angle, another child will know that the lines are perpendicular to each other, and a third child will only be able to relate a right angle to angles that are known to be perpendicular, such as the corner of an A4 sheet of paper. Terms such as obtuse or acute angles are then unnecessary to develop knowledge about these angles.

> The teacher asks if squares or isosceles triangles can be used to fill the surface. The pupils begin their inquiry and experience that when triangles and squares do not join together at a point, you cannot fill a surface with them. Even though they do not understand entirely why this is the case, they do have the feeling that it has something to do with the size of the angles.

A *solid* is a thing with a name, a shape, volume and a view. In the lower grades of primary school, pupils have acquired specific experience with solids, for example in building and construction activities, in activities with play materials in the gym and in investigation activities on volume. All kinds of figures have been given a name, sometimes a mathematical name, sometimes a common name. Common names include block instead of rectangular prism, tube or roll instead of cylinder, ball instead of sphere and dunce cap or pointy hat instead of cone. In contrast, the pupils become accustomed to words such as cube and pyramid fairly quickly. At the same time, a cube is also a block in the context of building blocks. During the upper grades of primary school, mathematical words such as cylinder, sphere and cone are introduced, in combination with previously used common names.

> Together with the pupils, sort various kinds of packaging materials. One of the sorting criteria will be shape. The pupils are given a sheet with shapes and their mathematical names, and they give names to objects. Other sorting criteria can include the type of material, the volume, whether or not they are stackable, and the aesthetic value.

Being involved with the shape means asking questions about physical properties such as: Is it stackable? Can you roll it or slide it? Is it rounded or pointed?

> The teacher is holding a contest; the pupils must make a structure with at least 12 and no more than 15 elements, with three stories, using one of the following basic elements: cube, sphere (tennis ball) and pyramid. The pupils start to work in groups. At the beginning, the builders using spheres have a lot of trouble with the structure collapsing. This continues until one of the pupils gets a rough mat to use as a foundation.

Volume is investigated with activities such as transferring liquid, seeking units and measuring. The *view* is two-dimensional *projection* of the object. In this projection, a solid becomes a plane.

> The pupils are given the assignment to design a 'shape box' which contains a rectangular prism, a pyramid, a cone and a cylinder. It must be possible to insert these solids through the lid into the box. A number of plane figures, such as a rectangle, a triangle, circle or a square, must be cut into the lid.

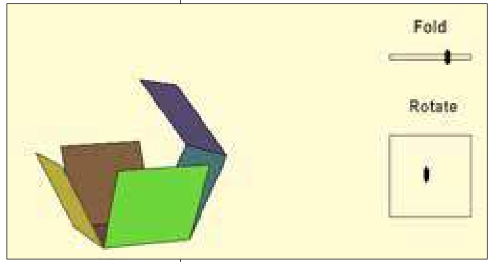
A shadow image of a solid on the ground is also a type of two-dimensional projection. This is nicely addressed during the activities where the cube is immersed in water.

Experiences with making solids from smaller basic figures have been frequently addressed during building situations. Children who have played with blocks and Legos are especially able to take part in this discussion. Most pupils only become aware of this aspect when the teacher begins a discussion and provides assignments that focus on discovering and describing relationships. The question 'How many individual cubes are there in a Rubik cube? makes the pupils aware of a world of cubes that are constructed of smaller cubes

Before they enter the upper grades of primary school, pupils have already acquired many experiences with the *relationship between solid and plane figures*. These activities include folding and pasting and other kinds of activities such as making solid shapes with play materials, fitting blocks with different shapes through a lid with holes, seeing the silhouettes of the solid objects, making something from a cutout and from construction material using construction drawings. There has been attention for are drawing a structure by means of a floor plan with heights (number of stacked blocks), or with views. The pupils have walked around a structure in order to see the changes in the view. During this process they learn to move into the standpoint from which a photograph was made. The shadow has also been a source of investivigation. The pupils in the upper grades of primary school therefore know that *solids* can be imaginarily constructed from flat surfaces and that *a foldout* can be made from a solid.

> *Making boxes*
> Pupils disassemble cardboard boxes and look at the resulting foldout. Based on a drawing of a foldout, they predict which solid the foldout represents. By using the foldout as a cutout, they then construct the solid. Based on this knowledge, they begin designing boxes themselves. This design is given a useful function, such as a candy box or a jewelry box.

A number of assignments using cutouts are available on www.reken-web.nl, where it is possible to directly compare a 3-D figure and the corresponding cutout (see figure 5.26 and 5.27). For example, on the computer, a cube can be folded together or folded out with the question: which drawing of a foldout corresponds with the cube and which does not?



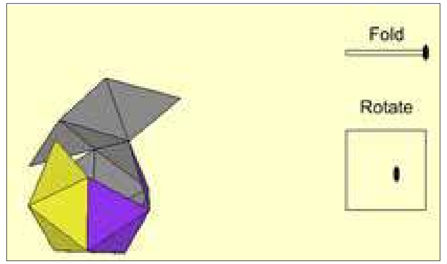


Figure 5.26 *Figure 5.27*

Of course, it becomes even more interesting when children design their own cutouts. One of the submissions for a contest based on the computer assignments was this scissors box, see figure 5.28.

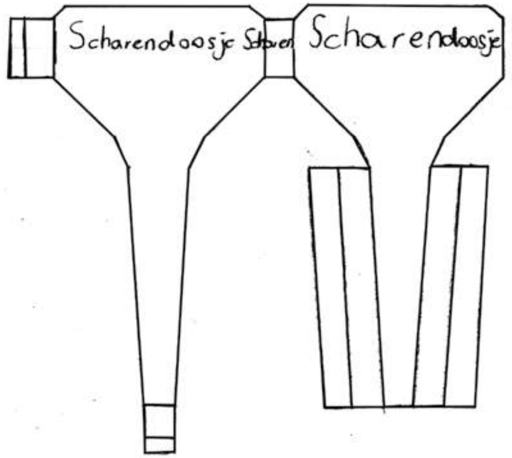


Figure 5.28

Constructing and spatial cognition

During the upper grades of primary school, *spatial cognition* is added. For example, the pupils think about how a solid appears as a fold out, or the reverse the process and mentally fold a *cutout* together.

Pupils think about how to make a fold out of the cardboard center of a toilet roll, they actually make the fold out, and then they check the design by rolling out the cardboard.

The pupils discover that all kinds of new solids can be created from another solid, for example by sawing off certain parts.

Pupils construct a cone from an A4 sheet of paper and then cut off the point. What does the point look like? What does the remaining figure look like? How can you close up the truncated cone? The pupils think about these questions and draw the fold out of a truncated cone.

During the ‘Cube sawing’ lesson described above, the pupils acquire experience with the possible strategies to find a *cross-sectional plane*; this is possible due to the various materials with which the cubes are made. An aim of the learning process is to work towards the situation where can make a mental cross-section by looking at a cube, where you can *imagine* not only how the *cut figure* will look, but also how the *cross-sectional plane* will look *from above*. The process of making a *mental* cross-section follows the process of actually cutting a cross section with a saw.

Preceding the experimentation with actually cutting a cross-section, attention is paid to the phenomenon of the *cross-sectional plane*. Attention is focused not only on the shape of a cross section, but also on the relationship between the solid, the method of making the cross-section and the shape of the cross-sectional plane.

While they are sawing the cube, the pupils are asked beforehand to think about the shape of the section that can result. Ivestigation questions give direction to the inquiry, such as: How can you obtain a triangle or a quadrangle? Is a square or rectangle also possible? What about pentagons and hexagons? And why can you never obtain a heptagon (a seven-sided figure)? The ultimate target of the Cube sawing activities is not only *reasoning* about the shape of a cross sectional plane, but also *visualizing* the cross-sections.

Representation: Bringing shadows into the picture

Introduction

For the third geometric terrain ‘Visualization and representation’, the example activities titled ‘Bringing shadows into the picture’ are described, analyzed and placed in a learning-teaching trajectory. The example activity is a rich investigation activity for grade 6, during which various geometric topics are addressed. This concerns topics such as vision lines, angles, shadows, the position of the sun with respect to the earth and proportions. The example activity, as in previous learning-teaching trajectories, is a possible basis for the learning-teaching trajectory description. The activity is used for bringing essential aspects of the learning-teaching trajectory into the picture. Other activities could take this same role.
In contrast with the other two learning teaching trajectories, ‘Spatial sense’ and ‘Plane and solid figures’, this learning-teaching trajectory involves attainment targets that go beyond geometry. It involves learning to make representations and visualizations that can be used to analyze problem situations and make reasoned solutions for these situations. Formulated more generally, it concerns developing geometric tools for making mathematical models of situations. In secondary education, such mathematical models will be frequently used. Moreover, the learning-teaching trajectory will elaborate upon various disciplines, not only physics and engineering, but also economics and geography.
We use both the word ‘representation’ and the word ‘visualization’. In the didactical literature, the word ‘visualization’ is often used in the sense of making an ‘appropriate’ image of a situation. For example, this image can be a graph, a model of a solid or a sketch of a situation. The activity of ‘visualization’ is characterized by reducing reality until something remains that contains only the essential information. In the visualization you can indicate imaginary lines, such as a vision line or a path taken by a ball. The

pupils make visualizations themselves, and are also provided with visualizations. ‘Representation’ is a more comprehensive concept which can also use language to make a suitable ‘picture’ of a situation. Examples of representations are formulas, schematics or a wooden cube as a model for the cube figure. These terms are used interchangeably in practice, so we will refer to ‘representation’ and ‘visualization’ in a single breath.
Before we look at the building blocks for a possible learning-teaching trajectory, we will describe the activity ‘Bringing shadows into the picture’. The following questions are keys to this process: ‘How do you learn to make a suitable picture of the situation?’, ‘Which questions help you to choose a suitable picture?’ and ‘What information can you obtain from a chosen picture?’

Target of the assignments

The activity has been developed with the intention of enabling pupils to discover how measuring the shadows of sticks can lead to all kinds of knowledge about the position of the sun with respect to the earth, the time aspect of shadows and reasoning about shadow images. During this activity they discover that two concepts are important: ‘view’ and ‘vision line’. You need a view (which can also be called a cross-sectional plane) in order to picture a stick and its shadow. You also use a view for reasoning about why a shadow becomes longer as the sun becomes lower. In this context of light and shadow, you use a vision line to picture the ray of sunlight. This of course concerns the ray of sunlight that passes along the top of the stick and consequently forms the border between the shadow area (behind the stick) and the lighted area (see figure 5.29).

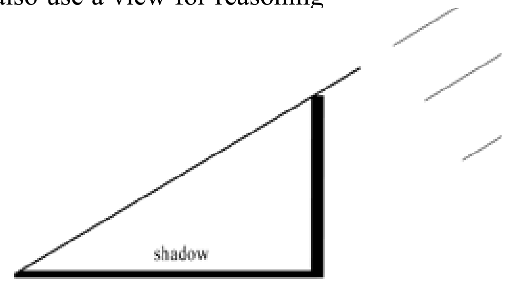


Figure 5.29

Besides learning to choose a suitable view and using a vision line, in this activity the students also come into contact with taking measurements of both the length of the shadow and the angle of the ray of sunlight. Finally, the shadow rotates around the stick due to the movement of the sun, and this provides a reason for additional investigations regarding the movement of the earth with respect to the sun, and as an extension of this topic, measuring the directions of the shadow on a compass rose.

Assignments:
For this activity, the following materials are required:
- two sticks, one 1.20 m long and one 0.70 m long;
- a measuring tape;
- several meters of string;
- sticky tape;
- 2 tent stakes;
- a compass;
- a shoebox pasted with an A4 sheet of paper;
- a set square (combination of protractor and ruler);
- a spirit level.

Choose a sunny day for this activity. Insert both sticks into the ground so they are exactly vertical, some distance apart. The long stick should protrude 1 m above the ground, the short stick 50 cm. Attach the end of the piece of string to the top of each stick using the sticky tape. The other end of the string is affixed to the ground with the tent stake, exactly at the endpoint of the shadow thrown by the stick.

Collect data about the shadow at least five times throughout the day.
- Use your *measuring tape* to measure the length of the shadows of both sticks.
- Use your *compass* to measure the direction of the shadows and the direction from which the sun is shining.
- Place the box on its side against the string with a point exactly at the endpoint of the shadow. To ensure that the box is exactly level, use your spirit level. Draw the angle of the rays of the sun on the underside of the box and use your *set square* to measure the angle.
- The photographs at figure 5.31 show what your measurement setup should look like.
- Record your observations on the table. Make sure you record the time of your observations.

Discuss your observations while taking the measurements and after studying your table in groups, and then write a report. Pay attention to the following aspects:
1. What does the direction of the shadow have to do with the direction of the sun?
2. How does the direction of the shadows change during the course of the day?
3. How does the length of the shadows change during the course of the day?
4. How does the angle of the rays of the sun change during the course of the day?
5. Compare the observations for the 1 m stick and the 50 cm stick. What do you notice?
6. Compare the angles of the rays of the sun with the 50 cm stick and the 1 m stick. What do you notice?
7. Can the shadows of the short stick and a long stick cross or touch each other?

		stick 50 cm			stick 1 m		
time	direction of the sun	direction of the shadow	length of the shadow (cm)	angle of the sun rays	direction of the shadow	length of the shadow (cm)	angle of the sun rays

Figure 5.30

Measuring shadows with the class

The teacher tells the pupils that they are going to measure shadows and takes them outside in order to make the measurement setup (see figure 5.31). The teacher has two sticks, one is 1.20 m long and the other is 70 cm.

Figure 5.31

The activity begins with 'planting' the two sticks in the ground so they are exactly vertical. For the pupils, this already leads to a number of questions. How can you make sure that the sticks are exactly vertical? Usually this is done by eye. While you move the stick slightly back and forth, you see immediately that this affects the length and direction of the shadow. If the stick is pointed a little too much towards the sun, then the shadow is shorter than if it is tilted in the other direction. If the stick is tipped a little too much to the left or the right, then the direction of the shadow is not entirely correct.

How close together should the sticks be planted? Does it matter for the length of the shadow if you place one of the sticks far away, further from the sun, or should they be placed neatly next to each other, both at the same distance from the sun? However, if you place the sticks very close together, later in the day you see that the shadows of the sticks fall together, because at a certain time they are exactly lined up with the direction of the rays of the sun. While you are planting the sticks, it appears that the shadows of the two sticks are parallel to each other. Is that correct?
When planting the sticks in the morning, should you also take account of how the shadows will appear later in the day? Perhaps the sticks will fall into the shadow of a building around noon, so that you will have to move them to take measurements later in the day.

The pupils are surprised that simply placing two sticks to measure their shadows has led to so many questions. The answers to the questions can be reasoned out, but a picture is required in order to clarify the reasoning. How could you make a picture of the situation?

Before the lengths of the shadows are measured, the endpoints of the shadows are marked with the tent stake, which is used to attach the string to the ground. What does this piece of string represent in the context of sunlight and shadow? While reasoning out the answer, the pupils realize that the string shows how a ray of sunlight touches the top of the stick and proceeds to the ground; other rays of sunlight bump into the stick, so to speak, and in this way they leave a shadow on the ground where the rays of the sun cannot penetrate the stick.

An important discovery has been made here. A string represents the ray of sunlight. This gives the pupils an impulse to draw the situation. If you look at the measurement setup from the side, you see a view of the stick with the ray of sunlight that marks the top of the shadow. The ray of sun is shown as a line, which then touches the ground after it passes along the stick at a specific angle.

The sticks cannot be inserted into pavement, so a piece of lawn must be chosen that is big enough for the lengths of the shadows. But a lawn is usually not entirely level. Then how would it be possible to measure the angle that the ray of sunlight makes with the ground as precisely as possible? The box can be used for this purpose, because then you can check (with the spirit level) whether the ground is level. If you move the box slightly back and forth, you see which effect this has on the size of the angle (see figure 5.32).

Figure 5.32

If the box is placed carefully on its side against the string, then the angle can be drawn on the underside of the box. But should the vertex be placed exactly where the tent stake is located, or can you also measure a little further up on the string? By sliding the box back and forth, the pupils discover that it doesn't matter regarding the size of the angle, as long as the horizontal line is parallel to the level ground. The angle is first drawn on the box, and the size of the angle can be measured later in degrees, but you can also place the set square exactly at the vertex and read the degrees directly (see figure 5.33).

Figure 5.33

Now the direction of the shadow must be determined. The compass is used for this purpose. The pupils discover fairly quickly that the direction from which the sun shines is the opposite of the direction towards which the shadow points. The direction can be measured with the wind rose, but it can also be measured in degrees. The latter measurement is more precise. The fact that the direction of the shadow is the same for both sticks is seen as self-evident by the pupils. Reasoning about this also requires a visualization. Now that the pupils have thought about the first measurement, new questions come into the picture about the subsequent activities.

What other measuring times are you going to choose during the day, besides taking account of the fact that you are bound to have other activities on the program? What are interesting times to take measurements? At midday, the sun is highest in the sky, but what time is it actually when the sun is at its highest point? What happens with daylight savings time? What time does the sun go down, and will it still be shining on this lawn at the end of the day? You can also simply decide to take another look in one hour. If you leave the strings from the first measurement in place, you can more easily see what has changed in the meantime. It is also interesting to predict what will change. Will the shadows become longer and in which direction will they point?

The system of measurement is more or less established by the first measurement. The pupils see that following the same system for all measurements will lead to comparable results. A new line on the box shows the new angle. An hour later in the morning, it turns out that the shadow has become shorter, the angle has become larger, and the sun is higher in the sky. After the second measurement, the pupils also discover that the shadow of the 1 m stick remains twice as long as that of the 50 cm stick. It is therefore no longer necessary to measure the shadows of both sticks. However, the teacher does ask, would be easier to measure the shadow of the shorter stick, or would it be more precise to measure the shadow of the longer stick?

On the blackboard, the teacher draws two right triangles that represent the stick-shadow relationship and the size of the angle. On the drawing, you see that the ray of sunlight passes more steeply along the stick during the second measurement (see figure 5.34).

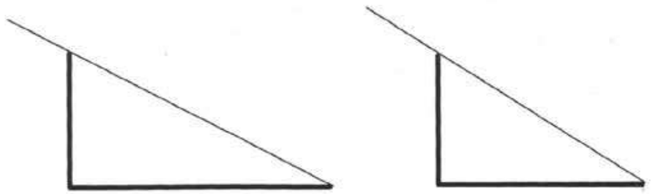


Figure 5.34

When you go outside for the third and fourth time to take measurements, you perceive what is happening with the movement of the sun: it began in the southeast and moved to the south and then towards the west. And the shadows therefore point in the opposite direction. You also see the trend for the length of the shadows: they began long, and at a certain time they were at their shortest. Have you indeed measured exactly at the time when they were at their shortest? At the end of the day they will probably be long again. The angle started small and became larger. What is the largest possible angle? At the end of the day, the angle will once again be small.

The direction therefore changes all day long. How can you explain this? How can you picture the shadow for a whole day? Can you do this in the same view that shows the stick-shadow relationship, or do you need another view? These questions are written down and are combined with the other questions that emerge during the activity.

The results of the measurements show the following picture (see figure 5.35) after one day:

		stick 50 cm			stick 1 m		
time	direction of the sun	direction of the shadow	length of the shadow (cm)	angle of the sun rays	direction of the shadow	length of the shadow (cm)	angle of the sun rays
09:00	east	west	220	18°	west	450	18°
10:00	southeast	northwest	138	27°	northwest	266	27°
13:00	south	north	50	45°	north	100	45°
13:30	south	north	60	42°	north	122	42°
15:30	southwest	northeast	75	37°	northeast	150	37°
16:30	southwest	northeast	90	22°	northeast	175	22°

Figure 5.35

To end the activities, the teacher holds a discussion with the children about whether these measurements would have provided different results if taken at different times or different locations on the earth.

Computer program Sun and Shadow

The activities involving the shadow thrown by a stick can elicit all kinds of new questions from the children. For example, they know that the sun comes up late in the winter and goes down early; are the shadows in the winter the same as those in the summer? And from there, an obvious question is, what are the shadows like in other countries?

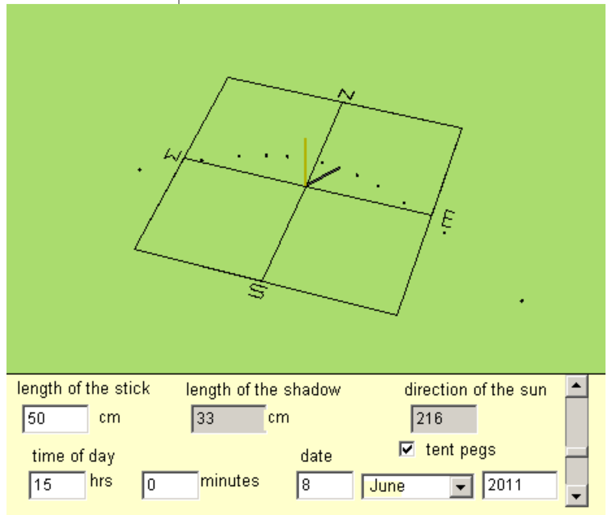


Figure 5.36

Pupils can investigate such questions with the computer program *Sun and shadow* that can be found at www.rekenweb.nl (see figure 5.36). The program can show the shadow thrown by a stick on every day of the year, at many locations on the earth. For example, pupils can compare the shadows on 21 June with those on 22 December – the longest and shortest days – or the shadows on 21 June in the Netherlands with those on that same day in the tropics.

By investigating shadows, the pupils in fact study the movement of the sun in the sky. This leads to questions such as: why are there seasons? And: why is it hotter in the tropics? These are important questions, but they are also difficult questions. By having pupils work with the computer program, their interest in such questions can be stimulated.

The activities in the perspective of a learning-teaching trajectory

An analysis of the questions that are elicited by the activities involved with measuring shadows provides a glimpse of the building blocks that can be used for a learning-teaching trajectory in the terrain of ‘Visualization and representation’. The building blocks are printed in bold. They can be found in the schematic in figure 5.40 where they have all been collected as a summary. We will explain each of the building blocks and provide suggestions for their possible application in education. During this process, we consider ‘Bringing shadows into the picture’ to be the center of the learning-teaching trajectory.

In geometric terms, the activities involved with measuring shadows concern a number of phenomena: the visualization of the ray of sunlight-stick-shadow situation, the sun and its shadow images, the angle of the incoming rays of sunlight and the relationship between the object and the shadow image. These phenomena are investigated by conducting systematic observations, and then geometric reasoning is applied. For the learning-teaching trajectory, we will focus on picturing the situation. With the aid of visualization, geometric reasoning can be used. In geometric reasoning, we encounter two important concepts: *view* (how do you picture the situation?) and *vision line* (how do you picture the ray of sunlight?). We will first examine the concept of view.

View

In the curriculum that leads to learning to make appropriately chosen visualizations, learning to interpret, explain, construct and apply views is an important step. This concerns learning to interpret views in the form of *sketches, drawings* and *photographs* that are made from *various standpoints*, followed by exploring the information that can be obtained from various standpoints. After this, the pupils independently make *sketches* and *drawings* of arrangements of a number of objects. Finally, this is followed by reasoning about the position you must take in order to obtain the information you are seeking.

During the lower grades of primary school, the pupils participate in a range of geometric activities with block structures. Based on building with blocks, the pupils must be able to picture the structure. At the same time, they are given a number of visualizations of the structure. Besides the *perspective* (3-D) representation, *front, side and top views* are introduced as visualizations. The pupils ultimately make their own suitable views of simple block structures. During this process, the question is how such a block structure can be portrayed in a construction drawing. The pupils work towards a *top view with heights (numbers of stacked blocks)* as the visualization that can portray the structure.

The pupils learn that a structure, which is built of identical construction elements, can be described completely or portrayed by a drawing. This drawing must meet certain requirements, but at the time the pupils are presented with this task and are working on it, these requirements are not crucial, and are therefore not discussed with them.
However, in the upper grades of primary school, it is exactly these requirements that are important. It is therefore important to go further with views. This concerns key questions such as: 'Should the drawn view show an *identical* representation of the 'real' view of the block structure?', 'Should the elements of the block structure all be the same size on the drawing?', and 'Should the floor plan be drawn to scale?' As an extension of the requirement to make images that are or are not *true-to-scale*, all kinds of measurement aspects come into the picture when working with views
These requirements are of course related to questions concerning the aim of the representation. These questions include:
- What will the picture be used for?
- What needs to be measured on the representation?
- What do you want to explain with the representation?
- Is this an exact representation?

Computer programs about taking a standpoint

At www.rekenweb.nl, there are many assignments concerning building with blocks (see figure 5.37). These programs were described in a previous Tal publication ('*Young Children Learn Measurement and Geometry*', 2004). The programs invite pupils to compare views from various sides. In the example given, an example structure and one that the pupils make themselves can be rotated so that the directions of view are the same, but the children quickly move on to reasoning in the manner of 'Here there are three, here there are two'.

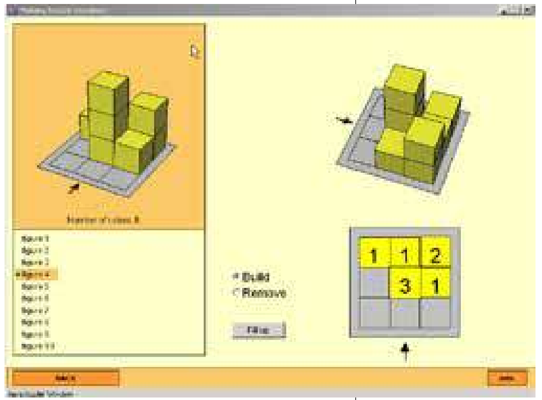


Figure 5.37

In *Schateiland* (Treasure Island) a ship sails around an island on which there are three towers. One assignment is to determine the location from which certain photos were taken. You choose a location on the map, and then in the telescope you can see the picture that corresponds with that location. There are also assignments without the telescope – only a photograph – which forces the children to reason more explicitly.

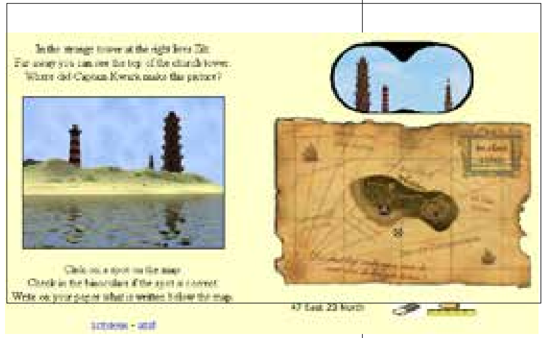


Figure 5.38

In the upper grades primary school, another aspect of views becomes important. In order to analyze, engage in *spatial cognition* and explain, it is often important to be able to make a two-dimensional representation of a situation. This is not only the case when explaining the shadow, but also when the task involves modeling a measurement situation or when you want to clearly explain some aspect. In that case, the pupils are asked to take a position from which a view can be made. Stated another way, a suitably chosen *cross-sectional plane* of the situation is made. For example, in the shadow lesson, pupils are asked to take a position in order to make a view of the measurement setup. The pupils take a viewpoint perpendicular to the setup with the stick, the ray of sunlight and the shadow image. This means that the stick, the ray of sunlight and the shadow image must fall within the chosen cross-sectional plane. In a sense, the pupils put that plane in front of themselves and look at it. On this plane, the pupils can measure the *angle*, and can take a true-to-scale measurement of the *stick/shadow proportion*. In every other cross-section that does not line up with the direction of the ray of sunlight and the shadow, the angle and the shadow image are distorted. Based on their knowledge of making the three views, the pupils can reason logically about how a suitable view to answer specific questions can be chosen. For example, the top view can be used to picture the *direction* of the shadow. A curriculum structure of views could therefore appear as follows:

- Analyzing and explaining drawings and photographs.
- Making views of solid figures from various positions
- Separate attention for the top view.
- Identifying an object from various standpoints.
- Using view as an explanatory model for situations.

Analyzing and explaining drawings and photographs

This concerns all kinds of images where questions are asked such as:

- 'What is shown here?'
- 'What information can you obtain from the image?'
- 'What can be measured on the image?'
- 'Where was the photographer standing?'

Various types of images are addressed such as those shown in perspective or parallel projection, but also views such as front, side and top views. Every view loses some information about the object. For example, in a top

view, all information about differences in height is lost. In a side view, the height is visible, but information about depth is lost. In this view, you cannot see what is closer and what is further away.

> The pupils in grade 4 are shown a photograph of train tracks that disappear into the distance. The teacher asks the children to trace the train tracks on the photograph onto paper. The pupils also make a top view of the train tracks. The question arises, why do the tracks on one drawing appear to merge together, while those on the top view do not?

Making views of solid figures from various positions

The pupils walk like a photographer around a solid figure and take 'snapshots'. The top view of the situation is used to show which position belongs to which picture. Depending on the aim, a sketch may be sufficient, or a true-to-scale drawing may be used. While drawing, attention is paid to issues such as: parallel in three dimensions also means parallel in two dimensions (on the plane). And: perpendicular lines on the cross-sectional plane are only visible as points on the cross-section.

> A table in the classroom holds three bottles, two lying their sides and one upright. The pupils in grade 4 draw the bottles from different positions. One of the pupils draws a top view. The drawings are compared with each other and then are displayed on the board. With each drawing, the question is asked: from which position was the drawing made? The position of each drawing is shown on a floor plan of the situation. Of course, the teacher asks the pupils to explain the relationship between the picture and the position.

Special attention for the top view

Top views (in the form of floor plans and maps) are very common in the upper grades primary school. Nevertheless, it is still a good idea to reflect with the pupils on what information is provided by a floor plan and by a map. This concerns questions such as: How true-to-scale is the information that is shown? and: What role is played by height differences on floor plans and maps?

The map of Texel is hanging on the wall. The teacher talks with the children about how this map could have been made. There was a map of the island even in the time when there were no aircraft from which a top view could be made. How could they have made that map? The class looks at the scale of the map, and the scale is used to determine the width of a road on the map. This leads to a brief discussion about why the width of the road is not shown true to scale. The question about the highest point of the island is discussed, in relation to measuring the length of a bicycle route on the map. The teacher and the pupils look at a satellite view of the island on Google-earth. This image is compared with the map.

Identifying an object from various standpoints

In these activities, the children take the step from drawing views to using the views in their reasoning. This concerns activities that focus on analyzing and thinking about the relationship between the standpoint of the photographer and the image of an object made from that location. The pupils must be able to mentally walk around an object and predict how the object will appear from a specific position.

The teacher has made a series of photographs of a block structure that is present in the classroom. The pupils make a top view of the structure based on the various photographs. They try to analyze the photographs to arrive at a reconstruction of the block structure. For each photograph, they reason out where it was taken.

Using view as an explanatory model for situations

Here we come onto the terrain of the activity where the view of the stick (with the ray of sunlight and the shadow image on the ground) is presented by means of a right triangle in which the angle of the incoming ray of light with the ground indicates the proportion between the length of the stick and the length of the shadow. Vision lines are used in both the 'Shadow activity' and in the example below. The difficulty in choosing a view that

is suitable for the problem lies in being able to mentally rotate a situation with the aim of bringing the correct view into the picture.

The teacher shows the pupils a photograph, taken on the beach, on which two ships can be seen. One ship is clearly visible, the other ship is much farther from the beach and only the top of the superstructure is visible. The assignment is to reason out why only the top of the ship in the distance can be seen.

Vision line

The vision line is the second concept that the 'Bringing shadows into the picture' activity addresses. *Observation* is part of doing geometry, and the observation will be closely examined. *Reasoning* out why you see some elements in your *field of view* or on an image and you don't see others requires a *visualization* of the field of view. In fact, this can be compared with the play of light and shadow. The *sun* beams its light to the earth, but at some locations this light is held back, and a *shadow* is created. Here as well, an explanation requires a visualization of the situation. For children, it has been clear almost from infancy that you can hide from someone else. With very young children, this takes place in an interesting fashion. You cover your own field of view with your hands in front of your eyes, and because you cannot see the other person, you think that the other person cannot see you. After this, children learn that you must hide behind something in order to not be seen. In an identical fashion, pupils experience that you cannot see the light source if you (and your eyes) are in the shadow. In fact, explanations for the above observations require pictures that visualize the situation.

The teacher has tipped a table on its side and asks a pupil to go sit behind it. All the other children are asked if they can see the pupil behind the table. Some pupils on the side of the room can look behind the table, but other pupils that are directly in front of the table cannot, of course. The teacher passes out a floor plan of the classroom and asks the pupils to draw the table with the pupil behind it on the floor plan. The teacher then asks the class to think about why some of them can see that pupil behind the table and others cannot. He emphatically refers to the floor plan as a way to visualize the situation. Hesitantly, some children draw a line from their own position to the pupil behind the table.

In this activity, it is important that vision lines (also known as lines of sight) are used in the discussion. A number of similar situations can be designed for this purpose. The cat that is hunting for mice hiding behind a few boxes, or the lamp that lights the room, but throws a shadow under the table. An interesting example where the vision line emphatically emerges is the situation where two objects are placed at a specific distance apart, and the teacher asks the pupils to place a third object on the line between the other two objects (see figure 5.39).

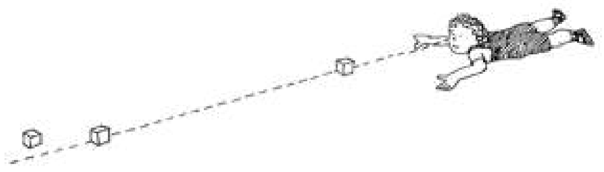


Figure 5.39

On the playground, two poles are placed 8 meters apart. The teacher asks one of the pupils to position another pupil between the poles in such a way that he is exactly in line with both poles. This situation is repeated on paper in the classroom. From a top view, the pupils use a ruler to draw the connection line between the two poles, and then place a mark on the line as the position for the pupil.

Pupils who watch sports on television have recently seen an increasing use of ‘vision lines’ for explaining game situations. A few examples: the simulation technology that is used in tennis matches to determine whether a ball has landed inbounds or out of bounds. A player who does not agree with the decision of the umpire can request a simulation. On a simulation of the tennis court, the path of the ball is shown, drawn as a line. The simulation program makes the decision about whether the ball is in or out. In the simulation, the drawn line of the tennis court is the vision line of the line judge. A similar technology is used for analyses of football matches on television to show whether a player was offside or not, or to determine whether a ball actually went over the goal line. The vision line of a line judge who is exactly on the correct line is shown on the simulation. With sports such as bicycle racing, speed skating and athletics, the finish line is the vision line of the jury, who determine the winner. A camera makes a finish photo, which the jury can use to make and justify a decision.

The pupils have set out part of an athletic track on the playground. Six children are going hold a foot race. One of the children is the starter, and two children are lying down on either side of the finish line, so they can look precisely along the finish line. Also beside the finish line is a third pupil with a stopwatch and a fourth pupil with a digital camera. When the starting shot has sounded, the children run towards the finish line. The observers are ready to record the finish as accurately as possible. Back in the classroom, a top view is drawn of the situation. The teacher asks the children to use the top view to think about what it would mean if the observers were not exactly lined up with the finish line. Of course, the speed of the pupils in the race is also calculated.

‘Bringing shadows into the picture’ leads to the following schematic representation (see figure 5.40): visualization and representation.

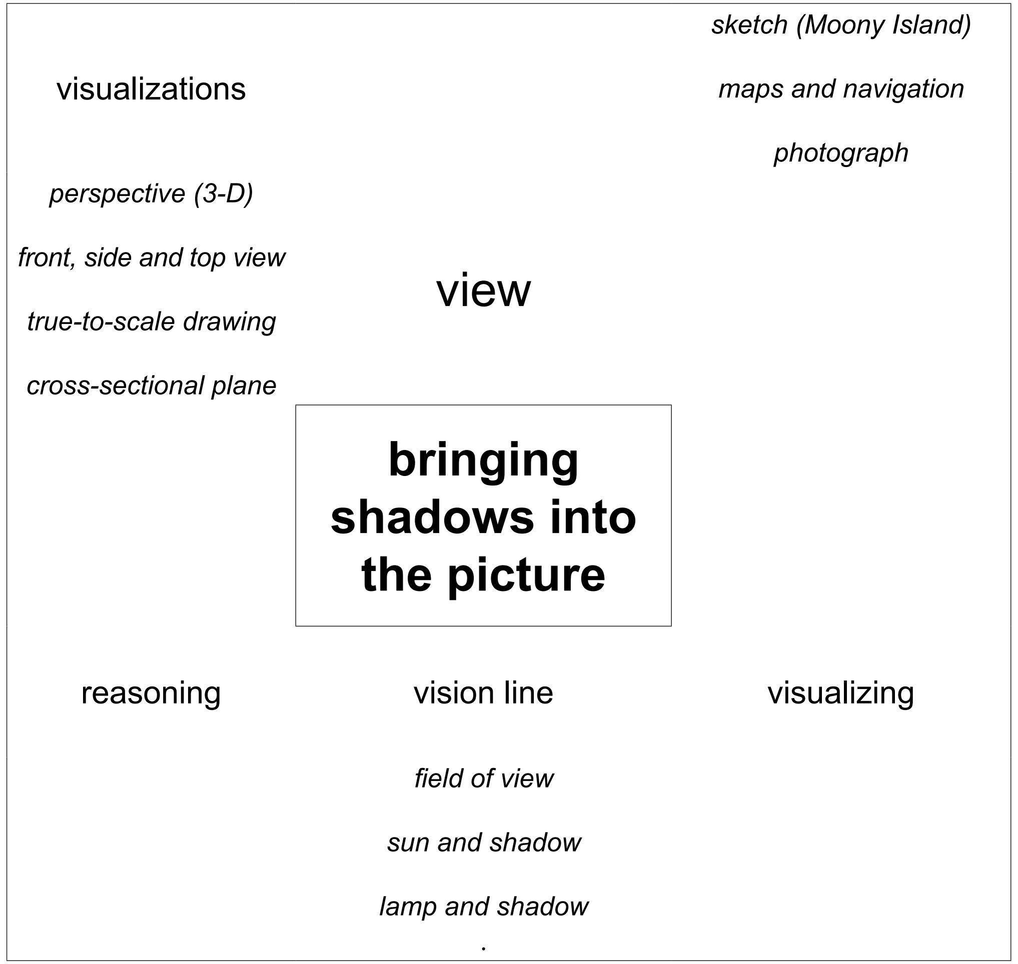


Figure 5.40. Schematic: ‘Bringing shadows intro the picture’ in the learning-teaching trajectory

In closing

In the above activities, the pupils are specifically asked to picture a situation, including imaginary vision lines, and to practice doing this. In many other situations, the visualization is provided to them. The pupils must use the visualization to solve problems. One example is the activity *Moony Island*, where pupils are required to draw and explain routes on a map of the island that is provided. Another example is the *Cube sawing* activity, where pupils are required to draw a plane in a perspective image of the cube. They are also asked to draw the sawn surface. Drawing this surface requires them to make an identical image of the sawn surface. Both aspects of dealing with visualizations and representations are important. Pupils are confronted with characteristics of a specific representation. Pupils learn which representation is most suitable to portray situations, and they especially learn to improve their reasoning about the position from which the situation must be pictured. In short, pupils learn to visualize, and they learn about the technique of visualization and representation.
Beginning with the activity on shadows, many follow-up activities can certainly be planned. For example, a completely different type of visualization could be used, such as a graph that shows the proportion between the length of the stick and the length of the shadow. There is also the question about shadows thrown by other light sources, such as a lamp. Does a shadow thrown by the sun behave differently than the shadow thrown by a lamp? See figure 5.41 and 5.42.

Figure 5.41

Figure 5.42

Reasoning out the moving shadow of the sun also awaits the pupils. In this context, the phases of the moon as well as shadow images can be explained.

6 Geometry attainment targets

Spatial sense

Grade 3/4

Localizing

- Pupils can use a simple map (schematic or otherwise) to find a location.

Taking a standpoint

- Pupils can describe or sketch how an object appears from a specific position.

Navigation

- On a floor plan or map of the immediate area, pupils can lay out a route and describe it with informal indications of distance and direction.

Grade 5/6

Localizing

- Pupils can interpret sketches, floor plans and maps that vary regarding the aspects of scale preservation, detail and orientation.
- Pupils can indicate the position of people or objects in space. During this process, they make use of sketches, floor plans and maps, and of informal and formal coordinate systems.

Taking a standpoint

- Pupils can reason out what you can see from a specific position. The other way around, pupils can reason out the position from which images or photographs were made. They can support their reasoning by making use of views and vision lines.

Navigation

- Pupils can describe routes with the aid of the following concepts: angle, turn, distance, unit, scale, direction and coordinates.

Some pupils can describe a direction change and measure the change in degrees instead of using cardinal directions. In this case, the measurement instrument is a compass rose with gradations in degrees.

Plane and solid figures

Grade 3/4

Properties of and relationships between figures

- Pupils can recognize and identify the solid figures of the sphere, cylinder, cone and cube.
- Pupils can recognize and identify the plane figures of the circle, triangle, square, quadrangle, rectangle, pentagon and hexagon.

Operations, transformations and constructions

- Pupils can use regular polygons such as a triangle, square or parallelogram to ‘tile’ a floor.
- Pupils can analyze a mosaic regarding forms of symmetry.

Grade 5/6

Properties of and relationships between figures

- Pupils know the names of the solid figures of the sphere, cylinder, cone, cube, rectangular prism and pyramid.
- Pupils know the names of the plane figures of the circle, triangle, equilateral triangle, right triangle, square, rectangle, rhombus, parallelogram, pentagon and hexagon.
- Pupils can recognize and identify the above figures in solid objects. They can also establish relationships with plane figures such as the circle, square and rectangle.

- Pupils are familiar with the concepts of perpendicular and parallel.
- Pupils can identify, name and use perspective drawings of simple solid figures to support their spatial reasoning.

With two similarly shaped figures, some pupils can identify which enlargement factor can be used to convert one figure into the other. Some pupils may also be able to establish a relationship between the enlargement factor and the concept of scale.

Operations, transformations and constructions

- Pupils can conduct operations on plane and solid figures (with objects, on paper or on the computer) as well as predict and analyze the consequences of these operations.
- Pupils can sketch the fold-out shapes of solid figures.
- Pupils can analyze and mentally fold together the pictured foldout shapes of simple solid figures.

Visualization and representation

For this terrain, no targets have been formulated for the lower grades.

Grade 5/6

Types of visualizations and representations

- Pupils are familiar with the following types of visualizations and representations: views (top view, side view and front view), perspective representations of a solid figure, schematic representations (such as a metro network), construction drawings, cut-outs or foldouts of a solid figure, and graphs.

Projections

- Pupils are aware that there are various projection methods that can be used to present three-dimensional reality in two dimensions.
- Pupils can relate a simple solid figure drawn in parallel projection to a concrete solid object that represents the figure.

- Pupils can use a simple solid figure drawn in parallel projection to indicate the lines and planes on the figure.

Some pupils can relate the difference between shadow images cast by the sun and those cast by a lamp to central and parallel projection.

Scale preservation

- Pupils can determine whether a representation is a true-to-scale image of reality.

Some pupils can determine whether a representation is or is not proportional or true-to-scale, and in those cases which are not proportional or true-to-scale, they can explain why not.

7 Graphs

Introduction

We see graphs everywhere: on television and in newspapers and periodicals. Graphs - which term we use here to include also diagrams and pie charts - allow you to present information in a compact and comprehensible fashion. An example is shown in the adjacent figure, where McDonalds explains the nutritional value of their French fries (medium). The graph shows the caloric value and the amounts of protein, fat, carbohydrate and sodium. For the consumer, however, this information is not sufficient by itself; how do you know if 5 grams of protein is a lot or a little? As a solution to this problem, the designer of the graph has shown the quantities as the percentages of these components that an adult requires per day. In this way, you see at a glance that 5 grams of protein is 7% of what an adult requires per day.

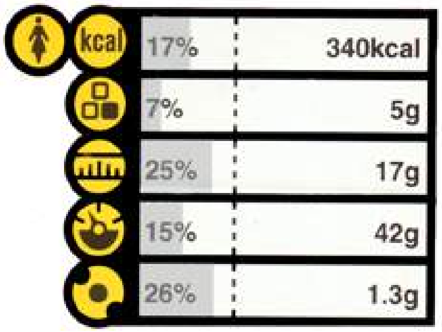


Figure 7.1

Graphs are representations that provide a large amount of information in a compact and succinct fashion. Because this concerns a representation, we could classify graphs as part of geometry. But graphs also have a clear measurement aspect. This concerns data from measurements or counts, which refer to a specific unit. We have therefore decided to discuss graphs as a separate topic.

Graphs play an important role in mathematics education in secondary school, where they are primarily used to portray relationships between different variables in order to study these relationships. This means that the preparatory value of graphs is an important reason to pay attention to this topic. In addition, the societal importance of graphs means this topic has a general educational and practical value in primary school. Children encounter graphs almost every day. They must be able to interpret these graphs and read the information they contain. This concerns the processing of information.

Types of graphs

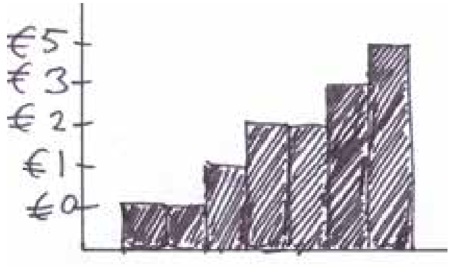


Figure 7.2

There are many different types of graphs. In this chapter we will discuss bar graphs, pie charts and line graphs as the three main categories, but there are variations within each main category, and different types of graphs are sometimes combined. The 'MacDonalds' graph shown above is not a standard bar graph, because bar graphs usually show quantities or counts instead of percentages.

It is important that pupils experience the importance of thinking carefully about the form of a graph. This determines the message provided by the graph and the clarity of the message. For example, with a graph indicating how much pocket money children receive, it seems obvious to place the zero at the bottom of the vertical axis and to maintain equal distances between the amounts. If we do not do that, the proportions become distorted. In the graph of figure 7.2, it appears as if the children who receive 'zero euros pocket money' still receive a small amount, and it also appears that the difference between € 3 and € 5 is equal to the difference between € 2 and € 3.

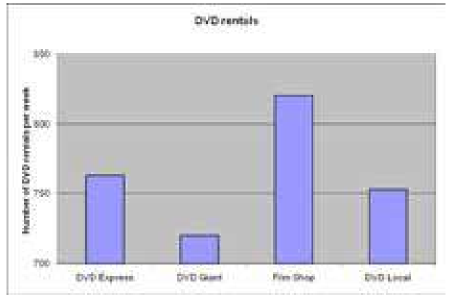


Figure 7.3

Such considerations have led to the convention that the axis must begin at zero, and that the distances on the axis must be equal. However, these conventions can be deliberately disregarded, as shown in the graph about DVD rentals of figure 7.3. Such a graph can lead to misunderstanding, however. In this graph, you can only compare the differences between the bars, but because we still have the tendency to compare the lengths of the bars, the graph suggests larger differences than those that actually exist. Sometimes a deliberate choice is made to deviate from graph conventions, and the pupils must learn to ascertain the motives of the graph designer in order to understand and correctly interpret the graph. For example, by using such

a graph, 'Film shop' (see figure 7.3) suggests a major difference with its competitors.

One problem with graphs is that drawing them with a pencil, ruler and compass is very time consuming. Most assignments in mathematics books therefore concern graphs that are presented ready-made in the book. This is unfortunate, because if we have the pupils draw graphs themselves, the discussion about useful conventions emerges automatically from this process. The text box on the following pages contains a description of an open assignment on graphs, with examples of the discussion points that can emerge from such an assignment.

Such discussions are central to education on graphs. The discussion concerns how you can best portray a specific situation, and how to interpret the graphs that you encounter. The computer is a suitable aid to allow children to make graphs themselves, because the computer takes over the drawing work. In fact, we cannot avoid using the computer when teaching about graphs. However, using the computer leads to other problems. At the end of this chapter we will discuss the role of computer programs.

Pupils do their own investigation

During the Big Mathematics Day in 2005, the topic was ‘Tallying, counting and drawing’. For the upper grades of primary school, the assigned activity on this day was for the pupils to make a questionnaire themselves, and then to show the results of the questionnaire on graphs. The examples below are from a tryout. They illustrate the discussion points that can result from such an open assignment.

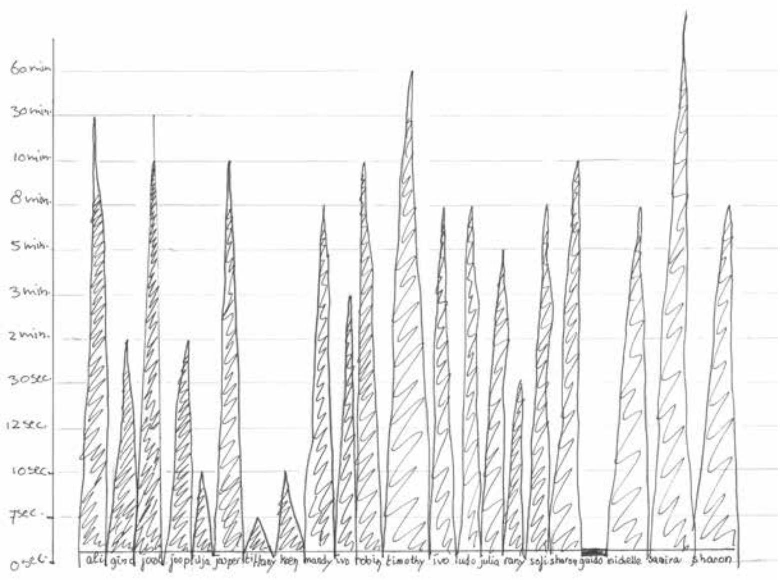


Figure 7.4

The scale on the axis

In graphs, the scale division on an axis is usually based on equal distances. The pupils who made a graph of the time that the children stood in front of a mirror did not keep to this convention: the distance between 0 and 7 seconds is the same as that between 30 and 40 minutes. This provides an inadequate summary of the information. See figure 7.4.

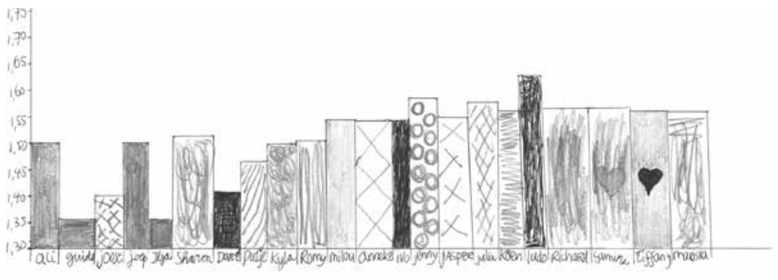


Figure 7.5

Does the axis always have to begin at zero?

When making a graph of your height, it is pointless to begin at 0 (nobody is less than 10 cm tall); it is better to begin at a height such as 120 cm. The distances between the remaining points on the axis remain the same. Because the axis is truncated, if one pupil’s height bar is twice as long as another pupil’s bar, this of course does not mean that the first pupil is twice as tall as the second. Essentially, a section of the bars has been cut off, which is often indicated with a zigzag line at the beginning of the axis. See figure 7.5.

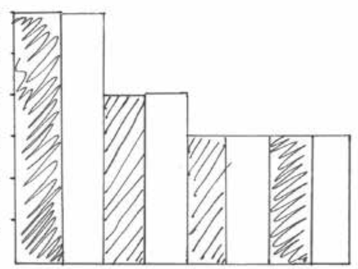


Figure 7.6

What happens if there are zero answers to a question?

Pupils have the tendency to think that the numbers on an axis belong to a specific line segment, instead of being points on a line. Based on this idea, they also give the zero a line segment on the axis, and they color in a bar if there are no answers instead of leaving it empty. By doing so, they probably want to show that zero is also a valid result, which must be included on the graph. In figure 7.6, the short bars are three units in height, but in fact they stand for ‘2’.

Bar graphs

Graphs show the relationship between two or more variables. The choice of a suitable graph is related to the nature of the variables:

- Often this concerns measurements on a scale, where the relationships between numbers have a meaning. An adult who is 180 cm tall is not only taller than a child who is 90 cm tall, but you can also say that the adult is twice as tall. Instead of a pair of jeans costing €60, you can buy two pairs of jeans costing €30; therefore the first pair of jeans is twice as expensive. In the class, there are 18 girls and 9 boys; the proportion between these numbers has meaning.
- Sometimes, the sequence on a scale has meaning, but the proportion between the numbers does not. In the graph of the number of pupils at a school during a number of years, the sequence 2000, 2001, 2002, 2003, 2004, 2005. would be maintained, but 2005 is not a specific proportion of 2000 (see figure 7.7).
- And sometimes the order is arbitrary. If we make a graph of holiday destinations, there is no compulsory reason to place Italy before Spain. Alphabetical sequence can be used, but the only meaning of this sequence is that it makes it easier to look things up. Sometimes the largest number comes first on the graph. This takes place with election results, for example.

In statistics, it is said that the first type of scale is a *ratio scale*, the second type is an *ordinal scale* and the third type is a *nominal scale*. Pupils do not have to know these distinctions, but they must be able to reason logically about the meaning of ratios and sequences.

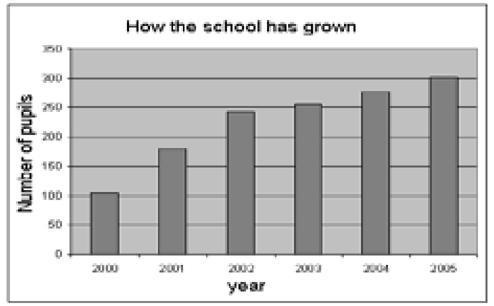


Figure 7.7

Bar graphs are used when the ratio between numbers has meaning for one of the variables, while this not the case with the other variables or if this ratio is unimportant. In the graph of numbers of pupils of figure 7.7, the proportion between the length of the bars indicates how quickly

the school has grown, but for the years along the horizontal axis, only the sequence is important. In a graph of holiday destinations, the numbers per country will also be shown with separate bars.
The bars can optionally be drawn horizontally or vertically. Graphs with vertical bars are the most common, but the graph of figure 7.8 showing cinema attendance is also clear.

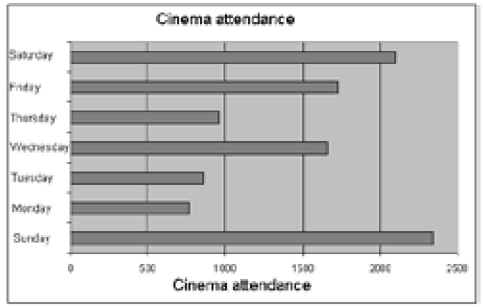


Figure 7.8

As stated previously, a portion of the axis on a bar graph can sometimes be truncated. In the graph of figure 7.9, of the number of minutes of television viewing per week, every bar begins at 750 minutes. However, there is a danger of incorrect interpretation. The magnitude of the difference between Arnoud and Fatma is still important, but if someone wants to imply that Fatma watches 'much more' television than Arnoud, they should also look at the second graph (figure 7.10), because it shows the magnitude of the differences with respect to the total. Both graphs indicate that that Fatma watches the most TV, but the second graph clearly shows that the differences are relatively small.

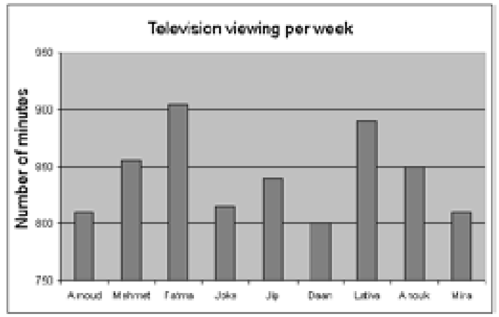


Figure 7.9

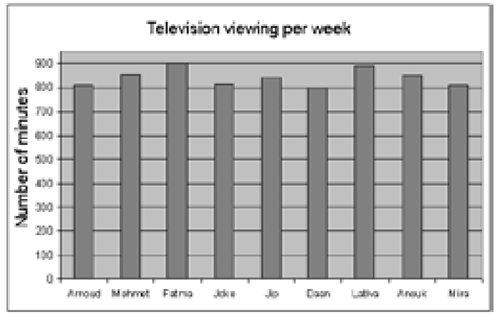


Figure 7.10

Pie charts

There are many situations that concern parts of a whole. In that case, a pie chart sometimes provides a clearer picture of the situation than a bar graph.

The pie chart of cinema attendance in figure 7.11 is based on the same numbers as the bar graph of figure 7.12. This pie chart still shows the relative proportions between the various days – on Saturday, there are approximately twice as many visitors as on Thursday – but it also shows the proportions with respect to the total. For example, more than half of the total cinema attendance is on the weekend (including Friday). Pie charts allow easier interpretation of proportions in terms of fractions or percentages.

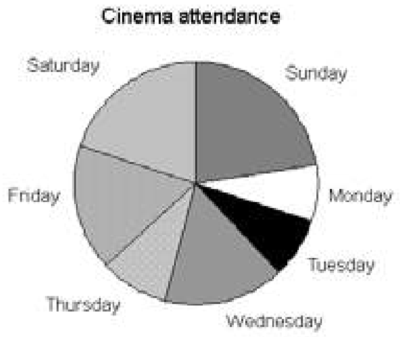


Figure 7.11

Figure 7.12

The differences between bar graphs and pie charts can be clearly illustrated by the way they are used to show election results. Bar graphs clearly show which political party is bigger, but if you want to show which coalition of parties has a majority, a pie chart is more useful.

In principle, colored bars can also be used to show part-whole relationships. However, the pie chart provides more support in this respect because we can easily estimate the size of the 'pie slices'. For example, we can clearly see if one section of a circle is exactly one third or if it deviates from this proportion. The relative size of the individual pieces in the adjacent bar graph (see figure 7.12) is much more difficult to interpret.

Pie charts are difficult to draw with a pencil, ruler and compass, even more difficult than bar graphs. However, when using the computer, this is no longer a problem. But it is still a good idea to occasionally ask the pupils construct a pie chart manually. They can do this by first coloring in sections on a strip of paper. The strip can be any length, so the bar graph can be made by simply coloring in as many sections as are required on a strip. The strip is then formed into a cylinder. Then, by drawing 'spokes', the divisions on the strip can be converted into the sections on a pie chart.

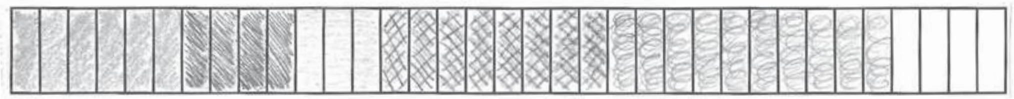


Figure 7.13

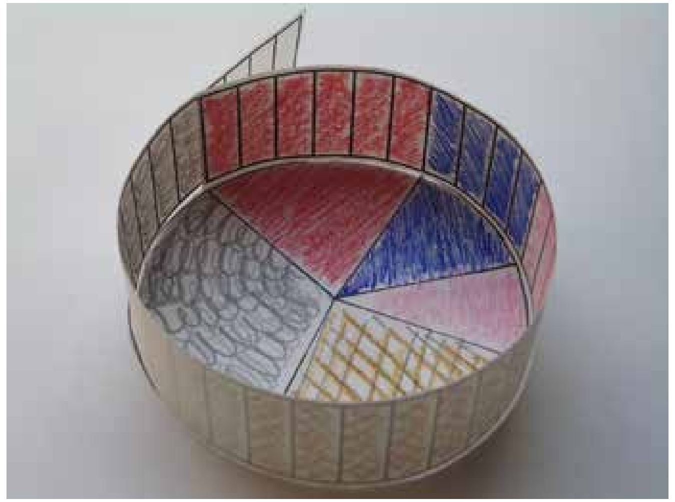


Figure 7.14

Line graphs

Often, the relationship between two variables on a graph is shown with a line. Strictly speaking, these must both be variables where each intermediate value on the scale has meaning. After all, if the line rises or falls, the graph suggests that the variable passes along all the intermediate values. In other words, it must concern variable with a continuous scale.

For example, growth in height provides a continuous scale, because between 156 cm and 157 cm there are also heights such as 156.3 cm, 156.48 cm and 156.9007 cm. The fact that height is not measured so precisely in practice does not take away from this fact; the intermediate values are meaningful in themselves. Time and temperature are other examples of continuous variables. An example of a discrete scale is the number of pupils attending a school; the school cannot have 212.8 pupils. This number could be an average, but not an actual count of pupils.

However, we must remain cautious even where continuous variables are concerned. After all, we measure only some of the points on a graph, and we complete the graph by drawing a smooth line through the intermediate points. This elicits the question of whether or not the line is actually so smooth in reality. This is not always the case. If we make a graph of the temperature during one day based on measurements taken at 03:00, 06:00, 09:00, 12:00, 15:00, 18:00, 21:00 and 00:00 hours, and if there is a rainstorm at 16:00 hours, then the dip in temperature caused by this rainstorm can probably not be seen on the graph.

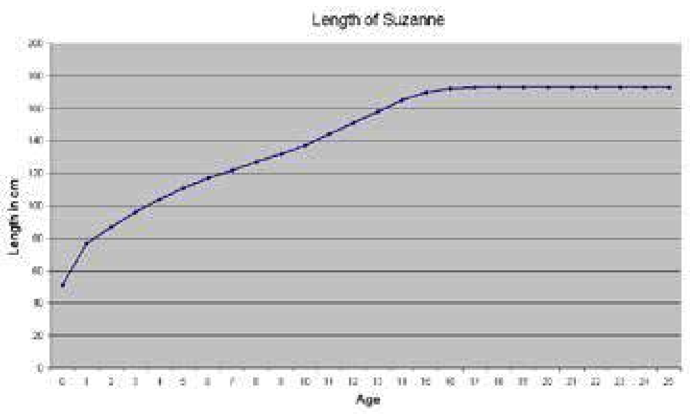


Figure 7.15

Line graphs often show how a specific variable changes in time. The growth chart of Suzanne (see figure 7.15) as an example. Suzanne was 51 cm at her birth, and 173 cm tall on her 18th birthday. From such a graph, you can not only read how tall Suzanne was at a specific age, but you can also determine whether her growth was fast or slow. The growth chart shows that Suzanne grew rapidly between her 11th and 15th birthdays: she had a 'growth spurt'.

Apparently, you can determine the rate of change by how steeply the line on the graph rises or falls. If the line rises very quickly, this indicates rapid change; at the other extreme, if there is no change at all, the line remains horizontal. Pupils will not find it easy to explain why this is the case. Such a line graph provides little help in this respect; it is a very abstract representation. In order to reason about steepness on a graph, it is a good idea to go back to a graph in which the measurements are shown as separate bars.

Assume that on every birthday we measure Suzanne by drawing a line on a wall of her bedroom. On her 18th birthday, we then have a series of separate measurements that we can show with a bar graph as in figure 7.16.

When did Suzanne grow rapidly? Between her 10th and 11th birthdays, she grew 7 cm in one year, while before this she grew approximately 5 cm per year. In this way, the difference between the length of sequential bars indicates how big the change is within a fixed unit of time. To explain why the steepness of the line on a line graph says something about the rate of change, pupils must be able to see the line graph as the result of separate measurements; essentially they must see the line graph as being composed of individual bars.

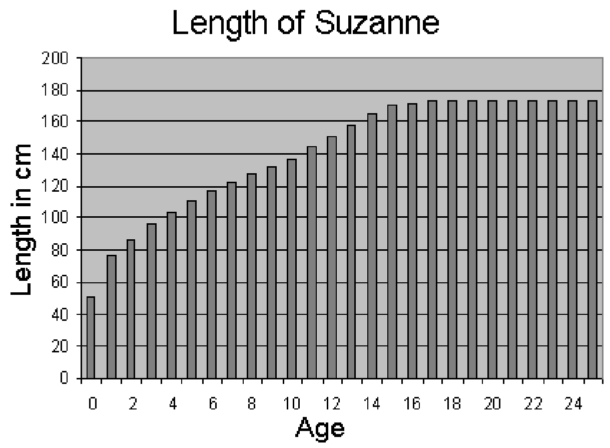


Figure 7.16

Graphs and the computer

We previously indicated that the computer is a valuable aid for drawing graphs. A spreadsheet program such as Excel can help in this process. The pupils can easily enter raw data in a table. The table structure provides a handhold when considering how data can be organized.
When children have conducted their own investigations and have entered the data in a clear fashion in Excel, the program then offers many possibilities to make a graph from the data. We can allow the pupils to independently choose what kind of graph they want to make. After this, questions are discussed such as: Why did you choose a specific type of graph? How can someone see what you want to show in the graph?
However, there are also objections to the computer program: there are a great many potential choices, and it requires significant computer skills to obtain exactly the graph that you intended. A more principle objection is that a program such as Excel is the final product of a long development process; the graph appears simply by pushing a button. Nevertheless, graphs are a relatively recent invention. The first graphs that showed a relationship between two variables appeared in the 18th century. But long before this, there was an entire development process that led to the graphs that we are now so familiar with. It is good for the pupils to go through a similar development themselves; they can independently 'rediscover' the core ideas behind graphs.

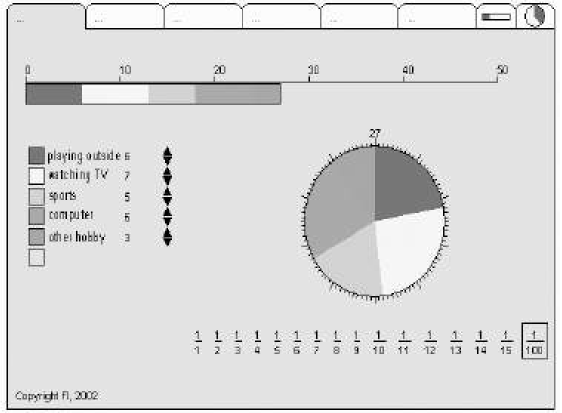


Figure 7.17

Let us take the line graph as an example, as a representation of the relationship between two continuous variables. In a certain sense, a line graph can be seen as a graph with a great many bars, of which only the very tops are shown. Interpreting points on the line becomes easier if pupils can 'visualize' a bar under a point. It would therefore be a good strategy if pupils, while working in a computer program, could experiment with the relationship between a bar graph and a line graph.

In addition, a line graph places demands on the x-axis and the y-axis: from the axes you must be able to read sequences as well as relationships. To encourage pupils to think about this, it is a good idea to have them work with computer programs in which they must independently make decisions on these types of issues.
In other words, besides Excel we need other computer programs that are designed with didactical intentions. One example is the computer program '*In Kaart*' (see www.rekenweb.nl, figure 7.7). This program draws pie charts from entered data and offers pupils the possibility of placing a fractional scale – including hundredths – over the pie chart. Such a program enables the pupils to experience how the same data can be described both with fractions and percentages. At the Freudenthal Institute the research into an educational computer program on graphs is being conducted as part of the project 'Mathematics education for the information society'. The institute developed a computer program on bar graphs and line graphs.

Education concerning graphs

Similar to mathematics education in general, education on the topic of graphs should give pupils sufficient room to experiment and contribute their own ideas. This means that we must periodically ask pupils to make a graph themselves using specific data, without stating exactly what the graph should look like. Although such activities require more time than answering questions about a pre-existing graph, the investment in time repays itself in terms of the insight that children develop during this process.
The most important aspect is that pupils learn to attribute meaning to graphs. By periodically challenging children to illustrate elements from their own environment by making a graph, they must go to work and attribute meaning to graphs. They must realize that it is useful to illustrate a phenomenon with a graph. The graph must contribute something to clarifying a situation. Children must also be able to explain this. They do this by discussing the graphs they have made with other children, which leads to them being asked to interpret and clarify the graphs.
By repeatedly engaging in such discussions, the children steadily improve their understanding of how you can best organize data on a graph. But this organization of data on a graph is the last step in making a graph. The process begins with collecting data and organizing the data in a table. But in

the meantime, the form of the graph is given a role, because the data must be collected and organized in accordance with the graph.

As we have shown previously, the topic of graphs appears at the interface between measurement and geometry. A graph creates a visual representation, where the form of the graph clarifies what the graph is meant to show. The measurement aspect is reflected in the units shown on the graph. The form of the graph and the units can be chosen so that they present a standpoint or statement in a compact form; graphs are widely used in the media for this purpose. In this way, we encounter the general educational value of mathematics. Graphs are addressed in education because they are important to society. Children encounter graphs almost everywhere, and they must also be capable of supporting ideas by using a graph.

However, as we stated previously, this is not the only reason to invest time in studying graphs. This topic is also part of the curriculum in secondary education and attention to graphs in primary school prepares children for this. This preparatory value applies primarily to line graphs.