

# Basics of Biomagnetism

Creation of Magnetic Fields in Biological Systems

Physics 656: Seminar Medical Physics

Physical Fundamentals of Medical Imaging

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## Biomagnetism

study of magnetic fields originating from biological systems

## Magneto-biology

study of effects of magnetic fields on an organism (e.g. orientation of birds guided by the magnetic field of the earth)

- Organism (e.g. human body) emits magnetic fields  $\mathbf{B}$  (often only measurable with highly sensitive devices such as SQUIDS (→ talk F. Bismarck))
- Organism produces also electric fields  $\mathbf{E}$ , which can be calculated by the measurement of potential differences, such as Electroencephalography (EEG) → not part of this talk
- The  $\mathbf{B}$ -fields originate from currents or contaminants in the tissue

- The measurement of the magnetic and electric fields can convey information about how the organism works: For example, certain active-areas of tissue in the cerebral cortex (outer-part of the brain) can be localised by the measurement technique of Magnetoencephalography (MEG) (description of MEG is not part of this talk)
- MEG measures fields, but this gives a problem: How to reconstruct the currents, which give rise to the measured fields, and their location in the tissue uniquely from the data of the MEG? (inverse Problem → talk A. Shani)
- In this talk we deal with the calculation of the  $\mathbf{B}$ -fields, given the currents in the tissue

# Magnetic Fields in the Human Body

## Occurrences of magnetic fields

- Magnetic constituents in the body (e.g. contaminants in the lung)
- Magnetic moments of molecules
- Ion currents in tissue

The magnetic fields, emitted by the human body, range approximately from nano- to femto-Tesla

Values of table taken from Hämäläinen et al, RevModPhys, 1993 and S.J. Williamson, L. Kaufman, Journal of Magnetism and Magnetic Materials, 1981

contaminants in the lung	$\sim 1 \text{ nT}$
heart-muscle	$\sim 50 \text{ pT}$
ion currents in the eyes	$\sim 10 \text{ pT}$
brain activity	fT-pT
optical brain stimulus	$\sim 20 \text{ fT}$
thermic magnetic noise (created by the thermic motion of atoms)	$\sim 1 \text{ fT}$

**E**- and **B**-fields are produced by currents. We introduce two types of currents:

## Primary current $\mathbf{j}^i$

Current-flow in active tissue (in intracellular-region also called 'impressed current', because  $\mathbf{j}^i$  gets 'impressed' by biological activity)

## Return current $\mathbf{j}^V$

Is evoked by the primary current: It 'neutralises' the primary current by transporting back the charge, which builds up through the primary current (also called 'volume current', because it transports back the charge through the surrounding volume (tissue), in which the primary current is embedded)

- Exact locations of the 'real' currents are not known → primary current and return current serve as theoretical models, representing the 'real' currents in the tissue
- In our calculations, the total current is represented by the sum of primary current and return current

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# Ion Mechanism in a Cell

- Cells are surrounded by a membrane, dividing the space into intracellular- and extracellular-space
- Ion-pump mechanism: Receptor molecules pump selected ions against concentration gradient of intra- and extracellular-space (e.g.  $\text{Na}^+$ - $\text{K}^+$ -pump: 3  $\text{Na}^+$  out, 2  $\text{K}^+$  into the cell). Details of this or other pump-mechanism are not discussed in this talk
- According to the concentration gradient we get a potential difference  $V_{\text{in}} - V_{\text{ext}}$  across the cell membrane  $\rightarrow$  for an equilibrium state, currents (for each type of ion) must be balanced
- Equilibrium: Concentration of certain ion-type  $C_{\text{ion}}$  determined by thermodynamics (Boltzmann-distribution)

$$C_{\text{ion}} \sim \exp\left(-\frac{|e|V_{\text{ion}}}{k_B T}\right) \Rightarrow V_{\text{ion,in}} - V_{\text{ion,ext}} = \frac{k_B T}{|e|} \ln\left(\frac{C_{\text{ion,ext}}}{C_{\text{ion,in}}}\right)$$

# Ion Mechanism

## Goldman's Equation

	$V_{\text{ion,in}} - V_{\text{ion,ext}}$
$\text{K}^+$	-89 mV
$\text{Cl}^-$	-48 mV
$\text{Na}^+$	52 mV

(Values of the table taken from S.J. Williamson, L. Kaufman, Journal of Magnetism and Magnetic Materials, 1981)

Take  $\text{K}^+$ ,  $\text{Cl}^-$ ,  $\text{Na}^+$  for the total membrane-potential  $V$  simultaneously into account (Goldman's equation):

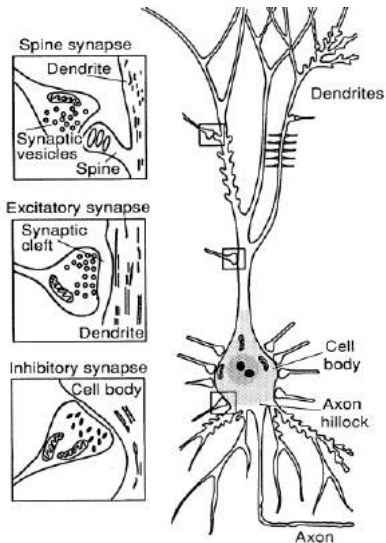
$$V = \frac{k_B T}{|e|} \ln \left( \frac{P(\text{K}^+)C_{\text{ext,K}^+} + P(\text{Na}^+)C_{\text{ext,Na}^+} + P(\text{Cl}^-)C_{\text{in,Cl}^-}}{P(\text{K}^+)C_{\text{in,K}^+} + P(\text{Na}^+)C_{\text{in,Na}^+} + P(\text{Cl}^-)C_{\text{ext,Cl}^-}} \right)$$

with  $P(\text{ion}) =$  permeability of the membrane with respect to that type of ion

This yields with typical values for humans and animals:  $V \approx -70$  mV

(Value taken from Hämäläinen et al, RevModPhys, 1993)

# Example: Pyramidal Neurons



(picture taken from Hämäläinen et al, RevModPhys, 1993)

## structure

- soma** Cell-body (processes signals)
- dendrites** Thread-like endings (receive stimuli from other cells)
- axon** Long fibre (carries nerve impulses)
- synapses** Connections to other neurons (release neurotransmitter for action potential (→ explained on the next slide))

# Action Potential (Pyramidal Neuron)

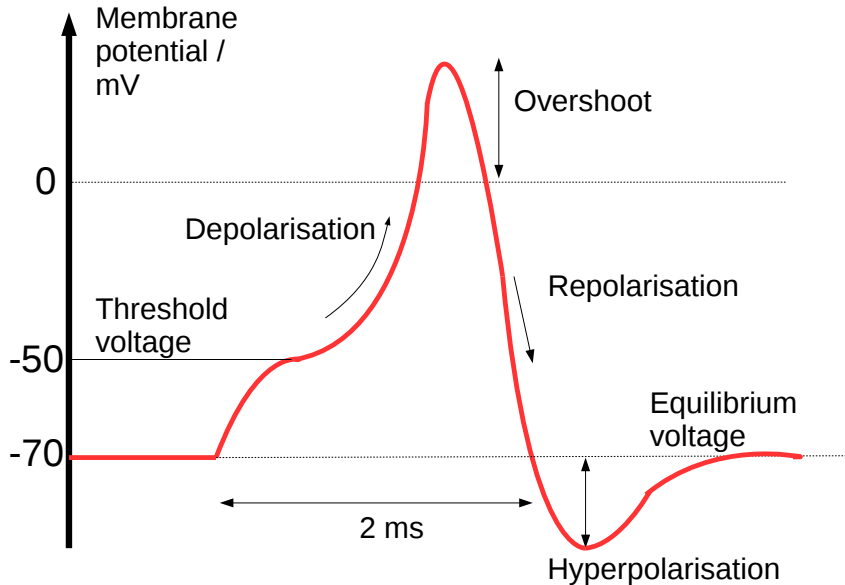
Action potential is an electric pulse, which travels from the axon-hill at the cell body along the axon to other neurons or cells

Action potentials are triggered by excitatory synaptic inputs. These are increasing the membrane potential

## Triggering mechanism

- 1 Synapses: Release neurotransmitters (some acid like glutamate)
- 2 Neurotransmitters bind to receptor molecules on cell membrane → change in membrane potential and permeability for a certain type of ions
- 3 Positive feedback loop: Rise of membrane potential opens ion channels, this in return increases the membrane potential
- 4 'Excitatory potential' propagates exponentially damped to the cell body → combined signal of several synapses must reach the axon-hill
- 5 Threshold  $\sim -55$  mV at the axon-hill triggers an action potential

# Potential course (picture adopted from <https://de.wikipedia.org/wiki/Aktionspotential>)



## Depolarisation and Repolarisation

- 1 Positive-feedback evokes a rapid raise in the membrane potential → sodium and potassium channel becomes maximally opened
- 2 Sodium channel slowly closes, membrane permeability of sodium becomes lower relative to the permeability of potassium → membrane voltage falls

Overshoot and hyperpolarisation are inertial-phenomena, i.e. in case of the overshoot, more ions are flooding into the intracellular-space than would be necessary for a zero membrane-voltage

The signal of the action potential propagates along the axon to other neurons, to trigger again an action potential

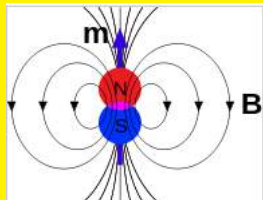
- Typical duration of an action potential of a neuron is 1 – 2 ms
- Within the phases of repolarisation and hyperpolarisation it is not possible to trigger an action potential anew (refractory period)
- Amplitude of the action potential is independent of triggering-strength ('all-or-non' principle), but not so the frequency → the stronger the synaptic input, the higher the frequency (limited by refractory period)
- Up to now it is unknown, how the entire process of an action potential conveys information

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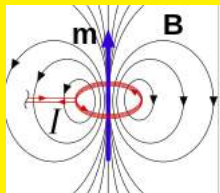


# Magnetic-Dipole

## pair of poles



## el. current (closed loop)



Magnetic-dipole can be represented by two models:

- Pair of (contrary charged) poles
- Closed loop (perfect circle) of electric current

Important field components:

- Direct field: Represented by those field line-pieces, which are orientated **parallel** to the magnetic moment  $\mathbf{m}$  ( $\mathbf{B} \cdot \mathbf{m} > 0$ )
- Return field: Represented by those field line-pieces, which are orientated **antiparallel** to  $\mathbf{m}$  ( $\mathbf{B} \cdot \mathbf{m} < 0$ )

(pictures taken from [https://en.wikipedia.org/wiki/Magnetic\\_dipole](https://en.wikipedia.org/wiki/Magnetic_dipole))

Magnetic fields in organic systems are often very complicated close to the tissue. But in most cases the fields are measured outside of the body, where the far-field approximation is sufficient. . .

# Far-Field Approximation for the Vector-Potential $\mathbf{A}$

For a magnetic-dipole, the vector potential  $\mathbf{A}(\mathbf{r})$  can be written as

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}'$$

Taylor-expansion of  $1/|\mathbf{r} - \mathbf{r}'|$  yields so called multipole-expansion for  $\mathbf{A}(\mathbf{r})$ :

$$\begin{aligned} \frac{1}{|\mathbf{r} - \mathbf{r}'|} &= \sum_{n=0}^{\infty} \frac{1}{n!} (\mathbf{r}' \cdot \nabla_{\bar{\mathbf{r}}})^n \frac{1}{|\mathbf{r} - \bar{\mathbf{r}}|} \Big|_{\bar{\mathbf{r}}=0} \\ &\approx \frac{1}{|\mathbf{r}|} + \frac{1}{|\mathbf{r}|^3} (\mathbf{r} \cdot \mathbf{r}') + \frac{1}{2|\mathbf{r}|^5} (3(\mathbf{r} \cdot \mathbf{r}')^2 - |\mathbf{r}|^2 |\mathbf{r}'|^2) + \dots \end{aligned}$$

$$\begin{aligned} \Rightarrow \mathbf{A}(\mathbf{r}) &\approx \frac{\mu_0}{4\pi} \int d\mathbf{r}' \mathbf{j}(\mathbf{r}') \left( \frac{1}{|\mathbf{r}|} + \frac{1}{|\mathbf{r}|^3} (\mathbf{r} \cdot \mathbf{r}') + \mathcal{O}\left(\frac{1}{|\mathbf{r}|^5}\right) \right) \\ &= \frac{\mu_0}{4\pi|\mathbf{r}|^2} \int d\mathbf{r}' \mathbf{j}(\mathbf{r}') (\mathbf{e}_{\mathbf{r}} \cdot \mathbf{r}') \end{aligned}$$

## Side Remark: Why No Magnetic Monopoles?

Define vector field (vanishing at infinity)  $\mathbf{V}_k = r'_k \mathbf{j}(\mathbf{r}')$ ,  $k = 1, 2, 3$

$$\begin{aligned}\nabla(\mathbf{V}_k(\mathbf{r}')) &= \nabla(r'_k \mathbf{j}(\mathbf{r}')) \\ &= \sum_{l=1}^3 \frac{\partial}{\partial r'_l} (r'_k j_l(\mathbf{r}')) \\ &= \sum_{l=1}^3 \left[ \delta_{kl} j_l(\mathbf{r}') + r'_k \frac{\partial}{\partial r'_l} j_l(\mathbf{r}') \right] \\ &= j_k(\mathbf{r}') + r'_k \nabla \mathbf{j}(\mathbf{r}')$$

Continuity equation ( $\partial \rho / \partial t + \nabla \cdot \mathbf{j}(\mathbf{r}') = 0$ ) for static charge-distribution:

$$\nabla \mathbf{j}(\mathbf{r}') = 0$$

$$\Leftrightarrow \int_{\Omega=\mathbb{R}^3} d\mathbf{r}' j_k(\mathbf{r}') = \int_{\Omega} d\mathbf{r}' \nabla(\mathbf{V}_k(\mathbf{r}')) = \oint_{\partial\Omega} \mathbf{V}_k(\mathbf{r}') d\mathbf{S}' = 0$$

# Impressed & Volume Current ( $\mathbf{j}^i$ & $\mathbf{j}^V$ )

## Impressed current $\mathbf{j}^i$

- Arise from biological activity, especially from diffusion of ions
- Establishes imbalance in  $C_{\text{ion}} \rightarrow$  evokes return current

## Volume current $\mathbf{j}^V$

- Is evoked by  $\mathbf{j}^i$ , prevents ionizing of the surrounding tissue
- Is dictated by distribution of conductivity  $\sigma$  and  $\mathbf{E}$ , where  $\mathbf{E}$  arises from the charge-transport of  $\mathbf{j}^i$
- Follows Ohm's law  $\mathbf{j}^V = \sigma(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r})$

For  $\mathbf{j}^V$  we need to know  $\sigma$ , but  $\sigma$  is unknown  $\rightarrow$  hint: From animal-experiments, it is known, that  $\sigma$  is likely to be highly anisotropic

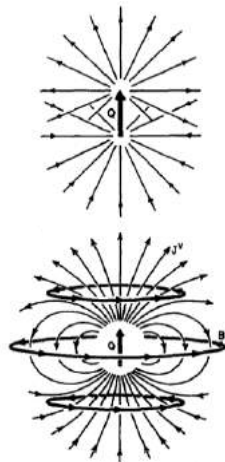
Total current  $\rightarrow \mathbf{j} = \mathbf{j}^i + \mathbf{j}^V$

$\mathbf{j}^i$  and  $\mathbf{j}^V$  can be combined to establish a 'current-dipole'...

Movement of localised charge over a short distance

$$\rightarrow Q = \text{current} \cdot \text{distance}$$

- Current-dipoles represent an unknown current pattern in terms of  $\mathbf{j}^i$  and  $\mathbf{j}^V$
- Battery: Biochemical processes impress flow of charge ( $= \mathbf{j}^i$ ) between  $\pm$ -terminals
- Back-flow ( $= \mathbf{j}^V$ ) prevents ionizing of tissue
- $\mathbf{j}^V$  = radially symmetric in- & out-flow. Pattern has same form as  $\mathbf{B}$  of a magnetic-dipole
- Orientation of the dipole in the direction of impressed current:  $Q \parallel \mathbf{j}^i$
- Also possible: Fixed charges as  $\pm$ -pole



(picture taken from S.J. Williamson and L. Kaufman, Journal of Magnetism and Magnetic Materials, 1981)

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# Quasi-Static Limit

- Low-frequency limit (< 1 kHz): All currents change in time synchronously  $\rightarrow$  situation can be viewed quasi-static, i.e.  $\mathbf{B} = \mathbf{B}(\mathbf{r})$  for each moment in time
- In quasi-static limit: Magnetic field described by Biot-Savart: (integration over unbounded, homogeneous medium,  $\mathbf{j}$  vanishes at infinity)

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d\mathbf{r}' \frac{\mathbf{j}(\mathbf{r}') \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}$$

Use the Maxwell-equations

$$\nabla \times \mathbf{B} = \mu_0(\mathbf{j} + \epsilon_0 \partial \mathbf{E} / \partial t)$$

$$\nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t$$

and the time-dependency of  $\mathbf{E}$  in the form  $\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0(\mathbf{r})e^{i\omega t}$  to justify quasi-static approximation:

## Side remark: Why is Quasi-Static Approximation Justified?

Insert  $\mathbf{j} = \sigma \mathbf{E} + \partial \mathbf{P} / \partial t$  and  $\mathbf{P} = (\epsilon - \epsilon_0) \mathbf{E}$  into

$$\nabla \times \mathbf{B} = \mu_0 (\mathbf{j} + \epsilon_0 \partial \mathbf{E} / \partial t)$$

$$\Rightarrow \nabla \times \mathbf{B} = \mu_0 (\sigma \mathbf{E} + \epsilon \partial \mathbf{E} / \partial t)$$

Constraint: Temporal part  $\epsilon \partial \mathbf{E} / \partial t$  must be smaller than  $\sigma \mathbf{E}$   
with  $\mathbf{E} = \mathbf{E}_0(\mathbf{r}) e^{i\omega t}$  and  $\omega = 2\pi f$

$$\Rightarrow |\epsilon \partial \mathbf{E} / \partial t| \ll |\sigma \mathbf{E}| \Leftrightarrow \omega \epsilon / \sigma \ll 1$$

with  $f \sim 100$  Hz,  $\sigma = 0.3 \Omega^{-1} \text{m}^{-1}$ ,  $\epsilon = 10^5 \epsilon_0 \rightarrow \omega \epsilon / \sigma = 2 \cdot 10^{-3} \ll 1$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \Leftrightarrow \nabla \times \nabla \times \mathbf{E} = -\mu_0 \frac{\partial}{\partial t} (\sigma + i\omega \epsilon) \mathbf{E} = -i\omega \mu_0 \sigma (1 + i\omega \epsilon / \sigma) \mathbf{E}$$

Spatial changes of solution  $\sim |\omega \mu_0 \sigma (1 + i\omega \epsilon / \sigma)|^{-1/2} \approx 65$  m

# Integral Formulas for $\mathbf{V}$ and $\mathbf{B}$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}(\mathbf{r}') \times \mathbf{R}}{R^3} d\mathbf{r}' \quad \mathbf{R} = \mathbf{r} - \mathbf{r}'$$

Rewrite integrand with ( $\nabla$  = Nabla-operator with respect to  $\mathbf{r}$ ,  $\nabla'$  = Nabla-operator with respect to  $\mathbf{r}'$ )

$$\mathbf{R}/R^3 = -\nabla(1/R) = \nabla'(1/R)$$

$$\mathbf{j} \times \nabla'(1/R) = (\nabla' \times \mathbf{j})/R - \nabla' \times (\mathbf{j}/R)$$

$$\begin{aligned} \hookrightarrow \mathbf{B}(\mathbf{r}) &= \frac{\mu_0}{4\pi} \left[ \int \frac{\nabla' \times \mathbf{j}(\mathbf{r}')}{R} d\mathbf{r}' - \underbrace{\int \nabla' \times \left( \frac{\mathbf{j}(\mathbf{r}')}{R} \right) d\mathbf{r}'}_{\text{Stokes} \rightarrow 0} \right] \\ &= \frac{\mu_0}{4\pi} \int \frac{\nabla' \times \mathbf{j}(\mathbf{r}')}{R} d\mathbf{r}' \end{aligned}$$

Because  $\nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t \approx 0 \Rightarrow \mathbf{E} = -\nabla V \Rightarrow \mathbf{j} = \mathbf{j}^i + \mathbf{E} \sigma = \mathbf{j}^i - \sigma \nabla V$

Use

$$\nabla' \times \mathbf{j} = \nabla \times \mathbf{j}^i - \nabla \times \sigma \nabla V = \nabla \times \mathbf{j}^i - \nabla \sigma \times \nabla V = \nabla \times \mathbf{j}^i + \nabla \times (V \nabla \sigma)$$

$$\begin{aligned} \mathbf{B}(\mathbf{r}) &= \frac{\mu_0}{4\pi} \int \frac{\nabla' \times (\mathbf{j}^i(\mathbf{r}') + V \nabla' \sigma)}{R} d\mathbf{r}' \\ &= \frac{\mu_0}{4\pi} \int (\mathbf{j}^i(\mathbf{r}') + V \nabla' \sigma) \times \frac{\mathbf{R}}{R^3} d\mathbf{r}' \end{aligned}$$

We still need an eq. to determine  $V$ !

Use in quasi-static approximation:

$$\underbrace{\nabla \cdot [\nabla \times \mathbf{B}]}_{=0} = \mu_0 \nabla \cdot (\mathbf{j} - \underbrace{\epsilon_0 \partial \mathbf{E} / \partial t}_{\approx 0}) \Leftrightarrow \nabla \cdot \mathbf{j} = 0$$

$$\hookrightarrow \nabla \cdot (\sigma \nabla V) = \nabla \cdot \mathbf{j}^i$$

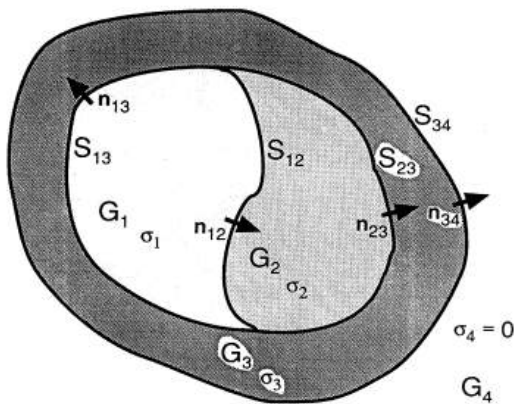
Choose proper boundary conditions, solve for  $V$  and plug into eq. for  $\mathbf{B}$

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# Piece-Wise Homogeneous Conductor

- $G_i$ : Area- or volume-part of the conductor
- $\sigma_i$ : Conductivity of the tissue in  $G_i$  (constant in  $G_i$ )
- $S_{ij}$ : Boundary-line/-surface, separating  $G_i$  from  $G_j$
- $\mathbf{n}_{ij}$ : Normal-vector of  $S_{ij}$ , pointing (conventionally) from  $G_i$  to  $G_j$

## Piece-wise homogeneous conductor



(picture taken from Hämäläinen et al, RevModPhys, 1993)

# Equation for $\mathbf{B}$ in Case of a Piece-wise Homogeneous Conductor

$\nabla' \sigma$  only non-zero at boundaries. Use ( $G = \cup_i G_i$ ) and  $\mathbf{j} = \mathbf{j}^i - \sigma \nabla V$

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_G \frac{\mathbf{j}(\mathbf{r}') \times \mathbf{R}}{R^3} d\mathbf{r}' = \mathbf{B}_0(\mathbf{r}) - \frac{\mu_0}{4\pi} \sum_i \sigma_i \int_{G_i} \nabla' V \times \frac{\mathbf{R}}{R^3} d\mathbf{r}'$$

with

$$\mathbf{B}_0(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_G \mathbf{j}^i \times \frac{\mathbf{R}}{R^3} d\mathbf{r}'$$

Let's rewrite the second term: Take a look at the vector identities:

$$\begin{aligned} \nabla \times [(1/R)\nabla V] &= [\nabla(1/R)] \times \nabla V + (1/R) [\nabla \times \nabla V] \\ 0 = \nabla \times \nabla(V/R) &= \nabla \times [(1/R)\nabla V + V\nabla(1/R)] \\ &= \nabla(1/R) \times \nabla V + \nabla \times V\nabla(1/R) \end{aligned}$$

Therefore

$$\nabla V \times \nabla(1/R) = \nabla \times V\nabla(1/R)$$

With this we get

$$\int_{G_i} \nabla' V \times \frac{\mathbf{R}}{R^3} d\mathbf{r}' = \int_{G_i} \nabla' V \times \nabla'(1/R) d\mathbf{r}' = \int_{G_i} \nabla' \times V \nabla'(1/R) d\mathbf{r}'$$

Use the vector analysis relationship  $\int_G \nabla \times \mathbf{a} d\mathbf{r} = -\int_{\partial G} \mathbf{a} \times d\mathbf{S}$  (see D. B. Geselowitz, IEEE Transactions on Magnetics, Vol. Mag-6, No 2, 1970):

$$\int_{G_i} \nabla' \times V \nabla'(1/R) d\mathbf{r}' = -\int_{\partial G_i} V \nabla'(1/R) \times d\mathbf{S}_i$$

The integral is now taken over boundaries of neighbouring  $G_i, G_j$  ( $d\mathbf{S}_i = -d\mathbf{S}_j$  on  $\partial G_i \cap \partial G_j$ )

$$\begin{aligned} \sum_i \sigma_i \int_{\partial G_i} (\dots) d\mathbf{S}_i &= \dots + \sigma_i \int_{\partial G_i} (\dots) d\mathbf{S}_i + \dots + \sigma_j \int_{\partial G_j} (\dots) d\mathbf{S}_j + \dots \\ &= \dots + (\sigma_i - \sigma_j) \int_{S_{ij}=\partial G_i \cap \partial G_j} (\dots) d\mathbf{S}_{ij} + \dots \\ &= \sum_{\langle i,j \rangle} (\sigma_i - \sigma_j) \int_{S_{ij}} (\dots) d\mathbf{S}_{ij} + \text{ext. boundary} \end{aligned}$$

$$\Rightarrow \mathbf{B}(\mathbf{r}) = \mathbf{B}_0(\mathbf{r}) + \frac{\mu_0}{4\pi} \sum_{ij} (\sigma_i - \sigma_j) \int_{S_{ij}} V(\mathbf{r}') \frac{\mathbf{R}}{R^3} \times d\mathbf{S}'_{ij}$$



# Integral Equation for $V$

For  $\mathbf{B}(\mathbf{r})$ , we need again an expression for  $V$ . Use Green's second identity

$$\int_G (\phi \nabla^2 \psi - \psi \nabla^2 \phi) d\mathbf{r} = \int_{S=\partial G} (\phi \nabla \psi - \psi \nabla \phi) d\mathbf{S}$$

which relates a volume-integral of an integrand involving two differentiable scalar-functions  $\psi$  and  $\phi$  and the  $\nabla$ -operator to an integral over the boundary-surface  $\leftrightarrow$  if we insert  $\psi = 1/R$ ,  $R = |\mathbf{r} - \mathbf{r}'|$  and  $\phi = V$ , we can derive an integral-equation for  $V$  in order to determine  $V$ .

This adopted to our problem reads:

$$\begin{aligned} \sum_i \sigma_i \int_{G_i} \left[ \frac{1}{R} \nabla'^2 V - V \nabla'^2 \frac{1}{R} \right] d\mathbf{r}' &= \sum_{ij} \int_{S_{ij}} \left[ \sigma_i \left[ \frac{1}{R} \nabla' V_i - V_i \nabla' \frac{1}{R} \right] \right. \\ &\quad \left. - \sigma_j \left[ \frac{1}{R} \nabla' V_j - V_j \nabla' \frac{1}{R} \right] \right] \cdot d\mathbf{S}'_{ij} \end{aligned}$$

Use continuity of current at boundaries (remember:  $\mathbf{j}^V = \sigma \cdot \mathbf{E} = -\sigma \cdot \nabla V$ ):

$$\begin{aligned} V_i(d\mathbf{S}'_{ij}) &= V_j(d\mathbf{S}'_{ij}) \\ \sigma_i \nabla' V_i \cdot d\mathbf{S}'_{ij} &= \sigma_j \nabla' V_j \cdot d\mathbf{S}'_{ij} \end{aligned}$$

as well as the Laplace-equation

$$\nabla'^2(1/R) = -4\pi\delta^{(3)}(\mathbf{R})$$

which in general can be stated in terms of Green's function  $G(\mathbf{a}, \mathbf{b})$

$$\nabla^2 G(\mathbf{a}, \mathbf{b}) = \delta^{(3)}(\mathbf{a} - \mathbf{b})$$

with the solution

$$G(\mathbf{a}, \mathbf{b}) = \frac{-1}{4\pi|\mathbf{a} - \mathbf{b}|}$$

$$\hookrightarrow \sum_i \sigma_i \int_{G_i} V \nabla'^2 \frac{1}{R} d\mathbf{r}' = 4\pi\sigma_0 V(\mathbf{R} = \mathbf{0})$$

Therefore:

$$\left( \sum_i \sigma_i \int_{G_i} \frac{1}{R} \nabla'^2 V d\mathbf{r}' \right) + 4\pi\sigma_0 V(\mathbf{R} = \mathbf{0}) = - \sum_{ij} (\sigma_i - \sigma_j) \int_{S_{ij}} V \nabla'(1/R) \cdot d\mathbf{S}'_{ij}$$

In quasi-static approximation:

$$\nabla \cdot \mathbf{j} = 0 \Leftrightarrow \nabla \cdot \sigma \nabla V = \nabla \cdot \mathbf{j}^i$$

we get

$$4\pi\sigma_0 V(\mathbf{R} = 0) = - \sum_i \int_{G_i} \frac{1}{R} \nabla' \cdot \mathbf{j}^i d\mathbf{r}' - \sum_{ij} (\sigma_i - \sigma_j) \int_{S_{ij}} \left( V \nabla' \frac{1}{R} \right) \cdot d\mathbf{S}'_{ij}$$

Rewrite the first term:

$$\sum_i \int_{G_i} \nabla' \cdot \left( \frac{\mathbf{j}^i}{R} \right) d\mathbf{r}' = \sum_{ij} \int_{S_{ij}} \frac{1}{R} \mathbf{j}^i \cdot d\mathbf{S}'_{ij} = \int_G \left( \mathbf{j}^i \cdot \nabla \frac{1}{R} + \frac{1}{R} \nabla \cdot \mathbf{j}^i \right) d\mathbf{r}'$$

Since  $\mathbf{j}^i$  vanishes on  $S_{ij}$ , we get

$$\int_G \frac{1}{R} \nabla \cdot \mathbf{j}^i d\mathbf{r}' = - \int_G \mathbf{j}^i \cdot \nabla \frac{1}{R} d\mathbf{r}'$$

Therefore, we obtain

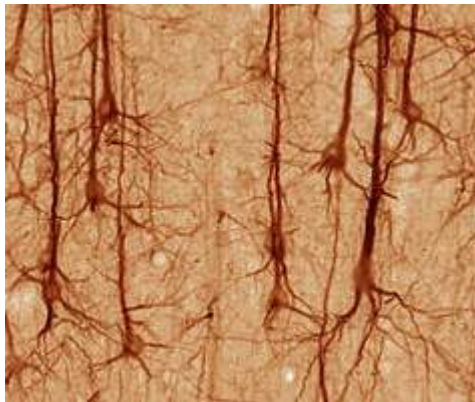
$$\begin{aligned}4\pi\sigma_0 V(\mathbf{R} = 0) &= \int_G \mathbf{j}^i \cdot \nabla \frac{1}{R} d\mathbf{r}' - \sum_{ij} (\sigma_i - \sigma_j) \int_{S_{ij}} \left( V \nabla' \frac{1}{R} \right) \cdot d\mathbf{S}'_{ij} \\ &= \underbrace{\int_G \mathbf{j}^i \cdot \frac{\mathbf{R}}{R^3} d\mathbf{r}'}_{4\pi\sigma_0 V_0(\mathbf{r})} - \sum_{ij} (\sigma_i - \sigma_j) \int_{S_{ij}} V \frac{\mathbf{R}}{R^3} \cdot d\mathbf{S}'_{ij}\end{aligned}$$

$$V(\mathbf{R} = |\mathbf{r} - \mathbf{r}'| = 0) = V_0(\mathbf{r}) - \frac{1}{4\pi\sigma_0} \sum_{ij} (\sigma_i - \sigma_j) \int_{S_{ij}} V \frac{\mathbf{R}}{R^3} \cdot d\mathbf{S}'_{ij}$$

↔ Integral equation for  $V(\mathbf{R} = 0)$  → determines  $V$  in the integration-area, which is exactly what we need in order to calculate  $\mathbf{B}$ ! ▶ B

# Spherically Symmetric Conductor

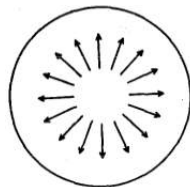
The axons of pyramidal neurons in the cerebral cortex tissue are approximately parallel to each other and perpendicular to the skull-surface



(picture taken from <https://en.wikipedia.org/wiki/Neuron>)

→ approximate head by an homogeneous spherically symmetric conductor → currents in the axons represented by

**radial-current pattern**



Approximation not realistic, but greatly simplifies solution for **B**:

# B for a Spherically Symmetric Conductor

$$\mathbf{B}(\mathbf{r}) = \mathbf{B}_0(\mathbf{r}) + \underbrace{\frac{\mu_0}{4\pi} \sum_{ij} (\sigma_i - \sigma_j) \int_{S_{ij}} V(\mathbf{r}') \frac{\mathbf{R}}{R^3} \times d\mathbf{S}'_{ij}}_{=0}$$

$\Rightarrow$  contribution of  $\mathbf{j}^V$  to the radial field component

$$B_r = \mathbf{B}(\mathbf{r}) \cdot \mathbf{e}_r = \mathbf{B}(\mathbf{r}) \cdot \mathbf{r}/|\mathbf{r}|$$

vanishes, since ( $d\mathbf{S} = \mathbf{n}(\mathbf{r}) \cdot dS$ , i.e. normal-vector of  $d\mathbf{S}$ )

$$(\mathbf{r} - \mathbf{r}') \times \mathbf{n}(\mathbf{r}') \cdot \mathbf{e}_r = (\mathbf{r} - \mathbf{r}') \times \frac{\mathbf{r}'}{|\mathbf{r}'|} \cdot \frac{\mathbf{r}}{|\mathbf{r}|} = 0$$

$\rightarrow$  calculate  $B_r$  with  $\mathbf{j}^i(\mathbf{r}') = \mathbf{Q}\delta^{(3)}(\mathbf{r}' - \mathbf{r}_Q)$  ( $\mathbf{r}_Q$  = location of the current-dipole  $\mathbf{Q}$ )

$$B_r = \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}^i(\mathbf{r}') \times \mathbf{R}}{R^3} d\mathbf{r}' = \frac{\mu_0}{4\pi} \frac{\mathbf{Q} \times \mathbf{r}_Q \cdot \mathbf{e}_r}{|\mathbf{r} - \mathbf{r}_Q|^3}$$

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- **Biomagnetism:** Magnetic fields (and also electric fields) arise from currents, impressed by biological activity
- **Primary- & return-current:** Represent real current pattern. Total current:  $\mathbf{j} = \mathbf{j}^i + \mathbf{j}^V$
- **Current-dipole:** Represented by a battery.  $\mathbf{j}^i$  connects  $\pm$ -terminals,  $\mathbf{j}^V$  symbolizes radial symmetric inflow pattern (alternatively: two charges, which evoke the same field pattern as a magnetic dipole)
- **Far-field approximation:**  $\mathbf{B}(\mathbf{r})$  can be approximated by the dipole moment (multipole expansion)
- **Quasi-static limit:** Basis for calculation of fields
  - all currents in tissue change synchronously
  - for each moment in time, neglect time-dependent parts in Maxwell-equations
  - $\mathbf{B}$  described by Biot-Savart

## General conductor

Applying quasi-static limit & concept of  $\mathbf{j}^i$  and  $\mathbf{j}^V$  to Biot-Savart  $\rightarrow$  get an equation for  $\mathbf{B}(\mathbf{r})$  and a differential equation for  $V$ :

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int (\mathbf{j}^i(\mathbf{r}') + V \nabla' \sigma) \times \frac{\mathbf{R}}{R^3} d\mathbf{r}'$$

$$\nabla \cdot (\sigma \nabla V) = \nabla \cdot \mathbf{j}^i$$

## Piece-Wise Homogeneous conductor

Split  $G$  into  $G_i$  with  $\sigma_i = \text{const.}$  Use vector identities and Stoke's Theorem:

$$\mathbf{B}(\mathbf{r}) = \mathbf{B}_0(\mathbf{r}) + \frac{\mu_0}{4\pi} \sum_{ij} (\sigma_i - \sigma_j) \int_{S_{ij}} V(\mathbf{r}') \frac{\mathbf{R}}{R^3} \times d\mathbf{S}'_{ij}$$

For  $V$ , we use a trick by applying Green's second identity and exploit the continuity of the currents at boundaries:





$$V(\mathbf{R} = |\mathbf{r} - \mathbf{r}'| = 0) = \sigma_0 V_0(\mathbf{r}) - \frac{1}{4\pi\sigma_0} \sum_{ij} (\sigma_i - \sigma_j) \int_{S_{ij}} V \frac{\mathbf{R}}{R^3} \cdot d\mathbf{S}'_{ij}$$

## Spherically symmetric conductor

$\mathbf{j}^V$  does not contribute to  $\mathbf{B}(\mathbf{r}) \rightarrow$  Choosing the limit of a point-like current ( $\mathbf{j}^i = \mathbf{Q}\delta^{(3)}(\mathbf{r} - \mathbf{r}_Q)$ ,  $\mathbf{r}_Q =$  position of the current-dipole),  $\mathbf{B}(\mathbf{r})$  simplifies to

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{j}^i(\mathbf{r}') \times \mathbf{R}}{R^3} d\mathbf{r}' = \frac{\mu_0}{4\pi} \frac{\mathbf{Q} \times \mathbf{r}_Q \cdot \mathbf{e}_r}{|\mathbf{r} - \mathbf{r}_Q|^3}$$

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