ELECTROMAGNETICS

GENERAL THEORY
OF THE
ELECTROMAGNETIC FIELD

CLASSICAL AND RELATIVISTIC
APPROACHES

THIRD EDITION

Revised and Augmented

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Andrei NICOLAIDE

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PREFACE TO THE PRESENT EDITION

The present book titled, *Electromagnetics: General theory of the electromagnetic field. Classical and relativistic approaches*, is an extended form of the previous two editions of the books titled *Electromagnetics: General theory of the electromagnetic field*.

The new book, at the difference of the previous ones, contains four new appendices, devoted to several topics, as follows: **a**. A study on the divergence of tensors related to the curvature of the space-time continuum; **b**. The energy-momentum tensor of the electromagnetic field in the theory of relativity; **c**. The Sagnac effect in The General theory of relativity; **d**. A new approach to the calculation of the magnetic field strength of a solenoid and to the introduction of magnetic quantities.

In this book, apart from some improvements, new results have been included, some of them belonging to the author. These last ones have been presented at the ICAEM *International Conference of Applied and Engineering Mathematics*, held in London the last four years (2008-2011). The book will appear in both forms electronic and print.

The volume has been built in order to avoid the reader to resort to books of mathematics, all mathematical developments being included in the book.

The purpose of this book has been to present in a legible manner some important subjects of the concerned topics. At the same time, the text has been so prepared that a reader not interested in the Special Theory of Relativity and General Theory of Relativity could read it, avoiding the text connected with the relativistic treatment.

The book is devoted to all readers interested in these topics.

PREFACE TO THE FIRST EDITION

In the present work the physical fundamentals of electromagnetic phenomena are studied having in view their technical applications.

The book contains the general theory of the electromagnetic field necessary for the study of the principal applications in the following domains: Electrostatics, Electrokinetics, Electrodynamics and Magnetostatics.

The general theory contains the introduction (i.e., the definition) of fundamental concepts among which: field and substance, electric charge, electric current, state quantities of electric and magnetic fields, as well as the study of laws and energy of the electromagnetic field.

The general theory is presented in four chapters. Further, three appendices are added.

For practical applications, the consideration of electromagnetic phenomena at a macroscopic scale is of special interest. However, in many applications, it is necessary to know the phenomena at a microscopic scale.

At the same time, it is useful to have in view that the physical model is, in many cases, relatively simple in the case of a microscopic study. For this reason, in this work, the

following procedure has been used: The various quantities and phenomena have been first examined at a microscopic scale, and then, by calculating the average values, the passage to macroscopic quantities describing the phenomena has been accomplished.

Concerning the presentation of the general laws, there are, in principle, two possibilities:

a. The introduction of these laws directly, as a generalization of experimental facts;

b. The derivation of these laws starting from the Coulomb law and the Special Theory of Relativity.

The Special Theory of Relativity has been used because it permits the derivation of the equations of the theory of electromagnetic field starting from a small number of general equations.

Appendix 3, which contains the main formulae of the Special Theory of Relativity, and the derivation of certain relations between forces, given by the author, facilitates to follow the calculations of Chapters 2 and 3.

Also, some relatively recent considerations on the theory of relativity have been mentioned in Introduction.

The text has been elaborated so that all references to the special theory of relativity may be omitted; however, in this case, the number of basic general equations that are not derived from more general relations is greater.

The study of the mentioned domains, namely Electrostatics, Electrokinetics, Electrodynamics, Magnetostatics, can be carried out by using the general laws of electromagnetic field for these various cases. Certain important problems concerning the mentioned domains are analysed in the present work.

A more detailed study of the mentioned domains can be found in several works devoted to these subjects, including the works of the author, mentioned in Bibliography.

The system of units used in this work is the International System of Units (SI) and all formulae are written in this rationalized system.

This work differs to some extent from many other usual textbooks and works by the attention paid to certain subjects like the passage from the microscopic theory to the macroscopic one, the way of using the Special Theory of Relativity, and the simplicity of the presentation.

Certain parts of this work, especially those related to the Theory of Relativity, represent the content of the lectures of an extra-course given by the author at the Université Bordeaux 1 (France) in the summer semester of 2001.

Andrei NICOLAIDE

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LIST OF SYMBOLS

$\boldsymbol{a}$	–	vector (p. 205).
A	–	linear current density, also called linear current sheet (p. 72).
$\boldsymbol{A}$	–	vector potential (magnetic) in the reference frame K (p. 129, 153).
$\boldsymbol{A}'$	–	vector potential in the reference frame K' in motion relatively to the reference frame K (p. 153).
$\boldsymbol{A}_o$, $\boldsymbol{A}_1$	–	vector potentials in the reference frames K_o and K_1 (p. 268).
$\boldsymbol{B}$	–	magnetic induction, also called magnetic flux density (p. 41, 108-110); magnetic induction in any reference frame K (p. 151, 152).
$\boldsymbol{B}'$	–	magnetic induction in any reference frame K' in motion relatively to the reference frame K (p. 152, 153).
$\boldsymbol{B}_i = \boldsymbol{M}_j$	–	intrinsic magnetic induction (p. 78).
B_{n1}, B_{n2}	–	normal components at two points, very near, situated on both sides of the separation surface of two media, in the same reference frame (p. 160).
$\boldsymbol{B}_{oP}$	–	magnetic induction at point P in the reference frame K_o (p. 114).
$\boldsymbol{B}_1$, $\boldsymbol{B}_2$	–	vector quantities at two points, very near, situated on both sides of the separation surface of two media, in the same reference frame (p. 160).
c	–	velocity of light in empty space, i.e., in vacuo (p. 102).
C_q	–	curve with an electric charge distribution (p. 48).
$\mathrm{d}\boldsymbol{l}$	–	line element (p. 32).
$\mathrm{d}\boldsymbol{S}$	–	surface element (p. 35, 214).
$\mathrm{d}v$	–	volume element (p. 48).
$\mathrm{d}v_o$, $\mathrm{d}v_1$	–	volume element in the reference frame K_o and K_1, respectively, (p.103).
$\boldsymbol{D}$	–	electric displacement (p. 41), also called electric flux density (p. 92) and electric induction (p. 123), in any reference frame K (p. 152).
$\boldsymbol{D}'$	–	electric displacement in any reference frame K' in motion relatively to the reference frame K (p. 152).
D_{n1}, D_{n2}	–	normal components at two points, very near, situated on both sides of the separation surface of two media, in the same reference frame (p. 156, 157).
$\boldsymbol{D}_o$	–	electric displacement in the reference frame K_o (p. 108).
$\boldsymbol{D}_1$, $\boldsymbol{D}_2$	–	vector quantities at two points, very near, situated on both sides of the separation surface of two media, in the same reference frame (p. 156).
e	–	electric charge of electron in absolute value (p. 38).
e	–	electromotive tension or electromotive force (p. 55).
$\boldsymbol{E}$	–	electric field strength, also called electric field intensity (p. 53); electric field strength, macroscopic value (p. 50); electric field strength in any reference frame K (p. 151, 153).
$\boldsymbol{E}'$	–	electric field strength in any reference frame K' in motion relatively to the reference frame K (p. 151, 153).
$\boldsymbol{E}_c$	–	Coulombian component of the electric field strength (p. 53).

$\boldsymbol{E}_i$ — impressed component of the electric field strength (p. 53).

$\boldsymbol{E}_i$ — electric field strength produced at any point by a point-like electric charge with the ordinal number i (p. 86).

$\boldsymbol{E}_l$ — electric field strength in the large sense (p. 52).

$\boldsymbol{E}_{\text{macro}}$ — macroscopic value of the electric field strength (p. 50).

$\boldsymbol{E}_{\text{micro}}(\boldsymbol{r}, t)$ — microscopic value of the electric field strength at a point at any moment (p. 50).

$\boldsymbol{E}_{\text{micro}}$ — microscopic value of the electric field strength (p. 50).

$\boldsymbol{E}_n$ — non-Coulombian electric field strength (p. 53).

$\boldsymbol{E}_{\text{o}}$ — electric field strength at a point at rest in the reference frame K_{o} (p. 97, 107).

$E_{\text{o}x}$, $E_{\text{o}y}$, $E_{\text{o}z}$ — component of the electric field strength at a point, along the $O_{\text{o}}x_{\text{o}}$, $O_{\text{o}}y_{\text{o}}$, $O_{\text{o}}z_{\text{o}}$ axes, in the reference frame K_{o} (p. 107).

$\boldsymbol{E}_r$ — induced electric field strength component (rotational, solenoidal or curl component) (p. 53).

E_{t1}, E_{t2} — tangential components at two points, very near, situated on both sides of the separation surface of two media, in the same reference frame (p. 159).

$\boldsymbol{E}_0$ — electric field intensity produced by external causes (p. 65).

$\boldsymbol{E}_1$ — electric field strength at a point at rest in the reference frame K_1 (p. 98).

$\boldsymbol{E}_1$, $\boldsymbol{E}_2$ — electric field strengths at two points, very near, situated on both sides of the separation surface, in the same reference frame (p. 158).

E_{1x}, E_{1y}, E_{1z} — component of the electric field strength at a point, along the O_1x_1, O_1y_1, O_1z_1 axes, in the reference frame K_1 (p. 106).

$\boldsymbol{E}_{21}$ — electric field strength at any point 2 produced by a point-like charge with index 1 (p. 86).

$\boldsymbol{F}$ — force in general, and force acting upon a point-like electric charge (p. 49).

$\boldsymbol{F}_{\text{el}}$ — force of electric nature acting on a point-like charge (p. 52).

$\boldsymbol{F}_E$ — force acting upon an electrically polarized small body (p. 68).

$\boldsymbol{F}_i$ — force acting upon any point-like electric charge q, and due to a point-like electric charge q_i (p. 86).

$\boldsymbol{F}_{\text{mag}}$ — force of magnetic nature acting upon a moving point-like electric charge (p. 110).

$\boldsymbol{F}_{\text{non-el}}$ — force of non-electric nature acting on a point-like charge (p. 52).

$\boldsymbol{F}_{\text{o}}$ — force exerted upon a point-like charge q, at rest in the reference frame K_{o} (p. 97).

$F_{\text{o}x}$, $F_{\text{o}y}$, $F_{\text{o}z}$ — components of the force along the $O_{\text{o}}x_{\text{o}}$, $O_{\text{o}}y_{\text{o}}$, $O_{\text{o}}z_{\text{o}}$ axes, of the force exerted upon a point-like charge q, at rest in the reference frame K_{o} (p. 106).

$\boldsymbol{F}_1$ — force exerted upon a point-like charge q, at rest in the reference frame K_1 (p. 98).

F_{1x}, F_{1y}, F_{1z}	–	components of the force along the O_1x_1, O_1y_1, O_1z_1 axes, exerted upon a point-like charge q, at rest in the reference frame K_1 (p. 106).
$\boldsymbol{F}_{21}$	–	force acting upon point-like charge 2 due to point-like charge 1 (p. 86).
$\boldsymbol{G}(\boldsymbol{r})$	–	vector function of a vector field (p. 32, 235).
$\boldsymbol{G}$	–	vector (p. 32); electromagnetic momentum of the electromagnetic field (p. 200).
$\boldsymbol{G}_p$	–	potential component of a vector function (p. 52).
$\boldsymbol{G}_r$	–	solenoidal component of a vector function (p. 52).
$\boldsymbol{h}$	–	height (oriented) (p. 62); oriented straight-line segment (p. 65).
$\boldsymbol{h}_d = \boldsymbol{h}$	–	distance oriented from the negative charge towards the positive charge of a dipole (p. 58).
$\boldsymbol{H}$	–	magnetic field strength, also called magnetic field intensity (p. 41, 108, 141).
$\boldsymbol{H}_{oP}$	–	magnetic field strength at point P in the reference frame K_o (p. 114).
$\boldsymbol{H}_p$	–	potential (non-curl or irrotational) component of the magnetic field strength (p. 144).
$\boldsymbol{H}_r$	–	curl (rotational or solenoidal) component of the magnetic field strength (p. 144).
H_{t1}, H_{t2}	–	tangential components at two points, very near, situated on both sides of the separation surface of two media, in the same reference frame (p. 163).
$\boldsymbol{H}_1$, $\boldsymbol{H}_2$	–	magnetic field strengths at two points, very near, situated on both sides of the separation surface, in the same reference frame (p. 162).
$\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$	–	unit vectors along the axes of a three-orthogonal system of co-ordinates.
i	–	intensity of electric current, strength of electric current (p. 68).
i_{Amper}	–	intensity of an Amperian (molecular) current (p. 77).
i_M	–	macroscopic intensity of the Amperian electric current (p. 77).
i_{mag}	–	intensity of magnetic current (p. 134).
i_P	–	intensity of polarization electric current (p. 74).
i_Σ	–	intensity of an electric current leaving a closed surface (p. 80).
$\boldsymbol{J}$	–	vector of the macroscopic conduction electric current density (p. 69, 70); density of conduction electric current in any reference frame K (p. 152, 153).
$\boldsymbol{J}'$	–	density of conduction electric current in any reference frame K' in motion relatively to the reference frame K (p. 152, 153).
$\boldsymbol{J}_A$	–	macroscopic density of the Amperian electric current (p. 79).
$\boldsymbol{J}_{conv}$	–	convection electric current density (p. 142).
$\boldsymbol{J}_e$	–	density of the resultant electric current produced by the electric free and bound charges in motion (p. 139).
$\boldsymbol{J}_l$	–	density of the electric current in the large sense (p. 142).
$\boldsymbol{J}_{macro}$	–	vector of the macroscopic electric current density (p. 70).
$\boldsymbol{J}_{micro}$	–	vector of the microscopic electric current density (p. 70).
J_{n1}, J_{n2}	–	normal components at two points, very near, situated on both sides of the separation surface of two media, in the same reference frame (p. 164).

$\boldsymbol{J}_P$	–	vector of macroscopic density of the electric polarization current (p. 73).
J_s	–	linear current density of a current sheet, also called linear current sheet (p. 72, 163).
$\boldsymbol{J}_1$, $\boldsymbol{J}_2$	–	vector quantities at two points, very near, situated on both sides of the separation surface of two media, in the same reference frame (p. 164).
K_o	–	inertial reference frame considered as original system (p. 97). The original reference frame and the quantities expressed in this frame are indicated by the suffix "o", in order to avoid any confusion with the suffix "0" of ε_0 and μ_0.
K_1	–	inertial reference frame in motion relatively to K_o (p. 98).
L	–	Lagrange function of an electric point-like charge moving in the electromagnetic field (p. 185).
m	–	mass (inertial) of a material point in motion relatively to any reference frame K (p. 184, 249).
$\boldsymbol{m}$	–	magnetic moment of a body (p. 41).
m_e	–	mass of electron (p. 38).
$\boldsymbol{m}_j$	–	Coulombian magnetic moment (p. 78).
m_p	–	mass of positron (p. 38).
$\boldsymbol{m}_0$	–	magnetic moment of an Amperian current, also called Amperian magnetic moment (p. 78).
m_0	–	mass of a material point at rest (rest mass) in the considered reference frame K_o (p. 249).
m_1	–	mass of a material point at rest in the reference frame K_1 (p. 249).
M	–	magnetization (p. 78); magnetization in any reference frame K (p. 152).
M'	–	magnetization in any reference frame K' in motion relatively to the reference frame K (p. 152).
M_j	–	magnetic polarization (p. 78).
M_{jp}	–	permanent magnetic polarization (p. 79).
M_{jt}	–	temporary magnetic polarization (p. 79).
M_p	–	permanent magnetization (p. 79).
M_t	–	temporary magnetization (p. 79).
$\boldsymbol{n}$	–	normal unit vector to a surface (p. 35, 214).
n_c	–	number of charge carriers per unit of volume (p. 69).
n_k	–	number of charge carriers of a certain type, indicated by the suffix k, per unit of volume (p. 70).
n_p	–	volume concentration of dipoles (p. 62).
n_0	–	concentration of Amperian currents (per unit of volume) (p. 77).
$\boldsymbol{n}_{12}$	–	unit vector of the normal to the separation surface, oriented from the medium 1 towards the medium 2 (p. 64, 156, 157).
$\boldsymbol{p}$	–	electric moment of a neutral system (p. 41); electric moment of a polarized dielectric body (p. 58, 59).
$\boldsymbol{p}_d$	–	electric moment of an electric dipole (p. 58).

$\boldsymbol{p}_i$ – electric moment of a multipole (p. 61).

p_{mk} – momentum (quantity of motion) of a material point in motion with the velocity $\boldsymbol{u}$ relatively to the reference frame K, expressed in the same reference frame, corresponding to the co-ordinate x_k (p. 185).

$\boldsymbol{p}_s$ – electric moment of a neutral system of point-like electric charges (p. 57).

$\boldsymbol{P}$ – Any flux density vector (p. 35).

$\boldsymbol{P}$ – electric polarization (p. 61);
electric polarization in any reference frame K (p. 152).

$\boldsymbol{P}'$ – electric polarization in any reference frame K' in motion relatively to the reference frame K (p. 152).

P_J – electromagnetic energy converted into heat in a body carrying electric current (p. 196).

$\boldsymbol{P}_o$ – electric polarization vector in the reference frame K_o (p. 120, 121).

$\boldsymbol{P}_p$ – permanent electric polarization vector (p. 62).

$\boldsymbol{P}_t$ – temporary electric polarization vector (p. 62).

q – electric charge, also called quantity of electricity (p. 41, 47, 49);
charge of an electric charge carrier (p. 69).

q_d – bound electric charge of an electric dipole (positive charge) (p. 58).

$q_{i\,\text{bound}}$ – bound electric charge with the ordinal number i of any distribution of a neutral system (p. 56, 58).

q_l – electric charge contained by a body in the case of a line distribution n (p. 48).

$q_{p\Sigma}$ – polarization electric charge within the closed surface Σ (p. 63).

q_S – electric charge contained by a body in the case of a surface distribution (p. 48).

q_v – electric charge contained by a body in the case of a volume distribution (p. 48).

q_Σ – electric charge contained in the volume bounded by the closed surface Σ (p. 47, 80).

$\boldsymbol{r}$ – position vector (p. 32, 208, 214).

$\boldsymbol{r}_{12}$ – vector having the starting end at point 1 and the terminal end (with arrow) at point 2 (p. 85).

$\boldsymbol{r}_{1AP}$ – vector with the origin (starting point) at the point A and the terminal end at the point P, expressed in the reference frame K_1 (p. 105, 249).

s_c – cross-section of an electric charge distribution (p. 110, 115).

S – Any open surface (p. 35, 214); area of any surface (p. 69).

S_d – separation surface of two media (p. 156, 157).

S_{ox} – area of a surface perpendicular to the Ox - axis (p. 103).

S_{oy} – area of a surface perpendicular to the Oy - axis (p. 103).

S_{oz} – area of a surface perpendicular to the Oz - axis (p. 103).

S_q – surface with an electric charge distribution (p. 48).

S_0 – surface attributed to the orbit of an Amperian current (p. 77).

S_Γ	–	simply connected open surface bounded by the closed curve Γ (p. 145, 214).
t	–	time in any reference frame.
$\boldsymbol{t}$	–	density of the surface tension (Maxwell stress tensor) (p. 200).
t_o	–	time in the reference frame K_o (p. 101).
t_1	–	time in the reference frame K_1 (p. 101).
$\boldsymbol{T}_E$	–	torque acting upon an electric dipole (p. 66).
$\boldsymbol{u}$	–	velocity of a material point in any reference frame K (p. 104, 249).
u_{AB}	–	electric tension, voltage, electric potential difference between the two points A and B (p. 56).
$u_{C_{AB}}$	–	electric tension between two points along a line (in the restricted sense) (p. 55).
$\left(u_{C_{AB}}\right)_l$	–	electric tension between two points along a line in the large sense (p. 55).
u_e	–	electromotive tension or electromotive force (p. 55).
$u_{e\,C_{AB}}$	–	electromotive tension or electromotive force along the open curve
	–	(p. 55).
$u_{e\Gamma}$	–	electromotive force along the closed curve Γ (p. 135).
$u_{m C_{AB}}$	–	magnetic tension along the curve C from the point A to the point B (p. 144).
$u_{mm C_{AB}}$	–	magnetomotive force along the curve C from the point A to the point B (p. 144).
$u_{mm\,\Gamma}$	–	magnetomotive force along the closed curve Γ (p. 145).
$\boldsymbol{v}$	–	velocity of a moving electric point-like electric charge relatively to any reference frame (p. 109).
$\boldsymbol{v}_k$	–	velocity component of charge carriers of a certain type, indicated by the suffix k, with respect to a reference frame (p. 70).
$\boldsymbol{v}_p$	–	velocity of positive bound charges (p. 75).
$\boldsymbol{v}_r$	–	velocity component of charge carriers, in ordinate motion with respect to a reference frame (p. 69); velocity of any point relatively to one reference frame (p. 232).
$\boldsymbol{v}_{1o}$	–	velocity of the reference frame K_1 with respect to the reference frame K_o (p. 101, 104).
$\boldsymbol{v}_{1o} = v$	–	symbol of velocity (p. 183).
$\boldsymbol{v}_{2o}$	–	velocity of the reference frame K_2 or of a material point denoted by 2, relatively to the reference frame K_o (p. 104).
V	–	electric potential (p. 53, 87).
V_D	–	volume of a dielectric body (p. 60).
V_o , V_1	–	electric potential in the reference frames K_o and K_1 (p. 183, 269).
V_q	–	domain with a volume electric charge distribution (p. 48).
V_Σ	–	volume bounded by the closed surface Σ (p. 90, 218).
w	–	number of turns of a coil (p. 129).
w_{em}	–	volume density of the electromagnetic energy (p. 192).

w_q	–	volume density of the energy transformed into internal energy of a body submitted to an electric or magnetic polarization cycle (p. 195).
W_e	–	electric energy of the electromagnetic field (p. 192).
W_{em}	–	energy of the electromagnetic field in a domain (p. 192).
W_L	–	work along a curve done by the forces of the considered field (p. 54).
W_m	–	magnetic energy of the electromagnetic field (p. 192).
x, y, z	–	co-ordinates of a three-orthogonal rectilinear system of co-ordinates (p. 33, 210).
x_k	–	co-ordinate (p. 185).
$\dot{x}_k$	–	velocity (p. 185).
x_o, y_o, z_o	–	co-ordinates in the system of reference of K_o (p. 97).
x_1, y_1, z_1	–	co-ordinates in the system of reference of K_1 (p. 97).
$\alpha = \dfrac{1}{\sqrt{1-\beta^2}}$	–	coefficient occurring in the Special Theory of Relativity in the case of reference frames K and K' (p. 153).
α_0	–	temperature coefficient (p. 84).
α_{1o}	–	coefficient occurring in the Special Theory of Relativity (p. 102).
$\beta = \dfrac{v}{c}$	–	coefficient occurring in the Special Theory of Relativity in the case of reference frames K and K' (p. 153).
β_{1o}	–	coefficient occurring in the Theory of Special Relativity (p. 102).
Γ	–	any closed curve (p. 35).
Δ	–	small increase of the quantity written after this sign (p. 47, 210).
Δq_Σ	–	electric charge contained in the physically infinitesimal volume Δv (p. 47).
ΔS	–	area of a surface element (p. 47).
Δt_0	–	interval of time (p. 50).
Δx_0, Δy_0, Δz_0	–	dimensions of a right parallelepiped (p. 50).
ε	–	electric permittivity (p. 124).
ε_r	–	relative electric permittivity, i.e., dielectric constant (p. 124).
ε_0	–	electric constant, i.e., electric permittivity of vacuum (p. 85, 107).
θ	–	temperature of a conductor (p. 83, 84).
Θ	–	current linkage, also termed ampere-turns (p. 145).
μ	–	magnetic permeability (p. 148).
μ_r	–	relative magnetic permeability (p. 148).
μ_0	–	magnetic constant, i.e., magnetic permeability of vacuum (p. 107, 139).
$\boldsymbol{\Pi}$	–	Poynting vector (p. 191).
ρ	–	electric resistivity (p. 83).
ρ_l	–	line density of the electric charge, macroscopic value (p. 47).
ρ_{pl}	–	line density of the polarization electric charge (p. 60).

ρ_{ps}	–	surface density of the polarization electric charge (p. 60).
ρ_{pv}	–	volume density of the polarization electric charge (p. 59).
ρ_{pvo}	–	volume density of the polarization electric charge in the reference frame K_o (p. 120).
ρ_s	–	surface density of the electric charge, macroscopic value (p. 47).
ρ_v	–	volume density of the electric charge, macroscopic value (p. 47).
ρ'_v	–	volume density of the electric charge in any reference frame K in motion relatively to the reference frame K' (p. 152).
$\rho_{v\ micro}$	–	volume density of the electric charge, microscopic value (p. 48).
ρ_{mv} , ρ_{Mv}	–	volume density of the fictive (fictitious) magnetic polarization, and magnetization charge, respectively (p. 79).
ρ_{vo}	–	volume density of the electric charge in the reference frame K_o (p. 119).
ρ_θ	–	electric resistivity at temperature θ (p. 84).
σ	–	electric conductivity (p. 83).
$\boldsymbol{\sigma}$	–	tensor of electric conductivity (p. 84).
Σ	–	closed surface (p. 35).
Σ_d	–	closed surface resulting from the intersection of two closed surfaces (p. 82).
Σ_o	–	closed surface in the reference frame K_o (p. 117).
Σ_1	–	closed surface in the reference frame K_1 (p. 117).
$\varphi(\boldsymbol{r})$	–	function of a scalar field of the position vector (p. 32, 208).
Φ	–	flux through any surface (p. 35, 214).
Φ	–	magnetic flux-turn (p. 129).
χ_e	–	electric susceptibility (p. 123).
χ_m	–	magnetic susceptibility (p. 148).
Ψ	–	magnetic flux-linkage (p. 129).
Ψ_{el}	–	electric flux , i.e., flux of the vector displacement through any surface, it is the surface-integral of the vector electric displacement (p. 122).
Ψ_E	–	flux of the vector electric field strength through any surface (p. 90).
Ψ_{S_Γ}	–	magnetic flux through the open surface bounded by the closed curve Γ (p. 128).
ω	–	angular velocity (p. 174).
ω_1	–	angular frequency of a quantity varying sinusoidally with time (p. 174).
Ω	–	solid angle subtended at any point by a surface (p. 91); angular velocity (p. 178).

INTRODUCTION

1. CONTENT OF ELECTROMAGNETICS

In this work, the foundations of Electromagnetics, including theory and applications are treated. It is useful to note that Electromagnetics and Electromagnetism can be considered synonyms.

The theory of electromagnetism includes the introduction (i.e., the definition) of several fundamental concepts among which: Field and substance, electric charge, electric current, state quantities of electric and magnetic fields. Also, it contains the study of forces acting upon electric charge carriers in motion, laws and energy of electromagnetic field. The applications concern the corresponding topics.

2. THE THEORIES USED IN THE STUDY OF ELECTROMAGNETISM

We recall that the Electromagnetism is a branch of Physics in which the electromagnetic phenomena are studied. It contains the study of physical bases and of the propagation of electromagnetic field. This work refers to physical bases only.

The principal domains of electromagnetism are the following ones: Electrostatics, Electrokinetics, Magnetostatics and Electrodynamics. These domains are very useful for the study of macroscopic phenomena and in practical applications.

The study of the domains above can be carried out by using the general laws of electromagnetism in these various cases. Certain important problems of the mentioned domains are analysed in the present work. A more detailed study of the mentioned domains can be found in several works devoted to these subjects, including the works of the author, mentioned in Bibliography.

In the study of electromagnetism, the following theories are utilized: *Theory of electromagnetic field* (Theory of Maxwell), *Theory of electrons* (Theory of Lorentz), *Theory of relativity* and *Quantum Mechanics*.

The theory of Maxwell is *the macroscopic theory of electromagnetic phenomena*. In the framework of this theory, relationships between the quantities that characterize the electric and magnetic state of the substance are given in the form of a set of differential equations. The theory refers to media at rest. An extension of this theory to moving media was made by Heinrich HERTZ.

The Theory of electrons is *the microscopic theory of electromagnetic phenomena*, which admits the existence of certain elementary charged particles, called electrons. The *electron* is characterized by its electric charge, mass, and magnetic moment. In the framework of this theory, the ponderomotive forces of the electromagnetic field are exclusively determined from the forces exerted upon particles and expressed by the

Lorentz formula. The electromagnetic field equations are obtained by applying the Maxwell equations for empty space (i.e., vacuum) at microscopic scale.

The theory of electrons can be presented in either *quantum* or *non-quantum* form, respectively. The non-quantum form of the theory of electrons has also two forms, namely: *non-relativistic* and *relativistic* one. In the framework of the non-relativistic form, the existence of a privileged reference frame is assumed. This reference frame is at rest with respect to the group of fixed stars and is referred to as *Lorentz inertial reference frame*. The theory of electrons refers to media at rest as well as to moving media.

The non-quantum theory of electrons cannot be put in accordance with some properties of elementary particles and the utilisation of Quantum Mechanics then becomes necessary.

Finally we shall recall that the fundamental physical interactions or forces, in nature, are of the following four types: Electromagnetic, Weak, Heavy and Gravitational.

3.   SHORT HISTORICAL SURVEY

In this Section, certain data of the history of the development of the knowledge of electromagnetic phenomena will be presented. The first knowledge about electric and magnetic phenomena refers to natural magnetism and to electrification by friction. Magnet and magnetism are so termed because the loadstone (iron ore) μάγνης (magnes) was originally found in the Thessalian Magnesia.

Also, in Antiquity the electrification by friction of amber, called in Ancient Greek κεχριμπάρι (kehrimpari, read kechrimpari) or ήλεκτρον (elektron, read ilektron), was known. This manner of electrification was described by THALES of Millet (640 ~ 547 BC).

The development of electromagnetism was related to a great extent to the discovery of the law of the force exerted between two *point-like bodies*, charged with electricity (i.e., having electric charge). The establishment of this law had several stages due to the research of Benjamin FRANKLIN (1706 – 1790), Joseph PRIESTLEY (1733 – 1804), John ROBISON (1739 – 1805), Henry CAVENDISH (1731 – 1810) and Charles-Augustin de COULOMB (1736 – 1806). Coulomb performed experiments by two different methods.

In the first method, he used a torsion balance and measured the angle proportional to the force exerted between electrified bodies.

In the second method, he used an apparatus with an oscillating device and determined the number of oscillations that depends on the force exerted between the electrified bodies; the results were published in 1785.

With respect to the previous experiments, he established the results more directly and also mentioned that the force is directly proportional to the product of the quantities of electricity (electric charges) of the two electrified bodies.

He established with a high precision that the force exerted between two electrified point-like bodies is inversely proportional to the square of the distance between them.

Carl Friedrich GAUSS (1777 – 1855) established important formulae in Electrostatics and Magnetostatics.

Hans Christian OERSTED (1777 – 1851) experimentally remarked the action exerted by an electrical conductor carrying an electric current, on a magnetic needle. This experiment was determined by the remark that the magnetic needle of a compass makes

oscillations during a storm. The result was published in 1820. This result has been of a great importance, because it has allowed the establishment of the relation between two classes of phenomena, previously independently treated.

At the same time, in the year 1820, Jean-Baptiste BIOT (1774 – 1862), Félix SAVART (1791 – 1841) and Pierre-Simon de LAPLACE (1749 – 1827) established the relation expressing the interaction between an element of electric current and a magnetic pole.

Continuing this research, André-Marie AMPÈRE (1775 – 1836) established the same year 1820, that forces are exerted between two conductors carrying electric currents. He also introduced the difference between electric potentials (potential difference, voltage, electric tension) and electric current. He showed that a permanent magnet in the form of a bar is equivalent to a coil carrying an electric current. It is worth noting that at present the Ampère conception lies at the base of the theory of magnetism.

Georg Simon OHM (1789 – 1854) established in 1826 the relationship between electric tension (voltage) and the intensity of the electric current.

Two very important discoveries lie on the ground of the theory of electromagnetic field.

The first one is the fundamental discovery made by Michael FARADAY (1791 – 1867) and consists in the fact that a magnetic field varying with time *induces* (i.e., produces) an electric field, what he experimentally established. A historical survey of the research carried out by several scientists on this subject can be found in literature [13], [25].

The second one belongs to James Clerk MAXWELL (1831 – 1879). Maxwell established in a theoretical way that, conversely, an electric field varying with time induces (i.e., produces) a magnetic field. Therefore, the electric fields varying with time have the same effect as the conduction currents concerning the production of the magnetic fields. Hence, the variation with time of the electric field may be considered as corresponding to an electric current, called by Maxwell *displacement current*.

To the previous two components of the electric current (i.e., *conduction current* and *displacement current*) it is to be added a third component namely *the convection current*. It is produced by the motion of electrified bodies with respect to a reference system. This component was studied by several scientists, among which Henry ROWLAND (1848 – 1901) and N. VASILESCU KARPEN (1870 – 1964).

At the same time, Faraday introduced the concept of line of force, in order to visualize the magnetic field and subsequently the electric field.

Faraday thought the seat of electric phenomena as going on a medium, whereas previously, mathematicians thought the same phenomena as being produced by centres of forces acting at a distance. The conception of Faraday allowed him to replace the concept of action at distance by the concept of a local interaction between electrified bodies and a field of forces, what has had a great importance for the subsequent development of the theory of electromagnetic field.

After six years of experimental researches, Faraday discovered in the year 1831 the phenomenon of electromagnetic induction mentioned above. In the first experiment, he utilized a soft iron ring having the cross section diameter of about 2.22 cm and the exterior diameter of about 15.24 cm. On this ring, there were wound two coils of insulated copper wires. The ends (terminals) of the first coil could be connected with an electric battery (of cells). The ends of the second coil were connected each other by a copper wire placed in the neighbourhood of a magnetic needle. When connecting or breaking the connection of the first coil with the battery, he remarked oscillations of the

magnetic needle that then ceased. This experiment led Faraday to the conclusion that the second coil carried during this time interval "an electricity wave". Hence, the phenomenon of *electromagnetic induction by transformation* was discovered.

In another experiment, he utilised a coil of wound wire forming a helix cylinder. When displacing, inside the coil, a permanent magnet in the form of a bar of about 1.905 cm in diameter, and of about 21.590 cm in length, he remarked that the needle of a galvanometer, connected with the ends of the coil, moved in different directions depending on the direction in which the permanent magnet had been displaced. Hence, the phenomenon of *electromagnetic induction by the relative motion* of a conductor with respect to the field lines produced by the permanent magnet was discovered. Therefore, *the law of electromagnetic induction* is also called the *Faraday law*.

The mathematical expression of the electromagnetic induction law was subsequently established by Maxwell.

Also, in 1831, Faraday invented the first *direct current generator* composed of a copper plate that could rotate between magnetic poles, and the external electrical circuit was connected between the centre and the rim of the plate. In 1851, he described a machine consisting of a rotating wire rectangle with an attached *commutator*, this being the prototype from which derived the direct current machines with commutator.

The *self-induction phenomenon* was discovered by Joseph HENRY (1797 – 1878) in the year 1832.

The conversion into heat of the energy due to electric currents flowing through conducting wires is called electro-heating effect.

James Prescott JOULE (1818 – 1889) carried out experimental research on the heat generated by electric currents. He established the relation expressing that the heat produced by electro-heating effect, in a given time, is proportional to the square of the current, and his results were published in 1840.

Heinrich Friedrich Emil LENZ (1804 – 1865) made investigations on the variation of the resistance of a conducting wire carrying an electric current and showed that the resistance increases with temperature, these results were reported in 1833. Afterwards, he performed research on the electro-heating effect.

He also established the statement that an electric current produced by the electromagnetic induction phenomenon, in any circuit, flows in a direction such that the effect of that current opposes the cause that produced the current. This statement is known as the Lenz rule.

Emil WARBURG (1846 – 1931) and John Henry POYNTING (1852 – 1914) established useful relations referring to the transformation and propagation of electromagnetic energy.

The study of electromagnetic field for the case of moving bodies was developed in the researches of Heinrich HERTZ (1857 – 1894), Hendrik Antoon LORENTZ (1853 – 1928), Hermann MINKOWSKI (1864 – 1909), Albert EINSTEIN (1879 – 1955).

Lorentz developed the theory of electrons which allowed the explanation of many electromagnetic phenomena; he also established the relation for the transformation of co-ordinates and of time, when passing from a reference frame to another, from the condition that the form of Maxwell equations remain unchanged.

Einstein developed the Special Theory of Relativity published in 1905, and the General Theory of Relativity formulated in the year 1916, a presentation of which can be found in [27].

The *Special Theory of Relativity*, is also referred to by one of the following denominations: *Theory of Special Relativity*, *Restricted Theory of Relativity, Theory of Restricted Relativity*. In the framework of the *Special Theory of Relativity*, Einstein obtained the Lorentz transformation relations, without utilizing the Maxwell equations.

Utilizing the theory of relativity, it has been possible to express the equations of the electromagnetic field in a general form for the case of moving bodies. The Theory of Relativity implies to assume a constant velocity of light in empty-space with respect to any reference frame. This assumption leads to a local time at the points taken in various reference systems.

An interesting interpretation of the Lorentz theory was given by Henri POINCARÉ (1854 – 1912), [39]. In this interpretation, he stated that when considering a body in motion, any perturbation propagates more rapidly along the direction of motion than along the cross direction and the wave surfaces would be no more spheres but ellipsoids. These considerations have been analysed by Édouard GUILLAUME but their development has not been continued [39].

It is interesting to be noted that Einstein and Poincaré obtained the same formula for the composition of velocities but with quite different derivations. The derivation of Einstein starts from relations of Mechanics and the postulates of the Special Theory of Relativity, whereas the derivation of Poincaré starts from the transformation relations of Lorentz.

After the special theory of relativity became known, it has been possible to derive the Maxwell equations starting from the Coulomb formula and the transformation relation of forces when passing from an inertial reference frame to another one.

Several mathematical explanations of the special theory of relativity can be found in literature, among which a derivation starting from the four-dimensional structure assumed for the universe [37].

The theory of relativity has been based on the postulate mentioned above, according to which the velocity of light in empty-space is constant with respect to any reference frame. This postulate was based on the experiments first carried out by Michelson in 1881, and repeated with improved accuracy by Michelson and Morley in 1887. These experiments concern the propagation of a monochromatic light, emitted from a source on the Earth, taking into account the revolution motion of the Earth around the Sun. For this purpose, an apparatus containing an interferometer was used.

From the mentioned experiments, it follows that the velocity of light on the Earth is not affected by the orbital motion of the Earth around the Sun.

Later, the above postulate was checked by several direct experiments. An example is mentioned in [18, p. 4] and refers to the experiment performed in 1964 by Alväger, Farley, Kjellman and Wallin. They determined the velocity of photons arisen from the decay of π^o - mesons. It is recalled that from the decay of each of these mesons, two photons arise. The velocity of the mesons above was found, using the equations of the Special Theory of Relativity, to be very close to the velocity of light. The velocity of the photons obtained as mentioned above was found to be very close to that of the mentioned mesons, except a very small deviation. Therefore, the velocity of photons was not added

to the velocity of mesons and, hence, the velocity of light was not surpassed. Thus, the mentioned postulate was verified in this case.

Despite the success of the theory of relativity it cannot be considered to be a complete one. Indeed, there are electrodynamic phenomena that cannot be satisfactorily explained by the known theories, the theory of relativity included. Further, an example of such a phenomenon will be given namely the experiment of G. Sagnac [28]-[31]. For a long time this phenomenon has been mentioned in literature, e.g., by Lucien FABRE [38], although not enough analysed.

The experiment carried out by Georges SAGNAC (1869 – 1928) in 1913, [31], is a very curious one. The experiment consists in achieving the interference of two light beams travelling in inverse directions along the same way. The light source, the interferometer and the reflecting mirrors which ensure the desired paths (ways) for the beams (namely approximately a circular trajectory), photographic plate, hence the set of apparatus is placed on a disc, outside which nothing related with the experiment occurs.

The light beams travelling around the same way but in opposite directions are reflected from the interferometer to a photographic plate. The disc can rotate with any angular velocity ω.

We recall that the *ether (aether)*, mentioned below, is the denomination of a certain substance assumed by certain scientists to fill all space (between particles of air and other substances) through which electromagnetic waves and light may be transmitted. However, according to several researches, among which the experiment of Albert A. MICHELSON and Edward W. MORLEY, the concept of ether appears as being non-consistent.

Sagnac obtained that the time for a light beam to travel around a way parallel to the disc surface differed, according to whether the travelling direction was with or against the rotation sense of the disc. Hence, the light beams had different velocities with respect to a reference frame fixed to the disc. The result, referred to as *Sagnac effect*, seems to be not in concordance with the Theory of Relativity. Indeed, the phenomenon appears, as if ether would exist at rest, independently of the existing motion [38, p. 111, 248-251]. This result determined many thorough analyses, one of the most recent and interesting being carried out in papers [34]-[36].

In these papers, A.G. KELLY made a thorough analysis examining the arguments for and against the theory of relativity by considering the Sagnac effect. His analysis is based on the most important studies and reports concerning this effect. His main remarks are the following ones:

1° Many experiments performed with a high precision, including laser light, have confirmed with a good accuracy the results of the Sagnac experiment. It can be mentioned a very precise experiment carried out by investigators using laser-light in a piping system filled with a helium-neon gas [35, p. 7]. In fact, the Sagnac effect proves that light does not travel with the same velocity in both directions relative to the interferometer on a spinning disc [35, p. 10].

2° According to the internationally agreed method of synchronizing clocks on Earth, using electromagnetic signals, the following three effects are considered [35, p. 10, 16]:

   a.   Correction calculated according to the special theory of relativity.

   b.   Correction calculated for the difference of the gravitational potential, according to the general theory of relativity.

c.   Correction for the rotation of the Earth about its axis. The last correction corresponds to the Sagnac effect (although it is not denominated as such). The last correction is necessary because light does not travel around the globe Eastward and Westward with the same velocity (i.e., in equal times).

3º The measurements of high precision made by several investigators showed that the velocity of light on the Earth is not influenced by the rotation of the Earth around the Sun but it is influenced by the rotation of the Earth about its axis.

4º Some authors, among which A.G. Kelly, have considered that the Sagnac effect could not be explained by the Theory of Relativity. This opinion has been justified, because the modification of the light velocity in the Sagnac effect is much greater than any relativistic effect, by a factor of the order of magnitude 10^7 [34, p. 8], [35, p. 14]. However, as we have proved, and also described in Appendix 9 of this book, the relations obtained using the relationships of the General Theory of Relativity are in good agreement with the results shown by the Sagnac effect.

5º Tests were carried out in order to determine the effect corresponding to the General Theory of Relativity on the time indicated by airborne clocks relative to a standard clock system fixed on the Earth. The clocks had to be carried Eastward, and Westward, respectively by aeroplane in both cases approximately at the same latitude. Atomic clocks with caesium were used. The results have not been conclusive because the clocks had not sufficient stability required by the experiment [36, p. 5].

According to paper [35, p. 22], the light moves on the Earth together with the gravitational field of the Earth.

Kelly has shown the importance of modifying the Theory of Relativity in order to avoid the mentioned discrepancies with respect to experiments.

Other remarks concerning the difficulties in using the Theory of Relativity can be found in paper [20].

Despite the difficulties encountered in utilizing the Theory of Relativity for explaining certain phenomena it can be considered as a very convenient mathematical and physical procedure for calculating the electromagnetic field state quantities in the case of moving bodies charged with electricity. Also, the Theory of Relativity proved a higher accuracy than any other known theory.

We should add that the derivation of the Maxwell equations starting from the Coulomb law, was performed by Leigh PAGE (1882 – 1952) in 1912, and developed subsequently by certain authors in several works, among which the following references (treatises and textbooks) of the Bibliography [6], [11], [13], [18], [23], [25].

Classical Mechanics and classical Electrodynamics when applied for the explanation of phenomena produced at atomic scale lead to results which are in contradiction with the experimental results. So, for the study of phenomena at atomic scale, the Quantum Mechanics also called Undulatory Mechanics has to be used. The bases of these Mechanics were built up by Louis-Victor-Pierre-Raymond de BROGLIE (1892 – 1987) in 1924 and Erwin SCHRÖDINGER (1887 – 1961) in 1926. It is recalled that in Quantum Mechanics, the notion of trajectory of a particle does not exist. This circumstance is expressed by the non-determination principle (incertitude principle) that was formulated by Werner HEISENBERG (1901 – 1976) in 1927 and is one of the fundamental principles of Quantum Mechanics.

It can be added that certain models proposed by R.L. VALLÉE [20], based on the analysis of trapped electromagnetic waves and some principles, allowed the find of certain results as by means of Quantum Mechanics.

It is to be noted that Quantum Mechanics is very useful for both theoretical and practical purposes. Indeed, most of the phenomena that occur in semiconductors and magnetic materials may be explained only by the use of this mechanics. Also, the achievement of new materials (semiconductors and magnetic materials) is related to Quantum Mechanics.

4.   THE SYSTEM OF UNITS OF MEASURE

There are several systems of units that can be used in electromagnetism [42]-[47]. For practical purposes at the "Congrès International des Électriciens" held in Paris in 1881, the following units of measure have been adopted: ohm, volt, ampère, farad.

Giovanni GIORGI (1871 – 1950) proposed in 1902 a system of units from which the International System of Units, SI (abbreviation from the French denomination *Système International d'Unités*), utilised at present, was derived. In 1935, the International Electrotechnical Commission (IEC or in French CEI - Commission Électrotechnique Internationale) recommended that preparation be made for the transition to the system of units suggested by Giorgi. In this system the basic units were the following: unit of length – the metre, unit of mass – the kilogram, unit of time – the second, magnetic permeability of free-space – μ_0 . This proposal was not universally accepted and instead of the magnetic permeability, it was suggested that the fourth basic unit was to be the ampere. The corresponding system of units was denoted MKSA. The International System of Units denoted SI was adopted by the eleventh Conférence générale des Poids et Mesures (General Conference of Weights and Measures) in 1960. The base (basic, fundamental) units of this system are the following: metre, kilogram, second, ampere, kelvin (for thermodynamic temperature), mole (for amount of substance), candela (for luminous intensity).

In respect to electrical quantities, the SI system differs from the MKSA system by the denomination of the unit of magnetic induction that is tesla (T) in the SI system and Wb/m^2 in the MKSA system.

It is useful to add the concept of rationalization. The rationalization of the equations of the electromagnetic field means the presentation of the main equations, i.e., the Maxwell equations, in a form not containing the factor 4π . In this way, certain symmetry in the equations connecting electric and magnetic quantities appears. In fact, the rationalization consists in adopting for ε_0 and μ_0 appropriate values in order to ensure the symmetry mentioned above. The SI system ensures the rationalization of the equations of the electromagnetic field, hence it is a rationalized system of units.

1. GENERALITIES ON THE THEORY OF THE ELECTROMAGNETIC FIELD AND ON THE STRUCTURE OF SUBSTANCE

1.1. FIELD AND SUBSTANCE

Field and substance are fundamental forms strictly connected in which matter exists. There are many varieties of fields. For the electromagnetic field, the following definition can be used. The electromagnetic field is a different form of existence of the substance of bodies, and exists in the regions of space in which ponderomotive actions (forces or torques) of electromagnetic nature can act on the bodies. By ponderomotive actions of electromagnetic nature we understand forces and torques exerted on bodies and which have not a cause of mechanical or thermal nature.

The major part of the properties of the electromagnetic field is indirectly studied by the effect that it produces (for instance, mechanical and thermal effects), because most of the manifestation manners of the electromagnetic field are not directly accessible to the human senses. Only the electromagnetic waves of certain wave-length within 0.4 μm and 0.76 μm are directly perceptible as light waves.

It has been established that the electric and magnetic phenomena are transmitted in space at a finite velocity even *in vacuo*, from a body to another. It follows that, in space, a physical system, termed field, exists and allows the transmission of ponderomotive actions in space and time. This statement is in accordance with the principle of continuity [38, p. 230, 231], [1, Vol. I, Arts. 7, 59, 60]; namely two distinct bodies can act to one another only by an inter-medium. Thus, the action is exerted not at distance, but through a medium. Therefore, all the laws could be expressed in a differential form between infinitely close points. At the same time, according to the principle of causality, the laws can contain only quantities that can be observed directly or indirectly.

In the study of the properties of bodies and generally of their substance, hence media presented by various bodies, we can distinguish: **a** – homogeneous media and non-homogeneous media; **b** – isotropic and anisotropic media.

Homogeneous medium shows the same properties at all points. *Isotropic* medium shows the same properties along any direction.

To any field of physical nature characterized by scalar or vector quantities, there corresponds a field with mathematical meaning, namely a *field of scalars* or *vectors*, respectively. It means that, to any point of the physical field, there corresponds a scalar or a vector. The physical field under consideration will be called scalar or vector field, respectively.

The vectors may be of polar or axial types; thus there are polar vectors and axial vectors. To any rotation motion of a body about an axis, a vector having the axis direction is attributed for representing the angular rotation velocity of the body about the axis. In this case, the vector designates the axis about which the rotation is accomplished. Such a vector is called an *axial vector*. The case is similar for the vector representing the torque acting on a body. To the gravitational force acting on a body, a vector is attributed for

representing the force. Such a vector is called a *polar vector*. Starting, for instance, from the expression of a torque it follows that a vector product $c = a \times b$ yields an axial vector if each vector from the right-hand side is a polar vector. From the mathematical form of a polar vector, it follows that its components, in a three-orthogonal rectilinear (Cartesian) right-handed system of co-ordinates, change their sign if the positive direction of every co-ordinate axis is inverted. From the mathematical form of the vector product, it follows that its components, in a system of co-ordinates described above, do not change in sign if the factors are polar vectors. Consequently, in a vector equation, all the terms must be vectors of the same type.

The scalar function of a scalar field or the vector function of a vector field depends on the position vector:

$$r = i\,x + j\,y + k\,z,\tag{1.1}$$

and may be written $\varphi(r)$ and $G(r)$, respectively:

$$\varphi(r) = \varphi(x, y, z),\tag{1.2 a}$$

$$G = G(r) = i\,G_x + j\,G_y + k\,G_z.\tag{1.2 b}$$

In the study of vector fields, the use of Vector Calculus is very convenient and will be utilized further on.

It is worth noting that the terms of direction have to be used in many sentences. *Direction* is the course taken by a moving person or thing. The word direction is used especially for straight-line ways (paths), but it is not compulsory. A direction normally shows two senses but the word *sense* will be considered to have the same meaning as direction. Each of them can be used.

Certain concepts that often occur in a study of the electromagnetic field will be further considered.

1.2. LINES OF FIELD. TUBES OF LINES OF FIELD. EQUIPOTENTIAL SURFACES. FLUXES.

1.2.1. Lines of Field

A *line of field* also termed a *line of force*, is a curve that is tangent at any point to the vector of the field strength at that point (Fig. 1.1). Hence, starting from any point of a vector field, and adding up, from this point, along the direction of the field strength an infinitely small straight-line segment, we obtain a next point, after the starting point, of the line of field. Continuing in this manner, we shall obtain the curve representing the line of field that passes through the given starting point.

In accordance with the definition given for the lines of field, it follows that the element of a line of field is:

$$d\boldsymbol{l} = \boldsymbol{i}\,d\,x + \boldsymbol{j}\,d\,y + \boldsymbol{k}\,d\,z.\tag{1.3}$$
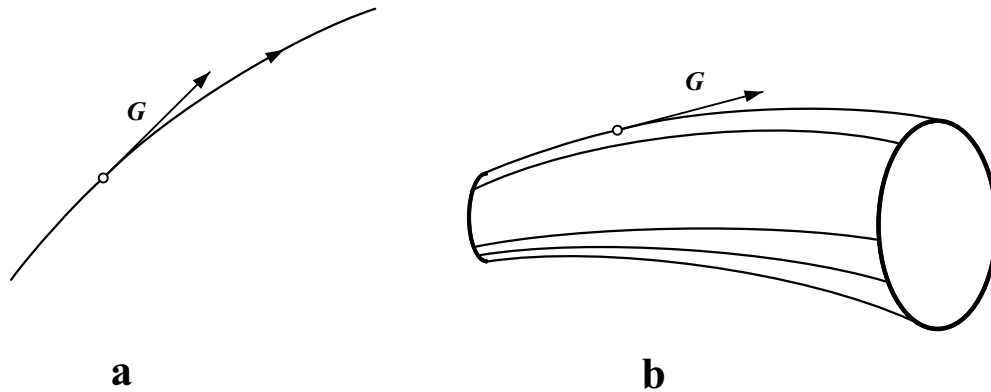
Fig. 1.1. Explanation concerning the lines of field:
a – line of field; **b** – tube of field lines.

The element of a line of field and the field strength vector at the same point are homo-parallel (i.e., parallel and of the same direction). It follows that in a three-orthogonal rectilinear system of co-ordinates we have:

$$\frac{\mathrm{d}\,x}{G_x} = \frac{\mathrm{d}\,y}{G_y} = \frac{\mathrm{d}\,z}{G_z}.$$ (1.4)

In the case of a scalar field, by line of field we understand the line of field of the vector field, the vector at any point being the gradient of the scalar field at the same point.

In the case of a vector field of the velocities of the particles of a fluid in motion, the vector at any point being the instantaneous velocity of the particle of the moving fluid, the field lines are called lines of current. In the case of a force field, they are called *lines of force* (denomination also used in the case of any vector field).

We may imagine each line of field represented by a thread (thin cord, thin rope) having at every point the direction (sense) of the field vector at that point. Moreover, the lines can also be supposed to be elastic. Then, they can suggest the forces that could appear in the considered field. As an example we may consider the lines of force between two bodies charged with electricity of opposite sign. The lines of force can suggest the attraction forces exerted between the two bodies.

Further on, we shall make some remarks concerning mathematical aspects.

The lines of field of a *potential field* with sources (i.e., the case of a field caharacterized by the curl equal to zero and the divergence different from zero, at any point) are open lines. These lines diverge from the positive sources of the field (i.e., the points where the divergence is positive) and converge towards the negative sources of the field (i.e., the points where the divergence is negative).

The lines of field of a *curl solenoidal field* (i.e., a field with the curl different from zero and the divergence equal to zero, at any point) can be: **a** – lines of a relatively simple form which are closed at finite distances or at infinity; **b** – lines of relatively complicated form, which may be closed lines as well as open lines, as certain conditions are fulfilled or not.

A case of type **b** given in literature [7] is that of the magnetic field lines corresponding to the following configuration of two electric currents: One current flows along a circumference, the other flows along a straight line perpendicular to the plane of the circumference at its centre.

The lines of field are also termed *flux lines*.

1.2.2. Tubes of Field Lines

A *tube of field lines* is called a surface formed by a set of field lines (of a vector field) which pass through all the points of a closed simply curve, and has the form of a tube.

The concept of tube of field lines has a geometrical meaning only in the case of the field lines of type **a**.

1.2.3. Equipotential Surface

Equipotential surface is called a surface, in a scalar field, formed by the set of points for which the field scalar has the same value.

For a given scalar field $\varphi(x, y, z)$, it is possible to deduce a gradient vector as $\boldsymbol{a}(x, y, z) = \pm \operatorname{grad} \varphi(x, y, z)$, where the plus or minus sign has to be taken according to the adopted convention. Therefore, the vector field $\boldsymbol{a}(x, y, z)$ derives from the potential $\varphi(x, y, z)$ of the scalar field. This is the case in which the concept of equipotential surface is important.

In the case above, *equipotential surface* is thus the surface formed by the set of points for which the potential has the same value:

$$\varphi(x, y, z) = \text{const.} \tag{1.5}$$

The equipotential surfaces serve to the study of a scalar field (e.g., the field of an electrostatic potential) from the qualitative point of view, because it permits to follow the directions along which the scalar function increases, decreases or remains constant. In addition, from a quantitative point of view, it permits to appreciate the rate of variation of the scalar field function.

The equipotential surfaces, as stated above, are surfaces on which the scalar potential (from which the vector field derives) shows the same value at any point. In this situation, the lines of the vector field are the orthogonal trajectories of the equipotential surfaces. Let us give some examples according to the nature of the vector field: Surfaces of equal electric potential (in electrostatic or in electrokinetic stationary fields); surfaces of equal magnetic potential (in magnetostatic fields); surfaces of equal potential of velocity (in fluids, in non-turbulent flow); surfaces of equal gravitational potential (in gravitational fields). The equipotential surfaces are also called *level surfaces*.

1.2.4. Flux

The concept of *flux* is generally used to characterize the transmission rate of a conservative quantity (for instance a liquid) through a surface. The flux through (or on) a surface is equal to the conservative quantity (e.g., quantity of fluid) that passes through the surface per unit of time.

Consequently, the flux is expressed by the surface-integral of a quantity also called *flux density*, and characterizes the transmission rate of the quantity through the surface under consideration. This quantity can be a conservative one. It may be a scalar or a vector one. Let us give examples of scalar and vector conservative quantities. Examples of scalar quantities: Mass, volume of incompressible fluids, energy. Examples of vector quantities: Momentum (also called quantity of motion), electric field strength, electric displacement (electric induction), magnetic induction. The flux of an incompressible volume of liquid is termed *flow rate*.

Let us calculate the flux Φ of any conservative scalar quantity W characterized by the flux density vectors $\boldsymbol{P}$, through a surface Σ:

$$\Phi = \int_{\Sigma} \boldsymbol{P} \cdot \mathrm{d}\boldsymbol{S} = \int_{\Sigma} \boldsymbol{P} \cdot \boldsymbol{n}\,\mathrm{d}S\,, \tag{1.6 a}$$

where the surface Σ may be an open or a closed one, according to the considered case, and $\boldsymbol{n}$ is the unit vector of the normal at any point of the surface.

Further on, the closed surfaces will be denoted by capital Greek letters, e.g., Σ, whereas the open surfaces by capital Latin letters and sometimes with an index that denotes the curve bounding the open surface, e.g., S_{Γ}, where Γ denotes the curve that bounds the surface *S*.

If no mention is made, by a *closed curve* Γ we shall understand a *simple closed curve*. If a point or a point-like body moving along a curve (also expressed around a curve, especially if the curve is a closed one) always in the same direction, after having started from any point of that curve, will arrive at the starting point passing only once through each point of the curve, this curve will be considered as a simple closed curve.

Also, if no mention is made, by an *open surface bounded by a closed curve* we shall understand a simply connected open surface (i.e., without holes).

Generally, in a field of vectors $\boldsymbol{G}(r)$, the flux is called the surface-integral of the component of the vector $\boldsymbol{G}$ along the direction of the normal to the surface Σ:

$$\Phi = \int_{\Sigma} G_n\,\mathrm{d}S = \int_{\Sigma} \boldsymbol{n} \cdot \boldsymbol{G}\,\mathrm{d}S = \int_{\Sigma} \boldsymbol{G} \cdot \mathrm{d}\boldsymbol{S} = \int_{\Sigma} \left(n_x\,G_x + n_y\,G_y + n_z\,G_z\right)\mathrm{d}S\,, \tag{1.6 b}$$

where the unit vector of the oriented normal also referred to as the unit vector of the *positive normal*, namely $\boldsymbol{n}$ at any point of the surface is adopted as explained below. For a closed surface, the positive normal is oriented outwards the surface. For an open surface bounded by a simple closed curve, the positive direction of the normal is associated, according to the right-handed screw rule, with the positive direction of travelling along the curve, the latter being adopted arbitrarily. Although the adoption of the travelling direction along the curve is arbitrary, however imprecision cannot appear, because in the concerned equations, both directions simultaneously occur (direction of travelling along the curve and direction of the normal to the surface bounded by the curve).

In Fig. 1.2, there are represented the directions that occur in the calculation of the flux for three cases: Closed surface, open surface, manifold open surface (i.e., formed by several sheets). Fig. 1.2 c concerns the calculation of the flux of a vector through an open

a   **b**   **c**
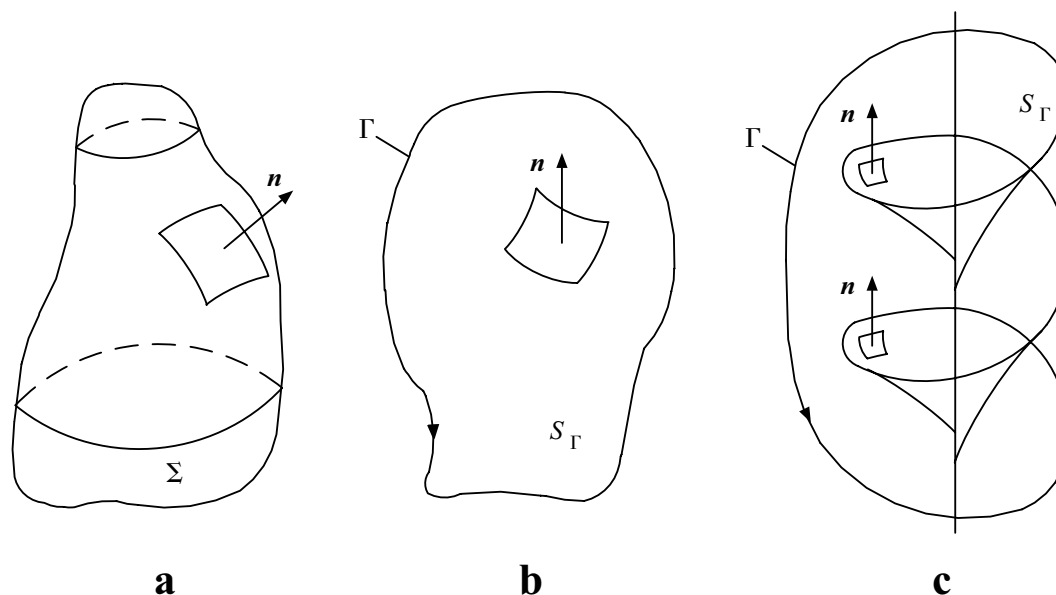
Fig. 1.2. The reference directions to the calculation of fluxes for:
a – closed surface; **b** – open surface; **c** – manifold open surface.

surface bounded by a curve forming several near loops (the case of a helix). The surface is a helical one.

We shall recall the generation of a helical surface. Consider a straight-line segment having one end at any point on one axis with which the segment forms a constant angle. Let the segment rotate about the axis, and simultaneously the point representing the end above to move along the axis with segments proportional to the arc of rotation of the segment. The curve described by each point of the segment will be a helix. The surface described by the segment will be a helical surface.

In the case of the helical surface, it follows that this flux is, in fact, equal to the sum of fluxes through every loop.

The flux corresponding to all loops is referred to as linked flux or *flux-linkage*. The flux through a single sheet is referred to as *flux-turn*.

Each tube of field lines containing a flux equal to the unit may be associated with a central line of field. This line may be referred to as unit field line or unit flux line. Then, the flux through any surface will be equal to the number representing the algebraic sum (i,e., taking into account the sense of the lines) of the unit field lines that pass through the surface.

1.3. PHYSICAL QUANTITIES. LAWS AND THEOREMS.

The characterization of physical states and phenomena is achieved by means of physical quantities. A detailed analysis referring to physical quantities has been made in

several works [1], [8], [11], [21], [22]. Further on, some principal aspects will be explained.

A kind of a physical quantity (in French, espèce de grandeurs physiques) is a class of physical properties susceptible of quantitative determination. For defining the kind of a physical quantity, it is necessary to know the measuring procedure and the unit of measure. The choice of the unit of measure is arbitrary. As examples, the following three kinds of quantities utilized in electromagnetism will be given: Electric charge, electric field strength, magnetic field strength.

A kind of physical quantity characterizes a common property of the elements (objects) of a set of physical objects. To identify a common property of a set of physical objects, it is necessary that a relation of order should exist between these objects.

A measure procedure is a repeatable experimental operation, by which, to each physical quantity it is possible to associate a mathematical quantity called *value* or *magnitude* in respect to a physical quantity termed unit.

According to the manner of introducing, the kinds of physical quantities can be divided into the following ones: *Kinds of primitive quantities* and *kinds of secondary (derived) quantities* (in French, espèces de grandeurs primitives et espèces de grandeurs secondaires ou dérivées)*,* also termed *primitive* and *secondary (derived) quantities*, respectively (in French, grandeurs primitives et, respectivement, grandeurs secondaires ou dérivées). The kinds of secondary (derived) quantities can be defined by means of other ones supposed as being known, hence introduced previously. The kinds of primitive quantities have to be introduced directly, by experimental way, and described by the measurement procedure, because they can no more be defined by means of quantities introduced previously.

In electromagnetism, apart from the primitive quantities of mechanics (length, time, mass, force), a series of new primitive quantities is necessary for characterizing from an electromagnetic point of view the state of bodies and of the electromagnetic field.

A system of units contains *fundamental* and *derived* units (in French, unités fondamentales ou de base et unités dérivées). The fundamental units have to be determined directly, experimentally (e.g., the metre). The derived units are derived by using the fundamental units (e.g., the square metre). The fundamental units must not be those of the primitive quantities, but those of the quantities that frequently appear in practice.

The laws express relations that are essentially necessary and repeatable between phenomena. In physics, laws are called the relations that express the most general knowledge on the phenomena of a research domain. They reflect the objective properties (of phenomena) that cannot be deduced by logical analysis (in the framework of the respective research field) from more general relations. The laws are established by the generalization of a great number of experimental results. In the theory of the electromagnetic field, there are *general laws* and *material laws* also called *constitutive laws*. The material laws differ from the general ones by the fact that they contain in their expression quantities specific to various materials, called *material quantities*.

The relations that can be deduced by logical analysis from other more general ones, and finally from laws, are called *theorems*.

It is useful to mention that there are relations that, at the time at which they were established, had law character but subsequently, after the progress of science, more general relations were discovered and the first ones could be derived or have represented

particular cases. Therefore, many relations that have a theorem character are called, by historical reason, laws. Further, for more clarity, when necessary, each denomination will be mentioned.

The examination of various quantities may be made in two ways, as the structure of bodies is taken into consideration. Consequently, two manners of studying quantities and phenomena naturally appear: The microscopic manner of the study and the macroscopic one. Correspondingly, there are a microscopic theory of the electromagnetic field and a macroscopic theory of the electromagnetic field.

1.4. MANNERS OF STUDYING THE THEORY OF THE ELECTROMAGNETIC FIELD

1.4.1. The Macroscopic Study of the Electromagnetic Field

The character of the *macroscopic* study (from the Ancient Greek; μακρός (makros) means long and σκοπέω (skopeo) means look at or examine) results from the fact that, in the framework of the study, the atomic structure of bodies is not taken into consideration; it is assumed that the substance is continuously distributed throughout the whole space. In the case of macroscopic study, all relations are obtained by an analysis of the mode in which the phenomena manifest themselves at the scale of the human senses.

1.4.2. The Microscopic Study of the Electromagnetic Field

The character of the *microscopic* study (from the Ancient Greek; μικρός (mikros) means small and σκοπέω (skopeo) means examine) results from the fact that, in the framework of the study, the atomic discontinuous structure is taken into account.

1.4.3. Generalities Concerning the Microscopic Study of the Electromagnetic Field

The microscopic study of the electromagnetic fields takes into account the atomic discontinuous structure of bodies.

It is recalled that all bodies are constituted of atoms, and an *atom* of each body is composed of relatively light bodies called *electrons* that have negative charge, and of a relatively heavy *nucleus*. The nucleus is essentially constituted of *protons*, particles that have a positive electric charge, and neutrons, particles that have no electric charge.

The electron is the smallest material particle with an indivisible negative charge. The electric charge of an electron is $-e = -1.602 \times 10^{-19}\,\mathrm{C}$, and its mass is $m_e = 0.9108 \times 10^{-30}\,\mathrm{kg}$.

The proton is a particle with a positive electric charge equal in absolute value to that of electron, and its mass is $m_p = 1.672 \times 10^{-27}\,\mathrm{kg}$, hence approximately 1836 times greater than the mass of electron. The *neutron* is a particle with zero electric charge, but with the mass approximately equal to that of the proton.

The substance is constituted of molecules that are formed of atoms. According to the Rutherford-Bohr-Sommerfeld atomic model, the substance presents itself in the form of planetary systems, each atom consisting of a nucleus with positive electric charge and of one or more electrons that turn about nucleus around closed orbits.

Concerning the form of a nucleus and of an electron, the simplest proposal is to consider them as being of spherical form. However, this supposition is not satisfactory and it is necessary to use the Quantum Mechanics. Also, there are possibilities of using models that permit to obtain results close to the various ones obtained experimentally [20, p. 64].

The dimensions of nucleus and electrons, in the framework of the simple physical model above, are so small, that in many phenomena, the atomic nuclei and the electrons can be considered of negligible dimensions. Hence, they can be considered as *material points* or *point-like bodies* with electric charge and mass. The study of the nucleus structure does not enter into the frame of the present work, and belongs to the domain of Nuclear Physics.

It is useful to be mentioned that, generally, one of the aims of Physics is the determination of the number, repartition and character of the particles charged with electricity which characterize the nature of bodies. Let us give the following example: The derivation of the laws of chemical and physical phenomena, by the aid of interaction laws of particles with electric charge.

The single important exception is represented by the phenomena in which an important function is represented by forces of mechanical nature (gravitation forces, elastic forces, capillary forces, friction forces, etc.) and nuclear forces, because only these forces cannot be reduced to the action of electric charges.

The characterization of the local state of the electromagnetic field in the macroscopic theory may be done by the help of local state quantities of the field: The electric field strength, the electric displacement (electric induction), the magnetic field strength, and the magnetic induction that will be further studied.

1.4.4. Macroscopic Average (Mean) Values

The macroscopic properties have to be described by means of macroscopic quantities. *Macroscopic quantities* are the quantities obtained by determining the average (mean) values of *microscopic quantities* for space domains and time intervals that are *physically infinitesimal quantities* (*physically infinitesimal* is in French, infiniment petit au sens physique [3, p. 408]). These mean values are called *macroscopic average (mean) values*.

By a *physical infinitesimal domain of space* (also called *of volume*) we understand a *domain* small enough, from a macroscopic point of view for, within it, the macroscopic quantities show a negligible variation with distance, and at the same time, great enough, from a microscopic point of view. The last condition means that the domain must contain a very great number of particles, i.e., molecules, atoms and elementary particles.

By a *physical infiniminal interval of time* we understand a *time interval* small enough, from a macroscopic point of view for, within it, the macroscopic time-dependent quantities show a negligible variation with time, and at the same time great enough, from a microscopic point of view. The last condition means that the time interval must have the duration much greater than the duration of processes occurring at microscopic scale, i.e., molecular or atomic scale.

The value of a physically infinitesimal volume depends on the nature of the substance and can be considered of about $(10^{-5}\,\text{cm})^3$. It can also be mentioned that the processes occurring at microscopic scale (e.g., the variation of the electric field strength) are depending on the period of the orbital motion of electrons [18, p. 50]. This period is of about 10^{-16} s. In fact, infinitesimal means infinitely small.

1.4.5. Manner of Studying Adopted in the Present Work

In the engineering practice, the study of phenomena at macroscopic scale is of a particular interest, however, in many cases, it is necessary to know the phenomena at the microscopic scale, as is the case of devices with semiconductor elements, elements of integrated circuits, devices with discharge in air, etc. Taking into account the physical model, often relatively simple in the microscopic study, the following procedure will be used: Firstly, the various quantities and phenomena at microscopic scale will be studied, then, by calculating the average (mean) values, the phenomena at macroscopic scale will be considered.

As starting point, the expression of the Coulomb force acting between two material points with electric charge will be taken into account.

Considering material points with electric charge, in motion with respect to various *systems of reference* (i.e., *reference frames*), and taking into account the transformation relations of the components of a force from the Special Theory of Relativity, the general relations of the Theory of the electromagnetic field can be obtained.

It is often stated that the observation of relativistic effects requires great velocities of moving bodies and measurements of high precision. However, in Electromagnetism, relativistic effects are encountered even in the case of small velocities compared to the velocity of light. The component of the force referred to as being of magnetic nature and which acts upon a moving material point with electric charge, is a relativistic component. It represents a supplementary confirmation of the importance presented by the theory of relativity.

The study of phenomena encountered in electromagnetism at microscopic scale requires the Quantum Mechanics. It is to be noted that by certain improvements of the models of various particles, the same results can be obtained for certain cases without utilizing the Quantum Mechanics [20].

1.4.6. Laws of the Theory of Electric and Magnetic Phenomena

From the explanation concerning the laws, it results that they can be grouped (classified) as follows. The macroscopic theory of electric and magnetic phenomena is considered to have twelve important laws, nine of them being general laws and three of them material laws.

The general laws are the following:
1. The law of electromagnetic induction.
2. The law of magnetic circuit (the magnetic circuital law).
3. The law of electric flux (the Gauss law).
4. The law of magnetic flux (the law of flux conservation).

5. The law of the relationship between electric displacement (electric induction), electric field strength and electric polarization.
6. The law of the relationship between magnetic induction, magnetic field strength, and magnetization.
7. The law of conservation of free (true) electric charge.
8. The law of the energy transformation in a body carrying conduction electric current.
9. Law of electrolysis.

The most important material laws are the following:
1. The law of temporary electric polarization.
2. The law of temporary magnetization.
3. The law of electric conduction.

To the general laws, the law of ponderomotive action upon a charged particle at rest can be added, and is referred to as *law of ponderomotive action*. However, it is included in the definition of the electric field strength.

The law of electrolysis will not be examined in this work. An explanation of this law can be found in works containing sections devoted to Electrochemistry [23], [48], [49].

The material laws exist only in the macroscopic theory. The material laws can be deduced from microscopic general laws under certain assumptions.

Four types of fundamental quantities macroscopically characterize the electromagnetic state of bodies: Electric charge q, electric moment p, density of the conduction electric current J, magnetic moment m.

The state of the electromagnetic field is macroscopically characterized by the following types of quantities: Electric field strength E, electric displacement (electric induction) D, magnetic field strength H, magnetic induction (magnetic flux density) B. These kinds of state quantities are introduced by the help of two kinds of fundamental quantities: Electric field strength *in vacuo* E and magnetic induction *in vacuo* B. The electromagnetic state of bodies and of the electromagnetic field is microscopically characterized by three kinds of fundamental quantities: Electric charge q, electric field strength E, magnetic induction B.

The microscopic theory of electric and magnetic phenomena has five general laws:
1. The law of electromagnetic induction.
2. The law of magnetic circuit or magnetic circuital law.
3. The electric flux law (Gauss law).
4. The magnetic flux law (the magnetic flux conservation law).
5. The law of ponderomotive action upon a moving electrically charged particle.

1.5. GENERAL CONSIDERATIONS ON THE STRUCTURE OF CONDUCTORS AND DIELECTRICS

1.5.1. Electrically Conductive Materials

The electrically conductive (conducting) materials are substances that, in normal conditions of use, have good electricity conducting properties (therefore they permit the rapid moving of particles charged with electricity within their interior).

The objects, pieces built of electrically conductive materials are called *electric conductors* or simply *conductors*.

There are two kinds of conductors:

a. Conductors of the first kind, in which, no chemical reaction occurs when they are carrying an electric current. This is the case of metals, metal alloys, carbon, semiconductors.

b. *Conductors of the second kind*, in which, chemical reactions occur when they are carrying an electric current. This is the case of electrolytes.

In metals, a part of the electrons that belong to the structure of an atom leaves the atom (hence they are delocalized), for this reason, these electrons are referred to as *free electrons*. These free electrons can move to large distances inside the conductor, ensuring the displacement of electric charges.

The conductors of the second kind contain in their structure ions. Every ion has an exceeding electric charge with respect to the neutral state. The ions can move to relatively large distances inside the conductor, ensuring the displacement of electric charges.

The earth also shows conducting properties. Therefore, in conductors there are particles that can displace themselves, carrying electric charges. They can be called *free charged particles* or *free electric charges*.

The most utilized materials in construction of installation and electrical equipment are copper and aluminium because of their great conductivity. Also, alloys of copper with other elements are used. The conductors are produced in form of wire or strip. The cross-section of a wire may be circular or rectangular.

The conductors are insulated for achieving various circuits and for avoiding defections or accidents. When an electric current flows through a conductor, this conductor heats and it is necessary to take into account that the utilization temperature is limited. For a non-insulated conductor, the limitation is determined by the mechanical strength of the equipment at high temperature. That is why, temperature of 200 – 300 ºC may not be surpassed.

For insulated conductors, the limitation is determined by the utilisation of the temperature of the insulation.

For achieving contacts, the following materials are utilized: Copper, silver, tungsten (wolfram), in pure form or alloys with other elements or in the form of sintered powders.

1.5.2. Dielectrics

Dielectrics are substances that in normal conditions in which they are utilized, are bad electricity conductors (i.e., they permit with difficulty the moving of electric charges),

because in general they do not contain free microscopic particles charged with electricity, which could displace themselves to relatively large distances.

Insulating materials are adopted from dielectrics. Any piece called *insulator* is achieved of an insulating material.

Dielectrics are solid, liquid or gaseous substances formed of systems of electric charges, namely neutral small domains (hence, the sum of electric charges of the interior of each domain is zero).

Dielectrics may be divided, from the point of view of the distribution of electric charges, into two fundamental groups: Polar and non-polar.

Dielectrics with *polar* and *non-polar* molecules have both positive and negative electric charges, equal in absolute value. The centre of mass of the particles with positive charge and with negative charge, respectively, may coincide (first case) or not (second case) in space, depending on the type of molecule, i.e., of the substance. In the first case the molecule has an electric moment only in the presence of an external electric field. In this case, it is a *non-polar dielectric*. In the second case, the molecule has an electric moment even in the absence of an external electric field. In this case, it is a *polar dielectric*.

Examples of non-polar dielectrics: Mono-atomic inert gases (He, Ne, Ar, Kr, Xe), diatomic gases (H_2, N_2, Cl_2), hydrocarbons and hydrocarbon composition (petroleum oils, polyethylene, polystyrene), carbon dioxide (CO_2), methane, benzene $\left(C_6H_6\right)$, NaCl (ionic crystals).

Examples of polar dielectrics: Hydrocarbon when some atoms or groups of atoms substitute hydrogen atoms like nitrobenzene ($C_6H_5 - NO_2$), methyl chloride $\left(CH_3Cl\right)$, water (a typical polar substance) having the electric moment of a molecule $p = 6.1 \times 10^{-30}$ C$\cdot$m. In the above cases, $p \approx q_d\,h$; $q_d \approx 10^{-19}$ C; $h \approx 10^{-10}$ m.

In the case of dielectrics, the particles charged with electricity cannot leave the system of particles, namely atoms, molecules, ions, to which they belong. Always, the sum of electric charge is zero for the system to which they belong, that is why these particles are called bound particles, and the corresponding electric charges – *bound electric charges*. Dielectrics can be charged by free electric charges, brought from outside.

Dielectrics may be divided from the point of view of the distribution of electric charges and types of molecules into the three following groups:

a. Dielectrics constituted by particles grouped in neutral molecules. This is the case for all dielectrics in gaseous or liquid form and for a part of those in solid form.

b. Dielectrics that besides the particles grouped in neutral molecules also contain ions, the last being fixed in certain equilibrium positions, for instance at the nodes of the crystalline lattice of the body. The ionic crystalline lattices are composed of elementary domains, and each domain is charged with positive and negative charges equal in absolute value, so that the domains globally appear as neutral. This is the case especially for crystalline dielectrics like quartz, mica, anhydrous oxide of aluminium, rutile (TiO_2).

c. Dielectrics that besides neutral molecules also contain molecules charged with positive and negative charges equal in absolute value, called dipolar molecules. This is the case of organic materials (cellulose, heat-convertible materials), glass, glass materials, certain crystalline dielectrics (ice).

Electrically insulating materials are used for the insulation of the parts of the installations that are at an electric potential difference (voltage), carry an electric current or are used in various electrical devices. The insulation of conductors is necessary, as shown, in the achievement of installations, and for avoiding damages and accidents.

The electrically insulating materials can be classified according to the allowable operation temperatures. These values have been established, taking into account the operating conditions, for obtaining satisfactory service duration.

The electrically insulating materials are in gaseous liquid or solid form:

1.   Gaseous form: Gaseous dielectrics.
2.   Liquid form: Petroleum oils, synthetic liquids like silicon liquids.
3.   Solid form: Polymers composing thermoplastic materials, cellulose, silk, thermosetting materials (thermohardening materials), varnishes, compounds, plastic compounds of binder (organic polymer capable of deformation) and powder like filler, fibrous or sheets of cotton, mica, paper, fibrous materials, glasses, ceramic materials.

1.6. ELECTRIC CHARGE. ELECTRIC FIELD STRENGTH *IN VACUO*

1.6.1. Electrification State. Electric Field.

First, the *state of electrification* of bodies will be considered. We shall consider a set composed of a piece of polished glass and a piece of resin, with no mechanical actions between them. Let the piece of glass and the piece of resin to be rubbed together, and then separated from each other. The two pieces will attract each other. We shall consider a second set also composed of a piece of polished glass and a piece of resin. The pieces of the second set will be subjected to the same operation as the pieces of the first set. We can observe that between the pieces above, ponderomotive actions are exerted as follows:

1.   Each piece of glass attracts each piece of resin.
2.   Each piece of resin attracts each piece of glass.
3.   Each piece of glass repels each other piece of glass.
4.   Each piece of resin repels each other piece of resin.
5.   Each piece of glass as well as each piece of resin does not exert a force on a set composed of a piece of glass and a piece of resin that have not been separated.

Therefore, ponderomotive actions that did not exist previously appear. In this situation, the system formed by certain or by all bodies above is considered to be *electrified*, and each of these bodies is considered to be an electrified body.

This type of electrification is called *electrification by friction*. Electrified state of bodies is called any state in which these bodies exert ponderomotive actions of electrical nature upon other bodies, i.e., actions of the same nature with the one exerted by the bodies electrified by friction.

If a body has the same behaviour as the piece of glass above, the body is considered as *vitreously electrified*. If a body has the same behaviour as the pieces of resin above, the body is considered to be *resinously electrified*. Conventionally, the vitreously electrification was called *positive electrification* and the resinously electrification was called *negative electrification*.

The positive electrification of a piece of glass can also be obtained by friction with a piece of cloth or texture. It is worth noting that the non-polished glass will not be electrified. Details are given in the work [12, tome III, p. 362].

The electrification by friction appears on the contact surfaces of two bodies, at least one of them being a dielectric (insulating material), as a result of friction when the two bodies are in relative motion. In the case of non-metallic bodies, electric charges, i.e., electrons, are delivered on a superficial film (pellicle) as a result of friction abrasion.

From a microscopic point of view, the electrification state of a body means that it is in the situation of having an excess or lack of electrons.

It was experimentally established that there is a great number of possibilities for electrifying a system of bodies: By contact (temporary or permanent) with electrified bodies, by deformation, by thermal action, by irradiation.

The electrification can be communicated from an electrified body to a non-electrified one, by contact or by influence. It is useful to mention that the electrification by influence means in fact a modification of the repartition of the electric charge, i.e., of the electrons in a conducting body under the influence of the charge of another body.

According to the transmission duration of the electrification state, the materials can be divided into three categories:

  a. *Electric conductors* or simpler *conductors* that transmit the electrification state in a very short time, of the order of 10^{-12} s, thus nearly instantaneously. The bodies made of such material are also called conductors.
  b. *Electric insulators* or simpler *insulator*s that transmit the electrification state in a very long time of the order of hours or days.
  c. *Semiconductors* that transmit the electrification state in an intermediary time, of the order of second fractions.

The ponderomotive actions that are exerted on the bodies situated in the neighbourhood of electrified bodies make evident the existence of the electrified system, in the space surrounding the electrified field.

In accordance with the general definition of Section 1.1, *electric field* is called the physical system that exists in the space regions in which ponderomotive actions (forces and torques) of electric nature can act on bodies.

The interactions between electrified bodies are produced by means of electric field produced by the electrified bodies. This fact was established by Faraday and Maxwell. The preceding statement represents, in fact, a general rule accepted in Physics and called principle of continuity: Two bodies can interact only by means of an intermediary medium (i.e., throughout the intermediary medium between the bodies). Hence all laws have to be expressed in a differential form concerning points infinitely close to each other.

1.6.2. True Electric Charge (Free Electric Charge)

The mechanical and thermal quantities do not suffice for the study of the electrification state and of the electric field. Therefore, it is necessary to introduce experimentally new physical quantities that will be primitive (fundamental) quantities in the framework of the theory of the electromagnetic field.

From Sub-section 1.6.1, it follows that there are two kinds of electrification state: positive and negative. The electrification state of an electrified body, namely the electrified piece of resin of Sub-section 1.6.1, will be considered as reference state. This electrification state is, as shown, negative.

To introduce the physical quantity called *true* (*free*) electric charge; the following measuring procedure can be used. All bodies under consideration in this procedure are considered to be very small, i.e., their dimensions are negligible compared with the distances between them.

Several small bodies in identical electrification state will be considered. All these bodies will be termed reference bodies.

As shown in Sub-section 1.6.1, no ponderomotive force acts on the system composed of a piece of glass and a piece of resin if the system is under the action of an electrified body, hence, if it is in an electric field. This system is called *neutral system*. The operation of bringing together two or several electrified bodies, so that the resulting system will not be subjected to a ponderomotive force, in an electric field, is called *neutralization*. Therefore, the *true electric charge* of any body is proportional to the number of reference bodies necessary for the neutralization of the given body.

The unit (of measure) of the true (free) electric charge may be adopted by convention that of one reference body, [8, Vol. I], [23, Vol. I, p. 46].

The proportionality constant is the negative unity. In this case, the true (free) electric charge of the given body is equal to the number of reference bodies and with opposite sign with respect to the last ones.

Hence, the electric charge has been introduced in a non-correlative way, i.e., regardless of the electric field strength. At the same time, it follows that the electric charge introduced in this way is independent of the reference frame (reference system) or of the velocity. The previous terms true and free can be considered synonyms, however certain authors are using the term true in the macroscopic approach and free in the microscopic approach.

It is to be noted that the electric charge of a moving body does not vary like the mass of the body with the velocity. This fact has been established experimentally observing that, in the case in which the number of protons of one atom nucleus is equal to the number of electrons turning about the nucleus, the neutral state of atoms is maintained, although the velocity of electrons and nuclei, respectively, are different. For this reason, hydrogen atoms or molecules are not deflected by an electric field. This fact is referred to as *constant charge principle* [18, p. 17].

It follows that the true electric charge is a property of bodies, characterized by a scalar quantity proportional to the number of electrons that are not neutralized by the positive charge of the nuclei of the constitution of bodies. The electric charge of a body is negative if it is an excess of electrons and positive if it is a lack of electrons. From the microscopic point of view, the electric charge is distributed among the microscopic particles of the respective body. Each microscopic particle of a certain type has always the same electric charge. From a microscopic point of view, the electric charge is distributed discontinuously in space.

From a macroscopic point of view, it is assumed that the true (free) electric charge, like the substance, is distributed continuously within the whole domain occupied by the considered body. Therefore, the macroscopic representation is an idealized one but

permits, in many applications, to simplify the calculations and ensures a very good accuracy.

The true electric charge and electric field strength *in vacuo* are primitive (fundamental) quantities in the framework of the microscopic theory, as well as in the framework of the macroscopic theory of the electromagnetic field. The particles with electric charge, like electrons and ions, which can move carrying electric charge, are called electric *charge carriers*.

In many cases, it is necessary to consider small bodies with electric charge. Since the electric charge, as mentioned above, is a property of bodies, the more exact wording is one of the following: A small body charged with the electric charge q; a *point-like charge q*; a *point charge q*.

The electric charge of a body is also denoted by q and can be termed *electric charge* or *quantity of electricity*.

The unit (of measure) of electric charge in the SI system of units is the coulomb (symbol C); its definition will be examined in Section 1.15.

1.6.3. Density of Electric Charge

The local state of electrification is characterized by the volume distribution of the electric charge. The volume density of the electric charge at any point of the body is defined as the ratio of the following quantities: The electric charge, denoted q_Σ, contained within the domain bounded by a small closed surface Σ including that point, and the volume Δv bounded by the above surface, which is a physically infinitesimal volume. As a macroscopic quantity, the electric charge may be considered continuously distributed in the space occupied by any body. In this case, the *macroscopic volume density of the electric charge* may be introduced in the form:

$$\rho_v = \lim_{\Delta v \to 0} \frac{\Delta q_\Sigma}{\Delta v} = \frac{\mathrm{d}\, q_\Sigma}{\mathrm{d}\, v}. \tag{1.7}$$

The electric charges may sometimes be distributed in a very thin layer over certain surfaces. Then, the idealized macroscopic case will be considered. In this case, the electric charge is distributed on these surfaces, considered as discontinuity surfaces. Therefore, the *macroscopic surface density of the electric charge* may be introduced in the form:

$$\rho_s = \lim_{\Delta S \to 0} \frac{\Delta q_S}{\Delta S} = \frac{\mathrm{d}\, q_S}{\mathrm{d}\, S}. \tag{1.8}$$

The electric charges may sometimes be distributed very non-uniformly and may be concentrated about certain lines. Also, the idealized macroscopic case will be considered. In this case, the electric charge is distributed along these lines, considered as discontinuity lines. The *macroscopic line density of the electric charge* may be introduced in the form:

$$\rho_l = \lim_{\Delta l \to 0} \frac{\Delta q_l}{\Delta l} = \frac{\mathrm{d}\, q_l}{\mathrm{d}\, l}. \tag{1.9}$$

The domains Δv, ΔS, Δl above utilized, in the denominators of relations (1.7), (1.8), (1.9), were firstly considered physically infinitesimal space quantities, but when assuming a continuous distribution of the electric charge they are assumed as tending to zero.

In many cases, it is necessary to calculate the average value with respect to space and time that may be denoted by the suffixes v and t respectively.

The quantity obtained by calculating the average value of the microscopic values may be designated by writing that microscopic quantity within the following angle brackets $\langle \ \rangle_{\text{suffix}}$ and the subscript of the second sign (here written suffix) indicates the quantity with respect to which the average value has to be calculated. If the average value has to be calculated with respect to two or more variables, then, the microscopic quantity will be written between two or more pairs of signs, as above, with the corresponding subscripts. If only a pair of signs is used and no suffix is written, it is considered that the average concerns both, space and time. More details and examples will be given in Sub-section 1.6.6.

The macroscopic electric charge densities above defined are the *average values* (*mean values*) and can be expressed in terms of the microscopic electric charge density:

$$\rho_v = \left\langle \left\langle \rho_{v \ \text{micro free}} \right\rangle_v \right\rangle_t ; \quad \rho_s = \left\langle \left\langle \rho_{s \ \text{micro free}} \right\rangle_s \right\rangle_t ; \quad \rho_l = \left\langle \left\langle \rho_{1 \ \text{micro free}} \right\rangle_1 \right\rangle_t . \quad (1.10 \text{ a, b, c})$$

The electric charges are also referred to as quantities of electricity, in the cases of volume, surface or line distribution, respectively, and are given by the following relations:

$$q_v = \int_{V_q} \rho_v \, \mathrm{d}v; \quad q_S = \int_{S_q} \rho_s \, \mathrm{d}S; \quad q_l = \int_{C_q} \rho_l \, \mathrm{d}l. \quad (1.11 \text{ a, b, c})$$

1.6.4. Conservation of the Free (True) Electric Charges

The total electric charge of an isolated system of bodies, i.e., situated *in vacuo* and surrounded by insulators, is constant. Certain explanation will be added.

At a given moment, it is possible that a certain electric charge appears in one region (zone) of the domain, where before no electric charge existed. However, at the same time, a certain electric charge disappears in another region of the domain. In such cases, the total electric charge is conserved, namely when in one region of the domain a charge $+ q$ appears, a charge $- q$ simultaneously appears in another region, because the number of charge carriers of the system of bodies remains the same.

1.6.5. The Electric Field Strength *in Vacuo*

Any reference frame will be considered. Let $\boldsymbol{F}$ be the force exerted *in vacuo* (i.e., empty space) on a small body with a small free electric charge q at rest in the reference frame in which the force is measured. The charge has been assumed to be small enough, in order of not disturbing the distribution of the other electric charges.

It can be experimentally established that the ratio between the two quantities does not depend on the electrification state of the above body, but depends only on the local electric state of the electric field. Therefore, the following relation can be written:

$$E = \frac{F}{q},$$
(1.12)

where the quantity E is independent of the true electric charge q. The quantity E is called the *electric field strength in vacuo*. Hence, the electric field strength at a point is defined in a certain reference frame, as the ratio of the force acting upon a point-like electric charge at rest at the considered point (with respect to the adopted reference frame), to the value of the electric charge.

In order to explore an electric field and thus find its strength at any point, a small body, satisfying the conditions below can be used. The conditions to be fulfilled by the body are the following:

a. Small enough dimensions, for permitting the measurement of the forces in very small regions of the field.

b. Small enough electric charge in order of not disturbing the electric state of the domain.

c. Build of conducting material, electrically very well insulated for permitting an easy transfer of electric charge.

The small body fulfilling the conditions above is referred to as *proof body* or *test body* (in French, corps d'épreuve). The test body will permit to determine the value of the macroscopic field strength of the electric field at a point in empty space or in a gaseous medium, according to the relation:

$$E = \lim_{q \to 0} \frac{F}{q},$$
(1.13)

where F is the force acting upon the test body and q is the electric charge of the body as mentioned below. In the case in which a continuous distribution of the electric charge is assumed, the value of q is supposed as tending to zero, actually, to a value small enough, different from zero.

Consequently, the following relation can be used in the case of point-like charges:

$$F = q\,E.$$
(1.14)

Formula (1.14) expresses a general relationship referred to as *expression* or *law of the ponderomotive action* upon one point-like electric charge at rest in the reference frame in which the force is expressed. It can also be considered as a relation used for defining the quantity E.

In accordance with relation (1.14), and with the definition of the lines of field of Sub-section 1.2.1, it follows that a line of field that passes through any point is just the trajectory of a particle with a very small electric charge (for not disturbing the state of the field) and with a very small velocity (otherwise the relation is no more valid), which passes through the considered point. The direction of the line of field is the direction along which a particle, positively charged, is moving.

1.6.6. The Macroscopic Electric Field Strength

The macroscopic electric field strength at a point is obtained by calculating the average value of the microscopic electric field strength. According to the definition, the macroscopic electric field strength $\boldsymbol{E}$ at a certain point $N(\boldsymbol{r})$, at the moment t, in any reference frame will be calculated.

For this purpose, a physically infinitesimal volume Δv having at its centre the considered point, will be chosen, and a physically infinitesimal time interval Δt_0, will be chosen so that the moment t will be at its middle. We shall denote by $\boldsymbol{E}_{\mathrm{micro}}(\boldsymbol{r}, t)$ the microscopic electric field strength at a point within the volume element Δv at a moment $t \in \left(t - \dfrac{\Delta t_0}{2}, t + \dfrac{\Delta t_0}{2} \right)$. The macroscopic electric field strength will be obtained by calculating the *average value* of the microscopic field strength by the relation:

$$\boldsymbol{E}_{\mathrm{macro}} = \boldsymbol{E}(\boldsymbol{r}, t) = \left\langle \left\langle \boldsymbol{E}_{\mathrm{micro}} \right\rangle_{\mathrm{v}} \right\rangle_{\mathrm{t0}} = \frac{1}{\Delta v\, \Delta t_0} \int\limits_{-\frac{\Delta t_0}{2}}^{\frac{\Delta t_0}{2}} \int\limits_{\Delta v} \boldsymbol{E}_{\mathrm{micro}}(\boldsymbol{r}, t + t_0)\, \mathrm{d} v\; \mathrm{d} t_0 \, . \qquad (1.15\ \mathrm{a})$$

In solids, a physically infinitesimal volume must be much greater than the volume attributed to an atom assumed to be bounded by a sphere having a radius of about 10^{-8} cm, but small enough at macroscopic scale. This small volume may be assumed to be a cube, or a sphere having a radius of about $10^{-4} - 10^{-5}$ cm. In the mentioned small volume, the number of atoms is of approximately 10^9. Another example, a cubic centimetre of copper contains approximately 10^{23} atoms and 8.5×10^{22} free electrons.

The physically infinitesimal time interval must be much greater than the period of time variation of atomic phenomena, namely the period of the orbital motion of electrons is of about 10^{-16} s, but small enough at macroscopic scale, therefore it may be assumed of $10^{-12} - 10^{-13}$ s.

If a three-orthogonal rectilinear system of co-ordinates is used, the volume Δv will be that of a right parallelepiped having the sides $\Delta x_0, \Delta y_0, \Delta z_0$, and it follows:

$$\boldsymbol{E}(\boldsymbol{r}, t) = \frac{1}{\Delta x_0\, \Delta y_0\, \Delta z_0\, \Delta t_0} \int\limits_{-\frac{\Delta t_0}{2}}^{\frac{\Delta t_0}{2}} \mathrm{d} t_0 \times$$

$$(1.15\ \mathrm{b})$$

$$\times \int\limits_{-\frac{\Delta x_0}{2}}^{\frac{\Delta x_0}{2}} \int\limits_{-\frac{\Delta y_0}{2}}^{\frac{\Delta y_0}{2}} \int\limits_{-\frac{\Delta z_0}{2}}^{\frac{\Delta z_0}{2}} \boldsymbol{E}_{\mathrm{micro}}(x + x_0, y + y_0, z + z_0, t + t_0)\, \mathrm{d} x_0\, \mathrm{d} y_0\, \mathrm{d} z_0 \, .$$

From the last relation, it results that [2], [3], [21]:

$$\frac{\partial}{\partial x}\boldsymbol{E}(\boldsymbol{r},t)=\left\langle\left\langle\frac{\partial}{\partial x}\boldsymbol{E}_{\text{micro}}(\boldsymbol{r},t)\right\rangle_{v}\right\rangle_{t},\qquad(1.15\ c)$$

and

$$\frac{\partial}{\partial t}\boldsymbol{E}(\boldsymbol{r},t)=\left\langle\left\langle\frac{\partial}{\partial t}\boldsymbol{E}_{\text{micro}}(\boldsymbol{r},t)\right\rangle_{v}\right\rangle_{t},\qquad(1.15\ d)$$

where the indices v and t refer to the averages with respect to volume (space) and time respectively.

The last two relations are useful in certain linear relations containing derivatives that have been obtained for microscopic quantities; the same relations will be valid for macroscopic quantities.

Within a physically infinitesimal volume and a physically infinitesimal time interval, the distribution of charges may be assumed as uniform. The average value of the sum of the forces acting upon the particles charged with electricity and belonging to the considered small volume will contain the following two terms. The first term will be the product of the average value of the electric charge q of the volume and the average value of the electric field strength. The second term will represent the average value of the product:

$$\left(\rho_{v\,\text{micro}}-\rho_{v\,\text{macro}}\right)\boldsymbol{E}_{\text{micro}}=\rho_{v\,\text{micro}}\left(\boldsymbol{E}_{\text{micro}}-\boldsymbol{E}_{\text{macro}}\right);$$

$$\rho_{v}=\rho_{v\,\text{macro}}\,;\qquad\qquad\qquad\qquad\qquad(1.16)$$

$$\boldsymbol{E}=\boldsymbol{E}_{\text{macro}}$$

that are not taken into consideration in the macroscopic studies.

It is interesting to remark that the microscopic field strength can vary very much from one point to another. This aspect will be highlighted by examining the electric field of electrons. A simple model of an electron is a sphere charged with electricity. The electron charge may be assumed as distributed over the surface or within the volume of the sphere. The radius of the sphere attributed to the volume occupied by the electron is of about 2.8×10^{-13} cm. The density of the electric charge of this model is very great. The electric field strength in the electron vicinity is very great and changes its sign at the extremities of each diameter of the sphere attributed to the electron, thus it shows considerable variations at distances of the order of 10^{-13} cm.

The average value of other physical quantities can be calculated in the same manner as above.

Further on, if no special mention is made concerning a quantity, that quantity will be considered as a macroscopic one.

1.7. ELECTRIC FIELD STRENGTH IN THE LARGE SENSE

If at a point of any medium, there is a particle with the electric charge q, very small so that it has a negligible effect on the resultant electric field, and at rest in the reference frame, the electric field strength at that point is given by the following relation (Fig. 1.3):

$$E_l = \frac{F}{q},$$

$$F = F_{el} + F_{non\text{-}el},$$
(1.17)

where F is the force acting on that particle, in the same reference frame. The component F_{el} is of electric nature and the component $F_{non\text{-}el}$ is of non-electric nature. The force F denotes the total force regardless of the nature of its components.

The particle with the electric charge q is assumed as belonging to the considered medium; it is no more a test body.

The macroscopic field strength at any point of a solid, a liquid or a gaseous medium, in the framework of the macroscopic theory of the electromagnetic field, is a primitive (fundamental) quantity. For this reason, it has to be introduced experimentally, hence directly. This introduction can be done using cavities of certain forms and taking into account the relations between the field quantities before and after having achieved these cavities.

The subject has been widely analysed in literature [1], [5], [8], [12], [21].

According to its nature, a field of vectors can have components of two types:

a – Potential component G_p, the curl of which is zero at any point of the space, but the divergence of which may be different from zero.

b – Solenoidal component G_r, the curl of which is different from zero at least within certain regions of the space, but the divergence of which is zero.
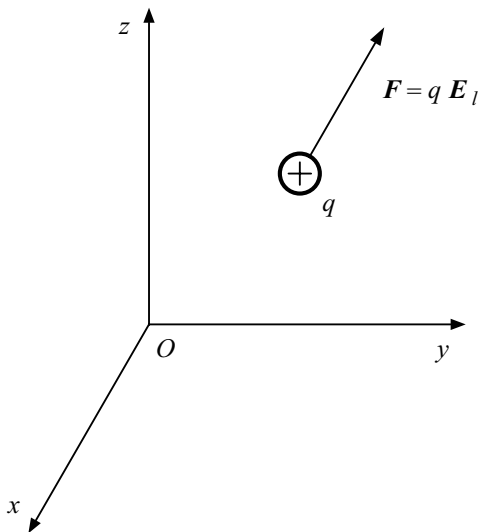
Fig. 1.3. Explanation concerning the electric field strength in the large sense.

This separation of components can also be used in the case of electric field. According to the nature of the components of the force F, the electric field strength *in the large sense*, E_l, may have components of three types:

1º The *Coulombian electric field strength*, E_c, which is produced by electric charges according to the Coulomb law. This is a potential component, hence $E_c = -\operatorname{grad} V$, where the function V is the potential function.

2º The *induced electric field strength* (also termed *rotational, curl,* or *solenoidal* field strength), E_r, which is produced by the electromagnetic induction phenomenon.

3º The *impressed electric field strength* also termed *extraneous electric field strength* [7, English edition, p. 12] (in French, champ électrique imprimé ou champ électromoteur [3, p. 135], in German, eingeprägte elektrische Feldstärke [22, p. 109]), E_i, which is not produced by electromagnetic causes, but by non-electromagnetic causes. Among these causes, the following ones can be mentioned:

a. Mechanical cause, for instance the acceleration of a metallic body that determines forces which produce a displacement of electrons with respect to the positive ions of the crystalline lattice of the metal.

b. Thermal or chemical phenomena determining forces which produce a displacement of electric charge carriers. To every force described above an electric field strength corresponds according to relation (1.17), and it is called *impressed electric field strength*.

Therefore, the expression of the electric field strength in the large sense is:

$$E_l = E_c + E_r + E_i \tag{1.18}$$

or

$$E_l = E_c + E_n , \tag{1.18 a}$$

$$E = E_c + E_r , \tag{1.18 b}$$

$$E_l = E + E_i , \tag{1.18 c}$$

where

$$E_n = E_r + E_i \tag{1.19}$$

is referred to as *the strength of the electromotive field* or the strength of *the non-Coulombian electric field*, and

$$E = E_c + E_r \tag{1.20}$$

represents *the strength of the electric field in the restricted sense* or simpler *the strength of the electric field*.

It can be added that the electric field strength is also referred to as *electric field intensity*.

The lines of the vector field E, i.e., lines of electric field strength, also called simply *electric field lines*, are the lines at each point of which the vector E is tangent.

1.8. LINE-INTEGRAL (CIRCULATION) OF THE ELECTRIC FIELD STRENGTH ALONG AN ARC OF CURVE. ELECTRIC POTENTIAL DIFFERENCE. ELECTRIC TENSION, VOLTAGE. ELECTROMOTIVE FORCE.

Generally, in a field of vectors, the line-integral of a field vector round a curve or an arc of curve, like in the case of a work, is also called *circulation* along the respective way. In the case of an electric field, this circulation is referred to as *electric tension* also termed *voltage*, in order to avoid any confusion with a mechanical meaning. This quantity is a derived one and will also be utilized in the study of the electric field.

A point-like electric charge that is moving with a very small velocity along any open curve C_{AB} in an electric field will be considered (Fig. 1.4). It is supposed that the charge q is small enough so that it will not sensibly modify the state of the electric field.

The resultant force $\boldsymbol{F}$, acting upon the considered particle, having one component of electric nature $\boldsymbol{F}_{\mathrm{el}}$ and another of non-electric nature $\boldsymbol{F}_{\mathrm{non\text{-}el}}$ does the work:

$$W_L = \int_{C_{AB}} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{l}. \tag{1.21}$$

Utilizing expressions (1.18) and (1.21), it follows:

$$W_L = \int_{C_{AB}} q\,\boldsymbol{E}_l \cdot \mathrm{d}\boldsymbol{l} = \int_{C_{AB}} q\left(\boldsymbol{E}_c + \boldsymbol{E}_r + \boldsymbol{E}_i\right)\mathrm{d}\boldsymbol{l}. \tag{1.21 a}$$

The ratio

$$\frac{W_L}{q} = \int_{C_{AB}} \boldsymbol{E}_l \cdot \mathrm{d}\boldsymbol{l} \tag{1.22}$$

is independent of the electric charge of the particle that is moving along the curve C_{AB} and characterizes the work done when the point-like charge is moving along the curve. The velocity of the point-like charge is assumed small enough so that the force acting on it will be given by relation (1.17).
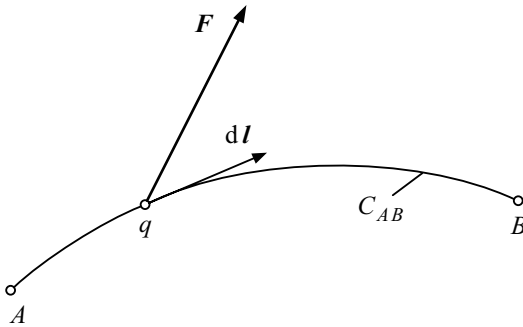
Fig. 1.4. Explanation to the calculation of the line-integral of the electric field strength.

The line-integral of the electric field strength in the large sense, along the considered curve, is called *electric tension in the large sense* along the considered curve and is given by the relation:

$$\left(u_{C_{AB}}\right)_l = \int\limits_{C_{AB}} \boldsymbol{E}_l \cdot \mathrm{d}\boldsymbol{l}. \tag{1.23 a}$$

If instead of the electric field strength in the large sense $\boldsymbol{E}_l$, the electric field strength in the restricted sense $\boldsymbol{E}$ is introduced in relation (1.23 a), the line-integral along the considered curve is also called *electric tension* and is given by the relation:

$$u_{C_{AB}} = \int\limits_{C_{AB}} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l}. \tag{1.23 b}$$

In accordance with relations (1.22) and (1.23 b), it follows:

$$W_L = q\, u_{C_{AB}} . \tag{1.23 c}$$

If instead of the electric field strength in the large sense $\boldsymbol{E}_l$, the non-Coulombian electric field strength $\boldsymbol{E}_n$ is introduced in relation (1.23 a), the line-integral along the considered curve is also called *electromotive tension* or *electromotive force* and is given by the relation:

$$u_{eC_{AB}} = \int\limits_{C_{AB}} \boldsymbol{E}_n \cdot \mathrm{d}\boldsymbol{l} . \tag{1.24}$$

If the integrals (1.23 a) and (1.23 b) are considered around a closed curve Γ, the obtained expressions is called *electromotive force* (e.m.f.) or *electromotive tension in the large sense* and *in the restricted sense*, respectively; or each of them is simply called *electromotive force* (abbreviation e.m.f.). Therefore, the following relations:

$$e = u_{el} = \oint\limits_{\Gamma} \boldsymbol{E}_l \cdot \mathrm{d}\boldsymbol{l} , \tag{1.24 a}$$

$$e = u_e = \oint\limits_{\Gamma} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} , \tag{1.24 b}$$

are obtained, respectively.

The electromotive force obtained by the line-integral of the electric field strength in the large sense around a closed curve coincides with the integral of the non-Coulombian component of the electric field strength along that closed curve. The reason is that the integral of the Coulombian component of the electric field strength is zero.

If instead of the electric field strength in the large sense $\boldsymbol{E}_l$, the Coulombian electric field strength $\boldsymbol{E}_c$ is introduced in relation (1.23 a), the line-integral along the considered curve C_{AB} yields:

$$\int_{C_{AB}} \boldsymbol{E}_c \cdot \mathrm{d}\boldsymbol{l} = \int_{C_{AB}} -\operatorname{grad} V \cdot \mathrm{d}\boldsymbol{l} = -\int_{C_{AB}} \left(\boldsymbol{i} \frac{\partial V}{\partial x} + \boldsymbol{j} \frac{\partial V}{\partial y} + \boldsymbol{k} \frac{\partial V}{\partial z} \right) \cdot \left(\boldsymbol{i}\,\mathrm{d}x + \boldsymbol{j}\,\mathrm{d}y + \boldsymbol{k}\,\mathrm{d}z \right) =$$

$$= -\int_{C_{AB}} \frac{\partial V}{\partial x}\mathrm{d}x + \frac{\partial V}{\partial y}\mathrm{d}y + \frac{\partial V}{\partial z}\mathrm{d}z = -\int_{C_{AB}} \mathrm{d}V = -\int_{A}^{B} \mathrm{d}V = V_A - V_B = u_{AB}$$

(1.24 c)

and is called *electric tension*, simply *tension* or *voltage* between the points *A* and *B* or *potential difference* (p.d.) between the points *A* and *B*. This potential difference between two points does not depend on the form of the curve C_{AB} but only on the position of the two points. If the curve C_{AB} is closed, the integral (1.24 c) is zero.

All the definitions given for electric tensions and electromotive forces refer to any medium, regardless of its nature.

The unit of electric tensions, potential difference and electromotive forces in the SI system of units (which is a rationalized system) is the volt (symbol V) and is explained in Sub-section 1.15.3.

1.9. POLARIZATION OF DIELECTRICS

1.9.1. The Polarization of Dielectrics Phenomenon. Polarization State of Dielectrics. Polarization Electric Charge (Bound Electric Charge). Electric Moment of a Neutral System of Electric Charges.

As shown in Sub-section 1.5.2, dielectrics consist of neutral systems (atoms, neutral molecules, neutral domains) of electric charges. The electrically charged particles of each system cannot move to relatively great distances, with respect to certain equilibrium positions. For this reason, these charges are called *bound electric charges*.

Although they have this denomination, they are of the same nature as the free electric charges. For each neutral system, the sum of all bound electric charges is zero. Therefore:

$$\sum_{i=1}^{n} q_{i\,\mathrm{bound}} = 0.$$

(1.25)

Under the action of an external electric field, the bound electrically charged particles are not dislodged from their places (if the electric field strength is not too great), but they are dislodged from their equilibrium into other neighbouring positions. Consequently, the electric positive charges move in the direction of the electric field, and the negative electric charges move in the opposite direction.

Therefore, the space distribution of the electric charges of the dielectric is modified. The new state of the dielectric is called *electric polarization state*. The respective phenomenon is termed *electric polarization*.

The metallic conductors are practically not polarizable.

A neutral system of electric charges can be characterized by a vector quantity called *electric moment* of the system. In any reference frame, let a neutral system be composed of *n* electrically charged particles that can be considered as point-like charges q_i, $(i = 1, ..., n)$, with the position vectors r_i, $(i = 1, ..., n)$ having their origin at the same point of the reference frame (Fig. 1.5). If one point-like electric charge is moving within the system around a closed orbit (as the case of electrons can be), when considering the average value with time, the end (extremity) of the vector r_i is considered as the action centre of the respective charge (i.e., the centre of the orbit).

The electric moment of a neutral system of point-like electric charges is:

$$\boldsymbol{p}_s = \sum_{i=1}^{n} q_{i\,\text{bound}} \, \boldsymbol{r}_i = \sum_{i=1}^{n} \boldsymbol{p}_i \, , \tag{1.26}$$

with

$$\sum_{i=1}^{n} q_{i\,\text{bound}} = 0. \tag{1.26 a}$$

The relation (1.26) can serve to the characterization of a neutral system of electric charges, because the electric moment $\boldsymbol{p}_s$ previously defined is independent of the choice of the reference frame. Indeed, if the origin of the reference frame is displaced from the point *O* to the point *O'* (Fig. 1.5 a) by any vector $\boldsymbol{a}$, then the electric moment of the neutral system in the new reference frame becomes:

$$\boldsymbol{p}_s' = \sum_{i=1}^{n} q_{i\,\text{bound}} \, \boldsymbol{r}_i' = \sum_{i=1}^{n} q_{i\,\text{bound}} \left(\boldsymbol{r}_i - \boldsymbol{a} \right) = \sum_{i=1}^{n} q_{i\,\text{bound}} \, \boldsymbol{r}_i - \boldsymbol{a} \sum_{i=1}^{n} q_{i\,\text{bound}} \, . \tag{1.26 b}$$

But

$$\boldsymbol{a} \sum_{i=1}^{n} q_{i\,\text{bound}} = 0. \tag{1.26 c}$$

It follows:

$$\boldsymbol{p}_s' = \sum_{i=1}^{n} q_{i\,\text{bound}} \, \boldsymbol{r}_i = \boldsymbol{p}_s \, . \tag{1.26 d}$$

Remark. It is useful to mention that the above definition of the electric moment has not an arbitrary character. It implicitly occurs in the expression of the electric potential produced by a neutral system of point-like charges at a relatively distant point and it remains only to denote it, as done above.

a   **b**
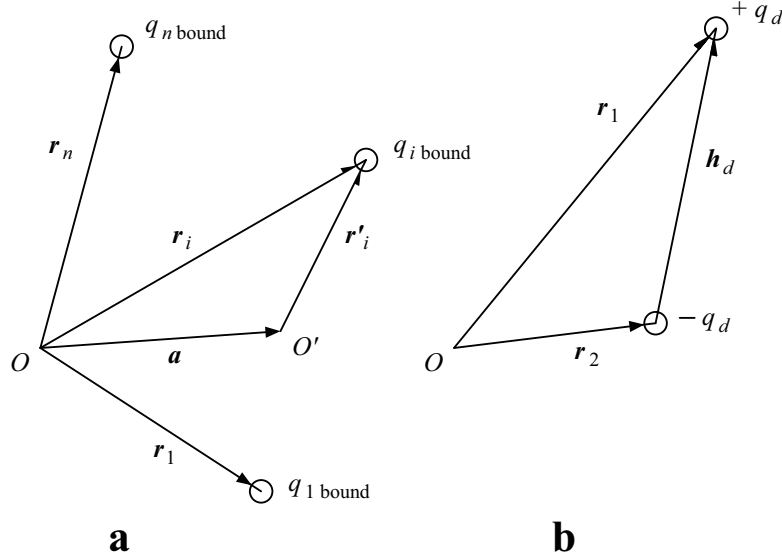
Fig. 1.5. Explanation concerning the electric moment of a neutral system of electric
charges: **a** – system with *n* charges; **b** – system with two charges (dipole).

The simplest neutral system of point-like electric charges is formed by two point-like
electric charges equal in absolute value but of opposite signs $+q$ and $-q$, like in Fig.
1.5. This neutral system is called *electric dipole or electric doublet,* or shortly *dipole* or
doublet. A neutral system formed of several electric charges is called *electric multipole* or
multipole.

The electric moment of an electric dipole is (Fig. 1.5 b):

$$p_d = \sum_{i=1}^{2} q_{i\,bound}\, r_i = q_d\, r_1 - q_d\, r_2 = q_d(r_1 - r_2) = q_d\, h_d\,, \tag{1.27}$$

where the vector h_d is always oriented as in Fig. 1.5. b, from the charge considered as
negative to the charge considered as positive.

1.9.2. The Macroscopic Electric Moment of a Polarized Body

The macroscopic electric moment of a polarized body can be obtained by performing
the sum of the electric moments p_s of all the neutral systems forming the body and
calculating the average value over a physically infinitesimal time interval.

The calculation of the average value with respect to time is necessary because the
charges move round their trajectories and also because of their thermal agitation.

It results that the electric moment of a polarized dielectric body of volume V_D is:

$$p = \left\langle \sum_{V_D} p_s \right\rangle_t = \left\langle \sum_{V_D} q_{i\,bound}\, r_i \right\rangle_t\,, \tag{1.28}$$

where the suffix t indicates that the calculation of the average value refers to time. The sum is extended over all bound electric charges, electrons and atomic nuclei that are contained in the volume V_D of the dielectric body. In the case of a dielectric dipole, relation (1.28) yields:

$$\boldsymbol{p} = \left\langle \sum_{V_D} \boldsymbol{p}_d \right\rangle_t = \left\langle q_d \, \boldsymbol{h}_d \right\rangle_t = q_d \, \boldsymbol{h}.$$
(1.28 a)

From a macroscopic point of view, because it is not possible to consider the structure of substance (electrons and nuclei); the electric moment of a body cannot be deduced as above and for this reason it is not a secondary (derived) quantity. It follows that, in the macroscopic study, the electric moment of a body has to be introduced experimentally as a primitive (fundamental) quantity. This procedure can be achieved by considering the ponderomotive actions exerted upon a small-polarized body, in an external electric field.

1.9.3. The Polarization Electric Charge. Density of the Polarization Electric Charge.

As shown, dielectrics are constituted of neutral systems. Hence the total electric charge due to the neutral systems inside the surface bounding the body is zero.

We shall consider (imagine) any macroscopic closed surface Σ situated inside a dielectric body. The sum of bound electric charges of the interior of the closed surface Σ is called *polarization electric charge* or *electric polarization charge* of the interior of that surface and is denoted by $q_{p\Sigma}$. Therefore, the polarization electric charge is a macroscopic charge. The manner of introducing this charge is analogous to that utilized for the free electric charge.

A dielectric is composed of numerous neutral systems. Hence, the neutral systems that are entirely in the interior of the surface yield a total charge equal to zero. Since the closed surface Σ intersects several neutral systems of the dielectric, some electric charges belonging to the neutral systems the centres of which are inside the surface Σ remain outside this surface. It results that the total electric charge that remains inside the surface may be different from zero. Therefore, the total electric charge of the interior of the closed surface is equal to the sum of the electric charges of the interior of the closed surface that belong to the neutral systems intersected by the closed surface.

As a macroscopic quantity, the polarization electric charge may be considered continuously distributed in the space occupied by the dielectric. In this case, a *volume density of the polarization electric charge* may be introduced:

$$\rho_{pv} = \lim_{\Delta v \to 0} \frac{\Delta q_{p\Sigma}}{\Delta v} = \frac{d\, q_{p\Sigma}}{d\, v}.$$
(1.29)

The neutral systems are not uniformly distributed, but they are more agglomerated in certain regions; hence, the volume density of the charge in these regions can have values different from zero. Hence, the volume density of the polarization electric charge represents the local excess of a charge of a certain sign, with respect to that of opposite sign.

The neutral systems may sometimes have a very non-uniform distribution and may be concentrated in a very thin sheet on certain surfaces. Then, the macroscopically idealized case may be considered. In this case, the polarization charges are distributed over these surfaces, considered as discontinuity surfaces. Therefore, the *surface density of the polarization electric charge* is:

$$\rho_{ps} = \lim_{\Delta S \to 0} \frac{\Delta q_{pS}}{\Delta S} = \frac{d\, q_{pS}}{d\, S}.$$ (1.30)

The neutral systems may sometimes have a very non-uniform distribution and may be concentrated about certain lines. Then, also the macroscopically idealized case may be considered. In this case, the polarization charges are distributed along these lines. Therefore, the *line density of the polarization electric charge* is:

$$\rho_{pl} = \lim_{\Delta l \to 0} \frac{\Delta q_{pl}}{\Delta l} = \frac{d\, q_{pl}}{d\, l}.$$ (1.31)

The domains Δv, ΔS, Δl above utilized, in the denominators of relations (1.29), (1.30), (1.31), are, like in relations (1.7), (1.8), (1.9), geometrically infinitesimal spaces.

The macroscopic densities of the polarization charges, previously defined, are the average values of the microscopic densities of the bound electric charges:

$$\rho_{pv} = \left\langle \left\langle \rho_{v\,\text{bound}} \right\rangle_v \right\rangle_t ; \quad \rho_{ps} = \left\langle \left\langle \rho_{s\,\text{bound}} \right\rangle_s \right\rangle_t ; \quad \rho_{pl} = \left\langle \left\langle \rho_{l\,\text{bound}} \right\rangle_l \right\rangle_t .$$ (1.32 a, b, c)

The polarization electric charges, in the case of a volume, surface or line distribution respectively, over a volume V_D, a surface S_D or a line C_D, respectively, are:

$$q_{pv} = \int_{V_D} \rho_{pv}\, d v; \quad q_{ps} = \int_{S_D} \rho_{ps}\, d S; \quad q_{pl} = \int_{C_D} \rho_{pl}\, d l.$$ (1.33 a, b, c)

1.9.4. Electric Polarization

The local polarization state at a point of a body can be characterized by a vector quantity called *electric polarization vector* or simpler *electric polarization*. If the electric polarization state of the body is uniform, then the electric polarization is equal to the vector sum of the electric moments of all multipoles of the unit of volume having at its centre the considered point.

Let the unit of volume contain n_p multipoles, uniformly distributed, each of them having the electric moment $\boldsymbol{p}$. Then the electric polarization is:

$$\boldsymbol{P} = n_p\, \boldsymbol{p}.$$ (1.34)

In general, the concentration of multipoles and their electric moments may differ from a point to another. Then, instead of summing up the electric moments of multipoles over

the unit of volume, the sum over a physically infinitesimal volume Δv containing at its centre the considered point has to be calculated.

Let p_i be the electric moment of a multipole with the ordinal number i, and Δp the vectorial sum of the electric moments of the considered volume. In this case, the electric polarization is:

$$P = \frac{\sum_{\Delta v} p_i}{\Delta v},$$ (1.35 a)

and relations (1.26), and (1.35) yield:

$$P = \frac{1}{\Delta v} \sum_{i=1}^{n} q_{pi}\, r_i,$$ (1.35 b)

the sum being extended to all electric charges of the neutral systems of the volume element Δv.

In any reference frame, with respect to which the substance is at rest, the macroscopic value of the *electric polarization vector* is given by the average relation:

$$P = \frac{1}{\Delta v} \left\langle \sum_{i=1}^{n} q_{pi}\, r_i \right\rangle_t,$$ (1.36 a)

where the sum refers to all the charges that form a neutral system in the physically infinitesimal volume.

The electric polarization defined by relation (1.36 a) represents the average value, over a physically infinitesimal time interval, of the average of the vector sum of the electric moments over a physically infinitesimal volume. The electric polarization represents the volume density of the electric dipoles or multipoles.

As a macroscopic quantity, the electric moments of multipoles denoted $p_i = p(r,t)$ may be considered continuously distributed in the space occupied by the dielectric. Then, the sizes and electric moments of multipoles are supposed infinitely small. In this case, the *electric polarization vector* may be introduced in the form:

$$P = \lim_{\Delta v \to 0} \frac{\sum_{\Delta v} p_i}{\Delta v} = \frac{d\, p}{d\, v}.$$ (1.36 b)

The state of the substance in this case is referred to, as mentioned, *polarization state*. From the microscopic point of view, the electric moment is a derived quantity. From the macroscopic point of view, the electric moment has to be introduced experimentally as a primitive (fundamental) quantity. The manner of introducing this quantity is based on the ponderomotive forces acting on an electrified small body situated in a hollow cavity within the substance.

The electric moment of a volume element Δv of a polarized dielectric material is:

$$\Delta p = P\, \Delta v.$$ (1.37)

The electric moment of a volume element V_D of a body of polarized dielectric material is:

$$p = \int_{V_D} P \, dv. \tag{1.38}$$

The electric polarization can be decomposed into two components. The first one exists even in the absence of certain external causes and is determined by the nature of the substance. This component is termed *permanent electric polarization* and is denoted by P_p. The second one appears only under the action of certain external causes (electric field) and is determined by these causes and by the nature of the substance. This component is termed *temporary electric polarization* and is denoted by P_t. It results that:

$$P = P_p + P_t. \tag{1.39}$$

In certain conditions each of this components may be zero.

1.9.5. Polarization Electric Charge of the Interior of a Closed Surface in a Dielectric

We shall consider a dielectric body (D) and in its interior, like in Fig. 1.6 a, we shall adopt (imagine) any closed surface Σ.

For the sake of simplicity, the multipoles will be replaced by identical equivalent dipoles with the corresponding electric moment according to relation (1.28 a).

As recalled in Sub-section 1.9.3, at the beginning, and as it can be seen in Fig. 1.6 a, the dipoles which are entirely inside the surface Σ give a total charge equal to zero.

Only the dipoles that are intersected by the surface Σ contribute to the total polarization electric charge (i.e., due to the bound charges) contained by the closed surface Σ. The centre of each dipole is represented in the figure by a dot. We shall calculate the electric charge belonging to the dipoles intersected by the surface element ΔS of the Fig. 1.6. In this region, we assume, for simplicity, that the electric moment vector of every dipole is parallel to the positive normal to the surface (outwards oriented) and of the same direction.

The total electric charge different from zero belongs to the intersected dipoles the centres of which are contained in the right parallelepiped of height h, the trace of which is represented by dashed line in the figure.

The volume of the parallelepiped is $h \cdot (n\,\Delta S) = h \cdot \Delta S$. As previously, the volume concentration (number of dipoles per unit of volume) of dipoles will be denoted by n_p.

The electric charge of the interior of the parallelepiped is:

$$\Delta q = -n_p \, q_d \, h \cdot (\Delta S), \tag{1.40 a}$$

but, according to relation (1.28 a), $q_d \, h = p$, and

$$\Delta q = -n_p \, p \cdot (\Delta S). \tag{1.40 b}$$
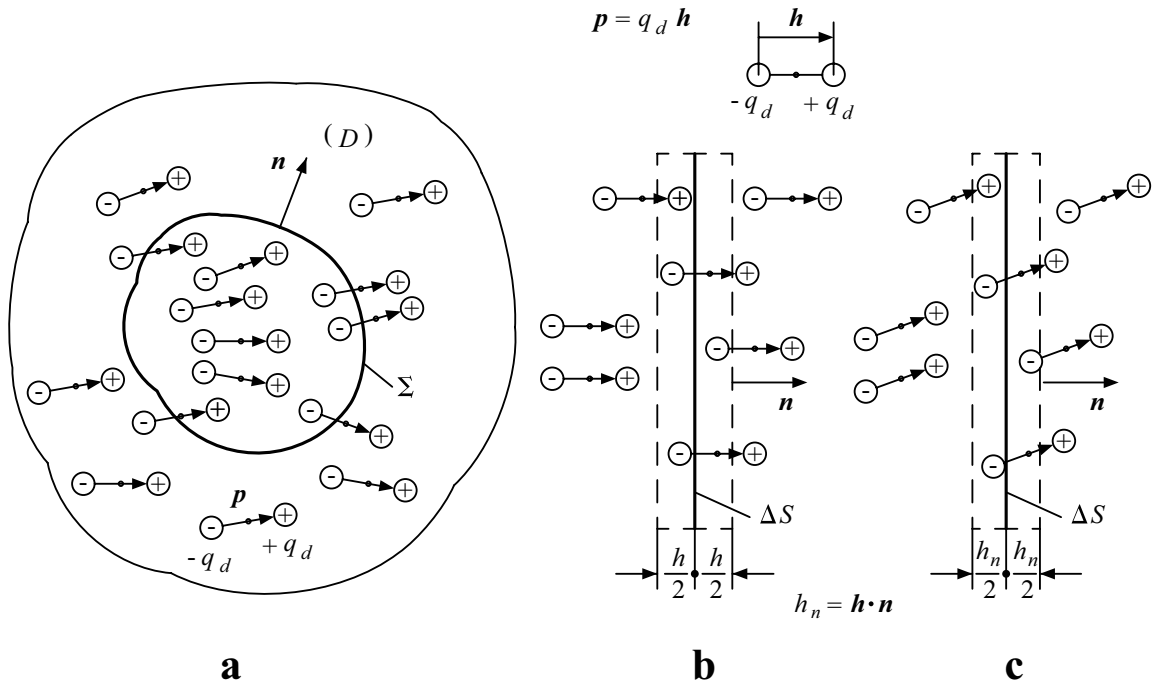
Fig. 1.6. Explanation to the calculation of the polarization
electric charge of the interior of a closed surface: **a**. Closed surface Σ inside a
dielectric body; **b**. Portion of the surface Σ intersected by dipoles situated
perpendicular to it; **c**. The same portion of surface intersected by dipoles forming any
angle with the normal to the surface. The centre of each dipole is marked by a dot.

According to relation (1.34), $n_p \, \boldsymbol{p} = \boldsymbol{P}$, and

$$\Delta q_p = -\boldsymbol{P} \cdot \left(\Delta \boldsymbol{S}\right). \tag{1.41}$$

In the case in which the positive normal to the surface is not parallel to the electric moment vectors $\boldsymbol{p}$ (Fig. 1.6 c), the electric charge of the interior of the parallelepiped constructed on the surface element $\Delta \boldsymbol{S}$ is given, as previously, by relation (1.41). Indeed, the volume of the parallelepiped is given by the same expression and consequently the expression of the electric charge does not change.

It follows that the polarization electric charge of the interior of the surface Σ is:

$$q_{p\Sigma} = -\int_{\Sigma} \boldsymbol{P} \cdot \mathrm{d}\boldsymbol{S}. \tag{1.42}$$

The polarization electric charge $q_{p\Sigma}$ can be expressed in terms of the *volume density of the electric polarization charge* according to relation (1.33 a):

$$q_{p\Sigma} = \int_{V_\Sigma} \rho_{pv}\, \mathrm{d}v.$$

(1.43)

Comparing relations (1.42) and (1.43), it follows that:

$$\int_{V_\Sigma} \rho_{pv}\, \mathrm{d}v = -\int_\Sigma \boldsymbol{P}\cdot \mathrm{d}\boldsymbol{S}.$$

(1.44)

By transforming the last surface integral into a volume integral, and taking into account that the relation holds for any closed surface, it follows:

$$\rho_{pv} = -\operatorname{div}\boldsymbol{P}.$$

(1.45)

Now, we shall examine the case of Fig. 1.7 where, on any discontinuity surface of the electric polarization vector $\boldsymbol{P}$, the distribution of multipoles is highly non-uniform. According to formula (1.45), on this surface, the density of the polarization electric charge becomes infinite. For obtaining in this case the relation between the electric polarization and the density of the polarization electric charge, we consider (Fig. 1.7) a right parallelepiped of height Δh that contains the surface element ΔS. The height Δh is taken very small as compared to the sizes of the surface element ΔS that is why the flux of a vector through the lateral surface of the parallelepiped can be neglected.
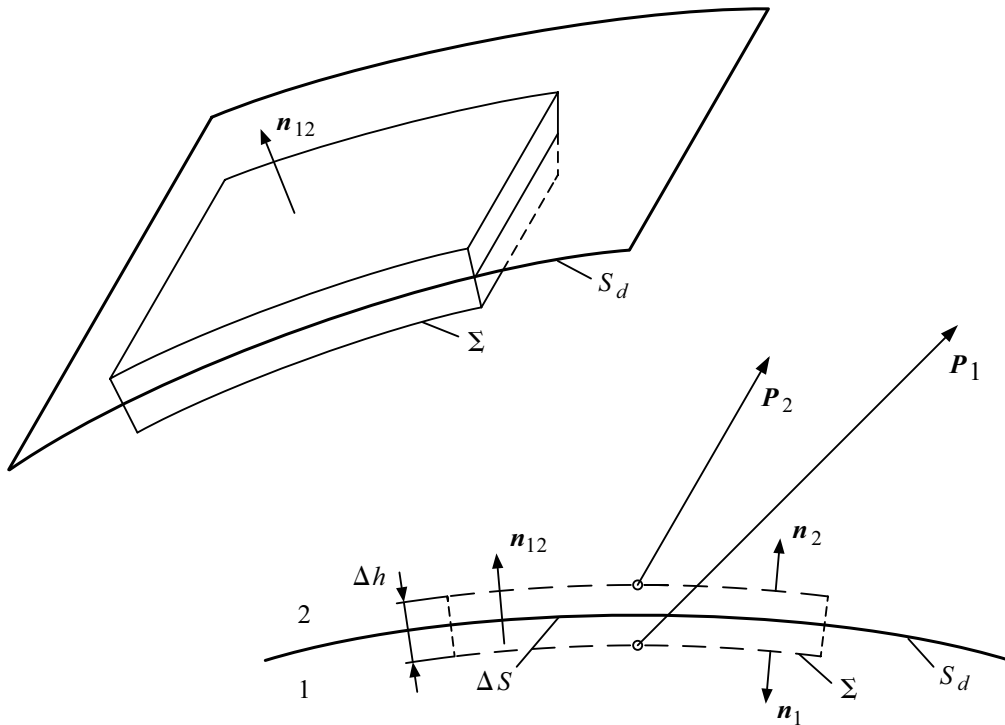
Fig. 1.7. Explanation to the calculation of the surface density of the
polarization electric charge.

By applying relation (1.44), we obtain:

$$\left(\rho_{pv}\,\Delta h\right)\boldsymbol{n}_{12}\cdot\left(\boldsymbol{n}_{12}\,\Delta S\right)=-\left(-\,\boldsymbol{P}_1\cdot\boldsymbol{n}_{12}+\boldsymbol{P}_2\cdot\boldsymbol{n}_{12}\right)\Delta S \tag{1.46}$$

or

$$\rho_{pv}\left(\Delta h\right)=\boldsymbol{n}_{12}\cdot\left(\boldsymbol{P}_1-\boldsymbol{P}_2\right). \tag{1.47}$$

The surface density of the electric charge is:

$$\rho_{ps}=\rho_{pv}\left(\Delta h\right), \tag{1.48 a}$$

and it results that:

$$\rho_{ps}=\boldsymbol{n}_{12}\cdot\left(\boldsymbol{P}_1-\boldsymbol{P}_2\right). \tag{1.48 b}$$

Therefore

$$\rho_{ps}=-\operatorname{div}_s\boldsymbol{P}=\boldsymbol{n}_{12}\cdot\left(\boldsymbol{P}_1-\boldsymbol{P}_2\right) \tag{1.49}$$

that represents the expression of the surface density of the polarization electric charge on a surface of discontinuity of the electric polarization, and $\operatorname{div}_s\boldsymbol{P}$ is referred to as the *surface divergence* of the vector $\boldsymbol{P}$.

1.9.6. Ponderomotive Actions Exerted upon a Polarized Body in an Electric Field

We consider a small electrically polarized body of electric moment $\boldsymbol{p}$ submitted to an external electric field of intensity $\boldsymbol{E}_0$. The equivalent electric dipole in the electric field will be examined (Fig. 1.8).

The force acting upon the dipole is given by the sum of forces acting upon the two point-like charges of the equivalent dipole:

$$\boldsymbol{F}_E=-q_d\,\boldsymbol{E}_0\!\left(\boldsymbol{r}-\frac{1}{2}\boldsymbol{h}\right)+q_d\,\boldsymbol{E}_0\!\left(\boldsymbol{r}+\frac{1}{2}\boldsymbol{h}\right), \tag{1.50}$$

where

$$\boldsymbol{h}=\boldsymbol{i}\,\Delta x+\boldsymbol{j}\,\Delta y+\boldsymbol{k}\,\Delta z, \tag{1.50 a}$$

$$\boldsymbol{E}_0\!\left(\boldsymbol{r}+\frac{1}{2}\boldsymbol{h}\right)=\boldsymbol{E}_0\!\left(x+\frac{1}{2}\Delta x,\,y+\frac{1}{2}\Delta y,\,z+\frac{1}{2}\Delta z\right). \tag{1.51}$$

By expanding in a series and neglecting the small quantities of higher order, it follows:

$$E_{0x}\left(r+\frac{1}{2}h\right)=E_{0x}(x,y,z)+$$

$$+\frac{\partial E_{0x}(x,y,z)}{\partial x}\cdot\frac{1}{2}\Delta x+\frac{\partial E_{0x}(x,y,z)}{\partial y}\cdot\frac{1}{2}\Delta y+\frac{\partial E_{0x}(x,y,z)}{\partial z}\cdot\frac{1}{2}\Delta z=$$

$$=E_{0x}(x,y,z)+$$

$$+\frac{1}{2}\Delta x\frac{\partial E_{0x}(x,y,z)}{\partial x}+\frac{1}{2}\Delta y\frac{\partial E_{0x}(x,y,z)}{\partial y}+\frac{1}{2}\Delta z\frac{\partial E_{0x}(x,y,z)}{\partial z}.$$

$$(1.52)$$

Analogous relations can be obtained for the Oy and Oz axes. It follows:

$$E_0\left(r+\frac{1}{2}h\right)=E_0(r)+\frac{1}{2}\Delta x\frac{\partial E_0(r)}{\partial x}+\frac{1}{2}\Delta y\frac{\partial E_0(r)}{\partial y}+\frac{1}{2}\Delta z\frac{\partial E_0(r)}{\partial z} \qquad (1.53)$$

or

$$E_0\left(r+\frac{1}{2}h\right)=E_0(r)+\left(\frac{1}{2}h\cdot\nabla\right)E_0(r). \qquad (1.54\ a)$$

Similarly, we get:

$$E_0\left(r-\frac{1}{2}h\right)=E_0(r)-\frac{\partial E_0(r)}{\partial h}\cdot\frac{1}{2}h=E_0(r)-\left(\frac{1}{2}h\cdot\nabla\right)E_0(r). \qquad (1.54\ b)$$

From relations (1.50) and (1.54), we obtain:

$$F_E=(q_d\,h\cdot\nabla)E_0(r) \qquad (1.55)$$

or, because $p=q_d\,h$, we have:

$$F_E=(p\cdot\nabla)E_0(r). \qquad (1.56)$$

From expression (1.56), it can be seen that the force, which acts upon the polarized body, is zero if the electric field is uniform (homogeneous).

The torque acting upon the dipole is given by the sum of the moments of the forces acting upon the two point-like electric charges, relatively to the axis that passes through the centre of the dipole and is perpendicular to the plane determined by the vectors h and E_0 (Fig. 1.9). The forces acting upon the two point-like electric charges differs of a very small quantity, so that it can be considered that upon the dipole only one couple of forces is acting. By using the known expressions of Mechanics for the moment of a force relatively to one axis, it results that their sum is:

$$T_E=\frac{1}{2}h\times q_d\,E_0\left(r-\frac{1}{2}h\right)+\frac{1}{2}h\times q_d\,E_0\left(r+\frac{1}{2}h\right). \qquad (1.57)$$

Fig. 1.8. The force acting upon an electrically polarized small body situated in electric field.
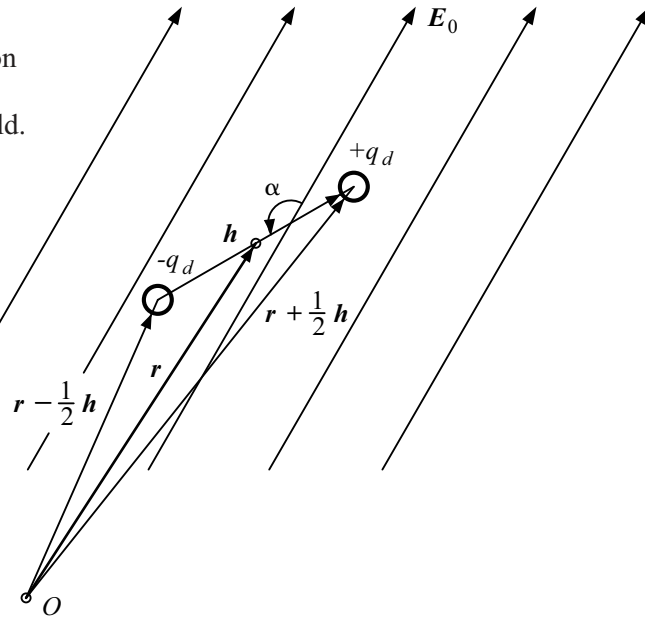
Fig. 1.9. The couple of forces acting upon an electrically polarized small body situated in electric field.
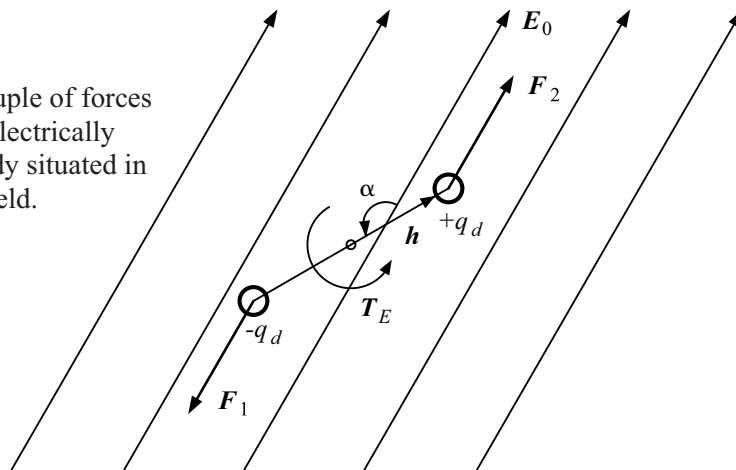
Expanding in a series in terms of $\boldsymbol{h}$ and neglecting the small quantities of higher order, the last relation yields:

$$T_E = \boldsymbol{p} \times \boldsymbol{E}_0(\boldsymbol{r}). \tag{1.58}$$

It follows that a force and a torque are acting upon a small electrically polarized body with the electric moment $\boldsymbol{p}$ and a free electric charge q, in an external electric field of intensity $\boldsymbol{E}_0(\boldsymbol{r})$, and the corresponding expressions are the following:

$$F_E = q\,E_0(r) + (p \cdot \nabla)E_0(r), \tag{1.59}$$

$$T_E = p \times E_0(r). \tag{1.60}$$

From relation (1.60), we can obtain the definition of the unit of the electric moment *coulomb metre* (symbol $C \cdot m$). The *coulomb metre* is the electric moment of a small electrically polarized small body upon that acts a maximum torque of $1\,N \cdot m$ in an external uniform electric field having the strength (intensity) of 1 V/m.

1.10.   THE ELECTRIC CURRENT

A set of electrically charged particles that are moving with respect to a reference frame represents an *electric current* in that frame.

The motion of electrically charged particles called electric charge carriers can occur in empty space or inside some bodies.

The electric current can also be represented by the motion, with respect to a reference frame, of some electrically charged bodies.

Taking into account the definition of the electric charge, the electrification state of a body is independent of the reference frame. However, the electric current depends, like the motion, on the reference frame to which it is related. Further on, excepting the case in which a special mention is made, only the electric current with respect to a reference frame fixed to the neighbouring substance will be considered.

According to the type of the charge carriers and their velocities, there are conduction electric current, convection electric current, polarization electric current, Amperian electric current.

1.10.1.   Electric Current Intensity.   Electric Current Density.

The electric current that crosses a surface S_Γ, generally an open one, is characterized by a scalar quantity i, called *electric current intensity (intensity of electric current)* [3, p. 113]*, electric current strength* [1, Vol. II, p. 152, Art. 495] or shorter *electric current* [44], and defined by the relation:

$$i = \lim_{\Delta t \to 0} \frac{\Delta q_{S_\Gamma}}{\Delta t} = \frac{d\,q_{S_\Gamma}}{d\,t}, \tag{1.61}$$

where the quantity Δq_{S_Γ} represents the free (true) electric charge that passes through the surface S_Γ in the physically infinitesimal time interval Δt, the latter, after assuming a continuous distribution of the electric charge, tends to zero.

The charges that pass through the surface in the direction of the positive normal to the surface are taken with their sign, and the charges that pass in opposite direction are taken with changed sign. In Fig. 1.10, we have supposed v_r and n of the same direction.

Conversely, the electric charge that passes through the surface is:

Fig. 1.10. Explanation concerning the electric current intensity in the case in which the velocity of charge carriers is perpendicular to the surface.
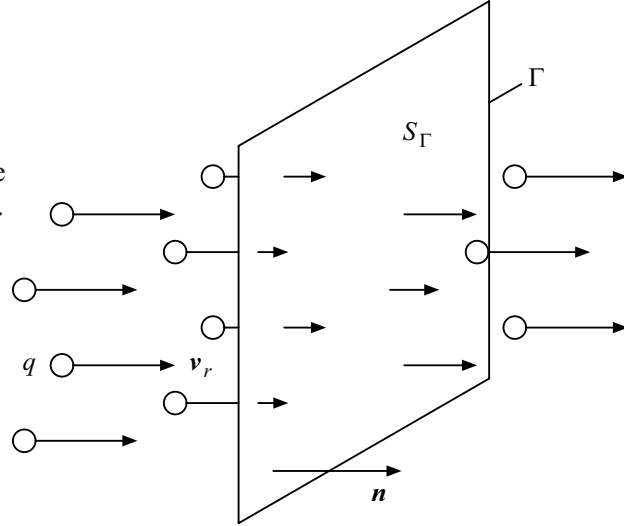
$$q_{S_\Gamma} = \int_0^t i \, \mathrm{d}t. \tag{1.62}$$

Concerning the positive normal to the surface, its direction is taken as follows. It is taken, as known, i.e., outwards, in the case of closed surfaces. It is taken in the direction associated according to the right-handed screw rule with the direction of travelling the curve bounding the surface (the latter direction being arbitrarily chosen) in the case of open surfaces. Let us consider that the unit of volume contains n_c charge carriers. Each of them is assumed to have the same electric charge q and in the adopted reference frame (generally a frame system fixed to the surface S_Γ), the same velocity v_r (relatively to the reference frame), normal to the surface S_Γ (Fig. 1.10).

In this case, the electric current intensity that crosses the surface S_Γ is:

$$i = n_c \, q \, v_r \, S, \tag{1.63}$$

where S denotes the area of the surface S_Γ.

The electric current intensity is a scalar quantity and for this reason it can have only sign but not direction. However, the notion of direction of a current is used.

The direction of an electric current through a surface is that of the normal to that surface for which the electric current intensity is positive. The sign of the electric current can be associated with the moving direction of the charge carriers. From relation (1.63), it results that the sign of the electric current through a surface with a certain positive normal is positive if the motion direction of positive electric charge carriers is that of that normal.

The intensity of the electric current that crosses the unit of surface is numerically equal to:

$$J = n_c \, q \, v_r \,, \tag{1.64}$$

and is called *electric current density*.
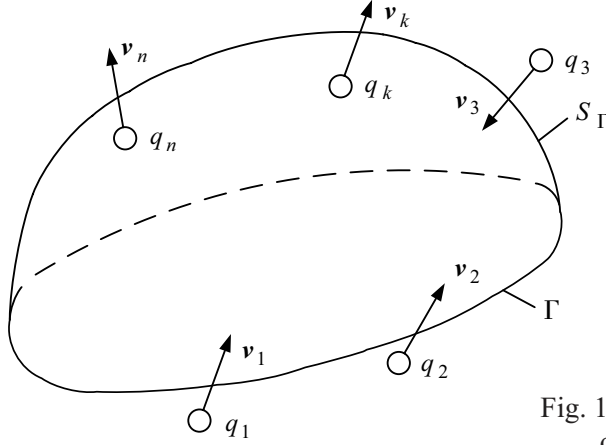
Fig. 1.11. Explanation to the calculation
of the electric current density.

Consequently, the electric current intensity passing through a surface S with a certain normal, is the quantity numerically equal to the electric charge passing through that surface in the unit of time in the direction of the normal. The electric current density, at any point, is the quantity numerically equal to the electric charge passing perpendicularly through the unit of surface in the unit of time. It is a vector quantity having the direction given by the velocity of the electric charge carriers and their sign. In the general case (Fig. 1.11), there are m types of electric charge carriers that cross the surface S.

Each type of electric charge carriers has: a certain velocity $\boldsymbol{v}_k$ with respect to a reference frame (generally fixed to the surface S_Γ), the electric charge q_k and the concentration n_k (number of electric charge carriers per unit of volume). The microscopic electric current density is expressed by the relation:

$$\boldsymbol{J}_{\text{micro}} = \sum_{k=1}^{m} n_k \, q_k \, \boldsymbol{v}_k \,.$$ (1.65)

Macroscopically, the electric current density is expressed by the macroscopic average (mean) of the microscopic quantity. The macroscopic quantity is obtained by calculating the average value with space (volume) and time, as shown in Sub-section 1.6.6. Thus:

$$\boldsymbol{J} = \boldsymbol{J}_{\text{macro}} = \left\langle \boldsymbol{J}_{\text{micro}} \right\rangle.$$ (1.66)

The electric current intensity that passes through (crosses) any surface S_Γ (generally open) is:

$$i = \int_{S_\Gamma} \boldsymbol{J} \cdot \mathrm{d}\,\boldsymbol{S} \,,$$ (1.67)

where the positive normal to the surface S_Γ is adopted as mentioned above.

If the electric current density is normal to the surface S_Γ, and does not vary from one point to another, it follows:

$$J = \frac{i}{S},$$ (1.68)

where S is the area of the surface.

Let us consider the electric charge of the charge carriers distributed in volume, with the density ρ_v, in the adopted reference frame. The intensity of the electric current that passes through the surface S_Γ is:

$$i = \int_{S_\Gamma} \rho_v \, \boldsymbol{v}_r \cdot \mathrm{d}\boldsymbol{S},$$ (1.69)

and the electric current density is:

$$\boldsymbol{J} = \rho_v \, \boldsymbol{v}_r.$$ (1.70)

In the microscopic theory of the electromagnetic field, the electric current intensity and the electric current density are derived quantities. In the macroscopic theory of the electromagnetic field, one of the two quantities electric current intensity or current density has to be introduced as a primitive (fundamental) quantity. For this purpose, the electrochemical effect can be used.

In stationary regime (stationary operating conditions), the quantities voltage and intensity of the electric current may be denoted by small letters as well as by capitals. However, in other regimes, the notation has certain meanings.

From a microscopic point of view, as shown, the electric current is produced by electrically charged microscopic particles in motion.

From a macroscopic point of view, it is accepted that the electric charge, as well as the substance, is distributed in space. The electric charges and therefore the electric current may be highly non-uniformly distributed on certain surfaces and may be concentrated on certain surfaces, in a very thin film. Then, the macroscopic idealized case is considered in which the electric current is distributed in the form of a current sheet on each of these surfaces. Fig. 1.12 shows the trace of the surface S_d with the current sheet on the figure plane.
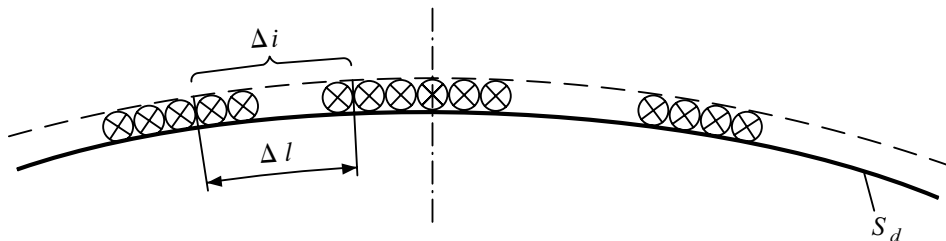
Fig. 1.12. Explanation to the calculation of the linear current density. The cross-sections of the conductors (round wires) show that the carried currents enter the figure plane.

Correspondingly, it is possible to define a *linear current density* also called *linear current sheet* and denoted by J_s or A:

$$J_s = A = \frac{\partial i}{\partial l}.$$ (1.71)

Usually, it may be expressed as the number of ampere-turns per metre.

1.10.2. Conduction Electric Current

The electric charge carriers of the interior of a body of any substance are called free (electric charge) carriers if they are not permanently located within a certain atom or molecule, and are called bound (electric charge) carriers if they are permanently located within one atom or molecule.

In certain cases, in dielectrics and semiconductors there are charge carriers (electrons or ions) which may have limited motions in the crystalline lattice. After such a motion, a free space remains, i.e., a space not occupied by particles of the crystalline lattice and it is called *hole*. After this free place is occupied by a neighbouring charge, a new neighbouring hole will appear.

This process can continue so that the row of holes, successively created, is equivalent to the displacement of the hole within the substance. The displacement of holes is analogous to the motion of a charge carrier having the sign opposite to that of the charges of the carriers the limited motions of which have produced the holes.

Therefore, if the holes are produced by limited displacements of certain electrons, the displacement of the hole is equivalent to the displacement of a positive charge. The free place is sometimes called *hole* or *lacuna* (pl. lacunae), as it is produced by the displacement of an electron or of an ion, respectively; however this distinction is not compulsory, moreover it is rarely used.

The *conduction electric current* in a body is represented by the oriented motion of free electrons with respect to a reference frame attached (fixed) to that body.

According to the nature of the electric charge carriers (electrons, ions, holes), the conduction electric current may be referred to as *electronic current*, *ion current*, *hole current*.

Correspondingly, the conduction of the body under consideration is called electronic conduction (e.g., for certain metals and semiconductors of N-type), ionic conduction (e.g., for electrolytes), and hole conduction (e.g., for semiconductors of P-type).

The macroscopic current density and the macroscopic current intensity can be expressed by relations (1.66), (1.67).

1.10.3. Convection Electric Current

The *convection electric current* is represented by the motion of electrically charged bodies with respect to a reference frame. Here, the electric current produced by the motion of a great number of electrically charged particles (for instance a flux of electrons or of protons) in empty space is included.

The macroscopic current density and the intensity of the convection electric current can also be expressed by relations (1.66) and (1.67).

1.10.4.   Polarization Electric Current

The *polarization electric current* is represented by the motion of bound electric charges with respect to the centre of mass of atoms and molecules. Further, the macroscopic density and intensity of the polarization electric current will be calculated.

An open surface S_Γ bounded by a simple curve Γ, both situated within a substance electrically polarized, will be considered. In the case of a variation of the electric field intensity, under the action of the forces determined by the electric field, the bound electric charge carriers will change their position. Therefore, electric charge carriers pass through the open surface S_Γ, to very small distances. Hence, an electric current passes through the surface, and it is called *polarization electric current*.

The charges that produce the polarization electric current are just the charges that produce the electric polarization.

At a point of the surface S_Γ, we shall consider the physically infinitesimal volume Δv having the centre of mass at that point and containing a neutral system of electric charges.

The electric polarization, in any reference frame, with respect to which the substance is at rest, is defined by relation (1.36 a) as:

$$ \boldsymbol{P} = \frac{1}{\Delta v} \left\langle \sum_{i=1}^{n} q_{pi}\, \boldsymbol{r}_i \right\rangle_t , \tag{1.72} $$

where the summation refers to all electric charges which form neutral systems within the physically infinitesimal volume Δv and the average value with respect to time is taken over a physically infinitesimal time interval. The quantities $\boldsymbol{r}_i$ represent the position vector in the considered reference frame. It is necessary to be mentioned that the position vector of the centre of mass of the substance in the volume element Δv remains unchanged even when the electric charge carriers change their position (because the positive and negative charges move in opposite directions). Also, the mean concentration of the bound carriers remains constant. At a modification of the electric field strength, as mentioned above, the positions of the electric bound charge carriers (the polarization charges) in the volume element Δv change. It follows:

$$ \frac{\partial \boldsymbol{P}}{\partial t} = \frac{1}{\Delta v} \left\langle \sum_{i=1}^{n} q_{pi}\, \frac{\partial \boldsymbol{r}_i}{\partial t} \right\rangle_t , \tag{1.73 a} $$

thus:

$$ \frac{\partial \boldsymbol{P}}{\partial t} = \frac{1}{\Delta v} \left\langle \sum_{i=1}^{n} q_{pi}\, \boldsymbol{v}_i \right\rangle_t . \tag{1.73 b} $$

The *macroscopic density of the polarization electric current* may be written:

$$ \boldsymbol{J}_P = \frac{1}{\Delta v} \left\langle \sum_{i=1}^{n} q_{pi}\, \boldsymbol{v}_i \right\rangle_t = \frac{\partial \boldsymbol{P}}{\partial t}, \tag{1.74} $$

and the intensity of the electric current through the surface S_Γ is:

$$i_P = \int\limits_{S_\Gamma} \boldsymbol{J}_P \cdot \mathrm{d}\,\boldsymbol{S} = \int\limits_{S_\Gamma} \frac{\partial \boldsymbol{P}}{\partial t} \cdot \mathrm{d}\,\boldsymbol{S}. \tag{1.75}$$

It is possible to obtain relation (1.75) in another way that permits the visualization of the phenomenon.

Therefore, we shall anew consider a substance electrically polarized and within this substance an open surface S_Γ bounded by a closed curve Γ.
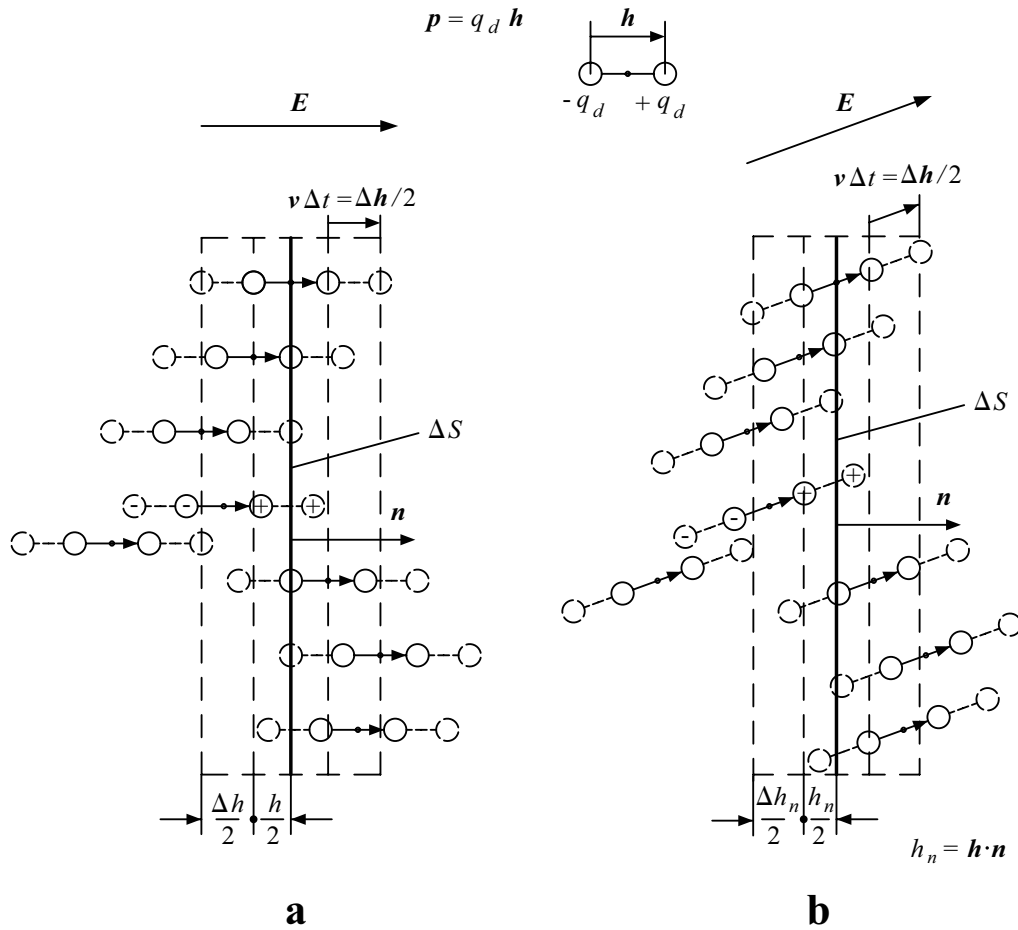
Fig. 1.13. Explanation of the polarization electric current produced by polarization electric charges (bound charges) that pass through a surface, by considering two positions **a** and **b** of bound electric dipoles. The centre of each dipole is marked by a dot. The dimension of Δh is magnified.

At a variation of the electric field strength, although the substance remains at rest with respect to the reference frame, the bound electric charges modify their position, passing through the surface S_Γ and therefore produce an electric current. We shall express the intensity of the electric current in terms of the electric polarization.

For this purpose, we consider an element $\Delta \mathbf{S}$ of the surface S_Γ. We also assume that the various multipoles are replaced by equivalent dipoles, all identical to each other (Fig. 1.13). The centre of each dipole is marked in the figure by a dot. This point remains at rest even when the bound electric charges move, since the substance is considered at rest.

We shall denote by n_p the concentration of dipoles (i.e., the number of dipoles per unit of volume) and by q_d the positive charge of the dipole. It follows that under the action of the electric field supposed as increasing in the direction of the arrow (Fig. 1.13), the electric bound charges will move by the distance $\frac{1}{2}\Delta h$ in the time interval Δt with the velocity $\frac{1}{2}\Delta h / \Delta t$. Therefore, the dipoles will be elongated.

We shall consider, on the left side of the surface element ΔS, the parallelepiped having one base situated at the distance $\frac{1}{2}h$ from the surface element ΔS and the height $\frac{1}{2}\Delta h$ oriented towards the left side. The positive electric charges belonging to dipoles that have their centres within the mentioned parallelepiped will pass through the surface ΔS towards the right side with the velocity $\mathbf{v}_p$. The positive charge of each dipole which has not its centre within the mentioned parallelepiped will not pass through the surface in the time interval Δt, and hence will not contribute in producing an electric current through that surface. Analogously, the negative electric charges from the right side of the surface ΔS will pass through the same surface towards the left side.

It results that through the surface $\Delta \mathbf{S} = \mathbf{n}\,\Delta S$, the following electric current produced by positive bound charges will pass in the interval of time Δt:

$$\Delta i_{\text{pos}} = n_p\, q_d\, \mathbf{v}_p \cdot \Delta \mathbf{S};$$

$$\mathbf{v}_p = \mathbf{v} = \frac{1}{2} \cdot \frac{\partial \mathbf{h}}{\partial t}. \tag{1.76}$$

For the negative charges, a similar expression can be obtained. It has to be taken into account that the passage of an electric current produced by a number of negative charges from the right to the left is equivalent to the passage of an electric current of an equal number of positive charges from the left to the right. It follows that the resultant polarization electric current through the surface $\Delta \mathbf{S}$, denoted by one of the symbols, i_P, i_p, is:

$$\Delta i_P = \left\langle n_p\, q_d\, \frac{\partial \mathbf{h}}{\partial t} \cdot \Delta \mathbf{S} \right\rangle_t. \tag{1.77}$$

Taking into account that the concentration of dipoles does not change because the centres of the dipoles remain at rest, it follows:

$$\Delta i_P = \left\langle \frac{\partial}{\partial t}\left(n_p\, q_d\, \boldsymbol{h}\right)\cdot \Delta \boldsymbol{S} \right\rangle_t = \frac{\partial \boldsymbol{P}}{\partial t}\cdot \Delta \boldsymbol{S}. \tag{1.78}$$

From relation (1.78), it follows that:

$$i_P = \int\limits_{S_\Gamma} \boldsymbol{J}_P \cdot \mathrm{d}\,\boldsymbol{S} = \int\limits_{S_\Gamma} \frac{\partial \boldsymbol{P}}{\partial t}\cdot \mathrm{d}\,\boldsymbol{S}. \tag{1.78 a}$$

The same result will be obtained if it is considered that a rotation of dipoles is produced under the action of the external electric field.

From the explanation above, it follows that the polarization current cannot be a steady-state direct current, since the displacement of bound charges is limited. Only polarization current varying with time can exist.

1.10.5.   Amperian Electric Current (Molecular Electric Current)

The motion of an electric charge carrier along a closed way of sub-molecular dimensions is referred to as *molecular* or *Amperian electric current*. The concept of molecular current was introduced by Ampère for the explanation of the magnetization of bodies based on electric currents, that is why the molecular currents are also called Amperian currents.

The following simple models can be considered. The Amperian (molecular) electric currents are essentially represented by the *orbital* and *spin motion* of electrons. Indeed, each electron describes an orbital trajectory about the nucleus of the atom. The orbital trajectory is assumed as having the shape of a circumference, or more precisely, that of an ellipse. Moreover, it is also assumed that each electron turns about an axis passing through its centre. This rotation may be of any sense, for some electrons, e.g., clockwise, of opposite sense, i.e., counter-clockwise, for the others. This electron movement is called spin motion.

Further on, the density and the intensity of the molecular current will be calculated. Details on the motion of electrons and more precise models can be found in works devoted to this subject. Certain details are given in the work [26]. For the sake of simplicity, all Amperian electric current will be supposed identical to each other having the intensity i_Amper and a circular trajectory of radius r_0 and normal $\boldsymbol{n}$.

Within a substance in which there are Amperian (molecular) electric currents, we consider (imagine) an open surface S_Γ bounded by a closed curve Γ (Fig. 1.14).

We shall calculate the *macroscopic intensity of the Amperian electric current* through this surface. The electric current intensity is:

$$i_M = \sum\limits_{S_\Gamma} i_\text{Amper}, \tag{1.79}$$

where the summation refers to all Amperian currents the trajectories of which intersect the surface S_Γ.

It is worth remarking that the intensity of a microscopic current occurs in the calculation of the sum of relation (1.79) only if that current crosses once the surface S_Γ (if it crossed twice the surface, its contribution to the total current would be zero). Hence, only the Amperian currents that link the curve Γ contribute to the sum. We shall calculate the macroscopic Amperian current given by the sum:

$$i_M = \sum_\Gamma i_{\text{Amper}}, \tag{1.80}$$

where the summation refers to all Amperian currents that link the curve Γ.

Let us consider that along a short portion Δl of the curve Γ, the orbits of the Amperian currents are parallel with each other and their surfaces perpendicular to the considered portion of the curve Γ. The Amperian current concentration (i.e., the number of Amperian currents per unit of volume) in this region will be denoted by n_0. The Amperian currents that link the arc Δl of the curve are contained inside a right cylinder having as axis the mentioned segment Δl approximated by a straight-line segment, and the diameter $2r_0$, where r_0 is the radius of the orbit of an Amperian current. Thus:

$$\sum_{\Delta l} i_{\text{Amper}} = n_0 \, i_{\text{Amper}} \, S_0 \, \Delta l, \tag{1.81}$$

where S_0 denotes both the surface of the orbit of an Amperian current and the area of the cross-section of the cylinder above. If the normal $\boldsymbol{n}$ to the surface of the orbit of the
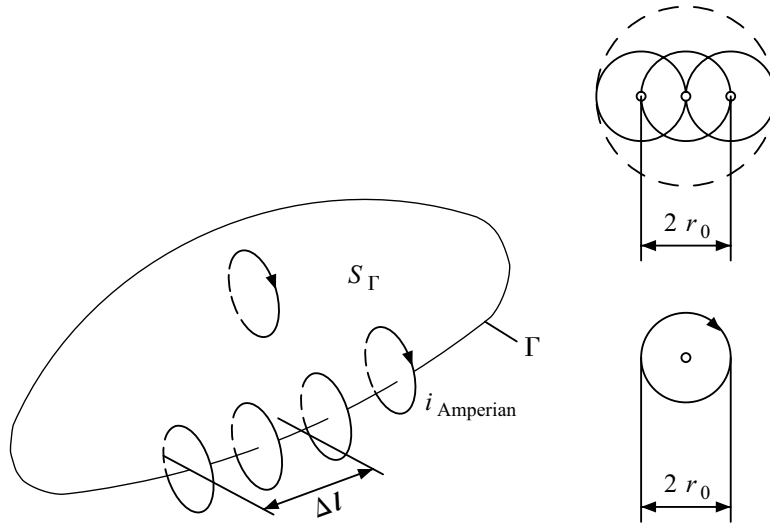
Fig. 1.14. Orbits of the Amperian (molecular) currents and the surface through which macroscopic electric currents are passing. For the sake of clearness, the orbits are magnified.

Amperian current forms a certain angle with the arc element $\Delta \boldsymbol{l}$, it follows:

$$\sum_{\Delta l} i_{\text{Amper}} = n_0 \, i_{\text{Amper}} \, \boldsymbol{S}_0 \cdot \Delta \boldsymbol{l}. \tag{1.81 a}$$

The calculation performed for a single curve Γ can be repeated for several closed curves very closed to each other. These curves form a bundle that occupies the domain in the form of a closed tube (pipe). The cross-section of this tube must be small enough, so that together with the line element $\Delta \boldsymbol{l}$, it shows a physically infinitesimal volume Δv.

We shall denote:

$$\boldsymbol{m}_0 = i_{\text{Amper}} \, \boldsymbol{S}_0 \,, \tag{1.82}$$

quantity called *magnetic moment of an Amperian current* or *Amperian magnetic moment*, and:

$$\boldsymbol{M} = \frac{1}{\Delta v} \left\langle \sum_{\Delta v} \boldsymbol{m}_0 \right\rangle_{\text{t}} = n_0 \, \boldsymbol{m}_0 \,, \tag{1.83}$$

where Δv represents the physically infinitesimal volume. The sum of magnetic moments of Amperian currents of a volume of any substance may be different from zero. The state of the substance in this case is referred to as *magnetization state*. From a microscopic point of view, the magnetic moment is a secondary (derived) quantity. From a macroscopic point of view, the magnetic moment has to be introduced experimentally as a primitive (fundamental) quantity. The manner of introducing this quantity is based on the ponderomotive forces acting on a magnetized small body situated in a hollow cavity within the substance. The vector quantity $\boldsymbol{M}$ defined by relation (1.83) is called *magnetization*. In the case of a uniform distribution, it represents, the average value, over a physically infinitesimal time interval, of the vector sum of magnetic moments of a unit of volume. Generally, it represents the ratio of the sum of magnetic moments, of a physically infinitesimal volume, to the magnitude of this volume.

Instead of the vector quantities, Amperian magnetic moment $\boldsymbol{m}$ and magnetization $\boldsymbol{M}$, corresponding to expressions (1.82) and (1.83), there are also used the quantities *Coulombian magnetic moment* $\boldsymbol{m}_j$ and *magnetic polarization* $\boldsymbol{M}_j$ defined by the following expressions that contain the constant μ_0:

$$\boldsymbol{m}_{0j} = \mu_0 \, \boldsymbol{m}_0 \,, \tag{1.84}$$

$$\boldsymbol{M}_j = \mu_0 \, \boldsymbol{M} \,, \tag{1.85}$$

where the quantity μ_0 represents a constant called the *magnetic constant* also termed *permeability of vacuum*, which will be explained in Section 2.4.

The quantity $\boldsymbol{M}_j$ is denoted, according to [44], by the symbol $\boldsymbol{J}$ but we shall not use it, in order to avoid any confusion, because in this work the electric current density denoted by the same symbol also occurs. Another symbol for denoting the same quantity, according to [44], is $\boldsymbol{B}_i$ and it is called *intrinsic magnetic induction*.

If the quantities $\boldsymbol{m}_j$ and $\boldsymbol{M}_j$ are used, then certain relations, referring to the magnetic polarization state, show a symmetrical form with the corresponding relations referring to the electric polarization state.

Correspondingly, we obtain the *volume density of the fictive (fictitious) magnetic charge*:

$$\rho_{Mv} = -\operatorname{div}\boldsymbol{M}, \tag{1.86}$$

called *volume density of the (fictive) magnetization charge*, and

$$\rho_{mv} = -\operatorname{div}\boldsymbol{M}_j, \tag{1.87}$$

called *volume density of the (fictive) magnetic polarization charge*.

The relations (1.81), (1.83) yield:

$$\sum_{\Gamma} i_{\text{Amper}} = \oint_{\Gamma}\boldsymbol{M}\cdot\mathrm{d}\boldsymbol{l}, \tag{1.88}$$

and taking into account relation (1.80), it results that:

$$i_M = \oint_{\Gamma}\boldsymbol{M}\cdot\mathrm{d}\boldsymbol{l}. \tag{1.89}$$

The *macroscopic density of the macroscopic Amperian electric current*, denoted by one of the symbols $\boldsymbol{J}_A$, $\boldsymbol{J}_a$, $\boldsymbol{J}_M$, $\boldsymbol{J}_m$, can be obtained by using the Stokes theorem as follows:

$$i_M = \int_{S_\Gamma}\boldsymbol{J}_A\cdot\mathrm{d}\boldsymbol{S} = \oint_{\Gamma}\boldsymbol{M}\cdot\mathrm{d}\boldsymbol{l} = \int_{S_\Gamma}\operatorname{curl}\boldsymbol{M}\cdot\mathrm{d}\boldsymbol{S}. \tag{1.90}$$

It results that:

$$\boldsymbol{J}_A = \boldsymbol{J}_a = \operatorname{curl}\boldsymbol{M} = \frac{1}{\mu_0}\operatorname{curl}\boldsymbol{M}_j. \tag{1.91}$$

The magnetization $\boldsymbol{M}$ can be decomposed into the sum of two components. The first one exists even in the absence of external causes and is determined by the nature of the substance. This component is called *permanent magnetization* and it is denoted by $\boldsymbol{M}_p$.

The second one appears only under certain external causes (e.g., a magnetic field) and is determined by these causes and the nature of the substance. This component is called *temporary magnetization* and it is denoted by $\boldsymbol{M}_t$. Therefore, for the components of the vectors magnetization and magnetic polarization, respectively, we obtain:

$$\boldsymbol{M} = \boldsymbol{M}_p + \boldsymbol{M}_t, \tag{1.92}$$

$$\boldsymbol{M}_j = \boldsymbol{M}_{jp} + \boldsymbol{M}_{jt}. \tag{1.93}$$

In certain situations, each of the two components above may be zero.

1.11. LAW OF FREE (TRUE) ELECTRIC CHARGE CONSERVATION

1.11.1. Integral Form of the Law

Let us consider any closed surface Σ passing only through insulating materials, so that the surface is not crossed by electric currents, then the total electric free charge of the interior of the surface Σ remains constant:

$$q_\Sigma = \text{const}. \tag{1.94}$$

This relation, called *law of electric charge conservation*, is valid regardless of the phenomena produced in the interior of the surface Σ.

If an electric conduction current of intensity i_Σ passes through the closed surface Σ, leaving this surface, then the integral form of the law is modified as follows:

$$i_\Sigma = -\frac{\mathrm{d}\,q_\Sigma}{\mathrm{d}\,t}. \tag{1.95}$$

This last expression of the integral form of the *law of electric charge conservation* also follows from the magnetic circuital law subsequently treated; however the first integral form of the law (1.94) does not result from the magnetic circuital law.

1.11.2. Local Form of the Law

The relation (1.95) can be modified as follows:

$$\int_\Sigma \boldsymbol{J} \cdot \mathrm{d}\,\boldsymbol{S} = -\frac{\mathrm{d}}{\mathrm{d}\,t} \int_{V_\Sigma} \rho_v \, \mathrm{d}\,v, \tag{1.96}$$

where the quantity $\boldsymbol{J}$ represents the density of the current passing through the surface Σ and the quantity ρ_v represents the volume density of the electric free charge.

Two cases will be examined: Surface Σ at rest with respect to the reference frame; surface Σ in motion with respect to the reference frame.

1° Surface at Rest Relatively to the Reference Frame

In a reference frame at rest with respect to the surface Σ, the operation of differentiation can be introduced under the integral sign and it follows:

$$\int_\Sigma \boldsymbol{J} \cdot \mathrm{d}\,\boldsymbol{S} = -\int_{V_\Sigma} \frac{\partial}{\partial t} \rho_v \, \mathrm{d}\,v. \tag{1.97}$$

By transforming the surface integral into a volume integral (Gauss-Ostrogradski theorem), we obtain:

$$\int_{V_\Sigma} \mathrm{div}\,\boldsymbol{J} \cdot \mathrm{d}\,\boldsymbol{S} = -\int_{V_\Sigma} \frac{\partial}{\partial t} \rho_v \, \mathrm{d}\,v. \tag{1.97 a}$$

Since the last relation is valid for any surface Σ, it results that:

$$\operatorname{div} \boldsymbol{J} = -\frac{\partial \rho_v}{\partial t}, \tag{1.98 a}$$

or

$$\frac{\partial \rho_v}{\partial t} + \operatorname{div} \boldsymbol{J} = 0. \tag{1.98 b}$$

Relation (1.98 b) represents the local form of the law of free (true) electric charge constancy for a medium to which the surface Σ is attached (fixed) and that is simultaneously at rest with respect to the reference frame.

2° Surface in Motion with Respect to the Reference Frame

If the surface Σ is moving with respect to the reference frame and the various points of this surface have any velocity $\boldsymbol{v}_r$ with respect to the reference frame, then when calculating the derivative:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{V_\Sigma} \rho_v \, \mathrm{d}v, \tag{1.99}$$

it is necessary to take into account that two terms will occur: One term is obtained supposing that the surface is at rest, and the quantity ρ_v varies with time; the second term is obtained supposing that the surface is moving and the quantity ρ_v does not vary with time, i.e., the procedure is the same as for the calculation of the derivative of composed functions.

The first term is:

$$\int_{V_\Sigma} \frac{\partial \rho_v}{\partial t} \, \mathrm{d}v. \tag{1.100 a}$$

The second term is given by the relation:

$$\frac{1}{\Delta t} \left[\int_{V_{\Sigma_2}} \rho_v \, \mathrm{d}v - \int_{V_{\Sigma_1}} \rho_v \, \mathrm{d}v \right], \tag{1.100 b}$$

where $\Sigma_1 = \Sigma$ and Σ_2 represent the two consecutive positions of the surface Σ (Fig. 1.15), and V_{Σ_1} and V_{Σ_2} are the domains (volumes) bounded by the surfaces Σ_1 and Σ_2, respectively. The surface Σ_1 denotes the first position and the surface Σ_2 denotes the second consecutive position after the displacement of every point of the surface Σ with the segment $\boldsymbol{v}_r \cdot \Delta t$.

Considering Fig. 1.15, it follows that the variation of the integral expressed by relation (100 b) is given by the integral:

$$\int_{V_{\Sigma_d}} \rho_v \, \mathrm{d}v, \tag{1.101}$$

that has to be calculated over the domain (volume) bounded by the closed surface Σ_d. The closed surface Σ_d is determined by the two consecutive positions Σ_1 and Σ_2 of the closed surfaces. Fig. 1.15 shows an element of the domain bounded by the surface Σ_d. This element is a curvilinear parallelepiped. From the examination of Fig.1.15, it follows:

$$\Delta v = \Delta \boldsymbol{S} \cdot \boldsymbol{v}_r \, \Delta t. \tag{1.102}$$

By replacing the volume element in expression (1.101) and integrating over all elements $\mathrm{d}\,\boldsymbol{S}$, hence over the whole surface Σ_d, we obtain:

$$\int_{V_{\Sigma_d}} \rho_v \, \mathrm{d}v = \int_{\Sigma_d} \rho_v \left(\mathrm{d}\,\boldsymbol{S} \right) \cdot \boldsymbol{v}_r \, \Delta t. \tag{1.103}$$

Therefore, the derivative of the considered integral is:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{V_\Sigma} \rho_v \, \mathrm{d}v = \int_{V_\Sigma} \frac{\partial \rho_v}{\partial t} \, \mathrm{d}v + \int_\Sigma \rho_v \, \boldsymbol{v}_r \cdot \mathrm{d}\,\boldsymbol{S}, \tag{1.104}$$

and by replacing in relation (1.96), it follows:

$$\int_\Sigma \boldsymbol{J} \cdot \mathrm{d}\,\boldsymbol{S} = -\int_{V_\Sigma} \frac{\partial \rho_v}{\partial t} \, \mathrm{d}v - \int_\Sigma \rho_v \, \boldsymbol{v}_r \cdot \mathrm{d}\,\boldsymbol{S}. \tag{1.105}$$

By transforming the surface integrals into volume integrals, the last relation becomes:

$$\int_{V_\Sigma} \operatorname{div} \boldsymbol{J} \, \mathrm{d}v = -\int_{V_\Sigma} \frac{\partial \rho_v}{\partial t} \, \mathrm{d}v - \int_{V_\Sigma} \operatorname{div}\left(\rho_v \, \boldsymbol{v}_r \right) \mathrm{d}v. \tag{1.106}$$

Since the last relation should be valid for any surface Σ, it results that:

$$\operatorname{div} \boldsymbol{J} = -\frac{\partial \rho_v}{\partial t} - \operatorname{div}\left(\rho_v \, \boldsymbol{v}_r \right). \tag{1.106 a}$$
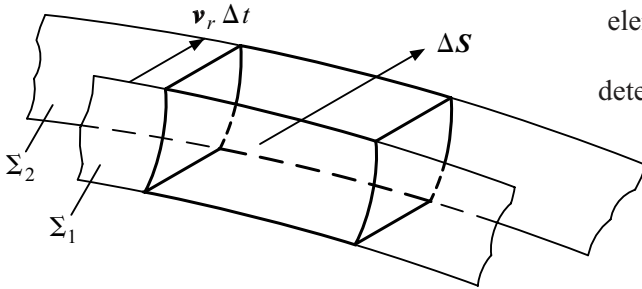
Fig. 1.15. The displacement of an element of the surface Σ in the two consecutive positions and determining the surfaces Σ_1 and Σ_2.

The last relation yields:

$$\frac{\partial \rho_v}{\partial t} + \mathrm{div}(\boldsymbol{J} + \rho_v\, \boldsymbol{v}_r) = 0, \tag{1.107}$$

that represents the local form, i.e., the differential form, for a medium (to which the surface Σ is attached) in motion with the velocity $\boldsymbol{v}_r$ with respect to the reference frame.

The quantity $\boldsymbol{J}$ represents the current density in a reference frame attached (fixed) to an element of the surface Σ and the derivative $\dfrac{\partial \rho_v}{\partial t}$ is expressed in the reference frame with respect to which the surface Σ is moving. Thus $\boldsymbol{J} + \rho_v\, \boldsymbol{v}_r$ represents the current density with respect to the reference frame at rest, relatively to which the surface Σ is moving. With this remark, relation (1.107) can be obtained directly from relation (1. 98 b) by replacing, correspondingly, the electric current density.

1.12. THE LAW OF ELECTRIC CONDUCTION. THE LOCAL FORM.

The magnitude of the density of the conduction electric current at a point of any medium depends on the strength of the electric field in the large sense at this point, and on the nature of the medium.

In the case of any isotropic medium, the density vector of the electric current, at a point, has the same direction as the vector of the electric field strength at the same point, and is given by the relation:

$$\boldsymbol{J} = \sigma\, \boldsymbol{E}_l, \tag{1.108}$$

where σ is a positive quantity. The quantity σ is called *electric conductivity* and is depending on the medium nature, thus a material quantity. The quantity:

$$\rho = \frac{1}{\sigma}, \tag{1.109}$$

is called *electric resistivity*.

Macroscopically, relation (1.108) can be obtained only in an experimental way. In the macroscopic theory, this relation is a *material law*, also termed *constitutive law*, and is referred to as the *electric conduction law* or the *general Ohm law*. In metallic conductors, the electric resistivity depends, generally, on temperature.

The expression of the electric resistivity can be expanded in a Taylor series in terms of temperature:

$$\rho_\theta = \rho(\theta) = \rho(\theta_0 + \Delta\theta) = \rho(\theta_0) + \frac{1}{1!}\left[\frac{\partial \rho(\theta)}{\partial \theta}\right]_{\theta_0}(\Delta\theta) + \frac{1}{2!}\left[\frac{\partial^2 \rho(\theta)}{\partial \theta^2}\right]_{\theta_0}(\Delta\theta)^2 + \ldots.$$

$$\tag{1.109 a}$$

For relatively small variation, by neglecting the small terms of higher order, the dependence is a linear one:

$$\rho_\theta = \rho_0 \left[1 + \alpha_0 \left(\theta - \theta_0\right)\right], \tag{1.110}$$

$$\alpha_0 = \frac{1}{\rho_0} \left(\frac{\partial \rho}{\partial \theta}\right)_{\theta_0}, \tag{1.110 a}$$

where the following symbols have been used:

ρ_θ −   electric resistivity at temperature θ;

ρ_0 −   electric resistivity at temperature θ_0;

α_0 −   raising coefficient of resistivity with temperature.

The values of the resistivity and of the raising coefficient of resistivity with temperature are given, according to the experimental results, in tables.

The resistivity of a metal usually increases with temperature, even during the metal melting. However, certain metals like bismuth, antimony, gallium show a decrease of their density during their melting, and their resistivity diminishes during their melting. In the microscopic theory of the electromagnetic field, relations (1.108) and (1.110) can be deduced starting from the general law of the electromagnetic field.

In the case of anisotropic media, the direction of the vector J is not the same with that of the vector E_l. However, in these media, it is possible to determine a system of three-orthogonal axes called principal axes, along which, the anisotropic medium behaves like an isotropic one. In the case of these media, the electric conduction law, in the local form, can be written in the form:

$$J = \sigma E_l, \tag{1.111}$$

where the quantity σ is the tensor of the electric conductivity. Relations (1.108) and (1.111) have a general character being valid in the cases when the quantities varies with time.

In dielectrics (substances with bad conducting properties) and in semiconductors, generally, the dependence between the density of the conduction electric current and the electric field strength is not a linear one. However, in these cases, the same manner of expressing the law is used. Then, the expression of the law is equivalent to the expression, for each considered substance, of the dependence $J = f(E_l)$ and $E_l = f(J)$.

The law is also applicable, with a good approximation, to electrolytes.

In the framework of the microscopic theory, using a simplified model, the relation (1.108), can be derived as follows. The electric current flowing in a body is produced by the oriented displacement of electric charge carriers under the action of external forces. The collision of the charge carriers with the ions and atoms of the body opposes to the oriented displacement of the charge carriers. Hence, the substance of the body presents a certain resistivity to the electric current. The relation (1.108) that expresses this phenomenon can be deduced directly from the equation of motion of the electric charge carriers moving under the action of the forces which occur [23, Vol. II, p. 30].

1.13. THE ELECTRIC FIELD STRENGTH OF ELECTRIC CHARGES *IN VACUO.* ELECTRIC CONSTANT (ELECTRIC PERMITTIVITY OF VACUUM).

1.13.1. COULOMB Formula

Let us consider two small bodies indicated by numbers 1 and 2, with the electric charges q_1 and q_2 respectively, at rest in a reference frame and situated at points A and P at a relatively great distance (compared with the dimensions of each body) from each other. Therefore, the charges q_1 and q_2 can be referred to as point-like charges.

The vector quantity $\boldsymbol{r}_{12}$ represents the distance between the two bodies, oriented from the body 1 towards the body 2, like in Fig. 1.16.

The force exerted upon the body 2 with the electric charge q_2 due to the body with the electric charge q_1 is given by the expression:

$$\boldsymbol{F}_{21} = k_0 \frac{q_1 q_2}{r_{12}^2} \cdot \frac{\boldsymbol{r}_{12}}{r_{12}}, \tag{1.112}$$

that represents *the Coulomb formula* in the case of empty space (i.e., vacuum).

At the time when it was experimentally established, relation (1.112) had a very general character and has been referred to as *the Coulomb law*. At present, it is also called *the Coulomb theorem*. From expression (1.112) and Fig. 1.16, it results that the direction of the force is so that the charges of the same sign repel each other, and the charges of opposite sign attract each other. The Coulomb formula can be used for introducing the notion of true (free) electric charge.

The quantity k_0 is a *universal constant* referring to vacuum (free space, empty space), i.e., a physical quantity independent of the nature of the bodies under consideration and of all other physical quantities, dependent on the chosen system of units.

In the SI system of units, which is a rationalized system, instead of the constant k_0 the quantity ε_0 is used, and the relation between the two quantities is:

$$k_0 = \frac{1}{4 \pi \varepsilon_0}. \tag{1.113}$$

The quantity ε_0 is also a *universal constant* referring to vacuum and is called *permittivity of vacuum* or *electric constant*, and in the SI system has the value:
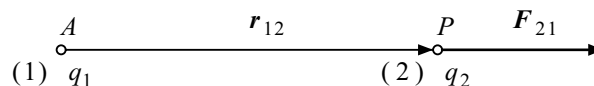
Fig. 1.16. Explanation concerning the Coulomb formula.

$$\varepsilon_0 = \frac{1}{4\pi 9 \cdot 10^9} \, \text{F/m}, \tag{1.114}$$

expressed in farad / metre (symbol F / m); the units of measure will be explained in Sub-section 3.13.3.

If instead of relation (1.114), other values are adopted for the constant of relation (1.112), other systems of units of measure of the electromagnetic quantities will be obtained. The expression of the Coulomb formula, *in vacuo*, in the SI system of units is:

$$\boldsymbol{F}_{21} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_1 q_2}{r_{12}^2} \cdot \frac{\boldsymbol{r}_{12}}{r_{12}}. \tag{1.115}$$

As mentioned in Sub-section 1.6.2, the unit of measure of electric charge is called *coulomb*. According to relation (1.115), the coulomb is the value of the point-like charge that exerts a force of $9 \cdot 10^9$ N upon another identical point-like charge placed at the distance of 1 m from the first point-like charge and very far from any other body.

According to relation (1.17), it results that the electric field strength produced by the charge q_1 at the point 2 (Fig. 1.16) has the expression:

$$\boldsymbol{E}_{21} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_1}{r_{12}^2} \cdot \frac{\boldsymbol{r}_{12}}{r_{12}}. \tag{1.116}$$

Relation (1.116) is also termed Coulomb formula, relation or theorem, respectively.

1.13.2.   Utilization of the Principle of Superposition

For obtaining the force acting on a point-like charge, in many cases it is useful to apply the principle of superposition. Therefore, for obtaining the force acting upon a point-like charge q, due to other point-like charges $q_i \, (i = 1, \ldots, n)$, the forces corresponding to the pairs of charges $q, q_1; \; q, q_2; \; \ldots; \; q, q_n$ will be calculated separately and the forces $\boldsymbol{F}_1, \boldsymbol{F}_2, \ldots, \boldsymbol{F}_n$ will be obtained. The resultant force exerted upon the point-like charge q situated at any point A is:

$$\boldsymbol{F} = \sum_{i=1}^{n} \boldsymbol{F}_i. \tag{1.117}$$

Correspondingly, the electric field strength at point A produced by the point-like electric charges $q_i \, (i = 1, \ldots, n)$, the charge q being absent, is:

$$\boldsymbol{E} = \sum_{i=1}^{n} \boldsymbol{E}_i. \tag{1.118}$$

1.13.3.   The Electric Potential Produced by Electric Charges at Rest

We shall calculate the electric field strength produced at any point N, called *observation point* or *field point*, by whatever point-like charge denoted here by q that is placed at any point M called *source point*. We shall denote $\boldsymbol{r} = \overrightarrow{MN}$. The electric field strength at the point N is given by relation (1.116) and can be written in the form:

$$\boldsymbol{E} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r^2} \cdot \frac{\boldsymbol{r}}{r} = -\operatorname{grad}\left(\frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r}\right) = -\operatorname{grad} V, \qquad (1.119)$$

where the following symbol has been used:

$$V = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r} + C, \qquad (1.120)$$

where C is an arbitrary constant. The scalar quantity V is called *potential* and because, in this case, it is produced by electric charges at rest, it is termed *electrostatic potential*. The minus sign in the right-hand side of relation (1.119) appears due to the following convention accepted in the theory of the electromagnetic field. According to this convention, the displacement direction of the particles with positive electric charge is oriented from the regions of a higher potential towards the regions of a lower potential.

Hence, the direction of the electric field strength vector and the direction of the lines of electric field are oriented from the regions of a higher potential towards the regions of a lower potential. This direction is opposite to that of the vector $\operatorname{grad} V$, since the vector $\operatorname{grad} V$ is oriented along the direction along which V increases most rapidly.

According to the principle of superposition, the electric field strength produced at any point N, by n point-like electric charges q_i $(i = 1, \ldots, n)$, each of them placed at the point M_i, is:

$$\boldsymbol{E} = \sum_{i=1}^{n} \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_i}{r_i^2} \cdot \frac{\boldsymbol{r}_i}{r_i} = \sum_{i=1}^{n} -\operatorname{grad}\left(\frac{1}{4\pi\varepsilon_0} \cdot \frac{q_i}{r_i}\right), \qquad (1.121)$$

where $\boldsymbol{r}_i = \overrightarrow{M_i N}$, and the potential is:

$$V = \sum_{i=1}^{n} V_i = \sum_{i=1}^{n} \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_i}{r_i} + C, \qquad (1.122)$$

where:

$$V_i = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_i}{r_i} + C_i. \qquad (1.122\text{ a})$$

If besides the point-like electric charges it is a *distributed electric charge*, then, by decomposing the electric charge distribution in elementary electric charges, the distribution of electric charges may be replaced by *sets of point-like charges* and relations of the form (1.121) and (1.122) can be applied.

In the general case, in which in the whole infinite empty space (i.e., *in vacuo*) there are n point-like electric charges and volume, surface and line electric charge distributions, the expression of the potential can be written:

$$V = \frac{1}{4\pi\varepsilon_0}\left(\sum_{i=1}^{n}\frac{q_i}{r_i} + \int_{V_q}\frac{\rho_v}{r}\,\mathrm{d}v + \int_{S_q}\frac{\rho_s}{r}\,\mathrm{d}S + \int_{C_q}\frac{\rho_l}{r}\,\mathrm{d}l\right) + C, \tag{1.123}$$

and the electric field strength is:

$$\boldsymbol{E} = -\mathrm{grad}\,V = \frac{1}{4\pi\varepsilon_0}\left(\sum_{i=1}^{n}\frac{q_i}{r_i^2}\cdot\frac{\boldsymbol{r}_i}{r_i} + \int_{V_q}\frac{\rho_v}{r^2}\cdot\frac{\boldsymbol{r}}{r}\,\mathrm{d}v + \int_{S_q}\frac{\rho_s}{r^2}\cdot\frac{\boldsymbol{r}}{r}\,\mathrm{d}S + \int_{C_q}\frac{\rho_l}{r^2}\cdot\frac{\boldsymbol{r}}{r}\,\mathrm{d}l\right). \tag{1.124}$$

The electrostatic potential is a scalar function introduced in order to simplify the study of the electrostatic field and generally has not a physical interpretation, excepting the difference of the potentials at two points that can be related to a work, as it will be further explained.

Therefore, in the applications in which the distribution of electric charges *in vacuo* is given, and the calculation of the electrostatic field is required, the following procedure is possible: To calculate successively the potential by relation (1.123) and then the gradient. This way represents, in many cases, a much simpler solution than the direct calculation by relation (1.124).

The electrostatic potential is useful for the calculation of the line-integral of the electrostatic field strength between two points, referred to as: a) *potential difference* between the two points, b) *voltage* between the two points or   c) *electric* (electrostatic, in the present case) *tension* between the two points.

The expression of the line integral is:

$$U_{MN\,(C)} = \int_{M}^{N}\boldsymbol{E}\cdot\mathrm{d}\boldsymbol{l}, \tag{1.125}$$

but, in the case of the electrostatic field, we may write:

$$\boldsymbol{E} = -\mathrm{grad}\,V = -\left(\boldsymbol{i}\,\frac{\partial V}{\partial x} + \boldsymbol{j}\,\frac{\partial V}{\partial y} + \boldsymbol{k}\,\frac{\partial V}{\partial z}\right), \tag{1.125 a}$$

and

$$\mathrm{d}\boldsymbol{l} = \boldsymbol{i}\,\mathrm{d}x + \boldsymbol{j}\,\mathrm{d}y + \boldsymbol{k}\,\mathrm{d}z. \tag{1.125 b}$$

Thus:

$$\boldsymbol{E}\cdot\mathrm{d}\boldsymbol{l} = -\frac{\partial V}{\partial x}\,\mathrm{d}x - \frac{\partial V}{\partial y}\,\mathrm{d}y - \frac{\partial V}{\partial z}\,\mathrm{d}z. \tag{1.125 c}$$

It follows:

$$U_{MN(C)} = -\int_{M}^{N} \left(\frac{\partial V}{\partial x} \, d\,x + \frac{\partial V}{\partial y} \, d\,y + \frac{\partial V}{\partial z} \, d\,z \right) = -\int_{M}^{N} dV,$$

(1.126)

and taking into account that the expression under the integral sign is a total differential, it results that:

$$U_{MN(C)} = V_M - V_N.$$

(1.127)

It results that, in the case of the electrostatic field, the potential difference (voltage) between two points is independent of the curve along which it is calculated and is equal to the difference of the electrostatic potentials at the two points.

1.14. THE ELECTRIC FLUX LAW *IN VACUO*

Let us consider a closed surface Σ situated *in vacuo*, and in its interior a point-like charge q (that may be referred to as source charge). The electric field strength at a point of the surface Σ can be obtained by relation (1.119), using the same symbols, and it is given by the following expression:

$$E = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r^2} \cdot \frac{r}{r}.$$

(1.128)

We shall calculate the flux of the electric field strength vector through the surface Σ.
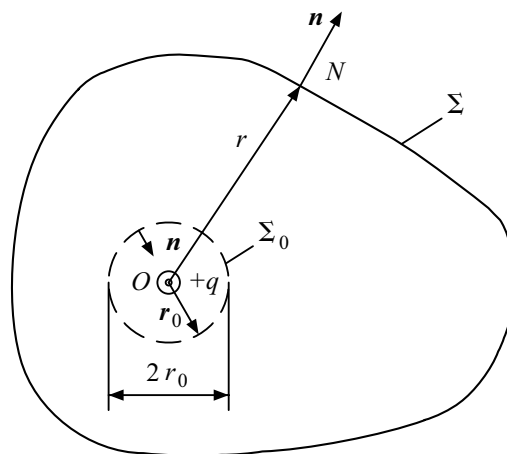
Fig. 1.17. Explanation concerning the electric flux law *in vacuo*.

We have:

$$\Psi_{\rm E} = \int_\Sigma \boldsymbol{E} \cdot {\rm d}\,\boldsymbol{S} = \int_\Sigma \frac{1}{4\,\pi\varepsilon_0} \cdot \frac{q}{r^2} \cdot \frac{\boldsymbol{r}}{r} \cdot {\rm d}\,\boldsymbol{S}. \tag{1.129}$$

We surround the point at which is the point-like charge, by a sphere of surface Σ_0 having its centre at that point and any radius r_0 (assumed generally small but this assumption is not necessary). Then, we shall consider the surface obtained by the union of surfaces Σ and Σ_0. We obtain:

$$\Sigma_{v0} = \Sigma \cup \Sigma_0. \tag{1.130}$$

We shall calculate the flux of the vector $\boldsymbol{E}$ through the surface Σ_{v0}. Having in view the positive direction of the positive normal to the last surface, as shown in Fig. 1.17, we obtain:

$$\Psi_{{\rm E}\,\Sigma_{v0}} = \int_{\Sigma_{v0}} \boldsymbol{E} \cdot {\rm d}\,\boldsymbol{S} = \int_\Sigma \boldsymbol{E} \cdot {\rm d}\,\boldsymbol{S} + \int_{\Sigma_0} \boldsymbol{E} \cdot {\rm d}\,\boldsymbol{S}. \tag{1.131}$$

For the exterior surface Σ, we have:

$$\Psi_{\rm E} = \int_\Sigma \boldsymbol{E} \cdot {\rm d}\,\boldsymbol{S}. \tag{1.131 a}$$

For the interior surface Σ_0, hence that of the sphere, we have $\boldsymbol{n} = -\dfrac{\boldsymbol{r}_0}{r_0}$, and consequently:

$$\Psi_{{\rm E}\,\Sigma_0} = \int_{\Sigma_0} \boldsymbol{E} \cdot {\rm d}\boldsymbol{S} = \int_{\Sigma_0} \boldsymbol{E} \cdot \boldsymbol{n}\,{\rm d}\,S = -\int_{\Sigma_0} \frac{1}{4\varepsilon_0} \cdot \frac{q}{r_0^2} \cdot \frac{\boldsymbol{r}_0}{r_0} \cdot \frac{\boldsymbol{r}_0}{r_0}\,{\rm d}\,S =$$

$$= -\frac{1}{4\,\pi\varepsilon_0} \cdot \frac{q}{r_0^2} \int_{\Sigma_0} {\rm d}\,S = -\frac{1}{4\,\pi\varepsilon_0} \cdot \frac{q}{r_0^2} 4\,\pi r_0^2 = -\frac{q}{\varepsilon_0}. \tag{1.131 b}$$

At the same time:

$$\Psi_{{\rm E}\,\Sigma_{v0}} = \int_{\Sigma_{v0}} \boldsymbol{E} \cdot {\rm d}\boldsymbol{S} = \int_{V_{\Sigma_{v0}}} {\rm div}\,\boldsymbol{E}\,{\rm d}\,v. \tag{1.132}$$

But in the considered domain V_{Σ_0}, which does not contain the singular point $r = 0$, we have:

$$\operatorname{div}\boldsymbol{E} = \frac{q}{4\,\pi\varepsilon_0} \operatorname{div} \frac{\boldsymbol{r}}{r^3} = 0. \tag{1.133}$$

Introducing into relation (1.131), the expressions (1.131 a) and (1.131 b), and taking into account relations (1.132) and (1.133), we obtain:

$$\int_{\Sigma} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \frac{q}{\varepsilon_0}. \tag{1.134}$$

Formula (1.134) expresses the integral form of the electric flux law *in vacuo*, in the case of a single point-like charge.

Formula (1.134) can also be directly established starting from relation (1.129), taking into account that the quantity:

$$\mathrm{d}\Omega = \frac{\boldsymbol{r}}{r^3} \cdot \mathrm{d}\boldsymbol{S} \tag{1.135}$$

represents the solid angle subtended at the point O (i.e., at $r = 0$) by the surface element $\mathrm{d}\boldsymbol{S}$. Therefore, the solid angle $\Delta\Omega$ is obtained as follows. Let O and $\Delta\boldsymbol{S}$ be the given point and surface, respectively. Let Σ_0 be a spherical surface having its centre at the point O and any radius r_0. Let us consider the conical surface obtained when a half-straight-line having its starting point at O is moving along the closed curve that bounds the surface $\Delta\boldsymbol{S}$. Let $\Delta\boldsymbol{S}_0$ be the surface determined by the intersection of the conical surface with the spherical surface. Then, the solid angle subtended at the point O by the surface $\Delta\boldsymbol{S}$ is defined as:

$$\Delta\Omega = \frac{\Delta S_0}{r_0^2}. \tag{1.136}$$

Immediately, it follows that if $\Delta\boldsymbol{S}$ is a closed surface containing the point O, then $\Delta\boldsymbol{S}_0$ will be the whole surface of the sphere Σ_0, and the solid angle will be 4π. It is the solid angle subtended at the point O by the surface of the sphere. If $\Delta\boldsymbol{S}_0$ were smaller than the whole surface of the sphere, the solid angle subtended at the point O would be less than 4π.

The concept of solid angle is an extension of the notion of plane angle, from the Plane Geometry, to the notion of solid angle, in the Solid Geometry. As shown, the definitions are analogous.

Also, as shown, if the point O is contained by the closed surface Σ, then the surface $\Delta\boldsymbol{S}$ is to be replaced by that whole surface and the solid angle subtended at the point O by the closed surface will be $\Omega = 4\pi$.

If the point O is just on the closed surface Σ, then the surface $\Delta\boldsymbol{S}$ is to be replaced by that whole surface and we shall consider the point O as tending from a neighbouring position to the final one, on the surface Σ. It results that, in this case, the solid angle subtended at the point O by the closed surface will be $\Omega = 2\pi$.

If the point O is outside the closed surface Σ, anew the surface $\Delta\boldsymbol{S}$ is to be replaced by that whole surface. In this case, the conical surface above will be tangent to the closed surface and consequently divides the whole closed surface into two parts. The value (area) of the surface on the sphere Σ_0 corresponding to those two parts will have the same absolute value but opposite signs. The different signs are determined by the

opposite directions of the positive normal to each of the two surface parts. Therefore, in this case, the solid angle subtended at the point O will be $\Omega = 0$.

It results that the flux of the vectors $\boldsymbol{E}$ through the surface element $\Delta \boldsymbol{S}$ is:

$$\int_{\Delta S} \boldsymbol{E} \cdot \mathrm{d}\,\boldsymbol{S} = \frac{q}{\varepsilon_0} \cdot \frac{\Omega}{4\pi}. \tag{1.137}$$

If inside the surface Σ, there are several point-like electric charges, then, in accordance with the principle of superposition and relation (1.118), expression (1.137) remains valid provided that, in the right-hand side, the sum of all point-like electric charges is introduced.

If inside the surface, there are distributed electric charges, then, by decomposing the electric charge distributions into elementary electric charges, the electric charge distributions can be replaced by sets of point-like electric charges and relation (1.137) remains valid.

It follows that, generally, for a closed surface Σ, the following expression can be written:

$$\int_{\Sigma} \boldsymbol{E} \cdot \mathrm{d}\,\boldsymbol{S} = \frac{q_{\Sigma}}{\varepsilon_0}, \tag{1.138}$$

or

$$\int_{\Sigma} \varepsilon_0\, \boldsymbol{E} \cdot \mathrm{d}\,\boldsymbol{S} = \int_{\Sigma} \boldsymbol{D} \cdot \mathrm{d}\,\boldsymbol{S} = q_{\Sigma}, \tag{1.139}$$

$$\boldsymbol{D} = \varepsilon_0\, \boldsymbol{E}, \tag{1.139 a}$$

where the quantity q_{Σ} represents the total free (true) electric charge of the interior of the surface Σ and $\boldsymbol{D}$ is called the electric flux density *in vacuo*.

The expressions (1.138) and (1.139) represent *the integral form of the electric flux law in vacuo*.

The electric charge of the inside of the surface Σ can be expressed in terms of the electric charge volume density:

$$q_{\Sigma} = \int_{V_{\Sigma}} \rho_v\, \mathrm{d}\,v, \tag{1.140}$$

and relation (1.139) becomes:

$$\varepsilon_0 \int_{\Sigma} \boldsymbol{E} \cdot \mathrm{d}\,\boldsymbol{S} = \int_{V_{\Sigma}} \rho_v\, \mathrm{d}\,v. \tag{1.141}$$

By the aid of the transformation relation of a volume integral into a surface integral (Gauss-Ostrogradski theorem), we obtain:

$$\varepsilon_0 \int_{V_{\Sigma}} \mathrm{div}\,\boldsymbol{E} \cdot \mathrm{d}\,v = \int_{V_{\Sigma}} \rho_v\, \mathrm{d}\,v. \tag{1.141 a}$$

Since the last relation can be written for any volume domain V_Σ, it results that:

$$\varepsilon_0 \operatorname{div} \boldsymbol{E} = \rho_v,\tag{1.142}$$

which is *the local form of the electric flux law in vacuo*.

By replacing relation (1.124) in (1.142), in the case in which only a volume electric charge distribution exists, it follows:

$$\operatorname{div} \operatorname{grad} V = -\frac{\rho_v}{\varepsilon_0}\tag{1.143 a}$$

or

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon_0}.\tag{1.143 b}$$

In the framework of the macroscopic theory, the electric flux law is a general law. In the framework of the microscopic theory, it can be derived as above, as a theorem. For this reason, the above law is also called the *electric flux theorem* as well as *the Gauss theorem*.

The integral form of the electric flux law can be expressed as follows: The flux of the electric field strength through a closed surface, situated *in vacuo*, is proportional to the sum of all electric charges belonging to the volume delimited by this surface.

In the case in which there is a single point-like electric charge, but situated on the surface Σ, then, in relation (1.138) or (1.139), the right-hand side will be multiplied by the factor $1/2$. The reason has been shown above (considering the solid angles) and it consists in the fact that instead of a sphere surrounding the point-like charge (Fig. 1.17), it is a half-sphere (hemisphere) that occurs.

The charges from the outside of the closed surface Σ do not occur in the expression of the electric flux law.

1.15.   THE SI UNITS OF: ELECTRIC CHARGE, ELECTRIC MOMENT, ELECTRIC TENSION, ELECTRIC FIELD STRENGTH, ELECTRIC CURRENT.

In the publications concerning the SI system of units, a set of definitions of the units of measure is given. For the same units, various definitions can be used as required. In the present work, each occurring unit is defined employing only relations and units previously introduced.

1.15.1.   The Unit of Electric Charge

The SI unit of electric charge may be established in accordance with the *Coulomb formula* and is called coulomb (symbol C). The electric charge of a conducting small body that exerts a force of $9 \cdot 10^9$ N upon another identical body having an identical

electric charge, *in vacuo*, at the distance of 1 m from the first body, and very far from any other bodies, is called *coulomb*. The definition of this unit has been examined in the Sub-section 1.13.1.

The units of volume, surface and line density corresponding to relations (1.7), (1.8) and (1.9), respectively, are the following: *coulomb per cubic metre* (symbol C/m^3), *coulomb per square metre* (symbol C/m^2) and *coulomb per metre* (symbol C/m).

1.15.2. The Unit of Electric Moment

The SI unit of electric moment results from relation (1.26) and is called coulomb-metre (symbol $C \cdot m$). Coulomb-metre is the electric moment of an electrically polarized small body upon which a maximum torque of $1 N \cdot m$ is exerted in an electric field having the strength of $1 V/m$. The definition of this unit has been examined in Sub-section 1.9.6.

The unit of electric induction and electric polarization is called *coulomb per square metre* (symbol C/m^2) and will be explained in Sub-section 3.13.1.

1.15.3. The Unit of Electric Tension

The SI unit of the quantities: electric potential difference, electric tension, electromotive force, and in general the integral of the electric field strength along a curve is the *volt* (symbol V).

The definition of this unit can be given according to relation (1.22). The line-integral of the electric field strength, along a curve, in any reference system, between two points, has the value of 1 volt, in the following case. Let a small body with the electric charge of 1 C move along that curve in an electric field, with a very small velocity. Also, during this motion, let the forces due to the electric field and exerted upon the body above do a work of 1 J. In this case, the integral above will be equal to 1 V.

The small velocity is required for the definition of the electric field to be valid in the considered reference frame.

After the unit of the electric potential difference has been adopted, according to relation (1.23 a), the unit of the electric field strength can be deduced and is called *volt per metre* (symbol V/m). The strength of a uniform (homogeneous) electric field is of 1 V/m if the electric potential difference between two points situated at a distance of 1 m along a direction parallel with the direction of the electric field strength is of 1 V.

1.15.4. The Unit of Electric Field Strength

At the same time, another definition of the electric field strength can also be given after having adopted the unit of electric charge. In accordance with relation (1.17), the electric field strength *in vacuo*, at any point, has the value of $1 V/m$ if at that point, the electric field exerts a force of $1 N$ upon a conducting small body having the electric charge of $1 C$.

1.15.5.   The Unit of Electric Current Intensity

The SI unit of the intensity (strength) of the electric current is called *ampere* (symbol A). The electric current constant with time that passes through any surface has the value of 1 A (ampere) if through that surface an electric charge of 1 coulomb passes in a time of 1 second:

$$1\,A = \frac{1\,C}{1\,s}. \tag{1.144}$$

Another form for expressing the unit in the SI system of a certain quantity will be shown for the case of the unit of electric current, as follows:

$$1[I]_{SI} = \frac{1[q]}{1[t]} = \frac{1\,C}{1\,s} = 1\,A. \tag{1.145}$$

The unit of the electric current density is called *ampere per square metre* (symbol A/m^2). Since this unit is small, in numerous practical applications, a multiple of this unit is used, namely ampere per square millimetre (symbol A/mm^2). When the electric current density is introduced in computation expressions in A/mm^2, it is necessary to make the involved transformations.

The unit of the linear current density is the ampere per metre (symbol A/m).

2. INTRODUCTION OF THE STATE QUANTITIES OF THE ELECTROMAGNETIC FIELD *IN VACUO*

The state quantities of the electromagnetic field can be introduced (i.e., defined) either by experimental way or by using the expression of transformation of forces when passing from one reference frame to another that is moving with respect to the first one. Firstly, the second way will be used, and further the first way will be presented. Each manner requires certain derivations. The manner of presentation used for the first way is based on several works [23], [11], [13], [25], [40], [41].

2.1. THE LAW OF PONDEROMOTIVE ACTION UPON A POINT-LIKE ELECTRIC CHARGE AT REST IN AN INERTIAL REFERENCE FRAME

The electric charge of any body, according to the non-correlative definition of Sub-section 1.6.2, will be considered as independent of the velocity of the body. This assumption is supported by experimental facts mentioned in the above Sub-section.

We shall consider a point-like electric charge *in vacuo* in a reference frame $K_o(x_o, y_o, z_o)$, considered as the original frame, and another reference frame $K_1(x_1, y_1, z_1)$ that is moving, at a constant velocity, relatively to the first reference frame. The original reference frame is indicated by the suffix "o", in order to avoid any confusion with the suffix "0" of ε_0 and μ_0.

Any reference frame may be considered, but further on, only inertial reference frames will be examined.

We shall recall the meaning of *inertial reference frame*, and we shall make some remarks.

Inertial reference frame is called any reference frame, with respect to which, three material points, very distant from each other and from other bodies, and moving in different, non- parallel planes, describe straight-line trajectories.

All inertial systems are equivalent to each other, from the point of view of mechanical phenomena. Indeed, all the laws of Classical Mechanics (e.g., inertial law, action and reaction law, fundamental law of Dynamics) have the same expression in any inertial reference frame. The inertial reference frames are in a uniform translation motion.

The expression of the law of ponderomotive action, in any reference frame *K*, exerted upon a point-like charge *q*, at rest, in this reference frame is:

$$\boldsymbol{F} = q\,\boldsymbol{E}\,. \tag{2.1}$$

The expression of the law of ponderomotive action is:

$$\boldsymbol{F}_o = q\,\boldsymbol{E}_o\,, \tag{2.2 a}$$

in the reference frame K_o if the point-like charge is at rest in this reference frame and:

$$\boldsymbol{F}_1 = q\,\boldsymbol{E}_1, \tag{2.2 b}$$

in the reference frame K_1 if the point-like charge is at rest in this reference frame.

The vector quantities $\boldsymbol{E}_\mathrm{o}$ and $\boldsymbol{E}_1$ represent the electric field strengths in the reference frames K_o and K_1, respectively.

If the point-like charge (or the body represented by this charge) is moving relatively to the reference frame, then the law of ponderomotive action is different from that presented above. In this case, the law can be established either by the generalization of experimental results or by using certain expressions from Mechanics established in the Special Theory of Relativity.

2.2. DERIVATION OF THE EXPRESSION OF THE LAW OF PONDEROMOTIVE ACTION UPON A POINT-LIKE ELECTRIC CHARGE THAT IS MOVING RELATIVELY TO AN INERTIAL REFERENCE FRAME

The expression of the law of ponderomotive action upon a point-like charge that is moving with respect to an inertial reference frame can be established by using certain assumptions (hypotheses) from Mechanics established in the Special Theory of Relativity.

The used hypotheses are the following:

1. In an inertial reference frame, the expression of the force acting upon a point-like electric charge that is in motion relatively to this reference frame, and produced by the interaction with another point-like electric charge, at rest with respect to the same reference frame, is given by the Coulomb law from Electrostatics.

2. For obtaining the expression of the force acting upon a point-like electric charge the principle of superposition will be used. So, in order to find the force acting upon a point-like electric charge q at rest or in motion, under the action of several point-like charges $q_i\ (i = 1, \ldots, n)$, we shall proceed as follows. We consider separately the pairs of charges q, q_1; q, q_2; $\ldots$; q, q_n and we obtain the forces $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, $\ldots$, $\boldsymbol{F}_n$. The resultant force exerted upon the point-like electric charge q is:

$$\boldsymbol{F} = \sum_{i=1}^{n} \boldsymbol{F}_i. \tag{2.3}$$

The subscript of point-like electric charges indicates only the ordinal number and has no relation with the subscript of the reference frame symbol. On the other hand, the definition of the electric charge does not depend on the reference frame.

3. For expressing the forces in another inertial reference frame, the transformation relation of forces, from Mechanics, established in the Special Theory of Relativity, will be used.

4. Firstly, only point-like charges will be considered. By this procedure, instead of elementary particles with electric and magnetic moments, only point-like electric charges without electric and magnetic moments will be considered.

2.3. THE TRANSFORMATION EXPRESSION (WHEN PASSING FROM AN INERTIAL SYSTEM TO ANOTHER) OF THE FORCE IN THE SPECIAL THEORY OF RELATIVITY

The following postulates lie at the base of the Special Theory of Relativity:

1. Relativity principle: There are an infinity of reference frames in uniform rectilinear relative motion, called inertial reference frames, in which all the laws of nature take the same form namely the simplest form, established for the case in which the reference frame would be at rest.

2. The principle of constant velocity of light in empty space: In all inertial reference frames, the velocity of light emitted by any source of light in empty space, measured with physically identical gauges and clocks, has the same value.

3. The mechanical momentum along any direction perpendicular to the direction of motion is conserved when passing from an inertial reference frame to another. The general expression of the momentum of a material point in motion [11, p. 666].

The first two postulates have been supported by several experiments. Einstein enounced these postulates. The last postulate is also included in the Special Theory of Relativity. The mentioned first two postulates lead to certain relations between lengths and times in two inertial reference frames in motion relatively to one another. These relations differ from those of Classical Mechanics.

Certain difficulties involved by these postulates have been mentioned in Introduction and are analysed in literature.

2.3.1. The Transformation Expressions of Co-ordinates and Time

We shall recall the notions of duration and time. The duration is a scalar quantity, expressing the relation between two events. This quantity is attached to the events as follows. Each event is related to the position of a moving point. We determine two positions occupied successively by a point corresponding to the two events. The duration between the two events is proportional to the length of the path (way) described by the point between the two positions.

For defining the duration and time in a reference frame, it is possible to do as follows. We consider a point that is in a uniform rectilinear motion in this frame, hence the point describes a straight-line trajectory. When a certain event occurs, we shall *mark the position of the moving point at that moment*. The duration between two events is thus a scalar quantity, proportional to the distance between the points marking the two events. The unit of measure of the duration is determined by adopting certain proportionality constant between distance and duration, for a given motion of the point above.

The duration measured with respect to an origin, and having the sign plus or minus as the second event is posterior or anterior to the origin-event, is called *time,* and may be

referred to as a temporal co-ordinate. The device for measuring the time is a clock of any type.

The standard unit of time is deduced from the duration of rotation of the Earth about its axis. The duration of a sidereal day is defined as the duration of the rotation of the Earth determined with a high precision by astronomical measurements relatively to stars. The duration of a solar day is defined as the duration of the rotation of the Earth determined with a high precision by astronomical measurements. The duration of the mean solar day is obtained from the duration of solar days taking into consideration the duration of the year. The unit of time adopted in Physics is the *second* deduced with respect of the mean solar day.

A more precise definition of the unit of time can be based on the consideration of the periodic duration of oscillation (vibration) of a certain kind of light the wave-length of which is used for defining the unit of time. The second is the duration of $9\,192\,631\,770$ periods of the radiation corresponding to the transition between the two hyperfine levels of the fundamental state of the atom of caesium 133. The clocks with caesium ensure a satisfactory precision for most applications.

The standard unit of length is the *metre* adopted firstly in France by the decree of the 1st August 1793 of the Convention as ten-millionth part of the quarter of an Earth meridian circle (according to the proposition of Borda, Condorcet, Laplace, Lagrange, Monge). Standard metres were manufactured.
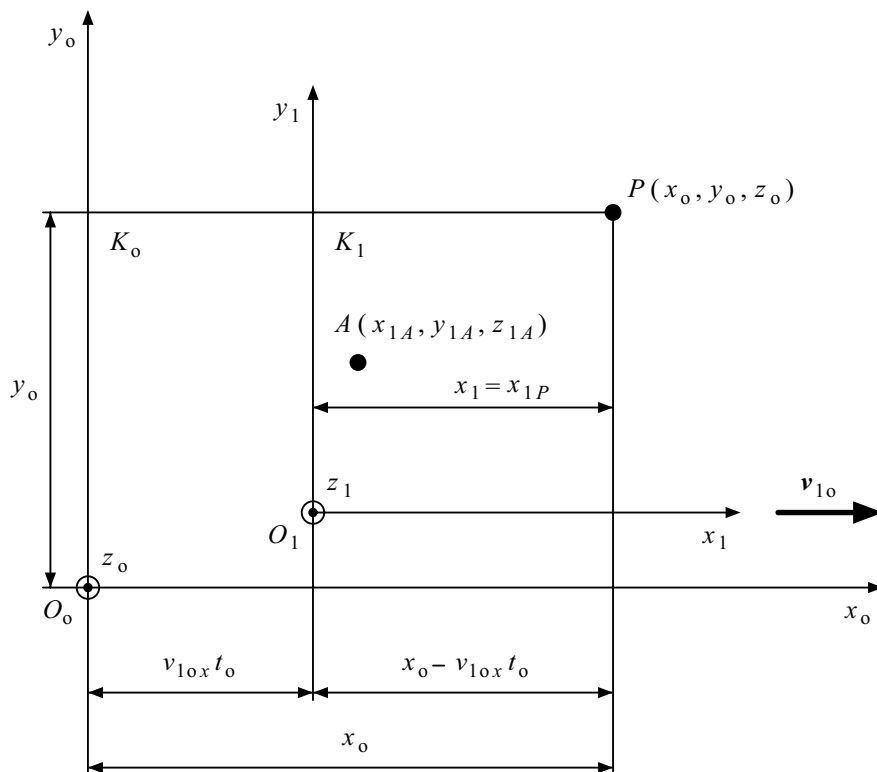
Fig. 2.1. The reference frames K_0 and K_1.

In 1887, Michelson proposed the utilization of optical interferometers for the measure of lengths and determined the length of the metre from the wave-length of cadmium. In 1960, the Conférence Générale des Poids et Mesures (CGPM) adopted the length of the metre from the wave-length of the radiation corresponding to the transition between two specified energy levels of the atom of krypton 96 *in vacuo*. In 1983, the CGPM adopted for the metre the following definition: the metre is the length covered by light *in vacuo* in $1/299\,792\,458$ second [46]. We shall consider the reference frames $K_o(x_o, y_o, z_o)$, and $K_1(x_1, y_1, z_1)$ of Fig. 2.1. Three-orthogonal rectilinear (Cartesian) right-handed systems of co-ordinates are used.

The co-ordinates of any point P seen by an observer in the frame K_o are denoted by x_o, y_o, z_o, and the co-ordinates of the same point seen by an observer in the frame K_1 are denoted by x_1, y_1, z_1.

As shown in Appendix 3, the unit vectors of the two systems of co-ordinates can be denoted by the sets $\boldsymbol{i}_o, \boldsymbol{j}_o, \boldsymbol{k}_o$, and $\boldsymbol{i}_1, \boldsymbol{j}_1, \boldsymbol{k}_1$, respectively. In the case in which the axes of co-ordinates of the two systems are parallel with each other, since the unit vectors are dimensionless, both sets of unit vectors may be denoted by the same set of symbols, namely $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$.

Hence, the unit vectors of the three axes in the frame K_o as well as in the frame K_1 are denoted by $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ and they are not affected by the transformation expressions of the co-ordinates, because they are dimensionless quantities that indicate only the directions.

The components of any vector, in the reference frame K_o, for instance $\boldsymbol{F}_o$, along the three axes of co-ordinates can be denoted by $F_{ox_o}, F_{oy_o}, F_{oz_o}$. If no confusion can appear, for the sake of brevity, the last suffix may be omitted, and then the components become F_{ox}, F_{oy}, F_{oz}. The components of any vector, in the reference frame K_1, for instance $\boldsymbol{F}_1$, along the three axes of co-ordinates can be denoted by $F_{1x_1}, F_{1y_1}, F_{1z_1}$ or F_{1x}, F_{1y}, F_{1z}, according to the case, as explained above.

The components of the velocity of the reference frame K_1 with respect to the reference frame K_o denoted for instance $\boldsymbol{v}_{1o}$, along the three axes of co-ordinates can be denoted by $v_{1ox_o}, v_{1oy_o}, v_{1oz_o}$ or $v_{1ox}, v_{1oy}, v_{1oz}$, according to the case, as explained above.

The reference frame K_1 is in a uniform rectilinear motion at velocity $\boldsymbol{v}_{1o} = \boldsymbol{i}\,v_{1o} = \boldsymbol{i}\,v_{1ox}$ with respect to the reference frame K_o that is supposed at rest.

The choice of the reference frames is arbitrary so that the adoption of the Ox_1- axis having the direction of the velocity vector $\boldsymbol{v}_{1o} = \boldsymbol{i}\,v_{1o} = \boldsymbol{i}\,v_{1ox}$ simplifies the mathematical expressions, without reducing the generality of the analysis.

In the reference frame K_o the time determined by a certain procedure (e.g., a clock), is denoted by t_o, and in the reference frame K_1 determined by the same procedure is denoted by t_1. The two determination procedures (usage of clocks) are assumed identical and synchronized. The term synchronization means, in this case, that at the moment when the origins of the two reference frames coincide, the time is chosen equal to zero. The

velocity of propagation of light, assumed to be *in vacuo* the same, in any reference frame, is denoted by c. In certain cases that here will not occur the symbol c_0 is also used.

On the base of the postulates of the Special Theory of Relativity, the transformation expressions that give the quantities in the frame K_1 in terms of the quantities in the reference frame K_o are as follows:

$$x_1 = \alpha_{1o} \left(x_o - v_{1ox} t_o \right),$$
(2.4 a)

$$y_1 = y_o \,,$$
(2.4 b)

$$z_1 = z_o \,,$$
(2.4 c)

$$t_1 = \alpha_{1o} \left(t_o - \frac{v_{1ox}}{c^2} x_o \right),$$
(2.4 d)

$$\alpha_{1o} = \frac{1}{\sqrt{1 - \beta_{1o}^2}}; \quad \beta_{1o} = \frac{v_{1ox}}{c}.$$
(2.4 e, f)

The relations (2.4 a, b, c, d, e, f) are called *Lorentz transformation relations*. The same relations have been established previously by Lorentz, on the base of other considerations than those previously presented (i.e., the postulates of the Special Theory of Relativity).

The transformation relations that give the quantities in the reference frame K_o in terms of those in the reference frame K_1 can be obtained analogously by solving the system of equations (2.4 a, b, c, d, e, f) and are:

$$x_o = \alpha_{o1} \left(x_1 + v_{1ox} t_1 \right),$$
(2.5 a)

$$y_o = y_1 \,,$$
(2.5 b)

$$z_o = z_1 \,,$$
(2.5 c)

$$t_o = \alpha_{o1} \left(t_1 + \frac{v_{1ox}}{c^2} x_1 \right),$$
(2.5 d)

$$\alpha_{o1} = \alpha_{1o} = \frac{1}{\sqrt{1 - \beta_{o1}^2}}; \quad \beta_{o1} = -\frac{v_{1ox}}{c} = -\beta_{1o} \,.$$
(2.5 e, f)

For every moment, the lengths can be calculated in each reference frame by the expression known from Geometry. For instance, we shall consider two points A and B belonging to a body at rest with respect to the reference frame K_1 and we shall express the distances between the two points in the two reference frames K_o and K_1.

We obtain:

$$l_{oAB} = \sqrt{\left(x_{oB} - x_{oA} \right)^2 + \left(y_{oB} - y_{oA} \right)^2 + \left(z_{oB} - z_{oA} \right)^2} \,,$$
(2.6 a)

$$l_{1AB} = \sqrt{(x_{1B} - x_{1A})^2 + (y_{1B} - y_{1A})^2 + (z_{1B} - z_{1A})^2} \,. \qquad (2.6\ b)$$

The observer in the reference frame K_o will locate the points A and B at the same moment t_o and then we obtain from relation (2.4 a):

$$(x_{1B} - x_{1A}) = \alpha_{1o} (x_{oB} - x_{oA}). \qquad (2.7)$$

Thus $l_{oAB} < l_{1AB}$ and this modification of the length due to the motion, for an observer in the reference frame K_o, is called *contraction of lengths* along the direction parallel with the motion.

If the distance between the two points is very small, the following relations can be written:

$$\mathrm{d}\,x_1 = \alpha_{1o}\,\mathrm{d}\,x_o\,, \qquad (2.8\ a)$$

$$\mathrm{d}\,y_1 = \mathrm{d}\,y_o\,, \qquad (2.8\ b)$$

$$\mathrm{d}\,z_1 = \mathrm{d}\,z_o\,, \qquad (2.8\ c)$$

or

$$\mathrm{d}\,\boldsymbol{r}_1 = \boldsymbol{i}\,\alpha_{1o}\,\mathrm{d}\,x_o + \boldsymbol{j}\,\mathrm{d}\,y_o + \boldsymbol{k}\,\mathrm{d}\,z_o\,. \qquad (2.8\ d)$$

Analogously, for the surface elements:

$$\mathrm{d}\,S_{1x} = \mathrm{d}\,y_1\,\mathrm{d}\,z_1 = \mathrm{d}\,S_{ox}\,,$$

$$\mathrm{d}\,S_{1y} = \mathrm{d}\,z_1\,\mathrm{d}\,x_1 = \alpha_{1o}\,\mathrm{d}\,S_{oy}\,, \qquad (2.9\ a,\ b,\ c)$$

$$\mathrm{d}\,S_{1z} = \mathrm{d}\,x_1\,\mathrm{d}\,y_1 = \alpha_{1o}\,\mathrm{d}\,S_{oz}\,,$$

and for the volume elements:

$$\mathrm{d}\,v_1 = \alpha_{1o}\,\mathrm{d}\,v_o\,. \qquad (2.10)$$

The relations by which the various geometrical quantities of the frame K_o are expressed in terms of those of the frame K_1 can be analogously written taking into consideration the corresponding transformation relations of co-ordinates. For instance:

$$\mathrm{d}\,S_{oy} = \alpha_{o1}\,\mathrm{d}\,S_{1y}\,. \qquad (2.11)$$

From relation (2.5 d), it results that between the duration Δt_1 between two events considered by an observer in the frame K_1 and the duration Δt_o between the same events, considered by an observer in the frame K_o, the following relation can be written:

$$\Delta t_o = \alpha_{1o}\,\Delta t_1\,. \qquad (2.12)$$

Thus $\Delta t_{\mathrm{o}} > \Delta t_1$ and this modification of the duration between the two events is called *dilatation of time* [13] or *dilation of time* [18].

From the analysis above, it follows: It is accepted that at every point in the reference frame K_{o}, there is a clock that defines a local time t_{o} and that at every point of the reference frame K_1 there is a clock that defines another local time t_1. It is not the case of general shift between the two reference frames, but it is a time difference that depends on the considered point. Then, the phenomena that simultaneously occur in the reference frame K_{o} are not generally simultaneous in the reference frame K_1 and inversely.

This aspect concerning the time is connected to another aspect concerning the space. Indeed, let us consider two points *A* and *B*, at rest in the reference frame K_1 and therefore moving relatively to the reference frame K_{o}. If in the reference frame K_1 the distance between them is measured by a ruler at rest, their position will be located at the same moment $t_{1A} = t_{1B}$. But in the reference frame K_{o}, the time $t_{\mathrm{o}A}$ differs from the time $t_{\mathrm{o}B}$.

2.3.2. The Transformation Expressions of Forces

We shall consider the reference frames K_{o} and K_1. The reference frame K_{o} is considered at rest, and the reference frame K_1 is considered in a uniform rectilinear motion, at the velocity $\boldsymbol{v}_{1\mathrm{o}} = \boldsymbol{i}\, v_{1\mathrm{o}x}$, relatively to the reference frame K_{o}.

We consider a material point that is moving at any velocity $\boldsymbol{v}_{2\mathrm{o}}$ (not necessary constant) relatively to the reference frame K_{o}, and at the velocity $\boldsymbol{v}_{21}$, relatively to the reference frame K_1.

The expression of the force acting upon a material point of mass *m* and velocity $\boldsymbol{u}$, in any reference frame *K* is given by the derivative of the momentum (quantity of motion):

$$\boldsymbol{F} = \frac{\mathrm{d}}{\mathrm{d}t}(m\,\boldsymbol{u}). \tag{2.13}$$

We shall denote by $\boldsymbol{F}_{\mathrm{o}}$ the force acting upon the material point in the reference frame K_{o} and by $\boldsymbol{F}_1$ the force acting upon the same point in the reference frame K_1.

In the framework of the Special Theory of Relativity, using the transformation expressions of co-ordinates and time, the transformation expression of the force can be obtained in the form given by relation (A.3.13) in Appendix 3:

$$F_{\mathrm{o}x} = F_{1x} + \boldsymbol{i} \cdot \left[\frac{1}{c^2} \boldsymbol{v}_{2\mathrm{o}} \times \left(\boldsymbol{v}_{1\mathrm{o}} \times \alpha_{1\mathrm{o}} \boldsymbol{F}_1 \right) \right], \tag{2.14 a}$$

$$F_{\mathrm{o}y} = \alpha_{1\mathrm{o}} F_{1y} + \boldsymbol{j} \cdot \left[\frac{1}{c^2} \boldsymbol{v}_{2\mathrm{o}} \times \left(\boldsymbol{v}_{1\mathrm{o}} \times \alpha_{1\mathrm{o}} \boldsymbol{F}_1 \right) \right], \tag{2.14 b}$$

$$F_{\mathrm{o}z} = \alpha_{1\mathrm{o}} F_{1z} + \boldsymbol{k} \cdot \left[\frac{1}{c^2} \boldsymbol{v}_{2\mathrm{o}} \times \left(\boldsymbol{v}_{1\mathrm{o}} \times \alpha_{1\mathrm{o}} \boldsymbol{F}_1 \right) \right], \tag{2.14 c}$$

$$\boldsymbol{F}_{\mathrm{o}} = \boldsymbol{i}\, F_{1x} + \boldsymbol{j}\, \alpha_{1\mathrm{o}} F_{1y} + \boldsymbol{k}\, \alpha_{1\mathrm{o}} F_{1z} + \frac{1}{c^2} \boldsymbol{v}_{2\mathrm{o}} \times \left(\boldsymbol{v}_{1\mathrm{o}} \times \alpha_{1\mathrm{o}} \boldsymbol{F}_1 \right). \tag{2.14 d}$$

2.3.3. The Manner of Adopting the Transformation Relations of Forces and Geometrical Quantities

We admit that there is a reference frame K_n having the following property: The force $\boldsymbol{F}_n$ acting in that reference frame upon an electrically charged particle moving with any velocity $\boldsymbol{v}_{pn}$ (not necessary constant) is independent of the velocity $\boldsymbol{v}_{pn}$. The last velocity must not be too great for avoiding the electromagnetic radiation of the charge and the corresponding braking. Such a case occurs when all charged particles are at rest in the reference frame K_n the particle above, of velocity $\boldsymbol{v}_{pn}$, excepted.

In any other reference frame, the forces can be expressed by relations of the type (2.14) in terms of the force $\boldsymbol{F}_n$ from the reference frame K_n. Therefore, $\boldsymbol{F}_n$ is considered as the *proper* value of the force in the reference frame K_n.

All geometrical elements (lines, surfaces, volumes) have their *proper* dimensions taken in the reference frame K_n. The sizes in the reference frame K_n can be expressed in terms of the sizes of any other reference frame. As a consequence, relations of the type (2.8) will be used. Hence the area ΔS_1 can be expressed in terms of the area ΔS_0. Consequently, relations (2.9 a, b, c) will be used.

With the usual denominations, the sizes are *contravariant* quantities (tensor), and the force components are *covariant* quantities. When writing various relations, it is necessary to keep in mind the kind of the involved quantities.

2.4. THE EXPRESSIONS OF THE FORCE AND ELECTRIC FIELD STRENGTH IN VARIOUS REFERENCE FRAMES. ELECTRIC DISPLACEMENT *IN VACUO* AND MAGNETIC INDUCTION *IN VACUO*. MAGNETIC CONSTANT (MAGNETIC PERMEABILITY OF VACUUM).

Let us consider that at a fixed point $A(x_{1A},\ y_{1A},\ z_{1A})$ of the reference frame K_1 there is a point-like electric charge q_1. The subscript of point-like electric charges has no relation with the subscript of the reference frame symbol, as explained in Section 2.2, assumption 2. The medium is considered to be vacuum.

We shall also consider that there is another point-like electric charge q. This charge is moving at the velocity $\boldsymbol{v}_{2o}$ (that is not necessary to be constant) relatively to the reference frame K_o, and $\boldsymbol{v}_{21}$ relatively to the reference frame K_1.

On the base of hypothesis 1 of Section 2.2, the force acting upon the point-like electric charge q at the point $P(x_1,\ y_1,\ z_1)$ in the reference frame K_1 is given by the Coulomb formula:

$$\boldsymbol{F}_1 = \frac{q_1\, q}{4\,\pi\,\varepsilon_0} \cdot \frac{\boldsymbol{r}_{1AP}}{r_{1AP}^3} = q\,\boldsymbol{E}_1\,, \tag{2.15}$$

where the quantity $\boldsymbol{E}_1$ represents, in accordance with relation (2.2 b), the electric field strength in the reference frame K_1.

The components of this force along the three axes of co-ordinates are:

$$F_{1x} = \frac{q_1 \, q}{4 \, \pi \, \varepsilon_0} \cdot \frac{x_1 - x_{1A}}{r_{1AP}^3} = q \, E_{1x} \, , \tag{2.16 a}$$

$$F_{1y} = \frac{q_1 \, q}{4 \, \pi \, \varepsilon_0} \cdot \frac{y_1 - y_{1A}}{r_{1AP}^3} = q \, E_{1y} \, , \tag{2.16 b}$$

$$F_{1z} = \frac{q_1 \, q}{4 \, \pi \, \varepsilon_0} \cdot \frac{z_1 - z_{1A}}{r_{1AP}^3} = q \, E_{1z} \, , \tag{2.16 c}$$

where:

$$r_{1AP} = r_1 = \left\{ \left(x_1 - x_{1A} \right)^2 + \left(y_1 - y_{1A} \right)^2 + \left(z_1 - z_{1A} \right)^2 \right\}^{\frac{1}{2}} . \tag{2.17}$$

Let us suppose the point P fixed to the reference frame K_o. By using the transformation expressions of co-ordinates and time (2.4), referring to Fig. 2.1, it follows:

$$F_{1x} = \frac{q_1 \, q}{4 \, \pi \, \varepsilon_0} \cdot \frac{\alpha_{1o} \left(x_o - v_{1ox} \, t_o \right) - x_{1A}}{r_{1AP}^3} = q \, E_{1x} \, , \tag{2.18 a}$$

$$F_{1y} = \frac{q_1 \, q}{4 \, \pi \, \varepsilon_0} \cdot \frac{y_o - y_{1A}}{r_{1AP}^3} = q \, E_{1y} \, , \tag{2.18 b}$$

$$F_{1z} = \frac{q_1 \, q}{4 \, \pi \, \varepsilon_0} \cdot \frac{z_o - z_{1A}}{r_{1AP}^3} = q \, E_{1z} \, , \tag{2.18 c}$$

where:

$$r_{1AP} = \left\{ \left[\alpha_{1o} \left(x_o - v_{1ox} \, t_o \right) - x_{1A} \right]^2 + \left(y_o - y_{1A} \right)^2 + \left(z_o - z_{1A} \right)^2 \right\}^{\frac{1}{2}} . \tag{2.19}$$

By using the transformation expressions of the force (2.14 a, . . ., d), we obtain the force in the reference frame K_o:

$$F_{ox} = q \left\{ E_{1x} + \boldsymbol{i} \cdot \left[\frac{1}{c^2} \boldsymbol{v}_{2o} \times \left(\boldsymbol{v}_{1o} \times \alpha_{1o} \, \boldsymbol{E}_1 \right) \right] \right\} , \tag{2.20 a}$$

$$F_{oy} = q \left\{ \alpha_{1o} \, E_{1y} + \boldsymbol{j} \cdot \left[\frac{1}{c^2} \boldsymbol{v}_{2o} \times \left(\boldsymbol{v}_{1o} \times \alpha_{1o} \, \boldsymbol{E}_1 \right) \right] \right\} , \tag{2.20 b}$$

$$F_{oz} = q \left\{ \alpha_{1o} \, E_{1z} + \boldsymbol{k} \cdot \left[\frac{1}{c^2} \boldsymbol{v}_{2o} \times \left(\boldsymbol{v}_{1o} \times \alpha_{1o} \, \boldsymbol{E}_1 \right) \right] \right\} . \tag{2.20 c}$$

In vector form, the expression of the force in the reference frame K_o is:

$$\boldsymbol{F}_o = \boldsymbol{i}\,F_{ox} + \boldsymbol{j}\,F_{oy} + \boldsymbol{k}\,F_{oz} = q\left(\boldsymbol{i}\,E_{1x} + \boldsymbol{j}\,\alpha_{1o}\,E_{1y} + \boldsymbol{k}\,\alpha_{1o}\,E_{1z}\right) +$$
$$+ q\left[\frac{1}{c^2}\,\boldsymbol{v}_{2o} \times \left(\boldsymbol{v}_{1o} \times \alpha_{1o}\,\boldsymbol{E}_1\right)\right]. \tag{2.20 d}$$

Relation (2.20 d) represents the expression of the force in the reference frame at rest K_o, in terms of the force in the moving reference frame K_1. We denote:

$$\boldsymbol{E}_o = \boldsymbol{i}\,E_{1x} + \boldsymbol{j}\,\alpha_{1o}\,E_{1y} + \boldsymbol{k}\,\alpha_{1o}\,E_{1z}. \tag{2.21}$$

The relation (2.21) represents the expression of the electric field strength in the reference frame at rest K_o, in terms of the electric field strength in the moving reference frame K_1.

Since $\boldsymbol{i} \times \boldsymbol{v}_{1o} = 0$, the direction of the Ox - axis coinciding with the direction of motion, we obtain:

$$\boldsymbol{F}_o = q\,\boldsymbol{E}_o + q\left[\boldsymbol{v}_{2o} \times \left(\frac{1}{c^2}\,\boldsymbol{v}_{1o} \times \boldsymbol{E}_o\right)\right]. \tag{2.22}$$

We denote $\dfrac{1}{c^2} = \varepsilon_0\,\mu_0$, where the constants ε_0 and μ_0 have already been introduced.

The constant ε_0 denotes, as previously, the (*electric*) *permittivity of vacuum*, also called *electric constant*. The constant μ_0 denotes the (*magnetic*) *permeability of vacuum*, also called *magnetic constant*. In the SI system of units (i.e., a rationalized system), the values of the two constants are:

$$\varepsilon_0 = \frac{1}{4\,\pi\,9\cdot 10^9}\,\frac{F}{m}\ (\text{farad}/\text{metre}), \tag{2.23}$$

$$\mu_0 = 4\,\pi\,10^{-7}\,\frac{H}{m}\ (\text{henry}/\text{metre}). \tag{2.24}$$

The definition of the units *farad* and *henry* will be given further in Sub-section 3.13.3. The following symbols are adopted:

$$\varepsilon_0\,\boldsymbol{E}_o = \boldsymbol{D}_o, \tag{2.25}$$

$$\frac{1}{c^2}\,\boldsymbol{v}_{1o} \times \boldsymbol{E}_o = \mu_0\left(\boldsymbol{v}_{1o} \times \boldsymbol{D}_o\right) = \boldsymbol{B}_o. \tag{2.26}$$

We may remark in relations (2.25) and (2.26), that the original reference frame is indicated by the suffix "o", in order to avoid any confusion with the suffix "0" of ε_0 and μ_0, as previously mentioned.

At the same time, the following quantity is also used:

$$\boldsymbol{H}_{\mathrm{o}} = \boldsymbol{v}_{\mathrm{1o}} \times \boldsymbol{D}_{\mathrm{o}} ; \quad \boldsymbol{B}_{\mathrm{o}} = \mu_0 \, \boldsymbol{H}_{\mathrm{o}} .$$ (2.27 a, b)

The vector quantity $\boldsymbol{D}_{\mathrm{o}}$ introduced by relation (2.25) is referred to by one of the following expressions: *Electric displacement, electric flux density*, and *electric induction, in vacuo*, in the reference frame K_{o}.

The vector quantity $\boldsymbol{B}_{\mathrm{o}}$ introduced above is referred to by one of the following expressions: *Magnetic induction*, and *magnetic flux density*, *in vacuo*, in the reference frame K_{o}.

The vector quantity $\boldsymbol{H}_{\mathrm{o}}$ is called *magnetic field strength* or *magnetic field intensity*, *in vacuo,* in the reference frame K_{o}.

In the SI system of units, the unit of electric displacement is, as mentioned, coulomb/square metre (symbol C/m^2), and the unit of magnetic induction is volt second per square metre, called tesla (symbol T). The definition of the unit of measure of the magnetic induction will be presented in Sub-section 3.13.4.

If instead of expression (2.23), other values are adopted for the constants included in the expression of c in relation (2.22), other units of measure are obtained for the units of measure of the electromagnetic quantities. With the adopted symbols, we obtain:

$$\boldsymbol{F}_{\mathrm{o}} = q\left(\boldsymbol{E}_{\mathrm{o}} + \boldsymbol{v}_{2\mathrm{o}} \times \boldsymbol{B}_{\mathrm{o}}\right).$$ (2.28)

This is the expression of the force in the reference frame K_{o} that acts upon a point-like electric charge, thus *the expression of the ponderomotive action upon a point-like electric charge in motion with respect to an inertial reference frame*. It is also referred to as the *Lorentz expression of the force*. It is necessary to add that in the last expression the quantities $\boldsymbol{E}_{\mathrm{o}}$ and $\boldsymbol{B}_{\mathrm{o}}$ are assumed to be not influenced by the electric charge q. For this assumption to be satisfied it is necessary that the velocity $\boldsymbol{v}_{2\mathrm{o}}$ be not too great. Indeed, otherwise the charge in motion produces an electromagnetic field that modifies the value of the quantities $\boldsymbol{E}_{\mathrm{o}}$ and $\boldsymbol{B}_{\mathrm{o}}$ in the preceding formula.

Hitherto, we have supposed that the $\boldsymbol{E}_{\mathrm{o}}$ and $\boldsymbol{B}_{\mathrm{o}}$ are produced by the point-like electric charge q_1 fixed in the reference frame K_1. If instead of a single point-like electric charge q_1, there are n point-like electric charges $q_i \, (i = 1, 2, \ldots, n)$, at rest in the reference frames $K_1, K_2, \ldots, K_n$, which are moving with the velocities $\boldsymbol{v}_i \, (i = 1, 2, \ldots, n)$ relatively to the reference frame K_{o}, each charge produces a force. According to the hypothesis 2 of Section 2.2, the principle of superposition and formula (2.28) remain valid in this case.

In the case in which the electric charge is distributed with the volume density ρ_{v1} in the reference frame K_1, then the expression of the electric field strength at a point $P(x_{1P}, y_{1P}, z_{1P})$, corresponding to relation (2.15), is replaced by:

$$\boldsymbol{E}_1 = \frac{1}{4\pi\varepsilon_0} \int\limits_{V_{\Sigma_1}} \rho_{v1}\left(r_{1M}\right) \frac{\boldsymbol{r}_{1MP}}{r_{1MP}^3} \, \mathrm{d}\,v_1 ,$$ (2.29)

$$r_1 = i\, x_{1M} + j\, y_{1M} + k\, z_{1M} , \tag{2.29 a}$$

$$r_{1MP} = \left[(x_{1P} - x_{1M})^2 + (y_{1P} - y_{1M})^2 + (z_{1P} - z_{1M})^2 \right]^{\frac{1}{2}} , \tag{2.29 b}$$

$$\mathrm{d}\, v_1 = \mathrm{d}\, x_1 \, \mathrm{d}\, y_1 \, \mathrm{d}\, z_1 , \tag{2.29 c}$$

where the vector r_{1M} has its origin at the origin $O_1(0, 0, 0)$ and its end at any point $M(x_{1M}, y_{1M}, z_{1M})$; the vector r_{1MP} has its origin at any source point $M(x_{1M}, y_{1M}, z_{1M})$ and its end at the observation point $P(x_{1P}, y_{1P}, z_{1P})$. The integral is extended over the whole space, in the reference frame K_1, and the index P refers to the observation point the co-ordinates of which are expressed in the same reference frame.

If there is a single point-like electric charge, then instead of any source point $M(x_{1M}, y_{1M}, z_{1M})$, there is a single point $A(x_{1A}, y_{1A}, z_{1A})$. In the approach above, the magnetic induction *in vacuo* B appears as a derived quantity. If the Special Theory of Relativity were not used, then the magnetic induction would be introduced as a primitive (fundamental) quantity, what can be done, as it will be shown in the next Section.

2.5. GENERAL EXPRESSIONS OF THE FORCE ACTING UPON A POINT-LIKE ELECTRIC CHARGE IN MOTION RELATIVELY TO AN INERTIAL REFERENCE FRAME. INTRODUCTION (DEFINITION) OF THE QUANTITIES: ELECTRIC FIELD STRENGTH E AND MAGNETIC INDUCTION B.

The general expression of the force acting upon a point-like electric charge in motion with respect to an inertial reference system, deduced in the preceding Section can also be established by the generalisation of certain experimental results.

We consider a point-like electric charge q that is moving with any velocity v (that is not necessary to be constant) with respect to any inertial reference frame K. The force acting upon a point-like electric charge (the Lorentz force) is:

$$F = q\left(E_l + v \times B \right), \tag{2.30}$$

where in the right-hand side, besides the velocity, other vector quantities occur, namely: E and B. For the sake of simplicity, we shall assume that $E_l = E$.

The vector quantity E, called *electric field strength*, can be introduced (in accordance with the explanation of Sub-section 1.6.5) as follows. The electric field strength at a point in any reference frame is the ratio between the force exerted at that point upon a point-like electric charge q, when it would be at rest in this reference frame, and the electric charge q.

It is to be noted that, in this case, the electric charge q should be small enough to not perturb the state of the field, hence, to not modify the distribution of the electric charge. This remark is significant only when introducing the vector E. In the general case, the

electric charge q of the point-like charge may be of any magnitude, hence q can modify the state of the electric field. However, in this case, the electric field strength at that point will be determined by the actual distribution of all electric charges. The relation (2.30) remains valid.

The vector quantity B, called *magnetic induction*, included in relation (2.30), can be introduced as follows: The magnetic induction at a point of an inertial reference frame is the vector quantity which multiplied by the velocity vector v and introduced into relation (2.30), where the quantity E is known, gives the force vector exerted upon a point-like electric charge q that is moving with any velocity v relatively to the inertial reference frame. Here, the remark concerning the necessity that the electric charge q be small enough to not modify the electric charge distribution remains still valid.

In the presented manner, the vector quantity B appears as a primitive (fundamental) quantity (if it is introduced in the previously way and if this way is considered to be the only possible for determining the vector quantity B).

If the vector quantity B is introduced via a certain deduction (for instance by certain relations of Mechanics, established in the Special Theory of Relativity) the vector quantity E remains a primitive (fundamental) quantity, and the vector quantity B becomes a secondary (derived) quantity. In this case, the previous introduction of the quantity B is called *measurement* or *verification*. The macroscopic magnetic induction is expressed by the macroscopic average value of the microscopic magnetic induction. The average value is obtained by using a multiple integral expressed by relation (1.15). The vector quantity B is a function of point or, in other words, B is a field vector (producing a field of vectors also called vector field).

The lines of the vector field B, shortly called *lines of magnetic induction* or *lines of magnetic field,* are lines to which the vector B is tangent at any point.

It is interesting to mention that the lines of magnetic field, like the lines of electric field, are only auxiliary notions that serve to describe the fields and are not material entities, the elements of which could be materialized or related to certain sources of the field.

The impossibility of such interpretation results, for instance, in the case of superposition of the fields of two coaxial cylindrical magnets, one fixed and the other in a rotating motion about its axis.

Magnetic flux tube is called the surface in a magnetic field bounded by the totality of lines of magnetic induction that passes through the points of a closed simple curve.

It is to be remarked that the component acting upon an electric charge in motion, which does not exist if the charge is at rest, is:

$$F_{\mathrm{mag}} = q\left(v \times B\right). \tag{2.30 a}$$

If the charge q is assumed to be continuously distributed in an infinitesimal volume $\mathrm{d}v$, then we can write in the reference frame K_1 the relation $q = \rho_v\,\mathrm{d}v$.

In the case of a thread-like (in French, filiforme) distribution of electric charge with the cross-section s_c in the reference frame K_1, the volume element $\mathrm{d}v$ becomes $s_c \cdot \mathrm{d}l$ (where the vectors $\mathrm{d}l$ and s_c are parallel). Then, taking into account that the quantity $J_o \cdot s_c$ represents an electric current i relatively to the reference frame K_o, the last formula can be written:

$$\mathrm{d}\,\boldsymbol{F}_{\mathrm{mag}} = i\,\mathrm{d}\,\boldsymbol{l} \times \boldsymbol{B}. \tag{2.30 b}$$

The last formula represents the force acting on a current element $i\,\mathrm{d}\,\boldsymbol{l}$ in a magnetic field of magnetic induction $\boldsymbol{B}$, and is referred to as the *Laplace formula*.

It is to be remarked that in relation (2.30), the force acting upon a point-like electric charge has two components: the first independent of the velocity and the second depending on the velocity of the electric charge carrier. For this reason, the two kinds of vector quantities $\boldsymbol{E}$ and $\boldsymbol{B}$ have different specific properties.

The vector $\boldsymbol{E}$ of relations (2.30 a, b) is not generally a quantity deriving from a potential. This fact can be established using relations (2.20) – (2.22). For this purpose, we shall calculate the curl of the vectors $\boldsymbol{E}$ of the various inertial reference frames.

The differential operators at a point of a reference frame will be indicated by the index corresponding to that reference frame in which the respective operator is calculated. Thus $\mathrm{curl}_{\mathrm{o}}\,\boldsymbol{E}$ means that the differential operator curl is calculated in the reference frame K_{o}.

The relations between the components of the vector quantities $\boldsymbol{E}_{\mathrm{o}}$ and $\boldsymbol{E}_1$ are given by relation (2.20 a, . . ., d). By replacing in the expanded expression of the curl the components of the vector $\boldsymbol{E}_1$, it results that:

$$\mathrm{curl}_1\,\boldsymbol{E}_1 = 0, \tag{2.31}$$

$$\boldsymbol{j}\cdot\left(\mathrm{curl}_1\,\boldsymbol{E}_1\right) = \frac{\partial E_{1x}}{\partial z_1} - \frac{\partial E_{1z}}{\partial x_1} = 0. \tag{2.31 a}$$

Taking into account the relation (2.21), it results that:

$$\mathrm{curl}_{\mathrm{o}}\,\boldsymbol{E}_{\mathrm{o}} \neq 0. \tag{2.32}$$

Indeed, from the expression of the curl, it follows:

$$\boldsymbol{j}\cdot\left(\mathrm{curl}_{\mathrm{o}}\,\boldsymbol{E}_{\mathrm{o}}\right) = \frac{\partial E_{\mathrm{o}x}}{\partial z_{\mathrm{o}}} - \frac{\partial E_{\mathrm{o}z}}{\partial x_{\mathrm{o}}}. \tag{2.32 a}$$

But, according to relations (2.21) and (2.20 d), we have:

$$E_{\mathrm{o}x} = E_{1x}, \tag{2.33 a}$$

$$E_{\mathrm{o}z} = \alpha_{1\mathrm{o}}\,E_{1z}. \tag{2.33 b}$$

We take into account that E_{1z} is a function of x_1, y_1, z_1. Relations (2.32 a), (2.4 c), (2.33 a), (2.4 a), (2.33 b), (2.31 a) yield:

$$\frac{\partial E_{\mathrm{o}x}}{\partial z_{\mathrm{o}}} - \frac{\partial E_{\mathrm{o}z}}{\partial x_{\mathrm{o}}} = \frac{\partial E_{1x}}{\partial z_1} - \alpha_{1\mathrm{o}}\frac{\partial E_{1z}}{\partial x_1}\cdot\frac{\partial x_1}{\partial x_{\mathrm{o}}} = \frac{\partial E_{1x}}{\partial z_1} - \alpha_{1\mathrm{o}}^2\frac{\partial E_{1z}}{\partial x_1} \neq 0. \tag{2.34}$$

The above component of the $\mathrm{curl}_{\mathrm{o}}$ being different from zero, relation (2.32) is established.

The electric field strength contains besides the Coulombian component, that is a potential one, a *curl (rotational) component,* also called *solenoidal component* or *induced component*:

$$\boldsymbol{E} = \boldsymbol{E}_c + \boldsymbol{E}_r .$$ (2.35)

Therefore, in the case of the reference frames K_o and K_1 considered previously, the strength of the resulting electric field in the reference frame K_o is $\boldsymbol{E}_\mathrm{o}$ and can be written:

$$\boldsymbol{E}_\mathrm{o} = \boldsymbol{E}_{\mathrm{o}c} + \boldsymbol{E}_{\mathrm{o}r} ,$$ (2.36)

where the quantity $\boldsymbol{E}_{\mathrm{o}c}$ represents the Coulombian component, and the quantity $\boldsymbol{E}_{\mathrm{o}r}$ represents the curl component (rotational, solenoidal or induced component) of the electric field strength.

The solenoidal component of the electric field strength and the magnetic induction at a point are produced by the electric charge in motion with respect to the considered reference frame.

2.6.   THE MAGNETIC FIELD

Firstly, we shall recall some generalities about magnetism and magnets. Various bodies, among which the loadstone (iron ore), have certain properties, explained below, and for this reason they are called magnets. A body, which after having been subjected to certain treatments gets such properties, becomes a magnet and the treatment represents the process of *magnetization.* The material of which the body is made represents a magnetic material. The properties mentioned above consist in actions or forces of magnetic origin. For instance, a body suspended near any part of the earth surface, so as to turn freely about a vertical axis, could tend to set itself in a certain position. If it is disturbed from this position, it could tend to come back in the previous position, after several oscillations. In the case in which the body satisfies the mentioned tendencies, then it is a magnetized body, and in the contrary case, it is an unmagnetized one. In the first case the body is a *magnet.* The study referring to actions of magnetic origin is called *magnetism.*

Between certain bodies, like those of magnetite ($\mathrm{Fe_2O_3}$), ponderomotive actions (forces and torques) are exerted. These actions are not of thermal, mechanical or electrical nature, but of magnetic nature.

Also, between conductors carrying macroscopic electric currents, ponderomotive actions are exerted. These actions are not of thermal, mechanical or electrical nature, but of magnetic nature. Therefore, upon bodies carrying electric currents, ponderomotive actions that are not of thermal, mechanical or electrical nature but of magnetic nature, can be exerted.

The ponderomotive actions of magnetic nature between magnetized bodies, or carrying electric currents, and other magnetized bodies, or carrying electric currents, are not directly exerted but by means of the magnetic field.

The *magnetic field* is the physical system that exists in the space regions in which ponderomotive actions (forces or torques) of magnetic nature can be exerted upon magnetized bodies or on bodies carrying macroscopic electric currents.

The magnetic field exists around magnetized bodies as well as in their interior and represents, in any reference frame, as a component of the electromagnetic field, namely the magnetic field. The magnetic field can be produced by various causes: Electric charges in motion, electric currents, variation with time of the electric field strength, variation with time of the electric polarization, and electrified bodies in motion relatively to the considered reference frame.

The electromagnetic field is a physical system that exists in the space regions, in which ponderomotive actions of electric nature as well as of magnetic nature are exerted upon bodies that are electrified, magnetized or crossed by electric currents.

2.7. TRANSFORMATION RELATION OF THE VOLUME DENSITY OF THE FREE (TRUE) ELECTRIC CHARGE

Let K_o and K_1 be two inertial reference frames. In any reference frame, the volume density of the true electric charge will be expressed in terms of the volume density of the true electric charge in the reference frame in which the electric charges are at rest.

We suppose that the electric charge is at rest in the reference frame K_1. At the same time, we take in view that in accordance with the non-correlative definition of the electric charge (of the Sub-section 1.6.2), the magnitude of the true electric charge is invariant relatively to the change of the inertial reference frame by another one. Then, the conservation law of the true electric charge of a volume element, in the two reference, frames K_o and K_1, can be expressed as follows:

$$\rho_{vo}\, \mathrm{d}\, v_o = \rho_{v1}\, \mathrm{d}\, v_1 \,, \tag{2.37}$$

and according to relation (2.10), it results that:

$$\rho_{vo} = \alpha_{1o}\, \rho_{v1} \,, \tag{2.38}$$

which is the transformation relation of the volume density of the true (free) electric charge.

The transformation relation of the volume density of the polarization electric charge has the same form.

2.8. THE EXPRESSIONS OF THE MAGNETIC FIELD STRENGTH PRODUCED AT A POINT BY A MOVING ELECTRIC CHARGE OR AN ELECTRIC CURRENT *IN VACUO*. THE BIOT-SAVART-LAPLACE FORMULA.

We shall consider the case in which the electric and magnetic field are produced by the charge q_1, *in vacuo*, mentioned at the beginning of Section 2.4. In this case, at any observation point $P(x_{1P}, y_{1P}, z_{1P})$, in the reference frame K_1, formula (2.15) yields:

$$\boldsymbol{E}_{1P} = \frac{q_1}{4\pi\varepsilon_0} \cdot \frac{\boldsymbol{r}_{1AP}}{r_{1AP}^3}, \tag{2.39}$$

and according to relation (2.21), we have in the reference frame K_o, at the same observation point $P(x_{oP}, y_{oP}, z_{oP})$, considered fixed in the reference frame K_o :

$$\boldsymbol{E}_{oP} = \frac{q_1}{4\pi\varepsilon_0} \left(\boldsymbol{i}\, \frac{x_{1P} - x_{1A}}{r_{1AP}^3} + \boldsymbol{j}\, \alpha_{1o}\, \frac{y_{1P} - y_{1A}}{r_{1AP}^3} + \boldsymbol{k}\, \alpha_{1o}\, \frac{z_{1P} - z_{1A}}{r_{1AP}^3} \right). \tag{2.40}$$

According to relation (2.26), we have at the same observation point $P(x_{1P}, y_{1P}, z_{1P})$:

$$\boldsymbol{B}_{oP} = \varepsilon_0\, \mu_0\, (\boldsymbol{v}_{1o} \times \boldsymbol{E}_{oP}) = \mu_0\, \frac{1}{4\pi}\, q_1\, \boldsymbol{v}_{1o} \times$$
$$\times \left(\boldsymbol{i}\, \frac{x_{1P} - x_{1A}}{r_{1AP}^3} + \boldsymbol{j}\, \alpha_{1o}\, \frac{y_{1P} - y_{1A}}{r_{1AP}^3} + \boldsymbol{k}\, \alpha_{1o}\, \frac{z_{1P} - z_{1A}}{r_{1AP}^3} \right), \tag{2.41}$$

$$\boldsymbol{H}_{oP} = \frac{1}{4\pi}\, q_1\, \boldsymbol{v}_{1o} \times \left(\boldsymbol{i}\, \frac{x_{1P} - x_{1A}}{r_{1AP}^3} + \boldsymbol{j}\, \alpha_{1o}\, \frac{y_{1P} - y_{1A}}{r_{1AP}^3} + \boldsymbol{k}\, \alpha_{1o}\, \frac{z_{1P} - z_{1A}}{r_{1AP}^3} \right). \tag{2.42}$$

For values of the ratio $\dfrac{v_{1o}}{c}$ small enough so that α_{1o} tends to unity, the last formula becomes:

$$\boldsymbol{H}_{oP} = \frac{1}{4\pi} \cdot \frac{q_1\, \boldsymbol{v}_{1o} \times \boldsymbol{r}_{1AP}}{r_{1AP}^3}. \tag{2.43}$$

If the charge is assumed to be continuously distributed around the point $A(x_{1A}, y_{1A}, z_{1A})$, in an infinitesimal volume $\mathrm{d}v_1$, then we can write, in the reference frame K_1, the relation $q_1 = \rho_{v1}\, \mathrm{d}v_1$ and it follows:

$$\mathrm{d}\boldsymbol{H}_{oP} = \frac{1}{4\pi} \cdot \frac{\rho_{v1}\, \boldsymbol{v}_{1o} \times \boldsymbol{r}_{1AP}}{r_{1AP}^3}\, \mathrm{d}v_1 \tag{2.44}$$

or, taking into account relation (1.70), we have:

$$\mathrm{d}\boldsymbol{H}_{\mathrm{o}P} = \frac{1}{4\pi} \cdot \frac{\boldsymbol{J}_{\mathrm{o}} \times \boldsymbol{r}_{1AP}}{r_{1AP}^{3}} \,\mathrm{d}v_1 \,. \tag{2.45}$$

In the case of a thread-like distribution of electric charge with the cross-section s_c in the reference frame K_1, the volume element $\mathrm{d}v_1$ becomes $\boldsymbol{s}_c \cdot \mathrm{d}\boldsymbol{l}$ (where the vectors $\mathrm{d}\boldsymbol{l}$ and $\boldsymbol{s}_c$ are parallel) and $\mathrm{d}\boldsymbol{l}$ is a length element. The quantity $i\,\mathrm{d}\boldsymbol{l}$ can be referred to as current element. Taking into account that the quantity $\boldsymbol{J}_{\mathrm{o}} \cdot \boldsymbol{s}_c$ represents an electric current i in respect to the reference frame K_{o}, the last formula can be written:

$$\mathrm{d}\boldsymbol{H}_{\mathrm{o}P} = \frac{1}{4\pi} \cdot \frac{i\,\mathrm{d}\boldsymbol{l} \times \boldsymbol{r}_{1AP}}{r_{1AP}^{3}} \,. \tag{2.46}$$

If we renounce the indices referring to the reference frames, the last relation yields:

$$\mathrm{d}\boldsymbol{H}_{P} = \frac{1}{4\pi} \cdot \frac{i\,\mathrm{d}\boldsymbol{l} \times \boldsymbol{r}_{AP}}{r_{AP}^{3}} \,. \tag{2.47}$$

The formula (2.47) is referred to as the *Biot-Savart-Laplace formula*. This formula was firstly established starting from certain experimental results.

In the case of a closed curve Γ, by integrating both sides of relation (2.47), it follows (Fig. 2.2):

$$\boldsymbol{H}(P) = \boldsymbol{H} = \frac{1}{4\pi} \oint_{\Gamma} \frac{i\,\mathrm{d}\boldsymbol{l} \times \boldsymbol{r}}{r^{3}} \,. \tag{2.48}$$
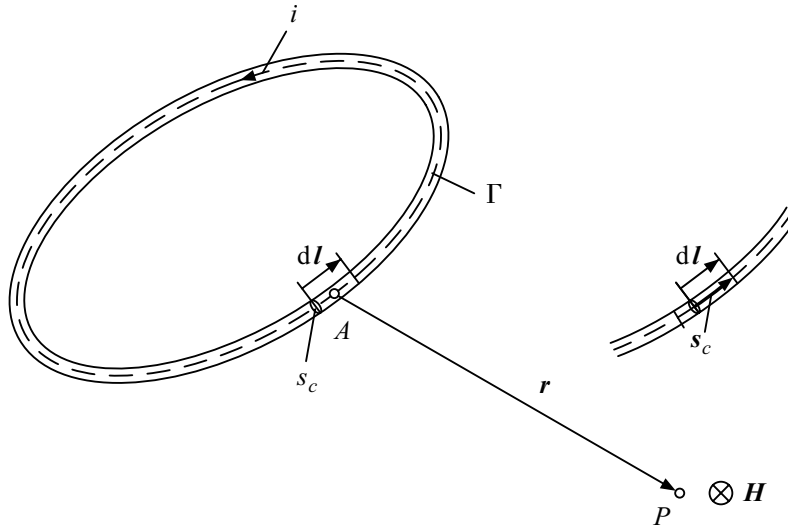
Fig. 2.2. Explanation to the calculation of the magnetic field strength produced by a circuit carrying an electric current.

It is interesting to make the following remark. Let us consider a two-dimensional orthogonal rectilinear system of co-ordinates xOy and two current elements, one of them placed along the Oy - axis and having its centre at the point O, and the second placed along the Ox - axis and having its centre at any point P on the same Ox - axis.

Let us calculate the magnetic field strength produced at the point P, by the current element having its centre at point O by formula (2.47). We shall obtain a vector $\boldsymbol{B}$ perpendicular at the point P, on the xOy - plane. Then, we calculate the force acting upon the current element with the centre at the point P, by formula (2.30 b). We obtain for this force a certain value different from zero. Hence, the force, calculated in this way, is different from zero.

If we calculate, in the same way, the force acting upon the current element having its centre at the point O, we obtain that this force is zero. Under these conditions the principle of action and reaction from the classical Mechanics is not satisfied.

For the remark above, in Classical Electromagnetism, it is considered that the formula (2.47) is valid only if it is applied for a closed contour, hence a closed electric current.

A special circumstance should be highlighted. The formulae used in calculation in Classical Electromagnetism concern electric conduction currents carried by conductors (wires). In a conductor, the electricity of charge carriers (electrons) is to a great extent compensated by the electricity of the lattice ions of the conductor (positive charge). For this reason, only the force of magnetic nature is calculated. That is not the case for the current elements previously examined, where only the moving charge carriers are considered. Therefore, both forces of magnetic and electric nature have to be calculated.

In fact, the remark above, relatively to the principle of action and reaction does not mean that the formulae concerning the current elements are not valid. Indeed, the formulae that have to be applied are the relations obtained from the Special Theory of Relativity, without the approximation assumed above. Therefore, the complete formulae (2.21), (2.22) or (2.28) must be used.

Let us consider that the current element along the Oy - axis contains the electric charge q_1 moving with the velocity v_1 in the positive direction of this axis, and also contains the immobile electric charge $-q_1$. The current element along the Ox - axis contains the electric charge q_2 moving with the velocity v_2 in the positive direction of this axis, and also contains the immobile electric charge $-q_2$.

The calculation carried out in [18, p. 263] has shown that concerning the principle of action and reaction of forces in the classical Mechanics, no deviation occurs along the Ox - axis, but a small deviation occurs along the Oy - axis, of the order of magnitude proportional to $\dfrac{v_1 v_2}{c^2}$.

Certain applications of formula (2.48) can be found in Section 3.15 and in papers [57], [58].

3. THE LAWS OF THE ELECTROMAGNETIC FIELD

3.1. THE LAW OF ELECTRIC FLUX

The expression of the law of electric flux can be established as a generalization of experimental results (i.e., the law being considered as correct, no experimental result has been found to contradict it) or deduced by utilizing certain relations established in the Special Theory of Relativity. We shall use both manners beginning with the second one.

3.1.1. The Expression of the Law of Electric Flux *in Vacuo*

Let K_o and K_1 be inertial reference frames. The reference frame K_1 is moving with the constant velocity $\boldsymbol{v}_{1o}$ relatively to the reference frame K_o.

Let Σ_1 be a closed surface in the reference frame K_1. The same surface, considered at the same position in space, but in the reference frame K_o, will be denoted by Σ_o, like in Fig. 3.1 a.

This surface contains in its interior a point-like electric charge q_1 situated *in vacuo* at the point A. We shall calculate the flux of the vectors $\varepsilon_0 \boldsymbol{E}_o$ through the closed surface Σ_o. Therefore:

$$\varepsilon_0 \int_{\Sigma_o} \boldsymbol{E}_o \cdot \mathrm{d}\boldsymbol{S}_o , \tag{3.1}$$

where:

$$\mathrm{d}\boldsymbol{S}_o = \boldsymbol{n}\,\mathrm{d}S_o . \tag{3.1 a}$$

By replacing the quantity $\boldsymbol{E}_o$ given by relation (2.21) and taking into account the relation (2.8 a, b, c), (2.9 a, b, c) and (2.21), we obtain:

$$\boldsymbol{E}_o \cdot \mathrm{d}\boldsymbol{S}_o = \left[\boldsymbol{i}\,E_{1x} + \boldsymbol{j}\,\alpha_{1o}\,E_{1y} + \boldsymbol{k}\,\alpha_{1o}\,E_{1z} \right] \cdot \left[\boldsymbol{i}\,\mathrm{d}y_o\,\mathrm{d}z_o + \boldsymbol{j}\,\mathrm{d}z_o\,\mathrm{d}x_o + \boldsymbol{k}\,\mathrm{d}x_o\,\mathrm{d}y_o \right] =$$

$$= E_{1x}\,\mathrm{d}y_o\,\mathrm{d}z_o + \alpha_{1o}\,E_{1y}\,\mathrm{d}z_o\,\mathrm{d}x_o + \alpha_{1o}\,E_{1z}\,\mathrm{d}x_o\,\mathrm{d}y_o = \tag{3.2 a}$$

$$= E_{1x}\,\mathrm{d}y_1\,\mathrm{d}z_1 + E_{1y}\,\mathrm{d}z_1\,\mathrm{d}x_1 + E_{1z}\,\mathrm{d}x_1\,\mathrm{d}y_1 .$$

It results that:

$$\boldsymbol{E}_o \cdot \mathrm{d}\boldsymbol{S}_o = \boldsymbol{E}_1 \cdot \mathrm{d}\boldsymbol{S}_1 . \tag{3.2 b}$$
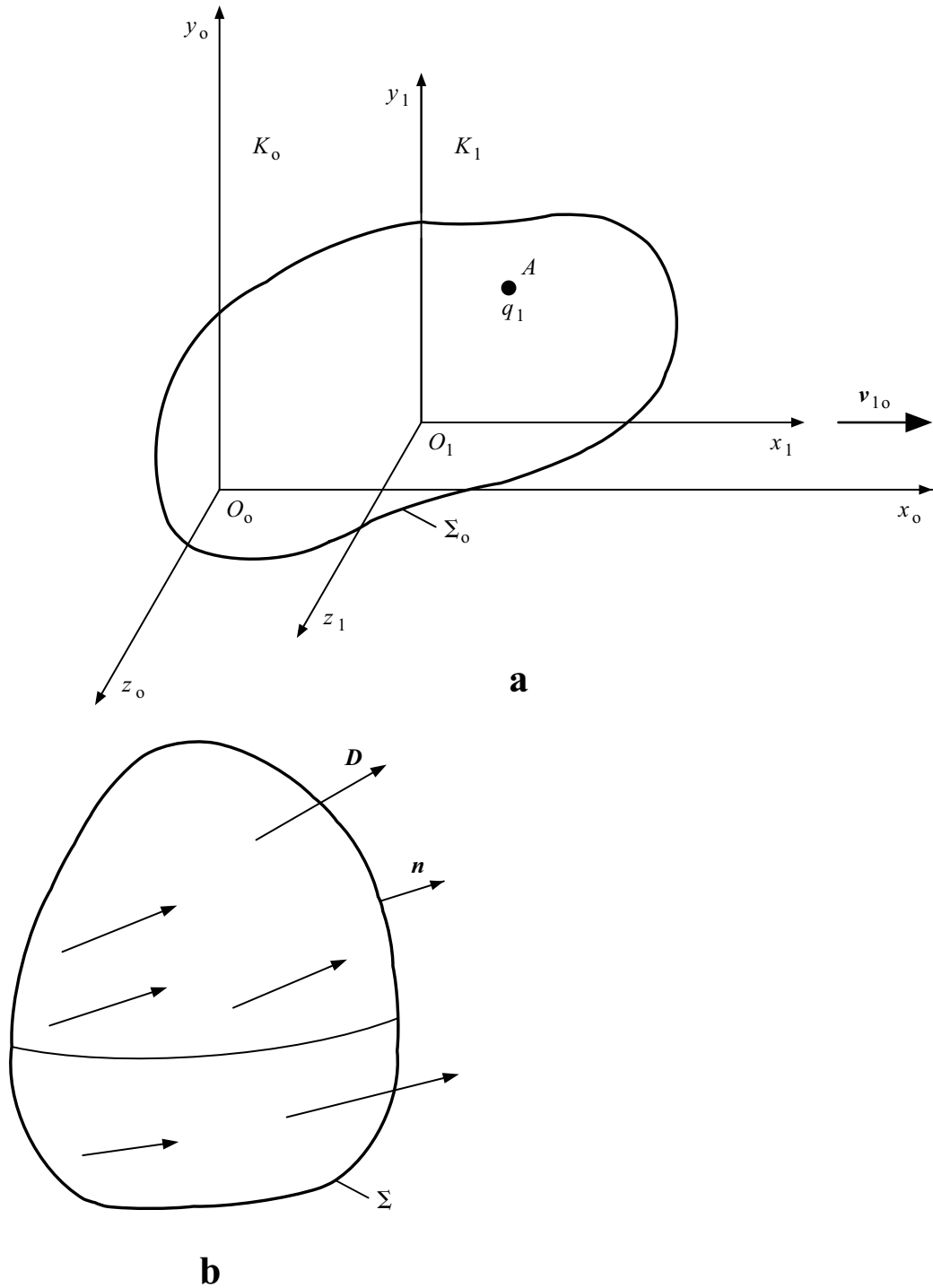
Fig. 3.1. Explanation concerning the integral form of the electric flux law in
vacuo referring to: **a** – deduction of expression; **b** – general expression.

But, in accordance with the law of electric flux *in vacuo* of Section 1.14, we can write:

$$\varepsilon_0 \int_{\Sigma_1} \boldsymbol{E}_1 \cdot \mathrm{d}\boldsymbol{S}_1 = q_1 . \tag{3.3}$$

It results that

$$\varepsilon_0 \int_{\Sigma_o} \boldsymbol{E}_o \cdot \mathrm{d}\boldsymbol{S}_o = q_1 . \tag{3.4}$$

In the case of several point-like electric charges, according to the hypothesis 2 of Section 2.2, we can apply the principle of superposition and expression (3.4) keeps the same form but instead of the charge q_1, the total charge q_{Σ_o} of the closed surface will be introduced.

The macroscopic quantities at a point, at any moment, are obtained by the computation of the average values (mean values) of the microscopic quantities over a physically infinitesimal volume, around the considered point and over a physically infinitesimal time interval containing the considered moment.

After the calculation of the average values, the form of relation 3.3 is not modified.

If instead of the point-like electric charge q_1, a distributed electric charge with the density ρ_{vo} in the volume V_{Σ_o} is given, then, by utilizing the transformation of the surface-integral into a volume-integral (Gauss-Ostrogradski theorem), it follows:

$$\varepsilon_0 \int_{\Sigma_o} \boldsymbol{E}_o \cdot \mathrm{d}\boldsymbol{S}_o = \int_{V_{\Sigma_o}} \rho_{vo} \, \mathrm{d}v_o = \int_{V_{\Sigma_o}} \mathrm{div}_o \, \varepsilon_0 \, \boldsymbol{E}_o \cdot \mathrm{d}v_o . \tag{3.5 a}$$

Since the preceding relation is valid for any surface Σ_o, it results that:

$$\mathrm{div}_o \, \varepsilon_0 \, \boldsymbol{E}_o = \rho_{vo} . \tag{3.5 b}$$

If we renounce the index o, and consider the total charge, relations (3.4) and (3.5 a) yield:

$$\int_{\Sigma} \varepsilon_0 \, \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = q_\Sigma , \tag{3.6}$$

where q_Σ is the total charge contained by the volume bounded by the surface Σ, and

$$\mathrm{div} \, \varepsilon_0 \, \boldsymbol{E} = \rho_v . \tag{3.7}$$

The expression established above remains valid for any inertial reference frame, since the differential operator divergence is invariant with respect to the transformation from one inertial referential frame to another, at rest relatively to the first one.

Relations (3.6) and (3.7) represent the integral form and the local form respectively, of the electric flux law for empty space (i.e., *in vacuo*) in any inertial reference frame.

3.1.2. The Expression of the Law of Electric Flux for Any Medium in the Case of Free (True) and Polarization Electric Charges

The expression of the law of electric flux for any medium, as previously mentioned, can be established either by the generalization of experimental results or using the results of the Special Theory of Relativity.

In the case of any medium, it is necessary to establish the relations between the macroscopic state quantities of the electromagnetic field. The macroscopic quantities at a point and at a certain moment can be obtained by calculating the mean values (average) of the microscopic quantities over a physically infinitesimal volume around the considered point and over a physically infinitesimal time interval that contains the considered moment.

In the case of any medium, both free (true) electric charges and bound electric charges can exist. If it is considered that only free electric charges exist, then, after the calculation of the mean value, the form of the relations between the various macroscopic quantities remains the same as for the microscopic quantities. If it is considered that both free (true) electric charges and bound electric charges exist, then, after the calculation of the mean value, the form of the relations between various macroscopic quantities is modified, as it will be shown further on. The symbols and the meaning of the quantities are the ones used in Chapter 1.

It will be considered that in the inertial system of reference frame K_o, in which the vector quantity $\boldsymbol{E}_\text{o}$ is expressed, there are both true (free) electric charges with the volume density $\rho_{v\text{o}}$ and polarization electric charges with the volume density $\rho_{pv\text{o}}$. In this case, after the calculation of the mean values, relation (3.4) yields:

$$\int_{\Sigma_\text{o}} \varepsilon_0\, \boldsymbol{E}_\text{o} \cdot \mathrm{d}\boldsymbol{S} = \int_{V_{\Sigma_\text{o}}} \left(\rho_{v\text{o}} + \rho_{pv\text{o}} \right) \mathrm{d}v_\text{o}\,, \tag{3.8 a}$$

thus:

$$\int_{\Sigma_\text{o}} \varepsilon_0\, \boldsymbol{E}_\text{o} \cdot \mathrm{d}\boldsymbol{S}_\text{o} - \int_{V_{\Sigma_\text{o}}} \rho_{pv\text{o}}\, \mathrm{d}v_\text{o} = \int_{V_{\Sigma_\text{o}}} \rho_{v\text{o}}\, \mathrm{d}v_\text{o}\,. \tag{3.8 b}$$

The volume density of the polarization electric charges $\rho_{pv\text{o}}$ in terms of the vector quantity $\boldsymbol{P}_\text{o}$, electric polarization, in the reference frame K_o, and also in any other inertial reference frame, according to relation (1.45), is:

$$\rho_{pv\text{o}} = -\operatorname{div}\boldsymbol{P}_\text{o}\,. \tag{3.9}$$

It results that:

$$\int_{\Sigma_\text{o}} \varepsilon_0\, \boldsymbol{E}_\text{o} \cdot \mathrm{d}\boldsymbol{S}_\text{o} + \int_{V_{\Sigma_\text{o}}} \operatorname{div}\boldsymbol{P}_\text{o}\, \mathrm{d}v_\text{o} = \int_{V_{\Sigma_\text{o}}} \rho_{v\text{o}}\, \mathrm{d}v_\text{o} \tag{3.10 a}$$

or, by transforming the first volume-integral into a surface-integral (Gauss-Ostrogradski theorem):

$$\int_{\Sigma_o} \varepsilon_0\, \boldsymbol{E}_o \cdot \mathrm{d}\boldsymbol{S}_o + \int_{\Sigma_o} \boldsymbol{P}_o \cdot \mathrm{d}\boldsymbol{S}_o = \int_{V_{\Sigma_o}} \rho_{vo}\, \mathrm{d}v_o\,. \tag{3.10 b}$$

Let us denote:

$$\boldsymbol{D}_o = \varepsilon_0\, \boldsymbol{E}_o + \boldsymbol{P}_o\,. \tag{3.11}$$

The vector quantity $\boldsymbol{D}_o$, is referred to by one of the terms: *Electric displacement*, *electric flux density* or *electric induction*, and is introduced in the reference frame in which the quantities $\boldsymbol{E}_o$ and $\boldsymbol{P}_o$ are expressed.

Therefore, in any inertial reference frame K_o, the following relations are satisfied:

$$\int_{\Sigma_o} \boldsymbol{D}_o \cdot \mathrm{d}\boldsymbol{S}_o = q_{\Sigma_o}\,, \tag{3.12}$$

$$q_{\Sigma_o} = \int_{V_{\Sigma_o}} \rho_{vo}\, \mathrm{d}v_o\,, \tag{3.12 a}$$

where q_{Σ_o} represents the free (true) electric charge of the interior of the surface Σ_o.

Taking into account that the two last relations are valid for any surface Σ_o and by utilizing the transformation theorem of a surface-integral into a volume-integral (Gauss and Ostrogradski theorem), relation (3.12), written in an integral form, yields the following relation, in differential (local) form:

$$\mathrm{div}_o\, \boldsymbol{D}_o = \rho_{vo}\,. \tag{3.13}$$

Relations (3.12) and (3.13) represent the *expressions of the integral* and *local form*, respectively, of *the law of electric flux for any medium*. Relation (3.12) is also called the *Gauss law* (*theorem*).

3.1.3. The General Expression of the Law of Electric Flux

The general expression of the law of electric flux, previously deduced, can be established as a generalization of experimental results, in the sense mentioned at the beginning of Section 3.1.

In an inertial reference frame, in any medium, the flux of the vector electric displacement $\boldsymbol{D}$ through a closed surface Σ is equal to the sum of electric true (free) charges of the inside of that surface (Fig. 3.1 b):

$$\int_{\Sigma} \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{S} = q_{\Sigma}\,. \tag{3.14}$$

Let us consider that the electric charge is macroscopically distributed with the volume density ρ_v.

By using the Gauss-Ostrogradski theorem and expression (1.11 a), relation (3.14), becomes:

$$\int_{V_\Sigma} \operatorname{div} \boldsymbol{D} \, \mathrm{d}v = \int_{V_\Sigma} \rho_v \, \mathrm{d}v. \tag{3.15}$$

Having in view that relation (3.15) is valid for any surface Σ, the following differential relation is obtained:

$$\operatorname{div} \boldsymbol{D} = \rho_v , \tag{3.16}$$

where the vector quantity $\boldsymbol{D}$ is the electric displacement at the considered point, and ρ_v is the volume density of the true (free) electric charge at the same point.

Relations (3.14) and (3.16) represent the integral and local form, respectively, of the electric flux law.

3.1.4. The Electric Flux through Various Surfaces

The electric flux Ψ_{el} through an open surface is the surface-integral of the vectors electric induction $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$. It follows:

$$\Psi_{\mathrm{el}} = \int_{S_\Gamma} \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{S}, \tag{3.17}$$

where S_Γ is any open surface bounded by the closed curve Γ. The positive direction of the normal to the surface S_Γ is associated to the right-handed screw rule with the travelling direction around the curve Γ.

For the calculation of the electric flux through a closed surface, that is the case of the law of electric flux, it is necessary to have in view that when calculating the surface-integral over a closed surface, the positive direction of the normal to the surface is outward.

3.2. THE RELATION BETWEEN THE ELECTRIC DISPLACEMENT, ELECTRIC FIELD STRENGTH AND ELECTRIC POLARIZATION

We shall examine, for the case of a medium containing any substance, the relation between the following state quantities of the electric field: Electric displacement, electric field strength and electric polarization.

The macroscopic state quantities of the electromagnetic field at a point and at a certain moment are obtained by calculating the mean values of the microscopic quantities over physically infinitesimal entities of space and time containing the considered point and moment, respectively.

In the case of any medium, there can be both true (free) electric charges and polarization (bound) electric charges.

If we consider that only free electric charge exists, then, after calculating the mean values, the form of relations between the values of the macroscopic quantities remain the same as for the microscopic quantities.

But if there are free electric charges as well as polarization charges, then, after calculating the mean values, the form of the relations between the values of the macroscopic quantities will be modified.

In the framework of the macroscopic theory of the electromagnetic field, the vector quantity referred to by one of the terms: *Electric displacement, electric flux density or electric induction* will be introduced by the following relation:

$$\boldsymbol{D} = \varepsilon_0 \, \boldsymbol{E} + \boldsymbol{P}, \tag{3.18}$$

that occurred in calculations of Sub-section 3.1.2, relation (3.11).

In the framework of the macroscopic theory, the vectors $\boldsymbol{D}$ and $\boldsymbol{E}$ are directly introduced as primitive (fundamental) quantities.

Relation (3.18) represents, in the framework of the macroscopic theory, *the expression of the law of relationship between the vectors: Electric displacement, electric field strength and electric polarization.*

3.3. THE LAW OF TEMPORARY ELECTRIC POLARIZATION

In the case of any isotropic medium (i.e., containing any isotropic substance), but without permanent electric polarization, the vector of temporary electric polarization is generally parallel with the electric field strength, and of the same direction. Hence:

$$\boldsymbol{P}_t = \varepsilon_0 \, \chi_e \, \boldsymbol{E}. \tag{3.19}$$

The quantity χ_e is termed *electric susceptibility* and depends on the nature of the medium, being a *material quantity*.

Relation (3.19) can be macroscopically established only by experimental way. This relation, in the macroscopic theory of the electromagnetic field, is a *material law* referred to as *the law of temporary electric polarization*.

Practically, for variations small enough of the quantity $\boldsymbol{E}$, the quantity χ_e is constant with respect to the vector quantity $\boldsymbol{E}$ for a part of the isotropic media that are called linear and isotropic media.

In the case of an anisotropic medium, the orientation of the vector $\boldsymbol{P}$ is not, generally, the same with that of the vector $\boldsymbol{E}$. However, in these media it is possible to determine generally, three-orthogonal axes, called *principal axes* or *eigenaxes* along which the anisotropic medium behaves like an isotropic one. For these media, the law of the temporary polarization is written in the form:

$$\boldsymbol{P}_t = \varepsilon_0 \, \chi_e \, \boldsymbol{E}, \tag{3.20}$$

where the *tensor quantity* χ_e represents the *electric susceptibility*.

For isotropic media without permanent electric polarization, the relation between the quantities electric induction, electric field strength and electric polarization (3.18) becomes:

$$\boldsymbol{D} = \varepsilon_0\,\boldsymbol{E} + \boldsymbol{P}_t = \varepsilon_0\left(1 + \chi_e\right)\boldsymbol{E}\,. \tag{3.21}$$

The following symbols are adopted:

$$\varepsilon_r = 1 + \chi_e\,, \tag{3.22}$$

$$\varepsilon = \varepsilon_0\,\varepsilon_r\,. \tag{3.23}$$

The quantity ε_r is called *relative electric permittivity* or *dielectric constant* of the medium and the quantity ε is called *electric permittivity* of the medium. With these symbols, relation (3.21) becomes:

$$\boldsymbol{D} = \varepsilon\,\boldsymbol{E}\,. \tag{3.24}$$

Taking into account the established relations that contain the quantities $\boldsymbol{E}$ and $\boldsymbol{D}$, certain general considerations concerning the electric field lines and electric induction lines will be made.
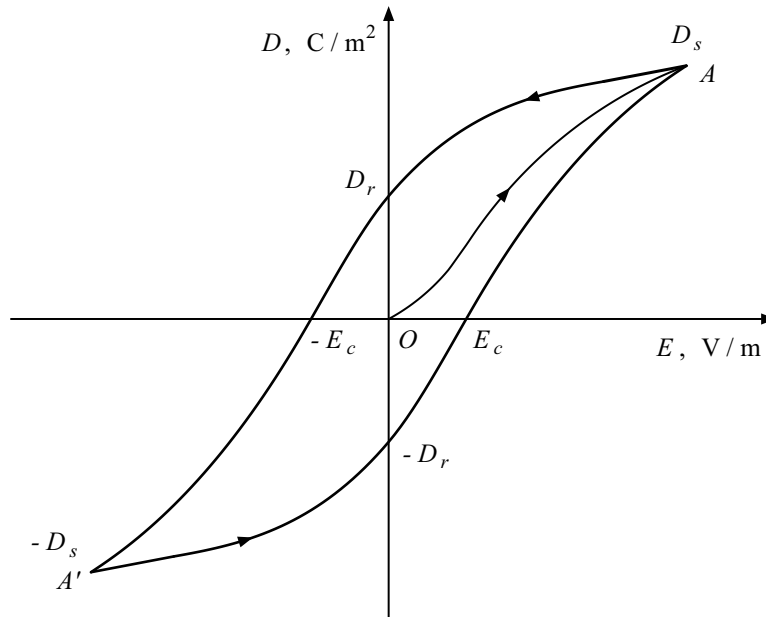
Fig. 3.2. Electric hysteresis loop, the direction of travelling along the curve (upward when the electric field strength increases, downward when the electric field strength decreases); at the middle, the branch starting from $E = 0$ and $D = 0$.

The line to which the vector E is tangent at any point is a *line of electric field*. The line to which the vector D is tangent at any point is a *line of electric displacement.*

In accordance with the explanations of Sub-section 1.2.1, the lines of electric field can be open lines that begin and end at points at which the divergence of the vector E is different from zero, or can be closed lines.

As it follows from relations (3.16) and (3.24), the quantity $\mathrm{div}\,E$ may be different from zero at points at which the density of the true electric charge is different from zero or at points at which the permittivity varies, thus at the points at which the quantity $\mathrm{grad}\,\varepsilon$ is different from zero (for example on the contact surface of two different dielectrics, because there $\mathrm{div}_s\,E \neq 0$).

In accordance with relation (3.16), the electric induction lines may be open lines that begin and finish at points at which the divergence of the vector D is different from zero or may be closed lines.

As it follows from relation (3.16), the quantity $\mathrm{div}\,D$ is different from zero only at points at which the density of electric charge is different from zero.

Similar considerations, concerning the lines of the electric field and the lines of electric displacement, may be extended also for the case in which permanent electric polarization exists. In this case, from relations (3.18), (3.24), (1.39), it follows:

$$D = \varepsilon\,E + P_p\,. \tag{3.25}$$

The dependence between the quantities D and E may be linear or non-linear.

Also, sometimes, the curve of D as a function of E may represent a closed curve (Fig. 3.2) called by one of the terms: *electric hysteresis loop, electric hysteresis cycle* or *electric hysteresis curve*. Certain characteristic points are marked on the figure.

3.4. THE LAW OF MAGNETIC FLUX

The expression of the magnetic flux can be established as a generalization of the experimental results (i.e., in the sense mentioned at the beginning of Section 3.1) or can be derived by using the relation established in the Special Theory of Relativity.

Further on, firstly, a derivation by using certain relations established in the Special Theory of Relativity will be presented.

3.4.1. The Expression of the Law of Magnetic Flux for Empty Space

Let us consider the inertial reference frames K_o and K_1. The reference frame K_1 is moving with a constant velocity $v_{1\mathrm{o}}$ relatively to the reference frame K_o. For the sake of simplicity, we assume $v_{1\mathrm{o}} = i\,v_{1\mathrm{o}x}$.

Let us also consider one point-like electric charge q situated in empty space (i.e., *in vacuo*) at a point $A(x_{1A}, y_{1A}, z_{1A})$. We shall calculate $\mathrm{div}_\mathrm{o}\,B_\mathrm{o}$ at any point P. We have:

$$\mathrm{div}_\mathrm{o}\,B_\mathrm{o} = \mathrm{div}_\mathrm{o}\left[\frac{1}{c^2}\,v_{1\mathrm{o}} \times E_\mathrm{o}\right] = \frac{1}{c^2}\left[E_\mathrm{o} \cdot \mathrm{curl}_\mathrm{o}\,v_{1\mathrm{o}} - v_{1\mathrm{o}} \cdot \mathrm{curl}_\mathrm{o}\,E_\mathrm{o}\right]. \tag{3.26}$$

From (2.20 d) and (2.21), it follows:

$$E_{oy} = \alpha_{1o} E_{1y} \, , \tag{3.26 a}$$

$$E_{oz} = \alpha_{1o} E_{1z} \, , \tag{3.26 b}$$

$$\boldsymbol{i} \cdot \left(\mathbf{curl}_o \, \boldsymbol{E}_o \right) = \frac{\partial E_{oz}}{\partial y_o} - \frac{\partial E_{oy}}{\partial z_o} = \alpha_{1o} \left(\frac{\partial E_{1z}}{\partial y_1} - \frac{\partial E_{1y}}{\partial z_1} \right) = 0 \,. \tag{3.26 c}$$

According to the last relation, the vector $\mathrm{curl}_o \, \boldsymbol{E}_o$ is perpendicular to the Ox - axis.

But $\mathrm{curl}_o \, \boldsymbol{v}_{1o} = 0$ because the velocity is constant, and $\boldsymbol{v}_{1o} \cdot \mathrm{curl}_o \, \boldsymbol{E}_o = 0$, since the quantities $\boldsymbol{v}_{1o}$ and $\mathrm{curl}_o \, \boldsymbol{E}_o$ are perpendicular to each other. Therefore, it follows:

$$\mathrm{div}_o \, \boldsymbol{B}_o = 0. \tag{3.27}$$

The left-hand side of relation (3.27) will be integrated over any closed surface Σ of the reference frame K_o. Utilizing the transformation of the volume-integral into a surface integral (Gauss-Ostrogradski theorem), it follows:

$$\int_{V_{\Sigma_o}} \mathrm{div}_o \, \boldsymbol{B}_o \, \mathrm{d} v_o = \int_{\Sigma_o} \boldsymbol{B}_o \cdot \mathrm{d} \boldsymbol{S}_o = 0. \tag{3.28}$$

Thus, the flux of the vector $\boldsymbol{B}_o$ through any closed surface is zero.

In the case of several point-like electric charges, according to the assumption 2 of Section 2.2, the principle of superposition can be applied, and for each of these charges a relation of the type (3.27) will be obtained. By summing up, side by side, these relations, it follows that the expression (3.27) remains valid. Moreover, the relation holds for any inertial reference frame (with any orientation of the three-orthogonal rectilinear system of co-ordinates), since the differential operator div is invariant relatively to the transformation of the co-ordinates when passing from one inertial frame to another, at rest, relatively to the first one.

Relations (3.27) and (3.28) represent the expressions of the local and integral form respectively, of the law of magnetic flux in empty space (i.e., in vacuo).

3.4.2. The Expression of the Law of Magnetic Flux for Any Medium

Expression (3.28), as mentioned, can be established as a generalization of experimental results (in the sense mentioned at the beginning of Section 3.1).

In the case of any medium (i.e., containing any type of substance), the aim is to establish the relations between the macroscopic state quantities of the electromagnetic field.

The macroscopic quantities at a point and at a certain moment are obtained by the calculation of the mean values of microscopic quantities over a physically infinitesimal volume, around the considered point and over a physically infinitesimal time interval containing the considered moment.
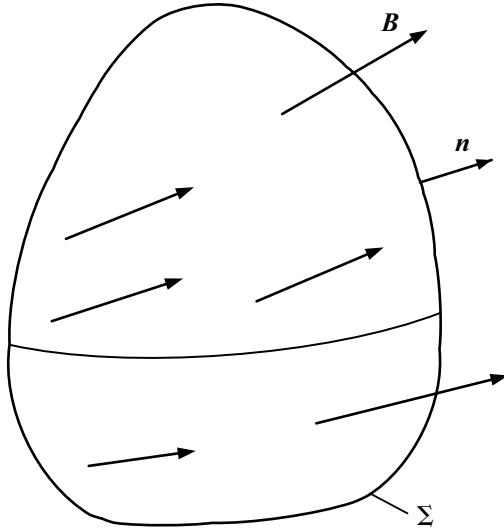
Fig. 3.3. Explanation referring to
the law of magnetic flux.
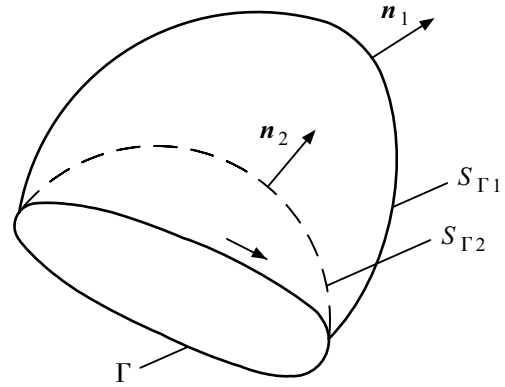
Fig. 3.4. Two open surfaces that are
bounded by the same closed curve Γ.

In the case in which electric charges do not directly occur in relations, then, after the calculation of average values, the form of relations between the macroscopic quantities remains the same as for the microscopic quantities. For this reason, relations (3.27) and (3.28) remain also valid for any medium.

3.4.3. The General Expression of the Law of Magnetic Flux (for Any Medium)

The general expression of the law of magnetic flux derived previously, as mentioned, can be established as a generalization of experimental results (in the sense mentioned at the beginning of Section 3.1).

In an inertial reference frame, in any medium, the flux of the vector magnetic induction $\boldsymbol{B}$ through a closed surface Σ is zero (Fig. 3.3):

$$\int_{\Sigma} \boldsymbol{B} \cdot \mathrm{d}\,\boldsymbol{S} = 0. \tag{3.29}$$

Utilizing the Gauss-Ostrogradski theorem, relation (3.29) becomes:

$$\int_{V_\Sigma} \mathrm{div}\,\boldsymbol{B}\,\mathrm{d}\,v = 0. \tag{3.30}$$

Taking into account that relation (3.30) remains valid for any closed surface Σ, the following relation in differential form will be obtained:

$$\mathrm{div}\,\boldsymbol{B} = 0. \tag{3.31}$$

Expressions (3.29) and (3.31) represent the *integral form* and *local form, respectively, of the law of magnetic flux for any medium*. The vector quantity $\boldsymbol{B}$ is introduced as shown in Section 2.5.

By comparing the local form of the law of electric flux with the local form of the law of magnetic flux, it results that the field of vectors $\boldsymbol{B}$ has no sources (because $\operatorname{div}\boldsymbol{B} = 0$). Therefore, the following statement is equivalent: The existence of magnetic forces, having the direction and the orientation of the vector $\boldsymbol{B}$ is not possible.

Certain investigators, for instance Dirac, in the year 1948 [18, p. 43] supposed that true magnetic charges referred to as *magnetic monopoles*, could exist or could be produced in accelerators of high energies.

If magnetic monopoles existed, then the lines of magnetic field would diverge from the magnetic monopoles like the lines of electric field diverge from the electric charges. For a system of magnetic monopoles, the divergence of the magnetic induction would not be zero, but equal to the volume density of the magnetic charge.

Up to now, the existence of magnetic monopoles, despite many investigations made during the recent years, has not been proved. Even if the existence of magnetic monopoles were established, the equations of the electromagnetic field would have to be modified only in quite special cases, in order to take into account the presence of magnetic monopoles.

It follows that it is justified to assume that true magnetic charges have not to be taken into consideration, and the divergence of the vector magnetic induction is zero.

In a vector field which has the divergence zero, the lines of field are always closed. Thus the lines of magnetic induction are always closed lines because in the magnetic field there are no sources (magnetic charges).

3.4.4. The Magnetic Flux through Various Surfaces

The magnetic flux Ψ through an open surface is the surface integral of the vector magnetic induction:

$$\Psi = \int_{S_\Gamma} \boldsymbol{B} \cdot \mathrm{d}\,\boldsymbol{S}, \tag{3.32}$$

where S_Γ is a simply connected open surface bounded by the closed curve Γ. The positive direction of the normal to the surface S_Γ is associated according to the right-handed screw rule with the travelling sense along the curve Γ.

The magnetic flux through two open simply connected surfaces bounded by the same closed curve Γ is the same. Indeed, if we consider two surfaces S_{Γ_1} and S_{Γ_2}, bounded by the same curve, their union forms a closed surface. In this case, by virtue of the law of magnetic flux, it follows (Fig. 3.4):

$$\Psi_{S_{\Gamma_1}} - \Psi_{S_{\Gamma_2}} = 0, \tag{3.32 a}$$

and hence (according to the explanation below), it follows:

$$\Psi_{S_{\Gamma_1}} = \Psi_{S_{\Gamma_2}}. \tag{3.33}$$

When applying the law of magnetic flux, it is necessary to take into account that at the calculation of a surface-integral over a closed surface, the positive direction of the normal to the surface is outward. Consequently, in the case of the preceding example, the positive direction of the normal to one of the open surfaces, in accordance with the chosen sense of travelling the bounding curve, will be opposite with respect to the positive direction of the normal to the closed surface, resulted from the union of the two open surfaces.

3.4.5. The Magnetic Flux-Turn. The Magnetic Flux-Linkage. Magnetic Vector Potential.

In the case in which the curve Γ that bounds the surface S_Γ through which the magnetic flux has to be calculated is taken along the conductor (wire) of a coil having w turns (Fig. 3.5), the notions of magnetic flux-turn and magnetic flux-linkage have to be introduced (defined). It is to be noted that the number of turns of a coil or generally of any winding can be denoted by w as well as by N, the last being utilized in several standards. However, we shall prefer the former symbol, because the latter has also other meanings in certain applications widely utilized.

Fig. 3.5 shows the coil conductor, two lines of magnetic induction, a helical surface (hatched) bounded by the coil contour and also the electric tension between the two points M and N of the coil.

The *magnetic flux-turn* is the magnetic flux that crosses the portion of the helical surface bounded by a single turn of the coil, according to relation:

$$\Phi = \int_{S_{\Gamma_w}} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S}. \tag{3.34}$$

The magnetic flux through the whole surface is calculated taking into account that a line of magnetic field intersects several times the surface.

The *magnetic flux-linkage*, also called *total magnetic flux*, is the magnetic flux that crosses the helical surface bounded by all the turns of the coil, according to relation:

$$\Psi = \int_{S_\Gamma} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S}. \tag{3.35}$$

The last two relations yield:

$$\Psi = w\Phi, \tag{3.36}$$

because the same line of magnetic field intersects w times the helical surface that is bounded by all the turns of the coil. If the magnetic flux-turn has not the same value for all the turns, then a mean value of the magnetic flux turn, also called *average magnetic flux-turn*, will be used.

According to relation (3.31), the divergence of the vector $\boldsymbol{B}$ is always zero. Consequently, it is possible to express the vector $\boldsymbol{B}$ as follows:

$$\boldsymbol{B} = \operatorname{curl} \boldsymbol{A}, \tag{3.37}$$

where $\boldsymbol{A}$ denotes the quantity called *magnetic vector potential*. Since the divergence of a curl is always zero, several vectors $\boldsymbol{A}$ can give the same vector magnetic induction $\boldsymbol{B}$.
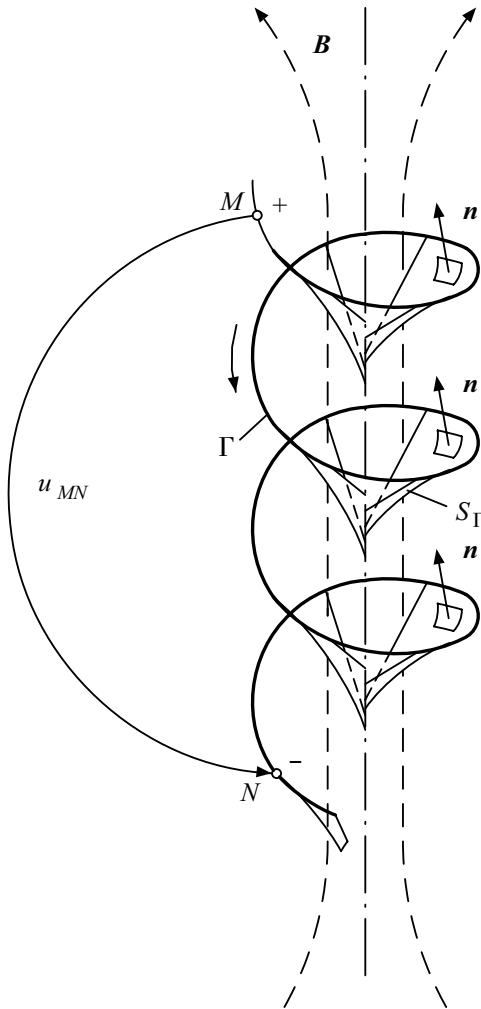
Fig. 3.5. Explanation to the calculation of magnetic fluxes: flux-turn and flux-linkage.

For example, the vectors $A = -i(yB)$ and $A = j(xB)$ give the same vector $B = kB$. This circumstance is of no interest, for only the value of B is important in this case.

By employing the last relation, the magnetic flux through an open surface, as previously, can be expressed in the form:

$$\Psi = \int_{S_\Gamma} B \cdot \mathrm{d}S = \int_{S_\Gamma} \operatorname{curl} A \cdot \mathrm{d}S,$$

(3.38)

that, by using the Stokes theorem, becomes:

$$\Psi = \int_{S_\Gamma} B \cdot \mathrm{d}S = \oint_\Gamma A \cdot \mathrm{d}l.$$

(3.39)

Therefore, the magnetic flux through a simply connected open surface bounded by a closed curve can be calculated by a line-integral along that curve. By using this

expression, the conclusions obtained above concerning the flux through surfaces related to some curves result directly.

3.5. THE LAW OF ELECTROMAGNETIC INDUCTION FOR MEDIA AT REST

The expression of the law of electromagnetic induction may be established by a generalization of experimental results (i.e., in the sense mentioned at the beginning of Section 3.1) or deduced by utilizing certain relations established in the Special Theory of Relativity. We shall use both manners beginning with the second one.

3.5.1. The Expression of the Law of Electromagnetic Induction for Empty Medium at Rest

Let K_o and K_1 be the two inertial reference frames. The reference frame K_1 is moving at a constant velocity $\boldsymbol{v}_{1o}$ relatively to the reference frame K_o.

We shall consider a point-like charge q_1, situated in empty space (i.e., *in vacuo*) at any point $A(x_{1A}, y_{1A}, z_{1A})$ fixed to the reference frame K_1, and we shall calculate $\mathrm{curl}_o\, \boldsymbol{E}_o$ at the point $P(x_o, y_o, z_o)$. It follows:

$$\mathrm{curl}_o\, \boldsymbol{E}_o = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x_o} & \dfrac{\partial}{\partial y_o} & \dfrac{\partial}{\partial z_o} \\ E_{ox} & E_{oy} & E_{oz} \end{vmatrix}. \tag{3.40}$$

The component along the $O_o x_o$ - axis is:

$$\frac{\partial E_{oz}}{\partial y_o} - \frac{\partial E_{oy}}{\partial z_o}, \tag{3.40 a}$$

and by using relations (2.21), (2.18 b, c) and (2.17), we obtain:

$$\frac{\partial E_{oz}}{\partial y_o} = \frac{q_1}{4\pi\varepsilon_0}\alpha_{1o}\left(y_o - y_{1A}\right)\left(z_o - z_{1A}\right)r_1^{-5}\left(-3\right), \tag{3.41 a}$$

$$\frac{\partial E_{oy}}{\partial z_o} = \frac{q_1}{4\pi\varepsilon_0}\alpha_{1o}\left(z_o - z_{1A}\right)\left(y_o - y_{1A}\right)r_1^{-5}\left(-3\right). \tag{3.41 b}$$

By replacing relations (3.41 a, b) into expression (3.40 a), it follows:

$$\frac{\partial E_{oz}}{\partial y_o} - \frac{\partial E_{oy}}{\partial z_o} = 0. \tag{3.42 a}$$

Analogously, it follows that the components along the axes $O_o y_o$ and $O_o z_o$ are:

$$\frac{\partial E_{ox}}{\partial z_o} - \frac{\partial E_{oz}}{\partial x_o} = \frac{q_1}{4\pi\varepsilon_0}[\alpha_{1o}(x_o - v_{1ox}t_o) - x_{1A}](z_o - z_{1A})3\alpha_{1o}^2\beta_{1o}^2 r_1^{-5}, \qquad (3.42\text{ b})$$

$$\frac{\partial E_{oy}}{\partial x_o} - \frac{\partial E_{ox}}{\partial y_o} = \frac{q_1}{4\pi\varepsilon_0}[\alpha_{1o}(x_o - v_{1ox}t_o) - x_{1A}](y_o - y_{1A})(-3)\alpha_{1o}^2\beta_{1o}^2 r_1^{-5}, \qquad (3.42\text{ c})$$

because

$$\alpha_{1o}^2 = \frac{1}{1 - \beta_{1o}^2}; \qquad (3.42\text{ d})$$

$$\beta_{1o} = \frac{v_{1o}}{c}; \qquad (3.42\text{ e})$$

$$1 - \alpha_{1o}^2 = 1 - \frac{1}{1 - \beta_{1o}^2} = -\alpha_{1o}^2\beta_{1o}^2. \qquad (3.42\text{ f})$$

Now, we calculate the vector quantity:

$$\frac{\partial \boldsymbol{B}_o}{\partial t_o} = \boldsymbol{i}\frac{\partial B_{ox}}{\partial t_o} + \boldsymbol{j}\frac{\partial B_{oy}}{\partial t_o} + \boldsymbol{k}\frac{\partial B_{oz}}{\partial t_o}. \qquad (3.43)$$

According to relation (2.26) and taking into account that $\varepsilon_0\mu_0 = \dfrac{1}{c^2}$, the vector quantity $\boldsymbol{B}_o$ is:

$$\boldsymbol{B}_o = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{1}{c^2}\boldsymbol{v}_{1ox} & 0 & 0 \\ E_{ox} & E_{oy} & E_{oz} \end{vmatrix} = -\boldsymbol{j}\cdot\left[\frac{1}{c^2}\boldsymbol{v}_{1ox}\alpha_{1o}E_{1z}\right] + \boldsymbol{k}\cdot\left[\frac{1}{c^2}\boldsymbol{v}_{1ox}\alpha_{1o}E_{1y}\right]. \qquad (3.44)$$

After the calculation of derivatives, we obtain the components:

$$\frac{\partial B_{ox}}{\partial t_o} = 0, \qquad (3.45\text{ a})$$

$$\frac{\partial B_{oy}}{\partial t_o} = -3\frac{1}{c^2}v_{1o}^2\alpha_{1o}^2\frac{q_1}{4\pi\varepsilon_0}[\alpha_{1o}(x_o - v_{1ox}t_o) - x_{1A}](z_o - z_{1A})r_1^{-5}, \qquad (3.45\text{ b})$$

$$\frac{\partial B_{oz}}{\partial t_o} = 3\frac{1}{c^2}v_{1o}^2\alpha_{1o}^2\frac{q_1}{4\pi\varepsilon_0}[\alpha_{1o}(x_o - v_{1ox}t_o) - x_{1A}](y_o - y_{1A})r_1^{-5}. \qquad (3.45\text{ c})$$

It follows that:

$$\frac{\partial E_{oz}}{\partial y_o} - \frac{\partial E_{oy}}{\partial z_o} = -\frac{\partial B_{ox}}{\partial t_o} = 0, \tag{3.46 a}$$

$$\frac{\partial E_{ox}}{\partial z_o} - \frac{\partial E_{oz}}{\partial x_o} = -\frac{\partial B_{oy}}{\partial t_o}, \tag{3.46 b}$$

$$\frac{\partial E_{oy}}{\partial x_o} - \frac{\partial E_{ox}}{\partial y_o} = -\frac{\partial B_{oz}}{\partial t_o}. \tag{3.46 c}$$

Thus:

$$\mathrm{curl}_o\, \boldsymbol{E}_o = -\frac{\partial \boldsymbol{B}_o}{\partial t_o}. \tag{3.47}$$

We shall integrate both sides of the last equation (3.47) over any simply connected open surface S_{Γ_o} bounded by the closed curve Γ_o, both at rest relatively to the reference frame K_o. By expressing this operation, it follows:

$$\int_{S_{\Gamma_o}} \mathrm{curl}_o\, \boldsymbol{E}_o \cdot \mathrm{d}\boldsymbol{S}_o = -\int_{S_{\Gamma_o}} \frac{\partial \boldsymbol{B}_o}{\partial t_o} \cdot \mathrm{d}\boldsymbol{S}_o, \tag{3.48}$$

and by transforming the surface-integral into a line-integral (Stokes theorem), it follows:

$$\oint_{\Gamma_o} \boldsymbol{E}_o \cdot \mathrm{d}\boldsymbol{l}_o = -\int_{S_{\Gamma_o}} \frac{\partial \boldsymbol{B}_o}{\partial t_o} \cdot \mathrm{d}\boldsymbol{S}_o. \tag{3.49}$$

In the case in which there are several point-like electric charges, according to the assumption 2 of Section 2.2, the principle of superposition may be applied and expression (3.49) remains valid also in this case. Expression (3.49) remains valid for each inertial system, since the differential operator curl is invariant with respect to the transformation from one reference frame to another one if the latter is at rest relatively to the former one.

Relations (3.47) and (3.49) represent the local and integral forms, respectively, of the law of electromagnetic induction for empty space, and reference frame at rest.

3.5.2. The Expression of the Law of Electromagnetic Induction for Any Medium at Rest

In the case of any medium (i.e., containing any substance, whatever it would be), it is necessary to establish the relations between the macroscopic state quantities of the electromagnetic field.

The macroscopic quantities at a point, at a certain instant, are obtained by the average of the microscopic quantities over a physically infinitesimal volume around the considered point, and over a physically infinitesimal time interval that contains the considered instant (time). As in Sub-section 3.4.2, the law keeps its previous form.

3.5.3. The General Expression of the Law of Electromagnetic Induction for Media at Rest

The general expression of the law of electromagnetic induction, previously derived, can be established by the generalization of experimental results (in the sense mentioned at the beginning of Section 3.4).

In an inertial reference frame, in any medium, the following relation exists between the vector quantities $\boldsymbol{E}$ and $\boldsymbol{B}$:

$$\oint_{\Gamma} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = -\int_{S_{\Gamma}} \frac{\partial \boldsymbol{B}}{\partial t} \cdot \mathrm{d}\boldsymbol{S}, \tag{3.50}$$

where S_{Γ} is a simply connected open surface bounded by the closed curve Γ, both at rest relatively to the considered inertial reference frame. The curve Γ may represent any contour in particular it can even be the contour of an electric circuit. The last situation is of great interest in applications.

The positive direction of the normal to the surface S_{Γ} is associated according to the right-handed screw rule with the travelling sense of the curve Γ (Fig. 3.6).

Relation (3.50) can also be written in the form:

$$\oint_{\Gamma} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = -i_{\mathrm{mag}}, \tag{3.51}$$

where the quantity i_{mag} is called, according to the work [3, p. 259], *intensity of magnetic current*.

By using the Stokes theorem, relation (3.50) becomes:

$$\int_{S_{\Gamma}} \mathrm{curl}\, \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = -\int_{S_{\Gamma}} \frac{\partial \boldsymbol{B}}{\partial t} \cdot \mathrm{d}\boldsymbol{S}. \tag{3.52}$$

Taking into account that the last relation holds for any surface S_{Γ}, the following relation in differential (local) form is obtained:

$$\mathrm{curl}\, \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}. \tag{3.53}$$

Relation (3.53) may also be written in the form [3, p. 367]:

$$\mathrm{curl}\, \boldsymbol{E} = -\boldsymbol{J}_{\mathrm{mag}}, \tag{3.54}$$

where the quantity $\boldsymbol{J}_{\mathrm{mag}}$ is called [3, p. 260] *magnetic current density*.

Relations (3.50) and (3.53) represent the *expressions of the integral form* and *local form*, respectively, of *the law of electromagnetic induction for any medium at rest*.

The vector quantities $\boldsymbol{E}$ and $\boldsymbol{B}$ can be introduced as shown in Section 2.5.

Fig. 3.6. Explanation to the
law of electromagnetic
induction.
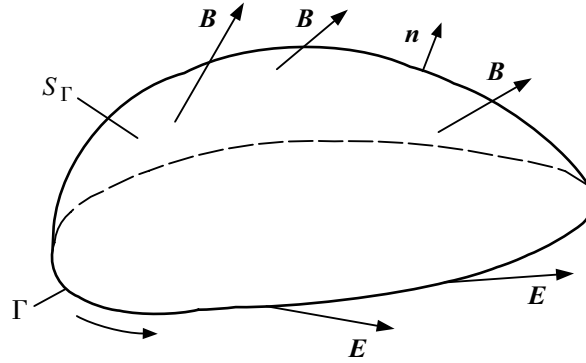
3.5.4. The Concise Integral Form of the Expression of the Law of Electromagnetic Induction for Media at Rest. Faraday Law.

The left-hand side of relation (3.50) represents the induced electromotive force (by the solenoidal component of the electric field) along the curve Γ:

$$u_{e\Gamma} = \oint_{\Gamma} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l}. \tag{3.55}$$

It is to be remarked that, in this case, the electric field strength $\boldsymbol{E}$ contains only the Coulombian and induced component, but not the impressed component, hence:

$$\boldsymbol{E} = \boldsymbol{E}_l - \boldsymbol{E}_i. \tag{3.55 a}$$

The integral of the Coulombian component along the closed curve is zero, since it derives from a potential.

Also, it can be seen that the right hand side of relation (3.50) may be written in the following form (since the surface S_Γ is at rest):

$$\int_{S_\Gamma} \frac{\partial \boldsymbol{B}}{\partial t} \cdot \mathrm{d}\boldsymbol{S} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\Psi_{S_\Gamma}\right), \tag{3.56}$$

where

$$\Psi_{S_\Gamma} = \int_{S_\Gamma} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S}, \tag{3.56 a}$$

represents the magnetic flux through any open surface S_Γ.

It follows:

$$u_{e\Gamma} = -\frac{\mathrm{d}\,\Psi_{S_\Gamma}}{\mathrm{d}t}, \tag{3.57}$$

that represents another manner of writing the *integral form of the law of electromagnetic induction for media at rest*. This law is also called the *Faraday law*.

The phenomenon of producing an electromotive force along a closed curve Γ by the variation with time of the magnetic flux which links that curve is called *electromagnetic induction*. The electromotive force produced in this way is called *induced electromotive force*, and the magnetic flux that produces the electromotive force is termed *inductive magnetic flux*.

The variation with time of the magnetic flux through any surface bounded by the curve Γ can have two causes:

a. Variation with time of the local magnetic induction (for instance due to the variation with time of the electric current carried by the circuits which produce the magnetic field);

b. Motion of the contour Γ.

The electromagnetic induction is referred to in the first case as *induction by transformation* and, in the second case, *induction by motion;* in particular it can be induction by rotation (i.e., rotation without translation) and induction by deformation.

The inductive magnetic flux can be produced by a permanent magnet or by electric currents. The inductive magnetic flux can be produced even by the electric current (if it exists) flowing along the circuit for the contour of which the electromotive force has to be calculated, and also by the currents flowing along other circuits. Correspondingly, the electromagnetic induction is referred to as *self-induction* and *mutual induction*, respectively.

3.6. THE LAW OF MAGNETIC CIRCUIT (MAGNETIC CIRCUITAL LAW) FOR MEDIA AT REST

The *law of magnetic circuit* also called *magnetic circuital law* can be established by the generalization of certain experimental results (in the sense mentioned at the beginning of Section 3.1) or derived by employing certain relations established in the Special Theory of Relativity. We shall use both manners beginning with the second one.

3.6.1. The Expression of the Law of Magnetic Circuit for Empty Medium at Rest

Let K_o and K_1 be two inertial reference frames. The reference frame K_1 is moving with the constant velocity $\boldsymbol{v}_{1o}$ with respect to the reference frame K_o. Let q_1 be a point-like electric charge fixed in empty space at any point $A(x_{1A}, y_{1A}, z_{1A})$.

We shall calculate the vector quantity $\operatorname{curl}_o \boldsymbol{B}_o$ at a point $P(x_o, y_o, z_o)$. Taking into account relation (2.26), we can write:

$$\operatorname{curl}_o \boldsymbol{B}_o = \operatorname{curl}_o \left(\mu_0 \, \boldsymbol{v}_{1o} \times \varepsilon_0 \, \boldsymbol{E}_o \right) = \varepsilon_0 \, \mu_0 \operatorname{curl}_o \left(\boldsymbol{v}_{1o} \times \boldsymbol{E}_o \right). \tag{3.58}$$

By using an expression of the Vector Calculus, namely:

$$\operatorname{curl}(\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{a} \operatorname{div} \boldsymbol{b} - (\boldsymbol{a} \cdot \nabla)\boldsymbol{b} + (\boldsymbol{b} \cdot \nabla)\boldsymbol{a} - \boldsymbol{b} \operatorname{div} \boldsymbol{a}, \tag{3.58 a}$$

it follows:

$$\text{curl}_{\text{o}}\, \boldsymbol{B}_{\text{o}} = \varepsilon_0\, \mu_0\, \boldsymbol{v}_{1\text{o}}\, \text{div}_{\text{o}}\, \boldsymbol{E}_{\text{o}} - \varepsilon_0\, \mu_0\, \boldsymbol{E}_{\text{o}}\, \text{div}_{\text{o}}\, \boldsymbol{v}_{1\text{o}} - \varepsilon_0\, \mu_0\, (\boldsymbol{v}_{1\text{o}} \cdot \nabla_{\text{o}})\boldsymbol{E}_{\text{o}} +$$
$$+ \varepsilon_0\, \mu_0\, (\boldsymbol{E}_{\text{o}} \cdot \nabla_{\text{o}})\boldsymbol{v}_{1\text{o}}\,. \tag{3.59}$$

But $\text{div}_{\text{o}}\boldsymbol{v}_{1\text{o}} = 0$ and $(\boldsymbol{E}_{\text{o}} \cdot \nabla_{\text{o}})\boldsymbol{v}_{1\text{o}} = 0$ because the velocity $\boldsymbol{v}_{1\text{o}}$ is constant. Hence, it follows:

$$\text{curl}_{\text{o}}\, \boldsymbol{B}_{\text{o}} = \varepsilon_0\, \mu_0\, \boldsymbol{v}_{1\text{o}}\, \text{div}_{\text{o}}\, \boldsymbol{E}_{\text{o}} - \varepsilon_0\, \mu_0\, (\boldsymbol{v}_{1\text{o}} \cdot \nabla_{\text{o}})\boldsymbol{E}_{\text{o}}\,. \tag{3.60}$$

From the known relation of Vector Calculus, it follows:

$$(\boldsymbol{v}_{1\text{o}} \cdot \nabla_{\text{o}})\boldsymbol{E}_{\text{o}} = v_{1\text{o}x}\, \frac{\partial \boldsymbol{E}_{\text{o}}}{\partial x_{\text{o}}}, \tag{3.61}$$

where we have taken into account that $\boldsymbol{v}_{1\text{o}} = \boldsymbol{i}\, v_{1\text{o}x}$.

Now, we calculate the following vector quantity:

$$\frac{\partial \boldsymbol{E}_{\text{o}}}{\partial x_{\text{o}}} = \boldsymbol{i}\, \frac{\partial E_{\text{o}x}}{\partial x_{\text{o}}} + \boldsymbol{j}\, \frac{\partial E_{\text{o}y}}{\partial x_{\text{o}}} + \boldsymbol{k}\, \frac{\partial E_{\text{o}z}}{\partial x_{\text{o}}}\,. \tag{3.62}$$

Since we have supposed that there is a single point-like electric charge q_1, it results that:

$$E_{\text{o}x} = E_{1x} = \frac{q_1}{4\pi\varepsilon_0} \cdot \frac{\alpha_{1\text{o}}\,(x_{\text{o}} - v_{1\text{o}x}\, t_{\text{o}}) - x_{1A}}{r_1^3}, \tag{3.63 a}$$

$$E_{\text{o}y} = \alpha_{1\text{o}}\, E_{1y} = \alpha_{1\text{o}}\, \frac{q_1}{4\pi\varepsilon_0} \cdot \frac{y_{\text{o}} - y_{1A}}{r_1^3}, \tag{3.63 b}$$

$$E_{\text{o}z} = \alpha_{1\text{o}}\, E_{1z} = \alpha_{1\text{o}}\, \frac{q_1}{4\pi\varepsilon_0} \cdot \frac{z_{\text{o}} - z_{1A}}{r_1^3}\,. \tag{3.63 c}$$

Taking into account all these relations (3. 63 a, b, c) in the general case of any system of electric charges, it results that:

$$(\boldsymbol{v}_{1\text{o}} \cdot \nabla)\boldsymbol{E}_{\text{o}} = v_{1\text{o}x}\, \frac{\partial \boldsymbol{E}_{\text{o}}}{\partial x_{\text{o}}} = -\frac{\partial \boldsymbol{E}_{\text{o}}}{\partial t_{\text{o}}}\,. \tag{3.64}$$

It follows:

$$\text{curl}_{\text{o}}\, \boldsymbol{B}_{\text{o}} = \varepsilon_0\, \mu_0\, \boldsymbol{v}_{1\text{o}}\, \text{div}_{\text{o}}\, \boldsymbol{E}_{\text{o}} + \varepsilon_0\, \mu_0\, \frac{\partial \boldsymbol{E}_{\text{o}}}{\partial t_{\text{o}}}\,. \tag{3.65}$$

In the case in which there are several point-like charges, according to the assumptions of Section 2.2, the principle of superposition can be applied, and the last relation keeps its form.

The macroscopic quantities at a point, at a certain moment, are obtained by the average of the microscopic quantities over a physically infinitesimal volume, around the

considered point, and over a physically infinitesimal time interval including the considered moment (time). After the calculation of the average (mean) values, the form of relation (3.65) remains unchanged.

Let us consider the free (true) electric charge continuously distributed in empty space, with the volume density ρ_{vo}, in the interior of the surface Σ_o. In accordance with the law of electric flux, it can be written:

$$\varepsilon_0 \, \mathrm{div}_o \, \boldsymbol{E}_o = \rho_{vo} \, . \tag{3.65 a}$$

Consequently, it follows:

$$\mathrm{curl}_o \, \boldsymbol{B}_o = \mu_0 \, \rho_{vo} \, \boldsymbol{v}_{1o} + \mu_0 \frac{\partial \, \varepsilon_0 \, \boldsymbol{E}_o}{\partial \, t_o} \tag{3.65 b}$$

or

$$\mathrm{curl}_o \, \frac{\boldsymbol{B}_o}{\mu_0} = \rho_{vo} \, \boldsymbol{v}_{1o} + \frac{\partial \, \varepsilon_0 \, \boldsymbol{E}_o}{\partial \, t_o} \, . \tag{3.66}$$

If we denote:

$$\boldsymbol{J}_{eo} = \rho_{vo} \, \boldsymbol{v}_{1o} \, , \tag{3.67}$$

the density of the current constituted by the electric charges in motion, the last relation yields:

$$\mathrm{curl}_o \, \frac{\boldsymbol{B}_o}{\mu_0} = \boldsymbol{J}_{eo} + \frac{\partial \, \varepsilon_0 \, \boldsymbol{E}_o}{\partial \, t_o} \, . \tag{3.68}$$

We shall integrate both sides of relation (3.68) over any open surface S_{Γ_o} bounded by the closed curve Γ_o, both in the reference frame K_o. It follows:

$$\int_{S_{\Gamma_o}} \mathrm{curl}_o \, \frac{\boldsymbol{B}_o}{\mu_0} \cdot \mathrm{d} \, \boldsymbol{S}_o = \int_{S_{\Gamma_o}} \boldsymbol{J}_{eo} \cdot \mathrm{d} \, \boldsymbol{S}_o + \int_{S_{\Gamma_o}} \frac{\partial \left(\varepsilon_0 \, \boldsymbol{E}_o \right)}{\partial \, t_o} \cdot \mathrm{d} \, \boldsymbol{S}_o \, . \tag{3.69}$$

By using the transformation of the surface-integral into a line-integral (Stokes theorem), it follows:

$$\oint_{\Gamma_o} \frac{\boldsymbol{B}_o}{\mu_0} \cdot \mathrm{d} \, \boldsymbol{l}_o = \int_{S_{\Gamma_o}} \boldsymbol{J}_{eo} \cdot \mathrm{d} \, \boldsymbol{S}_o + \int_{S_{\Gamma_o}} \frac{\partial \left(\varepsilon_0 \, \boldsymbol{E}_o \right)}{\partial \, t_o} \cdot \mathrm{d} \, \boldsymbol{S}_o \, . \tag{3.70}$$

Expression (3.70) remains valid for any inertial reference frame, for the differential operator curl is invariant with respect to the transformation from one inertial reference frame to another, at rest relatively to the previous one.

Relations (3.68) and (3.70) represent the expressions of the *local form* and *integral form*, respectively, of *the law of magnetic circuit for empty space* (i.e., *in vacuo*), and reference frame at rest.

3.6.2. The General Expression of the Law of Magnetic Circuit for Empty Space

The general expression (3.70), previously derived, can also be established as a generalization of experimental results (in the sense mentioned at the beginning of Section 3.1).

In an inertial reference frame, between the vector quantities: Magnetic induction $\boldsymbol{B}$, density of the resultant electric current (produced by the electric charges in motion) $\boldsymbol{J}_e$, and product of the permittivity of vacuum by the electric field strength $\varepsilon_0 \boldsymbol{E}$, the following relation exists in empty space:

$$\oint_{\Gamma} \frac{\boldsymbol{B}}{\mu_0} \cdot \mathrm{d}\boldsymbol{l} = \int_{S_{\Gamma}} \boldsymbol{J}_e \cdot \mathrm{d}\boldsymbol{S} + \int_{S_{\Gamma}} \frac{\partial(\varepsilon_0 \boldsymbol{E})}{\partial t} \cdot \mathrm{d}\boldsymbol{S}, \tag{3.71}$$

where S_{Γ} is a simply connected open surface, bounded by the closed curve Γ, both at rest with respect to the considered inertial reference frame. The positive direction of the normal to the surface S_{Γ} is associated, according to the right-handed screw rule, with the travelling sense along the curve Γ. The quantity μ_0 is a universal constant referring to vacuum, and termed *magnetic constant* or *permeability of vacuum*. In the SI system of units, it has, as mentioned in Section 2.4, the value:

$$\mu_0 = 4\pi 10^{-7} \ \mathrm{H/m}. \tag{3.72}$$

By using the Stokes theorem, relation (3.71) becomes:

$$\int_{S_{\Gamma}} \mathrm{curl} \frac{\boldsymbol{B}}{\mu_0} \cdot \mathrm{d}\boldsymbol{S} = \int_{S_{\Gamma}} \boldsymbol{J}_e \cdot \mathrm{d}\boldsymbol{S} + \int_{S_{\Gamma}} \frac{\partial(\varepsilon_0 \boldsymbol{E})}{\partial t} \cdot \mathrm{d}\boldsymbol{S}. \tag{3.73}$$

Taking into account that the last relation holds for any surface S_{Γ}, the following relation, in local (differential) form, is obtained:

$$\mathrm{curl} \frac{\boldsymbol{B}}{\mu_0} = \boldsymbol{J}_e + \frac{\partial \varepsilon_0 \boldsymbol{E}}{\partial t}. \tag{3.74}$$

Relations (3.71) and (3.74) represent the expressions of the integral form and local form, respectively, of the law of magnetic circuit for empty space, and frame at rest.

The vector quantities $\boldsymbol{B}$ and $\boldsymbol{E}$ can be introduced as shown in Section 2.5.

3.6.3. The Expressions of the Law of Magnetic Circuit and Magnetic Field Strength, for Any Medium at Rest, in the Case of all Types of Electric Charges and Currents

The expression of the law of magnetic circuit for any medium (i.e., containing whatever substance) at rest can be obtained by the generalization of certain experimental results (in the sense mentioned at the beginning of Section 3.1).

In the case of any medium, it is necessary to establish relations between the macroscopic state quantities of the electromagnetic field.

The macroscopic quantities at a point, at a certain moment, are obtained by calculating the average of the microscopic quantities over a physically infinitesimal volume, around the considered point, and over a physically infinitesimal time interval including the considered moment (time). In the case of a medium, in which only free charges can exist, after the calculation of the average (mean) values, the form of the relations between the various macroscopic quantities will not be modified (they will remain as in the case of microscopic values). If there are free electric charges, polarization electric charges, and Amperian electric currents, then, after the calculation of the average (mean) values, the form of the relations between the various macroscopic quantities will be modified.

The symbols and their meaning are the same as in Section 1.10.

The electric current density J_e (of relation (3.67), but without index o) in the case of macroscopic quantities can be expressed as the sum of three components: the *density of electric conduction currents* J, the *density of Amperian electric currents* J_a, and the *density of polarization electric currents* J_P. Consequently, the following relation is obtained:

$$J_e = J + J_a + J_P. \tag{3.75}$$

By replacing the quantities of relation (3.75) into relation (3.74), it results that:

$$\mathrm{curl}\,\frac{B}{\mu_0} = J + J_a + J_P + \frac{\partial(\varepsilon_0\,E)}{\partial t}. \tag{3.76}$$

If we integrate both sides of relation (3.76) over any open surface bounded by a closed curve Γ, and we utilize the transformation theorem of a surface-integral into a line-integral (Stokes theorem), the following relation between the macroscopic quantities is obtained:

$$\oint_\Gamma \frac{B}{\mu_0}\cdot \mathrm{d}\,l = \int_{S_\Gamma} J\cdot \mathrm{d}\,S + \int_{S_\Gamma} J_a\cdot \mathrm{d}\,S + \int_{S_\Gamma} J_P\cdot \mathrm{d}\,S + \int_{S_\Gamma} \frac{\partial(\varepsilon_0\,E)}{\partial t}\cdot \mathrm{d}\,S. \tag{3.77}$$

In the right-hand side, the second term represents the intensity of the Amperian electric current given by relation (1.90), and the third term represents the intensity of the polarization electric current given by relation (1.75). By performing the corresponding substitutions and grouping the terms, it follows:

$$\oint_\Gamma \frac{1}{\mu_0}\left(B - \mu_0\,M\right)\cdot \mathrm{d}\,l = \int_{S_\Gamma} J\cdot \mathrm{d}\,S + \int_{S_\Gamma} \frac{\partial}{\partial t}\left(\varepsilon_0\,E + P\right)\cdot \mathrm{d}\,S. \tag{3.78}$$

Taking into account relation (3.18), relation (3.78) can be written:

$$\oint_\Gamma H\cdot \mathrm{d}\,l = \int_{S_\Gamma} J\cdot \mathrm{d}\,S + \int_{S_\Gamma} \frac{\partial D}{\partial t}\cdot \mathrm{d}\,S, \tag{3.79}$$

where we have denoted:

$$H = \frac{1}{\mu_0} B - M \qquad (3.80\text{ a})$$

or

$$H = \frac{1}{\mu_0} B - \frac{1}{\mu_0} M_j . \qquad (3.80\text{ b})$$

The vector quantity H, defined by each of relation (3.80 a, b), is called *magnetic field strength* or *magnetic field intensity* and is a function of point or, in other words, it represents a vector field. The lines of the vector field H, called *lines of magnetic field strength,* are the lines to which the vector H is tangent at each point. Analogously, the lines of the vector field B, called *lines of magnetic induction* or *lines of magnetic field,* are the lines to which the vector B is tangent at each point.

A *tube of magnetic flux* is termed each surface in the magnetic field formed by the totality of lines of magnetic induction passing through the points of a simple closed curve.

In the SI system of units, the unit of measure of the electric field strength is the *ampere per metre*, symbol A/m. A definition of this unit will be given in Section 3.15.

Relation (3.79) can also be written in the form:

$$\oint_{\Gamma} H \cdot d l = \int_{S_{\Gamma}} J \cdot d S + \frac{d}{d t} \int_{S_{\Gamma}} D \cdot d S , \qquad (3.81)$$

because the position of the differential operator is indifferent in the case in which the derivative does not concern the variable with respect to which the integral is to be calculated, and the surface S_{Γ} is considered at rest.

By using the Stokes theorem, relation (3.79) becomes:

$$\int_{S_{\Gamma}} \text{curl} H \cdot d S = \int_{S_{\Gamma}} J \cdot d S + \int_{S_{\Gamma}} \frac{\partial D}{\partial t} \cdot d S . \qquad (3.82)$$

Having in view that the previous relation is valid for any surface S_{Γ}, the following relation in local (differential) form can be obtained:

$$\text{curl} H = J + \frac{\partial D}{\partial t} . \qquad (3.83)$$

Relations (3.81) and (3.83) represent the *expressions of the integral form* and *local form,* respectively, of *the law of magnetic circuit for any medium at rest, in the case in which, there are free electric charges, polarization electric charges and Amperian electric currents*.

A detailed analysis of the introduction of the physical quantities E, D, H, B can be found in papers [55], [56].

3.6.4. The General Expression of the Law of Magnetic Circuit for Any Medium at Rest

The general expression of the law of magnetic circuit for media at rest, derived previously, can be established by the generalization of certain experimental results.

In an inertial reference frame, and in any medium, the following relation exists between the vector quantities $\boldsymbol{H}$, $\boldsymbol{J}$, $\boldsymbol{D}$:

$$\oint_{\Gamma} \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = \int_{S_{\Gamma}} \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{S} + \int_{S_{\Gamma}} \frac{\partial \boldsymbol{D}}{\partial t} \cdot \mathrm{d}\boldsymbol{S}, \tag{3.84}$$

where S_{Γ} is any simply connected open surface bounded by the closed curve Γ, both at rest relatively to the reference frame. The positive direction of the normal to the surface S_{Γ} is associated with the travelling sense along the curve Γ by the right-handed screw rule.

If, in the considered space, there are bodies charged with electricity with the volume density ρ_{conv}, in motion at any velocity $\boldsymbol{v}_r$ relatively to the reference system, then, in the right-hand side of relation (3.84), another term occurs. Indeed, the motion of bodies charged with electricity represents a supplementary electric current called *convection electric current*. In this general case, expression (3.84) becomes:

$$\oint_{\Gamma} \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = \int_{S_{\Gamma}} \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{S} + \int_{S_{\Gamma}} \boldsymbol{J}_{\mathrm{conv}} \cdot \mathrm{d}\boldsymbol{S} + \int_{S_{\Gamma}} \frac{\partial \boldsymbol{D}}{\partial t} \cdot \mathrm{d}\boldsymbol{S}, \tag{3.84 a}$$

where the quantity $\boldsymbol{J}_{\mathrm{conv}} = \rho_{\mathrm{conv}} \boldsymbol{v}_r$ is the density of the convection electric current.

Relation (3.84 a) may also be written in the form:

$$\oint_{\Gamma} \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = i_l, \tag{3.85}$$

where the quantity i_l represents the *intensity of the electric current in the large sense*.

Having in view that relations (3.84) and (3.84 a) hold for any surface S_{Γ}, and using the Stokes theorem, the following two relations, in differential (local) form, are obtained:

$$\operatorname{curl} \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}, \quad \left(\boldsymbol{J}_{\mathrm{conv}} = 0 \right), \tag{3.86}$$

$$\operatorname{curl} \boldsymbol{H} = \boldsymbol{J} + \boldsymbol{J}_{\mathrm{conv}} + \frac{\partial \boldsymbol{D}}{\partial t}. \tag{3.86 a}$$

Relations (3.86) and (3.86 a) can also be written in the form:

$$\operatorname{curl} \boldsymbol{H} = \boldsymbol{J}_l, \tag{3.87}$$

$$\boldsymbol{J}_l = \boldsymbol{J} + \boldsymbol{J}_{\mathrm{conv}} + \frac{\partial \boldsymbol{D}}{\partial t}, \tag{3.87 a}$$

where the quantity J_l represents the density of the electric current in the large sense.

Relations (3.84), (3.84 a) and (3.86), (3.86 a) represent the *expressions of the integral form* and *local form, respectively, of the law of magnetic circuit, for any media at rest*.

The vector quantities H and D are defined by relations (3.80 b) and (3.18), respectively.

In the right-hand side of relation (3.84 a) the three terms have the following meaning, respectively: *intensity of the conduction electric current, intensity of the convection electric current,* and *intensity of the displacement electric current*. The intensity of the displacement electric current is the sum of the intensity of the displacement electric current *in vacuo* and the intensity of the polarization electric current. The sum of the terms of the right-hand side of relation (3.84 a) is referred to as *intensity of the electric current in the large sense*.

3.6.5. Conditions (Regimes) of the Electromagnetic Field. Law of Magnetic Circuit in Quasi-stationary Condition. Ampère Law (Theorem).

The electromagnetic field can be in one of the following conditions also called regimes: Static condition, Stationary condition, Non-stationary condition, Quasi-stationary condition.

Static condition is the regime in which no macroscopic quantity varies with time and no electric current exists.

Stationary condition is the regime in which no macroscopic state quantity varies with time and electric current exists but does not vary with time.

Non-stationary condition is the regime in which the macroscopic state quantities vary with time.

Quasi-stationary condition is the regime in which the macroscopic quantities vary with time but their variation is relatively slow, implying the following consequences. The value of the quantity $\dfrac{\partial D}{\partial t}$ may be neglected with respect to the quantity J, at any point

of a conducting medium or a medium in which a convection electric current exists; but the derivatives with respect to time of the other state quantities of the electromagnetic field cannot be neglected. One of the practical results of these remarks results in the fact that in an electric circuit in quasi-stationary condition, the intensity of the electric current is assumed to have the same value at any cross-section of the circuit wire. The higher the speed of variation of the state quantities (e.g., magnetic induction), the farther from the previous assumption the circuit will be.

In quasi-stationary condition, the expressions of the integral form and local form, respectively, of the law of magnetic circuit (without including the convection current) become:

$$\oint_{\Gamma} H \cdot \mathrm{d}l = \int_{S_{\Gamma}} J \cdot \mathrm{d}S, \tag{3.88}$$

$$\mathrm{curl}\, H = J, \tag{3.89}$$

and are referred to as *the integral form* and *the local form,* respectively*, of *the Ampère theorem* or of *the Ampère circuital law*.

Expression (3.88) may be obtained by employing the Biot-Savart-Laplace formula (2.47), and (2.48), if it is previously known. For this purpose, formula (2.48) will be applied for a closed curve Γ, and the expression of the solid angle, and certain vector transformations are to be used [23, p. 120].

3.6.6. The Components of the Magnetic Field Strength. Magnetic Tension. Magnetomotive Force.

The magnetic field strength has generally two components: a *potential* (*non-curl, irrotational*) *component* $\boldsymbol{H}_p$ and a *curl* (*rotational*, *solenoidal*) *component* $\boldsymbol{H}_r$. Hence:

$$\boldsymbol{H} = \boldsymbol{H}_p + \boldsymbol{H}_r. \tag{3.90}$$

Generally, in a field of vectors, the line-integral of the field vector along a curve can be referred to as tension; this tension is also introduced in the case of magnetic field. The definitions, in the case of the magnetic field, are analogous to those introduced for the electric field in Section 1.8.

The line-integral of the magnetic field strength along a curve is called *magnetic tension* and is given by the relation:

$$u_{m\,C_{AB}} = \int_{C_{AB}} \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l}. \tag{3.91}$$

If in relation (3.91), instead of the magnetic field strength $\boldsymbol{H}$, only the curl component $\boldsymbol{H}_r$ of the magnetic field strength is introduced, the *magnetomotive tension* calculated by employing a line-integral (along an open curve) is termed *magnetomotive force* (abbreviation m.m.f.) along that curve, and is obtained by the relation:

$$u_{mm\,C_{AB}} = \int_{C_{AB}} \boldsymbol{H}_r \cdot \mathrm{d}\boldsymbol{l}. \tag{3.92}$$

If the line-integrals (3.91) and (3.92) along a *closed curve* are considered, the obtained expression is called *magnetomotive force*, also termed *magnetomotive tension*, along that curve. The magnetomotive force obtained by integrating the magnetic field strength $\boldsymbol{H}$ along a closed curve coincides with the integral of the curl component of the magnetic field strength, because the line-integral of the potential component of the magnetic field strength is zero along a closed curve.

All the definitions of magnetic tensions refer to any medium, regardless of its state.

3.6.7. The Concise Integral Form of the Law of Magnetic Circuit for Media at Rest

The left-hand side of relation (3.84) represents the magnetomotive force along the closed curve Γ:

$$u_{mm\,\Gamma} = \oint_{\Gamma} \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l}. \tag{3.93}$$

It is to be remarked that the magnetomotive force is determined only by the curl (rotational) component of the magnetic field strength.

The first term of the right-hand side of relation (3.84 a), i.e., the flux of the vector of the conduction electric current density through the surface S_{Γ}, bounded by the closed curve Γ, is called *current-linkage* and is determined by the conduction electric current that crosses the surface S_{Γ}. The current-linkage is given by the relation:

$$\Theta = \int_{S_{\Gamma}} \boldsymbol{J} \cdot \mathrm{d}\boldsymbol{S}. \tag{3.94}$$

Let us consider a coil with w turns carrying an electric current. We shall adopt a curve linking the coil as it can be seen in Fig. 3.7.

In this case, a simply connected open surface S_{Γ}, bounded by the curve Γ, is crossed at w places by the conductor carrying the current i. At the other points of the surface, the density of the conduction electric current is zero. It follows:

$$\Theta = w\,i. \tag{3.95}$$

Analogously, for the coils of Fig. 3.8, the following result is obtained:

$$\Theta = w_1\,i_1 + w_2\,i_2. \tag{3.95 a}$$

In the case of coils, the current linkage is also expressed by the denomination *ampere-turns*.

The last term of the right-hand side of relations (3.84) and (3.84 a), i.e., the flux of the vector $\dfrac{\partial \boldsymbol{D}}{\partial t}$ through the surface S_{Γ} is called *displacement electric current*. If the considered condition is quasi-stationary, it will be disregarded.

It can be remarked that the expression of the displacement electric current of the right-hand side in relations (3.84) and (3.84 a) can be written:

$$\int_{S_{\Gamma}} \frac{\partial \boldsymbol{D}}{\partial t} \cdot \mathrm{d}\boldsymbol{S} = \frac{\mathrm{d}}{\mathrm{d}t} \int_{S_{\Gamma}} \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{S}, \tag{3.96}$$

because the order of the differential and integral operators is indifferent, since the derivation does not refer to the variable of integration, and the surface S_{Γ} is at rest.

But, the surface S_{Γ} being at rest, it results that:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{S_{\Gamma}} \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{S} = \frac{\mathrm{d}}{\mathrm{d}t} \left(\Psi_{el\,S_{\Gamma}} \right), \tag{3.97}$$
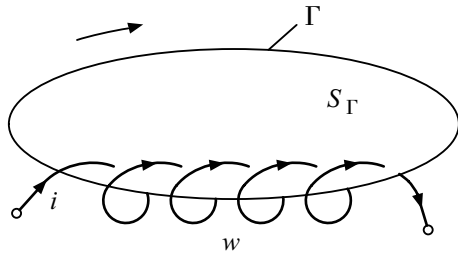
Fig. 3.7. Coil linked by a curve
of integration – curve Γ.
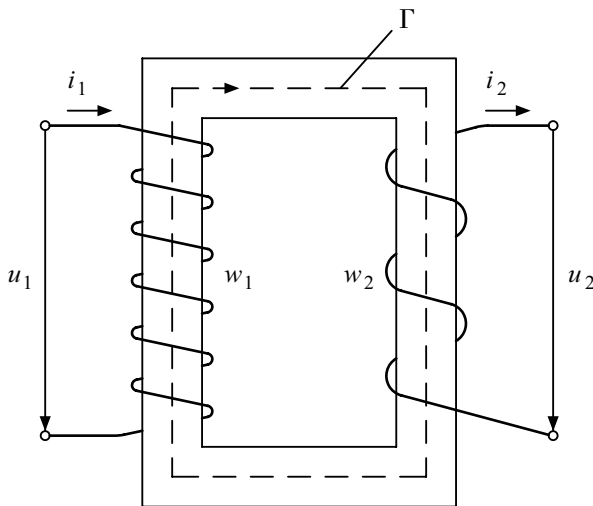
Fig. 3.8. Two coils disposed on a
closed core made of ferromagnetic
material and the curve Γ of
integration.

where

$$\Psi_{\mathrm{el}\, S_\Gamma} = \int_{S_\Gamma} \boldsymbol{D} \cdot \mathrm{d}\, \boldsymbol{S} \qquad (3.97\ \mathrm{a})$$

represents the electric flux through the open surface S_Γ.

Hence:

$$u_{mm\,\Gamma} = \Theta + i_{\mathrm{conv}} + \frac{\mathrm{d}}{\mathrm{d}\,t}\left(\Psi_{\mathrm{el}\, S_\Gamma}\right) \qquad (3.98)$$

that represents another way of writing the integral form of the law of magnetic circuit (magnetic circuital law) for media at rest.

Remark. It can be added that a coil constituted by a wire, wound on a cylinder, in the form of a helix is called solenoid (from the Ancient Greek σολήνας (solinas), tube).

3.6.8. Adoption of the Curves and Surfaces That Occur in the Expressions of the Laws of Electromagnetic Induction, and Magnetic Circuit

All differential operations $\left(\text{curl}, \text{div}, \dfrac{\partial}{\partial t} \right)$ are performed in the fixed reference system, i.e., assumed at rest.

The curve Γ and the surface S_Γ are taken in the fixed reference frame, but may also be taken in the moving reference frame. Concerning this circumstance, it is to be noted that, in both cases, the curve Γ and the surface S_Γ occur only by the elements $\mathrm{d}\boldsymbol{l}$ and $\mathrm{d}\boldsymbol{S}$ at any moment. The motion occurs by the velocity and by the determination of the various quantities in the two reference frames.

In most of cases, the laws are written for determining the quantities and relations in the moving reference frame with respect to the fixed one.

3.7. THE RELATIONSHIP BETWEEN MAGNETIC INDUCTION, MAGNETIC FIELD STRENGTH AND MAGNETIC POLARIZATION

In the case of a medium containing any substance, it is important to know the relation between the macroscopic state quantities of the magnetic field, i.e., magnetic induction, magnetic field strength and magnetization or magnetic polarization.

The macroscopic state quantities of the electromagnetic field at a point, at a given moment, are obtained by calculating the average value of the microscopic quantities over the physically infinitesimal of volume and of time interval, respectively, containing the point and time under consideration.

From relations (3.80 a, b), it follows that:

$$\boldsymbol{B} = \mu_0 \left(\boldsymbol{H} + \boldsymbol{M} \right) = \mu_0 \, \boldsymbol{H} + \boldsymbol{M}_j \,. \tag{3.99}$$

In the framework of the macroscopic theory, this formula constitutes *the expression of the law of relationship between the vectors: Magnetic induction, magnetic field strength and magnetization* (or *magnetic polarization*). In the framework of the macroscopic theory, the quantities above and their relationship have to be introduced as primitive (fundamental) quantities and law, respectively.

3.8. THE LAW OF TEMPORARY MAGNETIZATION

In the case of any isotropic medium (hence, containing any substance), but without permanent magnetization (or magnetic polarization), the temporary magnetization vector (temporary magnetic polarization vector) is parallel with the magnetic field strength vector and of the same direction, as follows:

$$\boldsymbol{M}_t = \chi_m \, \boldsymbol{H} \,, \tag{3.100 a}$$

$$\boldsymbol{M}_{jt} = \mu_0 \, \chi_m \, \boldsymbol{H} \,. \tag{3.100 b}$$

The quantity χ_m is called *magnetic susceptibility* and is depending on the nature of the medium, hence a material quantity.

In the framework of the macroscopic theory, relation (3.100 a) can be determined only experimentally. In the macroscopic theory of electromagnetic field, this relation is a *material law*, also termed *constitutive law*, referred to as *the law of temporary magnetization*, or *the law of temporary magnetic polarization*, as the reference concerns relation (3.100 a) or (3.100 b), respectively.

In fact, for variations slow enough of the quantity $\boldsymbol{H}$, the quantity χ_m is constant with respect to the vector quantity $\boldsymbol{H}$ for certain isotropic media called isotropic and linear media.

In the case of anisotropic media, the orientation of the vector $\boldsymbol{M}$ is not, generally, the same as that of the vector $\boldsymbol{H}$. However, in these media, it is possible to determine, generally, three-orthogonal axes, termed principal axes or eigenaxes, along which the anisotropic medium behaves like an isotropic medium. For these media, the law of temporary magnetization (or of the temporary magnetic polarization, respectively) is written in one of the following forms:

$$\boldsymbol{M}_t = \chi_m \, \boldsymbol{H} \,, \tag{3.101 a}$$

$$\boldsymbol{M}_{jt} = \mu_0 \, \chi_m \, \boldsymbol{H} \,, \tag{3.101 b}$$

where the quantity χ_m represents the *tensor of the magnetic susceptibility*.

For isotropic media without permanent magnetic polarization, the expression of the relationship between the magnetic induction, the magnetic field strength and magnetization (or magnetic polarization, respectively) (3.99) becomes:

$$\boldsymbol{B} = \mu_0 \left(\boldsymbol{H} + \boldsymbol{M}_t \right) = \mu_0 \left(1 + \chi_m \right) \boldsymbol{H} \,, \tag{3.102 a}$$

$$\boldsymbol{B} = \mu_0 \, \boldsymbol{H} + \boldsymbol{M}_{jt} = \mu_0 \left(1 + \chi_m \right) \boldsymbol{H} \,. \tag{3.102 b}$$

The following symbols are adopted:

$$\mu_r = 1 + \chi_m \,, \tag{3.103}$$

$$\mu = \mu_0 \, \mu_r \,. \tag{3.104}$$

The quantity μ_r is called *relative magnetic permeability* of the medium, and the quantity μ is *the magnetic permeability* of the medium.

With the adopted symbols, relations (3.102 a, b) become:

$$\boldsymbol{B} = \mu \, \boldsymbol{H} \,. \tag{3.105}$$

Taking into account the established relations that contain B and H, we shall make certain general considerations concerning the lines of magnetic induction and the lines of magnetic field strength.

The lines to which the vector H is tangent at any point, are *lines of magnetic field strength*. The lines to which the vector B is tangent at any point are *lines of magnetic induction*.

According to relation (3.29), the lines of magnetic induction are closed lines, because the divergence of the vector B is zero at every point.

In accordance with the explanations of Sub-section 1.2.1, the lines of magnetic field strength can be open lines that begin and finish at the points at which the divergence of the vector H is different from zero.

From relations (3.31) and (3.105), it follows that the quantity $\operatorname{div} H$ may be different from zero at the points at which the magnetic permeability varies.

Similar considerations concerning the lines of magnetic induction and the lines of magnetic field strength can be extended to the case in which permanent magnetization also exists. In this case, from relations (3.99), (3.102 a, b), (1.92), (1.93) it follows:

$$B = \mu H + \mu_0 M_p, \tag{3.106 a}$$

$$B = \mu H + M_{jp}. \tag{3.106 b}$$

The dependence between the quantities B and H may be linear or non-linear.
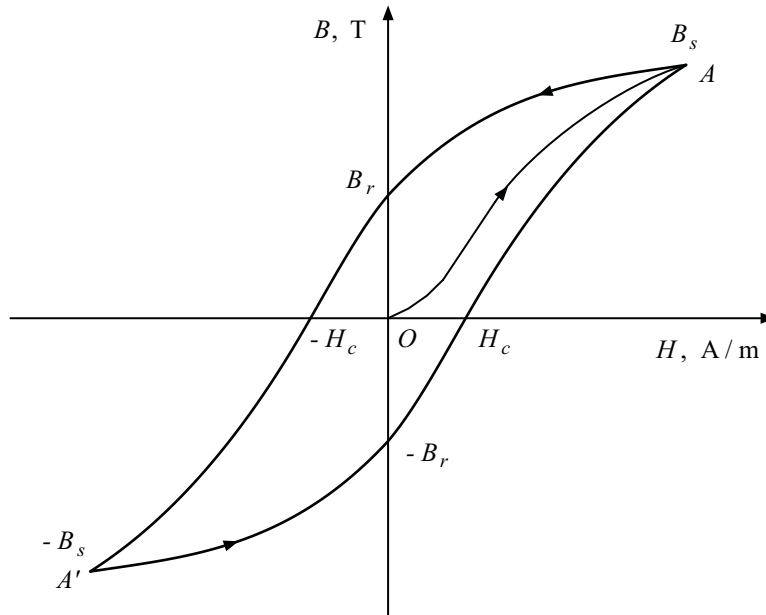
Fig. 3.9. Magnetic hysteresis loop, the direction of travelling along the curve (upward when the magnetic field strength increases, downward when the magnetic field strength decreases); at the middle, the branch starting from $H = 0$ and $B = 0$.

Also, sometimes, the curve of B as a function of H may represent a closed curve (Fig. 3.9) referred to by one of the terms: *hysteresis loop*, *hysteresis cycle*, *hysteresis curve*. Certain characteristic points are marked on the curve.

3.9. DERIVATION OF THE FUNDAMENTAL EQUATIONS OF THE ELECTROMAGNETIC FIELD THEORY IN THE GENERAL CASE. MAXWELL EQUATIONS.

In the preceding Sections, we have derived the fundamental laws of the theory of the electromagnetic field in the case in which the velocities of particles that produce the electromagnetic field are constant with respect to time in the reference frame in which the equations are expressed. The experience has shown that these equations are also valid when the velocities of the mentioned particles vary with time.

There are possibilities to derive the equations for the general case of fields showing forces with a central symmetry (radial symmetry). Here we include the theory of the electromagnetic field, starting from the field of Coulombian forces that also show a central symmetry. Also, it is possible, under certain assumptions, to derive the equations in the case in which the particles that produce the field are moving with velocities varying with time [11], [18], [23], [40], [41].

The equations (3.53), (3.16), (3.18), (3.86 a), (3.31), (3.99) will be written as follows:

$$\operatorname{curl} \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{3.107 a}$$

$$\operatorname{div} \boldsymbol{D} = \rho_v, \tag{3.107 b}$$

$$\boldsymbol{D} = \varepsilon_0 \, \boldsymbol{E} + \boldsymbol{P}, \tag{3.107 c}$$

$$\operatorname{curl} \boldsymbol{H} = \boldsymbol{J} + \boldsymbol{J}_{\text{conv}} + \frac{\partial \boldsymbol{D}}{\partial t}, \tag{3.107 d}$$

$$\operatorname{div} \boldsymbol{B} = 0, \tag{3.107 e}$$

$$\boldsymbol{B} = \mu_0 \left(\boldsymbol{H} + \boldsymbol{M} \right) = \mu_0 \, \boldsymbol{H} + \boldsymbol{M}_j, \tag{3.107 f}$$

and represent the laws in local (differential) form of the electromagnetic field, in the macroscopic theory for media at rest, and are referred to as *the laws of the electromagnetic field for media at rest*.

For isotropic and homogeneous media and without permanent electric and magnetic polarization, according to relations (3.24), (3.105), it follows:

$$\boldsymbol{D} = \varepsilon \, \boldsymbol{E}, \tag{3.107 g}$$

$$\boldsymbol{B} = \mu \, \boldsymbol{H}. \tag{3.107 h}$$

Relations (3.107 a, . . ., f) are also referred to as *Maxwell equations*.

The relations that represent the equations of the electromagnetic field contain, apart from the relations mentioned above, also other equations that are related to Electrokinetics and Electrodynamics.

Maxwell established a system of equations in two forms: 32 scalar equations for the first form and 10 vector equations plus two scalar equations for the second form, respectively. In the latter form, the equations have been designated in his work [1, Arts. 591, 598, 603, 606-614] by capital letters from (A) to (L). These equations link scalar and vector quantities. The quantities represent the electromagnetic field state quantities and the bodies state quantities. In his work, these equations have been written in a form very close to that utilized at present. For denoting the vectors, he used letters of the Gothic alphabet.

3.10.   RELATIONS BETWEEN THE STATE QUANTITIES OF THE ELECTROMAGNETIC FIELD IN VARIOUS INERTIAL REFERENCE FRAMES

Let K and K' be two inertial frames, the first at rest, the latter moving at a constant velocity v relatively to the former one.

Also, let one point-like electric charge be in motion relatively to the two inertial reference frames.

The relations between the various state quantities of electromagnetic field can be established in several ways. These ways are as follows: **a**. Comparison between the forces acting upon a point-like electric charge in motion in the two inertial reference frames;
b. Change of variables, i.e., co-ordinates and time that leave unchanged the form of the equations of the electromagnetic field. The equations that have the same mathematical form in all inertial reference frames after a transformation of the variables, i.e., co-ordinates and time, are considered to be *covariant in Lorentz sense*.

From the comparison of the forces components expressed in the two inertial reference frames, in terms of the various state quantities of the electromagnetic field, we obtain relations between the various state quantities of the electromagnetic field. For the other quantities, the relations are obtained by putting the condition the equations of electromagnetic field to be satisfied in each of the two reference frames. Detailed calculations are given in Appendix 3.

The most general case can be considered in which there are several point-like electric charges, each of them moving at any velocity relatively to both reference frames. These charges produce the electromagnetic field. For the sake of simplicity, we shall assume $v_{1o} = i\,v_{1ox} = v$.

The obtained relations between the state quantities of electromagnetic field are the following:

$$E'_{x'} = E_x + i \cdot \alpha \left(v \times B \right), \tag{3.108 a}$$

$$E'_{y'} = \alpha \left[E_y + j \cdot \left(v \times B \right) \right], \tag{3.108 b}$$

$$E'_{z'} = \alpha \left[E_z + k \cdot \left(v \times B \right) \right]. \tag{3.108 c}$$

$$B'_{x'} = B_x - \boldsymbol{i} \cdot \alpha \left(\frac{1}{c^2} \boldsymbol{v} \times \boldsymbol{E} \right),$$

(3.109 a)

$$B'_{y'} = \alpha \left[B_y - \boldsymbol{j} \cdot \left(\frac{1}{c^2} \boldsymbol{v} \times \boldsymbol{E} \right) \right],$$

(3.109 b)

$$B'_{z'} = \alpha \left[B_z - \boldsymbol{k} \cdot \left(\frac{1}{c^2} \boldsymbol{v} \times \boldsymbol{E} \right) \right].$$

(3.109 c)

$$D'_{x'} = D_x + \boldsymbol{i} \cdot \alpha \left(\frac{1}{c^2} \boldsymbol{v} \times \boldsymbol{H} \right),$$

(3.110 a)

$$D'_{y'} = \alpha \left[D_y + \boldsymbol{j} \cdot \left(\frac{1}{c^2} \boldsymbol{v} \times \boldsymbol{H} \right) \right],$$

(3.110 b)

$$D'_{z'} = \alpha \left[D_z + \boldsymbol{k} \cdot \left(\frac{1}{c^2} \boldsymbol{v} \times \boldsymbol{H} \right) \right].$$

(3.110 c)

$$H'_{x'} = H_x - \boldsymbol{i} \cdot \alpha \left(\boldsymbol{v} \times \boldsymbol{D} \right),$$

(3.111 a)

$$H'_{y'} = \alpha \left[H_y - \boldsymbol{j} \cdot \left(\boldsymbol{v} \times \boldsymbol{D} \right) \right],$$

(3.111 b)

$$H'_{z'} = \alpha \left[H_z - \boldsymbol{k} \cdot \left(\boldsymbol{v} \times \boldsymbol{D} \right) \right].$$

(3.111 c)

$$P'_{x'} = P_x - \boldsymbol{i} \cdot \alpha \left(\frac{1}{c^2} \boldsymbol{v} \times \boldsymbol{M} \right),$$

(3.112 a)

$$P'_{y'} = \alpha \left[P_y - \boldsymbol{j} \cdot \left(\frac{1}{c^2} \boldsymbol{v} \times \boldsymbol{M} \right) \right],$$

(3.112 b)

$$P'_{z'} = \alpha \left[P_z - \boldsymbol{k} \cdot \left(\frac{1}{c^2} \boldsymbol{v} \times \boldsymbol{M} \right) \right].$$

(3.112 c)

$$M'_{x'} = M_x + \boldsymbol{i} \cdot \alpha \left(\boldsymbol{v} \times \boldsymbol{P} \right),$$

(3.113 a)

$$M'_{y'} = \alpha \left[M_y + \boldsymbol{j} \cdot \left(\boldsymbol{v} \times \boldsymbol{P} \right) \right],$$

(3.113 b)

$$M'_{z'} = \alpha \left[M_z + \boldsymbol{k} \cdot \left(\boldsymbol{v} \times \boldsymbol{P} \right) \right].$$

(3.113 c)

$$J_x = \alpha \left(J'_{x'} + \boldsymbol{i} \cdot \boldsymbol{v} \rho'_v \right),$$

(3.114 a)

$$J_y = J'_{y'},$$

(3.114 b)

$$J_z = J'_{z'}.$$

(3.114 c)

$$V' = \alpha \left(V - v A_x \right), \tag{3.115}$$

$$A'_{x'} = \alpha \left(A_x - \frac{v}{c^2} V \right); \quad A'_{y'} = A_y; \quad A'_{z'} = A_z; \tag{3.116 a, b, c,}$$

where

$$\alpha = \frac{1}{\sqrt{1 - \beta^2}}, \tag{3.117 a}$$

$$\beta = \frac{|v|}{c}. \tag{3.117 b}$$

If the velocities are relatively small $|v| \ll c$, then it follows that $\alpha \approx 1$ and the following relations will be obtained:

$$E' = E + v \times B, \tag{3.118}$$

$$B' = B - \frac{1}{c^2} v \times E, \tag{3.119}$$

$$D' = D + \frac{1}{c^2} v \times H, \tag{3.120}$$

$$H' = H - v \times D, \tag{3.121}$$

$$P' = P - \frac{1}{c^2} v \times M, \tag{3.122}$$

$$M' = M + v \times P, \tag{3.123}$$

$$J = J' + \rho'_v v. \tag{3.124}$$

3.11. EXPRESSIONS OF THE LAWS OF THE ELECTROMAGNETIC INDUCTION AND MAGNETIC CIRCUIT FOR MOVING MEDIA

If the differential operator curl is applied to both sides of relation (3.118), it follows:

$$\operatorname{curl} E' = \operatorname{curl} E + \operatorname{curl}(v \times B), \tag{3.125}$$

but, using relation (3.53) and (3.37):

$$\operatorname{curl} E = -\frac{\partial B}{\partial t}, \qquad B = \operatorname{curl} A, \tag{3.126 a, b}$$

and it follows that:

$$\operatorname{curl} \boldsymbol{E}' = -\frac{\partial \boldsymbol{B}}{\partial t} + \operatorname{curl}(\boldsymbol{v} \times \boldsymbol{B}).$$ (3.127)

If the differential operator curl is applied to both sides of relation (3.121), it follows that:

$$\operatorname{curl} \boldsymbol{H}' = \operatorname{curl} \boldsymbol{H} - \operatorname{curl}(\boldsymbol{v} \times \boldsymbol{D})$$ (3.128)

but using relation (3.86), it results that:

$$\operatorname{curl} \boldsymbol{H}' = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} - \operatorname{curl}(\boldsymbol{v} \times \boldsymbol{D})$$ (3.129)

or, if $\boldsymbol{J}_{\text{conv}} \neq 0$, in accordance with relation (3.86 a), it follows:

$$\operatorname{curl} \boldsymbol{H}' = \boldsymbol{J} + \rho_{\text{conv}} \boldsymbol{v} + \frac{\partial \boldsymbol{D}}{\partial t} - \operatorname{curl}(\boldsymbol{v} \times \boldsymbol{D}).$$ (3.129 a)

By adding up to each side of relation (3.86), the quantity $\operatorname{curl} \boldsymbol{M}$ and multiplying by the quantity μ_0, it results that:

$$\operatorname{curl}\left[\mu_0 \left(\boldsymbol{H} + \boldsymbol{M} \right) \right] = \mu_0 \left[\boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} + \operatorname{curl} \boldsymbol{M} \right].$$ (3.130)

We shall consider the case in which the moving reference frame is fixed to the moving substance. Also, we shall suppose that the magnetic polarization in the moving substance is zero, $\boldsymbol{M}' = 0$. In this case, from relation (3.123), it follows:

$$\boldsymbol{M} = -\boldsymbol{v} \times \boldsymbol{P}.$$ (3.131)

For the case under consideration, it results that:

$$\operatorname{curl} \boldsymbol{B} = \mu_0 \left[\boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} - \operatorname{curl}(\boldsymbol{v} \times \boldsymbol{P}) \right]$$ (3.132)

and

$$\boldsymbol{P} \approx \boldsymbol{P}'.$$ (3.133)

We shall consider in the moving reference frame an open surface bounded by any closed curve. At the moment of observation in the fixed reference frame, this surface will be denoted by S_{Γ}.

Integrating both sides of relations (3.125) and (3.129) over the surface S_{Γ} and using the transformation of surface-integrals into line-integrals (the Stokes theorem), it follows successively:

$$\int_{S_{\Gamma}} \operatorname{curl} \boldsymbol{E}' \cdot \mathrm{d} \boldsymbol{S} = -\int_{S_{\Gamma}} \frac{\partial \boldsymbol{B}}{\partial t} \cdot \mathrm{d} \boldsymbol{S} + \int_{S_{\Gamma}} \operatorname{curl}(\boldsymbol{v} \times \boldsymbol{B}) \cdot \mathrm{d} \boldsymbol{S},$$

$$\oint_{\Gamma} \boldsymbol{E}' \cdot d\boldsymbol{l} = -\int_{S_{\Gamma}} \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{S} + \int_{S_{\Gamma}} \operatorname{curl}(\boldsymbol{v} \times \boldsymbol{B}) \cdot d\boldsymbol{S},$$

$$\oint_{\Gamma} \boldsymbol{E}' \cdot d\boldsymbol{l} = -\int_{S_{\Gamma}} \frac{\partial \boldsymbol{B}}{\partial t} \cdot d\boldsymbol{S} + \oint_{\Gamma} (\boldsymbol{v} \times \boldsymbol{B}) \cdot d\boldsymbol{l}. \tag{3.134}$$

$$\int_{S_{\Gamma}} \operatorname{curl} \boldsymbol{H}' \cdot d\boldsymbol{S} = \int_{S_{\Gamma}} \boldsymbol{J} \cdot d\boldsymbol{S} + \int_{S_{\Gamma}} \frac{\partial \boldsymbol{D}}{\partial t} \cdot d\boldsymbol{S} - \int_{S_{\Gamma}} \operatorname{curl}(\boldsymbol{v} \times \boldsymbol{D}) \cdot d\boldsymbol{S},$$

$$\oint_{\Gamma} \boldsymbol{H}' \cdot d\boldsymbol{l} = \int_{S_{\Gamma}} \boldsymbol{J} \cdot d\boldsymbol{S} + \int_{S_{\Gamma}} \frac{\partial \boldsymbol{D}}{\partial t} \cdot d\boldsymbol{S} - \oint_{\Gamma} (\boldsymbol{v} \times \boldsymbol{D}) \cdot d\boldsymbol{l}. \tag{3.135}$$

Relations (3.127), (3.134) and (3.129), (3.135) represent the *expressions of the local* and *integral forms of the laws of electromagnetic induction* and *of magnetic circuit,* respectively*, for any moving medium.*

The vector quantities $\boldsymbol{E}$ and $\boldsymbol{B}$ are introduced as explained in Section 2.5, the vector quantity $\boldsymbol{H}$ is introduced according to relation (3.80) and the vector quantity $\boldsymbol{D}$ is introduced as shown in Section 3.2, relation (3.18).

Expressions (3.134) and (3.135), for moving media, may be obtained from relations (3.56) and (3.96). For this purpose, instead of quantities $\boldsymbol{E}$ and $\boldsymbol{H}$, we shall introduce the quantities $\boldsymbol{E}'$ and $\boldsymbol{H}'$ from a moving reference frame fixed to the moving medium, and we shall calculate the derivatives with respect to of the magnetic and electric fluxes. Hence, we have to calculate $\dfrac{d}{dt}\int_{S_{\Gamma}} \boldsymbol{B} \cdot d\boldsymbol{S}$ and $\dfrac{d}{dt}\int_{S_{\Gamma}} \boldsymbol{D} \cdot d\boldsymbol{S}$, and the quantities $\boldsymbol{B}$ and $\boldsymbol{D}$ are expressed in the fixed reference frame. The calculation of the derivative can be performed taking into account the following two remarks:

   1° – The vector function, which has to be integrated, varies with time even in the case in which the surface S_{Γ} is at rest;

   2° – The surface S_{Γ} varies in form and position relatively to the system of co-ordinates even in the case in which the function above does not vary. The derivation of this formula is given in Vector Calculus and is sometimes called the derivative of the flux.

3.12.   THE RELATIONS BETWEEN THE COMPONENTS OF THE STATE QUANTITIES OF ELECTROMAGNETIC FIELD IN THE CASE OF DISCONTINUITY SURFACES

The relations between the components, expressed in the same reference frame, of the quantities $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{B}$ and $\boldsymbol{H}$, of both sides of the discontinuity surface will be examined.

3.12.1. The Relation between the Normal Components of Electric Displacement Vectors

Let S_d be the separation surface of two insulating media having different electric permittivity. Let ΔS be a portion of the separation surface S_d. The dimensions of ΔS are assumed infinitesimals of the first order. The dimensions of ΔS are taken small enough for to be assumed as plane. The portion ΔS of the separation surface will be dressed by a very flat surface Σ having the form of a right cylinder or a parallelepiped (Fig. 3.10) the height of which is an infinitesimal of a higher order relatively to the dimensions of the cross-section ΔS.

Let the vector quantities $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ be the electric displacements at two points very near situated on both sides of the separation surface.

The true (free) electric charge is supposed to be distributed in space with a finite volume density.

The law of electric flux, in integral form, expressed by relation (3.14) will be applied for the case of the closed surface Σ. The true electric charge in the interior of the parallelepiped, represented by the right-hand side of expression (3.14) of the electric flux law, is an infinitely small quantity of higher order, proportional to the height of the parallelepiped and tends to zero together with that height.

Neglecting the flux through the lateral surface of the parallelepiped, because its height is much smaller than the dimensions of its bases, the following relations will be obtained:

$$\int_{\Sigma} \boldsymbol{D} \cdot \mathrm{d}\,\boldsymbol{S} = q_{\Sigma} = 0, \tag{3.136}$$

$$\boldsymbol{D}_1 \cdot \left(-\boldsymbol{n}_{12}\right)\Delta S + \boldsymbol{D}_2 \cdot \boldsymbol{n}_{12}\,\Delta S = 0, \tag{3.137}$$

where the vector $\boldsymbol{n}_{12}$ is the unit vector of the normal to the separation surface S_d, oriented from the medium 1 towards the medium 2, hence $\boldsymbol{n}_1 = -\boldsymbol{n}_{12} = -\boldsymbol{n}_2$.
Hence:

$$\boldsymbol{n}_{12} \cdot \left(\boldsymbol{D}_2 - \boldsymbol{D}_1\right) = 0. \tag{3.138}$$

It follows that, in the case of the true (free) electric charge distributed in space, at the passage through a separation surface of two media, the normal component of the electric induction is conserved, thus:

$$D_{n1} = D_{n2}, \tag{3.139 a}$$

$$\boldsymbol{D}_{n1} = \boldsymbol{D}_{n2}. \tag{3.139 b}$$

The last relation expresses *the theorem of conservation of the normal component of the electric displacement.*

In the case in which there is true electric charge distributed with a finite surface density ρ_s just on the separation surface, say on the portion ΔS, then $q_{\Sigma} = \rho_s\,\Delta S$ and instead of relation (3.138) we obtain:
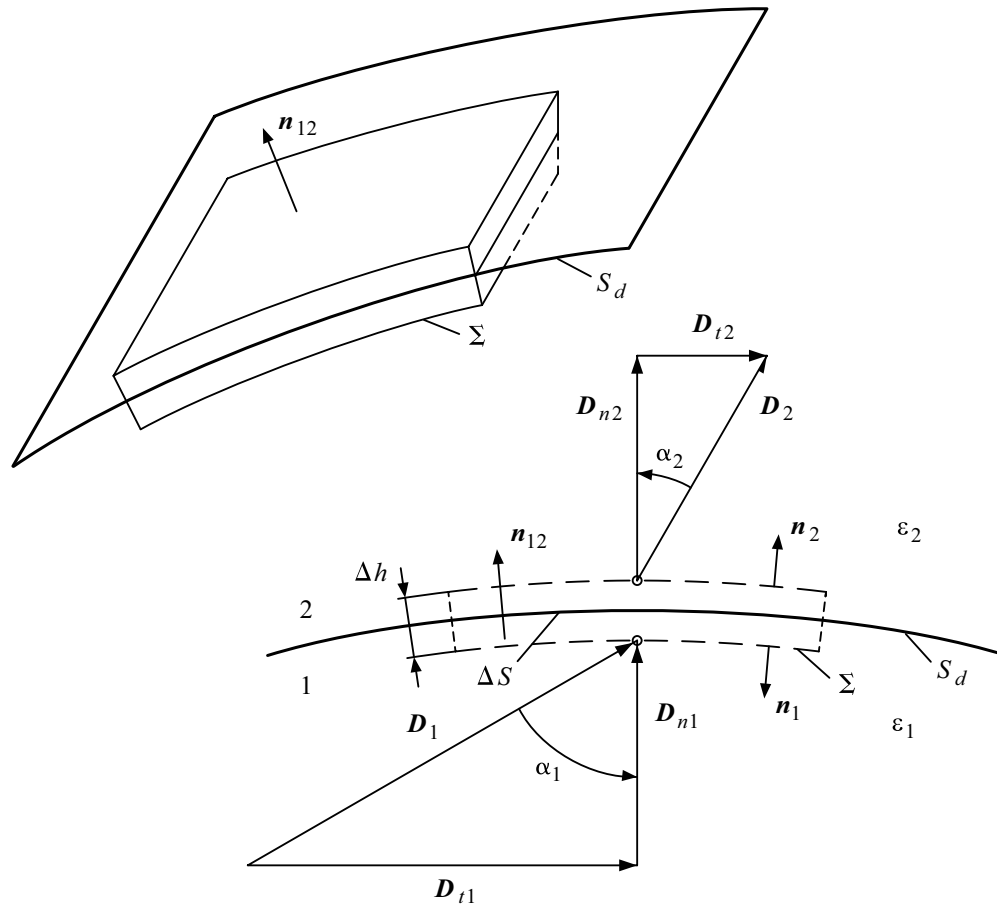
Fig. 3.10. Explanation to the computation of the normal components of the vector
electric displacement at the passage from one medium to another.
The vector D_1 has been displaced, being brought with its end point to
its starting point for avoiding superposition in the figure.

$$n_{12} \cdot (D_2 - D_1) = \text{div}_s\, D = \rho_s \,, \tag{3.140}$$

thus:

$$D_{n2} - D_{n1} = \rho_s \,. \tag{3.141}$$

3.12.2. The Relation between the Tangential Components of Electric Field Strength Vectors

Let S_d be the separation surface of two insulating media having different electric permittivity. Also, it will be assumed that the electric field strength has finite values on the separation surface and within its vicinity. Also, let be a small rectangular plane curve

Γ having each of its long sides on one of the two sides, respectively, of the separation surface, parallel and very near to that surface (Fig. 3.11). The length of one long side of the rectangular curve, parallel to the separation surface will be denoted by Δl and the height, parallel with the normal to the separation surface, will be denoted by Δh. The length of Δl is taken small enough for to be considered as a straight-line segment. The quantity Δl is assumed as an infinitesimal of the first order, and the quantity Δh as an infinitesimal of a higher order.

Let the vector quantities E_1 and E_2 be the electric field strengths at two points very near situated on both sides of the separation surface.

The law of electromagnetic induction in integral form, expressed by relation (3.50) will be applied for the closed curve Γ that has a rectangular form (Fig. 3.11). The travelling direction along the curve of Fig. 3.11 results from the direction of the unit vectors t_1 and t_2. The magnetic flux represented by the surface-integral of the right-hand side of expression (3.50) of the law of electromagnetic induction, is an infinitely small quantity of higher order, proportional to the height of the rectangle, and tends to zero together with that height.
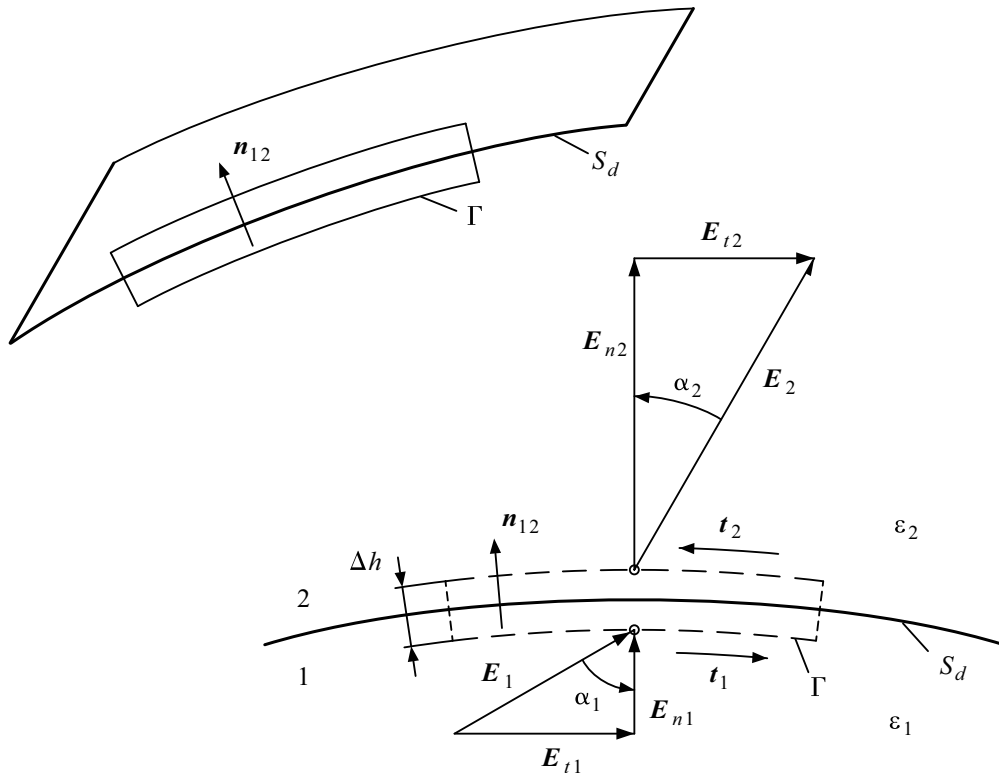
Fig. 3.11. Explanation to the calculation of tangential components of electric field strength vectors at the passage from one medium to another.
The vector E_1 has been displaced, being brought with its end point to its starting point for avoiding superposition in the figure.

By neglecting the line-integrals along the height, because the height Δh is much smaller than the length Δl, it follows:

$$E_1 \cdot t\, \Delta l + E_2 \cdot \left(-t\right)\Delta l = 0, \tag{3.142}$$

where the vector quantity t is the unit vector of the tangent to the separation surface along the side of length Δl of the rectangle. Thus:

$$t \cdot \left(E_1 - E_2\right) = 0. \tag{3.143}$$

It results that at the passage through a separation surface between media, the tangential component of the electric field strength is conserved:

$$E_{t1} = E_{t2}, \tag{3.144 a}$$

$$E_{t1} = E_{t2}. \tag{3.144 b}$$

The last relation expresses *the theorem of conservation of the tangential component of the electric field strength.*

3.12.3. The Theorem of Refraction of the Lines of Electric Field in the Case of Insulating Media

Let us consider a separation surface between two isotropic media of different electric permittivity and we shall assume that the electric charge is distributed in space, i.e., it has a volume distribution.
We can write:

$$D_1 = \varepsilon_1\, E_1, \tag{3.145 a}$$

$$D_2 = \varepsilon_2\, E_2. \tag{3.145 b}$$

At the passage through the separation surface above, the vectors electric displacement and electric field strength vary in magnitude and direction, i.e., they are refracted. From Fig. 3.10, it follows:

$$\tan \alpha_1 = \frac{D_{t1}}{D_{n1}} = \frac{\varepsilon_1\, E_{t1}}{D_{n1}}, \tag{3.145 c}$$

$$\tan \alpha_2 = \frac{D_{t2}}{D_{n2}} = \frac{\varepsilon_2\, E_{t2}}{D_{n2}}, \tag{3.145 d}$$

$$\frac{\tan \alpha_1}{\tan \alpha_2} = \frac{\varepsilon_1\, E_{t1}}{D_{n1}} \cdot \frac{D_{n2}}{\varepsilon_2\, E_{t2}}, \tag{3.145 e}$$

or by using the conservation theorems (3.139) and (3.144), it results that:

$$\frac{\tan \alpha_1}{\tan \varepsilon_2} = \frac{\varepsilon_1}{\varepsilon_2}. \tag{3.146}$$

Relation (3.146) represents the theorem of refraction of the lines of electric field in the case in which the true (free) electric charge has a volume distribution. In this case, there are not discontinuity surfaces of the true electric charge distribution. Therefore, on the separation surface, there is not electric charge distributed with a finite surface density.

The refraction theorem expressed by relation (3.146) is similar, to some extent, to that of Optics. However, it differs from the latter because relation (3.146) is not compatible with the total reflection phenomenon, since the trigonometric tangent function occurs instead of sine function in Optics. Hence, the refraction ratio (3.146) cannot have negative values like in Optics.

3.12.4. The Relation between the Normal Components of Magnetic Induction Vectors

Let S_d be the separation surface of two insulating media having different magnetic permeability. Let ΔS be a portion of the separation surface S_d. The dimensions of ΔS are assumed very small, hence infinitesimal entities of the first order. The portion ΔS of the separation surface will be dressed by a surface Σ having the form of a right cylinder or parallelepiped (Fig. 3.12) the height of which is an infinitesimal of a higher order relatively to the dimensions of the cross-section ΔS that may be considered as a plane surface element. The bases of the parallelepiped that is a closed surface have the same dimensions like the cross-section. Let the vector quantities $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ be the magnetic inductions at two points very near situated on both sides of the separation surface.

The law of magnetic flux, in integral form, expressed by relation (3.29), will be applied for the case of the closed surface Σ. Neglecting the flux through the lateral surface of the parallelepiped, because its height is much smaller than the dimensions of its bases, the following relations are obtained:

$$\int_{\Sigma} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = 0, \tag{3.147}$$

$$\boldsymbol{B}_1 \cdot \left(-\boldsymbol{n}_{12}\right)\Delta S + \boldsymbol{B}_2 \cdot \boldsymbol{n}_{12}\, \Delta S = 0, \tag{3.148}$$

where the vector $\boldsymbol{n}_{12}$ is the unit vector of the normal to the separation surface S_d, oriented from the medium 1 towards the medium 2, hence: $\boldsymbol{n}_1 = -\boldsymbol{n}_{12} = -\boldsymbol{n}_2$.

Hence:

$$\boldsymbol{n}_{12} \cdot \left(\boldsymbol{B}_2 - \boldsymbol{B}_1\right) = \mathrm{div}_s\, \boldsymbol{B} = 0. \tag{3.149}$$

It follows that at the passage through a separation surface of two media, the normal component of the magnetic induction is conserved, thus:

$$B_{n1} = B_{n2}, \tag{3.150 a}$$

$$\boldsymbol{B}_{n1} = \boldsymbol{B}_{n2}. \tag{3.150 b}$$

The last relation expresses *the theorem of conservation of the normal component of the magnetic induction.*
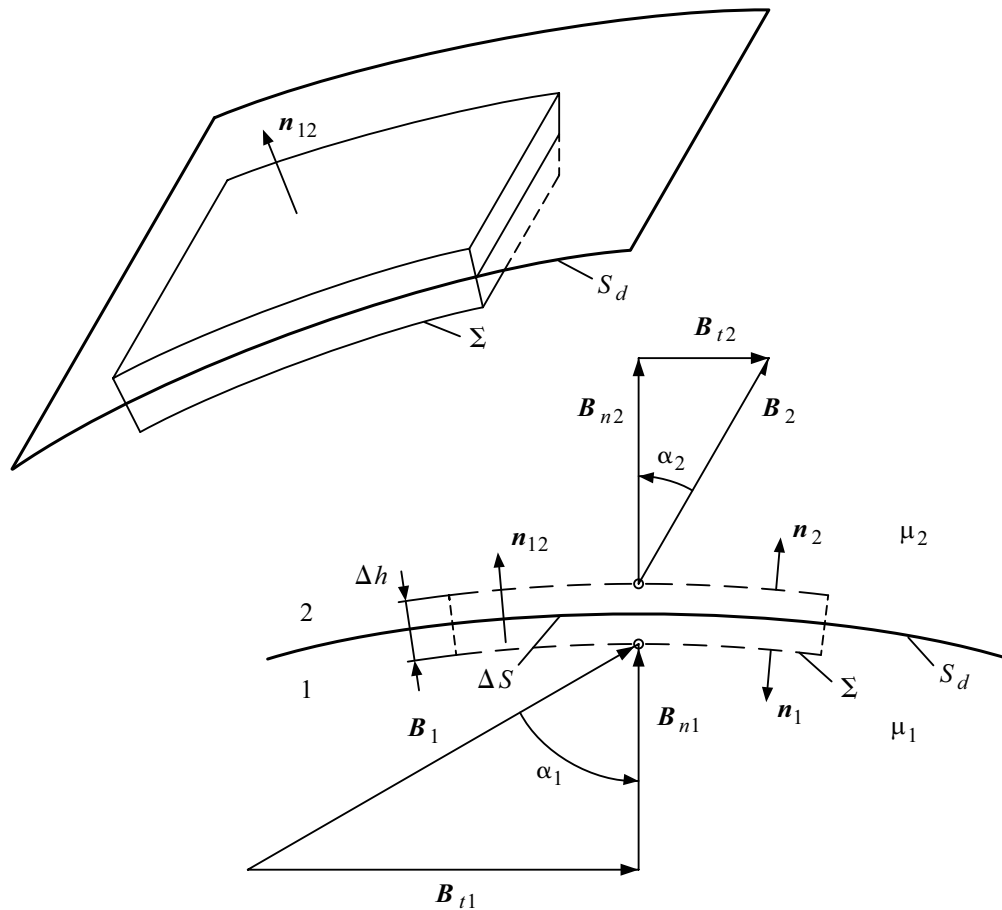
Fig. 3.12. Explanation to the computation of the normal components of the vector
magnetic induction at the passage from one medium to another.
The vector B_1 has been displaced, being brought with its end point to
its starting point for avoiding superposition in the figure.

3.12.5. The Relation between the Tangential Components of the Magnetic Field Strength Vectors

Let S_d be the separation surface of two insulating media having different magnetic permeability. Also, it will be assumed that the electric current density has finite values on the separation surface, hence, the electric current is not discontinuously distributed in the form of a current sheet on the separation surface. In addition, it will be assumed that the electric displacement has finite values.

Also, let Γ be a small rectangular plane curve having each of its long sides on one of the two sides, respectively, of the separation surface, parallel and very near that surface (Fig. 3.13). The length of one long side of the rectangular curve, parallel to the separation
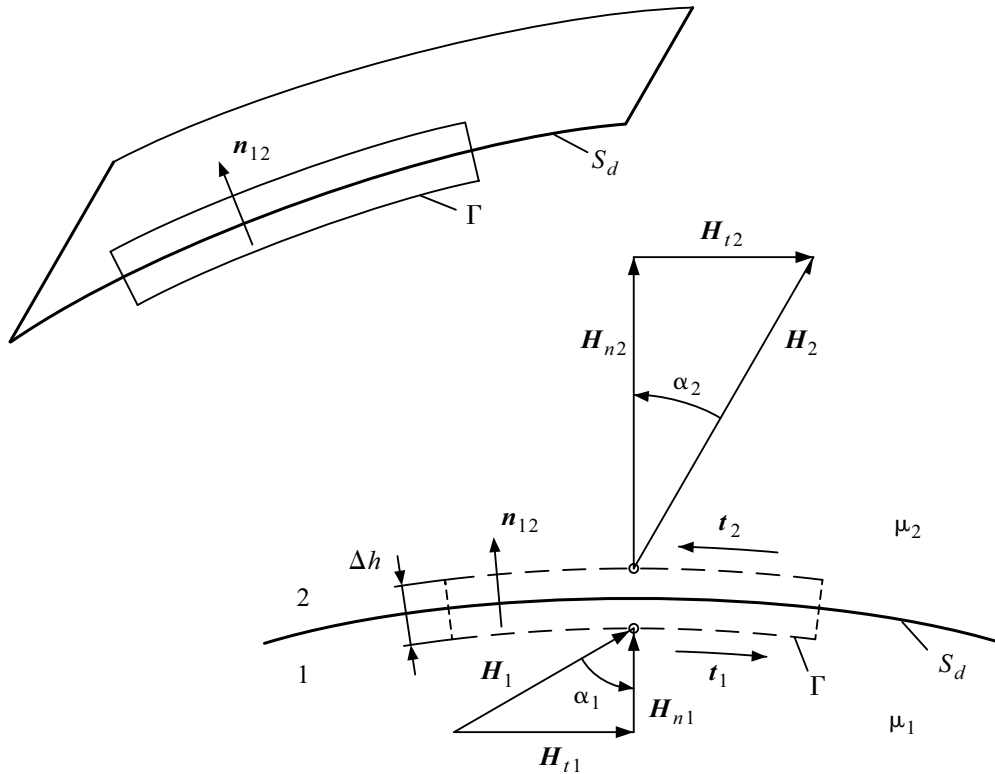
Fig. 3.13. Explanation to the calculation of tangential components of the magnetic
field strength vectors at the passage from one medium to another.
The vector H_1 has been displaced, being brought with its end point to
its starting point for avoiding superposition in the figure.

surface will be denoted by Δl, and the height, parallel with the normal to the separation surface, will be denoted by Δh. The length of Δl is taken small enough for to be considered as a straight-line segment. The quantity Δl is assumed as an infinitesimal of the first order, and the quantity Δh as an infinitesimal of a higher order.

Let the vector quantities H_1 and H_2 be the magnetic field strengths at two points very close one to the other situated on both sides of the separation surface.

The law of magnetic circuit in integral form, expressed by relation (3.84) will be applied for the closed curve Γ that has a rectangular form (Fig. 3.13). The travelling direction along the curve Γ results from the direction of the unit vectors t_1 and t_2.

The electric flux represented by the surface-integral of the right-hand side of expression (3.81) of the law of magnetic circuit, is an infinitely small quantity of higher order, proportional to the height of the rectangle, and tends to zero together with that height.

By neglecting the line-integrals along the height, because the height Δh is much smaller than the length Δl, it follows:

$$\boldsymbol{H}_1 \cdot \boldsymbol{t}\,\Delta l + \boldsymbol{H}_2 \cdot \left(-\boldsymbol{t}\right)\Delta l = 0, \tag{3.151}$$

where the vector quantity $\boldsymbol{t}$ is the unit vector of the tangent to the separation surface along the side of length Δl of the rectangle. Thus:

$$\boldsymbol{t} \cdot \left(\boldsymbol{H}_1 - \boldsymbol{H}_2\right) = 0. \tag{3.152}$$

It results that at the passage through a separation surface between media, the tangential component of the magnetic field is conserved:

$$H_{t1} = H_{t2}, \tag{3.153 a}$$

$$\boldsymbol{H}_{t1} = \boldsymbol{H}_{t2}. \tag{3.153 b}$$

The last relation expresses *the theorem of conservation of the tangential component of the magnetic field strength.*

We shall consider the case in which an electric current is distributed on the separation surface in the form of a current sheet (Fig. 1.12), of linear density J_s of direction perpendicular to the plane of the figure at the points of the trace of the surface S_d in the Fig. 1.12. In this case, the current linkage is $J_s \cdot \Delta l$ and instead of relation (3.152), we obtain:

$$\boldsymbol{t} \cdot \left(\boldsymbol{H}_1 - \boldsymbol{H}_2\right) = J_s, \tag{3.154}$$

thus:

$$H_{t1} - H_{t2} = J_s. \tag{3.155}$$

3.12.6. The Theorem of Refraction of the Lines of Magnetic Field at the Passage through the Separation Surface of Two Media

Let us consider a separation surface between two isotropic media of different magnetic permeability and we shall assume that the electric current has a volume distribution.

We can write:

$$\boldsymbol{B}_1 = \mu_1 \, \boldsymbol{H}_1, \tag{3.156 a}$$

$$\boldsymbol{B}_2 = \mu_2 \, \boldsymbol{H}_2. \tag{3.156 b}$$

At the passage through the separation surface above, the vectors magnetic induction and magnetic field strength vary in magnitude and direction, i.e., they are refracted. From Fig. 3.13, it follows:

$$\tan \alpha_1 = \frac{B_{t1}}{B_{n1}} = \frac{\mu_1 \, H_{t1}}{B_{n1}}, \tag{3.156 c}$$

$$\tan \alpha_2 = \frac{B_{t2}}{B_{n2}} = \frac{\mu_2 \, H_{t2}}{B_{n2}}, \tag{3.156 d}$$

$$\frac{\tan \alpha_1}{\tan \alpha_2} = \frac{\mu_1 H_{t1}}{B_{n1}} \cdot \frac{B_{n2}}{\mu_2 H_{t2}} \qquad (3.156\ e)$$

or, by using the conservation theorems (3.150) and (3.153), it follows:

$$\frac{\tan \alpha_1}{\tan \alpha_2} = \frac{\mu_1}{\mu_2}. \qquad (3.157)$$

Relation (3.157) represents the theorem of refraction of the lines of magnetic field in the case in which the electric current has a volume distribution. Hence, there are not discontinuity surfaces carrying electric current sheets.

3.12.7. The Relation between the Normal Components of Electric Displacement Vectors and Electric Current Densities

Let us consider the separation surface S_d between two media having different electric properties, namely different electric permittivity and conductivity. The surface is assumed carrying no electric current sheet. The very flat parallelepiped surface of Sub-section 3.12.1 (Fig. 3.10) will be considered. For the parallelepiped base of medium 1, bounded by a rectangular curve, by applying relation (3.79), we can write:

$$\oint_{\Gamma_1} \boldsymbol{H}_1 \cdot \mathrm{d}\boldsymbol{l} = \int_{S_{\Gamma_1}} \left(\boldsymbol{J}_1 + \frac{\partial \boldsymbol{D}_1}{\partial t} \right) \cdot \mathrm{d}\boldsymbol{S} = \int_{S_{\Gamma_1}} \left(J_{n1} + \frac{\partial D_{n1}}{\partial t} \right) \mathrm{d}\,S. \qquad (3.158\ a)$$

For the parallelepiped base of medium 2, bounded by a rectangular curve, by applying relation (3.79), we can write:

$$\oint_{\Gamma_2} \boldsymbol{H}_2 \cdot \mathrm{d}\boldsymbol{l} = \int_{S_{\Gamma_2}} \left(\boldsymbol{J}_2 + \frac{\partial \boldsymbol{D}_2}{\partial t} \right) \cdot \mathrm{d}\boldsymbol{S} = \int_{S_{\Gamma_2}} \left(J_{n2} + \frac{\partial D_{n2}}{\partial t} \right) \mathrm{d}\,S. \qquad (3.158\ b)$$

The curves Γ_1 and Γ_2 that bound the two bases tend to the same curve Γ of very small dimensions placed on the separation surface. The quantities $\boldsymbol{H}_1 \cdot \mathrm{d}\boldsymbol{l} = \boldsymbol{H}_{t1} \cdot \mathrm{d}\boldsymbol{l}$ and $\boldsymbol{H}_2 \cdot \mathrm{d}\boldsymbol{l} = \boldsymbol{H}_{t2} \cdot \mathrm{d}\boldsymbol{l}$ are equal, for according to relation (3.153), $H_{t1} = H_{t2}$.

The last two relations (3.158 a) and (3.158 b) yield:

$$J_{n1} + \frac{\partial D_{n1}}{\partial t} = J_{n2} + \frac{\partial D_{n2}}{\partial t}. \qquad (3.159)$$

and, as mentioned, the following relation

$$H_{t1} = H_{t2}$$

subsists.

3.13.   THE SI UNITS OF MEASURE OF ELECTRIC AND MAGNETIC QUANTITIES: ELECTRIC FLUX, ELECTRIC DISPLACEMENT, ELECTRIC RESISTIVITY, MAGNETIC FLUX, MAGNETIC INDUCTION, MAGNETIC FIELD STRENGTH.

3.13.1.   The Units of Electric Flux and Displacement

In the SI system, the unit of measure of the flux of electric displacement, also called electric flux, results from relation (3.14) and it is the coulomb, symbol C, the same as for the electric charge. From the same relation, it follows that the unit of measure of the electric displacement (electric flux density) is C/m^2. From relation (3.18) it follows that the unit of measure of the electric polarization is the same as for the electric displacement.

3.13.2.   The Unit of Electric Resistivity

In the SI system, the unit of measure of the electric resistivity results from relations (1.108) and (1.109). This unit of measure is given by the ratio of E_l and J as follows:

$$1[\rho]_{SI} = \frac{1[E]}{1[J]} = \frac{1\,V/m}{1\,A/m^2} = \frac{1\,V}{1\,A}\,1\,m.$$

But in Electrokinetics, the ratio between one volt and one ampere is called ohm, symbol Ω, thus:

$$\frac{1\,V}{1\,A} = 1\,\Omega.$$

Therefore the unit of measure of the electric resistivity is Ωm, as follows:

$$1[\rho] = 1\,\Omega\,m.$$

3.13.3.   The Unit of Magnetic Flux

A definition of the unit of measure of magnetic flux may be given by using the law of electromagnetic induction in the integral form (3.57):

$$u_e = -\frac{d\,\Psi}{d\,t}.$$

In the SI system of units, the unit of measure of magnetic flux is the *weber*, symbol Wb.

The weber is the magnetic flux that cancelling by a linear variation after a time of one second induces in a single turn linked by it, the electromotive force of 1 V.

Indeed, for the linear variation with time of the magnetic flux, mentioned above, and shown in Fig. 3.14, it follows:

$$u_e = -\frac{\mathrm{d}\,\Psi}{\mathrm{d}\,t} = -\frac{0 - \Psi_0}{1} = \frac{\Psi_0}{1}.$$

If the variation of the magnetic flux were not a linear one then the electromotive force induced at any instant, in the turn, would be given by the derivative of the magnetic flux. In this case, the value of the magnetic flux is more difficult to be determined than in the case above.

3.13.4. The Unit of Magnetic Induction

In the SI system of units, the unit of measure of magnetic induction is the *tesla*, symbol T. A definition of the unit of measure of magnetic induction can be obtained by using the defining relation of magnetic flux (3.32):

$$\Psi = \int_{S_\Gamma} \boldsymbol{B} \cdot \mathrm{d}\,\boldsymbol{S}.$$

In the case of a homogeneous magnetic field, the last relation becomes:

$$\Psi = \boldsymbol{B} \cdot \boldsymbol{S},$$

and if the vector quantity $\boldsymbol{B}$ is perpendicular to the considered surface, i.e., the vectors $\boldsymbol{B}$ and $\boldsymbol{S}$ are parallel to each other, it results that:

$$\Psi = B\,S.$$

The *tesla* is the magnetic induction of a homogeneous magnetic field (i.e., with a uniform distribution), the flux of which, through a surface having the area of $1\,\mathrm{m}^2$, and perpendicular to the lines of the magnetic field, is of $1\,\mathrm{Wb}$.
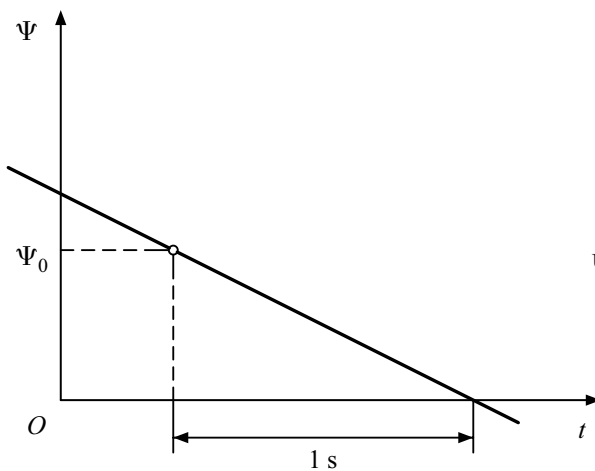
Fig. 3.14. Explanation concerning the definition of the unit of measure of the magnetic flux.

3.13.5. The Unit of Magnetic Field Strength

The unit of measure of the magnetic field strength can be obtained by using the expression of the magnetic field strength at the centre of a circular turn carrying a direct current:

$$H = \frac{i}{2a}.$$

This expression will be established in Sub-section 3.15.2.

In the SI system, the unit of measure of the magnetic field strength is the ampere per metre, symbol A/m.

Ampere per metre is the magnetic field strength at the centre of a circular turn with the diameter of 1 m, carrying a direct current of 1 A.

The unit of measure of the magnetic tension and magnetomotive force, in the SI system of units, is the *ampere*, symbol A, hence the same unit of measure as for the electric current intensity.

3.13.6. The Units of Electric and Magnetic Constants

The unit of measure of the electric constant, in the SI system of units, can be easily obtained from relation (3.6). The unit of measure of the left-hand side of this relation is that of the product of the quantities E, S and ε_0, hence $(V/m)\,m^2 \cdot [\varepsilon_0]$, thus $V \cdot m \cdot [\varepsilon_0]$. The unit of measure of the right-hand side of the same relation is that of the electric charge, thus C. By equating the two units of measure, we obtain:

$$[\varepsilon_0] = \frac{C}{V} \cdot \frac{1}{m}.$$

But in Electrostatics, the ratio C/V is defined as farad, symbol F. Therefore the unit of the electric constant is F/m.

The unit of measure of the magnetic constant can be easily obtained from relation (2.27 b). From these relations, it follows that *in vacuo*:

$$\boldsymbol{B} = \mu_0 \, \boldsymbol{H},$$

where, for the sake of simplicity, the subscript o indicating the reference frame has been omitted.

In the last relation, the unit of measure of the left-hand side is that of B. The unit of measure of the right-hand side of the same relation is that of the product of the quantities H, and μ_0, hence $(A/m) \cdot [\mu_0]$. But, as shown above, namely by the last relation of Sub-section 3.13.4, this unit of B can be expressed in the form Wb/m^2. By equating the two units of measure, we obtain:

$$[\mu_0] = \frac{Wb}{A} \cdot \frac{1}{m}.$$

But in Electrodynamics, the ratio $\mathrm{Wb/A}$ is defined as henry, symbol H. Therefore the unit of the electric constant is $\mathrm{H/m}$. The same results could be obtained, e.g., by using relations (3.23), (3.24) and (3.104), (3.105), respectively.

3.13.7.   Remark on the Various Systems of Units of Measure in Electromagnetism

Before the adoption of the International System of Units of Measure, and the MKSA which preceded it, two systems of units of measure played an important role, having a scientific value: The Electrostatic System of Units and the Electromagnetic System of Units [1, Arts. 625, 628].

The Electrostatic System of Units has been based on the definition of the unit of electric charge according to the Coulomb formula for electric charges *in vacuo*, with the constant of proportionality dimensionless and equal to unity. The two charges occurring in formula are taken equal to each other. Hence the formula will contain only the quantities: force, electric charge q and a length r.

The Electromagnetic System of Units has been based on the definition of the unit of magnetic charge (i.e., the fictitious magnetic charge) according to the Coulomb formula for Amperian magnetic charges with the constant of proportionality dimensionless and equal to unity. Further on, the formula of the electromagnetic force, like that exerted between two coils carrying electric currents, including one proportionality constant, has also been established using the same system of units. As shown in [1, Vol. II, Art. 628], the ratio between the unit of electric charge in the Electromagnetic System of Units, and that of the Electrostatic System of Units, denoted by indices uem and ues, has the dimension of a velocity, and was found to be approximately equal to $c = 3 \times 10^{10}$ cm/s. It is just the velocity of light in empty space (vacuum) or air. The apparatus conceived and used by Maxwell for the experimental determination can be found in [4, 372].

The same result may be directly obtained by remaking the calculations concerning formulae (2.22) – (2.24), taking $\mu_0 = 4\pi$ and dimensionless. Using formula (1.113) there follows:

$$\varepsilon_0 = \frac{1}{c^2} \cdot \frac{1}{\mu_0} = \frac{1}{c^2\, 4\pi}; \quad k_0 = c^2; \quad c\, q_{\mathrm{uem}} = q_{\mathrm{ues}}; \quad 1[q_{\mathrm{uem}}] = c[q_{\mathrm{ues}}],$$

and then we obtain the ratio above.

A combination of the two systems of units, called the Gaussian System of Units, has also been used [3], [7], [8], [11], [22], [23], [24].

3.14.   THE LAWS OF ELECTROMAGNETIC FIELD IN THE CASE OF EXISTENCE OF MAGNETIC MONOPOLES

We shall further examine the modifications occurring in the laws of the theory of the electromagnetic field, in the case of existence of magnetic monopoles. The existence of these monopoles has been supposed, but up to now no doubtless experiment has proved their existence.

3.14.1. Expression of the Interaction Force between Two Magnetic Monopoles

The expression of the interaction force between two point-like magnetic charges, i.e., monopoles, is assumed to be of the same form as the expression of the interaction force between two point-like electric charges, hence an expression of the form:

$$F_{21} = \frac{1}{4\pi\mu_0^{-1}} \cdot \frac{q_{M1}\, q_{M2}}{r_{12}^2} \cdot \frac{r_{12}}{r_{12}} \tag{3.160}$$

or

$$F_{21} = \frac{1}{4\pi\mu_0} \cdot \frac{q_{m1}\, q_{m2}}{r_{12}^2} \cdot \frac{r_{12}}{r_{12}}, \tag{3.161}$$

which is *the Coulomb formula for magnetic charges* in the case of empty space (i.e., vacuum). The quantities introduced by relation (3.160) are called *Amperian magnetic charges*, whereas the quantities introduced by relation (3.161) are called *Coulombian magnetic charges*. Between these quantities we have the following relation:

$$q_m = \mu_0\, q_M . \tag{3.162}$$

If for the constant of the right-hand side of relations (3.160), (3.161), other value were adopted, different units of measure would be obtained for the electromagnetic quantities.

Analogously as for the electric charges, we can consider volume distributions of magnetic monopoles with the volume density of the magnetic charge ρ_{vM} or ρ_{vm}, respectively, including the suffix M or m. Also, we can consider magnetic charge with surface distribution with the surface density ρ_{sM} or ρ_{sm}, and line distribution with the line density ρ_{lM} or ρ_{lm}. The monopoles can be at rest or in motion.

Similarly as in the case of electric charges, the following force produced by any point-like magnetic charge q_{m1} placed at any point, called source point, is acting on the point-like magnetic charge q_m placed at another point, called field point or observation point:

$$F_m = q_M\, B_m = q_m\, H_m . \tag{3.163}$$

If we denote by r_{12} the vector having its origin at the source point and its end at the field point (observation point), it follows:

$$H_m = \frac{1}{4\pi\mu_0^{-1}} \cdot \frac{q_{M1}}{r_{12}^2} \cdot \frac{r_{12}}{r_{12}} = \frac{1}{4\pi\mu_0} \cdot \frac{q_{m1}}{r_{12}^2} \cdot \frac{r_{12}}{r_{12}}, \tag{3.164}$$

$$B_m = \mu_0\, H_m . \tag{3.165}$$

3.14.2. Electric Field Produced by Moving Magnetic Monopoles

We shall consider the same reference frames K_o and K_1, of Sub-section 2.3.2, the latter moving at the velocity v_{1o} with respect to the former. At any fixed point, A, in K_1, there is a point-like magnetic charge q_{m1} (the subscript 1 of q_{m1} has no relation with the

subscript 1 of K_1), and another magnetic point-like magnetic charge q_m, at any point P, moving at the velocity $\boldsymbol{v}_{2\mathrm{o}}$ relatively to K_o. We shall consider the magnetic field strength produced by q_{m1} at the point P, in empty space.

We shall remake the calculations of Section 2.4, but instead of relations (2.18 a, b, c) which are based on formula (1.112), we shall use the relations based on one of formulae (3.160) or (3.161). In order to distinguish the field state quantities due to the magnetic point-like charge, those quantities will have the suffix m. We obtain:

$$\boldsymbol{D}_{\mathrm{o}m} = \varepsilon_0 \, \boldsymbol{E}_{\mathrm{o}m}; \quad \boldsymbol{D}_{\mathrm{o}m} = -\frac{1}{c^2} \boldsymbol{v}_{1\mathrm{o}} \times \boldsymbol{H}_{\mathrm{o}m}; \qquad\qquad (3.166 \text{ a, b})$$

$$\boldsymbol{F}_{\mathrm{o}m} = q_m \boldsymbol{H}_{\mathrm{o}m} + q_m \left[\boldsymbol{v}_{2\mathrm{o}} \times \left(\frac{1}{c^2} \boldsymbol{v}_{1\mathrm{o}} \times \boldsymbol{H}_{\mathrm{o}m} \right) \right]. \qquad\qquad (3.167)$$

3.14.3.   The Expressions of the Laws of the Theory of the Electromagnetic Field in the Case of Magnetic Monopoles

We shall make derivations similar to those for expressions: (3.5 b), (3.27), (3.47), (3.68). The calculations include: $\mathrm{div}_\mathrm{o} \, \mu_0 \, \boldsymbol{H}_{\mathrm{o}m}$, $\mathrm{curl}_\mathrm{o} \, \boldsymbol{H}_{\mathrm{o}m}$, $\dfrac{\partial}{\partial t_\mathrm{o}} \boldsymbol{D}_{\mathrm{o}m}$, $\mathrm{curl}_\mathrm{o} \, \boldsymbol{D}_{\mathrm{o}m}$, in K_o.

The field state quantities obtained from the last relations for magnetic charges will be added, side by side, with those previously obtained for the electric charges, as follows:

$$\boldsymbol{D}_\mathrm{o} := \boldsymbol{D}_\mathrm{o} + \boldsymbol{D}_{\mathrm{o}m}; \quad \boldsymbol{E}_\mathrm{o} := \boldsymbol{E}_\mathrm{o} + \boldsymbol{E}_{\mathrm{o}m}; \quad \boldsymbol{B}_\mathrm{o} := \boldsymbol{B}_\mathrm{o} + \boldsymbol{B}_{\mathrm{o}m}; \quad \boldsymbol{H}_\mathrm{o} := \boldsymbol{H}_\mathrm{o} + \boldsymbol{H}_{\mathrm{o}m}. \qquad (3.\,168)$$

It results that the form of relations (3.5 b) and (3.27) is not modified, while relations (3.27) and (3.68) become:

$$\mathrm{div}_\mathrm{o} \, \boldsymbol{B}_\mathrm{o} = \mu_0 \, \rho_{voM}; \; \mathrm{curl}_\mathrm{o} \, \boldsymbol{E}_\mathrm{o} = -\boldsymbol{J}_{\mathrm{mag}\,\mathrm{o}}, \; \boldsymbol{J}_{\mathrm{mag}\,\mathrm{o}} = \frac{\partial \boldsymbol{B}_\mathrm{o}}{\partial t_\mathrm{o}} + \mu_0 \, \rho_{voM} \, \boldsymbol{v}_m, \; \boldsymbol{v}_m = \boldsymbol{v}_{1\mathrm{o}}, \quad (3.169 \text{ a-d})$$

where $\boldsymbol{J}_{\mathrm{mag}}$ may be called the *magnetic current density*.

3.15.   APPLICATION OF THE BIOT-SAVART-LAPLACE FORMULA TO THE CALCULATION OF THE MAGNETIC FIELD STRENGTH

3.15.1.   Expression of the Magnetic Field Strength Produced by a Thread-Like Rectilinear Conductor Carrying a Constant Electric Current

Let us consider a rectilinear thread-like conductor of length l and constant cross-section s_c carrying an electric current of intensity i shown in Fig. 3.15, in empty space. In this case, the magnetic field strength is given by relation (2.48). It follows:

$$\boldsymbol{H}(P) = \boldsymbol{H} = \frac{1}{4\pi} \int_{\Gamma} \frac{i\,\mathrm{d}\boldsymbol{l} \times \boldsymbol{r}}{r^3}. \tag{3.170}$$

The vector $\boldsymbol{H}$ at the point P is perpendicular to the plane determined by the straight-line segment and the point P, and oriented so that it enters into the plane of the figure. We shall consider the modulus of the vector $\boldsymbol{H}$. According to the symbols of Fig. 3.15, we have:

$$\mathrm{d}\,H(P) = \mathrm{d}\,H = \frac{1}{4\pi} \cdot \frac{i(\mathrm{d}\,l)r\sin(\mathrm{d}\boldsymbol{l},\boldsymbol{r})}{r^3}, \tag{3.171}$$

but

$$\sin(\mathrm{d}\boldsymbol{l},\boldsymbol{r}) = \sin\left[\pi - \left(\frac{\pi}{2} - \theta\right)\right] = \cos\theta \tag{3.172}$$

and

$$l = a \cdot \tan\theta; \qquad \mathrm{d}\,l = \frac{a}{\cos^2\theta}\,\mathrm{d}\,\theta; \qquad r = \frac{a}{\cos\theta}. \tag{3.173}$$

It follows:

$$H = \frac{1}{4\pi}i \int_{-\theta_1}^{\theta_2} \frac{\dfrac{a}{\cos^2\theta}\cos\theta}{\dfrac{a^2}{\cos^2\theta}}\,\mathrm{d}\,\theta = \frac{1}{4\pi a}i(\sin\theta_1 + \sin\theta_2). \tag{3.174}$$

For an infinitely long conductor, i.e., $b \to \infty$, we have:

$$\theta_1 = \frac{\pi}{2}; \qquad \theta_2 = \frac{\pi}{2}. \tag{3.175}$$

Fig. 3.15. Explanation to the calculation of the magnetic field strength produced by a rectilinear thread-like conductor carrying an electric current.
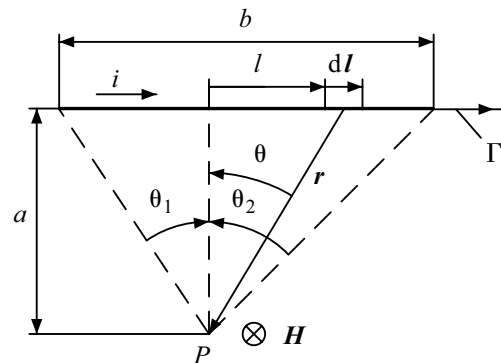
It results that:

$$H = \frac{i}{2\pi a}.$$ (3.176)

The same result can also be obtained directly, by applying the magnetic circuital law.

The direction of the vector magnetic field strength follows from the Biot-Savart-Laplace formula. From this expression, without performing computations, we verify that the vector H at the point P is perpendicular on the plane determined by the rectilinear conductor and the point P like in Fig. 3.16, and situated so that it lies in the plane of that figure. From geometrical reasons of symmetry, the lines of magnetic field are circles with their centre on the rectilinear thread-like conductor and perpendicular to it.

We shall apply the magnetic circuital law for a curve Γ that is just the magnetic field line passing through the point P as shown in Fig. 3.16. It follows:

$$\oint_\Gamma H \cdot \mathrm{d}l = \int_{S_\Gamma} J \cdot \mathrm{d}S.$$ (3.177)

The vectors H and $\mathrm{d}l$ are parallel (as shown in the figure), J (oriented like i in the figure) and $\mathrm{d}S$ are also parallel (both have the direction of the arrow that indicates in the figure the sense of the current i). The modulus of the magnetic field strength is constant along the same line of field. The current i passes only once through the surface S. Therefore:

$$H\,2\pi a = i,$$ (3.178)

whence:

$$H = \frac{i}{2\pi a}.$$ (3.179)
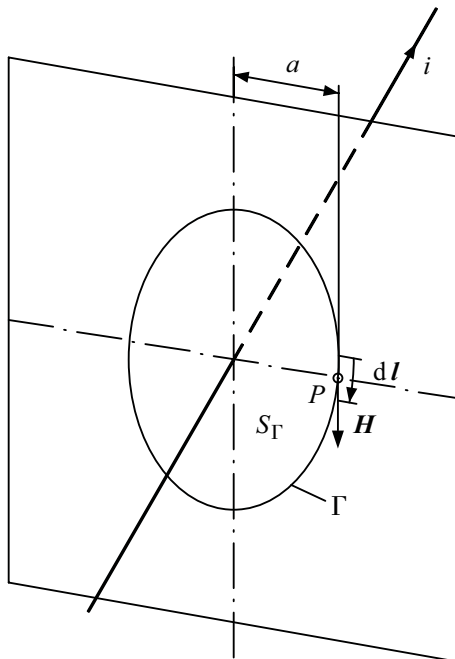
Fig. 3.16. Explanation to the calculation of the magnetic field strength produced by a rectilinear thread-like conductor carrying a constant electric current, by using the magnetic circuital law.

3.15.2. Expression of the Magnetic Field Strength Produced at a Point on the Axis of a Circular Turn Carrying a Constant Electric Current

Let us consider a circular turn constituted by a thread-like conductor, of constant cross-section, carrying a constant electric current. Let a be the radius of the turn and i the intensity of the electric current. For calculating the magnetic field strength produced by that current, relation (2.48) will be used.

The vector quantity $\mathrm{d}\boldsymbol{H}$ of Fig. 3.17 is perpendicular to the plane determined by the position vector $\boldsymbol{r}$ and the vector quantity $\mathrm{d}\boldsymbol{l}$, according to the Biot-Savart-Laplace formula (2.48).

By virtue of geometrical reasons, it results that the vector quantity $\boldsymbol{H}$ has the same direction as the turn axis, because the perpendicular components to this axis cancel each other. Also, according to Fig. 3.17, it follows:

$$\left|\mathrm{d}\boldsymbol{H}\right| = \left|\mathrm{d}\boldsymbol{H}_\theta\right|\sin\theta . \tag{3.180}$$

Formula (2.48), in this case, can be written in the form:

$$H = \frac{i}{4\pi}\oint_\Gamma \frac{\left|\mathrm{d}\boldsymbol{l}\times\boldsymbol{r}\right|}{r^3}\sin\theta , \tag{3.181}$$

$$\left|\mathrm{d}\boldsymbol{l}\times\boldsymbol{r}\right| = (\mathrm{d}l)r\sin\frac{\pi}{2} . \tag{3.182}$$

According to Fig. 3.17, it results that:

$$\mathrm{d}l = a\,\mathrm{d}\varphi , \tag{3.183}$$
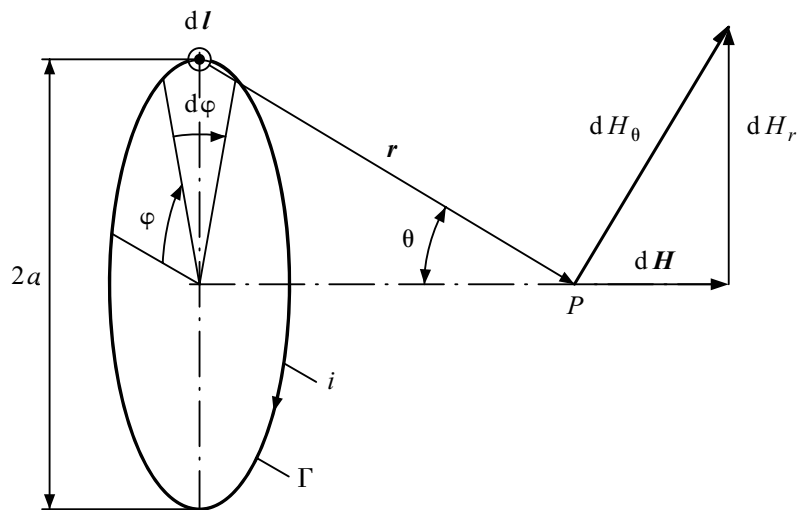
Fig. 3.17. Explanation to the calculation of the magnetic field strength at a point on the axis of a circular turn carrying a constant electric current.

$$r = \frac{a}{\sin \theta}. \tag{3.184}$$

After performing the substitutions and the calculation of the integral, it follows:

$$H = \frac{i}{4\pi} \int_0^{2\pi} \frac{a \, \mathrm{d}\varphi}{a^2} \sin^3 \theta = \frac{i}{2a} \sin^3 \theta. \tag{3.185}$$

In particular, the magnetic field strength at the centre of the turn is:

$$H = \frac{i}{2a} \sin^3 \frac{\pi}{2} = \frac{i}{2a}. \tag{3.186}$$

The magnetic field strength produced by a circular turn carrying an electric current, at a point that is not placed on the axis of the turn, can also be calculated by the relation (2.48). In this case, elliptic integrals occur in calculation [15, p. 30].

3.16.   APPLICATION OF BOTH FORMS OF THE LAW OF ELECTROMAGNETIC INDUCTION FOR MEDIA AT REST AND IN MOTION

3.16.1.   Calculation of the Electromotive Force Induced in a Coil in Rotational Motion in a Uniform Magnetic Field

A typical example of application of various forms of the law of electromagnetic induction is the calculation of the induced electromotive forces.

We shall calculate the electromotive force induced in a coil in rotational motion in a magnetic field having the magnetic induction uniformly distributed in space. Firstly, let the coil be constituted by a single turn.

Let

$$B = B_M \sin(\omega_1 t + \gamma_1), \tag{3.187}$$

be the expression of the magnetic induction that varies with time, in a sinusoidal form, with the angular frequency ω_1. The considered configuration, composed of the inductive pair of poles that produces the magnetic field and the turn in rotational motion, is shown in Fig. 3.18. The turn is in rotational motion, about one of its symmetry axes, with an angular velocity (speed) ω. The sizes R and l of the turn are shown in Fig. 3.18.

The electromotive force induced in the turn can be collected by a system of rings and brushes. We shall denote by Γ a curve taken along the conductor (wire) of which the turn is made.

The law of electromagnetic induction for media at rest will be utilized. The reference frame will be taken fixed to the turn in rotational motion (Fig. 3.18 a). In accordance with relation (3.118), we can assume that in the adopted moving reference frame, the magnetic

Fig. 3.18. Explanation to the calculation of the induced
electromotive force in a coil in rotational motion.

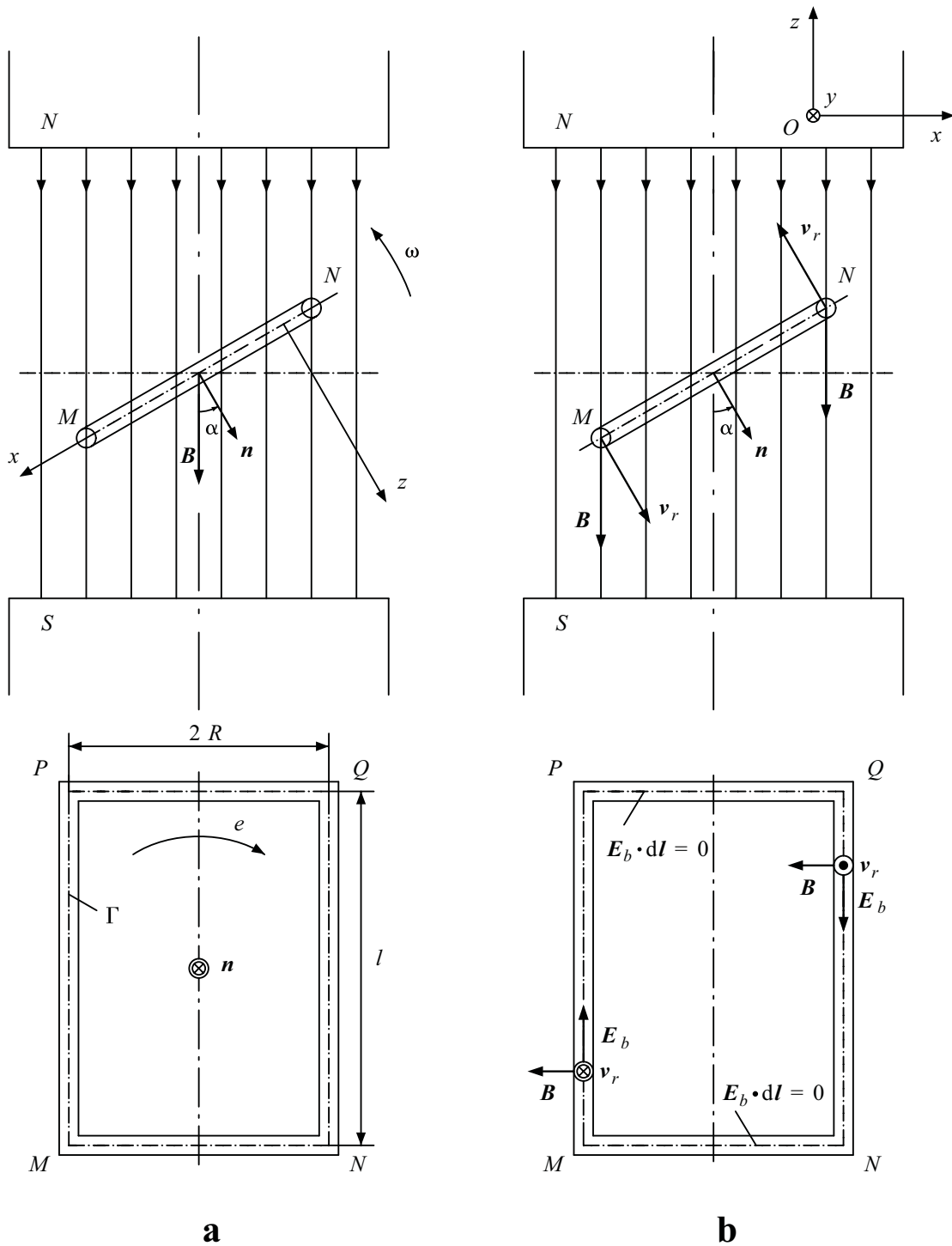induction has the same value as in the reference system considered at rest. We can arbitrarily adopt the travelling direction along the curve Γ, and the direction of the positive normal to the surface S_Γ will be associated according to the right-handed screw rule. The angle determining the initial position of the turn has been denoted by γ_0.

The magnetic flux linked by the turn is:

$$\varphi = 2\,R\,l\,B\cos\alpha, \tag{3.188}$$

$$\alpha = \omega t + \gamma_0, \tag{3.189 a}$$

$$\Psi = \varphi = \Phi_M \sin(\omega_1 t + \gamma_1)\cos(\omega t + \gamma_0), \tag{3.189 b}$$

$$\Phi_M = 2\,R\,l\,B_M. \tag{3.189 c}$$

By using relation (3.50), it follows:

$$
\begin{aligned}
e = u_e = \oint_\Gamma \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{l} = -\frac{\mathrm{d}\Psi}{\mathrm{d}t} &= -\omega_1\,\Phi_M\cos(\omega_1 t + \gamma_1)\cos(\omega t + \gamma_0) + \\
&\quad + \omega\,\Phi_M\sin(\omega_1 t + \gamma_1)\sin(\omega t + \gamma_0).
\end{aligned}
\tag{3.190}
$$

Further on, we shall use the law of electromagnetic induction for moving media. The reference frame will be considered fixed to the inductive poles (Fig. 3.18 b).

By applying relation (3.134) to this case, it follows:

$$
\begin{aligned}
e = u_e = \oint_\Gamma \boldsymbol{E}'\cdot\mathrm{d}\boldsymbol{l} &= -\int_{S_\Gamma}\frac{\partial\boldsymbol{B}}{\partial t}\cdot\mathrm{d}\boldsymbol{S} + \oint_\Gamma (\boldsymbol{v}_r\times\boldsymbol{B})\cdot\mathrm{d}\boldsymbol{l} = \\
&= -\int_{S_\Gamma}\omega_1\,B_M\cos(\omega_1 t + \gamma_1)\cos(\omega t + \gamma_0)\,\mathrm{d}S + \\
&\quad + \int_\Gamma v_r\,B_M\sin(\omega_1 t + \gamma_1)\sin(\omega t + \gamma_0)\,\mathrm{d}l
\end{aligned}
\tag{3.191}
$$

or

$$
\begin{aligned}
e = u_e &= -\omega_1\,B_M\,S\cos(\omega_1 t + \gamma_1)\cos(\omega t + \gamma_0) + \\
&\quad + 2\,v_r\,B_M\,l\sin(\omega_1 t + \gamma_1)\sin(\omega t + \gamma_0),
\end{aligned}
\tag{3.192}
$$

where

$$S = 2\,R\,l; \qquad v_r = \omega R. \tag{3.192 a, b}$$

If the coil of Fig. 3.18 had w turns instead of one turn, then all the induced electromotive forces would be w times greater.

From relation (3.134), it follows that the induced electromotive force (e.m.f.) has two components: The first is a transformation component and the second is a moving component (in this case a rotation component). In Fig. 3.18 b, the moving component of the electric field strength has been denoted by $\boldsymbol{E}_b = \boldsymbol{v}_r\times\boldsymbol{B}$. Also, from Fig. 3.18 b, it

follows that, in the front parts of the coil, the vector of the moving component of the electric field strength E_b is perpendicular to the conductor, hence:

$$E_b \cdot \mathrm{d}\boldsymbol{l} = (v_r \times B) \cdot \mathrm{d}\boldsymbol{l} = 0,\qquad\qquad (3.192\ \mathrm{c})$$

so that it does not contribute to the electromotive force.

3.16.2. Calculation of the Electromotive Force Induced by the Rotation of a Magnet about Its Axis

Another typical example of application of the law of electromagnetic induction is the calculation of the electromotive force induced between two points of one cylinder of a material with magnetic and conducting properties (permanent magnet), and which turns about its axis.
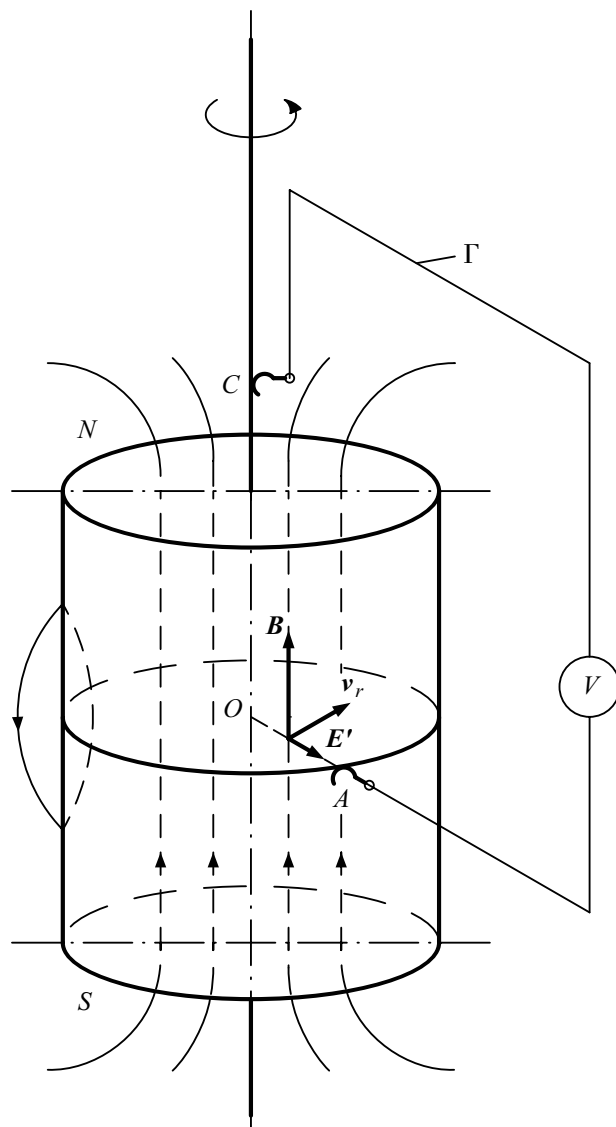
Fig. 3.19. Explanation to the calculation of the electromotive force induced by the rotation of one permanent magnet in the form of a cylinder.

The corresponding configuration is shown in Fig. 3.19. As it can be seen in Fig. 3.19, the two points C and A are situated one on the axis of the cylinder, the other on one of its generating straight lines. The electromotive force induced between the two points can be collected by a system of sliding contacts.

Let Ω be the angular velocity of the rotational motion, and r the cylinder radius. We shall consider a reference frame at rest relatively to which the cylinder is in rotational motion and a moving reference frame fixed to the cylinder. At any instant, we consider a closed curve composed of a portion CO along the cylinder axis, a radius OA and a line that closes along a conductor placed in air between the two points A and C.

At any point of the reference frame at rest, the magnetic induction does not vary. In this case, according to formula (3.134), it follows:

$$e = u_e = \oint_\Gamma \boldsymbol{E}' \, \mathrm{d}\boldsymbol{l} = -\int_{S_\Gamma} \frac{\partial \boldsymbol{B}}{\partial t} \, \mathrm{d}\boldsymbol{S} + \oint_\Gamma \left(\boldsymbol{v}_r \times \boldsymbol{B}\right) \cdot \mathrm{d}\boldsymbol{l} = \oint_\Gamma \left(\boldsymbol{v}_r \times \boldsymbol{B}\right) \cdot \mathrm{d}\boldsymbol{l}, \tag{3.193}$$

where $v_r = \Omega r$. If the magnetic field is assumed to be uniform, it follows:

$$e = u_e = \frac{1}{2}\Omega r^2 \, B = v_{r \, \text{average}} \, r \, B. \tag{3.194}$$

After the discovery of the law of electromagnetic induction (examined in Sub-section 3.5.4 and Section 3.11), many discussions took place, especially in the 19$^{\text{th}}$ century, relatively to the possibility of interpreting the phenomenon of electromagnetic induction, in the case of a right circular cylindrical permanent magnet which turns about its axis as being determined by the fact that the lines of magnetic field turn together with the magnet. According to the mention of Section 2.5, that the lines of field are only an auxiliary notion and not material forms, such an interpretation is not justified.

3.17.   ELECTRODYNAMIC POTENTIALS

The general equations of the electromagnetic field in local form, established above, are:

$$\operatorname{curl} \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \tag{3.195 a}$$

$$\operatorname{div} \boldsymbol{D} = \rho_v, \tag{3.195 b}$$

$$\boldsymbol{D} = \varepsilon_0 \, \boldsymbol{E} + \boldsymbol{P}, \tag{3.195 c}$$

$$\operatorname{curl} \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}, \tag{3.195 d}$$

$$\operatorname{div} \boldsymbol{B} = 0, \tag{3.195 e}$$

$$\boldsymbol{B} = \mu_0 \, \boldsymbol{H} + \boldsymbol{M}_j. \tag{3.195 f}$$

The equations above are called the *equations of the theory of electromagnetic field for media at rest* or the *Maxwell equations*.

In many cases, for solving this system of equations, it is convenient to introduce certain substitution functions namely: Electrodynamic scalar potential and vector potential, respectively. This introduction is not arbitrary, it results from the examination of the equations above. Concerning the fifth equation, it can be remarked that the divergence of any vector that is a curl of a field vector is always zero. Hence:

$$\operatorname{div} \boldsymbol{B} = \operatorname{div}\left(\operatorname{curl} \boldsymbol{A}\right) = 0 \tag{3.196}$$

and

$$\boldsymbol{B} = \operatorname{curl} \boldsymbol{A}. \tag{3.197}$$

The quantity $\boldsymbol{A}$ is called *vector electrodynamic potential*.

From the equation (3.195 a), it follows:

$$\operatorname{curl} \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = \operatorname{curl} \boldsymbol{E} + \frac{\partial}{\partial t} \operatorname{curl} \boldsymbol{A} = \operatorname{curl}\left(\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t}\right) = 0. \tag{3.198}$$

In the last equation, the derivative operator with respect to t, and the operator curl that contains the derivatives with respect to co-ordinates can be replaced one by the other, because the two operators refer to different variables.

Concerning the last equation, it can be remarked that the operator curl of any vector that is a gradient of a scalar field is always zero. Hence:

$$\operatorname{curl}\left(\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t}\right) = \operatorname{curl}\left(-\operatorname{grad} V\right) = 0. \tag{3.199 a}$$

It follows:

$$\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t} = -\operatorname{grad} V \tag{3.199 b}$$

and

$$\boldsymbol{E} = -\operatorname{grad} V - \frac{\partial \boldsymbol{A}}{\partial t}. \tag{3.200}$$

The quantity V of the last relation is called *scalar electrodynamic potential*.

It is worth noting that in relation (3.199 a) we have introduced the sign minus like in the case of potential produced by point-like charges at rest. Indeed, if the state quantities do not vary with time, the relations (3.200) and (1.119) must coincide.

3.18.   THE SCALAR AND VECTOR ELECTRODYNAMIC POTENTIALS PRODUCED BY ONE POINT-LIKE ELECTRIC CHARGE MOVING AT CONSTANT VELOCITY

Let K_o be a reference frame assumed at rest, K_1 a moving reference frame with respect to the previous one and a material point, with a point-like electric charge q_1, fixed in the reference frame K_1 at its origin. The velocity of the reference frame K_1 and that of the point-like electric charge q_1, fixed in the same reference frame, is assumed to be constant $\boldsymbol{v}_{1o} = \boldsymbol{i}\, v_{1ox} = \text{const.}$ The medium will be considered empty space (vacuum).

The electric field strength produced by the point-like charge q_1 at any point $P(x_{1P}, y_{1P}, z_{1P})$ in the reference frame K_1 is:

$$\boldsymbol{E}_{1P} = \frac{1}{4\,\pi\,\varepsilon_0} \cdot \frac{q_1}{r_{1P}^3}\, \boldsymbol{r}_{1P}\,; \tag{3.201}$$

$$E_{1Px} = \frac{1}{4\,\pi\,\varepsilon_0} \cdot \frac{q_1\, x_{1P}}{\left(x_{1P}^2 + y_{1P}^2 + z_{1P}^2\right)^{\frac{3}{2}}}\,, \tag{3.201 a}$$

$$x_{1P} = \alpha_{1o}\left(x_{oP} - v_{1o}\, t_o\right);$$

$$E_{1Py} = \frac{1}{4\,\pi\,\varepsilon_0} \cdot \frac{q_1\, y_{1P}}{\left(x_{1P}^2 + y_{1P}^2 + z_{1P}^2\right)^{\frac{3}{2}}}\,, \tag{3.201 b}$$

$$y_{1P} = y_{oP}\,;$$

$$E_{1Pz} = \frac{1}{4\,\pi\,\varepsilon_0} \cdot \frac{q_1\, z_{1P}}{\left(x_{1P}^2 + y_{1P}^2 + z_{1P}^2\right)^{\frac{3}{2}}}\,, \tag{3.201 c}$$

$$z_{1P} = z_{oP}\,.$$

The electric field strength produced by the same point-like charge q_1 at the same point $P(x_{oP}, y_{oP}, z_{oP})$ in the reference frame K_o, according to relation (2.21), is:

$$\boldsymbol{E}_o = \boldsymbol{i}\, E_{1x} + \boldsymbol{j}\,\alpha\, E_{1y} + \boldsymbol{k}\,\alpha\, E_{1z}\,. \tag{3.202}$$

According to relation (2.26), the magnetic induction will be:

$$B_{ox} = 0\,, \tag{3.203 a}$$

$$B_{oy} = -\frac{1}{c^2} \cdot \frac{1}{4\pi} \cdot \frac{\alpha_{1o} \, q_1 \, z_{1P} \, v_{1ox}}{\left(x_{1P}^2 + y_{1P}^2 + z_{1P}^2\right)^{\frac{3}{2}}} \cdot \frac{1}{\varepsilon_0}, \tag{3.203 b}$$

$$B_{oz} = \frac{1}{c^2} \cdot \frac{1}{4\pi} \cdot \frac{\alpha_{1o} \, q_1 \, y_{1P} \, v_{1ox}}{\left(x_{1P}^2 + y_{1P}^2 + z_{1P}^2\right)^{\frac{3}{2}}} \cdot \frac{1}{\varepsilon_0}. \tag{3.203 c}$$

According to expressions (3.203 a, b, c), it results that, in this case, the quantity $\boldsymbol{B}_o$ has components only along the $O_o y_o$ and $O_o z_o$ axes. According to relation (3.203 a) and (3.197), we can write that the vector potential $\boldsymbol{A}$ has a single component, namely along the $O_o x_o$ - axis. We obtain successively:

$$\boldsymbol{B}_o = \mathrm{curl}_o \, \boldsymbol{A} = \mathrm{curl}_o \left(\boldsymbol{i} \, A_o\right); \tag{3.204 a}$$

$$B_{ox} = 0; \quad B_{oy} = \frac{\partial A_o}{\partial z_o}; \quad B_{oz} = -\frac{\partial A_o}{\partial y_o}; \tag{3.204 b}$$

$$A_{ox} = \frac{1}{4\pi} \cdot \frac{1}{c^2} \cdot \frac{\alpha_{1o} \, q \, v_{1ox}}{r_{1P}} \cdot \frac{1}{\varepsilon_0}. \tag{3.204 c}$$

At the initial moment, both reference frames are supposed to coincide. The point-like charge q_1 is fixed at the origin of the reference frame K_1. The observation point denoted by P is fixed in the reference frame K_o. After a time t_o, the origin of the reference frame K_1 will be on the $O_o x_o$ - axis at the distance $v_{1ox} t_o$ from the origin of the reference frame K_o. The distance between the point-like charge fixed at the origin of the reference frame K_1 and the point P marked in the same reference frame will also be modified.

Let r_o be the distance between the point-like charge at the initial moment and the point P marked in the reference frame K_o. Also, let t_o be equal to the time necessary for the field wave starting from the material point with a point-like charge that initially is at the origin, to reach the point P.

Hence, we can write:

$$t_o = \frac{r_o}{c}; \; r_o^2 = x_{oP}^2 + y_{oP}^2 + z_{oP}^2. \tag{3.205}$$

At the same time, we can write the relations:

$$x_{1P} = \alpha_{1o} \left(x_{oP} - v_{1ox} t_o\right), \tag{3.206 a}$$

$$r_{1P} = \left[\alpha_{1o}^2 \left(x_{oP} - v_{1ox} t_o\right)^2 + y_{oP}^2 + z_{oP}^2\right]^{\frac{1}{2}}, \tag{3.206 b}$$

$$r_{1P} = \alpha_{1o} \left[\left(x_{oP} - v_{1ox} t_o \right)^2 + \frac{1}{\alpha_{1o}^2} \left(y_{oP}^2 + z_{oP}^2 \right) \right]^{\frac{1}{2}} =$$

$$= \alpha_{1o} \left[x_{oP}^2 - 2 v_{1ox} x_{oP} t_o + v_{1ox}^2 t_o^2 + \left(1 - \beta^2 \right) \left(y_{oP}^2 + z_{oP}^2 \right) \right]^{\frac{1}{2}} =$$

$$= \alpha_{1o} \left[r_o^2 - 2 v_{1ox} x_{oP} \frac{r_o}{c} + \frac{v_{1ox}^2}{c^2} r_o^2 - \frac{v_{1ox}^2}{c^2} \left(y_{oP}^2 + z_{oP}^2 \right) \right]^{\frac{1}{2}} =$$

$$= \alpha_{1o} \left[r_o^2 - 2 v_{1ox} x_{oP} \frac{r_o}{c} + \frac{v_{1ox}^2}{c^2} x_{oP}^2 \right]^{\frac{1}{2}}.$$

(3.206 c)

Hence:

$$r_{1P} = \alpha_{1o} \left[r_o^2 - 2 r_o \frac{\mathbf{v}_{1o} \cdot \mathbf{r}_o}{c} + \left(\frac{\mathbf{v}_{1o} \cdot \mathbf{r}_o}{c} \right)^2 \right]^{\frac{1}{2}} = \alpha_{1o} \left(r_o - \frac{\mathbf{v}_{1o} \cdot \mathbf{r}_o}{c} \right).$$

(3.207)

Therefore, relation (3.204 c) becomes:

$$A_o = \frac{1}{4\pi} \cdot \frac{1}{c^2} \cdot \frac{q_1 \mathbf{v}_{1o}}{r_o - \dfrac{\mathbf{v}_{1o} \cdot \mathbf{r}_o}{c}} \cdot \frac{1}{\varepsilon_0}.$$

(3.208)

From the expression (3.200) of the electric field strength in terms of the scalar and vector potential, we obtain:

$$E_{ox} = -\left(\text{grad}_o V_o \right)_x - \left(\frac{\partial A_o}{\partial t_o} \right)_x;$$

(3.209 a)

$$E_{oy} = -\left(\text{grad}_o V_o \right)_y; \qquad \left(\frac{\partial A_o}{\partial t_o} \right)_y = 0;$$

(3.209 b)

$$E_{oz} = -\left(\text{grad}_o V_o \right)_z; \qquad \left(\frac{\partial A_o}{\partial t_o} \right)_z = 0.$$

(3.209 c)

The expression of V_o can be obtained easily from relations (3. 209 b, c):

$$-\left(\text{grad}_o V_o \right)_y = E_{oy} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{\alpha_{1o} q_1 y_{1P}}{r_{1P}^3} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{\alpha_{1o} q_1 y_{oP}}{r_{1P}^3}.$$

(3.210 a)

It follows:

$$-\left(\text{grad}_o V_o \right)_z = E_{oz} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{\alpha_{1o} q_1 z_{1P}}{r_{1P}^3} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{\alpha_{1o} q_1 z_{oP}}{r_{1P}^3}.$$

(3.210 b)

The relations (3.210 a, b) yield:

$$V_\mathrm{o} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_1}{\dfrac{1}{\alpha_{1\mathrm{o}}} r_{1P}}.$$

(3.211)

It follows:

$$V_\mathrm{o} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_1}{r_\mathrm{o} - \dfrac{\boldsymbol{v}_{1\mathrm{o}} \cdot \boldsymbol{r}_\mathrm{o}}{c}}.$$

(3.212)

If we omit the index referring to the reference frame, we can write:

$$V = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_1}{r_\mathrm{o} - \dfrac{\boldsymbol{v}_{1\mathrm{o}} \cdot \boldsymbol{r}_\mathrm{o}}{c}},$$

(3.213)

$$\boldsymbol{A} = \frac{1}{4\pi} \cdot \frac{1}{c^2} \cdot \frac{q_1 \boldsymbol{v}_{1\mathrm{o}}}{r_\mathrm{o} - \dfrac{\boldsymbol{v}_{1\mathrm{o}} \cdot \boldsymbol{r}_\mathrm{o}}{c}} \cdot \frac{1}{\varepsilon_0}.$$

(3.214)

The last two relations represent the expressions of the scalar and vector electrodynamic potential, respectively, at any observation point, also called field point, in empty space (vacuum).

The field wave starting from the point-like charge is propagated with the velocity c. Let the point-like charge be at any point M at the moment of observation. The vector $\boldsymbol{r}_\mathrm{o}$ in formula above has its end at the observation point P, but its origin is not at the point M at which the point-like charge is at the same moment but at the point at which the point-like charge was at a previous moment, say N. In the calculation above, this previous point N was chosen at the origin. The distance from the point N to the point P is equal to the distance covered by the field wave during the time necessary to the point-like charge for moving from the point N to the point M. Therefore the perturbation at point N manifests at any point P not at the same moment, but later, and for this reason, the potentials obtained above are referred to as *retarded potentials*.

The scalar and vector potential above, produced by a moving point-like charge are also referred to as LIÉNARD-WIECHERT potentials.

3.19.   THE SCALAR AND VECTOR ELECTRODYNAMIC POTENTIALS PRODUCED BY ONE POINT-LIKE ELECTRIC CHARGE MOVING AT NON-CONSTANT VELOCITY

Let us consider a set of two material points, with the electric charges q and q_1, namely point-like electric charges, in the same inertial reference frame, say K_o. We suppose that the charge q has the velocity $\boldsymbol{v}_{2\mathrm{o}} = \boldsymbol{v}_r$ and the charge q_1 has the velocity $\boldsymbol{v}_{1\mathrm{o}} = \boldsymbol{i}\, v_{1\mathrm{o}} = \boldsymbol{v}_1$. Firstly, we shall suppose that the point-like charge that produces the

electromagnetic field is moving at a constant velocity v_1. The equation of motion of a material point of inertial mass m, with the electric charge q, in the electromagnetic field, giving the expression of the force acting upon the material point, established for the case in which $v_1 = \text{const}$, is:

$$F = \frac{\mathrm{d}}{\mathrm{d}t}(m v_r) = q(E + v_r \times B)$$
(3.215 a)

or

$$F = \frac{\mathrm{d}}{\mathrm{d}t}(m v_r) = q\left[\left(-\operatorname{grad}V - \frac{\partial A}{\partial t}\right) + v_r \times \operatorname{curl}A\right],$$
(3.215 b)

where V and A are the retarded electrodynamic scalar and vector potentials, also called Liénard-Wiechert potentials produced by one moving point-like electric charge that we could term source electric charge. As shown above by relations (3.213) and (3.214), the expressions of these potentials at an observation (field) point, in the case in which the source charge is moving with constant velocity, can be brought to the form:

$$V = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_1}{r - \dfrac{v_1 \cdot r}{c}}; \qquad A = \frac{\mu_0}{4\pi} \cdot \frac{q_1 v_1}{r - \dfrac{v_1 \cdot r}{c}},$$
(3.216 a, b)

where r is the vector having its origin (source point) at the point N, as explained after relation (3.214), and its end at the observation point (field point) P, at any moment t. Therefore, the quantities V and A of the left-hand side are given at any moment t by the point-like electric charge q_1 considered at the position it had at the moment $t - \dfrac{r}{c}$.

The forces acting upon a moving point-like charge q can be obtained from the Lagrange equation, where the generalized co-ordinates s_k are used:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{s}_k}\right) - \frac{\partial L}{\partial s_k} = 0.$$
(3.217)

For a system of n material points that contains k connection conditions, there are $f = 3n - k$ equations. Let the Lagrange function be adopted in the form:

$$L = T - U = T - qV + q(v_r \cdot A),$$
(3.218)

with

$$T = -m_0 c^2 \sqrt{1 - \frac{v_r^2}{c^2}}.$$
(3.219)

In the case above, the Lagrange function of the system of material points with electric charge (3.218) is equal to the difference of two terms. The first one is the Lagrange

function T of the material point with the electric charge q when the electromagnetic field is zero. The second one is the supplementary Lagrange function U of the same material point with the electric charge q due to its interaction with other electric charges.

The quantity T represents, in the case of small velocities, the *kinetic energy*, apart from a constant, in other words, within to a constant. This remark can be obtained as a result of the expansion in McLaurin series of the root square. The quantity U represents the *potential energy*. These two terms do not depend on acceleration.

It can be added that, instead of the previous expression of the quantity T, also another expression could be used, for example:

$$T = m_0\, c^2 \left(1 - \sqrt{1 - \frac{v_r^2}{c^2}} \right), \tag{3.219 a}$$

however, the further results will not be affected.

Now, we shall consider the case in which the velocity $\boldsymbol{v}_1$ of the point-like charge q_1 that produces the electromagnetic field is not constant with time. The Lagrange function is assumed to not depend on acceleration. As a consequence, in the case in which the electric charge q_1 that produces the electromagnetic field has a variable velocity $\boldsymbol{v}_1$, the expression of the term U of relation (3.218) remains the same as in the case of a constant velocity above. Therefore, the expressions of the quantities V and $\boldsymbol{A}$ (3.216 a, b) remain valid in this case.

It is possible to show that for given values of the quantities V and $\boldsymbol{A}$, the force acting upon a moving charge in electromagnetic field depends only on the derivatives of these quantities, regardless of the form of variation of $\boldsymbol{v}_1$.

The derivation of this force can be achieved by adopting, under a certain modification, the derivation of the work [50, Vol. I, Ch. V, Par. 60]. The modification is constituted by the utilization of another expression of the term T and by the consideration that the inertial mass of the point-like charge q is depending on its velocity.

We shall consider only the system of co-ordinates of the reference frame K_o. Therefore, no index will be used for indicating the reference frame. The Lagrange equation, in the known form is:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}_k} \right) - \frac{\partial L}{\partial x_k} = 0, \quad \left(k = 1, 2, 3 \right);$$

$$x_1 = x; \quad x_2 = y; \quad x_3 = z, \tag{3.220}$$

where the over-script dot denotes the derivative.

We shall denote by p_{mk} the component of the momentum $\boldsymbol{p}_m$ corresponding to the co-ordinate x_k. The components of the momentum are given by the known relation:

$$p_{mk} = \frac{\partial L}{\partial \dot{x}_k}, \quad \left(k = 1, 2, 3 \right);$$

$$x_1 = x; \quad x_2 = y; \quad x_3 = z. \tag{3.221}$$

From relations (3.218), (3.219) and (3.221), we obtain the following partial derivative:

$$p_{mk} = \frac{\partial L}{\partial \dot{x}_k} = m_0 c^2 \frac{\dot{x}_k}{\left(1 - \frac{v_r^2}{c^2}\right)^{\frac{1}{2}}} \cdot \frac{1}{c^2} + q A_k = m \dot{x}_k + q A_k ; \quad (k = 1, 2, 3);$$

$$x_1 = x; \quad x_2 = y; \quad x_3 = z;$$

$$\boldsymbol{v}_r = \boldsymbol{i} v_{rx} + \boldsymbol{j} v_{ry} + \boldsymbol{k} v_{rz} = \boldsymbol{i} \dot{x}_1 + \boldsymbol{j} \dot{x}_2 + \boldsymbol{k} \dot{x}_3 = \boldsymbol{i} \dot{x} + \boldsymbol{j} \dot{y} + \boldsymbol{k} \dot{z}.$$

$$(3.222)$$

The derivative of the momentum is:

$$D_k = \frac{\mathrm{d} p_{mk}}{\mathrm{d} t} = \frac{\mathrm{d}}{\mathrm{d} t}(m \dot{x}_k) + q \frac{\mathrm{d} A_k}{\mathrm{d} t} =$$

$$= \frac{\mathrm{d}}{\mathrm{d} t}(m \dot{x}_k) + q \left(\frac{\partial A_k}{\partial t} + \frac{\partial A_k}{\partial x_1} \cdot \frac{\mathrm{d} x_1}{\mathrm{d} t} + \frac{\partial A_k}{\partial x_2} \cdot \frac{\mathrm{d} x_2}{\mathrm{d} t} + \frac{\partial A_k}{\partial x_3} \cdot \frac{\mathrm{d} x_3}{\mathrm{d} t} \right), \quad (3.223)$$

$$(k = 1, 2, 3); \quad x_1 = x; \quad x_2 = y; \quad x_3 = z.$$

In the case in which $k = 1$ then, $x_1 = x$, $A_1 = A_x$ and we obtain:

$$D_x = \frac{\mathrm{d} p_{mx}}{\mathrm{d} t} = \frac{\mathrm{d}}{\mathrm{d} t}(m \dot{x}) + q \frac{\mathrm{d} A_x}{\mathrm{d} t} =$$

$$= \frac{\mathrm{d}}{\mathrm{d} t}(m \dot{x}) + q \left(\frac{\partial A_x}{\partial t} + \dot{x} \frac{\partial A_x}{\partial x} + \dot{y} \frac{\partial A_x}{\partial y} + \dot{z} \frac{\partial A_x}{\partial z} \right). \quad (3.224)$$

From the Lagrange equations (3.220) and (3.221), we obtain:

$$\frac{\mathrm{d} p_{mk}}{\mathrm{d} t} = \frac{\partial L}{\partial x_k}; \quad (k = 1, 2, 3). \quad (3.225)$$

We shall assume that the term T does not depend on the quantities of the form x_k but only on $\dot{x}_k$. In the case in which $k = 1$ then, as above, $x_1 = x$, $A_1 = A_x$ and we get:

$$D_x = \frac{\mathrm{d} p_{mx}}{\mathrm{d} t} = \frac{\partial L}{\partial x} = -q \frac{\partial V}{\partial x} + q \left(\dot{x} \frac{\partial A_x}{\partial x} + \dot{y} \frac{\partial A_y}{\partial x} + \dot{z} \frac{\partial A_z}{\partial x} \right). \quad (3.226)$$

By equating the right-hand sides of relations (3.224) and (3.226), we obtain:

$$\frac{\mathrm{d}}{\mathrm{d} t}(m \dot{x}) + q \left(\frac{\partial A_x}{\partial t} + \dot{x} \frac{\partial A_x}{\partial x} + \dot{y} \frac{\partial A_x}{\partial y} + \dot{z} \frac{\partial A_x}{\partial z} \right) =$$

$$= -q \frac{\partial V}{\partial x} + q \left(\dot{x} \frac{\partial A_x}{\partial x} + \dot{y} \frac{\partial A_y}{\partial x} + \dot{z} \frac{\partial A_z}{\partial x} \right). \quad (3.227)$$

The force is:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m\dot{x}\right) = q\left(-\frac{\partial V}{\partial x} - \frac{\partial A_x}{\partial t}\right) + q\left[\dot{y}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - \dot{z}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\right]. \tag{3.228}$$

By adding up, side by side, the relations for the three axes, it follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(m\,\boldsymbol{v}_r\right) = q\left(-\operatorname{grad}V - \frac{\partial \boldsymbol{A}}{\partial t}\right) + q\left[\boldsymbol{v}_r \times \operatorname{curl}\boldsymbol{A}\right]. \tag{3.229}$$

Therefore, in accordance with relations (3.200) and (3.197), we have the expressions:

$$\boldsymbol{E} = -\operatorname{grad}V - \frac{\partial \boldsymbol{A}}{\partial t}; \qquad \boldsymbol{B} = \operatorname{curl}\boldsymbol{A}; \tag{3.230 a, b}$$

where V and $\boldsymbol{A}$ are the expressions of retarded electrodynamic potentials, given by relations (3.213) and (3.214) which have been considered to be valid for any variation with time of the velocity $\boldsymbol{v}_1$. The expressions of the quantities $\boldsymbol{E}$ and $\boldsymbol{B}$ given by the last two expressions (3.230 a, b) and (3.228) yield relation (2.30) which represents the expression of the force acting upon a point-like electric charge in electromagnetic field. Therefore, there are reasons to assume that the laws established for a constant velocity $\boldsymbol{v}_1$ may also be considered valid for a varying velocity.

4. THE ENERGY OF THE ELECTROMAGNETIC FIELD

We recall that, the *energy of a physical system* in any state, with respect to one *reference state*, arbitrarily chosen, is the sum of the equivalent work of all actions exerted upon the other systems, when the examined physical system passes, in any manner, from the state under consideration to the reference state.

Also we are recalling that the *equivalent work* of any action is the work necessary to produce the respective action, or the work that would be obtained by suppressing the respective action.

The energy satisfies a conservation law as follows. The same value of the energy of a physical system in the actual state (present state), with respect to the reference state, will be obtained, regardless of the sequence of transformations undergone by the physical system when passing from the reference state to the actual state (final state). Therefore, the energy of a physical system can be expressed only in terms of the state quantities that determine the actual state (final state) and the reference state.

4.1. THE EXPRESSION OF THE ENERGY OF THE ELECTROMAGNETIC FIELD. POYNTING VECTOR.

We shall establish the expressions of the energy of the electromagnetic field in terms of macroscopic quantities.

From a macroscopic point of view, let us consider the electric charge having a volume distribution with the volume density ρ_v. The set of particles of the volume element dv is charged with the quantity of electricity $dq = \rho_v \, dv$. We consider that the centre of mass of the set of particles electrically charged of the volume element is moving at the velocity v_r in the electromagnetic field of a domain Σ of Fig. 4.1. The macroscopic state quantities of the field will be denoted by E and B. The force acting upon the volume element containing the electric charge dq is given by relation (2.30):

$$d\boldsymbol{F} = (d q)(\boldsymbol{E}_l + \boldsymbol{v}_r \times \boldsymbol{B}), \tag{4.1}$$

where

$$\boldsymbol{E}_l = \boldsymbol{E} + \boldsymbol{E}_i, \tag{4.1 a}$$

$$d q = \rho_v \, dv, \tag{4.1 b}$$

and

$$d\boldsymbol{F} = (\boldsymbol{E}_l + \boldsymbol{v}_r \times \boldsymbol{B})(\rho_v \, dv). \tag{4.2}$$
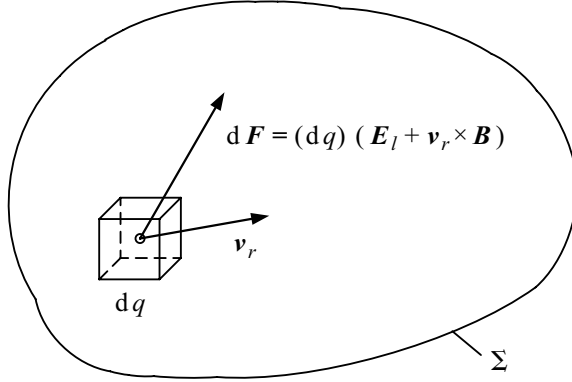
Fig. 4.1. Explanation concerning the
derivation of the expression
of the energy of electromagnetic field.

The solenoidal component of the electric field strength at a point and the magnetic induction, are produced by the electrically charged particles in motion relatively to the respective reference frame.

The work done by the field forces, in the case of the displacement of the system of particles with the charge $\mathrm{d}q$, along the line element $\mathrm{d}\boldsymbol{l}$, in the time interval $\mathrm{d}t$, is:

$$\mathrm{d}^2 W = \mathrm{d}\boldsymbol{F} \cdot \mathrm{d}\boldsymbol{l} = \mathrm{d}\boldsymbol{F} \cdot \boldsymbol{v}_r \, \mathrm{d}t. \tag{4.3}$$

From relations (4.3) and (4.2), if follows:

$$\mathrm{d}^2 W = \rho_v \, \boldsymbol{v}_r \cdot \left(\boldsymbol{E}_l + \boldsymbol{v}_r \times \boldsymbol{B}\right)\mathrm{d}v\,\mathrm{d}t \tag{4.4}$$

or, after performing the calculations:

$$\mathrm{d}^2 W = \rho_v \, \boldsymbol{v}_r \cdot \boldsymbol{E}_l \, \mathrm{d}v\,\mathrm{d}t, \tag{4.5}$$

because the product of vectors $\boldsymbol{v}_r \times \boldsymbol{B}$ and $\boldsymbol{v}_r$ is zero, the vectors being perpendicular to each other, thus producing no work.

Taking into account relation (4.5), (1.18 c) and (1.70), it follows:

$$\mathrm{d}^2 W = \boldsymbol{J} \cdot \boldsymbol{E}_l \, \mathrm{d}v\,\mathrm{d}t = \boldsymbol{J} \cdot \left(\boldsymbol{E} + \boldsymbol{E}_i\right)\mathrm{d}v\,\mathrm{d}t. \tag{4.6}$$

The work done for the whole volume is:

$$\mathrm{d}W = \int_{V_\Sigma}\left(\boldsymbol{J} \cdot \boldsymbol{E}_l\right)\mathrm{d}t\,\mathrm{d}v = \int_{V_\Sigma}\left(\boldsymbol{J} \cdot \boldsymbol{E}\right)\mathrm{d}t\,\mathrm{d}v + \int_{V_\Sigma}\left(\boldsymbol{J} \cdot \boldsymbol{E}_i\right)\mathrm{d}t\,\mathrm{d}v. \tag{4.7}$$

Relation (3.86) yields:

$$\boldsymbol{J} = \operatorname{curl}\boldsymbol{H} - \frac{\partial \boldsymbol{D}}{\partial t}. \tag{4.8}$$

From relations (4.7) and (4.8), it results that:

$$\mathrm{d}\,W = \int\limits_{V_\Sigma} \left(\boldsymbol{E} \cdot \operatorname{curl} \boldsymbol{H} - \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{J} \cdot \boldsymbol{E}_i \right) \mathrm{d}\,t\,\mathrm{d}\,v. \tag{4.9}$$

Taking into account the vector relation:

$$\boldsymbol{E} \cdot \operatorname{curl} \boldsymbol{H} = -\operatorname{div}(\boldsymbol{E} \times \boldsymbol{H}) + \boldsymbol{H} \cdot \operatorname{curl} \boldsymbol{E},$$

it follows:

$$\mathrm{d}\,W = -\int\limits_{V_\Sigma} \operatorname{div}(\boldsymbol{E} \times \boldsymbol{H}) \mathrm{d}\,t\,\mathrm{d}\,v + \int\limits_{V_\Sigma} \left(\boldsymbol{H} \cdot \operatorname{curl} \boldsymbol{E} - \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{J} \cdot \boldsymbol{E}_i \right) \mathrm{d}\,t\,\mathrm{d}\,v. \tag{4.10}$$

Taking into account relation (3.53):

$$\operatorname{curl} \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t},$$

we have:

$$\mathrm{d}\,W = -\int\limits_{V_\Sigma} \operatorname{div}(\boldsymbol{E} \times \boldsymbol{H}) \mathrm{d}\,t\,\mathrm{d}\,v - \int\limits_{V_\Sigma} \left(\boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{H} \cdot \frac{\partial \boldsymbol{B}}{\partial t} \right) \mathrm{d}\,t\,\mathrm{d}\,v + \int\limits_{V_\Sigma} (\boldsymbol{J} \cdot \boldsymbol{E}_i) \mathrm{d}\,t\,\mathrm{d}\,v, \tag{4.11}$$

or, performing the transformation of a volume integral into a surface integral (Gauss-Ostrogradski theorem), the preceding relation yields:

$$\frac{\mathrm{d}\,W}{\mathrm{d}\,t} = -\int\limits_{\Sigma} (\boldsymbol{E} \times \boldsymbol{H}) \cdot \mathrm{d}\,\boldsymbol{S} - \int\limits_{V_\Sigma} \left(\boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{H} \cdot \frac{\partial \boldsymbol{B}}{\partial t} \right) \mathrm{d}\,v + \int\limits_{V_\Sigma} (\boldsymbol{J} \cdot \boldsymbol{E}_i) \mathrm{d}\,v. \tag{4.12}$$

In the case of an isotropic and linear medium, we have the relations (3.24), (3.105):

$$\boldsymbol{D} = \varepsilon\,\boldsymbol{E}; \qquad \boldsymbol{B} = \mu\,\boldsymbol{H},$$

and relation (4.12) gets the simpler form:

$$\frac{\mathrm{d}\,W}{\mathrm{d}\,t} = -\int\limits_{\Sigma} (\boldsymbol{E} \times \boldsymbol{H}) \cdot \mathrm{d}\,\boldsymbol{S} - \frac{\partial}{\partial t} \int\limits_{V_\Sigma} \left(\frac{1}{2} \boldsymbol{E} \cdot \boldsymbol{D} + \frac{1}{2} \boldsymbol{H} \cdot \boldsymbol{B} \right) \mathrm{d}\,v + \int\limits_{V_\Sigma} (\boldsymbol{J} \cdot \boldsymbol{E}_i) \mathrm{d}\,v. \tag{4.13}$$

In relations (4.11), (4.12) and (4.13), the following symbols will be used:

$$\boldsymbol{\Pi} = \boldsymbol{E} \times \boldsymbol{H}, \tag{4.14}$$

$$W_e = \int\limits_{V_\Sigma} \frac{1}{2} \boldsymbol{E} \cdot \boldsymbol{D} \,\mathrm{d}\,v \tag{4.15 a}$$

or

$$W_e = \int\limits_{V_\Sigma} \int\limits_{D_0}^{D} E \cdot \mathrm{d}\, D \,\mathrm{d}\, v\,, \tag{4.15 b}$$

$$W_m = \int\limits_{V_\Sigma} \frac{1}{2} H \cdot B \,\mathrm{d}\, v \tag{4.16 a}$$

or

$$W_m = \int\limits_{V_\Sigma} \int\limits_{B_0}^{B} H \cdot \mathrm{d}\, B \,\mathrm{d}\, v\,, \tag{4.16 b}$$

$$W_{em} = W_e + W_m\,, \tag{4.17}$$

$$P_r = \int\limits_{\Sigma} \Pi \cdot \mathrm{d}\, S\,. \tag{4.18}$$

Using the symbols above, relation (4.13) becomes:

$$\frac{\mathrm{d}\, W}{\mathrm{d}\, t} = -\int\limits_{\Sigma} \Pi \cdot \mathrm{d}\, S - \frac{\partial W_e}{\partial t} - \frac{\partial W_m}{\partial t} + \int\limits_{V_\Sigma} (J \cdot E_i)\mathrm{d}\, v\,. \tag{4.19}$$

By equating the expressions of $\dfrac{\mathrm{d}\, W}{\mathrm{d}\, t}$ from relations (4.7) and (4.19), it follows:

$$\int\limits_{V_\Sigma} (J \cdot E)\mathrm{d}\, v = -\int\limits_{V_\Sigma} \Pi \cdot \mathrm{d}\, S - \frac{\partial W_e}{\partial t} - \frac{\partial W_m}{\partial t} \tag{4.20}$$

or

$$\int\limits_{V_\Sigma} (J \cdot E_i)\mathrm{d}\, v - \frac{\partial W_e}{\partial t} - \frac{\partial W_m}{\partial t} = \frac{\mathrm{d}\, W}{\mathrm{d}\, t} + P_r\,. \tag{4.21}$$

In the relations above, only macroscopic quantities occur.

In the case of bodies moving with any velocity v_s relative to the reference frame, the quantities J and E of relation (4.20) can be replaced by using relation (3.124) and (3.118). Hence, it is possible to highlight terms of the form $(J \times B) \cdot v_s \,\mathrm{d}\, v$ that represent the work done in the unit of time, by the force produced by the magnetic field and which acts upon the volume element $\mathrm{d}\, v$. This work is different from zero if vectors J and v_s are not in the same direction.

In the case in which microscopic quantities are considered, relations of the same type will be obtained, with the mention that the macroscopic quantities have to be replaced by the microscopic ones. If no special mention is made, it will be assumed that the macroscopic energy is considered.

If the domain contains the whole space, and the electric charges, at rest or in motion, that produce the electromagnetic field are at finite distance, then the state quantities on

the surface Σ at infinity are zero and the term given by relation (4.18), in relation (4.19), vanishes. The simple case in which the strength of the impressed electric field is zero, hence $\boldsymbol{E}_i = 0$, will be considered. In this case, relation (4.19) represents the law of energy conservation for the physical system of domain Σ, which can be expressed as follows. The work of the forces produced by the electromagnetic field, done in the unit of time, is equal to the decrease of the energy of the electromagnetic field, given by relation (4.19), in the unit of time.

In the case in which the strength of the impressed electric field is different from zero, the relationship of the sentence above has to be correspondingly completed.

The energy of *the electromagnetic field* W_{em} contains a term W_e that contains only state quantities of the electric field, given by relation (4.15), and a term W_m that contains only state quantities of the magnetic field, given by relation (4.16). For this reason, the first term W_e of relation (4.17) is called *energy of the electric field*, and the second term W_m of relation (4.17) is called *energy of the magnetic field*.

It is interesting to be added that if the energy of electromagnetic field of any domain is expressed in two inertial reference frames, the obtained values are not equal each other [11, p. 412].

As previously mentioned, the expressions of the macroscopic energy of the electromagnetic field and of the microscopic energy have the same form. As shown in Sub-section 1.6.6, by calculating the average values of the microscopic state quantities of the electromagnetic field, the macroscopic state quantities of electromagnetic field will be obtained. In the expression of the microscopic energy of electromagnetic field, the product of two microscopic quantities occurs. Hence, it is no more possible to state that, in general, by calculating the average value of this product, the above expression of the energy in terms of the macroscopic state quantities will result. For this reason, in general, the macroscopic energy will not be obtained by calculating the average value of the microscopic energy.

The relation (4.19), for $\boldsymbol{E}_i = 0$, expresses, generally, the law of energy conservation of the physical system of the domain Σ, which can be enounced as follows. The decrease of the energy of the electromagnetic field in the unit of time is equal to the work done by the forces produced by the electromagnetic field in the unit of time, plus a supplementary term that depends only on the state quantities on the domain boundary surface, and which represents the radiated energy through the surface in the unit of time.

The vector $\boldsymbol{\Pi} = \boldsymbol{E} \times \boldsymbol{H}$ is termed *flux density vector of the electromagnetic energy* or *Poynting vector*.

According to relations (4.15 a), (4.16 a) and (4.17), *the volume density of the electromagnetic energy* (for linear media) is:

$$w_{em} = \frac{1}{2}\left(\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{H} \cdot \boldsymbol{B}\right). \tag{4.22}$$

The quantities

$$P_j = \int_{V_\Sigma} \boldsymbol{J} \cdot \boldsymbol{E}\, \mathrm{d}v, \tag{4.23}$$

and

$$p_j = \frac{\mathrm{d}^2 W}{\mathrm{d}\,t\,\mathrm{d}\,v} = \boldsymbol{J} \cdot \boldsymbol{E} \tag{4.24}$$

represent the power and the volume density of the same power transferred to the bodies of the domain, due to the action of the forces of electromagnetic nature of the field.

The relation (4.21) represents the *theorem of electromagnetic energy*, for linear media. This theorem can be expressed as follows.

The decrease speed of the electromagnetic field energy of a domain bounded by a closed surface, plus the power delivered by the sources of the impressed electric field is equal to the sum of the power transferred to the bodies contained inside the domain (due to the work of the forces of electromagnetic nature) and the flux of electromagnetic power through the domain boundary surface.

It is no possible to give an exact and general expression of the macroscopic electromagnetic energy in the case of substances with irreversible electric or magnetic polarization because, in this case, the polarization process is accompanied by local non-electromagnetic transformations.

The case of media containing substances with reversible polarization will be considered. Also, it will be assumed that $\boldsymbol{E}_i = 0$. In this case, relation (4.11) yields:

$$\mathrm{d}W = -(\mathrm{d}\,t)\int_{\Sigma} (\boldsymbol{E} \times \boldsymbol{H}) \cdot \mathrm{d}\,\boldsymbol{S} - \int_{V_{\Sigma}} \boldsymbol{E} \cdot \mathrm{d}\,\boldsymbol{D}\,\mathrm{d}\,v - \int_{V_{\Sigma}} \boldsymbol{H} \cdot \mathrm{d}\,\boldsymbol{B}\,\mathrm{d}\,v. \tag{4.25}$$

The last two terms of the right-hand side of relation (4.25) represent the increase of the internal energy if the transformations take place at constant entropy, but it is more convenient to consider transformations at constant temperature; then the respective terms represent the increase of the free energy (12, tome 3, p. 222).

From relations (4.25) and (4.7), the following two relations can be obtained:

$$\frac{\mathrm{d}W}{\mathrm{d}\,t} = -\int_{\Sigma} \boldsymbol{\Pi} \cdot \mathrm{d}\,\boldsymbol{S} - \frac{\mathrm{d}}{\mathrm{d}\,t}\int_{V_{\Sigma}} \left[\int_{(S_0)} \boldsymbol{E} \cdot \mathrm{d}\,\boldsymbol{D} + \int_{(S_0)} \boldsymbol{H} \cdot \mathrm{d}\,\boldsymbol{B} \right] \mathrm{d}\,v, \tag{4.26}$$

$$\int_{V_{\Sigma}} \boldsymbol{J} \cdot \boldsymbol{E}\,\mathrm{d}\,v = -\int_{\Sigma} \boldsymbol{\Pi} \cdot \mathrm{d}\,\boldsymbol{S} - \frac{\mathrm{d}}{\mathrm{d}\,t}\int_{V_{\Sigma}} \left[\int_{(S_0)} \boldsymbol{E} \cdot \mathrm{d}\,\boldsymbol{D} + \int_{(S_0)} \boldsymbol{H} \cdot \mathrm{d}\,\boldsymbol{B} \right] \mathrm{d}\,v, \tag{4.27}$$

where the corresponding integrals are calculated starting from the values that characterize the reference state denoted by S_0.

The reference state for the electromagnetic energy is generally characterized, by the values zero of the state quantities of the electromagnetic field.

In order to bring a physical system from this reference state to another state (present state or actual state, i.e., final state) that differs from the reference state only by the quantities of the local state of the electromagnetic field, it is necessary to do, from the exterior of the system, actions, the equivalent work of which must be equal to the electromagnetic energy in present (final) state.

In accordance with the concept of field, the electromagnetic energy is distributed throughout the space with a certain volume density that depends on the values of the local state quantities of the electromagnetic field.

The reference state for calculating the macroscopic electromagnetic energy is chosen according to the properties of the medium.

In media in which the couple of vector quantities E, D and H, B, respectively, cancel simultaneously (i.e., in media with temporary reversible polarization or *in vacuo*), the reference state is characterized by the following values of the state quantities $E = 0$, $D = 0$, $H = 0$, $B = 0$.

In media with permanent polarization, the reference state is characterized by the following values of the field state quantities: $E = 0$, $H = 0$, $D \neq 0$, $B \neq 0$. In the right-hand side of expression (4.27), the first term represents the expression of the radiated power; the first term within square brackets represents the energy of the electric field, and the second term, within the same brackets, the energy of the magnetic field.

4.2.   THE THEOREM OF IRREVERSIBLE TRANSFORMATION OF ELECTROMAGNETIC ENERGY IN THE CASE OF HYSTERESIS PHENOMENON

The case of bodies submitted to an electric or magnetic irreversible polarization, like in Fig. 3.2 and Fig. 3.9 will be considered. In this case, a part of the energy is transformed irreversibly in internal energy of the body. The volume density of the energy w_q that is transformed into internal energy of a body submitted to an electric or magnetic polarization cycle, also termed loop, is proportional to the area of the loop, i.e., the area of the hysteresis loop, as follows:

$$w_q = w_{qe} + w_{qm} = \oint_{\text{Cycle E}} E \cdot \mathrm{d}\,D + \oint_{\text{Cycle M}} H \cdot \mathrm{d}\,B\,, \tag{4.28}$$

where the symbols Cycle E and Cycle M mean the cycle (also called loop) of electric and magnetic polarization, respectively. In relation (4.28), each integral is proportional to the area of the corresponding hysteresis loop.

The relation (4.28) expresses *the theorem of irreversible transformation of electromagnetic energy due to polarization loops*. This theorem is also called *Warburg theorem* [22, p. 464].

4.3.   THE THEOREM OF IRREVERSIBLE TRANSFORMATION OF ELECTROMAGNETIC ENERGY INTO HEAT

From the microscopic point of view, using a simplified model, when an electric current flows in a metallic body, for instance along a conducting wire, the forces acting upon the moving charge carriers of the current, i.e., the electrons, do an internal work. The electrons cede energy by collision to the crystalline lattice of the body, and therefore

the thermal agitation of ions and atoms of the crystalline lattice will increase and a *thermal effect* occurs.

The expression of the electromagnetic power converted into heat can be obtained using formulae (4.6), (1.108), (1.109). We shall consider the case in which no other phenomenon exists, the electric current flowing in a conducting metallic body excepted. The formulae (1.108) and (1.109) can be written in the form:

$$\boldsymbol{E}_l = \rho \boldsymbol{J} \,. \tag{4.29}$$

Formulae (4.6) and (4.29) yield:

$$\mathrm{d}^2 W = \boldsymbol{J} \cdot \boldsymbol{E}_l \, \mathrm{d}v \, \mathrm{d}t = \rho \boldsymbol{J}^2 \, \mathrm{d}v \, \mathrm{d}t \,. \tag{4.30}$$

We introduce the symbol P_J in relation (4.30) for the energy converted in the body, per unit of time, i.e., the power, into heat and we get:

$$p_J \, \mathrm{d}v = \mathrm{d}P_J = \frac{\mathrm{d}^2 W}{\mathrm{d}t} = \boldsymbol{J} \cdot \boldsymbol{E}_l \, \mathrm{d}v = \rho \boldsymbol{J}^2 \, \mathrm{d}v \,. \tag{4.30 a}$$

It results that the electromagnetic energy converted into heat per unit of time, i.e., the power converted when an electric conductive body of volume V_Σ carries electric currents, is given by the expression:

$$P_J = \int_{V_\Sigma} \rho \boldsymbol{J}^2 \, \mathrm{d}v \,, \tag{4.31}$$

and the volume density of this power is given by the relation:

$$p_J = \rho \boldsymbol{J}^2 \,. \tag{4.32}$$

The two last formulae express the integral form and the local form of the *law of the electro-heating effect* or *electro-calorific effect*. This law is referred to as the *Joule law* and is also known as the *Joule-Lenz law*.

The examined transformation is an irreversible one.

The law above can be derived in the framework of the microscopic theory by calculating directly the collision forces and the corresponding work [23, Vol. II, p. 46].

4.4.   THE THEOREM OF ELECTROMAGNETIC MOMENTUM

Let us consider a system of n bodies or particles each of mass m_k and velocity $\boldsymbol{v}_k$, $(k = 1, 2, \ldots, n)$, electrically charged and which are moving in empty space, in any electromagnetic field. Let us denote by Σ any closed surface that contains the system of bodies or particles. We shall calculate the resultant force acting upon the system of particles. In order to simplify the relations we suppose that the impressed field does not exist, i.e., $\boldsymbol{E}_i = 0$. If impressed fields existed, the final relation could be correspondingly completed.

If the considered system contains conducting bodies, the macroscopic quantities have to be introduced. However, the form of relations will be the same.

We shall denote by v_r the velocity of the substance from the volume element, in which the volume density of the electric charge is ρ_v.

By relation (4.2), for $E_i = 0$, we obtain:

$$F = \int_{V_\Sigma} \left(\rho_v\, E + \rho_v\, v_r \times B \right) d v, \tag{4.33}$$

but according to relation (1.70), $J = \rho_v\, v_r$. Hence, relation (4.33) can be written:

$$F = \int_{V_\Sigma} \left(\rho_v\, E + J \times B \right) d v. \tag{4.34}$$

By replacing the force by the derivative of momentum, it follows:

$$\frac{d}{dt} \sum_{k=1}^{n} m_k\, v_k = \int_{V_\Sigma} \left(\rho_v\, E + J \times B \right) d v, \tag{4.35}$$

or

$$\frac{d}{dt} \sum_{k=1}^{n} m_k\, v_k = \int_{V_\Sigma} f\, d v, \tag{4.36}$$
$$f = \rho_v\, E + J \times B.$$

By replacing the vector quantity J with its expression from relation (3.86), it follows:

$$f = \rho_v\, E + \left(\operatorname{curl} H - \frac{\partial D}{\partial t} \right) \times B$$

or

$$f = \rho_v\, E - B \times \operatorname{curl} H - \frac{\partial D}{\partial t} \times B. \tag{4.37}$$

The case of empty space, i.e., vacuum, will be considered; according to relations (3.24), (3.105), we have $D = \varepsilon_0\, E$ and $B = \mu_0\, H$.

Relation (4.37) becomes:

$$f = \rho_v\, E - \mu_0\, H \times \operatorname{curl} H - \varepsilon_0\, \mu_0\, \frac{\partial}{\partial t} \left(E \times H \right) + \varepsilon_0\, \mu_0\, E \times \frac{\partial H}{\partial t}. \tag{4.38}$$

By using relation (3.53), it results that:

$$f = \rho_v\, E - H \times \operatorname{curl} B - \varepsilon_0\, E \times \operatorname{curl} E - \varepsilon_0\, \frac{\partial}{\partial t} \left(E \times B \right) \tag{4.39}$$

or, by using relation (3.16), we obtain:

$$\boldsymbol{f} = \boldsymbol{E} \operatorname{div}(\varepsilon_0 \, \boldsymbol{E}) - \varepsilon_0 \, \boldsymbol{E} \times \operatorname{curl} \boldsymbol{E} - \frac{1}{\mu_0} \boldsymbol{B} \times \operatorname{curl} \boldsymbol{B} - \varepsilon_0 \frac{\partial}{\partial t} (\boldsymbol{E} \times \boldsymbol{B}). \tag{4.40}$$

Taking into account relation (3.31) applied to the present case in which $\boldsymbol{B} = \mu_0 \, \boldsymbol{H}$, we can write:

$$\boldsymbol{f} = \varepsilon_0 \left(\boldsymbol{E} \operatorname{div} \boldsymbol{E} - \boldsymbol{E} \times \operatorname{curl} \boldsymbol{E} \right) + \mu_0 \left(\boldsymbol{H} \operatorname{div} \boldsymbol{H} - \boldsymbol{H} \times \operatorname{curl} \boldsymbol{H} \right) - \varepsilon_0 \, \mu_0 \frac{\partial}{\partial t} (\boldsymbol{E} \times \boldsymbol{H}). \tag{4.41}$$

By replacing expression (4.41) into relation (4.36), we obtain:

$$\frac{\mathrm{d}}{\mathrm{d} t} \sum_{k=1}^{n} m_k \, \boldsymbol{v}_k = \int\limits_{V_\Sigma} \boldsymbol{f}_1 \, \mathrm{d} v + \int\limits_{V_\Sigma} \boldsymbol{f}_2 \, \mathrm{d} v + \int\limits_{V_\Sigma} \boldsymbol{f}_3 \, \mathrm{d} v, \tag{4.42}$$

$$\boldsymbol{f}_1 = \varepsilon_0 \left(\boldsymbol{E} \operatorname{div} \boldsymbol{E} - \boldsymbol{E} \times \operatorname{curl} \boldsymbol{E} \right),$$

$$\boldsymbol{f}_2 = \mu_0 \left(\boldsymbol{H} \operatorname{div} \boldsymbol{H} - \boldsymbol{H} \times \operatorname{curl} \boldsymbol{H} \right),$$

$$\boldsymbol{f}_3 = -\varepsilon_0 \, \mu_0 \frac{\partial}{\partial t} (\boldsymbol{E} \times \boldsymbol{H}). \tag{4.43 a, b, c}$$

We shall examine the first integral of the right-hand side of relation (4.42):

$$F_1 = \int\limits_{V_\Sigma} \boldsymbol{f}_1 \, \mathrm{d} v. \tag{4.44}$$

This integral can be transformed into a surface integral. For this purpose, we multiply both sides of relation (4.44) by any constant vector $\boldsymbol{C}$, and we obtain:

$$\boldsymbol{C} \cdot \boldsymbol{F}_1 = \int\limits_{V_\Sigma} \boldsymbol{C} \cdot \boldsymbol{f}_1 \, \mathrm{d} v. \tag{4.45}$$

The quantity under the last integral can be written:

$$\boldsymbol{C} \cdot \boldsymbol{f}_1 = \varepsilon_0 \left[\boldsymbol{C} \cdot \boldsymbol{E} \operatorname{div} \boldsymbol{E} - \boldsymbol{C} \cdot \left(\boldsymbol{E} \times \operatorname{curl} \boldsymbol{E} \right) \right]. \tag{4.46}$$

We shall take into consideration the vector relation (A.1.70) for this case:

$$\operatorname{div} \left[(\boldsymbol{C} \cdot \boldsymbol{a}) \boldsymbol{a} \right] = (\boldsymbol{C} \cdot \boldsymbol{a}) \operatorname{div} \boldsymbol{a} + \boldsymbol{a} \cdot \operatorname{grad} (\boldsymbol{C} \cdot \boldsymbol{a}).$$

But, taking into account that the vector $\boldsymbol{C}$ is constant, relation (A.1.74) yields:

$$\operatorname{grad} (\boldsymbol{C} \cdot \boldsymbol{a}) = (\boldsymbol{C} \cdot \nabla) \boldsymbol{a} + \boldsymbol{C} \times \operatorname{curl} \boldsymbol{a}.$$

Hence:

$$\operatorname{div} \left[(\boldsymbol{C} \cdot \boldsymbol{a}) \boldsymbol{a} \right] = (\boldsymbol{C} \cdot \boldsymbol{a}) \operatorname{div} \boldsymbol{a} + \boldsymbol{a} \cdot \left[(\boldsymbol{C} \cdot \nabla) \boldsymbol{a} \right] + \boldsymbol{a} \cdot (\boldsymbol{C} \times \operatorname{curl} \boldsymbol{a}).$$

But

$$a \cdot [(C \cdot \nabla)a] = \frac{1}{2} C \cdot \operatorname{grad} a^2 .$$

Therefore:

$$\operatorname{div}[(C \cdot a)a] = (C \cdot a)\operatorname{div} a + \frac{1}{2} C \cdot \operatorname{grad} a^2 - C \cdot (a \times \operatorname{curl} a)$$

or

$$(C \cdot a)\operatorname{div} a - C \cdot (a \times \operatorname{curl} a) = \operatorname{div}[(C \cdot a)a] - C \cdot \operatorname{grad}\left(\frac{1}{2} a^2\right).$$

But, according to relation (A.1.70), taking into account that $C = \operatorname{const}$, we have:

$$C \cdot \operatorname{grad}\left(\frac{1}{2} a^2\right) = \operatorname{div}\left[\left(C \frac{1}{2} a^2\right)\right].$$

It follows:

$$(C \cdot a)\operatorname{div} a - C \cdot (a \times \operatorname{curl} a) = \operatorname{div}[(C \cdot a)a] - \operatorname{div}\left(C \frac{1}{2} a^2\right). \tag{4.47}$$

By using the vector relation (4.47), relation (4.45) becomes:

$$\int_{V_\Sigma} C \cdot f_1 \, dv = \int_{V_\Sigma} \varepsilon_0 \operatorname{div}\left[(C \cdot E)E - C \frac{1}{2} E^2\right] dv$$

or

$$\int_{V_\Sigma} C \cdot f_1 \, dv = \varepsilon_0 \int_\Sigma \left[(C \cdot E)E - C \frac{1}{2} E^2\right] \cdot dS = \varepsilon_0 \int_\Sigma \left[(C \cdot E)E \cdot n - \frac{1}{2} E^2 C \cdot n\right] dS.$$

It results that:

$$C \cdot \int_{V_\Sigma} f_1 \, dv = C \cdot \int_\Sigma \left[\varepsilon_0 (E \cdot n)E - \frac{1}{2} \varepsilon_0 E^2 n\right] dS. \tag{4.48}$$

It follows:

$$\int_{V_\Sigma} f_1 \, dv = \int_\Sigma \left[\varepsilon_0 (E \cdot n)E - \frac{1}{2} \varepsilon_0 E^2 n\right] dS. \tag{4.49}$$

The second integral of the right-hand side of relation (4.42) can be analogously modified:

$$\int_{V_\Sigma} \boldsymbol{f}_2 \, \mathrm{d}\,v = \int_\Sigma \left[\mu_0 (\boldsymbol{H} \cdot \boldsymbol{n}) \boldsymbol{H} - \frac{1}{2} \mu_0 \, H^2 \, \boldsymbol{n} \right] \mathrm{d}\,S . \qquad (4.50)$$

The third integral of the right-hand side of relation (4.42) cannot be transformed into a surface-integral.

By replacing relations (4.49) and (4.50) into relation (4.42), we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}\,t} \sum_{k=1}^n m_k \, \boldsymbol{v}_k = \int_\Sigma \left[\varepsilon_0 (\boldsymbol{E} \cdot \boldsymbol{n}) \boldsymbol{E} + \mu_0 (\boldsymbol{H} \cdot \boldsymbol{n}) \boldsymbol{H} - \left(\frac{1}{2} \varepsilon_0 \, E^2 + \frac{1}{2} \mu_0 \, H^2 \right) \boldsymbol{n} \right] \mathrm{d}\,S -$$

$$- \frac{\mathrm{d}}{\mathrm{d}\,t} \int_{V_\Sigma} \varepsilon_0 \, \mu_0 \, (\boldsymbol{E} \times \boldsymbol{H}) \mathrm{d}\,v .$$

$$(4.51)$$

Also, it follows:

$$- \frac{\mathrm{d}}{\mathrm{d}\,t} \left[\sum_{k=1}^n m_k \, \boldsymbol{v}_k + \boldsymbol{G} \right] = \int_\Sigma - \left[\varepsilon_0 (\boldsymbol{E} \cdot \boldsymbol{n}) \boldsymbol{E} + \mu_0 (\boldsymbol{H} \cdot \boldsymbol{n}) \boldsymbol{H} - \left(\frac{1}{2} \varepsilon_0 \, E^2 + \frac{1}{2} \mu_0 \, H^2 \right) \boldsymbol{n} \right] \mathrm{d}\,S ,$$

$$(4.52)$$

and

$$- \frac{\mathrm{d}}{\mathrm{d}\,t} \left[\sum_{k=1}^n m_k \, \boldsymbol{v}_k + \boldsymbol{G} \right] = \int_\Sigma - (\boldsymbol{t}) \mathrm{d}\,S , \qquad (4.53)$$

where

$$\boldsymbol{G} = \int_{V_\Sigma} \boldsymbol{g} \, \mathrm{d}\,v = \int_{V_\Sigma} \varepsilon_0 \, \mu_0 \, (\boldsymbol{E} \times \boldsymbol{H}) \mathrm{d}\,v , \qquad (4.54 \ \mathrm{a})$$

$$\boldsymbol{g} = \varepsilon_0 \, \mu_0 \, (\boldsymbol{E} \times \boldsymbol{H}), \qquad (4.54 \ \mathrm{b})$$

$$\boldsymbol{t} = \varepsilon_0 (\boldsymbol{E} \cdot \boldsymbol{n}) \boldsymbol{E} + \mu_0 (\boldsymbol{H} \cdot \boldsymbol{n}) \boldsymbol{H} - \left(\frac{1}{2} \varepsilon_0 \, E^2 + \frac{1}{2} \mu_0 \, H^2 \right) \boldsymbol{n} . \qquad (4.54 \ \mathrm{c})$$

The quantity $\boldsymbol{G}$ is termed *electromagnetic momentum* or *electromagnetic quantity of motion*, and the quantity $\boldsymbol{g}$ is called *volume density of the electromagnetic momentum.*

The quantity $\boldsymbol{t}$ is called *density of the surface tension* (also called the *Maxwell stress tensor*) with two components, an electric one (which contains only electric state quantities) and another magnetic one (which contains only magnetic state quantities).

The quantity given by relation (4.54 a) can be physically interpreted taking into account the following considerations. Let us consider the system of bodies or particles together with the corresponding electromagnetic field as representing an isolated physical system. In this case, the surface-integral of the right-hand side of relation (4.53) vanishes.

It follows that for the isolated system constituted by bodies or particles, the conserved quantity is not the mechanical momentum, like in Classical Mechanics, but the sum of the mechanical momentum and the quantity $\boldsymbol{G}$, which can be interpreted as a momentum associated with the electromagnetic field, called *total momentum*.

Relation (4.52) or (4.53) represents *the theorem of electromagnetic momentum*. It follows that, for a system of bodies or particles together with the field, the sum of the tensions on the surface Σ which contains this system is equal to the velocity of the decrease of the total momentum (mechanical plus electromagnetic), i.e., $\sum_{k=1}^{n} m_k \, \boldsymbol{v}_k + \boldsymbol{G}$, located within the volume V_Σ.

A detailed and general analysis of the theorem of energy and electromagnetic momentum (also called quantity of motion) for electrically and magnetically polarized media can be found in works [59], [60].

APPENDICES

APPENDIX 1

VECTOR CALCULUS

In this Appendix, we shall recall some definitions and certain more used relations. At the same time, certain more important parts of the proofs will be given.

A.1.1. VECTOR ALGEBRA

A quantity that can be characterized by a single number, in any system of units, is called *scalar quantity*.

The most typical scalar is an abstract number. Other examples: temperature, mass, density, energy.

A *vector* is a quantity characterized by a number, termed its magnitude, expressed in certain units of measure, and a certain direction in space that includes a certain orientation (sense). There are also other definitions of vector quantities [54].

The simplest example of a vector is constituted by a straight-line segment denoted $\overrightarrow{AB}$ having its magnitude equal to its length (i.e., the length of $\overrightarrow{AB}$) and a direction oriented from the point A towards the point B as shown in Fig. A.1.1.

Other examples of vector quantities: the force, the velocity (speed), the acceleration. A vector may be denoted by one of the following symbols: $\overrightarrow{AB}$, $\boldsymbol{a}$, $\vec{a}$ or $\overline{a}$. We shall use the second one. The magnitude of a vector is called the *modulus* of the vector and is denoted:

$$\left| \boldsymbol{a} \right| = a. \tag{A.1.1}$$

According to their application point, the vectors can be grouped into three categories: free vectors, sliding vectors and bound vectors.

The free vectors have a given magnitude and direction, but their application point is arbitrary, for instance: the moment of a couple of forces.

The sliding vectors have a given magnitude and direction, but their application point can be arbitrarily chosen along a given straight line called support straight line the direction of which coincides with that of the vector, for instance a force.

Fig. A.1.1. Representation of a vector.
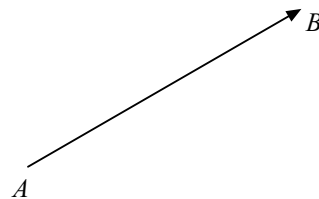
The bound vectors have a given magnitude, direction, and application point. For instance, a vector field, (field of vectors) is a field of bound vectors.

For the analytical definition of a vector in space, it is necessary to know the following data. For a free vector: 3 quantities, namely its projections along the axes of co-ordinates. For the sliding vectors: 5 quantities, the projections along the axes of co-ordinates and the parameters of the direction of the support straight line. For the bound vectors: 6 quantities, the projections along the axes of co-ordinates and the co-ordinates of the application point.

The vector having its magnitude equal to unity is called *unit vector*. The unit vector of any vector has its magnitude equal to unity, the same direction as the considered vector, and has no physical dimension. The unit vector of a vector $\boldsymbol{a}$ can be expressed as $\dfrac{\boldsymbol{a}}{a}$.

Two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are equal to each other if their magnitudes are equal and their unit vectors have the same direction.

A vector can be expressed in a three-orthogonal rectilinear system of co-ordinates (Cartesian system of co-ordinates) in the form:

$$\boldsymbol{a} = \boldsymbol{i}\, a_x + \boldsymbol{j}\, a_y + \boldsymbol{k}\, a_z \,, \tag{A.1.2}$$

where the quantities $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ are the unit vectors of the axes of co-ordinates Ox, Oy, Oz, and the quantities a_x, a_y, a_z are the components of the vector $\boldsymbol{a}$ along the axes of co-ordinates Ox, Oy, Oz.

a. The *scalar product* $\boldsymbol{a} \cdot \boldsymbol{b}$ of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$:

$$\boldsymbol{a} = \boldsymbol{i}\, a_x + \boldsymbol{j}\, a_y + \boldsymbol{k}\, a_z \,, \tag{A.1.3}$$

$$\boldsymbol{b} = \boldsymbol{i}\, b_x + \boldsymbol{j}\, b_y + \boldsymbol{k}\, b_z \tag{A.1.4}$$

is:

$$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a} = a\, b \cos(\boldsymbol{a}, \boldsymbol{b}) = a_x\, b_x + a_y\, b_y + a_z\, b_z \,. \tag{A.1.5}$$

b. The *vector product* $\boldsymbol{a} \times \boldsymbol{b}$ of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is the vector perpendicular to vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ and having its modulus equal to the area of the parallelogram constructed on these vectors, hence:

$$\left| \boldsymbol{a} \times \boldsymbol{b} \right| = a\, b \sin(\boldsymbol{a}, \boldsymbol{b}); \tag{A.1.6}$$

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \boldsymbol{i}\left(a_y\, b_z - a_z\, b_y \right) + \boldsymbol{j}\left(a_z\, b_x - a_x\, b_z \right) + \boldsymbol{k}\left(a_x\, b_y - a_y\, b_x \right); \tag{A.1.7}$$

$$\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a} \,. \tag{A.1.8}$$

The orientation (sense) of the vector $\boldsymbol{a} \times \boldsymbol{b}$ is determined from the condition that the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{a} \times \boldsymbol{b}$ to constitute a right-handed system, like in Fig. A.1.2.
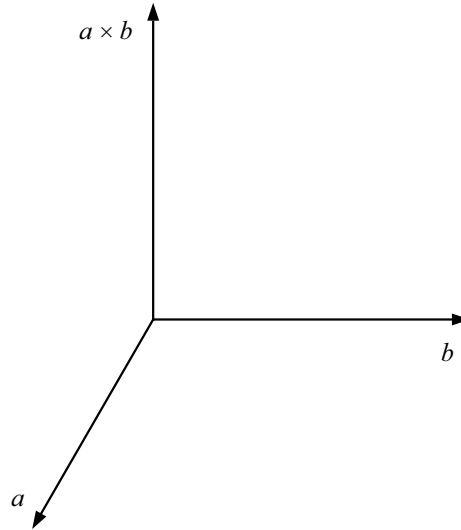
Fig. A.1.2. The vector product $c = a \times b$.

The three-orthogonal rectilinear systems of co-ordinates utilized in the present work are right-handed co-ordinates systems.

The direction of the vector $c = a \times b$ can be obtained by means of the *right-handed screw rule* as follows: The right-handed screw will be placed perpendicular to the plane formed by vectors a and b their points of application being brought to the same point. The screw fixed to the vector a will be turned so that the vector a will turn towards the vector b along the shortest way. The direction in which the right-handed screw moves is just the direction of the vector $c = a \times b$.

c. The *triple scalar product* (the scalar product of three vectors), called also *mixed product* (of three vectors) of vectors a, b, c is a scalar, numerically equal to the volume of the parallelepiped constructed on these vectors:

$$a \cdot (b \times c) = b \cdot (c \times a) = c \cdot (a \times b) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}, \tag{A.1.9}$$

and it also follows:

$$a \cdot (b \times c) = -b \cdot (a \times c) = -c \cdot (b \times a) = -a \cdot (c \times b). \tag{A.1.9 a}$$

As it can be seen, the expressions (A.1.9) and (A.1.9 a) can be deduced one from the other by circular permutations of letters a, b and c disposed, for instance round a circle, in the sequence of the trigonometric sense.

d. The *triple vector product* (of three vectors), also called *double vector product* (of three vectors) of vectors a, b, c is:

$$a \times (b \times c) = b (a \cdot c) - c (a \cdot b). \tag{A.1.10}$$

e. Calculation of the derivative of a vector with respect to a parameter. If the vectors are functions of a scalar variable, by applying the usual rules of differentiation, it follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}(a+b) = \frac{\mathrm{d}a}{\mathrm{d}t} + \frac{\mathrm{d}b}{\mathrm{d}t}, \qquad\qquad\qquad (\text{A.1.11 a})$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\varphi a) = \varphi\frac{\mathrm{d}a}{\mathrm{d}t} + \frac{\mathrm{d}\varphi}{\mathrm{d}t}a, \qquad\qquad\qquad (\text{A.1.11 b})$$

$$\frac{\mathrm{d}(a\cdot b)}{\mathrm{d}t} = b\cdot\frac{\mathrm{d}a}{\mathrm{d}t} + a\cdot\frac{\mathrm{d}b}{\mathrm{d}t}, \qquad\qquad\qquad (\text{A.1.11 c})$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(a\times b) = \frac{\mathrm{d}a}{\mathrm{d}t}\times b + a\times\frac{\mathrm{d}b}{\mathrm{d}t}. \qquad\qquad\qquad (\text{A.1.11 d})$$

A.1.2. VECTOR ANALYSIS

A.1.2.1. Scalar and Vector Fields

By a scalar or a vector field, is meant a domain of space where a scalar or a vector quantity is associated with every point. Each point of space is determined by the position vector. Therefore, the characterization of a scalar or vector field is equivalent with the adoption of a scalar function $\varphi(r)$ or a vector function $a(r)$ that are depending on the position vector r. A scalar field and a vector field can also be referred to as a field of scalars and a field of vectors, respectively. For scalar and vector, the plural form has been used since, at every point of the considered domain, there is a certain scalar and vector, respectively. The functions $\varphi(r)$ and $a(r)$ can depend apart from the position vector r, also on other scalar arguments, for example on time. The functions $\varphi(r)$ and $a(r)$ are considered continuous and differentiable with respect to all arguments. We shall consider the scalar function $\varphi(r) = \varphi(x, y, z)$. Examples of such fields: the temperature field of a body non-uniformly heated, the density field of a non-homogeneous body, the field of an electrostatic potential, the field of a magnetic potential.

A domain is called connected domain, if any two points belonging to the domain can be joined by a continuous curve that belongs in totality to this domain.

A domain is called simply connected domain, if any closed curve that belongs in totality to this domain can be reduced to a single point by continuous deformations without overstepping the domain. The domains that have not this property are called multiply connected domains (doubly connected, triply connected, etc.).

Examples of simply connected domains: the interior of a sphere, the interior of a cylinder of finite length, the exterior of a cylinder of finite length.

Examples of multiply connected domains: the exterior of a cylinder of infinite length, the interior of a torus.

A.1.2.2. The Derivative of a Scalar Function in Regard to a Given Direction

We shall consider a spectrum composed by the lines of field, represented by continuous lines, and traces of potential surfaces, represented by dashed lines, like in Fig. 1.3, where the direction (sense) in which we have chosen to travel the lines of field has been represented by an arrow.

Let us consider that the scalar function φ has the value φ_0 at any point M_0 and after one displacement $\Delta s = \overrightarrow{M_0 M_s}$ along the direction of a vector s, the point goes from the point M_0 to the point M_s, where the scalar has the value φ_s, like in Fig. A.1.4.

The increment of the quantity φ, after this displacement, is:

$$\Delta\varphi = \varphi_s - \varphi_0 .$$

The limit of the ratio of this increment to the magnitude of the displacement Δs is denoted by $\dfrac{\partial \varphi}{\partial s}$ and is called the derivative of the scalar φ at the point M_0 in regard to the direction s :

$$\frac{\partial \varphi}{\partial s} = \lim_{\Delta s \to 0} \frac{\varphi_s - \varphi_0}{\Delta s} . \tag{A.1.12}$$

The value of this derivative depends on the direction of the vector s.

For this reason, the derivative of a scalar function along a given direction must not be confused with a usual partial derivative with respect to any variable s.
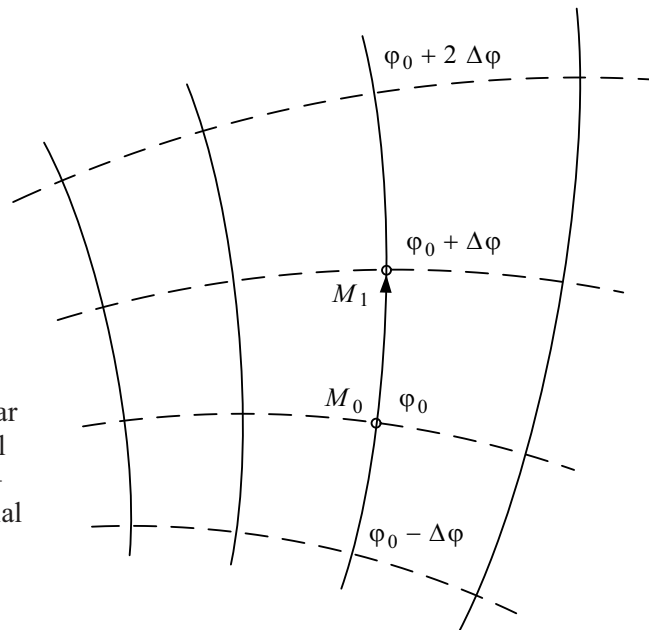

Fig. A.1.3. Plane section perpendicular to a configuration with plane parallel symmetry, spectrum: lines of field – continuous lines; traces of equipotential surfaces – dashed lines.
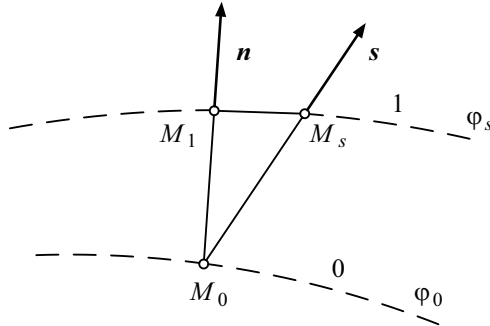
Fig. A.1.4. Explanation to the calculation of the derivative of a scalar function with respect to a certain direction.

In order to study the dependence of the value of the derivative $\dfrac{\partial \varphi}{\partial s}$ on the direction of differentiation s, we shall consider the points of the field at which the function φ has the same value for instance φ_0. In the general case, the set of these points forms a surface that is termed level surface or equipotential surface. This surface has the equation:

$$\varphi(x, y, z) = \varphi_0 . \qquad (A.1.13)$$

Fig. A.1.3 shows one section made by a plane through a set of equipotential surfaces corresponding to certain values of the function φ equal to φ_0, $\varphi_0 \pm \Delta\varphi$, $\varphi_0 + 2\Delta\varphi$.

A.1.2.3.   The Gradient

We shall denote by $\boldsymbol{n}$ the unit vector of the normal to the level surface $\varphi = \varphi_0$, at a point M_0 of Fig. A.1.4, oriented in the direction assumed as corresponding to the ascending values of the function φ. We shall show that in the case, in which the value of the derivative $\dfrac{\partial \varphi}{\partial s}$ with respect to the direction of this normal is known, it is possible to calculate the value of the derivative of the scalar function φ with respect to any direction $\boldsymbol{s}$. We consider the equipotential surface having the ordinal number 1 that passes through the point M_s situated on the direction $\boldsymbol{s}$, like in Fig. A.1.4.

Let us consider two neighbouring points $M_0(x, y, z)$ and $M_0'(x + \Delta x, y + \Delta y, z + \Delta z)$, both on the equipotential surface $\varphi(x, y, z) = \varphi_0 = \text{const}$. Then, we have:

$$\varphi(x + \Delta x, y + \Delta y, z + \Delta z) - \varphi(x, y, z) = 0, \qquad (A.1.14)$$

both functions of the left-hand side having the same value, since they are taken for two points belonging to the same equipotential surface.

Therefore, the expression of the total (exact) differential of the function $\varphi = \varphi_0$, calculated at any point $M_0(x, y, z)$, as the three quantities Δx, Δy, Δz tend to zero, is:

$$\frac{\partial \varphi}{\partial x} d x + \frac{\partial \varphi}{\partial y} d y + \frac{\partial \varphi}{\partial z} d z = 0.$$ (A.1.15)

The last expression can be considered as the scalar product of two vectors:

$$\boldsymbol{a} = \boldsymbol{i} \frac{\partial \varphi}{\partial x} + \boldsymbol{j} \frac{\partial \varphi}{\partial y} + \boldsymbol{k} \frac{\partial \varphi}{\partial z},$$ (A.1.16 a)

$$\boldsymbol{b} = \boldsymbol{i} d x + \boldsymbol{j} d y + \boldsymbol{k} d z.$$ (A.1.16 b)

Since the scalar product of the two vectors is zero, it follows that the two vectors are perpendicular to each other. The second one is a very small vector contained by the surface $\varphi(x, y, z) = \varphi_0 = \text{const}.$ Therefore, the first vector is perpendicular to the second one. The second vector, with the origin at the point $M_0(x, y, z)$, can have any direction, provided it remains contained by the surface $\varphi(x, y, z) = \varphi_0 = \text{const}$. According to relation (A.1.15), it follows that the first vector is perpendicular on the surface $\varphi(x, y, z) = \varphi_0 = \text{const}$ at the point $M_0(x, y, z)$; hence it has the direction of the positive normal to the surface at the same point. The positive direction is considered that along which the value of the function $\varphi(x, y, z)$ is assumed to increase. This vector, denoted above by $\boldsymbol{a}$, is called the *gradient* of the function φ. The usual symbol is:

$$\text{grad}\, \varphi = \boldsymbol{i} \frac{\partial \varphi}{\partial x} + \boldsymbol{j} \frac{\partial \varphi}{\partial y} + \boldsymbol{k} \frac{\partial \varphi}{\partial z}.$$ (A.1.17)

Let us calculate the *derivative of the function* $\varphi(x, y, z)$ at the same point $M_0(x, y, z)$ like above *along any direction* $\boldsymbol{s}$. The expression of this derivative is:

$$\frac{\partial \varphi}{\partial s} = \lim_{\Delta s \to 0} \frac{\varphi(M_s) - \varphi(M_0)}{\Delta s},$$ (A.1.18 a)

where:

$$\Delta \boldsymbol{s} = \boldsymbol{s} \, \Delta s,$$ (A.1.18 b)

$$\varphi(M) = \varphi(\boldsymbol{r}) = \varphi(x, y, z) = \varphi_0,$$ (A.1.18 c)

$$\varphi(M_s) = \varphi(\boldsymbol{r}_s) = \varphi(\boldsymbol{r} + \Delta \boldsymbol{s}) = \varphi(x + \Delta x, y + \Delta y, z + \Delta z) = \varphi_s.$$ (A.1.18 d)

Therefore:

$$\frac{\partial \varphi}{\partial s} = \lim_{\Delta s \to 0} \frac{\varphi(x + \Delta s_x, y + \Delta s_y, z + \Delta s_z) - \varphi(x, y, z)}{\Delta s},$$ (A.1.18 e)

where the vector $\boldsymbol{s}$ is the unit vector of $\Delta \boldsymbol{s}$.

By expanding in a series and retaining only the small quantities of the first order, we get:

$$\frac{\partial \varphi}{\partial s} = \frac{\partial \varphi}{\partial x} \cdot \frac{\mathrm{d} s_x}{\mathrm{d} s} + \frac{\partial \varphi}{\partial y} \cdot \frac{\mathrm{d} s_y}{\mathrm{d} s} + \frac{\partial \varphi}{\partial z} \cdot \frac{\mathrm{d} s_z}{\mathrm{d} s}.$$
(A.1.19 a)

Therefore:

$$\frac{\partial \varphi}{\partial s} = \frac{\partial \varphi}{\partial x}(\boldsymbol{i} \cdot \boldsymbol{s}) + \frac{\partial \varphi}{\partial y}(\boldsymbol{j} \cdot \boldsymbol{s}) + \frac{\partial \varphi}{\partial z}(\boldsymbol{k} \cdot \boldsymbol{s}).$$
(A.1.19 b)

Hence:

$$\frac{\partial \varphi}{\partial s} = \left(\boldsymbol{i}\frac{\partial \varphi}{\partial x} + \boldsymbol{j}\frac{\partial \varphi}{\partial y} + \boldsymbol{k}\frac{\partial \varphi}{\partial z} \right) \cdot \left(\boldsymbol{i}\, s_x + \boldsymbol{j}\, s_y + \boldsymbol{k}\, s_z \right) = (\operatorname{grad} \varphi) \cdot \boldsymbol{s}.$$
(A.1.19 c)

From the last relation, it follows that the greatest value of the left hand-side, at a point, of this relation is obtained in the case in which the vector $\boldsymbol{s}$ has the direction of the gradient vector, hence the one of the normal at the same point of the surface, thus $\boldsymbol{s} = \boldsymbol{n}$. Therefore:

$$\frac{\partial \varphi}{\partial n} = (\operatorname{grad} \varphi) \cdot \boldsymbol{n}; \qquad \operatorname{grad} \varphi = \frac{\partial \varphi}{\partial n}\boldsymbol{n}.$$
(A.1.20 a, b)

Therefore:

$$\frac{\partial \varphi}{\partial s} = \frac{\partial \varphi}{\partial n}\boldsymbol{n} \cdot \boldsymbol{s}.$$
(A.1.21)

It follows that the greatest value of the derivative of a function of point is obtained if the derivative of that function is calculated along the direction of the positive normal to the equipotential surface that passes through the considered point. Otherwise, the value would be smaller, because the scalar product $\boldsymbol{n} \cdot \boldsymbol{s}$ is a cosine.

The vector $\dfrac{\partial \varphi}{\partial n}\boldsymbol{n}$ oriented along the normal to the equipotential surface in the direction of the increasing values of the function φ, as said above, is called *the gradient of the scalar function* φ and is expressed by the relation above.

Therefore, the derivative of the function φ with respect to the direction $\boldsymbol{s}$ is equal to the projection of the vector $\operatorname{grad} \varphi$ along the direction $\boldsymbol{s}$.

In a three-orthogonal rectilinear system of co-ordinates, according to relations (A.1.20) and (A.1.21), it follows:

$$(\operatorname{grad} \varphi)_x = \frac{\partial \varphi}{\partial x}; \quad (\operatorname{grad} \varphi)_y = \frac{\partial \varphi}{\partial y}; \quad (\operatorname{grad} \varphi)_z = \frac{\partial \varphi}{\partial z},$$
(A.1.22 a, b, c)

hence:

$$|\operatorname{grad} \varphi| = \sqrt{\left(\frac{\partial \varphi}{\partial x} \right)^2 + \left(\frac{\partial \varphi}{\partial y} \right)^2 + \left(\frac{\partial \varphi}{\partial z} \right)^2}.$$
(A.1.22 d)

From expressions (A.1.20 a) and (A.1.20 b), it follows that the vector grad does not depend on the choice of the system of co-ordinates; hence it is one invariant with respect to the system of co-ordinates.

The gradient of a scalar function may be also expressed in other types of systems of co-ordinates. From relation (A.1.21) it follows that the direction along which the scalar function φ has the most rapid increase is the direction of the normal $\boldsymbol{n}$ to the level surface passing through the point under consideration.

If the scalar field φ is known, then at every point of this field, the vector $\operatorname{grad}\varphi$ can be determined and it is perpendicular to the level surfaces of this field. Let us consider a set of lines perpendicular to the level surfaces, i.e., a set of orthogonal trajectories of the level surfaces; at every point of the field, the direction of the gradient will coincide with the direction of one orthogonal trajectory. For this reason, the orthogonal trajectories of the level surfaces are termed gradient lines or lines of field.

If the level surfaces are represented like in Fig. A.1.3 so that the value of the scalar function φ on the respective consecutive surfaces should be φ_0, $\varphi_0 \pm \Delta\varphi$, $\varphi_0 + 2\Delta\varphi$, it is possible to write for $\Delta\varphi$ small enough:

$$\Delta\varphi = \frac{\partial \varphi}{\partial n}\Delta n = \left| \operatorname{grad}\varphi \right| \Delta n, \tag{A.1.23}$$

and in the case in which $\Delta\varphi$ is constant:

$$\left| \operatorname{grad}\varphi \right| = \frac{\text{const}}{\Delta n}. \tag{A.1.24}$$

Therefore, if we represent a level surface, with a constant variation of the scalar function between two neighbouring surfaces, the density of the level surfaces gives an approximate indication about the numerical value of the gradient. Therefore, in a region of space in which the level surfaces are denser, the value of the gradient is greater.

At the same time, it is important to be noted, that the calculation of the derivative of a scalar φ that is a function of another scalar ψ, with respect to any variable, gives:

$$\left(\operatorname{grad}\varphi \right)_s = \frac{\partial \varphi}{\partial s} = \frac{\partial \varphi}{\partial \psi}\cdot\frac{\partial \psi}{\partial s} = \frac{\partial \varphi}{\partial \psi}\left(\operatorname{grad}\psi \right)_s. \tag{A.1.25}$$

Hence, in the case of the gradient, the usual rule of the calculation of a derivative of a function of function is maintained.

A.1.2.3.1. *The Gradient of the Magnitude of the Position Vector*

The magnitude of the position vector $\boldsymbol{r}$ is a scalar function that is depending on the position of two points: The origin and the extremity (end) of the position vector. The former of these points is called *source point* and the latter *observation point* or *field point*.

Two cases will be considered: 1. The calculation of the gradient with respect to the field point, say N, denoted $\operatorname{grad}_N r$. 2. The calculation of the gradient with respect to the source point, say O, denoted $\operatorname{grad}_O r$.

We shall take into account that the function under consideration is:

$$\varphi = \varphi(x_N, y_N, z_N) = r = \left[(x_N - x_O)^2 + (y_N - y_O)^2 + (z_N - z_O)^2\right]^{\frac{1}{2}}. \qquad (A.1.26)$$

Likewise, let us denote:

$$\boldsymbol{r} = \boldsymbol{i}(x_N - x_O) + \boldsymbol{j}(y_N - y_O) + \boldsymbol{k}(z_N - z_O). \qquad (A.1.27)$$

In the first case, we have to calculate the derivative with respect to the co-ordinates of the point N. We obtain:

$$\operatorname{grad}_N r = \frac{\boldsymbol{r}}{r}. \qquad (A.1.28)$$

In the second case, we have to calculate the derivative with respect to the co-ordinates of the point O. We obtain:

$$\operatorname{grad}_O r = -\frac{\boldsymbol{r}}{r}. \qquad (A.1.29)$$

A proof of the relations above, based on geometrical considerations for the calculation of the gradient is also possible [23, Vol. I, p. 461].

A.1.2.4. The Flux of a Vector through a Surface

Let us consider the surface-integral of a field vector $\boldsymbol{a}(\boldsymbol{r})$ through any open or closed surface.

The flux $\mathrm{d}\Phi$ of any vector $\boldsymbol{a}$ of a field of vectors $\boldsymbol{a}(\boldsymbol{r})$, through the surface element $\mathrm{d}S$, is the quantity given by the expression:

$$\mathrm{d}\Phi = a_n\,\mathrm{d}S = a\cos(\boldsymbol{a}, \boldsymbol{n})\mathrm{d}S = \boldsymbol{a}\cdot\boldsymbol{n}\,\mathrm{d}S = \boldsymbol{a}\cdot\mathrm{d}\boldsymbol{S}, \qquad (A.1.30)$$

where the quantity $\boldsymbol{a}$ is the value of the vector at a point of the surface element $\mathrm{d}S$ (e.g., at its middle), the quantity a_n is the component of the vector $\boldsymbol{a}$ along the direction $\boldsymbol{n}$, and the normal $\boldsymbol{n}$ to the surface element $\mathrm{d}S$ has the positive direction. In the case of any closed surface Σ, the direction of the normal is considered as positive if it is oriented outwards the surface, like in Fig. A.1.5 a. In the case of a simply connected open surface S_Γ, for a travelling sense of the closed curve Γ by which the surface is bounded, the normal is considered as positive if it is associated with the sense of travelling, according to the right-handed screw rule, like in Fig. A.1.5 b. The flux of a vector $\boldsymbol{a}$ through a surface S, closed as well as open, like in Fig. A.1.5 c, is the sum of the fluxes through the surface elements and it is given by the relation:

$$\Phi = \int_S a_n\,\mathrm{d}S = \int_S \boldsymbol{a}\cdot\boldsymbol{n}\,\mathrm{d}S = \int_S \boldsymbol{a}\cdot\mathrm{d}\boldsymbol{S};$$

$$\mathrm{d}\boldsymbol{S} = \boldsymbol{n}\,\mathrm{d}S. \qquad (A.1.31)$$
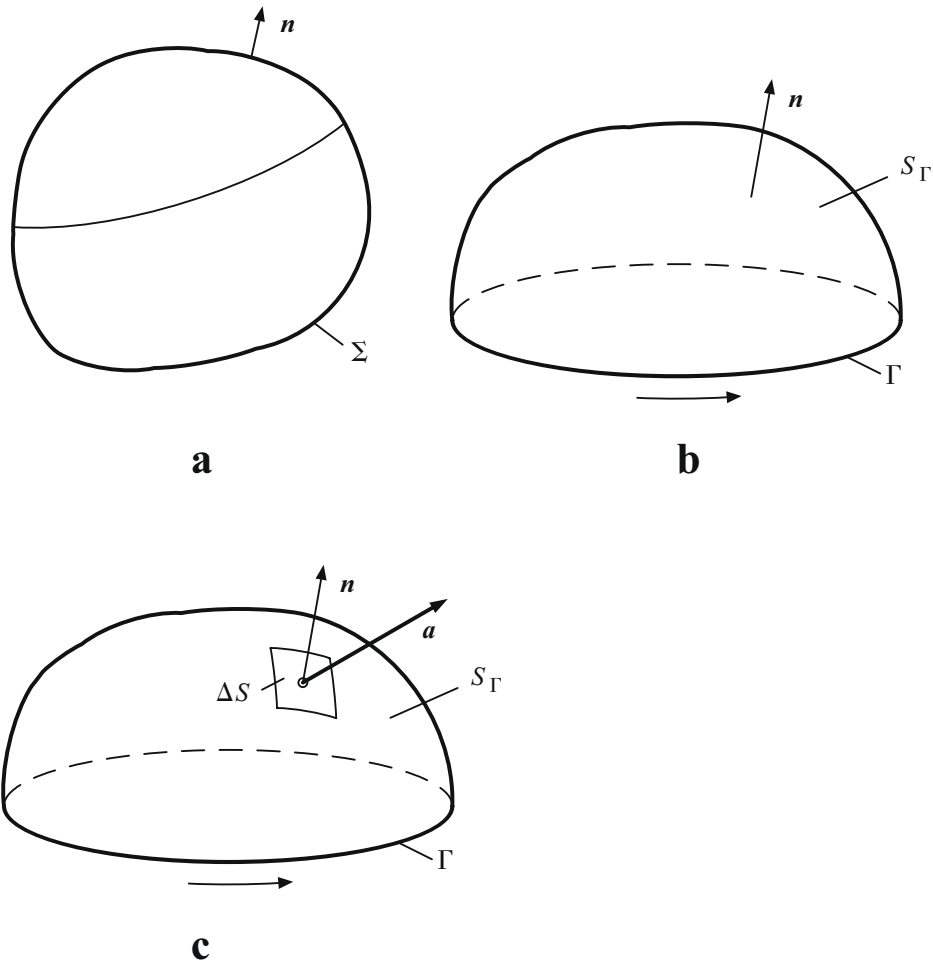
Fig. A.1.5. The flux of a vector through a surface: **a** – the positive normal in the case of a closed surface; **b** – the positive normal in the case of an open surface; **c** – explanation to the calculation of the flux through a surface.

A.1.2.5. The Gauss-Ostrogradski Theorem. The Divergence of a Vector.

Let us consider the surface-integral of the vector $\boldsymbol{a}$ through a closed surface situated in a field of vectors $\boldsymbol{a}(\boldsymbol{r})$:

$$\int_{\Sigma} \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{S} = \int_{\Sigma} a_n \cdot \mathrm{d}S. \tag{A.1.32}$$

The surface-integral of relation (A.1.32), where Σ is a closed surface, can be transformed into a volume integral.
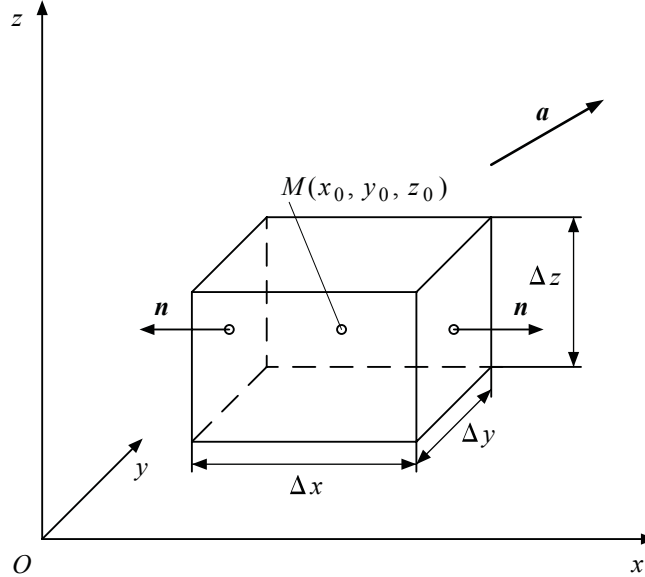
Fig. A.1.6. The flux of a vector field through a parallelepiped surface.

The transformation relation represents the GAUSS-OSTROGRADSKI *theorem (formula)*. Let us consider this transformation for a right infinitely small parallelepiped of surface $\Sigma_{\Delta v}$. A field of vectors $a = a(r)$ is considered. We shall calculate the surface-integral of the normal component of vector a over the surface $\Sigma_{\Delta v}$. The surface-integral will be calculated for the six faces of the parallelepiped with reference to Fig. A.1.6.

We shall denote by x_0, y_0, z_0 the co-ordinates of the centre of the parallelepiped. Firstly, we consider the pair of faces perpendicular to the Ox - axis indicated by the ordinal numbers 1 and 2. The quantity $a(x_0, y_0, z_0)$ represents the vector a at the centre of the parallelepiped. The flux of the vector a through the surface 2 is:

$$\Delta\Phi_2 = a_{2x}\,\Delta S = a_x\left(x_0 + \frac{1}{2}\Delta x, y_0, z_0\right)\Delta y\,\Delta z, \qquad\qquad \text{(A.1.33 a)}$$

where the quantity a_{2x} is the mean value (average) of the component along the Ox - axis of the vector a_n on the face with the ordinal number 2, or the value at its middle.

The flux of the vector a through the face 1 is:

$$\Delta\Phi_1 = -a_{1x}\,\Delta S = -a_x\left(x_0 - \frac{1}{2}\Delta x, y_0, z_0\right)\Delta y\,\Delta z, \qquad\qquad \text{(A.1.33 b)}$$

where the quantity a_{1x} is the mean (average) value of the component along the Ox - axis of the vector a_n on the face with the ordinal number 1, or the value at its middle.

The total flux through the faces 1 and 2 is:

$$\left[a_x\left(x_0 + \frac{1}{2}\Delta x, y_0, z_0 \right) - a_x\left(x_0 - \frac{1}{2}\Delta x, y_0, z_0 \right) \right]\Delta y\,\Delta z , \qquad \text{(A.1.33 c)}$$

and this sum, established with a precision up to small quantities of the third order, is equal to:

$$\Delta\Phi_1 + \Delta\Phi_2 = \left(\frac{\partial a_x}{\partial x} \right)_0 \Delta x\,\Delta y\,\Delta z , \qquad \text{(A.1.33 d)}$$

where the index 0 shows that the derivative is calculated at the centre of the parallelepiped.

For the pairs of faces perpendicular to the Oy and Oz axes, and indicated by the ordinal numbers 3, 4, and 5, 6, respectively, the following sums of fluxes are obtained:

$$\Delta\Phi_3 + \Delta\Phi_4 = \left(\frac{\partial a_x}{\partial y} \right)_0 \Delta x\,\Delta y\,\Delta z \qquad \text{(A.1.33 e)}$$

and

$$\Delta\Phi_5 + \Delta\Phi_6 = \left(\frac{\partial a_x}{\partial z} \right)_0 \Delta x\,\Delta y\,\Delta z . \qquad \text{(A.1.33 f)}$$

Summing up, side by side, the preceding expressions, the total flux is obtained:

$$\Delta\Phi = \int_{\Sigma_{\Delta v}} a_n\,\mathrm{d}S = \left(\frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} \right)\Delta x\,\Delta y\,\Delta z . \qquad \text{(A.1.34)}$$

The sum within parentheses is called the divergence of the vector $\boldsymbol{a}$ or the divergence of $\boldsymbol{a}$ and is denoted:

$$\operatorname{div}\boldsymbol{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} . \qquad \text{(A.1.35)}$$

The volume element will be denoted by:

$$\mathrm{d}v = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z . \qquad \text{(A.1.36)}$$

With this symbol, the expression of the flux through the surface of an infinitely small rectangular parallelepiped can be written:

$$\mathrm{d}\Phi = \operatorname{div}\boldsymbol{a}\,\mathrm{d}v . \qquad \text{(A.1.37)}$$

We shall consider a domain of volume V_Σ bounded by any closed surface Σ. This domain can be decomposed by three sets of parallel planes, perpendicular to each other, in a set of rectangular infinitely small parallelepipeds. The marginal volume elements neighbouring the surface Σ are not in general of parallelepiped form. However, by a very fine decomposition, it is possible to make the marginal elements to coincide with the required precision with the surface Σ. Calculating the flux of the vector $\boldsymbol{a}$ through the

surface of each parallelepiped inside the surface Σ and adding up, side by side, the obtained expressions, it follows:

$$\sum_{V_\Sigma} \mathrm{d}\,\Phi = \sum_{V_\Sigma} \operatorname{div} \boldsymbol{a}\,\mathrm{d}\,v = \int_{V_\Sigma} \operatorname{div} \boldsymbol{a}\,\mathrm{d}\,v, \tag{A.1.38}$$

the summation is extended over all volume elements, and namely over their surfaces.

In the sum $\displaystyle\sum_{V_\Sigma} \mathrm{d}\,\Phi$, the flux of vectors through each internal surface occurs twice, namely the first time in the calculation of the flux through the surface of the parallelepiped situated on one side of this face, and the second time, in the calculation of the flux through the surface of the parallelepiped situated on the other side of this face. Since the positive normal of the considered face belonging to the first parallelepiped is of opposite direction with regard to the positive normal of the same face belonging to the second parallelepiped, the fluxes through this surface are of opposite sign. Therefore, all the terms of the sum above, which refer to the internal faces, cancel each other and the sum reduces to the sum of fluxes of vectors $\boldsymbol{a}$ only through the marginal surfaces of the parallelepipeds that coincide with the elements of the surface Σ.

Hence:

$$\Phi = \int_\Sigma \boldsymbol{a} \cdot \mathrm{d}\,\boldsymbol{S} = \int_{V_\Sigma} \operatorname{div} \boldsymbol{a}\,\mathrm{d}\,v. \tag{A.1.39}$$

For the relation (A.1.39) to be valid, the vector function must be continuous and differentiable at all the points of the domain V_Σ.

The relation (A.1.39) expresses the Gauss-Ostrogradski theorem and can be enounced as follows: The flux of a vector $\boldsymbol{a}$ through any closed surface Σ is equal to the volume integral of the divergence of the same vector over the volume bounded by this closed surface.

If the surface Σ is so small that at all points of its inside the quantity $\operatorname{div} \boldsymbol{a}$ may be assumed as constant, then, in relation (A.1.39) the quantity $\operatorname{div} \boldsymbol{a}$ can be placed before the integral sign.

Hence:

$$\Delta\Phi = \operatorname{div} \boldsymbol{a}\,\Delta v, \tag{A.1.40}$$

where the quantity Δv represents the volume of the domain V_Σ. It results that:

$$\operatorname{div} \boldsymbol{a} = \lim_{\Delta v \to 0} \frac{\displaystyle\int_{\Sigma_{\Delta v}} \boldsymbol{a} \cdot \mathrm{d}\,\boldsymbol{S}}{\Delta v}, \tag{A.1.41}$$

and this expression can be considered as the definition of the divergence of a vector. From expression (A.1.41), it results that $\operatorname{div} \boldsymbol{a}$ does not depend on the choice of the system of co-ordinates.

The divergence of a vector can be expressed also in other systems of co-ordinates [12, tome I, p. 44].

A.1.2.6.   The Line-integral of a Vector along a Curve. Circulation.

Let us consider the line-integral around an open curve C or a closed curve Γ situated in a field of vectors $\boldsymbol{a}(\boldsymbol{r})$. The curve Γ will be decomposed into very small line elements $\Delta\boldsymbol{l}$ the direction of which coincide with the travelling sense considered as positive of the curve. We shall express the scalar product of each element $\mathrm{d}\boldsymbol{l}$ and the vector $\boldsymbol{a}$ at the corresponding point:

$$\boldsymbol{a} \cdot \mathrm{d}\boldsymbol{l} = a_l\,\mathrm{d}l. \tag{A.1.42}$$

The limit of the sum of the products of relation (A.1.42) along the curve is called the line-integral of the vector $\boldsymbol{a}$ along the curve C :

$$\int_C \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{l} = \int_C a_l\,\mathrm{d}l, \tag{A.1.43}$$

where a_l is the component of the vector $\boldsymbol{a}$ along the direction of $\mathrm{d}\boldsymbol{l}$.

The line-integral of the vector $\boldsymbol{a}$ along the curve C is referred to as *the circulation of vector $\boldsymbol{a}$ along that curve*. That curve may be an open curve or a closed one [52, p. 138], [54, p. 123].

A.1.2.7. The Stokes Theorem. The Curl of a Vector.

Let us consider the line-integral of the vector $\boldsymbol{a}$ around a closed curve situated in a field of vectors $\boldsymbol{a}(\boldsymbol{r})$:

$$\oint_\Gamma \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{l} = \int_\Gamma a_l\,\mathrm{d}l. \tag{A.1.44}$$

The line-integral of relation (A.1.44), where Γ is a closed curve, can be transformed into a surface-integral. The transformation relation represents the STOKES theorem.

Let us consider this transformation for an infinitely small rectangle. A field of vectors $\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{r})$ is considered. We shall calculate the line-integral of the vector $\boldsymbol{a}$ along the contour of a rectangle, perpendicular to the Oz - axis, hence in the plane xOy, like in Fig. A.1.7. We have to calculate the line-integral decomposed for the four sides of the rectangle with reference to Fig. A.1.7.

The travelling sense along the curve Γ has been chosen associated, according to the right-handed screw rule, with the positive direction of the Oz - axis, hence, in the figure, the counter-clockwise sense.

We shall denote by x_0, y_0, z_0 the co-ordinates of the centre of the rectangle.

The quantity $\boldsymbol{a}(x_0, y_0, z_0)$ represents the vector $\boldsymbol{a}$ at the centre of the rectangle. The line-integral of the vector $\boldsymbol{a}$ along the side 1 is:

$$\Delta C_1 = \boldsymbol{i}\,a_{1x} \cdot \Delta\boldsymbol{l} = a_x\left(x_0, y_0 - \frac{1}{2}\Delta y, z_0\right)\Delta x, \tag{A.1.45 a}$$
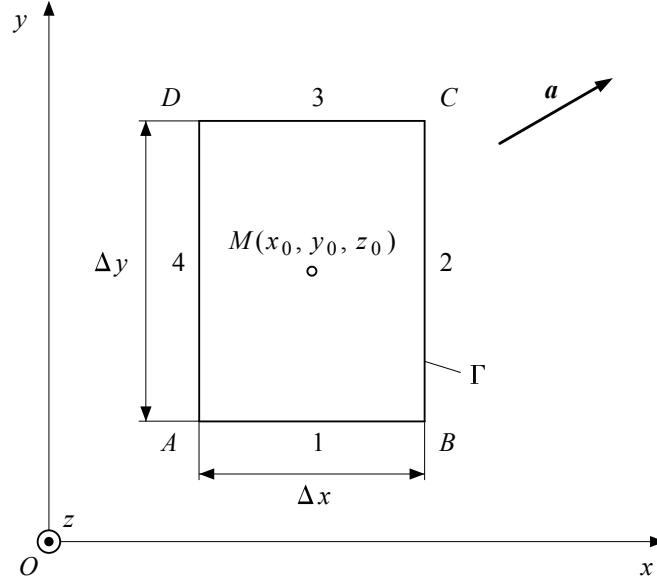
Fig. A.1.7. Circulation of a vector around a curve Γ of a rectangle.

where the quantity a_{1x} is the mean value of the component along the Ox - axis of the vector $\boldsymbol{a}$ on the side having the ordinal number 1, or the value at its middle.

The line-integral along the side 3 is:

$$\Delta C_3 = \boldsymbol{i} \, a_{3x} \cdot \Delta \boldsymbol{l} = a_x \left(x_0, y_0 + \frac{1}{2} \Delta y, z_0 \right) \Delta x, \qquad \text{(A.1.45 b)}$$

where the quantity a_{3x} is the mean value of the component along the Ox - axis of the vector $\boldsymbol{a}$ on the side having the ordinal number 3, or the the value at its middle.

The line-integral along the sides 1 and 3 is:

$$\Delta C_1 + \Delta C_3 = \left[a_x \left(x_0, y_0 - \frac{1}{2} \Delta y, z_0 \right) - a_x \left(x_0, y_0 + \frac{1}{2} \Delta y, z_0 \right) \right] \Delta x, \qquad \text{(A.1.45 c)}$$

and after expanding in a series, it results that the sum including the small quantities of the second order, is equal to:

$$\Delta C_1 + \Delta C_3 = - \left(\frac{\partial a_x}{\partial y} \right)_0 \Delta x \, \Delta y, \qquad \text{(A.1.45 d)}$$

where the index zero denotes that the derivative is calculated at the centre of the rectangle.

The line-integral along the sides 2 and 4 is:

$$\Delta C_2 + \Delta C_4 = \left(\frac{\partial a_y}{\partial x}\right)_0 \Delta y\, \Delta x. \qquad \text{(A.1.45 e)}$$

By summing up, side by side, the relations established above, the following expression of the integral is obtained:

$$\Delta C = \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)_0 \Delta x\, \Delta y. \qquad \text{(A.1.45 f)}$$

It can be added that the quantity dC is not a total differential of C. We consider the face having the normal $\boldsymbol{n}$ parallel with the Oz - axis. Denoting the area of the rectangle by $dS_z = \Delta x\, \Delta y$, i.e., after passing to limit, it follows:

$$\left(\boldsymbol{n}\, \| \, Oz\right) \quad \Delta C = \int_\Gamma \boldsymbol{a} \cdot d\boldsymbol{l} = \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right) dS_z. \qquad \text{(A.1.46 a)}$$

Analogously, for rectangles perpendicular to the other two axes, the following relations are obtained:

$$\left(\boldsymbol{n}\, \| \, Ox\right) \quad \Delta C = \int_\Gamma \boldsymbol{a} \cdot d\boldsymbol{l} = \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right) dS_x, \qquad \text{(A.1.46 b)}$$

$$\left(\boldsymbol{n}\, \| \, Oy\right) \quad \Delta C = \int_\Gamma \boldsymbol{a} \cdot d\boldsymbol{l} = \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right) dS_y. \qquad \text{(A.1.46 c)}$$

The combinations of the derivatives of the components of the vector $\boldsymbol{a}$ of expressions (A.1.46 a, b, c) are considered to be the components of a vector called curl or rotational of the vector $\boldsymbol{a}$ or curl of $\boldsymbol{a}$ and denoted $\mathrm{curl}\,\boldsymbol{a}$.

In this case, it follows:

$$\left(\mathrm{curl}\,\boldsymbol{a}\right)_x = \frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}, \qquad \text{(A.1.47 a)}$$

$$\left(\mathrm{curl}\,\boldsymbol{a}\right)_y = \frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}, \qquad \text{(A.1.47 b)}$$

$$\left(\mathrm{curl}\,\boldsymbol{a}\right)_z = \frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}. \qquad \text{(A.1.47 c)}$$

The vector $\mathrm{curl}\,\boldsymbol{a}$ can be put in the form of a symbolic determinant:

$$\operatorname{curl}\boldsymbol{a} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}. \qquad\qquad (A.1.48)$$

By the aid of the symbol (A.1.48), the expressions (A.1.46 a, b, c) can be written in the form:

$$\mathrm{d}C = \operatorname{curl}\boldsymbol{a}\cdot\mathrm{d}\boldsymbol{S}, \qquad\qquad (A.1.49)$$

where the unit vector $\boldsymbol{n}$ of the vector $\mathrm{d}\boldsymbol{S}$ is the positive normal to the surface $\mathrm{d}\boldsymbol{S}$, associated with the travelling direction along the contour of the surface, according to the right-handed screw rule. Making successively the vector $\boldsymbol{n}$ parallel with the axes Ox, Oy, Oz, the expressions (A.1.46 a, b, c) are obtained. Hence, for a rectangle in each of the three positions, relation (A.1.49) is valid.

A simply connected surface bounded by a closed curve is considered. Let us take three sets of parallel planes, each plane of a set is parallel to one plane of the three planes of the three-orthogonal rectilinear system of co-ordinates. The planes of each set are distanced by the steps Δx, Δy, Δz, respectively. If only the planes nearest the surface are considered, a surface constituted only by rectangles will be obtained. If the steps are very small and tend to zero, the surface constituted by rectangles tends to the given surface.

But for each rectangle of the surface formed by rectangles, relation (A.1.49) is valid. The relation (A.1.49) will be written for each rectangle, and these relations will be summed up, side by side. At each margin of two neighbouring rectangles, the common side will be travelled twice in opposite directions, like in Fig. A.1.8, so that the sum $\Sigma\,\boldsymbol{a}\cdot\mathrm{d}\boldsymbol{l}$ will contain the two terms of the form:

$$\int_{A}^{B} \boldsymbol{a}\cdot\mathrm{d}\boldsymbol{l} \quad\text{and}\quad \int_{B}^{A} \boldsymbol{a}\cdot\mathrm{d}\boldsymbol{l}, \qquad\qquad (A.1.50)$$

the sum of which is zero. Hence, the sum $\Sigma\,\boldsymbol{a}\cdot\mathrm{d}\boldsymbol{l}$ becomes equal to the sum of terms corresponding only to the external boundary of the surface, hence to the integral of the vector $\boldsymbol{a}$ along the contour of the surface. It follows:

$$C = \oint_{\Gamma} \boldsymbol{a}\cdot\mathrm{d}\boldsymbol{l} = \int_{S_{\Gamma}} \operatorname{curl}\boldsymbol{a}\cdot\mathrm{d}\boldsymbol{S}. \qquad\qquad (A.1.51)$$

For the relation (A.1.51) to be valid, the vector function $\boldsymbol{a}$ must be continuous and differentiable at all the points of the surface S. Relation (A.1.51) expresses the theorem of transformation of a line-integral into a surface-integral, called Stokes theorem that can be enounced as follows: The circulation of an arbitrary vector $\boldsymbol{a}$ along a closed curve Γ is equal to the flux of the curl of this vector through the surface S_{Γ} bounded by the curve Γ.

From relation (A.1.51), it follows that in the case in which S_{Γ} is a closed surface:

Fig. A.1.8. The travelling of a vector
along the elements of an open
surface bounded by a curve of any form.
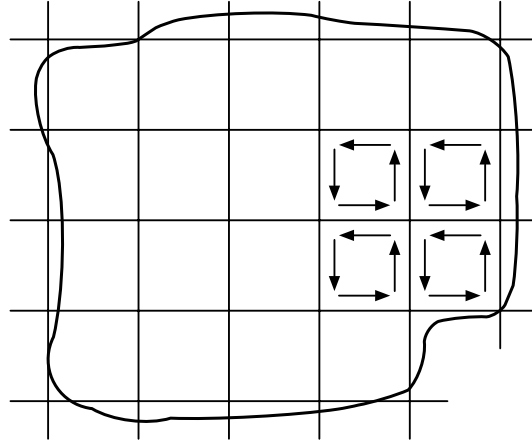
$$\int_{S_\Gamma} \operatorname{curl} \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{S} = 0,$$ (A.1.52)

because if S_Γ is a closed surface, the contour reduces to a point and:

$$\oint_\Gamma \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{l} = 0.$$ (A.1.53)

In the case in which S_Γ is a surface small enough for to be considered plane, and for all its points the quantity $\operatorname{curl}\boldsymbol{a}$ to remain constant, the quantity $\operatorname{curl}\boldsymbol{a}$ may be brought out of the integral sign. Hence:

$$\Delta C = \operatorname{curl}\boldsymbol{a} \cdot \Delta\boldsymbol{S},$$ (A.1.54)

where ΔS represents the area of the surface S_Γ.

In this case, relation (A.1.54) can be written:

$$\Delta C = \left(\operatorname{curl}\boldsymbol{a}\right)_n \Delta S,$$ (A.1.55)

where the quantity $\left(\operatorname{curl}\boldsymbol{a}\right)_n$ is the component of the vector $\boldsymbol{a}$ along the positive normal to the surface ΔS. Therefore:

$$\left(\operatorname{curl}\boldsymbol{a}\right)_n = \lim_{\Delta S \to 0} \frac{\oint_\Gamma \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{l}}{S_\Gamma}.$$ (A.1.56)

From expression (A.1.56), it results that the vector $\operatorname{curl}\boldsymbol{a}$ does not depend on the choice of the system of co-ordinates, hence it is an invariant with respect to the system of co-ordinates.

The curl of a vector can also be expressed in various types of systems of co-ordinates [12, tome I, p. 44].

A.1.2.8.   Nabla Operator. Hamilton Operator.

Previously it has been established that the gradient of a scalar function φ is a vector oriented along the direction of the maximal increase of the function φ, with the magnitude equal to the derivative of the function along this direction. Among the various symbols of the gradient, one of the most utilized is $\nabla \varphi$.

The sign ∇ is read *nabla* or also *del* and it denotes a differential operator. With this symbol, it is possible to write:

$$\nabla \varphi = \boldsymbol{i} \frac{\partial \varphi}{\partial x} + \boldsymbol{j} \frac{\partial \varphi}{\partial y} + \boldsymbol{k} \frac{\partial \varphi}{\partial z}. \tag{A.1.57}$$

From this expression, it can be seen that the operator ∇ can be considered as a differential operator:

$$\nabla = \boldsymbol{i} \frac{\partial}{\partial x} + \boldsymbol{j} \frac{\partial}{\partial y} + \boldsymbol{k} \frac{\partial}{\partial z}, \tag{A.1.58}$$

that, being applied to the scalar function φ, gives the quantity $\operatorname{grad}\varphi$. This operator can be considered as a symbolic vector and it is also called *the Hamilton operator*.

A.1.2.9.   The Derivative of a Vector along a Direction

Let us consider an arbitrary point $M(\boldsymbol{r})$ in a field of vectors $\boldsymbol{a}(\boldsymbol{r})$ like in Fig. A.1.9. Let $M(\boldsymbol{r} + \Delta \boldsymbol{s})$ be another point in the same vector field. In particular cases, the arbitrary point M can be considered on a curve the tangent of which at the point M has the direction of the unit vector $\boldsymbol{s}$, but it is not a condition. The unit vector $\boldsymbol{s}$ can be written:

$$\boldsymbol{s} = \boldsymbol{i}\, s_x + \boldsymbol{j}\, s_y + \boldsymbol{k}\, s_z . \tag{A.1.59 a}$$
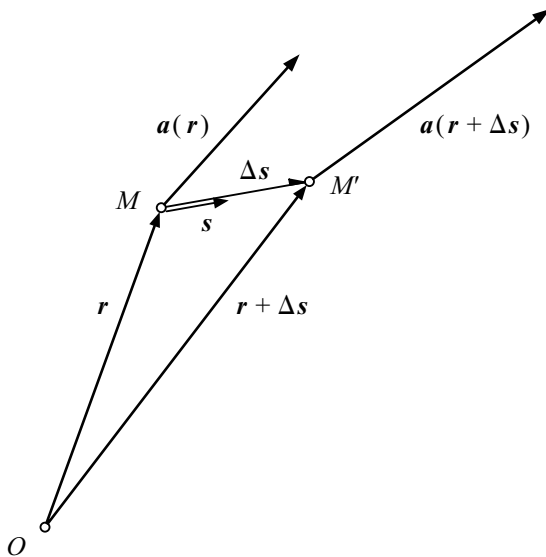
Fig. A.1.9. Explanation to the calculation of the derivative of a vector along a given direction.

The limit of the ratio

$$\frac{a(r + \Delta s) - a(r)}{\Delta s}$$

(A.1.59 b)

as $\Delta s \to 0$, is termed the derivative of the vector a along the direction of the unit vector s at the considered point M and is denoted:

$$\frac{\partial a}{\partial s} = \lim_{\Delta s \to 0} \frac{a(r + \Delta s) - a(r)}{\Delta s}.$$

(A.1.60)

Introducing the co-ordinates, we obtain:

$$\frac{\partial a}{\partial s} = \lim_{\Delta s \to 0} \frac{a(x + \Delta x, y + \Delta y, z + \Delta z) - a(x, y, z)}{\Delta s}.$$

(A.1.61 a)

By expanding in a series, and retaining only the small quantities of the first order, we obtain:

$$\frac{\partial a}{\partial s} = \frac{\partial a}{\partial x} \cdot \frac{dx}{ds} + \frac{\partial a}{\partial y} \cdot \frac{dy}{ds} + \frac{\partial a}{\partial z} \cdot \frac{dz}{ds},$$

(A.1.61 b)

where

$$\frac{dx}{ds} = s_x = s \cdot i; \quad \frac{dy}{ds} = s_y = s \cdot j; \quad \frac{dz}{ds} = s_z = s \cdot k.$$

(A.1.61 c)

It follows that:

$$\frac{\partial a}{\partial s} = (s \cdot i)\frac{\partial a}{\partial x} + (s \cdot j)\frac{\partial a}{\partial y} + (s \cdot k)\frac{\partial a}{\partial z}.$$

(A.1.62)

Since we have $|s| = 1$, the last relation is the scalar product $(s \cdot \nabla)a$ of the following scalar operator:

$$s \cdot \nabla = (s \cdot i)\frac{\partial}{\partial x} + (s \cdot j)\frac{\partial}{\partial y} + (s \cdot k)\frac{\partial}{\partial z}$$

(A.1.63 a)

and the vector a, and gives just $\dfrac{\partial a}{\partial s}$, so that the symbol $(s \cdot \nabla)a$ appears as justified.

Now, the more general case is examined, in which in expression $s \cdot \nabla$ instead of the unit vector s, any vector v will be considered. Hence, the scalar operator obtained by performing the scalar product of the arbitrary vector $v = s\,v$ and the symbolic vector ∇, is:

$$\boldsymbol{v} \cdot \nabla = v(\boldsymbol{s} \cdot \boldsymbol{i})\frac{\partial}{\partial x} + v(\boldsymbol{s} \cdot \boldsymbol{j})\frac{\partial}{\partial y} + v(\boldsymbol{s} \cdot \boldsymbol{k})\frac{\partial}{\partial z},$$
$$\boldsymbol{v} = \boldsymbol{s}\, v.$$

(A.1.63 b)

Therefore:

$$(\boldsymbol{v} \cdot \nabla)\boldsymbol{a} = v\left[(\boldsymbol{s} \cdot \boldsymbol{i})\frac{\partial \boldsymbol{a}}{\partial x} + (\boldsymbol{s} \cdot \boldsymbol{j})\frac{\partial \boldsymbol{a}}{\partial y} + (\boldsymbol{s} \cdot \boldsymbol{k})\frac{\partial \boldsymbol{a}}{\partial z} \right],$$

(A.1.63 c)

$$v_x = \boldsymbol{v} \cdot \boldsymbol{i} = v(\boldsymbol{s} \cdot \boldsymbol{i}); \quad v_y = \boldsymbol{v} \cdot \boldsymbol{j} = v(\boldsymbol{s} \cdot \boldsymbol{j}); \quad v_z = \boldsymbol{v} \cdot \boldsymbol{k} = v(\boldsymbol{s} \cdot \boldsymbol{k}).$$

(A.1.63 d)

Hence:

$$(\boldsymbol{v} \cdot \nabla)\boldsymbol{a} = v_x \frac{\partial \boldsymbol{a}}{\partial x} + v_y \frac{\partial \boldsymbol{a}}{\partial y} + v_z \frac{\partial \boldsymbol{a}}{\partial z}$$

(A.1.64)

or, in another compact form:

$$(\boldsymbol{v} \cdot \nabla)\boldsymbol{a} = v\frac{\partial \boldsymbol{a}}{\partial s},$$
$$\boldsymbol{v} = \boldsymbol{s}\, v.$$

(A.1.64 a)

In the case in which $v = 1$, the operation above represents the derivative of the vector $\boldsymbol{a}$ with respect to the direction of the vector $\boldsymbol{v}$. Hence, the expression $(\boldsymbol{v} \cdot \nabla)\boldsymbol{a}$ represents the derivative of the vector $\boldsymbol{a}$ with respect to the direction of the vector $\boldsymbol{v}$ multiplied by the modulus of the last vector.

A.1.2.10. Expressing the Divergence and the Curl of a Vector by Means of the Nabla Operator

The divergence can also been expressed, formally, as the scalar product of the symbolic vector ∇ by the vector $\boldsymbol{a}$:

$$\boldsymbol{a} = \boldsymbol{i}\, a_x + \boldsymbol{j}\, a_y + \boldsymbol{k}\, a_z .$$

(A.1.65 a)

By performing this product according the formula of the scalar product of two vectors

$$\boldsymbol{b} \cdot \boldsymbol{a} = b_x\, a_x + b_y\, a_y + b_z\, a_z ,$$

(A.1.65 b)

and putting

$$b_x = \frac{\partial}{\partial x}; \quad b_y = \frac{\partial}{\partial y}; \quad b_z = \frac{\partial}{\partial z},$$

(A.1.65 c)

it follows:

$$\nabla \cdot \boldsymbol{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}. \tag{A.1.66}$$

Analogously, the curl of a vector can be considered as the vector product of the symbolic vector ∇ by the vector $\boldsymbol{a}$:

$$\operatorname{curl}\boldsymbol{a} = \nabla \times \boldsymbol{a} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}, \tag{A.1.67 a}$$

$$\operatorname{curl}\boldsymbol{a} = \nabla \times \boldsymbol{a} = \boldsymbol{i}\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right) + \boldsymbol{j}\left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right) + \boldsymbol{k}\left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right). \tag{A.1.67 b}$$

A.1.2.11.   Differential Operations by the Nabla Operator

The differential operator:

$$\nabla = \boldsymbol{i}\frac{\partial}{\partial x} + \boldsymbol{j}\frac{\partial}{\partial y} + \boldsymbol{k}\frac{\partial}{\partial z} \tag{A.1.68}$$

is constituted by three partial derivatives relatively to the axes of co-ordinates. In the case in which it is applied to a product, the operation is the same as in the case of the calculation of the derivative of a product. The first term will be calculated considering the first factor as variable and the other factors as constants. The second term will be calculated considering the second factor as variable and the other factors as constants, etc. In the case of the application of the operator ∇, always when confusions could appear, all the vectors that will be considered momentary to be constant, will be denoted by the index c (constant). The computing method by the operator ∇ can be explained as follows. All vectors occurring in the expression under consideration, excepting one considered as variable are supposed as being constant. Then, the expression will be transformed, so that, all constant vectors may be arranged before the operator ∇, and the one considered as variable, after this operator. If from the point of view of Vector Algebra two variants are valid, the variant giving a result different from zero will be kept.

Sometimes, the derivation of the relations by using the operator ∇, due to its formal character, could be considered as being rather a mnemonic rule than a rigorous proof. For this reason, the usage of the symbolic operator requires certain precaution.

Derivatives of Products

1. $\operatorname{grad}(\varphi\psi) = \nabla(\varphi\,\psi) = \nabla(\varphi_c\,\psi) + \nabla(\varphi\psi_c) = \varphi_c\,\nabla\psi + \psi_c\,\nabla\varphi;$

    $\operatorname{grad}(\varphi\psi) = \varphi\operatorname{grad}\psi + \psi\operatorname{grad}\varphi.$ (A.1.69)

2. $\operatorname{div}(\varphi\boldsymbol{a}) = \nabla\cdot(\varphi\boldsymbol{a}) = \nabla\cdot(\varphi_c\,\boldsymbol{a}) + \nabla(\varphi\boldsymbol{a}_c) = \varphi_c\,\nabla\cdot\boldsymbol{a} + \boldsymbol{a}_c\cdot\nabla\varphi;$

    $\operatorname{div}(\varphi\boldsymbol{a}) = \varphi\operatorname{div}\boldsymbol{a} + \boldsymbol{a}\cdot\operatorname{grad}\varphi.$ (A.1.70)

3. $\operatorname{curl}(\varphi\boldsymbol{a}) = \nabla\times(\varphi\boldsymbol{a}) = \nabla\times(\varphi_c\,\boldsymbol{a}) + \nabla\times(\varphi\boldsymbol{a}_c) =$

    $\qquad = \nabla\times(\varphi_c\,\boldsymbol{a}) - \varphi\,\boldsymbol{a}_c\times\nabla =$

    $\qquad = \varphi_c\,\nabla\times\boldsymbol{a} - \boldsymbol{a}_c\times(\nabla\varphi);$

    $\operatorname{curl}(\varphi\boldsymbol{a}) = \varphi\operatorname{curl}\boldsymbol{a} - \boldsymbol{a}\times\operatorname{grad}\varphi.$ (A.1.71)

4. $\operatorname{div}(\boldsymbol{a}\times\boldsymbol{b}) = \nabla\cdot(\boldsymbol{a}\times\boldsymbol{b}) = \nabla\cdot(\boldsymbol{a}_c\times\boldsymbol{b}) + \nabla\cdot(\boldsymbol{a}\times\boldsymbol{b}_c) =$

    $\qquad = \nabla\cdot(\boldsymbol{a}_c\times\boldsymbol{b}) + \boldsymbol{b}_c\cdot(\nabla\times\boldsymbol{a}) =$

    $\qquad = -\nabla\cdot(\boldsymbol{b}\times\boldsymbol{a}_c) + \boldsymbol{b}_c\cdot(\nabla\times\boldsymbol{a}) = -\boldsymbol{a}_c\cdot(\nabla\times\boldsymbol{b}) + \boldsymbol{b}_c\cdot(\nabla\times\boldsymbol{a});$

    $\operatorname{div}(\boldsymbol{a}\times\boldsymbol{b}) = \boldsymbol{b}\cdot\operatorname{curl}\boldsymbol{a} - \boldsymbol{a}\cdot\operatorname{curl}\boldsymbol{b}.$ (A.1.72)

5. $\operatorname{grad}(\boldsymbol{a}\cdot\boldsymbol{b}) = \nabla(\boldsymbol{a}\cdot\boldsymbol{b}) = \nabla(\boldsymbol{a}_c\cdot\boldsymbol{b}) + \nabla(\boldsymbol{a}\cdot\boldsymbol{b}_c);$ (A.1.73 a)

but

$\boldsymbol{A}\times(\boldsymbol{B}\times\boldsymbol{C}) = \boldsymbol{B}(\boldsymbol{A}\cdot\boldsymbol{C}) - \boldsymbol{C}(\boldsymbol{A}\cdot\boldsymbol{B});$

$\boldsymbol{C}(\boldsymbol{A}\cdot\boldsymbol{B}) = (\boldsymbol{A}\cdot\boldsymbol{C})\boldsymbol{B} + \boldsymbol{A}\times(\boldsymbol{C}\times\boldsymbol{B}).$ (A.1.73 b)

By substituting

$\boldsymbol{C} = \nabla;\quad \boldsymbol{A} = \boldsymbol{a}_c;\quad \boldsymbol{B} = \boldsymbol{b};$ (A.1.73 c)

it follows

$\nabla(\boldsymbol{a}_c\cdot\boldsymbol{b}) = (\boldsymbol{a}_c\cdot\nabla)\boldsymbol{b} + \boldsymbol{a}_c\times(\nabla\times\boldsymbol{b}).$ (A.1.73 d)

Analogously:

$\nabla(\boldsymbol{a}\cdot\boldsymbol{b}_c) = (\boldsymbol{b}_c\cdot\nabla)\boldsymbol{a} + \boldsymbol{b}_c\times(\nabla\times\boldsymbol{a}).$ (A.1.73 e)

Therefore:

$\nabla(\boldsymbol{a}\cdot\boldsymbol{b}) = (\boldsymbol{a}\cdot\nabla)\boldsymbol{b} + (\boldsymbol{b}\cdot\nabla)\boldsymbol{a} + \boldsymbol{a}\times(\nabla\times\boldsymbol{b}) + \boldsymbol{b}\times(\nabla\times\boldsymbol{a});$

$\operatorname{grad}(\boldsymbol{a}\cdot\boldsymbol{b}) = (\boldsymbol{a}\cdot\nabla)\boldsymbol{b} + (\boldsymbol{b}\cdot\nabla)\boldsymbol{a} + \boldsymbol{a}\times\operatorname{curl}\boldsymbol{b} + \boldsymbol{b}\times\operatorname{curl}\boldsymbol{a}.$ (A.1.74)

6.　　$\operatorname{curl}(a \times b) = \nabla \times (a \times b) = \nabla \times (a_c \times b) + \nabla(a \times b_c),$ 　　　　(A.1.75 a)

but

$$A \times (B \times C) = B(A \cdot C) - C(A \cdot B) = B(A \cdot C) - (B \cdot A)C. \qquad \text{(A.1.75 b)}$$

By substituting

$$A = \nabla; \qquad B = a_c; \qquad C = b; \qquad\qquad\qquad \text{(A.1.75 c)}$$

it follows:

$$\nabla \times (a_c \times b) = a_c(\nabla \cdot b) - (a_c \cdot \nabla)b. \qquad\qquad \text{(A.1.75 d)}$$

Analogously:

$$\nabla \times (a \times b_c) = (b_c \cdot \nabla)a - b_c(\nabla \cdot a). \qquad\qquad \text{(A.1.75 e)}$$

Therefore:

$$\nabla \times (a \times b) = a(\nabla \cdot b) + (b \cdot \nabla)a - (a \cdot \nabla)b - b(\nabla \cdot a);$$

$$\operatorname{curl}(a \times b) = a \operatorname{div} b - b \operatorname{div} a + (b \cdot \nabla)a - (a \cdot \nabla)b. \qquad \text{(A.1.76)}$$

　The utilization of the nabla operator simplifies much the calculation of the *derivatives of the first*, *second* and *higher orders* of scalar and vector quantities. In this way, important relations of higher orders can be obtained as follows.

　Analogously to the relations of Vector Algebra as $b \cdot (b\varphi) = b^2\varphi$, for $b = \nabla$, the following relations can be obtained:

$$\nabla \cdot (\nabla\varphi) = \operatorname{div} \operatorname{grad} \varphi = \nabla^2\varphi; \qquad\qquad\qquad \text{(A.1.77 a)}$$

but

$$\nabla^2 = \frac{\partial}{\partial x} \cdot \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \cdot \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \cdot \frac{\partial}{\partial z}. \qquad\qquad \text{(A.1.77 b)}$$

Therefore:

$$\operatorname{div} \operatorname{grad} \varphi = \nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2}. \qquad\qquad\qquad \text{(A.1.78)}$$

　By applying relation (A.1.78) for $\varphi = \dfrac{1}{r}$, where r is the modulus of the position vector, it follows:

1.　　$\nabla^2 \dfrac{1}{r} = 0;$

　　　$r = \sqrt{x^2 + y^2 + z^2}$ 　　　　　　　　　　　　　(A.1.79)

and

$$\nabla r = \frac{\boldsymbol{r}}{r}. \tag{A.1.80}$$

2.

$$\operatorname{grad} \operatorname{div} \boldsymbol{a} = \nabla(\nabla \cdot \boldsymbol{a}) = \left(\boldsymbol{i} \frac{\partial}{\partial x} + \boldsymbol{j} \frac{\partial}{\partial y} + \boldsymbol{k} \frac{\partial}{\partial z} \right) \left(\frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} \right). \tag{A.1.81}$$

3.

$$\nabla^2 \boldsymbol{a} = (\nabla \cdot \nabla)\boldsymbol{a} = \frac{\partial^2 \boldsymbol{a}}{\partial x^2} + \frac{\partial^2 \boldsymbol{a}}{\partial y^2} + \frac{\partial^2 \boldsymbol{a}}{\partial z^2}. \tag{A.1.82}$$

4.

$$(\nabla \varphi)^2 = (\operatorname{grad} \varphi)^2 = \left(\frac{\partial \varphi}{\partial x} \right)^2 + \left(\frac{\partial \varphi}{\partial y} \right)^2 + \left(\frac{\partial \varphi}{\partial z} \right)^2. \tag{A.1.83}$$

Analogously to relations of Vector Algebra:

5. $\boldsymbol{b} \times (\boldsymbol{b}\,\varphi) = 0, \quad \boldsymbol{b} \cdot (\boldsymbol{b} \times \boldsymbol{a}) = 0, \quad \boldsymbol{b} \times (\boldsymbol{b} \times \boldsymbol{a}) = \boldsymbol{b}(\boldsymbol{b} \cdot \boldsymbol{a}) - (\boldsymbol{b} \cdot \boldsymbol{b})\boldsymbol{a}$

and for $\boldsymbol{b} = \nabla$, the following relations are obtained:

$$\nabla \times \nabla \varphi = \operatorname{curl} \operatorname{grad} \varphi = 0; \tag{A.1.84}$$

$$\nabla \cdot (\nabla \times \boldsymbol{a}) = \operatorname{div} \operatorname{curl} \boldsymbol{a} = 0; \tag{A.1.85}$$

$$\nabla \times (\nabla \times \boldsymbol{a}) = \nabla(\nabla \cdot \boldsymbol{a}) - (\nabla \cdot \nabla)\boldsymbol{a} = \operatorname{grad} \operatorname{div} \boldsymbol{a} - \nabla^2 \boldsymbol{a}. \tag{A.1.86}$$

All relations above can be verified by direct calculation, using the co-ordinates of a three-orthogonal rectilinear system of reference.

A.1.2.12.   Integral Transformations Using the Nabla Operator

1° *Scalar integral relations*. GREEN *theorem.* In relation (A.1.39) that expresses the Gauss-Ostrogradski theorem:

$$\int\limits_{V_\Sigma} \operatorname{div} \boldsymbol{a}\, \mathrm{d}v = \int\limits_\Sigma \boldsymbol{a} \cdot \mathrm{d}\boldsymbol{S},$$

the following substitution will be performed:

$$\boldsymbol{a} = \psi \operatorname{grad} \varphi, \tag{A.1.87 a}$$

where φ and ψ are two arbitrary scalars and:

$$\boldsymbol{b} = \nabla \varphi. \tag{A.1.87 b}$$

It follows:

$$\operatorname{div} \boldsymbol{a} = \nabla \cdot [\psi (\nabla \varphi)] = \nabla \cdot (\psi_c\, \boldsymbol{b}) + \nabla \cdot (\psi\, \boldsymbol{b}_c) = \psi_c \operatorname{div} \boldsymbol{b} + \boldsymbol{b}_c \cdot \operatorname{grad} \psi. \tag{A.1.87 c}$$

Hence:

$$\operatorname{div}\boldsymbol{a} = \psi\operatorname{div}\operatorname{grad}\varphi + \left(\operatorname{grad}\psi\right)\left(\operatorname{grad}\varphi\right). \tag{A.1.87 d}$$

Therefore:

$$\int_{V_\Sigma}\operatorname{div}\boldsymbol{a}\,\mathrm{d}v = \int_{V_\Sigma}\left[\psi\nabla^2\varphi + (\nabla\psi)(\nabla\varphi)\right]\,\mathrm{d}v = \int_\Sigma \boldsymbol{a}\cdot\mathrm{d}\boldsymbol{S} = \int_\Sigma\psi\left(\operatorname{grad}\varphi\right)\cdot\boldsymbol{n}\,\mathrm{d}S. \tag{A.1.87 e}$$

But:

$$\boldsymbol{n}\cdot\operatorname{grad}\varphi = \frac{\partial\varphi}{\partial n}. \tag{A.1.87 f}$$

It follows:

$$\int_{V_\Sigma}\left[\psi\nabla^2\varphi + (\nabla\psi)(\nabla\varphi)\right]\mathrm{d}v = \int_\Sigma\psi\frac{\partial\varphi}{\partial n}\,\mathrm{d}S. \tag{A.1.88}$$

Analogously, replacing the quantities φ and ψ by each other, it follows:

$$\int_{V_\Sigma}\left[\varphi\nabla^2\psi + (\nabla\varphi)(\nabla\psi)\right]\mathrm{d}v = \int_\Sigma\varphi\frac{\partial\psi}{\partial n}\,\mathrm{d}S. \tag{A.1.89}$$

By subtracting the two last relations, it follows:

$$\int_{V_\Sigma}\left[\psi\nabla^2\varphi - \varphi\nabla^2\psi\right]\mathrm{d}v = \int_\Sigma\left(\psi\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial\psi}{\partial n}\right)\mathrm{d}S. \tag{A.1.90}$$

The relations (A.1.88), (A.1.89) and (A.1.90) are the three forms of the Green theorem.

2° *Vector integral relations*. The following volume integral will be calculated:

$$\int_{V_\Sigma}\operatorname{curl}\boldsymbol{a}\,\mathrm{d}v. \tag{A.1.91 a}$$

For this purpose, this integral will be multiplied by any constant vector $\boldsymbol{c}$:

$$\boldsymbol{c}\cdot\int_\Sigma\operatorname{curl}\boldsymbol{a}\,\mathrm{d}v = \int_{V_\Sigma}\boldsymbol{c}\cdot\operatorname{curl}\boldsymbol{a}\,\mathrm{d}v. \tag{A.1.91 b}$$

According to formula (A.1.72), taking into account that $\boldsymbol{c} = \text{const}$, it follows:

$$\operatorname{div}\left(\boldsymbol{a}\times\boldsymbol{c}\right) = \boldsymbol{c}\cdot\operatorname{curl}\boldsymbol{a}. \tag{A.1.91 c}$$

The two last equations yield:

$$\int_{V_\Sigma} \mathrm{div}(\boldsymbol{a} \times \boldsymbol{c})\,\mathrm{d}v = \int_{V_\Sigma} \boldsymbol{c} \cdot \mathrm{curl}\,\boldsymbol{a}\,\mathrm{d}v. \tag{A.1.91 d}$$

The last relation, taking into account relation (A.1.39), becomes:

$$\int_\Sigma (\boldsymbol{a} \times \boldsymbol{c}) \cdot \mathrm{d}\boldsymbol{S} = \int_{V_\Sigma} \boldsymbol{c} \cdot \mathrm{curl}\,\boldsymbol{a}\,\mathrm{d}v, \tag{A.1.91 e}$$

and hence:

$$\int_\Sigma (\boldsymbol{a} \times \boldsymbol{c}) \cdot \boldsymbol{n}\,\mathrm{d}S = \int_{V_\Sigma} \boldsymbol{c} \cdot \mathrm{curl}\,\boldsymbol{a}\,\mathrm{d}v. \tag{A.1.91 f}$$

The last relation, taking into account relation (A.1.9 a), becomes:

$$\int_\Sigma \boldsymbol{c} \cdot (\boldsymbol{n} \times \boldsymbol{a})\,\mathrm{d}S = \int_{V_\Sigma} \boldsymbol{c} \cdot \mathrm{curl}\,\boldsymbol{a}\,\mathrm{d}v.$$

Taking into account that the last relation holds whatever the constant vector $\boldsymbol{c}$ would be, it follows:

$$\int_\Sigma (\boldsymbol{n} \times \boldsymbol{a})\,\mathrm{d}S = \int_{V_\Sigma} \mathrm{curl}\,\boldsymbol{a}\,\mathrm{d}v. \tag{A.1.92}$$

A.1.2.13.   Substantial Derivative of a Scalar with Respect to Time

In a reference frame supposed to be at rest, let us consider any point $M(\boldsymbol{r})$ moving at velocity $\boldsymbol{v}_r$ relatively to this frame, and the scalar function $f(\boldsymbol{r}, t)$ that is depending on the point M and that is varying with time.

In various cases, it is necessary to calculate the derivative of the form:

$$\frac{\mathrm{d}}{\mathrm{d}t} f(M, t), \tag{A.1.93}$$

in another reference frame that is moving at the same velocity $\boldsymbol{v}_r$ as the point M and hence with respect to which the point is at rest. In cases of interest, the point M is fixed to the surrounding substance that is moving together with the point M. For this reason the considered derivative is referred to as *substantial derivative*.

The variation with time of the function f in the moving reference frame is determined by two causes: **a**. The modification of the position of the point M in the reference frame at rest; **b**. The variation with time of the function at each point of the reference frame at rest.

The derivative is obtained from the expression:

$$\frac{\mathrm{d}}{\mathrm{d}t} f(M, t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}\left[f(x + \Delta x, y + \Delta y, z + \Delta z, t + \Delta t) - f(x, y, z, t)\right]. \tag{A.1.94}$$

It follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}f(x,y,z,t)=\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left[\frac{\mathrm{d}f}{\mathrm{d}x}\Delta x+\frac{\mathrm{d}f}{\mathrm{d}y}\Delta y+\frac{\mathrm{d}f}{\mathrm{d}z}\Delta z+\frac{\mathrm{d}f}{\mathrm{d}t}\Delta t\right] \qquad (A.1.95)$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}f(x,y,z,t)=\frac{\mathrm{d}f}{\mathrm{d}x}v_{rx}+\frac{\mathrm{d}f}{\mathrm{d}y}v_{ry}+\frac{\mathrm{d}f}{\mathrm{d}z}v_{rz}+\frac{\mathrm{d}f}{\mathrm{d}t}, \qquad (A.1.96)$$

where the components of the velocity at which the point $M(x,y,z)$ is moving relatively to the reference frame at rest have been denoted v_{rx}, v_{ry}, v_{rz}. Therefore:

$$\frac{\mathrm{d}}{\mathrm{d}t}f(\boldsymbol{r},t)=\frac{\mathrm{d}f}{\mathrm{d}t}+\boldsymbol{v}_r\cdot\mathrm{grad}\,f. \qquad (A.1.97)$$

A.1.2.14. Substantial Derivative of a Volume Integral of a Scalar Function with Respect to Time

In various cases, it is necessary to calculate the derivative of the form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_\Sigma}f\,\mathrm{d}v, \qquad (A.1.98)$$

where $f=f(M,t)=f(\boldsymbol{r},t)$ is a scalar function with space, i.e., depends on the moving point M. In the most general case, when both the function f and the volume V_Σ are varying with time, the derivative of the given form is referred to as substantial derivative with respect to time. The reason of the usage of the term substantial has been mentioned above.

As in the preceding Sub-section, the derivative can be decomposed as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_\Sigma}f\,\mathrm{d}v=\left[\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_\Sigma}f\,\mathrm{d}v\right]_{V_\Sigma=\mathrm{const}}+\left[\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_\Sigma}f\,\mathrm{d}v\right]_{f=\mathrm{const}}. \qquad (A.1.99)$$

By differentiating the first term of the right-hand side, it follows:

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_\Sigma}f\,\mathrm{d}v\right]_{V_\Sigma=\mathrm{const}}=\int_{V_\Sigma}\left(\frac{\partial f}{\partial t}+\frac{\partial f}{\partial x}\cdot\frac{\mathrm{d}x}{\mathrm{d}t}+\frac{\partial f}{\partial y}\cdot\frac{\mathrm{d}y}{\mathrm{d}t}+\frac{\partial f}{\partial z}\cdot\frac{\mathrm{d}z}{\mathrm{d}t}\right)\mathrm{d}v=$$

$$(A.1.100\ a)$$

$$=\int_{V_\Sigma}\left(\frac{\partial f}{\partial t}+\boldsymbol{v}_r\cdot\mathrm{grad}\,f\right)\mathrm{d}v,$$

where $\boldsymbol{v}_r$ represents the velocity of both the moving point and the substance relatively to the reference frame at rest.

The second term of the right-hand side will be differentiated considering the variation of the volume. The final value of the volume after the time interval Δt is:

$$V'_\Sigma = V_\Sigma + \int_\Sigma \boldsymbol{v}_r \cdot \mathrm{d}\boldsymbol{S}\,\Delta t, \tag{A.1.100 b}$$

where $\boldsymbol{v}_r \cdot \mathrm{d}\boldsymbol{S}$ represents the volume described by the surface element $\mathrm{d}\boldsymbol{S}$ in the unit of time.

Using the relation (A.1.39) for $\boldsymbol{a} = \boldsymbol{v}_r$, the last relation becomes:

$$\int_\Sigma \boldsymbol{v}_r \cdot \mathrm{d}\boldsymbol{S} = \int_{V_\Sigma} \operatorname{div}\boldsymbol{v}_r\,\mathrm{d}v. \tag{A.1.100 c}$$

Relations (A.1.100 b) and (A.1.100 c) yield:

$$V'_\Sigma = V_\Sigma + \int_\Sigma \operatorname{div}\boldsymbol{v}_r\,\Delta t\,\mathrm{d}v, \tag{A.1.100 d}$$

$$\Delta V_\Sigma = V'_\Sigma - V_\Sigma. \tag{A.1.100 e}$$

It follows:

$$\left[\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_\Sigma} f\,\mathrm{d}v\right]_{f=\mathrm{const}} = f\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_\Sigma}\mathrm{d}v = f\lim_{\Delta t\to 0}\frac{V'_\Sigma - V_\Sigma}{\Delta t} = f\int_{V_\Sigma}\operatorname{div}\boldsymbol{v}_r\,\mathrm{d}v. \tag{A.1.100 f}$$

By substituting expressions (A.1.100 a) and (A.1.100 f) in relation (A.1.99), it follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_\Sigma} f\,\mathrm{d}v = \int_{V_\Sigma}\left(\frac{\partial f}{\partial t} + \boldsymbol{v}_r \cdot \operatorname{grad} f + f\operatorname{div}\boldsymbol{v}_r\right)\mathrm{d}v. \tag{A.1.101}$$

Finally, it follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_\Sigma} f\,\mathrm{d}v = \int_{V_\Sigma}\left[\frac{\partial f}{\partial t} + \operatorname{div}\left(\boldsymbol{v}_r f\right)\right]\mathrm{d}v. \tag{A.1.102}$$

A.1.2.15. Derivative with Respect to Time of the Flux through a Moving Open Surface

Let us calculate the derivative with respect to time of the flux of a vector $G(r,t)$ through a simply connected open surface bounded by any closed simple curve Γ, when the surface S_Γ is moving relatively to a reference frame considered at rest.

The velocity of any point of the surface S_Γ or of the curve Γ will be denoted by v_r and may have various values at different points. It is assumed that at any point of the considered domain, the vector $G(r,t)$ is differentiable with respect to time and with the co-ordinates of the reference frame. The corresponding flux is:

$$\Phi = \int_{S_\Gamma} G(r,t) \cdot d\,S.$$
(A.1.103)

The variation with time of the quantity Φ will be determined by two causes: **a**. The variation with time of the vector $G(r,t)$, at each point of the reference frame at rest; **b**. The modification with time of the position and sizes of the surface S_Γ. After an interval of time Δt, the surface S_Γ occupies another position and becomes the surface $S'_{\Gamma'}$, as can be seen in Fig. A.1.10.

The derivative with respect to time of the flux Φ is given by the expression:

$$\frac{d\,\Phi}{d\,t} = \frac{d}{d\,t} \int_{S_\Gamma} G(r,t) \cdot d\,S = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\{ \int_{S'_{\Gamma'}} G(r,t+\Delta t) \cdot d\,S - \int_{S_\Gamma} G(r,t) \right\}.$$
(A.1.104)

The first integral of the last side, if only the small quantities of the first order are kept, can be expanded in the form:

$$\int_{S'_{\Gamma'}} G(r,t+\Delta t) \cdot d\,S = \int_{S'_{\Gamma'}} G(r,t) \cdot d\,S + (\Delta t) \int_{S'_{\Gamma'}} \frac{\partial G(r,t)}{\partial t} \cdot d\,S.$$
(A.1.105 a)

In the last integral of the right-hand side, which is a small quantity of the first order, the influence of the variation of $S'_{\Gamma'}$ will be neglected producing a small quantity of higher order. It results that:

$$\frac{d}{d\,t} \int_{S_\Gamma} G(r,t) \cdot d\,S = \int_{S_\Gamma} \frac{\partial G(r,t)}{\partial t} \cdot d\,S +$$

$$+ \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\{ \int_{S'_{\Gamma'}} G(r,t) \cdot d\,S - \int_{S_\Gamma} G(r,t) \cdot d\,S \right\}.$$
(A.1.105 b)

It follows that the derivative with respect to time of the flux of a vector through an open surface that is moving relatively to a reference frame (supposed at rest) is given by the sum of two terms: **a** – the derivative of the flux supposing the open surface at rest

relatively to the reference frame; **b** – the derivative of the flux supposing that the vector that produces the flux does not vary with time but the position and dimensions of the surface are modified with time. The first term has been directly obtained from calculation. The second term will be further calculated.

It can be remarked, in Fig. A.1.10, that the surfaces S_Γ and $S'_{\Gamma'}$ together with the lateral surface S_{lat} form a closed surface $\Sigma = S_\Gamma \cup S'_{\Gamma''} \cup S_{\text{lat}}$. The flux of the vector $G(r, t)$ through the closed surface Σ, according to relation (A.1.38), is:

$$\int_\Sigma G(r, t) \cdot \mathrm{d}\,S = \int_{V_\Sigma} \operatorname{div} G(r, t)\mathrm{d}\,v, \tag{A.1.105 c}$$

where $\mathrm{d}\,S$ represents the element of the closed surface Σ.

But the term of the left-hand side of the last relation can be decomposed, and taking into account the positive sense of the normal adopted in Fig. A.1.10, it follows:
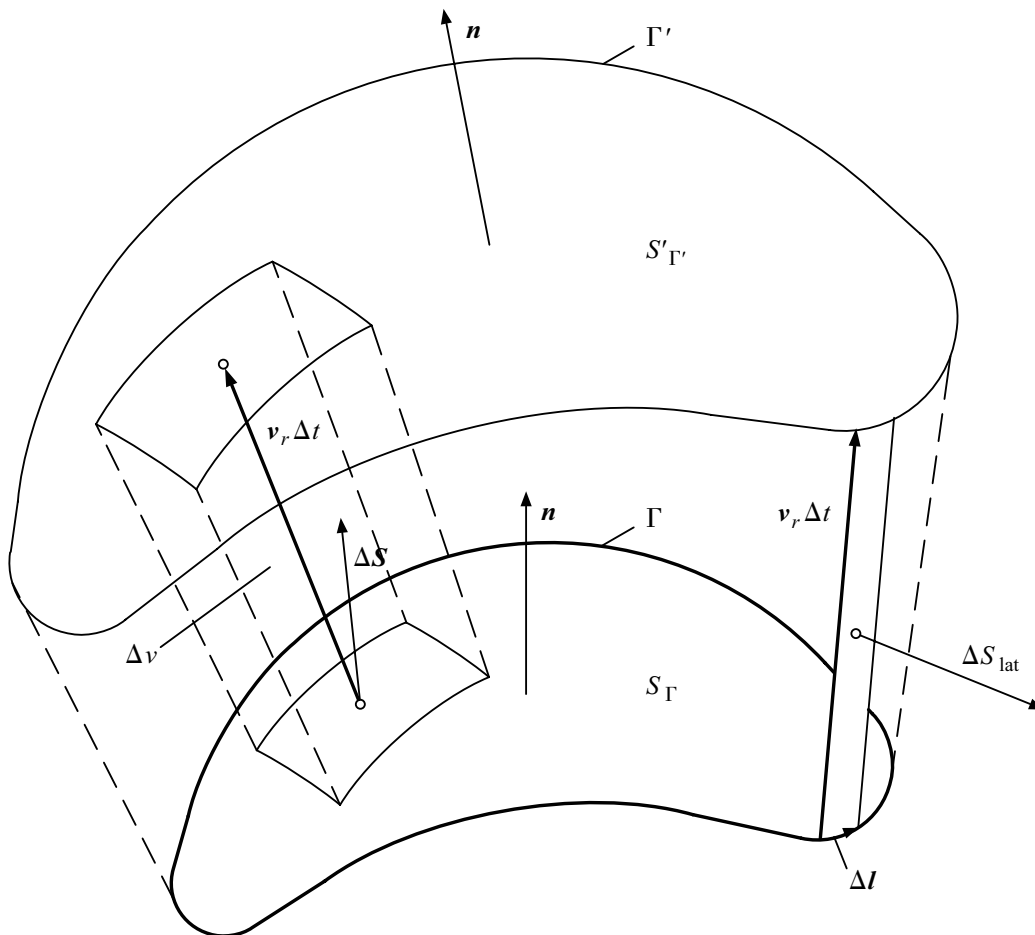
Fig. A.1.10. Explanation to the calculation of the
derivative of the flux through an open surface.

$$\int_{\Sigma} \boldsymbol{G}(\boldsymbol{r},t)\cdot \mathrm{d}\,\boldsymbol{S} = -\int_{S_\Gamma} \boldsymbol{G}(\boldsymbol{r},t)\cdot \mathrm{d}\,\boldsymbol{S} + \int_{S'_{\Gamma'}} \boldsymbol{G}(\boldsymbol{r},t)\cdot \mathrm{d}\,\boldsymbol{S} + \int_{S_{\mathrm{lat}}} \boldsymbol{G}(\boldsymbol{r},t)\mathrm{d}\,\boldsymbol{S}_{\mathrm{lat}}\,. \tag{A.1.105 d}$$

The element of the surface S_{lat}, with the symbols of Fig. A.1.10, is:

$$\Delta \boldsymbol{S}_{\mathrm{lat}} = \Delta \boldsymbol{l} \times \boldsymbol{v}_r\,\Delta t\,. \tag{A.1.105 e}$$

The volume element of the domain V_Σ, with the symbols of Fig. A.1.10, is:

$$\Delta v = \left(\boldsymbol{v}_r\,\Delta t\right)\cdot \Delta \boldsymbol{S}\,. \tag{A.1.105 f}$$

From relations (A.1.105 d) and (A.1.105 e), it follows:

$$\int_{\Sigma} \boldsymbol{G}(\boldsymbol{r},t)\cdot \mathrm{d}\,\boldsymbol{S} = -\int_{S_\Gamma} \boldsymbol{G}(\boldsymbol{r},t)\cdot \mathrm{d}\,\boldsymbol{S} + \int_{S'_{\Gamma'}} \boldsymbol{G}(\boldsymbol{r},t)\cdot \mathrm{d}\,\boldsymbol{S} - $$
$$- \left(\Delta t\right)\oint_{\Gamma} \boldsymbol{G}(\boldsymbol{r},t)\cdot \left(\boldsymbol{v}_r \times \mathrm{d}\,\boldsymbol{l}\right). \tag{A.1.105 g}$$

From relations (A.1.105 c), (A.1.105 f), (A.1.105 g), it follows:

$$\int_{S'_{\Gamma'}} \boldsymbol{G}(\boldsymbol{r},t)\cdot \mathrm{d}\,\boldsymbol{S} - \int_{S_\Gamma} \boldsymbol{G}(\boldsymbol{r},t)\cdot \mathrm{d}\,\boldsymbol{S} = \left(\Delta t\right)\int_{S_\Gamma} \boldsymbol{v}_r\,\mathrm{div}\,\boldsymbol{G}(\boldsymbol{r},t)\cdot \mathrm{d}\,\boldsymbol{S} + $$
$$+ \left(\Delta t\right)\oint_{\Gamma} \boldsymbol{G}(\boldsymbol{r},t)\cdot \left(\boldsymbol{v}_r \times \mathrm{d}\,\boldsymbol{l}\right). \tag{A.1.105 h}$$

Relations (A.1.105 b) and (A.1.105 h) yield:

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\int_{S_\Gamma} \boldsymbol{G}(\boldsymbol{r},t)\cdot \mathrm{d}\,\boldsymbol{S} = \int_{S_\Gamma} \frac{\partial\,\boldsymbol{G}(\boldsymbol{r},t)}{\partial\,t}\cdot \mathrm{d}\,\boldsymbol{S} - \oint_{\Gamma} \left[\boldsymbol{v}_r \times \boldsymbol{G}(\boldsymbol{r},t)\right]\cdot \mathrm{d}\,\boldsymbol{l} + $$
$$+ \oint_{\Gamma} \boldsymbol{v}_r\,\mathrm{div}\,\boldsymbol{G}(\boldsymbol{r},t)\cdot \mathrm{d}\,\boldsymbol{S}\,. \tag{A.1.106}$$

APPENDIX 2

EXPRESSIONS OF THE DIFFERENTIAL OPERATORS IN CURVILINEAR CO-ORDINATES

A.2.1. GENERAL CONSIDERATIONS

In applications, besides the three-orthogonal rectilinear system of co-ordinates, other systems of co-ordinates are also of importance. Further on, the expressions of the differential operators for various systems of co-ordinates will be recalled [12, tome I, p. 44].

In the three-orthogonal system with rectilinear axes, the position of any point is determined by the distances of that point to the planes determined by the axes of co-ordinates.

Hence, any point $M(x_M, y_M, z_M)$ can be considered as the intersection of the planes:

$$x = x_M, \qquad y = y_M, \qquad z = z_M, \qquad \text{(A.2.1 a, b, c)}$$

parallel with the planes yOz, zOx, xOy, respectively.

The intersection of planes $y = \text{const}$ and $z = \text{const}$ represents the line of variation of the quantity x. The line $y = 0$; $z = 0$ represents the line of co-ordinate x. Analogously, the lines of the co-ordinates y and z can be obtained. The three lines obtained in this way are straight lines.

The quantities x, y, z represent the co-ordinates of the three-orthogonal system of co-ordinates with rectilinear axes.

Generally, if the following three sets of surfaces are considered:

$$x_1(x, y, z) = C_1, \qquad x_2(x, y, z) = C_2, \qquad x_3(x, y, z) = C_3, \qquad \text{(A.2.2 a, b, c)}$$

and if various values are given to the constants C_1, C_2, C_3, it can be considered that each point M of the space is determined by a set of three values corresponding to the three constants.

The intersection of the surfaces $x_2 = C_2$ and $x_3 = C_3$ represents the line of variation of the quantity x_1. The line $x_2 = 0$; $x_3 = 0$ represents the line of the co-ordinate x_1. Analogously, it is possible to obtain the lines of variation of the quantities x_2 and x_3.

In the general case, the lines of variation are not straight lines but curvilinear ones. For this reason, in the general case, a system of curvilinear co-ordinates is obtained as in Fig. A.2.1.

The quantities x_1, x_2, x_3 represent the co-ordinates of a curvilinear system of co-ordinates.
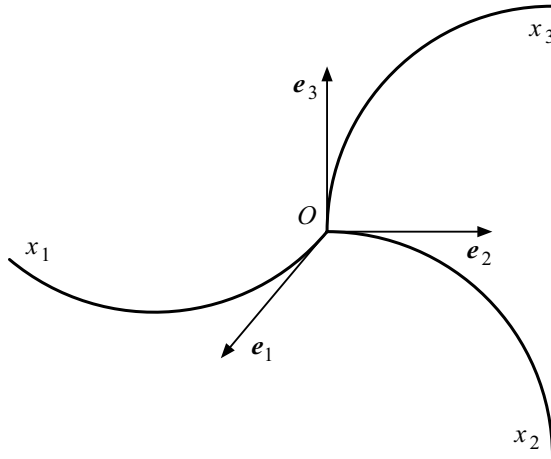
Fig. A.2.1. Curvilinear system of co-ordinates.

Between the co-ordinates x_1, x_2, x_3 of a curvilinear system of co-ordinates and the co-ordinates x, y, z of a three-orthogonal rectilinear system of co-ordinates, there are the following relations:

$$x_1 = f_1(x, y, z); \qquad x = \varphi_1(x_1, x_2, x_3);$$

$$x_2 = f_2(x, y, z); \qquad y = \varphi_2(x_1, x_2, x_3); \qquad \text{(A.2.3 a, \ldots, f)}$$

$$x_3 = f_3(x, y, z); \qquad z = \varphi_3(x_1, x_2, x_3).$$

The position of a point M can be defined by the corresponding position vector $\boldsymbol{r}$.

At any point $M(x_1, x_2, x_3)$ of the curvilinear system of co-ordinates, it is possible to construct three fundamental vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$, tangent to the lines of co-ordinates x_1, x_2, x_3, and each of them having the length equal to unity. Hence, the three fundamental vectors above are unit vectors. The three vectors form a trihedron that can have a different position at each point. In addition, it should be noted that the unit vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$, tangent to the lines of co-ordinates, are oriented in the sense in which the co-ordinates x_1, x_2, x_3 increase.

The chief difference between the curvilinear system of co-ordinates and the rectilinear system of co-ordinates resides as mentioned in the fact that, in the case of a curvilinear system, the directions of vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ depend on the point at which these vectors are taken.

The expression of a vector in this system of curvilinear co-ordinates is:

$$\boldsymbol{a} = \boldsymbol{e}_1\, a_1 + \boldsymbol{e}_2\, a_2 + \boldsymbol{e}_3\, a_3. \qquad \text{(A.2.4)}$$

Further on, we shall consider only those systems of curvilinear co-ordinates for which the three fundamental vectors are perpendicular to each other. These are called

rectangular or orthogonal systems of curvilinear co-ordinates. In this work, only systems of co-ordinates the fundamental vectors of which form a right trihedron are used. They are termed right-angled systems of co-ordinates.

The displacement of a point $M(x_1, x_2, x_3)$ at a variation dx_1, of the co-ordinate x_1, the other two co-ordinates being constant, is ds_1.

Analogously, the following variations will be obtained: ds_2 at a variation dx_2 and ds_3 at a variation dx_3. The expressions for the length of the element of arc, the area of the surface elements and the volume of an element, taking into account that the system is orthogonal, and neglecting the small quantities of higher order, are:

$$ds^2 = ds_1^2 + ds_2^2 + ds_3^2 . \qquad (A.2.5)$$

$$dS_1 = ds_2\, ds_3 ,$$

$$dS_2 = ds_3\, ds_1 , \qquad (A.2.6\ a,\ b,\ c)$$

$$dS_3 = ds_1\, ds_2 .$$

$$dV = ds_1\, ds_2\, ds_3 . \qquad (A.2.7)$$

We shall take into account that the partial derivative of the position vector $\boldsymbol{r}$ with respect to the co-ordinate x_1 (the co-ordinates x_2 and x_3 remain constant) is:

$$\frac{\partial \boldsymbol{r}}{\partial x_1} = \boldsymbol{e}_1\, h_1 , \qquad (A.2.8)$$

where:

$$\left| \frac{\partial \boldsymbol{r}}{\partial x_1} \right| = h_1 . \qquad (A.2.9)$$

It follows:

$$\boldsymbol{e}_1\, ds_1 = \frac{\partial \boldsymbol{r}}{\partial x_1} dx_1 = \boldsymbol{e}_1\, h_1\, dx_1 , \qquad (A.2.10)$$

hence:

$$ds_1 = h_1\, dx_1 . \qquad (A.2.11)$$

Analogously, it is possible to obtain the corresponding expressions of ds_2 and ds_3. The following expressions are obtained:

$$ds^2 = h_1^2\, dx_1^2 + h_2^2\, dx_2^2 + h_3^2\, dx_3^2 . \qquad (A.2.12)$$

$$dS_1 = h_2\, h_3\, dx_2\, dx_3 ; \quad dS_2 = h_3\, h_1\, dx_3\, dx_1 ; \quad dS_3 = h_1\, h_2\, dx_1\, dx_2 ; \qquad (A.2.13\ a,\ b,\ c)$$

$$\mathrm{d}V = h_1\, h_2\, h_3\, \mathrm{d}\, x_1\, \mathrm{d}\, x_2\, \mathrm{d}\, x_3\, ; \tag{A.2.14}$$

$$h_i = \left| \frac{\partial\, \boldsymbol{r}}{\partial\, x_i} \right| ; \tag{A.2.15}$$

$$h_i = \sqrt{\left(\frac{\partial\, x}{\partial x_i} \right)^2 + \left(\frac{\partial\, y}{\partial x_i} \right)^2 + \left(\frac{\partial\, z}{\partial x_i} \right)^2 } . \tag{A.2.16}$$

The quantities of the form h_i are called LAMÉ coefficients.

We shall express the differential operators taking into account the relations above. The gradient of a scalar function $\Phi(x_1, x_2, x_3)$ has the expression:

$$\mathrm{grad}\,\Phi = \boldsymbol{e}_1\, \frac{1}{h_1} \cdot \frac{\partial\,\Phi}{\partial\, x_1} + \boldsymbol{e}_2\, \frac{1}{h_2} \cdot \frac{\partial\,\Phi}{\partial\, x_2} + \boldsymbol{e}_3\, \frac{1}{h_3} \cdot \frac{\partial\,\Phi}{\partial\, x_3}. \tag{A.2.17}$$

The divergence of a vector $\boldsymbol{a}(x_1, x_2, x_3)$ has the expression:

$$\mathrm{div}\,\boldsymbol{a} = \frac{1}{h_1\, h_2\, h_3} \left[\frac{\partial}{\partial\, x_1}\left(a_1\, h_2\, h_3\right) + \frac{\partial}{\partial\, x_2}\left(a_2\, h_3\, h_1\right) + \frac{\partial}{\partial\, x_3}\left(a_3\, h_1\, h_2\right) \right]. \tag{A.2.18}$$

The curl of a vector $\boldsymbol{a}(x_1, x_2, x_3)$ has the expression:

$$\mathrm{curl}\,\boldsymbol{a} = \boldsymbol{e}_1\, \frac{1}{h_2\, h_3} \left[\frac{\partial}{\partial\, x_2}\left(a_3\, h_3\right) - \frac{\partial}{\partial\, x_3}\left(a_2\, h_2\right) \right] + \boldsymbol{e}_2\, \frac{1}{h_3\, h_1} \left[\frac{\partial}{\partial\, x_3}\left(a_1\, h_1\right) - \frac{\partial}{\partial\, x_1}\left(a_3\, h_3\right) \right] +$$
$$+ \boldsymbol{e}_3\, \frac{1}{h_1\, h_2} \left[\frac{\partial}{\partial\, x_1}\left(a_2\, h_2\right) - \frac{\partial}{\partial\, x_2}\left(a_1\, h_1\right) \right].$$

$$\tag{A.2.19}$$

The Laplace operator applied to a scalar function, $\nabla^2\Phi$, has the expression:

$$\nabla^2\Phi = \frac{1}{h_1\, h_2\, h_3} \left[\frac{\partial}{\partial\, x_1}\left(\frac{h_2\, h_3}{h_1} \cdot \frac{\partial\,\Phi}{\partial\, x_1} \right) + \frac{\partial}{\partial\, x_2}\left(\frac{h_3\, h_1}{h_2} \cdot \frac{\partial\,\Phi}{\partial\, x_2} \right) + \frac{\partial}{\partial\, x_3}\left(\frac{h_1\, h_2}{h_3} \cdot \frac{\partial\,\Phi}{\partial\, x_3} \right) \right].$$
$$\tag{A.2.20}$$

A.2.2. FORMULAE FOR THREE-ORTHOGONAL RECTILINEAR, CYLINDRICAL AND SPHERICAL CO-ORDINATES

Further on, the following three-orthogonal systems of co-ordinates will be considered: Three-orthogonal rectilinear system of co-ordinates, cylindrical system of co-ordinates, spherical polar system of co-ordinates.
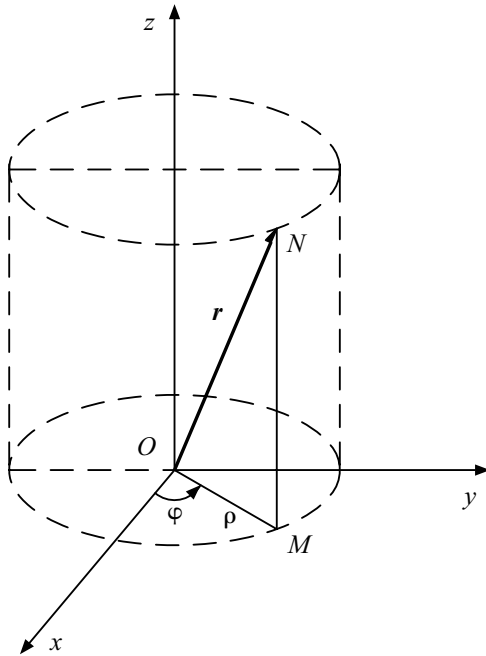
Fig. A.2.2. System of cylindrical
co-ordinates.
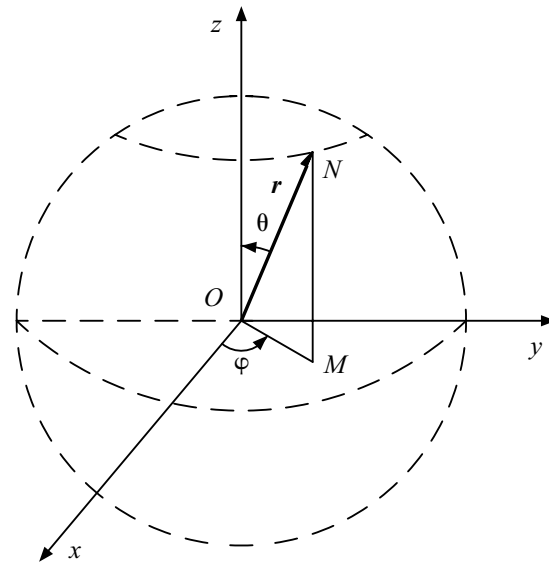
Fig. A.2.3. System of spherical polar
co-ordinates.

Besides these, also other systems of three-orthogonal systems are used, for instance the elliptical system of co-ordinates.

For a three-orthogonal system of co-ordinates with rectilinear axes, it follows:

$$x_1 = x; \qquad x_2 = y; \qquad x_3 = z;$$
$$d s^2 = (d x)^2 + (d y)^2 + (d z)^2;$$
$$h_1 = 1; \qquad h_2 = 1; \qquad h_3 = 1.$$

(A.2.21 a, . . ., g)

For a cylindrical system of co-ordinates (Fig. A.2.2), it follows:

$$x_1 = \rho; \qquad x_2 = \varphi; \qquad x_3 = z;$$
$$x = \rho \cos\varphi; \quad y = \rho \sin\varphi; \quad z = x_3;$$
$$d s^2 = (d\rho)^2 + (\rho\, d\varphi)^2 + (d z)^2;$$
$$h_1 = 1; \qquad h_2 = \rho; \qquad h_3 = 1.$$

(A.2.22 a, . . ., j)

For a spherical system of co-ordinates (Fig. A.2.3), it follows:

$$x_1 = r; \qquad\qquad x_2 = \theta; \qquad\quad x_3 = \varphi;$$
$$x = r \sin\theta \cos\varphi; \quad y = r \sin\theta \sin\varphi; \quad z = r\cos\theta;$$
$$\mathrm{d}s^2 = (\mathrm{d}r)^2 + (r\,\mathrm{d}\theta)^2 + (r\sin\theta\,\mathrm{d}\varphi)^2 ;$$
$$h_1 = 1; \qquad\qquad h_2 = r; \qquad\quad h_3 = r\sin\theta.$$

(A.2.23 a, . . ., j)

The relations of differential operators in cylindrical co-ordinates:

$$\left(\operatorname{grad}\Phi\right)_\rho = \frac{\partial\Phi}{\partial\rho}; \qquad \left(\operatorname{grad}\Phi\right)_\varphi = \frac{1}{\rho}\cdot\frac{\partial\Phi}{\partial\varphi}; \qquad \left(\operatorname{grad}\Phi\right)_z = \frac{\partial\Phi}{\partial z}. \qquad \text{(A.2.24 a, b, c)}$$

$$\operatorname{div}\boldsymbol{a} = \frac{1}{\rho}\cdot\frac{\partial}{\partial\rho}\left(\rho\,a_\rho\right) + \frac{1}{\rho}\cdot\frac{\partial a_\varphi}{\partial\varphi} + \frac{\partial a_z}{\partial z}. \qquad\qquad \text{(A.2.25)}$$

$$\left(\operatorname{curl}\boldsymbol{a}\right)_\rho = \frac{1}{\rho}\cdot\frac{\partial a_z}{\partial\varphi} - \frac{\partial a_\varphi}{\partial z}; \quad \left(\operatorname{curl}\boldsymbol{a}\right)_\varphi = \frac{\partial a_\rho}{\partial z} - \frac{\partial a_z}{\partial\rho}; \quad \left(\operatorname{curl}\boldsymbol{a}\right)_z = \frac{1}{\rho}\cdot\frac{\partial}{\partial\rho}\left(\rho\,a_\varphi\right) - \frac{1}{\rho}\cdot\frac{\partial a_\rho}{\partial\varphi}.$$

(A.2.26)

$$\nabla^2\Phi = \frac{1}{\rho}\cdot\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\cdot\frac{\partial^2\Phi}{\partial\varphi^2} + \frac{\partial^2\Phi}{\partial z^2}. \qquad\qquad \text{(A.2.27)}$$

The relations of differential operators in spherical co-ordinates:

$$\left(\operatorname{grad}\Phi\right)_r = \frac{\partial\Phi}{\partial r}; \qquad \left(\operatorname{grad}\Phi\right)_\theta = \frac{1}{r}\cdot\frac{\partial\Phi}{\partial\theta}; \qquad \left(\operatorname{grad}\Phi\right)_\varphi = \frac{1}{r\sin\theta}\cdot\frac{\partial\Phi}{\partial\varphi}.$$

(A.2.28 a, b, c)

$$\operatorname{div}\boldsymbol{a} = \frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left(r^2\,a_r\right) + \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(a_\theta\sin\theta\right) + \frac{\partial a_\varphi}{\partial\varphi}\right]. \qquad \text{(A.2.29)}$$

$$\left(\operatorname{curl}\boldsymbol{a}\right)_r = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(a_\varphi\sin\theta\right) - \frac{\partial a_\theta}{\partial\varphi}\right];$$

$$\left(\operatorname{curl}\boldsymbol{a}\right)_\theta = \frac{1}{r\sin\theta}\cdot\frac{\partial a_r}{\partial\varphi} - \frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r\,a_\varphi\right); \qquad\qquad \text{(A.2.30 a, b, c)}$$

$$\left(\operatorname{curl}\boldsymbol{a}\right)_\varphi = \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\,a_\theta\right) - \frac{\partial a_r}{\partial\theta}\right].$$

$$\nabla^2\Phi = \frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\frac{\partial\Phi}{\partial\theta}\sin\theta\right) + \frac{1}{r^2}\cdot\frac{1}{\sin^2\theta}\cdot\frac{\partial^2\Phi}{\partial\varphi^2}. \qquad \text{(A.2.31)}$$

A.2.3. ELLIPSOIDAL CO-ORDINATES

We shall recall some more important results concerning the ellipsoidal co-ordinates.

The relation between ellipsoidal co-ordinates and those of a right three-orthogonal system of co-ordinates with rectilinear axes is given by the equation of a surface of the second order:

$$\frac{x^2}{a^2+u}+\frac{y^2}{b^2+u}+\frac{z^2}{c^2+u}=1;$$

$$(a>b>c). \tag{A.2.32}$$

Relation (A.2.32) represents an equation of the third degree with respect to u. This equation has for each point of co-ordinates x, y, z, three real distinct roots ξ, η, ζ situated respectively within the intervals:

$$\xi \in \left[-c^2,+\infty\right), \tag{A.2.33 a}$$

$$\eta \in \left[-b^2,-c^2\right], \tag{A.2.33 b}$$

$$\zeta \in \left[-a^2,-b^2\right]. \tag{A.2.33 c}$$

The geometrical meaning of the roots ξ, η, ζ consists in the following. The surfaces corresponding to equation (A.2.33), in which the quantity u is successively replaced by one of the quantities ξ, η, ζ, represent ellipsoids, hyperboloids with a sheet, hyperboloids with two sheets. These surfaces of the second order are confocal with the ellipsoid given by equation (A.2.32) for $u=0$.

Through each point of the space, one of the surfaces belonging to the three families is passing. The three surfaces are orthogonal to each other. The relations for passing from the ellipsoidal co-ordinates to those of the three-orthogonal system of co-ordinates with rectilinear axes are obtained by solving the system of three equations with three unknowns x, y, z, which is obtained by replacing in equation (A.2.32) the quantity u successively by ξ, η, ζ. It follows:

$$x=\pm\left[\frac{\left(\xi+a^2\right)\left(\eta+a^2\right)\left(\zeta+a^2\right)}{\left(b^2-a^2\right)\left(c^2-a^2\right)}\right]^{\frac{1}{2}}, \tag{A.2.34 a}$$

$$y=\pm\left[\frac{\left(\xi+b^2\right)\left(\eta+b^2\right)\left(\zeta+b^2\right)}{\left(c^2-b^2\right)\left(a^2-b^2\right)}\right]^{\frac{1}{2}}, \tag{A.2.34 b}$$

$$z = \pm \left[\frac{\left(\xi + c^2\right)\left(\eta + c^2\right)\left(\zeta + c^2\right)}{\left(a^2 - c^2\right)\left(b^2 - c^2\right)} \right]^{\frac{1}{2}} . \tag{A.2.34 c}$$

To each point of co-ordinates x, y, z, of the three-orthogonal system of co-ordinates with rectilinear axes, there corresponds a point of co-ordinates ξ, η, ζ, in the ellipsoidal system of co-ordinates.

The length element expressed in ellipsoidal co-ordinates is:

$$\mathrm{d}s^2 = h_\xi^2 \left(\mathrm{d}\xi\right)^2 + h_\eta^2 \left(\mathrm{d}\eta\right)^2 + h_\zeta^2 \left(\mathrm{d}\zeta\right)^2 , \tag{A.2.35}$$

where the following symbols have been used:

$$h_\xi = \frac{\sqrt{(\xi - \eta)(\xi - \zeta)}}{2\,R_\xi}; \quad h_\eta = \frac{\sqrt{(\eta - \zeta)(\eta - \xi)}}{2\,R_\eta}; \quad h_\zeta = \frac{\sqrt{(\zeta - \xi)(\zeta - \eta)}}{2\,R_\zeta}; \tag{A.2.36 a, b, c}$$

$$R_u = \left[(u + a)^2 + (u + b)^2 + (u + c)^2\right]^{\frac{1}{2}};$$
$$(u = \xi, \eta, \zeta). \tag{A.2.37}$$

The expression of the quantity $\nabla^2 V$ in ellipsoidal co-ordinates is:

$$\nabla^2 V = \frac{1}{(\xi - \eta)(\zeta - \xi)(\eta - \zeta)} \left[(\eta - \zeta) R_\xi \frac{\partial}{\partial \xi}\left(R_\xi \frac{\partial V}{\partial \xi} \right) + (\zeta - \xi) R_\eta \frac{\partial}{\partial \eta}\left(R_\eta \frac{\partial V}{\partial \eta} \right) + \right.$$
$$\left. + (\xi - \eta) R_\zeta \frac{\partial}{\partial \zeta}\left(R_\zeta \frac{\partial V}{\partial \zeta} \right) \right]^{\frac{1}{2}} .$$

$$\tag{A.2.38}$$

APPENDIX 3

GENERAL RELATIONS DEDUCED FROM THE SPECIAL THEORY OF RELATIVITY

In this Appendix, we shall first recall some basic relations of the Theory of Special Relativity concerning Mechanics. Then, starting from these relations, we shall deduce certain general relations concerning forces in various reference frames. These relations can be used in the Theory of Electromagnetic Field as well as in the case of other fields of forces. The derivation is based on previous papers of the author [23], [40], [41].

A.3.1. RELATIONS OF MECHANICS IN THE SPECIAL THEORY OF RELATIVITY

Let us consider the inertial reference frames $K_\mathrm{o}(x_\mathrm{o}, y_\mathrm{o}, z_\mathrm{o})$ and $K_1(x_1, y_1, z_1)$ having the corresponding axes of co-ordinates parallel to each other. Three-orthogonal rectilinear (Cartesian) right-handed systems of co-ordinates are used.

The time in two reference frames will be denoted by t_o and t_1, respectively. The unit vectors of the two systems of co-ordinates can be denoted by the sets $\boldsymbol{i}_\mathrm{o}, \boldsymbol{j}_\mathrm{o}, \boldsymbol{k}_\mathrm{o}$, and $\boldsymbol{i}_1, \boldsymbol{j}_1, \boldsymbol{k}_1$, respectively.

In the case in which the axes of co-ordinates of the two systems are parallel with each other, since the unit vectors are dimensionless, both sets of unit vectors can be denoted by the same set of symbols, namely $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$.

The components of any vector, in the reference frame K_o, for instance $\boldsymbol{F}_\mathrm{o}$, along the three axes of co-ordinates can be denoted by $F_{\mathrm{o}x_\mathrm{o}}, F_{\mathrm{o}y_\mathrm{o}}, F_{\mathrm{o}z_\mathrm{o}}$. If no confusion may appear, for the sake of brevity, the last suffix may be suppressed and then the components become $F_{\mathrm{o}x}, F_{\mathrm{o}y}, F_{\mathrm{o}z}$.

The components of any vector, in the reference frame K_1, for instance $\boldsymbol{F}_1$, along the three axes of co-ordinates can be denoted by $F_{1x_1}, F_{1y_1}, F_{1z_1}$ or F_{1x}, F_{1y}, F_{1z}, according to the case, as explained above.

The components of the velocity $\boldsymbol{v}_{1\mathrm{o}}$ of the reference frame K_1 with respect to the reference frame K_o, along the three axes of co-ordinates, can be denoted by $v_{1\mathrm{o}x_\mathrm{o}}, v_{1\mathrm{o}y_\mathrm{o}}, v_{1\mathrm{o}z_\mathrm{o}}$ or $v_{1\mathrm{o}x}, v_{1\mathrm{o}y}, v_{1\mathrm{o}z}$, according to the case, as explained above.

The used symbols are given at the end of this appendix. At the initial moment $t = 0$, the two systems of co-ordinates above are assumed to coincide.

If no mention is made, the medium is the vacuum (i.e., empty space).

In order to facilitate the understanding of the treatment, firstly we shall recall the relations of Mechanics in the Special Theory of Relativity, brought in a convenient form, in accordance with the following treatment.

We consider the relation in the Special Theory of Relativity as being obtained directly, hence without resorting to the equations of Electromagnetism.

A.3.1.1. General Relations of Mechanics in the Special Theory of Relativity

Between the quantities of Mechanics given in the list of symbols at the end of this appendix, the relations below have been established and will be given using three-dimensional vectors (Fig. A.3.1). For the sake of simplicity, we denote $r_{1A} = r_1$, $r_{oP} = r_o$. If no mention is made, we shall also assume $v_{1o} = i\, v_{1o} = i\, v_{1ox}$. The quantities in the reference frame K_1, expressed in terms of the quantities of the reference frame K_o, are:

$$r_1 = r_o - v_{1o}\, t_o + (\alpha_{1o} - 1)v_{1o} \cdot (r_o - v_{1o}\, t_o)\frac{v_{1o}}{v_{1o}^2}, \tag{A.3.1}$$

$$\alpha_{1o} = \frac{1}{\sqrt{1 - \beta_{1o}^2}}; \qquad \beta_{1o} = \frac{v_{1o}}{c}, \tag{A.3.1 a, b}$$

$$\Delta l_1 = \Delta l_o + (\alpha_{1o} - 1)(v_{1o} \cdot \Delta l_o)\frac{v_{1o}}{v_{1o}^2}, \tag{A.3.2}$$

$$\Delta S_1 = \alpha_{1o}\, \Delta S_o + (1 - \alpha_{1o})\frac{1}{v_{1o}^2}(v_{1o} \cdot \Delta S_o)v_{1o}, \tag{A.3.3}$$

$$\Delta V_1 = \alpha_{1o}\, \Delta V_o, \tag{A.3.4}$$

$$t_1 = \alpha_{1o}\left(t_o - \frac{v_{1o} \cdot r_o}{c^2}\right), \tag{A.3.5}$$

$$v_{21} = \frac{v_{2o} - v_{1o}}{\alpha_{1o}(1 - \beta_{2o} \cdot \beta_{1o})} + \frac{\alpha_{1o} - 1}{\alpha_{1o}(1 - \beta_{2o} \cdot \beta_{1o})} \cdot \frac{\beta_{2o} \cdot \beta_{1o} - \beta_{1o}^2}{\beta_{1o}^2}v_{1o}, \tag{A.3.6 a}$$

$$v_{2o} = \frac{v_{21} + v_{1o}}{\alpha_{1o}(1 + \beta_{21} \cdot \beta_{1o})} + \frac{\alpha_{1o} - 1}{\alpha_{1o}(1 + \beta_{21} \cdot \beta_{1o})} \cdot \frac{\beta_{21} \cdot \beta_{1o} + \beta_{1o}^2}{\beta_{1o}^2}v_{1o}. \tag{A.3.6 b}$$

The last two relations can be brought in a form sometimes suitable in applications, for instance the latter:

$$v_{2o} = \frac{i(v_{21} + v_{1o})_x}{1 + \beta_{21} \cdot \beta_{1o}} + \frac{j(v_{21} + v_{1o})_y}{\alpha_{1o}(1 + \beta_{21} \cdot \beta_{1o})} + \frac{k(v_{21} + v_{1o})_z}{\alpha_{1o}(1 + \beta_{21} \cdot \beta_{1o})}, \tag{A.3.6 c}$$
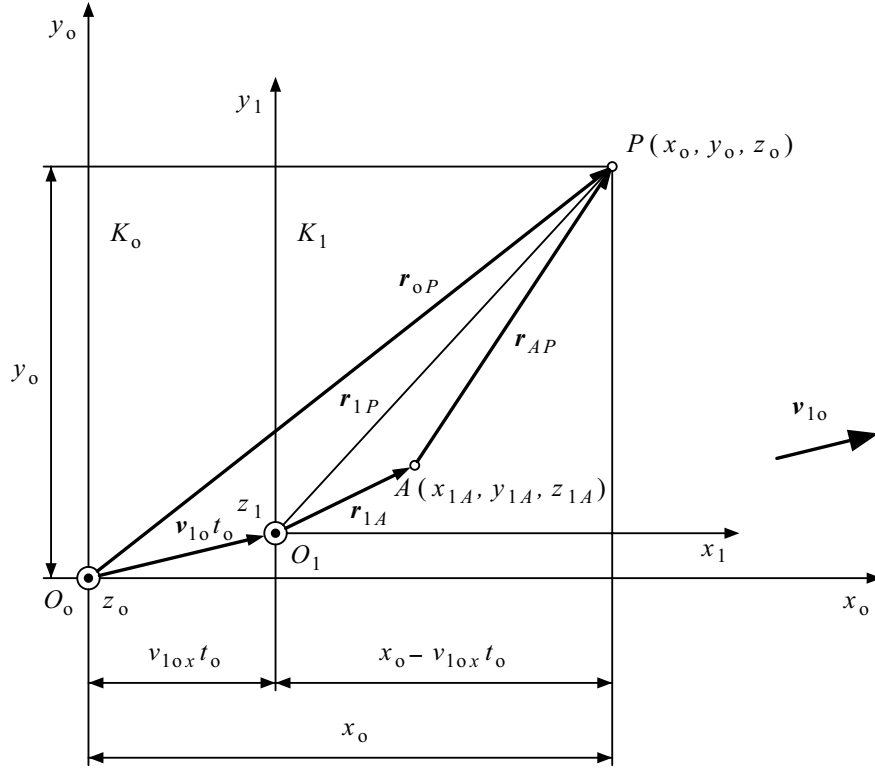
Fig. A.3.1. The systems of co-ordinates $K_o(x_o, y_o, z_o)$, $K_1(x_1, y_1, z_1)$, and a position vector.

$$\boldsymbol{a}_{21} = \frac{\alpha_{1o}\left(1 - \dfrac{\boldsymbol{v}_{2o} \cdot \boldsymbol{v}_{1o}}{c^2}\right)\boldsymbol{a}_{2o} + (1 - \alpha_{1o})(\boldsymbol{v}_{1o} \cdot \boldsymbol{a}_{2o})\dfrac{\boldsymbol{v}_{1o}}{v_{1o}^2} + \alpha_{1o}\dfrac{\boldsymbol{v}_{1o} \cdot \boldsymbol{a}_{2o}}{c^2}\boldsymbol{v}_{2o}}{\alpha_{1o}^3\left(1 - \dfrac{\boldsymbol{v}_{2o} \cdot \boldsymbol{v}_{1o}}{c^2}\right)^3}, \qquad \text{(A.3.7)}$$

$$m_1 = \alpha_{1o}\left(1 - \frac{\boldsymbol{v}_{2o} \cdot \boldsymbol{v}_{1o}}{c^2}\right)m, \qquad \text{(A.3.8)}$$

$$m = \frac{m_0}{\sqrt{1 - \dfrac{u^2}{c^2}}}; \qquad \boldsymbol{u} = \boldsymbol{v}_{2o}, \qquad \text{(A.3.9)}$$

where the quantity m is the relativistic mass in the reference frame K_o of a material point moving at the velocity $\boldsymbol{u} = \boldsymbol{v}_{2o}$ relative to this reference frame, whereas m_1 is the relativistic mass in the reference frame K_1, and m_0 is the mass of the material point at rest in K_o.

$$F_1 = \frac{1 - \alpha_{1o}\,\boldsymbol{\beta}_{2o} \cdot \boldsymbol{\beta}_{1o}}{\alpha_{1o}\left(1 - \boldsymbol{\beta}_{2o} \cdot \boldsymbol{\beta}_{1o}\right)}\,F_o + \frac{\alpha_{1o} - 1}{\alpha_{1o}\left(1 - \boldsymbol{\beta}_{2o} \cdot \boldsymbol{\beta}_{1o}\right)} \cdot \frac{F_o \cdot v_{1o}}{v_{1o}^2}\,v_{1o} - \frac{1}{c^2} \cdot \frac{v_{2o} \times \left(v_{1o} \times F_o\right)}{1 - \boldsymbol{\beta}_{2o} \cdot \boldsymbol{\beta}_{1o}},$$

$$\text{(A.3.10)}$$

$$F_o = \frac{1 + \alpha_{1o}\,\boldsymbol{\beta}_{21} \cdot \boldsymbol{\beta}_{1o}}{\alpha_{1o}\left(1 + \boldsymbol{\beta}_{21} \cdot \boldsymbol{\beta}_{1o}\right)}\,F_1 + \frac{\alpha_{1o} - 1}{\alpha_{1o}\left(1 + \boldsymbol{\beta}_{21} \cdot \boldsymbol{\beta}_{1o}\right)} \cdot \frac{F_1 \cdot v_{1o}}{v_{1o}^2}\,v_{1o} + \frac{1}{c^2} \cdot \frac{v_{21} \times \left(v_{1o} \times F_1\right)}{1 + \boldsymbol{\beta}_{21} \cdot \boldsymbol{\beta}_{1o}},$$

$$\text{(A.3.11)}$$

$$\boldsymbol{\beta}_{1o} = \frac{v_{1o}}{c}; \qquad \boldsymbol{\beta}_{2o} = \frac{v_{2o}}{c}; \qquad \boldsymbol{\beta}_{21} = \frac{v_{21}}{c}. \qquad \text{(A.3.11 a, b, c)}$$

We assume that there is a certain inertial reference frame K_o that has the following property. The force acting in this reference frame upon a particle p that is moving at any velocity $v_{pn} \neq \text{const}$, relative to the reference frame K_n is independent of the velocity v_{pn}.

For instance, in the given reference frame K_1, the force F_1 can be independent of the velocity v_{21}. Such an example occurs in the case in which there is a point-like electric charge q_1 at rest in the reference frame, and F_1 is the force exerted upon any point-like charge q_2 moving at the velocity v_{21} relatively to the reference frame K_1. The value of a force in the reference frame K_n and for instance in the reference frame K_1 in the example above is referred to as *proper value of the force*.

The forces in any reference frame, say K_1, can be expressed in terms of the forces in a certain reference frame, say, by means of a relation of the type (A.3.10).

At the same time, we assume that all the geometrical elements (lines, surfaces) have their proper dimensions (sizes) in the reference frame K_n. The geometrical dimensions (sizes) in the reference frame K_n can be expressed in terms of the geometrical dimensions (sizes) of any other reference frame by means of the relations above (A.3.2) and (A.3.3).

With the usual denominations, the geometrical dimensions are contravariant quantities, whereas the forces are covariant quantities. When writing various relations, it is useful to take into account this remark.

It is possible to express the relation between the differential operators when passing from one reference frame to another [23, Vol. I, p. 504]. These operators concern gradient, divergence, curl and derivative with respect to time.

Remark. In the case in which v_{1o} is not of the form $i\,v_{1ox}$, any vector $r_o\left(x_o, y_o, z_o\right)$ or $r_1\left(x_1, y_1, z_1\right)$ can be decomposed into three components: One parallel to the direction of v_{1o} and the other two components perpendicular to the direction of v_{1o}, chosen so that, together with the component parallel to the direction of v_{1o}, they form a Cartesian right-handed reference frame. Then, the transformation relations can be applied for each component. Therefore, the relations of Sub-section A.3.1.1 subsist.

A.3.2.   RELATIONS CONCERNING THE FORCE VECTORS IN VARIOUS REFERENCE FRAMES

A.3.2.1. Transformation of the Force when Passing from a Reference Frame to Another

By multiplying both sides of relation (A.3.6 a) by $\dfrac{v_{1o}}{c^2}$ and adding the unity to both sides, we can obtain:

$$\alpha_{1o}\left(1+\boldsymbol{\beta}_{21}\cdot\boldsymbol{\beta}_{1o}\right)=\frac{1}{\alpha_{1o}\left(1-\boldsymbol{\beta}_{2o}\cdot\boldsymbol{\beta}_{1o}\right)}. \tag{A.3.12}$$

Taking into account relation (A.3.12), expressions (A.3.10) and (A.3.11) can be written in the form:

$$\boldsymbol{F}_o = \alpha_{1o}\,\boldsymbol{F}_1 - \left(\alpha_{1o}-1\right)\frac{\boldsymbol{F}_1\cdot\boldsymbol{v}_{1o}}{v_{1o}^2}\,\boldsymbol{v}_{1o} + \alpha_{1o}\frac{1}{c^2}\,\boldsymbol{v}_{2o}\times\left(\boldsymbol{v}_{1o}\times\boldsymbol{F}_1\right), \tag{A.3.13}$$

$$\boldsymbol{F}_1 = \alpha_{1o}\,\boldsymbol{F}_o - \left(\alpha_{1o}-1\right)\frac{\boldsymbol{F}_o\cdot\boldsymbol{v}_{1o}}{v_{1o}^2}\,\boldsymbol{v}_{1o} - \alpha_{1o}\frac{1}{c^2}\,\boldsymbol{v}_{21}\times\left(\boldsymbol{v}_{1o}\times\boldsymbol{F}_o\right). \tag{A.3.14}$$

The derivation of the last two formulae, namely relations (A.3.13), (A.3.14) can be achieved starting from relations (A.3.10), (A.3.11) as follows. We consider relation (A.3.13) and deduce it starting from relation (A.3.11). The aim is to express the last term of that relation in terms of the velocity $\boldsymbol{v}_{2o}$ instead of $\boldsymbol{v}_{21}$. Hence, we have as a factor the velocity of the point-like charge subjected to that force with respect to the reference frame in which the force is expressed.

The concerned factor of relation (A.3.11), taking into account relation (A.3.12), becomes:

$$\frac{1}{c^2}\cdot\frac{\boldsymbol{v}_{21}}{1+\boldsymbol{\beta}_{21}\cdot\boldsymbol{\beta}_{1o}} = \frac{1}{c^2}\,\alpha_{1o}^2\,\boldsymbol{v}_{21}\left(1-\boldsymbol{\beta}_{2o}\cdot\boldsymbol{\beta}_{1o}\right). \tag{A.3.15}$$

From relation (A.3.6 a), it results that:

$$\frac{1}{c^2}\,\alpha_{1o}\,\boldsymbol{v}_{21}\left(1-\boldsymbol{\beta}_{2o}\cdot\boldsymbol{\beta}_{1o}\right) = \frac{1}{c^2}\left[\boldsymbol{v}_{2o}-\boldsymbol{v}_{1o}+\left(\alpha_{1o}-1\right)\frac{\boldsymbol{\beta}_{2o}\cdot\boldsymbol{\beta}_{1o}-\beta_{1o}^2}{\beta_{1o}^2}\,\boldsymbol{v}_{1o}\right], \tag{A.3.16 a}$$

and:

$$\frac{1}{c^2}\cdot\frac{\boldsymbol{v}_{21}}{1+\boldsymbol{\beta}_{21}\cdot\boldsymbol{\beta}_{1o}} = \frac{1}{c^2}\,\alpha_{1o}^2\,\boldsymbol{v}_{21}\left(1-\boldsymbol{\beta}_{2o}\cdot\boldsymbol{\beta}_{1o}\right) =$$

$$= \frac{1}{c^2}\,\alpha_{1o}\left[\boldsymbol{v}_{2o}-\boldsymbol{v}_{1o}+\left(\alpha_{1o}-1\right)\frac{\boldsymbol{\beta}_{2o}\cdot\boldsymbol{\beta}_{1o}-\beta_{1o}^2}{\beta_{1o}^2}\,\boldsymbol{v}_{1o}\right], \tag{A.3.16 b}$$

or:

$$\frac{1}{c^2} \alpha_{1o}^2 \, v_{21} \left(1 - \boldsymbol{\beta}_{2o} \cdot \boldsymbol{\beta}_{1o}\right) = \frac{1}{c^2} \alpha_{1o} \, v_{2o} + \alpha_{1o} \left[\left(\alpha_{1o} - 1\right) \frac{\boldsymbol{\beta}_{2o} \cdot \boldsymbol{\beta}_{1o}}{\beta_{1o}^2} - \alpha_{1o} \right] \frac{1}{c^2} \, v_{1o} .$$

(A.3.17)

Taking into account relation (A.3.15), the first term of relation (A.3.11) can be written:

$$\frac{1 + \alpha_{1o} \, \boldsymbol{\beta}_{21} \cdot \boldsymbol{\beta}_{1o}}{\alpha_{1o} \left(1 + \boldsymbol{\beta}_{21} \cdot \boldsymbol{\beta}_{1o}\right)} \boldsymbol{F}_1 = \frac{1 + \alpha_{1o} \, \boldsymbol{\beta}_{21} \cdot \boldsymbol{\beta}_{1o} + \alpha_{1o} - \alpha_{1o}}{\alpha_{1o} \left(1 + \boldsymbol{\beta}_{21} \cdot \boldsymbol{\beta}_{1o}\right)} \boldsymbol{F}_1 =$$

$$= \left[1 + \frac{1 - \alpha_{1o}}{\alpha_{1o} \left(1 + \boldsymbol{\beta}_{21} \cdot \boldsymbol{\beta}_{1o}\right)} \right] \boldsymbol{F}_1 = \left[1 + \left(1 - \alpha_{1o}\right) \alpha_{1o} \left(1 - \boldsymbol{\beta}_{2o} \cdot \boldsymbol{\beta}_{1o}\right) \right] \boldsymbol{F}_1 .$$

(A.3.18)

Similarly, the second term of relation (A.3.11) can be written:

$$\frac{\alpha_{1o} - 1}{\alpha_{1o} \left(1 + \boldsymbol{\beta}_{21} \cdot \boldsymbol{\beta}_{1o}\right)} \cdot \frac{\boldsymbol{F}_1 \cdot \boldsymbol{v}_{1o}}{v_{1o}^2} \, \boldsymbol{v}_{1o} = \left(\alpha_{1o} - 1\right) \alpha_{1o} \left(1 - \boldsymbol{\beta}_{2o} \cdot \boldsymbol{\beta}_{1o}\right) \frac{\boldsymbol{F}_1 \cdot \boldsymbol{v}_{1o}}{v_{1o}^2} \, \boldsymbol{v}_{1o} .$$

(A.3.19)

Having in view relations (A.3.18), (A.3.19), (A.3.16 b), relation (A.3.11) becomes:

$$\boldsymbol{F}_o = \left[1 + \left(1 - \alpha_{1o}\right) \alpha_{1o} \left(1 - \boldsymbol{\beta}_{2o} \cdot \boldsymbol{\beta}_{1o}\right) \right] \boldsymbol{F}_1 +$$

$$+ \left(\alpha_{1o} - 1\right) \alpha_{1o} \left(1 - \boldsymbol{\beta}_{2o} \cdot \boldsymbol{\beta}_{1o}\right) \frac{\boldsymbol{F}_1 \cdot \boldsymbol{v}_{1o}}{v_{1o}^2} \, \boldsymbol{v}_{1o} +$$

$$+ \frac{1}{c^2} \alpha_{1o} \, \boldsymbol{v}_{2o} \times \left(\boldsymbol{v}_{1o} \times \boldsymbol{F}_1\right) + \frac{1}{c^2} \alpha_{1o} \left[-1 + \left(\alpha_{1o} - 1\right) \frac{\boldsymbol{\beta}_{21} \cdot \boldsymbol{\beta}_{1o} - \beta_{1o}^2}{\beta_{1o}^2} \right] \boldsymbol{v}_{1o} \times \left(\boldsymbol{v}_{1o} \times \boldsymbol{F}_1\right) .$$

(A.3.20)

The vector $\boldsymbol{F}_o$ will be considered to be composed of two terms:

$$\boldsymbol{F}_{oI} = \left[1 + \left(1 - \alpha_{1o}\right) \alpha_{1o} \left(1 - \boldsymbol{\beta}_{2o} \cdot \boldsymbol{\beta}_{1o}\right) \right] \boldsymbol{F}_1 +$$

$$+ \left(\alpha_{1o} - 1\right) \alpha_{1o} \left(1 - \boldsymbol{\beta}_{2o} \cdot \boldsymbol{\beta}_{1o}\right) \frac{\boldsymbol{F}_1 \cdot \boldsymbol{v}_{1o}}{v_{1o}^2} \, \boldsymbol{v}_{1o} +$$

$$+ \frac{1}{c^2} \alpha_{1o} \left[-1 + \left(\alpha_{1o} - 1\right) \frac{\boldsymbol{\beta}_{2o} \cdot \boldsymbol{\beta}_{1o} - \beta_{1o}^2}{\beta_{1o}^2} \right] \boldsymbol{v}_{1o} \times \left(\boldsymbol{v}_{1o} \times \boldsymbol{F}_1\right) ,$$

(A.3.21)

$$\boldsymbol{F}_{oII} = \frac{1}{c^2} \alpha_{1o} \, \boldsymbol{v}_{2o} \times \left(\boldsymbol{v}_{1o} \times \boldsymbol{F}_1\right) .$$

(A.3.22)

At this stage, the calculations can be shortened as follows. Let us assume $v_{1o} = i\,v_{1ox}$. In this case, along the Ox - axis, from relation (A.3.21), we obtain:

$$F_{oIx} = \left[1 + (1 - \alpha_{1o})\alpha_{1o}(1 - \boldsymbol{\beta}_{2o} \cdot \boldsymbol{\beta}_{1o})\right]F_{1x} +$$

$$+ (\alpha_{1o} - 1)\alpha_{1o}(1 - \boldsymbol{\beta}_{2o} \cdot \boldsymbol{\beta}_{1o})\frac{F_{1x}\,v_{1ox}}{v_{1ox}^2}\,v_{1ox} +$$

$$+ \frac{1}{c^2}\alpha_{1o}\left[-1 + (\alpha_{1o} - 1)\frac{\boldsymbol{\beta}_{2o} \cdot \boldsymbol{\beta}_{1o} - \boldsymbol{\beta}_{1o}^2}{\boldsymbol{\beta}_{1o}^2}\right]\left[v_{1ox}\left(v_{1ox}\,F_{1x}\right) - F_{1x}\left(v_{1ox}^2\right)\right] = F_{1x}\,.$$

$$\text{(A.3.23)}$$

Similarly, along the Oy - axis, from relation (A.3.21), we obtain:

$$F_{oIy} = \left[1 + (1 - \alpha_{1o})\alpha_{1o}(1 - \boldsymbol{\beta}_{2o} \cdot \boldsymbol{\beta}_{1o})\right]F_{1y} +$$

$$+ \frac{1}{c^2}\alpha_{1o}\left[-1 + (\alpha_{1o} - 1)\frac{\boldsymbol{\beta}_{2o} \cdot \boldsymbol{\beta}_{1o} - \boldsymbol{\beta}_{1o}^2}{\boldsymbol{\beta}_{1o}^2}\right]\left[-F_{1y}\left(v_{1ox}^2\right)\right] =$$

$$= \left\{1 + (1 - \alpha_{1o})\alpha_{1o}(1 - \boldsymbol{\beta}_{2o} \cdot \boldsymbol{\beta}_{1o}) + \left[-\alpha_{1o} - (1 - \alpha_{1o})\alpha_{1o}\frac{\boldsymbol{\beta}_{2o} \cdot \boldsymbol{\beta}_{1o} - \boldsymbol{\beta}_{1o}^2}{\boldsymbol{\beta}_{1o}^2}\right]\left(-\boldsymbol{\beta}_{1o}^2\right)\right\}F_{1y} =$$

$$= \left[1 + \alpha_{1o} - \alpha_{1o}^2 + \alpha_{1o}^2\,\boldsymbol{\beta}_{1o}^2\right]F_{1y} = \alpha_{1o}\,F_{1y}\,,$$

$$\text{(A.3.24)}$$

since:

$$\alpha_{1o}^2\left(1 - \boldsymbol{\beta}_{1o}^2\right) = 1. \qquad\qquad \text{(A.3.24 a)}$$

Similarly, we obtain

$$F_{oIz} = \alpha_{1o}\,F_{1z}\,. \qquad\qquad \text{(A.3.25)}$$

Finally, we have:

$$\boldsymbol{F}_{oI} = \boldsymbol{i}\,F_{oIx} + \boldsymbol{j}\,F_{oIy} + \boldsymbol{k}\,F_{oIz}\,, \qquad\qquad \text{(A.3.26 a)}$$

$$\boldsymbol{F}_{oI} = \boldsymbol{i}\,F_{1x} + \boldsymbol{j}\,\alpha_{1o}\,F_{1y} + \boldsymbol{k}\,\alpha_{1o}\,F_{1z} \qquad\qquad \text{(A.3.26 b)}$$

or in compact form:

$$\boldsymbol{F}_{oI} = \alpha_{1o}\,\boldsymbol{F}_1 - (\alpha_{1o} - 1)\frac{\boldsymbol{F}_1 \cdot \boldsymbol{v}_{1o}}{\boldsymbol{v}_{1o}^2}\,\boldsymbol{v}_{1o}\,. \qquad\qquad \text{(A.3.27)}$$

Consequently, the sum of the expressions (A.3.27) and (A.3.22):

$$\boldsymbol{F}_\mathrm{o} = \boldsymbol{F}_\mathrm{oI} + \boldsymbol{F}_\mathrm{oII} \tag{A.3.28}$$

will give just the expression (A.3.13).

A.3.2.2.   Expressions of the Force Acting on a Material Point Moving in Any Reference Frame

Each of the two relations (A.3.13) and (A.3.14) can be modified as shown below. For instance, first, the former becomes:

$$\boldsymbol{F}_\mathrm{o} = \boldsymbol{i}\, F_{1x} + \boldsymbol{j}\, \alpha_{1\mathrm{o}}\, F_{1y} + \boldsymbol{k}\, \alpha_{1\mathrm{o}}\, F_{1z} + \alpha_{1\mathrm{o}}\, \frac{1}{c^2}\, \boldsymbol{v}_{2\mathrm{o}} \times (\boldsymbol{v}_{1\mathrm{o}} \times \boldsymbol{F}_1). \tag{A.3.29}$$

We assume that, in the reference frame K_1, the force acting upon a material point moving at the velocity $\boldsymbol{v}_{21} \neq \mathrm{const}$, is $\boldsymbol{F}_1$. The force in the reference frame K_o will be denoted, as above, by $\boldsymbol{F}_\mathrm{o}$. The expression of the force $\boldsymbol{F}_\mathrm{o}$ in the reference frame K_o can be written in the form:

$$\boldsymbol{F}_\mathrm{o} = \boldsymbol{F}_{or} + \boldsymbol{F}_{ov}, \tag{A.3.30}$$

where:

$$
\begin{aligned}
\boldsymbol{F}_{or} &= \alpha_{1\mathrm{o}}\, \boldsymbol{F}_1 - (\alpha_{1\mathrm{o}} - 1)\frac{\boldsymbol{F}_1 \cdot \boldsymbol{v}_{1\mathrm{o}}}{v_{1\mathrm{o}}^2}\, \boldsymbol{v}_{1\mathrm{o}} = \boldsymbol{F}_1 - (\alpha_{1\mathrm{o}} - 1)\frac{1}{v_{1\mathrm{o}}^2}\, \boldsymbol{v}_{1\mathrm{o}} \times (\boldsymbol{v}_{1\mathrm{o}} \times \boldsymbol{F}_1) = \\
&= \boldsymbol{i}\, F_{1x} + \boldsymbol{j}\, \alpha_{1\mathrm{o}}\, F_{1y} + \boldsymbol{k}\, \alpha_{1\mathrm{o}}\, F_{1z},
\end{aligned}
\tag{A.3.31}
$$

and

$$\boldsymbol{F}_{ov} = \boldsymbol{v}_{2\mathrm{o}} \times \frac{\alpha_{1\mathrm{o}}}{c^2}(\boldsymbol{v}_{1\mathrm{o}} \times \boldsymbol{F}_1). \tag{A.3.32 a}$$

Multiplying the first both sides of relation (A.3.31) by $\boldsymbol{v}_{1\mathrm{o}}$, we get:

$$\boldsymbol{v}_{1\mathrm{o}} \times \boldsymbol{F}_{or} = \alpha_{1\mathrm{o}}\, \boldsymbol{v}_{1\mathrm{o}} \times \boldsymbol{F}_1. \tag{A.3.32 b}$$

Hence, according to relations (A.3.31), (A.3.32 b), (A.3.30):

$$\boldsymbol{F}_\mathrm{o} = \boldsymbol{F}_{or} + \boldsymbol{v}_{2\mathrm{o}} \times \left(\frac{1}{c^2}\, \boldsymbol{v}_{1\mathrm{o}} \times \boldsymbol{F}_{or} \right). \tag{A.3.33}$$

We denote:

$$\boldsymbol{G}_{ob} = \frac{1}{c^2}\, \boldsymbol{v}_{1\mathrm{o}} \times \boldsymbol{F}_{or}, \tag{A.3.34}$$

and it follows:

$$F_o = F_{or} + v_{2o} \times G_{ob} .$$ (A.3.35)

If in a certain reference frame K_1, the force vector F_1 acting upon a material point is independent of the velocity v_{21} then, in any reference frame K_o, the components F_{or} and G_{ob} will also be independent of the velocity v_{21} of the material point.

Analogously to relation (A.3.35), the expression (A.3.14) can be written in the form:

$$F_1 = F_{1r} + v_{21} \times G_{1b} .$$ (A.3.36)

The components F_{1r} and G_{1b} are:

$$F_{1r} = \alpha_{1o} F_{or} - (\alpha_{1o} - 1) \frac{F_{or} \cdot v_{1o}}{v_{1o}^2} v_{1o} + \alpha_{1o} v_{1o} \times G_{ob} ,$$ (A.3.37)

$$G_{1b} = \alpha_{1o} G_{ob} - (\alpha_{1o} - 1) \frac{G_{ob} \cdot v_{1o}}{v_{1o}^2} v_{1o} - \alpha_{1o} \frac{1}{c^2} v_{1o} \times F_{or} .$$ (A.3.38)

A.3.2.3. Derivation of the Components Entering into the Transformation Expressions of the Force Acting on a Moving Material Point when Passing from a Reference Frame to Another

The aim of this Sub-subsection is to establish the relations (A.3.37) and (A.3.38), between the quantities F_{or} and F_{1r} and also between G_{or} and G_{1r}, respectively. We shall give a direct derivation, this way being more conclusive in comparison with other ones.

We suppose that the vector F_o in the reference frame K_o is given and, as shown, it can be expressed in the form:

$$F_o = F_{or} + v_{2o} \times G_{ob} .$$ (A.3.39)

We shall replace this expression into relation (A.3.14) written in the form:

$$F_1 = i F_{ox} + j \alpha_{1o} F_{oy} + k \alpha_{1o} F_{oz} - \alpha_{1o} \frac{1}{c^2} v_{21} \times (v_{1o} \times F_o).$$ (A.3.40)

We obtain:

$$F_1 = i (F_{or} + v_{2o} \times G_{ob})_x + j \alpha_{1o} (F_{or} + v_{2o} \times G_{ob})_y + k \alpha_{1o} (F_{or} + v_{2o} \times G_{ob})_z -$$

$$- \alpha_{1o} \frac{1}{c^2} v_{21} \times [v_{1o} \times (F_{or} + v_{2o} \times G_{ob})].$$

(A.3.41)

We consider the case, further called *case A*, in which we assume that:

$$F_{or} = i F_{orx} ,$$ (A.3.42 a)
$$G_{ob} = i G_{obx} .$$ (A.3.42 b)

Taking into account that, in this case, the vector $v_{2o} \times G_{ob}$ has components only along the Oy and Oz axes, hence the two terms containing the two components of the corresponding force when passing from the reference frames K_o to K_1 will contain the factor α_{1o}. The component along the Ox - axis vanishes, hence $(v_{2o} \times G_{ob})_x = 0$. Since the vector $v_{2o} \times G_{ob}$ has components only along the Oy and Oz axes, the vector v_{2o} will occur only with the last two terms of relation (A.3.6 c).

In this case, denoted *case A*, relation (A.3.41) yields successively:

$$F_{1A} = i\, F_{orx} + j\, \alpha_{1o} \left(F_{or} + v_{2o} \times G_{ob}\right)_y + k\, \alpha_{1o} \left(F_{or} + v_{2o} \times G_{ob}\right)_z -$$
$$- \alpha_{1o} \frac{1}{c^2} v_{21} \times \left[v_{1o} \times \left(F_{or} + v_{2o} \times G_{ob}\right)\right], \tag{A.3.43 a}$$

$$F_{1A} = i\, F_{orx} + \alpha_{1o} \left(v_{2o} \times G_{ob}\right) - \alpha_{1o} \frac{1}{c^2} v_{21} \times \left[v_{1o} \times \left(F_{or} + v_{2o} \times G_{ob}\right)\right]. \tag{A.3.43 b}$$

After the substitution of expression (A.3.6 c) into the last relation, and taking into account that in this case $v_{1o} = i\, v_{1ox}$, $v_{1o} \times G_{ob} = i\, v_{1ox} \times i\, G_{obx} = 0$ and $v_{1o} \times F_{or} = i\, v_{1ox} \times i\, F_{orx} = 0$, it follows:

$$F_{1A} = i\, F_{orx} + \alpha_{1o} \frac{(v_{21} + v_{1o}) \times i\, G_{obx}}{\alpha_{1o} \left(1 + \boldsymbol{\beta}_{21} \cdot \boldsymbol{\beta}_{1o}\right)} - \alpha_{1o} \frac{1}{c^2} v_{21} \times \left\{ v_{1o} \times \left[F_{or} + \frac{(v_{21} + v_{1o}) \times i\, G_{obx}}{\alpha_{1o} \left(1 + \boldsymbol{\beta}_{21} \cdot \boldsymbol{\beta}_{1o}\right)} \right] \right\}, \tag{A.3.44}$$

and further:

$$F_{1A} = i\, F_{orx} + \alpha_{1o} \frac{v_{21} \times i\, G_{obx}}{\alpha_{1o} \left(1 + \boldsymbol{\beta}_{21} \cdot \boldsymbol{\beta}_{1o}\right)} - \alpha_{1o} \frac{1}{c^2} v_{21} \times \left\{ v_{1o} \times \left[\frac{v_{21} \times i\, G_{obx}}{\alpha_{1o} \left(1 + \boldsymbol{\beta}_{21} \cdot \boldsymbol{\beta}_{1o}\right)} \right] \right\}. \tag{A.3.45}$$

The last relation can be written:

$$F_{1A} = i\, F_{orx} + \alpha_{1o} \frac{v_{21} \times G_{ob}}{\alpha_{1o} \left(1 + \boldsymbol{\beta}_{21} \cdot \boldsymbol{\beta}_{1o}\right)} - \alpha_{1o} \frac{1}{c^2} v_{21} \times \left\{ v_{1o} \times \left[\frac{v_{21} \times G_{ob}}{\alpha_{1o} \left(1 + \boldsymbol{\beta}_{21} \cdot \boldsymbol{\beta}_{1o}\right)} \right] \right\}. \tag{A.3.46}$$

By transforming the expressions inside braces, we get:

$$F_{1A} = i\, F_{orx} + \alpha_{1o} \frac{v_{21} \times G_{ob}}{\alpha_{1o} \left(1 + \boldsymbol{\beta}_{21} \cdot \boldsymbol{\beta}_{1o}\right)} - \alpha_{1o} \frac{1}{c^2} v_{21} \times \left\{ \frac{v_{21} \left(v_{1o} \cdot G_{ob}\right) - G_{ob} \left(v_{21} \cdot v_{1o}\right)}{\alpha_{1o} \left(1 + \boldsymbol{\beta}_{21} \cdot \boldsymbol{\beta}_{1o}\right)} \right\}. \tag{A.3.47}$$

Within the braces of the last term of relation (A.3.47), the term containing in numerator v_{21} can be cancelled because $v_{21} \times v_{21} = 0$. Therefore, we obtain:

$$F_{1A} = i\, F_{orx} + \alpha_{1o} \frac{v_{21} \times G_{ob}}{\alpha_{1o} \left(1 + \boldsymbol{\beta}_{21} \cdot \boldsymbol{\beta}_{1o}\right)} - \alpha_{1o} \frac{1}{c^2} v_{21} \times \left\{ \frac{- G_{ob} \left(v_{21} \cdot v_{1o}\right)}{\alpha_{1o} \left(1 + \boldsymbol{\beta}_{21} \cdot \boldsymbol{\beta}_{1o}\right)} \right\}. \tag{A.3.48}$$

By summing up the factors preceded by v_{21} and containing $G_{ob} = i\,G_{obx}$, we get:

$$\alpha_{1o}\,\frac{G_{ob}}{\alpha_{1o}\left(1 + \boldsymbol{\beta}_{21} \cdot \boldsymbol{\beta}_{1o}\right)} + \alpha_{1o}\,\frac{1}{c^2} \cdot \frac{G_{ob}\left(v_{21} \cdot v_{1o}\right)}{\alpha_{1o}\left(1 + \boldsymbol{\beta}_{21} \cdot \boldsymbol{\beta}_{1o}\right)} = G_{ob}\,. \qquad (A.3.49)$$

Therefore:

$$F_{1A} = i\,F_{orx} + v_{21} \times i\,G_{obx}\,. \qquad (A.3.50)$$

We consider the case, further called *case B*, in which we assume that:

$$F_{or} = j\,F_{ory}\,, \qquad (A.3.51\ a)$$

$$G_{ob} = j\,G_{oby}\,. \qquad (A.3.51\ b)$$

In this case, denoted *case B*, relation (A.3.41) yields:

$$F_{1B} = i\left(v_{2o} \times j\,G_{oby}\right)_x + j\,\alpha_{1o}\,F_{ory} + j\,\alpha_{1o}\left(v_{2o} \times j\,G_{oby}\right)_y + k\,\alpha_{1o}\left(v_{2o} \times j\,G_{oby}\right)_z -$$

$$- \alpha_{1o}\,\frac{1}{c^2}\,v_{21} \times \left[v_{1o} \times \left(j\,F_{ory} + v_{2o} \times j\,G_{oby}\right)\right].$$

$$(A.3.52)$$

We shall replace v_{2o} in the form:

$$v_{2o} = i\,v_{2ox} + j\,v_{2oy} + k\,v_{2oz}\,, \qquad (A.3.53\ a)$$

and

$$v_{21} = i\,v_{21x_1} + j\,v_{21y_1} + k\,v_{21z_1} = i\,v_{21x} + j\,v_{21y} + k\,v_{21z}\,. \qquad (A.3.53\ b)$$

According to relation (A.3.6 c), we get:

$$v_{2o} = i\,\frac{v_{21x} + v_{1o\,x}}{1 + \boldsymbol{\beta}_{21} \cdot \boldsymbol{\beta}_{1o}} + j\,\frac{v_{21y}}{\alpha_{1o}\left(1 + \boldsymbol{\beta}_{21} \cdot \boldsymbol{\beta}_{1o}\right)} + k\,\frac{v_{21z}}{\alpha_{1o}\left(1 + \boldsymbol{\beta}_{21} \cdot \boldsymbol{\beta}_{1o}\right)}\,. \qquad (A.3.54)$$

From relations (A.3.53) and (A.3.54), we obtain:

$$v_{2o} \times j\,G_{oby} = -i\,\frac{v_{21z}\,G_{oby}}{\alpha_{1o}\left(1 + \boldsymbol{\beta}_{21} \cdot \boldsymbol{\beta}_{1o}\right)} + k\,\frac{\left(v_{21x} + v_{1o\,x}\right)G_{oby}}{1 + \boldsymbol{\beta}_{21} \cdot \boldsymbol{\beta}_{1o}}\,. \qquad (A.3.55)$$

From relations (A.3.52) and (A.3.55), we obtain:

$$F_{1B} = j\,\alpha_{1o}\,F_{ory} - i\,\frac{v_{21z}\,G_{oby}}{\alpha_{1o}\left(1 + \boldsymbol{\beta}_{21} \cdot \boldsymbol{\beta}_{1o}\right)} + k\,\alpha_{1o}\,\frac{v_{21x}\,G_{oby}}{1 + \boldsymbol{\beta}_{21} \cdot \boldsymbol{\beta}_{1o}} + k\,\alpha_{1o}\,\frac{v_{1ox}\,G_{oby}}{1 + \boldsymbol{\beta}_{21} \cdot \boldsymbol{\beta}_{1o}} -$$

$$- \alpha_{1o}\,\frac{1}{c^2}\,v_{21} \times \left(v_{1o} \times j\,F_{ory}\right) - \alpha_{1o}\,\frac{1}{c^2}\,v_{21} \times \left[v_{1o} \times \left(v_{2o} \times j\,G_{oby}\right)\right],$$

$$(A.3.56\ a)$$

and:

$$\boldsymbol{F}_{1B} = \boldsymbol{j}\,\alpha_{1o}\,F_{ory} - \boldsymbol{i}\,\frac{v_{21z}\,G_{oby}}{\alpha_{1o}\left(1+\boldsymbol{\beta}_{21}\cdot\boldsymbol{\beta}_{1o}\right)} + \boldsymbol{k}\,\alpha_{1o}\,\frac{v_{21x}\,G_{oby}}{1+\boldsymbol{\beta}_{21}\cdot\boldsymbol{\beta}_{1o}} + \boldsymbol{k}\,\alpha_{1o}\,\frac{v_{1ox}\,G_{oby}}{1+\boldsymbol{\beta}_{21}\cdot\boldsymbol{\beta}_{1o}} -$$

$$- \alpha_{1o}\,\frac{1}{c^2}\,\boldsymbol{v}_{21}\times\left(\boldsymbol{v}_{1o}\times\boldsymbol{j}\,F_{ory}\right) -$$

$$- \alpha_{1o}\,\frac{1}{c^2}\,\boldsymbol{v}_{21}\times\left\{\boldsymbol{v}_{1o}\times\left[\left(-\boldsymbol{i}\,\frac{v_{21z}}{\alpha_{1o}\left(1+\boldsymbol{\beta}_{21}\cdot\boldsymbol{\beta}_{1o}\right)} + \boldsymbol{k}\,\frac{v_{21x}+v_{1ox}}{1+\boldsymbol{\beta}_{21}\cdot\boldsymbol{\beta}_{1o}}\right)G_{oby}\right]\right\}.$$

$$\text{(A.3.56 b)}$$

By substituting in the last relation $\boldsymbol{v}_{1o} = \boldsymbol{i}\,v_{1ox}$, we obtain:

$$\boldsymbol{F}_{1B} = \boldsymbol{j}\,\alpha_{1o}\,F_{ory} - \boldsymbol{i}\,\frac{v_{21z}\,G_{oby}}{\alpha_{1o}\left(1+\boldsymbol{\beta}_{21}\cdot\boldsymbol{\beta}_{1o}\right)} + \boldsymbol{k}\,\alpha_{1o}\,\frac{v_{21x}\,G_{oby}}{1+\boldsymbol{\beta}_{21}\cdot\boldsymbol{\beta}_{1o}} + \boldsymbol{k}\,\alpha_{1o}\,\frac{v_{1ox}\,G_{oby}}{1+\boldsymbol{\beta}_{21}\cdot\boldsymbol{\beta}_{1o}} -$$

$$- \alpha_{1o}\,\frac{1}{c^2}\,\boldsymbol{v}_{21}\times\left(\boldsymbol{v}_{1o}\times\boldsymbol{j}\,F_{ory}\right) -$$

$$- \alpha_{1o}\,\frac{1}{c^2}\,\boldsymbol{v}_{21}\times\left\{-\boldsymbol{j}\,v_{1ox}\,\frac{v_{21x}+v_{1ox}}{1+\boldsymbol{\beta}_{21}\cdot\boldsymbol{\beta}_{1o}}\,G_{oby}\right\}.$$

$$\text{(A.3.57)}$$

For obtaining the result in a compact form, we shall express the various terms as follows.

1° The term having as coefficient the quantities $\boldsymbol{i}$, G_{oby}, v_{21z}.

The corresponding factor is:

$$-\frac{1}{\alpha_{1o}}\cdot\frac{1}{1+\boldsymbol{\beta}_{21}\cdot\boldsymbol{\beta}_{1o}} - \alpha_{1o}\,\frac{\boldsymbol{\beta}_{21}\cdot\boldsymbol{\beta}_{1o}+\boldsymbol{\beta}_{1o}^2}{1+\boldsymbol{\beta}_{21}\cdot\boldsymbol{\beta}_{1o}} =$$

$$\text{(A.3.57 a)}$$

$$= -\frac{1+\alpha_{1o}^2\,\boldsymbol{\beta}_{21}\cdot\boldsymbol{\beta}_{1o}+\alpha_{1o}^2\,\boldsymbol{\beta}_{1o}^2}{\alpha_{1o}\left(1+\boldsymbol{\beta}_{21}\cdot\boldsymbol{\beta}_{1o}\right)} = -\alpha_{1o}\,,$$

since

$$\alpha_{1o}^2\,\beta_{1o}^2 = \alpha_{1o}^2 - 1.$$

2° The term having as coefficient the quantities $\boldsymbol{k}$, G_{oby}, v_{21x}.

The corresponding factor is:

$$\alpha_{1o}\,\frac{1}{1+\boldsymbol{\beta}_{21}\cdot\boldsymbol{\beta}_{1o}} + \alpha_{1o}\,\frac{\boldsymbol{\beta}_{21}\cdot\boldsymbol{\beta}_{1o}}{1+\boldsymbol{\beta}_{21}\cdot\boldsymbol{\beta}_{1o}} = \alpha_{1o}\,. \qquad\qquad \text{(A.3.57 b)}$$

3° The term having as coefficient the quantities $\boldsymbol{k}$, G_{oby}, v_{1ox}.

The corresponding factor is:

$$\alpha_{1o}\frac{1}{1+\boldsymbol{\beta}_{21}\cdot\boldsymbol{\beta}_{1o}}+\alpha_{1o}\frac{\boldsymbol{\beta}_{21}\cdot\boldsymbol{\beta}_{1o}}{1+\boldsymbol{\beta}_{21}\cdot\boldsymbol{\beta}_{1o}}=\alpha_{1o}. \tag{A.3.57 c}$$

By replacing the terms corresponding to factors given by relations (A.3.57 a, b, c) into expression (A.3.57), it follows:

$$\boldsymbol{F}_{1B}=\boldsymbol{j}\,\alpha_{1o}\,F_{ory}-\alpha_{1o}\frac{1}{c^2}\boldsymbol{v}_{21}\times\left(\boldsymbol{v}_{1o}\times\boldsymbol{j}\,F_{ory}\right)-\boldsymbol{i}\,\alpha_{1o}\,v_{21z}\,G_{oby}+$$

$$+\boldsymbol{k}\,\alpha_{1o}\,v_{21x}\,G_{oby}+\boldsymbol{k}\,\alpha_{1o}\,v_{1ox}G_{oby}. \tag{A.3.58}$$

By introducing the vector products, the last relation can be written in the form:

$$\boldsymbol{F}_{1B}=\boldsymbol{j}\,\alpha_{1o}\,F_{ory}+\alpha_{1o}\left(\boldsymbol{v}_{1o}\times\boldsymbol{j}\,G_{oby}\right)+\boldsymbol{v}_{21}\times\left[\boldsymbol{j}\,\alpha_{1o}\,G_{oby}-\alpha_{1o}\frac{1}{c^2}\left(\boldsymbol{v}_{1o}\times\boldsymbol{j}\,F_{ory}\right)\right]. \tag{A.3.59}$$

Therefore:

$$\boldsymbol{F}_{1B}=\boldsymbol{F}_{1r}+\boldsymbol{v}_{21}\times\boldsymbol{G}_{1b}. \tag{A.3.60}$$

Similarly, we can obtain:

$$\boldsymbol{F}_{1C}=\boldsymbol{k}\,\alpha_{1o}\,F_{orz}+\alpha_{1o}\left(\boldsymbol{v}_{1o}\times\boldsymbol{k}\,G_{obz}\right)+\boldsymbol{v}_{21}\times\left[\boldsymbol{k}\,\alpha_{1o}\,G_{obz}-\alpha_{1o}\frac{1}{c^2}\left(\boldsymbol{v}_{1o}\times\boldsymbol{k}\,F_{orz}\right)\right]. \tag{A.3.61}$$

Hence:

$$\boldsymbol{F}_{1C}=\boldsymbol{F}_{1r}+\boldsymbol{v}_{21}\times\boldsymbol{G}_{1b}. \tag{A.3.62}$$

By summing up, side by side, the relations (A.3.50), (A.3.59), (A.3.61), it follows:

$$\boldsymbol{F}_1=\boldsymbol{i}\,F_{orx}+\boldsymbol{j}\,\alpha_{1o}\,F_{ory}+\boldsymbol{k}\,\alpha_{1o}\,F_{orz}+\alpha_{1o}\left(\boldsymbol{v}_{1o}\times\boldsymbol{G}_{ob}\right)+$$

$$+\boldsymbol{v}_{21}\times\left[\left(\boldsymbol{i}\,G_{obx}+\boldsymbol{j}\,\alpha_{1o}\,G_{oby}+\boldsymbol{k}\,\alpha_{1o}\,G_{obz}\right)-\alpha_{1o}\frac{1}{c^2}\left(\boldsymbol{v}_{1o}\times\boldsymbol{F}_{or}\right)\right]. \tag{A.3.63}$$

Therefore:

$$\boldsymbol{F}_1=\boldsymbol{F}_{1r}+\boldsymbol{v}_{21}\times\boldsymbol{G}_{1b}. \tag{A.3.64}$$

By comparing relations (A.3.63) with (A.3.64), it follows that the quantities $\boldsymbol{F}_{1r}$ and $\boldsymbol{G}_{1b}$ are expressed in terms of the quantities $\boldsymbol{F}_{or}$ and $\boldsymbol{G}_{ob}$ just by the relations (A.3.37) and (A.3.38).

A.3.3.   INTEGRAL AND LOCAL FORMS OF RELATIONS CONTAINING THE VECTORS IN VARIOUS REFERENCE FRAMES

A.3.3.1.   The Fluxes of Vectors F_{or} and F_{1r} through a Surface

We may consider as known the resultant force F_1 in the reference frame K_1 acting upon a moving material point and we assume that it is independent of the velocity v_{21}. Hence, according to relation (A.3.36), the vector G_{1b} is zero.

We need to calculate the flux of the vectors F_{or} and F_{1r} through a given surface, open or closed, in the reference frames K_o and K_1, respectively. Firstly, we shall calculate the flux in the reference frame K_1.

For obtaining the surface element ΔS_1, we start from the quantity ΔS_o in the reference frame K_o where the flux has to be calculated. For calculating the vector F_{or}, we start from the vector F_1 in the reference frame K_1 where the resultant force has been considered. In this way, as we shall show immediately, the flux remains unchanged in the two reference frames. The conservation of the flux through a surface is equivalent with the geometrical interpretation that the number of lines of field, through the considered surface, remains unchanged.

In the reference frame K_o, we obtain the flux:

$$F_{or} \cdot \Delta S_o .$$

$$(A.3.65)$$

Taking into account relation (A.3.36) and $G_{1b} = 0$ of above, it results that $F_1 = F_{1r}$. Hence, in the reference frame K_1, we obtain the flux:

$$F_{1r} \cdot \Delta S_1 = \alpha_{1o}\, F_1 \cdot \Delta S_o + \left(1 - \alpha_{1o}\right)\frac{1}{v_{1o}^2}\left(v_{1o} \cdot \Delta S_o\right)v_{1o} \cdot F_1 .$$

$$(A.3.66)$$

But in the reference frame K_o, as shown above, by relation (A.3.31), we have:

$$F_{or} = \alpha_{1o}\, F_1 + \left(1 - \alpha_{1o}\right)\frac{1}{v_{1o}^2}\left(v_{1o} \cdot F_1\right)v_{1o} .$$

$$(A.3.67)$$

Multiplying both sides of the last equation by ΔS_o, it follows:

$$F_{1r} \cdot \Delta S_1 = F_{or} \cdot \Delta S_o .$$

$$(A.3.68)$$

For a closed surface, we obtain:

$$\int_{\Sigma_1} F_{1r} \cdot \mathrm{d}\, S_1 = \int_{\Sigma_o} F_{or} \cdot \mathrm{d}\, S_o ,$$

$$(A.3.69\ a)$$

in integral form, and:

$$\mathrm{div}_1 \boldsymbol{F}_{1r} = \alpha_{1o}\, \mathrm{div}_o\, \boldsymbol{F}_{or}\,, \tag{A.3.69 b}$$

in differential (local) form, respectively. The index of the operator div indicates the reference frame in which this operator has been calculated.

A.3.3.2. The Flux of the Vector $\boldsymbol{G}_{ob}$ through a Closed Surface

We need to calculate the fluxes of vectors $\boldsymbol{G}_{ob}$ given by relation (A.3.34) through a given surface, a closed or open one, in the reference frame K_o :

$$
\begin{aligned}
\boldsymbol{G}_{ob} \cdot \Delta \boldsymbol{S}_o &= \frac{1}{c^2} \Delta \boldsymbol{S}_o \cdot \left(\boldsymbol{v}_{1o} \times \boldsymbol{F}_{or}\right) = \frac{1}{c^2}\left\{ \boldsymbol{v}_{1o} \times \left[\alpha_{1o}\, \boldsymbol{F}_1 - \left(\alpha_{1o} - 1\right)\frac{1}{v_{1o}^2}\left(\boldsymbol{F}_1 \cdot \boldsymbol{v}_{1o}\right)\boldsymbol{v}_{1o} \right] \right\} \cdot \Delta \boldsymbol{S}_o = \\
&= \frac{1}{c^2} \alpha_{1o}\left(\boldsymbol{v}_{1o} \times \boldsymbol{F}_1\right) \cdot \Delta \boldsymbol{S}_o\,.
\end{aligned}
\tag{A.3.70}
$$

Taking into account relation (A.3.3) and that $\left(\boldsymbol{v}_{1o} \times \boldsymbol{F}_1\right) \cdot \boldsymbol{v}_{1o} = 0$, it follows:

$$
\begin{aligned}
\boldsymbol{G}_{ob} \cdot \Delta \boldsymbol{S}_o &= \frac{1}{c^2}\left(\boldsymbol{v}_{1o} \times \boldsymbol{F}_{or}\right) \cdot \Delta \boldsymbol{S}_o = \frac{1}{c^2}\left\{ \boldsymbol{v}_{1o} \times \left[\alpha_{1o}\, \boldsymbol{F}_1 - \left(\alpha_{1o} - 1\right)\frac{1}{v_{1o}^2}\left(\boldsymbol{F}_1 \cdot \boldsymbol{v}_{1o}\right)\boldsymbol{v}_{1o} \right] \right\} \cdot \Delta \boldsymbol{S}_o = \\
&= \frac{1}{c^2}\left(\boldsymbol{v}_{1o} \times \boldsymbol{F}_1\right) \cdot \left[\alpha_{1o}\, \Delta \boldsymbol{S}_o + \left(1 - \alpha_{1o}\right)\frac{1}{v_{1o}^2}\left(\boldsymbol{v}_{1o} \cdot \Delta \boldsymbol{S}_o\right)\boldsymbol{v}_{1o} \right] = \\
&= \frac{1}{c^2}\left(\boldsymbol{v}_{1o} \times \boldsymbol{F}_1\right) \cdot \Delta \boldsymbol{S}_1\,.
\end{aligned}
\tag{A.3.71}
$$

It follows:

$$
\int_{\Sigma_o} \boldsymbol{G}_{ob} \cdot \mathrm{d}\boldsymbol{S}_o = \frac{1}{c^2} \boldsymbol{v}_{1o} \cdot \int_{\Sigma_1} \boldsymbol{F}_1 \times \mathrm{d}\boldsymbol{S}_1 = -\frac{1}{c^2} \boldsymbol{v}_{1o} \cdot \int_{V_{\Sigma_1}} \mathrm{curl}_1\, \boldsymbol{F}_1\, \mathrm{d}V_1\,, \tag{A.3.72}
$$

in integral form, and

$$\alpha_{1o}\, \mathrm{div}_o\, \boldsymbol{G}_{ob} = -\frac{1}{c^2} \boldsymbol{v}_{1o} \cdot \mathrm{curl}_1\, \boldsymbol{F}_1\,, \tag{A.3.73}$$

in differential (local) form. The indices of the operators div and curl indicate the reference frame in which these operators have been calculated.

A.3.3.3.   The Circulation of the Vector F_{or} along a Closed Curve in the Case of a Field of Vectors with Central Symmetry

We need to calculate the circulation of the vector F_{or} round a closed curve in the reference frame K_o. We consider the case of any *field of forces with central symmetry* in the reference frame K_1. We also assume the same condition as at the beginning of Sub-section A.3.3.1, and hence $G_{1b} = 0$. As previously, we adopt the axes of the system of co-ordinates in a convenient manner, namely so that the vector velocity v_{1o} be parallel and of the same direction with the Ox - axis. The general expression of the forces in the considered case is:

$$F_1 = f(r_{1AP})r_{1AP}, \tag{A.3.74}$$

where the position vector $r_{1AP} = r_{AP}$ has the origin at any point $A(x_{1A}, y_{1A}, z_{1A})$, source point, and the extremity at the point $P(x_o, y_o, z_o)$, observation or field point, at which the mentioned force has to be calculated at any moment t_1 in the reference frame K_1. The position vector can be expressed in the form:

$$r_{1AP} = i\left[\alpha_{1o}\left(x_o - v_{1ox}t_o\right) - x_{1A}\right] + j\left(y_o - y_{1A}\right) + k\left(z_o - z_{1A}\right), \tag{A.3.74 a}$$

or, generally, for any direction of v_{1o}:

$$r_{1AP} = r_o - v_{1o}t_o + (\alpha_{1o} - 1)v_{1o} \cdot (r_o - v_{1o}t_o)\frac{v_{1o}}{v_{1o}^2} - r_{1A}. \tag{A.3.74 b}$$

We shall use the expression of the force (A.3.31) in the form:

$$F_{or} = i\,F_{1x} + j\,\alpha_{1o}\,F_{1y} + k\,\alpha_{1o}F_{1z}. \tag{A.3.75}$$

The circulation of the vector F_{or} round a closed curve Γ is:

$$\oint_{\Gamma_o} F_{or} \cdot d\boldsymbol{l} = \int_{S_{\Gamma_o}} \mathrm{curl}_o\, F_{or} \cdot d\boldsymbol{S}_o, \tag{A.3.76}$$

where S_{Γ_o} is a simply connected open surface bounded by the curve Γ_o. It is assumed that on the surface S_{Γ_o} the function $F_{or}(r_1)$ is differentiable at each point. Taking into account the relations (A.3.74), (3.75), (A.3.76), after performing the calculations of components of the curl applied to that function, we obtain:

$$\mathrm{curl}_o\, F_{or} = -\frac{\partial}{\partial t_o}\left[i \cdot 0 - j\frac{1}{c^2}v_{1ox}\,F_{orz} + k\frac{1}{c^2}v_{1ox}\,F_{ory}\right]. \tag{A.3.77}$$

Hence, taking into account relation (A.3.34), we get:

$$\text{curl}_\text{o}\, \boldsymbol{F}_{or} = -\frac{\partial}{\partial t_\text{o}}\left[\frac{1}{c^2}\boldsymbol{v}_{1\text{o}}\times\boldsymbol{F}_{or}\right] = -\frac{\partial}{\partial t_\text{o}}\boldsymbol{G}_{ob}\,. \tag{A.3.78}$$

Since the operator curl_o and the time t_o are invariant with respect to the transformation for passing from one three-orthogonal rectilinear system of co-ordinates to another three-orthogonal rectilinear system co-ordinates at rest relative to the first one, it follows that relation (A.3.78) remains valid for any direction of the velocity $\boldsymbol{v}_{1\text{o}}$. Since relations (A.3.77), (A.3.75), (A.3.74) are linear relatively to vectors $\boldsymbol{F}_{or}$, $\boldsymbol{G}_{ob}$, $\boldsymbol{r}_1$, it follows that relation (A.3.78) is valid also in the case in which the right-hand side of relation (A.3.74) is a linear combination of terms of the same form. It follows:

$$\oint_{\Gamma_\text{o}} \boldsymbol{F}_{or}\cdot\text{d}\boldsymbol{l}_\text{o} = -\frac{\partial}{\partial t_\text{o}}\int_{S_{\Gamma_\text{o}}}\boldsymbol{G}_{ob}\cdot\text{d}\boldsymbol{S}_\text{o}\,, \tag{A.3.79}$$

in integral form, and

$$\text{curl}_\text{o}\, \boldsymbol{F}_{or} = -\frac{\partial}{\partial t_\text{o}}\boldsymbol{G}_{ob}\,, \tag{A.3.80}$$

in differential (local) form.

It can be remarked the analogy of the two last relation with the expression of the law of electromagnetic induction.

A.3.3.4. The Circulation of the Vector $\boldsymbol{G}_{ob}$ along a Closed Curve in the Case of a Field of Vectors with Central Symmetry

We need to calculate the circulation of the vector $\boldsymbol{G}_{ob}$ round (along) a closed curve in the reference frame K_o. We consider the case of a field of vectors of central symmetry in the reference frame K_1 and under the same conditions as in Sub-section A.3.3.1.

The circulation of the vector $\boldsymbol{G}_{ob}$ along any closed curve Γ_o is:

$$\oint_{\Gamma_\text{o}} \boldsymbol{G}_{ob}\cdot\text{d}\boldsymbol{l}_\text{o} = \int_{S_{\Gamma_\text{o}}}\text{curl}_\text{o}\,\boldsymbol{G}_{ob}\cdot\text{d}\boldsymbol{S}_\text{o}\,, \tag{A.3.81}$$

where S_{Γ_o} is the open surface bounded by the curve Γ_o. Taking into account relation (A.3.34), we can write:

$$\text{curl}_\text{o}\, \boldsymbol{G}_{ob} = \text{curl}_\text{o}\left(\frac{1}{c^2}\boldsymbol{v}_{1\text{o}}\times\boldsymbol{F}_{or}\right) = \frac{1}{c^2}\boldsymbol{v}_{1\text{o}}\,\text{div}_\text{o}\,\boldsymbol{F}_{or} - \frac{1}{c^2}\left(\boldsymbol{v}_{1\text{o}}\cdot\nabla_\text{o}\right)\boldsymbol{F}_{or}\,. \tag{A.3.82}$$

Taking into account relations (A.3.74), (A.3.75), after performing the calculations of the components of each vector, we shall obtain:

$$-\frac{1}{c^2}\left(\boldsymbol{v}_{1o} \cdot \nabla_o \right)\boldsymbol{F}_{or} = \frac{1}{c^2} \cdot \frac{\partial \boldsymbol{F}_{or}}{\partial t_o}. \tag{A.3.83}$$

Therefore, relation (A.3.82) becomes:

$$\mathrm{curl}_o\, \boldsymbol{G}_{ob} = \frac{1}{c^2}\boldsymbol{v}_{1o}\, \mathrm{div}_o\, \boldsymbol{F}_{or} + \frac{1}{c^2} \cdot \frac{\partial}{\partial t_o}\boldsymbol{F}_{or}. \tag{A.3.84}$$

The operator curl_o, the operator div_o, and the time t_o are invariant with respect to the transformation from one right-handed three-orthogonal rectilinear system of co-ordinates to another right-handed three-orthogonal rectilinear system of co-ordinates, at rest relative to the first one. It follows that relation (A.3.83) is valid for any direction of the velocity $\boldsymbol{v}_{1o}$. For similar reasons to those of Sub-section A.3.3.3, relation (A.3.84) is valid also in the case in which the right-hand side of relation (A.3.74) is a linear combination of terms of the same form. It follows:

$$\oint_{\Gamma_o} \boldsymbol{G}_{ob} \cdot \mathrm{d}\boldsymbol{l}_o = \int_{S_{\Gamma_o}} \frac{1}{c^2}\boldsymbol{v}_{1o}\, \mathrm{div}_o\, \boldsymbol{F}_{or} \cdot \mathrm{d}\boldsymbol{S}_o + \frac{\partial}{\partial t_o} \int_{S_{\Gamma_o}} \frac{1}{c^2}\boldsymbol{F}_{or} \cdot \mathrm{d}\boldsymbol{S}_o, \tag{A.3.85}$$

in integral form, and

$$\mathrm{curl}_o\, \boldsymbol{G}_{ob} = \frac{1}{c^2}\boldsymbol{v}_{1o}\, \mathrm{div}_o\, \boldsymbol{F}_{or} + \frac{1}{c^2} \cdot \frac{\partial \boldsymbol{F}_{or}}{\partial t_o}, \tag{A.3.86}$$

in differential (local) form.

It can be remarked the analogy of the two last relation with the expression of the law of magnetic circuit.

A.3.3.5. The Relation between the Volume Densities of a Scalar Function when Passing from One Reference Frame to Another

We consider the volume density of a scalar function the magnitude of which does not change when passing from one inertial reference frame to another. This is the case of the electric charge. In any reference frame, the volume density of the electric charges is expressed in terms of the volume density of the electric charges of the reference frame in which there are at rest. We consider the reference frames $K_o(x_o, y_o, z_o)$, $K_1(x_1, y_1, z_1)$, $K_2(x_2, y_2, z_2)$. We shall have in view that the value of the electric charge of any body is invariant with respect to the transformation of the axes of co-ordinates in the reference frame and we shall suppose that the electric charges are at rest in the reference frame K_o. We express the conservation of the electric charge of a volume element:

$$\rho_{v2}\, \mathrm{d}V_{\Sigma_2} = \rho_{v1}\, \mathrm{d}V_{\Sigma_1} = \rho_{vo}\, \mathrm{d}V_{\Sigma o}. \tag{A.3.87}$$

For an observer in the reference frame K_2, taking into account expression (A.3.4), the following relations can be written:

$$\mathrm{d}V_{\Sigma_2} = \alpha_{21}\,\mathrm{d}V_{\Sigma_1} = \alpha_{2\mathrm{o}}\,\mathrm{d}V_{\Sigma\mathrm{o}}\,. \tag{A.3.88}$$

The two last relations yield:

$$\rho_{v2} = \frac{1}{\alpha_{21}}\rho_{v1} = \frac{1}{\alpha_{2\mathrm{o}}}\rho_{v\mathrm{o}}\,. \tag{A.3.89}$$

Having in view the expressions of the quantities of the form α_{ij} and $\boldsymbol{\beta}_{ij}$, relation (A.3.89), as shown below, yields:

$$\rho_{v2} = \frac{1}{\alpha_{1\mathrm{o}}\,\alpha_{2\mathrm{o}}} \cdot \frac{1}{1 - \boldsymbol{\beta}_{2\mathrm{o}} \cdot \boldsymbol{\beta}_{1\mathrm{o}}}\rho_{v1} = \frac{1}{\alpha_{2\mathrm{o}}}\rho_{v\mathrm{o}}\,. \tag{A.3.90}$$

Therefore, the relation between the volume densities is:

$$\rho_{v1} = \alpha_{1\mathrm{o}}\left(1 - \boldsymbol{\beta}_{2\mathrm{o}} \cdot \boldsymbol{\beta}_{1\mathrm{o}}\right)\rho_{v\mathrm{o}}\,. \tag{A.3.91}$$

A.3.3.6. The Derivation of the Relation between the Volume Densities of a Scalar Function when Passing from One Reference Frame to Another

We have to calculate the coefficient α_{21} of the preceding formula (A.3.89) for the case $\boldsymbol{v}_{1\mathrm{o}} = \boldsymbol{i}\,v_{1\mathrm{o}x} = v_{1\mathrm{o}}$. This quantity is:

$$\alpha_{21} = \frac{1}{\sqrt{1 - \beta_{21}^2}}\,. \tag{A.3.92}$$

From relation (A.3.6 a) written in the same form as (A.3.6 c), it follows:

$$\boldsymbol{\beta}_{21} = \boldsymbol{i}\,\frac{\beta_{2\mathrm{o}x} - \beta_{1\mathrm{o}x}}{1 - \boldsymbol{\beta}_{2\mathrm{o}} \cdot \boldsymbol{\beta}_{1\mathrm{o}}} + \boldsymbol{j}\,\frac{\beta_{2\mathrm{o}y}}{\alpha_{1\mathrm{o}}\left(1 - \boldsymbol{\beta}_{2\mathrm{o}} \cdot \boldsymbol{\beta}_{1\mathrm{o}}\right)} + \boldsymbol{k}\,\frac{\beta_{2\mathrm{o}z}}{\alpha_{1\mathrm{o}}\left(1 - \boldsymbol{\beta}_{2\mathrm{o}} \cdot \boldsymbol{\beta}_{1\mathrm{o}}\right)}\,. \tag{A.3.93}$$

Hence:

$$1 - \beta_{21}^2 = 1 - \frac{\alpha_{1\mathrm{o}}^2\left(\beta_{2\mathrm{o}x} - \beta_{1\mathrm{o}x}\right)^2}{\alpha_{1\mathrm{o}}^2\left(1 - \boldsymbol{\beta}_{2\mathrm{o}} \cdot \boldsymbol{\beta}_{1\mathrm{o}}\right)^2} - \frac{\beta_{2\mathrm{o}y}^2 + \beta_{2\mathrm{o}z}^2}{\alpha_{1\mathrm{o}}^2\left(1 - \boldsymbol{\beta}_{2\mathrm{o}} \cdot \boldsymbol{\beta}_{1\mathrm{o}}\right)^2} =$$

$$= \frac{\alpha_{1\mathrm{o}}^2\left(1 - \beta_{2\mathrm{o}x}\,\beta_{1\mathrm{o}x}\right)^2 - \alpha_{1\mathrm{o}}^2\left(\beta_{2\mathrm{o}x} - \beta_{1\mathrm{o}x}\right)^2 - \beta_{2\mathrm{o}}^2 + \beta_{2\mathrm{o}x}^2}{\alpha_{1\mathrm{o}}^2\left(1 - \boldsymbol{\beta}_{2\mathrm{o}} \cdot \boldsymbol{\beta}_{1\mathrm{o}}\right)^2} =$$

$$= \frac{\alpha_{1o}^2 - 2\alpha_{1o}^2 \beta_{2ox} \beta_{1ox} + \alpha_{1o}^2 \beta_{2ox}^2 \beta_{1ox}^2 - \alpha_{1o}^2 \beta_{2ox}^2 - \alpha_{1o}^2 \beta_{1ox}^2 + 2\alpha_{1o}^2 \beta_{2ox} \beta_{1ox} - \beta_{2o}^2 + \beta_{2ox}^2}{\alpha_{1o}^2 \left(1 - \boldsymbol{\beta}_{2o} \cdot \boldsymbol{\beta}_{1o}\right)^2}$$

$$\text{(A.3.94)}$$

But:

$$\alpha_{1o}^2 - \alpha_{1o}^2 \beta_{1ox}^2 = 1. \tag{A.3.94 a}$$

Consequently, we obtain successively:

$$1 - \beta_{21}^2 = \frac{\alpha_{1o}^2 \left(1 - \beta_{1ox}^2\right) + \beta_{2ox}^2 \alpha_{1o}^2 \left(\beta_{1ox}^2 - 1\right) - \beta_{2o}^2 + \beta_{2ox}^2}{\alpha_{1o}^2 \left(1 - \beta_{21x} \beta_{1ox}\right)^2}, \tag{A.3.95 a}$$

$$1 - \beta_{21}^2 = \frac{1 - \beta_{2ox}^2 - \beta_{2o}^2 + \beta_{2ox}^2}{\alpha_{1o}^2 \left(1 - \beta_{21x} \beta_{1ox}\right)^2} = \frac{1}{\alpha_{2o}^2 \alpha_{1o}^2 \left(1 - \beta_{21x} \beta_{1ox}\right)^2}, \tag{A.3.95 b}$$

$$\alpha_{21} = \frac{1}{\sqrt{1 - \beta_{21}^2}} = \frac{\alpha_{1o} \left(1 - \boldsymbol{\beta}_{2o} \cdot \boldsymbol{\beta}_{1o}\right)}{\sqrt{1 - \beta_{2o}^2}} = \alpha_{1o} \alpha_{2o} \left(1 - \boldsymbol{\beta}_{2o} \cdot \boldsymbol{\beta}_{1o}\right). \tag{A.3.96}$$

A.3.3.7.   The Relation between the Densities of the Flow Rate of a Scalar Quantity when Passing from One Reference Frame to Another

We consider the flowing rate of a scalar quantity, namely of a quantity that does not change when passing from one inertial reference frame to another. This is the case of the vector electric current density. Let us consider the reference frames K_o and K_1. We suppose that a set of electric charges is moving at the velocity $\boldsymbol{v}_{21}$ relatively to the reference frame K_1. The electric current density in the reference frame K_1 is:

$$\boldsymbol{J}_1 = \rho_{v1} \boldsymbol{v}_{21}. \tag{A.3.97}$$

Taking into account relations (A.3.6 a, c), (A.3.91) and (A.3.97), it follows:

$$\boldsymbol{J}_1 = \rho_{vo} \boldsymbol{v}_{2o} + \rho_{vo} \left[(\alpha_{1o} - 1) \frac{\boldsymbol{\beta}_{2o} \cdot \boldsymbol{\beta}_{1o}}{\boldsymbol{\beta}_{1o}^2} - \alpha_{1o} \right] \boldsymbol{v}_{1o}. \tag{A.3.98}$$

A.3.4. RELATIONS BETWEEN THE DIFFERENTIAL OPERATORS WHEN PASSING FROM ONE REFERENCE FRAME TO ANOTHER

We consider the inertial reference frames K_o and K_1. The general relation between the differential operators when passing from one reference frame to another can be obtained by assuming, for instance, that the velocity of the reference frame K_1 relative to the reference frame K_o is parallel and of the same direction with the axes $O_o x_o$ and $O_1 x_1$. Since $\boldsymbol{v}_{1o} = \boldsymbol{i}\, v_{1ox} = \boldsymbol{i}\, v_{1o}$, we obtain:

$$x_o = \alpha_{1o}\left(x_1 + v_{1o}\, t_1\right); \qquad y_o = y_1; \qquad z_o = z_1;$$

$$t_o = \alpha_{1o}\left(t_1 + \frac{v_{1o}}{c^2}\, x_1\right);$$

$$x_1 = \alpha_{1o}\left(x_o - v_{1o}\, t_o\right); \qquad y_1 = y_o; \qquad z_1 = z_o;$$

(A.3.99 a, . . ., h)

$$t_1 = \alpha_{1o}\left(t_o - \frac{v_{1o}}{c^2}\, x_o\right);$$

and for a differentiable function $f\left(x_o, y_o, z_o, t_o\right)$:

$$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_o} \cdot \frac{\partial x_o}{\partial x_1} + \frac{\partial f}{\partial t_o} \cdot \frac{\partial t_o}{\partial x_1}; \qquad \frac{\partial f}{\partial t_1} = \frac{\partial f}{\partial x_o} \cdot \frac{\partial x_o}{\partial t_1} + \frac{\partial f}{\partial t_o} \cdot \frac{\partial t_o}{\partial t_1};$$

(A.3.100 a, . . ., d)

$$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial x_o} \alpha_{1o} + \frac{\partial f}{\partial t_o} \alpha_{1o} \frac{v_{1o}}{c^2}; \qquad \frac{\partial f}{\partial t_1} = \frac{\partial f}{\partial x_o} \alpha_{1o}\, v_{1o} + \frac{\partial f}{\partial t_o} \alpha_{1o}.$$

We shall establish the relations between the differential nabla operators expressed in the two reference frames, namely:

$$\nabla_o = \boldsymbol{i}\,\frac{\partial}{\partial x_o} + \boldsymbol{j}\,\frac{\partial}{\partial y_o} + \boldsymbol{k}\,\frac{\partial}{\partial z_o}; \qquad \nabla_1 = \boldsymbol{i}\,\frac{\partial}{\partial x_1} + \boldsymbol{j}\,\frac{\partial}{\partial y_1} + \boldsymbol{k}\,\frac{\partial}{\partial z_1}. \qquad \text{(A.3.101 a, b)}$$

Taking into account that: $\boldsymbol{v}_{o1} = -\boldsymbol{v}_{1o}$, it follows:

$$\nabla_1 = \nabla_o + \frac{1}{v_{1o}^2}\left(\alpha_{1o} - 1\right)\left(\boldsymbol{v}_{1o} \cdot \nabla_o\right)\boldsymbol{v}_{1o} + \alpha_{1o}\frac{\boldsymbol{v}_{1o}}{c^2} \cdot \frac{\partial}{\partial t_o};$$

(A.3.102 a, b)

$$\nabla_o = \nabla_1 + \frac{1}{v_{1o}^2}\left(\alpha_{1o} - 1\right)\left(\boldsymbol{v}_{1o} \cdot \nabla_1\right)\boldsymbol{v}_{1o} - \alpha_{1o}\frac{\boldsymbol{v}_{1o}}{c^2} \cdot \frac{\partial}{\partial t_1}.$$

For example:

$$\nabla_1 \cdot \boldsymbol{A} = \nabla_{\mathrm{o}} \cdot \boldsymbol{A} + \frac{1}{v_{1\mathrm{o}}^2}(\alpha_{1\mathrm{o}} - 1)(\boldsymbol{v}_{1\mathrm{o}} \cdot \nabla_{\mathrm{o}})(\boldsymbol{v}_{1\mathrm{o}} \cdot \boldsymbol{A}) + \alpha_{1\mathrm{o}} \frac{\boldsymbol{v}_{1\mathrm{o}}}{c^2} \cdot \frac{\partial \boldsymbol{A}}{\partial t_{\mathrm{o}}};$$

$$\nabla_1 \times \boldsymbol{A} = \nabla_{\mathrm{o}} \times \boldsymbol{A} + \frac{1}{v_{1\mathrm{o}}^2}(\alpha_{1\mathrm{o}} - 1)(\boldsymbol{v}_{1\mathrm{o}} \cdot \nabla_{\mathrm{o}})(\boldsymbol{v}_{1\mathrm{o}} \times A) + \alpha_{1\mathrm{o}} \frac{\boldsymbol{v}_{1\mathrm{o}}}{c^2} \times \frac{\partial \boldsymbol{A}}{\partial t_{\mathrm{o}}}.$$

(A.3.103 a, b)

Also, it is useful to mention that for a scalar or vector function that depends only on x_1, y_1, z_1, for example $f(x_1, y_1, z_1)$ or $\boldsymbol{A}(x_1, y_1, z_1)$, we obtain:

$$\frac{\partial f}{\partial t_1} = 0; \qquad \frac{\partial \boldsymbol{A}}{\partial t_1} = 0; \qquad\qquad\qquad \text{(A.3.104 a, b)}$$

and

$$\frac{\partial f}{\partial t_{\mathrm{o}}} = -\boldsymbol{v}_{1\mathrm{o}} \cdot (\nabla_{\mathrm{o}} f); \qquad \frac{\partial \boldsymbol{A}}{\partial t_{\mathrm{o}}} = -(\boldsymbol{v}_{1\mathrm{o}} \cdot \nabla_{\mathrm{o}})\boldsymbol{A}. \qquad \text{(A.3.104 c, d)}$$

A.3.5. APPLICATION TO THE TRANSFORMATION OF SCALAR AND VECTOR POTENTIALS WHEN PASSING FROM ONE REFERENCE FRAME TO ANOTHER

The aim is to establish the relation between the pairs V_1, $\boldsymbol{A}_1$ and V_{o}, $\boldsymbol{A}_{\mathrm{o}}$, respectively. Let K_{o} and K_1 be reference frames and the considered vectors:

$$\boldsymbol{E}_1 = \boldsymbol{i}\,E_{\mathrm{o}x_{\mathrm{o}}} + \boldsymbol{j}\,\alpha_{1\mathrm{o}}\,E_{\mathrm{o}y_{\mathrm{o}}} + \boldsymbol{k}\,\alpha_{1\mathrm{o}}\,E_{\mathrm{o}z_{\mathrm{o}}} + \alpha_{1\mathrm{o}}(\boldsymbol{v}_{1\mathrm{o}} \times \boldsymbol{B}_{\mathrm{o}});$$

$$E_{1x_1} = E_{\mathrm{o}x_{\mathrm{o}}} + \alpha_{1\mathrm{o}}\,\boldsymbol{i} \cdot (\boldsymbol{v}_{1\mathrm{o}} \times \boldsymbol{B}_{\mathrm{o}});$$

$$E_{1y_1} = \alpha_{1\mathrm{o}}\,E_{\mathrm{o}y_{\mathrm{o}}} + \alpha_{1\mathrm{o}}\,\boldsymbol{j} \cdot (\boldsymbol{v}_{1\mathrm{o}} \times \boldsymbol{B}_{\mathrm{o}}); \qquad \text{(A.3.105 a, . . .,d)}$$

$$E_{1z_1} = \alpha_{1\mathrm{o}}\,E_{\mathrm{o}z_{\mathrm{o}}} + \alpha_{1\mathrm{o}}\,\boldsymbol{k} \cdot (\boldsymbol{v}_{1\mathrm{o}} \times \boldsymbol{B}_{\mathrm{o}});$$

and for the reference frame K_{o}, according to formula (3.200), of Chapter 3, we have:

$$\boldsymbol{B}_{\mathrm{o}} = \mathrm{curl}_{\mathrm{o}}\,\boldsymbol{A}_{\mathrm{o}};$$

$$\boldsymbol{E}_{\mathrm{o}} = -\mathrm{grad}_{\mathrm{o}}\,V_{\mathrm{o}} - \frac{\partial \boldsymbol{A}_{\mathrm{o}}}{\partial t_{\mathrm{o}}}; \qquad\qquad\qquad \text{(A.3.106 a, b, c)}$$

$$\boldsymbol{v}_{1\mathrm{o}} \times \mathrm{curl}_{\mathrm{o}}\,\boldsymbol{A}_{\mathrm{o}} = -\boldsymbol{j}\,v_{1\mathrm{o}}\left(\frac{\partial A_{\mathrm{o}y_{\mathrm{o}}}}{\partial x_{\mathrm{o}}} - \frac{\partial A_{\mathrm{o}x_{\mathrm{o}}}}{\partial y_{\mathrm{o}}}\right) + \boldsymbol{k}\,v_{1\mathrm{o}}\left(\frac{\partial A_{\mathrm{o}x_{\mathrm{o}}}}{\partial z_{\mathrm{o}}} - \frac{\partial A_{\mathrm{o}z_{\mathrm{o}}}}{\partial x_{\mathrm{o}}}\right).$$

As above, we have taken $\boldsymbol{v}_{1\mathrm{o}} = \boldsymbol{i}\,v_{1\mathrm{o}x_{\mathrm{o}}} = \boldsymbol{i}\,v_{1\mathrm{o}}$.

Now, we consider the Ox - axis. From relations like (3.200), we obtain:

$$E_{1x_1} = -\frac{\partial V_1}{\partial x_1} - \frac{\partial A_{1x_1}}{\partial t_1}; \qquad \text{(A.3.107 a)}$$

$$E_{ox_o} = -\frac{\partial V_o}{\partial x_o} - \frac{\partial A_{ox_o}}{\partial t_o}. \qquad \text{(A.3.107 b)}$$

According to equation (A.3.105 b), we need $\boldsymbol{v}_{1o} \times \boldsymbol{B}_o = \boldsymbol{v}_{1o} \times \text{curl}_o \, \boldsymbol{A}_o$. As previously, we take $\boldsymbol{v}_{1o} = \boldsymbol{i}\, v_{1ox_o} = \boldsymbol{i}\, v_{1o}$. We get:

$$\boldsymbol{i} \cdot \left(\boldsymbol{v}_{1o} \times \boldsymbol{B}_o\right) = 0. \qquad \text{(A.3.107 c)}$$

By replacing expressions (A.3.107 a, b, c) into relation (A.3.105 b), we get:

$$-\frac{\partial V_1}{\partial x_1} - \frac{\partial A_{1x_1}}{\partial t_1} = -\frac{\partial V_o}{\partial x_o} - \frac{\partial A_{ox_o}}{\partial t_o}. \qquad \text{(A.3.108)}$$

Using relations (A.3.99 a, . . ., h) and (A.3.100 a, . . ., d) with other indices, we get:

$$\frac{\partial V_o}{\partial x_o} = \frac{\partial V_o}{\partial x_1} \cdot \frac{\partial x_1}{\partial x_o} + \frac{\partial V_o}{\partial t_1} \cdot \frac{\partial t_1}{\partial x_o} = \alpha_{1o} \frac{\partial V_o}{\partial x_1} - \alpha_{1o} \frac{v_{1o}}{c^2} \cdot \frac{\partial V_o}{\partial t_1}. \qquad \text{(A.3.109)}$$

$$\frac{\partial A_{o1x_o}}{\partial t_o} = \frac{\partial A_{ox_o}}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_o} + \frac{\partial A_{ox_o}}{\partial t_1} \cdot \frac{\partial t_1}{\partial t_o} = \alpha_{1o} \frac{\partial A_{ox_o}}{\partial t_1} - \alpha_{1o} v_{1o} \frac{\partial A_{ox_o}}{\partial x_1}. \qquad \text{(A.3.110)}$$

After replacing the expressions (A.3.109), (A.3.110), into relation (A.3.108), and performing the calculations, it results that:

$$-\frac{\partial V_1}{\partial x_1} - \frac{\partial A_{1x_1}}{\partial t_1} = -\alpha_{1o} \frac{\partial V_o}{\partial x_1} + \alpha_{1o} \frac{v_{1o}}{c^2} \cdot \frac{\partial V_o}{\partial t_1} - \alpha_{1o} \frac{\partial A_{ox_o}}{\partial t_1} + \alpha_{1o} v_{1o} \frac{\partial A_{ox_o}}{\partial x_1}. \qquad \text{(A.3.111)}$$

By equating the derivatives with respect to the same variables of both sides of the last relation, we have:

$$V_1 = \alpha_{1o}\left(V_o - v_{1ox} A_{ox_o}\right),$$

$$A_{1x_1} = \alpha_{1o}\left(A_{ox_o} - \frac{v_{1ox}}{c^2} V_o\right). \qquad \text{(A.3.112 a, b)}$$

Now, we consider the Oy - axis. From relation (3.200), we get:

$$E_{1y_1} = -\frac{\partial V_1}{\partial y_1} - \frac{\partial A_{1y_1}}{\partial t_1}; \qquad \text{(A.3.113 a)}$$

$$E_{oy_o} = -\frac{\partial V_o}{\partial y_o} - \frac{\partial A_{oy_o}}{\partial t_o}. \qquad \text{(A.3.113 b)}$$

According to equation (A.3.105 c), we need $\boldsymbol{v}_{1o} \times \boldsymbol{B}_o = \boldsymbol{v}_{1o} \times \text{curl}_o\, \boldsymbol{A}_o$. As previously, we take $\boldsymbol{v}_{1o} = \boldsymbol{i}\, v_{1ox_o} = \boldsymbol{i}\, v_{1o}$.

We get:

$$\boldsymbol{k} \left(\text{curl}_o\, \boldsymbol{A}_o\right)_z = \boldsymbol{k} \left(\frac{\partial A_{oy_o}}{\partial x_o} - \frac{\partial A_{ox_o}}{\partial y_o} \right);$$

$$\boldsymbol{v}_{1o} \times \text{curl}_o\, \boldsymbol{A}_o = -\boldsymbol{j}\, v_{1o} \left(\frac{\partial A_{oy_o}}{\partial x_o} - \frac{\partial A_{ox_o}}{\partial y_o} \right). \qquad \text{(A.3.113 c, d)}$$

By replacing expressions (A.3.113 a, b, c) into relation (A.3.105 c), we have:

$$-\frac{\partial V_1}{\partial y_1} - \frac{\partial A_{1y_1}}{\partial t_1} = -\alpha_{1o} \frac{\partial V_o}{\partial y_o} - \alpha_{1o} \frac{\partial A_{oy_o}}{\partial t_o} - \alpha_{1o}\, v_{1o} \left(\frac{\partial A_{oy_o}}{\partial x_o} - \frac{\partial A_{ox_o}}{\partial y_o} \right). \qquad \text{(A.3.114)}$$

Using relations (A.3.99 a, . . ., h) and (A.3.100 a, . . ., d), we get:

$$\frac{\partial V_o}{\partial y_o} = \frac{\partial V_o}{\partial y_1}; \qquad \frac{\partial A_{oy_o}}{\partial y_o} = \frac{\partial A_{oy_o}}{\partial y_1}. \qquad \text{(A.3.115 a, b)}$$

$$\frac{\partial A_{oy_o}}{\partial t_o} = \frac{\partial A_{oy_o}}{\partial t_1} \cdot \frac{\partial t_1}{\partial t_o} + \frac{\partial A_{oy_o}}{\partial x_1} \cdot \frac{\partial x_1}{\partial t_o} = \alpha_{1o} \frac{\partial A_{oy_o}}{\partial t_1} - \alpha_{1o}\, v_{1o} \frac{\partial A_{oy_o}}{\partial x_1}. \qquad \text{(A.3.116)}$$

$$\frac{\partial A_{oy_o}}{\partial x_o} = \frac{\partial A_{oy_o}}{\partial x_1} \cdot \frac{\partial x_1}{\partial x_o} + \frac{\partial A_{oy_o}}{\partial t_1} \cdot \frac{\partial t_1}{\partial x_o} = \alpha_{1o} \frac{\partial A_{oy_o}}{\partial x_1} - \alpha_{1o} \frac{v_{1o}}{c^2} \cdot \frac{\partial A_{oy_o}}{\partial t_1}. \qquad \text{(A.3.117)}$$

After replacing the expressions (A.3.115 a, b), (A.3.116), (A.3.117) into relation (A.3.114), and performing the calculations, it results that:

$$-\frac{\partial V_1}{\partial y_1} - \frac{\partial A_{1y_1}}{\partial t_1} = -\alpha_{1o} \frac{\partial V_o}{\partial y_1} - \alpha_{1o} \left(\alpha_{1o} \frac{\partial A_{oy_o}}{\partial t_1} - \alpha_{1o}\, v_{1o} \frac{\partial A_{oy_o}}{\partial x_1} \right) -$$

$$-\alpha_{1o}\, v_{1o} \left(\alpha_{1o} \frac{\partial A_{oy_o}}{\partial x_1} - \alpha_{1o} \frac{v_{1o}}{c^2} \cdot \frac{\partial A_{oy_o}}{\partial t_1} \right) + \alpha_{1o}\, v_{1o} \frac{\partial A_{ox_o}}{\partial y_1}. \qquad \text{(A.3.118)}$$

Therefore:

$$-\frac{\partial V_1}{\partial y_1} - \frac{\partial A_{1y_1}}{\partial t_1} = -\alpha_{1o}\frac{\partial V_o}{\partial y_1} + \frac{\partial A_{oy_o}}{\partial t_1}\left(-\alpha_{1o}^2 + \alpha_{1o}^2\frac{v_{1o}^2}{c^2}\right) +$$

$$+\frac{\partial A_{oy_o}}{\partial x_1}\left(\alpha_{1o}^2\,v_{1o} - \alpha_{1o}^2\,v_{1o}\right) + \alpha_{1o}\,v_{1o}\frac{\partial A_{ox_o}}{\partial y_1} = \qquad\text{(A.3.119)}$$

$$= -\alpha_{1o}\frac{\partial V_o}{\partial y_1} - \frac{\partial A_{oy_o}}{\partial t_1} + \alpha_{1o}\,v_{1o}\frac{\partial A_{ox_o}}{\partial y_1}.$$

By equating the derivatives with respect to the same variables of both sides of the last relation, we get besides relations (A.3.112 a, b):

$$V_1 = \alpha_{1o}\left(V_o - v_{1o}\,A_{ox_o}\right);$$
$$A_{1y_1} = A_{oy_o}\,. \qquad\qquad\text{(A.3.120 a, b)}$$

A similar relation can be obtained for the Oz - axis of the reference frames.

$$V_1 = \alpha_{1o}\left(V_o - v_{1o}\,A_{ox_o}\right);$$
$$A_{1z_1} = A_{oz_o}\,. \qquad\qquad\text{(A.3.121 a, b)}$$

Hence, in compact form

$$V_1 = \alpha_{1o}\left(V_{1o} - \boldsymbol{v}_{1o}\cdot\boldsymbol{A}_o\right);$$

$$\boldsymbol{A}_1 = \alpha_{1o}\left(\boldsymbol{A}_o - \frac{\boldsymbol{v}_{1o}}{c^2}V_o\right). \qquad\text{(A.3.122 a, b)}$$

List of Symbols Used in Appendix 3

$\boldsymbol{a}_{21}$ — acceleration in the reference frame K_1 of a material point at rest with respect to reference frame K_2;

$\boldsymbol{A}_o$ — vector potential in the reference frame K_o;

$\boldsymbol{A}_1$ — vector potential in the reference frame K_1;

A_{ox_o} — component of the vector $\boldsymbol{A}_o$ along the $O_o x_o$ - axis;

A_{1x_1} — component of the vector $\boldsymbol{A}_1$ along the $O_1 x_1$ - axis;

c — velocity of light *in vacuo*;

$\mathrm{d}\boldsymbol{l}$ — vector length element;

$\boldsymbol{F}_i$ — force acting in the reference frame K_i upon a material point j that is moving with the velocity $\boldsymbol{v}_{ji}$ with respect to the reference frame K_i;

$\boldsymbol{F}_{ir}$ –   rest component of the force $\boldsymbol{F}_i$ corresponding to the case in which the material point would be at rest with respect to the reference frame K_i;

$\boldsymbol{F}_{iv}$ –   motion component of the force $\boldsymbol{F}_i$ arising in the case in which the material point is moving with the velocity $\boldsymbol{v}_{ji}$ with respect to the reference frame K_i;

K_i –   symbol of an inertial reference frame, where the space co-ordinates have been denoted x_i, y_i, z_i, and the time t_i;

m –   relativistic mass of the material point in motion with the velocity $\boldsymbol{u}$, with respect to any inertial reference frame K;

m_0 –   rest mass of a material point, at rest with respect to any reference frame K, expressed in the same reference frame; if no other mention is made, then by mass is understood the rest mass;

q –   electric charge of a point-like body;

$\boldsymbol{r}_{\mathrm{o}}$ –   position vector of a point in the reference frame K_{o}, with its origin at the origin of the same reference frame;

$\boldsymbol{r}_1$ –   position vector of a point in the reference frame K_1, with its origin at the origin of the same reference frame;

t_{o} –   time in the reference frame K_{o};

t_1 –   time in the reference frame K_1;

$\boldsymbol{u}$ –   velocity of a material point in any reference frame K;

$\boldsymbol{v}$ –   velocity of any reference frame relative to the reference frame K;

$\boldsymbol{v}_{1\mathrm{o}}$ –   velocity of the reference frame K_1 relatively to the reference frame K_{o};

$\boldsymbol{v}_{2\mathrm{o}}$ –   velocity of the reference frame K_2 or of a material point denoted by 2, relatively to the reference frame K_{o};

$\boldsymbol{v}_{ji}$ –   velocity of a point denoted by j or of any reference frame K_j relatively to the reference frame K_i;

V_{o} –   scalar potential in the reference frame K_{o}, p. 268; volume, p. 248;

V_1 –   scalar potential in the reference frame K_1, p. 268; volume, p. 248, 261;

$x_{\mathrm{o}}, y_{\mathrm{o}}, z_{\mathrm{o}}$ –   co-ordinates in the system of co-ordinates of K_{o};

x_1, y_1, z_1 –   co-ordinates in the system of co-ordinates of K_1;

$$\alpha_{ij} = \frac{1}{\sqrt{1 - \boldsymbol{\beta}_{ij}^2}} = \frac{1}{\sqrt{1 - \beta_{ij}^2}};$$

$$\boldsymbol{\beta}_{ij} = \frac{\boldsymbol{v}_{ij}}{c};$$

$\Delta\boldsymbol{S}_i$ –   vector surface element in the reference frame K_i;

ρ_{vi} –   volume density of the electric charge in the reference frame K_i.

The symbols with one index refer to a certain reference frame, for instance K_1.

The symbols with two indices refer to quantities concerning two systems, for instance $\boldsymbol{v}_{2\mathrm{o}}$ represents the velocity of the reference frame K_2 relatively to the reference frame K_{o}.

APPENDIX 4

DEDUCING THE GENERAL RELATIONS
OF THE SPECIAL THEORY OF RELATIVITY

In this Appendix, we shall present a deduction of certain basic relations of the Theory of Special Relativity concerning Mechanics, which we have used in Appendix 3. The deductions differ from the usual ones and are closed to the concerned applications.

A.4.1. DERIVING THE CO-ORDINATE TRANSFORMATION
RELATIONS FOR PASSING FROM ONE SYSTEM
OF CO-ORDINATES TO ANOTHER ONE

We shall consider two systems of Cartesian co-ordinates, denoted K and K'. The axes of co-ordinates of the two systems are parallel to each other. The time and space co-ordinates of K will be t, x, y, z, and c will represent the velocity of light *in vacuo*, and will be assumed to be constant. The system K will be considered as fixed and the system K' will be considered in translation motion, with the constant velocity v, along the Ox - axis of K. At the first moment, i.e., at $t = 0$, we shall suppose that the origins O and O' of the two systems of co-ordinates coincide.

The fundamental relation assumed in the Special and General Theory of Relativity is represented by one invariant, namely the constant value of the space-time interval:

$$\mathrm{d}s^2 = g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j;\quad \forall i \in [0,3];\quad x^0 = ct;\quad x^1 = x;\quad x^2 = y;\quad x^3 = z;\qquad \text{(A.4.1 a-e)}$$

where g_{ij} represents the general term of the fundamental tensor, regardless of the system of reference. The relation (A.4.1) is called the *fundamental time-space* relation or *fundamental form of the metric relation* of the considered space.

In the case of a Galilean system of co-ordinates:

$$\mathrm{d}s^2 = c^2\,\mathrm{d}t^2 - \left(\mathrm{d}x^i\right)^2,\ \forall i \in [1,3];\quad x^1 = x;\quad x^2 = y;\quad x^3 = z;\qquad \text{(A.4.2 a-d)}$$

and then, the corresponding quadratic form (A.4.1) becomes (A.4.2) and is called normal or quasi-Euclidean form.

Since we have considered the motion only along the Ox - axis, we shall take into consideration, apart from the time co-ordinate, only the x co-ordinate involved in the motion, whereas the co-ordinates y and z are not affected. The co-ordinates of the system K' will be denoted like those of K, but with the sign prime.

We shall look for the relations between the co-ordinates in the two systems of co-ordinates of above. So far, in the previous published works, these relations have been usually obtained through relatively complicated geometrical deductions, which we shall avoid. For this purpose we shall express the involved co-ordinates of the system K versus those of K', namely x and t.

A linear dependence will be assumed, as follows:

$$x = a\,x' + b\,t',$$
$$t = p\,t' + q\,x'.$$
$$\text{(A.4.3 a, b)}$$

From the condition of the constant value of the space-time interval (A.4.2 a), we get the following relation:

$$\left(c^2 p^2 - b^2\right)(\mathrm{d}\,t')^2 + 2\left(c^2 pq - ab\right)\mathrm{d}\,x'\,\mathrm{d}\,t' - \left(a^2 - c^2 q^2\right)(\mathrm{d}\,x')^2 = 0 . \qquad \text{(A.4.4)}$$

By identifying the coefficients, there follows:

$$c^2 p^2 - b^2 = c^2 ,$$
$$c^2 q^2 - a^2 = -1,$$
$$c^2 pq = ab.$$
$$\text{(A.4.5 a, b, c)}$$

By replacing p and q from (A.4.5 a) and (A.4.5 b) into (A.4.5 c), we get:

$$c^2\left(a^2 - 1\right)\left(b^2 + c^2\right)\frac{1}{c^4} = a^2 b^2 ;$$
$$b = c\sqrt{a^2 - 1}\,;$$
$$p = a; \quad q = \frac{b}{c^2}.$$
$$\text{(A.4.6 a-d)}$$

For the square root, we have taken the sign plus and we shall examine further the case which corresponds to the chosen sign.

We shall consider any point fixed in the system K', for instance the origin O', with $x' = 0$, its velocity in the system K will be v along the direction of the increasing values of the co-ordinate x'.

Considering relations (A.4.3 a, b), and replacing $x' = 0$, there follows:

$$\frac{\mathrm{d}\,x}{\mathrm{d}\,t} = \frac{b}{p} = v; \quad \alpha = a = \frac{1}{\sqrt{1 - \beta^2}}; \quad \beta = \frac{v}{c};$$
$$b = \alpha v; \quad p = \alpha; \quad q = \alpha\frac{v}{c^2}.$$
$$\text{(A.4.7 a-f)}$$

Using relations (A.4.3) and (A.4.7), and including also the relations containing the co-ordinates unaffected by the co-ordinate transformation, there follows:

$$x = \alpha\left(x' + vt'\right);$$

$$t = \alpha\left(t' + \frac{v}{c^2}x'\right);. \qquad\qquad \text{(A.4.8 a-d)}$$

$$y = y'; \quad z = z'.$$

For the inverse transformation, by solving the system given by (A.4.8), we obtain:

$$x' = \alpha\left(x - vt\right);$$

$$t' = \alpha\left(t - \frac{v}{c^2}x\right); \qquad\qquad \text{(A.4.9 a-c)}$$

$$y' = y; \quad z' = z;$$

where we can remark the change of sign. From these sets of relations, it results directly that simultaneous events in frame K' are not simultaneous in frame K, and conversely.

Also, from relation (A.4.8 b), it follows that for a clock located at any fixed point of the frame K', one obtains $\Delta t = \alpha\,\Delta t'$, hence $\Delta t > \Delta t'$, so that we deal with a dilation of time.

In addition, from relation (A.4.9 a), it follows that, for a ruler placed in K' along the $O'x'$ - axis, at any given time t, one obtains $\Delta x' = \alpha\Delta x$, hence $\Delta x' > \Delta x$, so that we now deal with a contraction of length.

A.4.2.   DERIVING THE EXPRESSION OF THE ADDITION OF VELOCITIES

Further on, in order to give more suggestive explanations, we shall denote the reference frames K and K', by K_o and K_1, respectively, where the original frame K, assumed as immobile, is indicated by the upright subscript o, whereas the moving frame is indicated by the subscript 1. We shall similarly denote the quantities belonging to these systems. The constant velocity of K_1 relatively to K_o will be denoted $\boldsymbol{v}_{1o} = \boldsymbol{i}\, v_{1ox}$.

Let us consider a point P, moving with the velocity $\boldsymbol{v}_{21}$ relatively to the reference frame K_1. At any time, if x_1 is the co-ordinate of a point in K_1, the corresponding co-ordinates in K_o are consequently:

$$x_o = \alpha_{1o}\left(x_1 + v_{1ox}t_1\right); \quad \alpha_{1o} = \frac{1}{\sqrt{1 - \beta_{1o}^2}}; \quad \beta_{1o} = \frac{v_{1ox}}{c};$$

$$\qquad\qquad \text{(A.4.10 a-d)}$$

$$t_0 = \alpha_{1o}\left(t_1 + \frac{v_{1ox}}{c^2}x_1\right).$$

The velocity v_{2ox} of point P in K_o will be obtained by using relations (A.4.8 a, b) with the modified symbols as explained above:

$$v_{2ox} = \frac{\mathrm{d}\,x_o}{\mathrm{d}\,t_o} = \frac{\mathrm{d}}{\mathrm{d}\,t_o}\left[\alpha_{1o}\left(x_1 + v_{1ox}t_1\right)\right] = \frac{\mathrm{d}\left[\alpha_{1o}\left(x_1 + v_{1ox}t_1\right)\right]}{\mathrm{d}\left[\alpha_{1o}\left(t_1 + \dfrac{v_{1ox}}{c^2}x_1\right)\right]} =$$

$$= \frac{\mathrm{d}\,x_1 + v_{1ox}\,\mathrm{d}\,t_1}{\mathrm{d}\,t_1 + \dfrac{v_{1ox}}{c^2}\,\mathrm{d}\,x_1} = \frac{\dfrac{\mathrm{d}\,x_1}{\mathrm{d}\,t_1} + v_{1ox}}{1 + \dfrac{v_{1ox}}{c^2}\cdot\dfrac{\mathrm{d}\,x_1}{\mathrm{d}\,t_1}}, \qquad\qquad (A.4.11\ a,\ b)$$

$$v_{2ox} = \frac{v_{21x} + v_{1ox}}{1 + \dfrac{v_{1ox}v_{21x}}{c^2}}.$$

Further on, we shall take into account that no difference occurs between y_o and z_o relatively to y_1 and z_1.

As previously shown for v_{2ox}, there follows:

$$v_{2oy} = \frac{\mathrm{d}\,y_o}{\mathrm{d}\,t_o} = \frac{\mathrm{d}\,y_1}{\mathrm{d}\left[\alpha_{1o}\left(t_1 + \dfrac{v_{1ox}}{c^2}x_1\right)\right]} = \frac{\dfrac{\mathrm{d}\,y_1}{\mathrm{d}\,t_1}}{\alpha_{1o}\left(1 + \dfrac{v_{1ox}}{c^2}\cdot\dfrac{\mathrm{d}\,x_1}{\mathrm{d}\,t_1}\right)};$$

$$v_{2oy} = \frac{v_{21y}}{\alpha_{1o}\left(1 + \dfrac{v_{1ox}v_{21x}}{c^2}\right)}; \quad v_{2oz} = \frac{v_{21z}}{\alpha_{1o}\left(1 + \dfrac{v_{1ox}v_{21x}}{c^2}\right)}. \qquad (A.4.12\ a\text{-}c)$$

Calculating the derivatives with respect to time, like previously, we shall get the expressions of the accelerations.

A.4.3. RELATIONSHIP BETWEEN MASS AND VELOCITY

According to one of the postulates of the Theory of Special Relativity (Chapter 2, p. 99), the mechanical momentum (quantity of motion) along any direction perpendicular to the direction of motion is kept (conserved) when passing from an inertial reference frame K_o to another K_1. Therefore, for any material point 2, with the mass m_o, we can write the following relations:

$$m_o v_{2oy} = m_1 v_{21y}; \quad m_o\frac{v_{21y}}{\alpha_{1o}\left(1 + \dfrac{v_{1ox}v_{21x}}{c^2}\right)} = m_1 v_{21y}, \qquad (A.4.13)$$

where the factor v_{21y} may be simplified. We shall, now, consider the material point at rest in the reference frame K_1, and in this case, $v_{21x} = 0$, and the mass of the material point in a reference frame in which the point is at rest, called *proper mass*, will be denoted by m_0 (the subscript is zero not o, as for other symbols, $m_0 \neq m_o$). By replacing the value of the velocity $v_{21x} = 0$, we get:

$$m = m_1 = \frac{1}{\sqrt{1 - \dfrac{v_{1o}^2}{c^2}}} m_0 .$$

(A.4.14)

A similar relation may be written for any system of reference.

A.4.4. RELATIONS BETWEEN THE FORCES OF TWO SYSTEMS OF REFERENCE

The components of the force will be calculated by the derivatives with respect to time of the components of the momentum (quantity of motion).

A.4.4.1. Expression of the Components along the Ox - axis

The expression of the force acting on a material point P, along the Ox - axis, will be:

$$F_{ox} = \frac{\mathrm{d}}{\mathrm{d}\,t_o}\left[m_0 \left(1 - \frac{v_{2o}^2}{c^2}\right)^{-\frac{1}{2}} v_{2ox} \right],$$

(A.4.15)

where m_0 (the index of m is zero not o) denotes, like above, the proper mass of the point, considered as a material point, in a reference frame in which the point is at rest. By using relations (A.4.11) and (A.4.12), we shall obtain:

$$\left(1 - \frac{v_{2o}^2}{c^2}\right)^{\frac{1}{2}} = \left(1 - \frac{v_{21}^2}{c^2}\right)^{\frac{1}{2}} \frac{1}{\alpha_{1o}\left(1 + \dfrac{v_{1ox} v_{21x}}{c^2}\right)} .$$

(A.4.16)

For obtaining relation (A.4.16), have had to replace (A.4.11 b), (A.4.12 b), (A.4.12 c) in the left-hand side of relation (A.4.16), hence in (A.4.16 a), below:

$$v_{2o}^2 = v_{2ox}^2 + v_{2oy}^2 + v_{2oz}^2 ; \quad v_{21}^2 = v_{21x}^2 + v_{21y}^2 + v_{21z}^2 .$$

(A.4.16 a)

By performing the calculations, we have obtained as an intermediate result the following expression:

$$
1 - \frac{v_{2o}^2}{c^2} =
$$
$$
= \frac{c^4 + v_{1ox}^2 v_{1ox}^2 - c^2 v_{1ox}^2 - c^2 v_{21x}^2 - c^2 v_{21y}^2 + v_{1ox}^2 v_{v21y}^2 - c^2 v_{21z}^2 + v_{1ox}^2 v_{21z}^2}{c^2 + v_{1ox} v_{21x}}, \tag{A.4.16 b}
$$

and grouping the terms of the numerator in formula (A.4.16 b), we obtain the right-hand side of relation (A.4.16).

The component of the force along the Ox - axis in the considered systems of reference K_o, results as follows:

$$
F_{ox} = \frac{\mathrm{d}}{\mathrm{d}\, t_o} \left[m_0 \left(1 - \frac{v_{2o}^2}{c^2} \right)^{-\frac{1}{2}} v_{2ox} \right] = \frac{\mathrm{d}\, S}{\mathrm{d}\, t_o};
$$

$$
S = m_0 \left(1 - \frac{v_{2o}^2}{c^2} \right)^{-\frac{1}{2}} v_{2ox} = m_0 \left(1 - \frac{v_{21}^2}{c^2} \right)^{-\frac{1}{2}} \alpha_{10} \left(1 + \frac{v_{1ox} v_{21x}}{c^2} \right) \frac{v_{21x} + v_{1ox}}{1 + \frac{v_{1ox} v_{21x}}{c^2}};
$$

$$
F_{ox} = \frac{\mathrm{d}\, S}{\mathrm{d}\, t_o} = \tag{A.4.17 a-c}
$$

$$
= \frac{1}{\alpha_{1o}} \cdot \frac{\alpha_{10}}{\mathrm{d}\, t_1 + \frac{v_{1ox}}{c^2} \mathrm{d}\, x_1} \left\{ \mathrm{d} \left[m_0 \left(1 - \frac{v_{21}^2}{c^2} \right)^{-\frac{1}{2}} v_{21x} \right] + \mathrm{d} \left[m_0 \left(1 - \frac{v_{21}^2}{c^2} \right)^{-\frac{1}{2}} v_{1ox} \right] \right\} =
$$

$$
= \frac{1}{1 + \frac{v_{1ox}}{c^2} \cdot \frac{\mathrm{d}\, x_1}{\mathrm{d}\, t_1}} \left\{ \frac{\mathrm{d}}{\mathrm{d}\, t_1} \left[m_0 \left(1 - \frac{v_{21}^2}{c^2} \right)^{-\frac{1}{2}} v_{21x} \right] + \frac{\mathrm{d}}{\mathrm{d}\, t_1} \left[m_0 \left(1 - \frac{v_{21}^2}{c^2} \right)^{-\frac{1}{2}} v_{1ox} \right] \right\}.
$$

Therefore:

$$F_{ox} = \left\{ \frac{d}{dt_1}\left[m_0\left(1 - \frac{v_{21}^2}{c^2}\right)^{-\frac{1}{2}} v_{21x} \right] + \frac{d}{dt_1}\left[m_0\left(1 - \frac{v_{21}^2}{c^2}\right)^{-\frac{1}{2}} v_{1ox} \right] \right\} \frac{1}{1 + \frac{v_{1ox}v_{21x}}{c^2}};$$

$$F_{ox} = \frac{1}{1 + \frac{v_{1ox}v_{21x}}{c^2}}(F_{1x} + R); \quad F_{1x} = \frac{d}{dt_1}\left[m_0\left(1 - \frac{v_{21}^2}{c^2}\right)^{-\frac{1}{2}} v_{21x} \right]; \quad \text{(A.4.18 a-c)}$$

$$R = \frac{d}{dt_1}\left[m_0\left(1 - \frac{v_{21}^2}{c^2}\right)^{-\frac{1}{2}} v_{1ox} \right].$$

There remains to calculate the term R. We can write:

$$R = \frac{d}{dt_1}\left[m_0\left(1 - \frac{v_{21}^2}{c^2}\right)^{-\frac{1}{2}} v_{1ox} \right] =$$

$$= v_{1ox}\frac{1}{c^2} m_0\left(1 - \frac{v_{21}^2}{c^2}\right)^{-\frac{3}{2}}\left(v_{21x}\dot{v}_{21x} + v_{21y}\dot{v}_{21y} + v_{21z}\dot{v}_{21z} \right) = \quad \text{(A.4.19)}$$

$$= v_{1ox}\frac{1}{c^2} m_0\left(1 - \frac{v_{21}^2}{c^2}\right)^{-\frac{3}{2}}(\mathbf{v}_{21} \cdot \dot{\mathbf{v}}_{21}),$$

where, for simplicity, we have denoted $\dot{\mathbf{v}}_{21} = \dfrac{d\mathbf{v}_{21}}{dt_1}$.

The expression of R is, to some extent, similar to that of F_{1x} but includes a scalar product. Therefore, we shall express F_{1x} using also a vector form:

$$\mathbf{F}_1 = \frac{d}{dt_1}\left[m_0\left(1 - \frac{v_{21}^2}{c^2}\right)^{-\frac{1}{2}} \mathbf{v}_{21} \right] =$$

$$= m_0\left(1 - \frac{v_{21}^2}{c^2}\right)^{-\frac{1}{2}}\left(\frac{\frac{v_{21}^2}{c^2}}{1 - \frac{v_{21}^2}{c^2}} \right)\dot{\mathbf{v}}_{21} + m_0\left(1 - \frac{v_{21}^2}{c^2}\right)^{-\frac{1}{2}}\dot{\mathbf{v}}_{21}. \quad \text{(A.4.20)}$$

In order to separate a common factor in both expressions, namely (A.4.19) and (A.4.20) eliminate $\dot{v}_{21}$, we multiply the last one by v_{21}. We shall obtain:

$$\boldsymbol{F}_1 \cdot \boldsymbol{v}_{21} = m_0 \left(1 - \frac{v_{21}^2}{c^2}\right)^{-\frac{1}{2}} \left(\frac{\dfrac{v_{21}^2}{c^2}}{1 - \dfrac{v_{21}^2}{c^2}} + 1\right) \dot{v}_{21} v_{21} =$$

$$= m_0 \left(1 - \frac{v_{21}^2}{c^2}\right)^{-\frac{3}{2}} \dot{v}_{21} \boldsymbol{v}_{21}.$$

(A.4.21)

By comparing relation (A.4.21) with (A.4.19), it results that:

$$R = \frac{1}{c^2} v_{1ox}\, \boldsymbol{v}_{21} \cdot \boldsymbol{F}_1 = \frac{1}{c^2} v_{1ox} \left(v_{21x} F_{1x} + v_{21y} F_{1y} + v_{21z} F_{1z}\right);$$

$$F_{ox} = \frac{1}{1 + \dfrac{1}{c^2} v_{1ox} v_{21x}} F_{1x} + \frac{\dfrac{1}{c^2} v_{1ox} \left(v_{21x} F_{1x} + v_{21y} F_{1y} + v_{21z} F_{1z}\right)}{1 + \dfrac{1}{c^2} v_{1ox} v_{21x}} =$$

$$= \frac{1 + \dfrac{1}{c^2} v_{1ox} v_{21x}}{1 + \dfrac{1}{c^2} v_{1ox} v_{21x}} F_{1x} + \frac{\dfrac{1}{c^2} v_{1ox} \left(v_{21y} F_{1y} + v_{21z} F_{1z}\right)}{1 + \dfrac{1}{c^2} v_{1ox} v_{21x}} =$$

(A.4.22 a, b)

$$= F_{1x} + \frac{\dfrac{1}{c^2} v_{1ox} \left(v_{21y} F_{1y} + v_{21z} F_{1z}\right)}{1 + \dfrac{1}{c^2} v_{1ox} v_{21x}}.$$

A.4.4.2.   Expression of the Components along the Oy - and Oz - axes

The expression of the force acting on a material point P, along the Oy - axis, according to formulae (A.4.12) and (A.4.13), will be:

$$F_{\mathrm{o}y} = \frac{\mathrm{d}}{\mathrm{d}t_{\mathrm{o}}}\left[m_0\left(1-\frac{v_{2\mathrm{o}}^2}{c^2}\right)^{-\frac{1}{2}} v_{2\mathrm{o}y}\right] = \frac{\mathrm{d}S}{\alpha_{1\mathrm{o}}\left(\mathrm{d}t_1 + \frac{v_{1\mathrm{o}x}}{c^2}\mathrm{d}x_1\right)};$$

$$S = m_0\left(1-\frac{v_{2\mathrm{o}}^2}{c^2}\right)^{-\frac{1}{2}} v_{2\mathrm{o}y} =$$

$$= m_0\left(1-\frac{v_{21}^2}{c^2}\right)^{-\frac{1}{2}} \alpha_{1\mathrm{o}}\left(1+\frac{v_{1\mathrm{o}x}v_{21x}}{c^2}\right)\frac{v_{21y}}{\alpha_{1\mathrm{o}}\left(1+\frac{v_{1\mathrm{o}x}v_{21x}}{c^2}\right)} = m_0\left(1-\frac{v_{21}^2}{c^2}\right)^{-\frac{1}{2}} v_{21y}; \quad \text{(A.4.23 a-c)}$$

$$F_{\mathrm{o}y} = \frac{1}{\alpha_{1\mathrm{o}}}\cdot\frac{\mathrm{d}S}{\mathrm{d}t_1 + \frac{v_{1\mathrm{o}x}}{c^2}\mathrm{d}x_1} = \frac{1}{\alpha_{1\mathrm{o}}}\cdot\frac{\dfrac{\mathrm{d}S}{\mathrm{d}t_1}}{1+\frac{v_{1\mathrm{o}x}}{c^2}\cdot\frac{\mathrm{d}x_1}{\mathrm{d}t_1}};$$

$$F_{\mathrm{o}y} = \frac{1}{\alpha_{1\mathrm{o}}}\cdot\frac{1}{1+\frac{v_{1\mathrm{o}x}v_{21x}}{c^2}}\cdot\frac{\mathrm{d}}{\mathrm{d}t_1}\left[m_0\left(1-\frac{v_{21}^2}{c^2}\right)^{-\frac{1}{2}} v_{21y}\right] = \frac{1}{\alpha_{1\mathrm{o}}}\cdot\frac{1}{1+\frac{v_{1\mathrm{o}x}v_{21x}}{c^2}}\cdot F_{1y}.$$

Similarly, we can obtain the expression of $F_{\mathrm{o}z}$.

A.4.4.3. Expressions of All Components

For obtaining compact expressions in view of many applications, we shall use the symbols and expressions below:

$$\boldsymbol{\beta}_{1\mathrm{o}} = \frac{v_{1\mathrm{o}}}{c}; \quad \boldsymbol{\beta}_{2\mathrm{o}} = \frac{v_{2\mathrm{o}}}{c}; \quad \boldsymbol{\beta}_{21} = \frac{v_{21}}{c}. \quad \text{(A.4.24 a-c)}$$

With these symbols, and assuming that $\boldsymbol{\beta}_{1\mathrm{o}} = i\,v_{1\mathrm{o}x}$, relations (A.4.22) and (A.4.23) becomes:

$$F_{\mathrm{o}x} = F_{1x} + \frac{\beta_{1\mathrm{o}x}\beta_{21y}}{1+\beta_{1\mathrm{o}x}\cdot\beta_{21x}}F_{1y} + \frac{\beta_{1\mathrm{o}x}\beta_{21z}}{1+\beta_{1\mathrm{o}x}\cdot\beta_{21x}}F_{1z};$$

$$\text{(A.4.25 a-c)}$$

$$F_{\mathrm{o}y} = \frac{1}{\alpha_{1\mathrm{o}}}\cdot\frac{1}{1+\beta_{21x}\cdot\beta_{1\mathrm{o}x}}F_{1y}; \quad F_{\mathrm{o}z} = \frac{1}{\alpha_{1\mathrm{o}}}\cdot\frac{1}{1+\beta_{21x}\cdot\beta_{1\mathrm{o}x}}F_{1z}.$$

Combining the formulae (A.4.24) and (A.4.25), we can write all relations in the following vector form:

$$\boldsymbol{\beta}_{1o} = \frac{\boldsymbol{v}_{1o}}{c}; \quad \boldsymbol{\beta}_{2o} = \frac{\boldsymbol{v}_{2o}}{c}; \quad \boldsymbol{\beta}_{21} = \frac{\boldsymbol{v}_{21}}{c};$$

$$F_{ox} = \frac{1}{1 + \boldsymbol{\beta}_{1o} \cdot \boldsymbol{\beta}_{21}} F_{1x} + \frac{v_{1ox}}{c} \cdot \frac{\boldsymbol{\beta}_{21} \cdot \boldsymbol{F}_1}{1 + \boldsymbol{\beta}_{1o} \cdot \boldsymbol{\beta}_{21}};$$

$$F_{oy} = \frac{1}{\alpha_{1o}} \cdot \frac{1}{1 + \boldsymbol{\beta}_{1o} \cdot \boldsymbol{\beta}_{21}} F_{1y}; \quad F_{oz} = \frac{1}{\alpha_{1o}} \cdot \frac{1}{1 + \boldsymbol{\beta}_{1o} \cdot \boldsymbol{\beta}_{21}} F_{1z};$$

$$\boldsymbol{F}_o = \frac{1 + \alpha_{1o}\, \boldsymbol{\beta}_{21} \cdot \boldsymbol{\beta}_{1o}}{\alpha_{1o}\left(1 + \boldsymbol{\beta}_{21} \cdot \boldsymbol{\beta}_{1o}\right)} \boldsymbol{F}_1 + \frac{\alpha_{1o} - 1}{\alpha_{1o}\left(1 + \boldsymbol{\beta}_{21} \cdot \boldsymbol{\beta}_{1o}\right)} \cdot \frac{\boldsymbol{v}_{1o}}{v_{1o}^2} \boldsymbol{v}_{1o} \cdot \boldsymbol{F}_1 + \frac{1}{c^2} \cdot \frac{\boldsymbol{v}_{21} \times \left(\boldsymbol{v}_{1o} \times \boldsymbol{F}_1\right)}{1 + \boldsymbol{\beta}_{21} \cdot \boldsymbol{\beta}_{1o}}.$$

(A.4.26 a-g)

APPENDIX 5

THE EQUATIONS OF THE ELECTROMAGNETIC FIELD IN THE GENERAL THEORY OF RELATIVITY

In this Appendix, we shall present the equations of the electromagnetic field in the Special and General Theory of Relativity. We shall begin with the covariant forms of the equations, so that the passage from the Special to the General Theory of Relativity could be performed without difficulties. The bibliographic mentions [1]-[12], specific for this Appendix, are at the end of this Appendix. Also, a list of symbols is given at the end of this Appendix. In the known literature various methods are used [1]-[11]. We shall have in view the works [8], [9], which present certain advantages for the purpose of this Appendix.

A.5.1. THE FOUR POTENTIAL TENSOR OF RANK 1 AND THE TENSORS OF RANK 2

From the formulae of the general theory of the electromagnetic field (Chapter 3, p. 129, 134, 142, 150, 182), in the usual vector form, we have:

$$\operatorname{curl} \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}; \quad \boldsymbol{B} = \operatorname{curl} \boldsymbol{A}; \quad \boldsymbol{E} = -\operatorname{grad} V - \frac{\partial \boldsymbol{A}}{\partial t};$$

$$\operatorname{curl} \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}; \quad \boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}; \quad \boldsymbol{B} = \mu_0 \boldsymbol{H} + \mu_0 \boldsymbol{M}. \tag{A.5.1 a, ..., f}$$

We shall rewrite the equations (A.5.1) considering the general case of non-linear media:

$$\operatorname{curl}(\mu_0 \boldsymbol{H} + \mu_0 \boldsymbol{M}) = \mu_0 \boldsymbol{J} + \varepsilon_0 \mu_0 \frac{\partial \boldsymbol{E}}{\partial t} + \mu_0 \frac{\partial \boldsymbol{P}}{\partial t} + \mu_0 \operatorname{curl} \boldsymbol{M};$$

$$\operatorname{curl} \boldsymbol{B} = \operatorname{curl} \operatorname{curl} \boldsymbol{A} = \mu_0 \boldsymbol{J} + \varepsilon_0 \mu_0 \frac{\partial}{\partial t}\left(-\operatorname{grad} V - \frac{\partial \boldsymbol{A}}{\partial t}\right) + \mu_0\left(\frac{\partial \boldsymbol{P}}{\partial t} + \operatorname{curl} \boldsymbol{M}\right); \tag{A.5.2 a-c}$$

$$\operatorname{grad} \operatorname{div} \boldsymbol{A} - \nabla^2 \boldsymbol{A} = \mu_0 \boldsymbol{J} - \varepsilon_0 \mu_0\left(\operatorname{grad} \frac{\partial V}{\partial t} + \frac{\partial^2 \boldsymbol{A}}{\partial t^2}\right) + \mu_0\left(\frac{\partial \boldsymbol{P}}{\partial t} + \operatorname{curl} \boldsymbol{M}\right).$$

Rearranging the terms of (A.5.2 c), we get:

$$\nabla^2 A - \varepsilon_0 \mu_0 \frac{\partial^2 A}{\partial t^2} = -\mu_0 \left(J + \frac{\partial P}{\partial t} + \operatorname{curl} M \right) + \operatorname{grad}\left(\operatorname{div} A + \varepsilon_0 \mu_0 \frac{\partial V}{\partial t} \right). \qquad (A.5.3)$$

Since only the curl of vector A is imposed, the divergence can be chosen by using the L. V. Lorenz (not to be confused with H. A. Lorentz) gauge condition:

$$\operatorname{div} A + \varepsilon_0 \mu_0 \frac{\partial V}{\partial t} = 0 . \qquad (A.5.4)$$

The last relation may be written in the form:

$$\frac{\partial A_i}{\partial x^i} + \frac{1}{c} \cdot \frac{\partial V}{\partial x^0} = 0, \quad i \in [1, 3]; \quad x^0 = c\,t; \quad c^2 = \frac{1}{\varepsilon_0 \mu_0} . \qquad (A.5.4 \text{ a, b, c})$$

The components along any axis of a Cartesian system of co-ordinates will be:

$$E_i = -\frac{\partial V}{\partial x^i} - \frac{\partial A_i}{\partial t}; \quad B_i = \frac{\partial A_k}{\partial x^j} - \frac{\partial A_j}{\partial x^k}, \quad \forall j \neq k; \quad \forall i, j, k \in [1, 3]. \qquad (A.5.5 \text{ a, b})$$

The relations (A.5.1 a, b) may be written using a set of four quantities A_i as follows:

$$E_i = \frac{\partial A_0}{\partial x^i} - \frac{\partial A_i}{\partial t}, \quad B_i = \frac{\partial A_k}{\partial x^j} - \frac{\partial A_j}{\partial x^k}, \quad \forall i, j, k \in [1, 3]; \quad A_0 = -V. \qquad (A.5.6 \text{ a, b, c})$$

A.5.2. ESTABLISHING THE NATURE OF THE TENSORS

A tensor is in fact a geometrical object, the vector being a particular case. By the nature of a tensor is meant its character, namely whether when passing from one system of reference to another one, the tensor will be transformed as a contravariant or as a covariant one.

If the geometrical object, which will be defined as a tensor, is considered separately, with no relation with other tensors, each of the two variants may be chosen. If it is considered in relation with other tensors, its nature being established, the situation is different. For example, it is necessary to mention that if the product of two tensors yields a result of a certain nature, say a scalar (tensor of rank 0), the product should give the same result in any other system of reference.

If the character of one tensor is established and we have to choose the character of a geometrical object that will constitute the other tensor, one can use the known procedures: the *tensor quotient law* [5, p. 167] or the *theorem of tensor classification* [10, p. 376]. We shall extend the last one, using integral operators, in order to facilitate the procedure.

In order to fix the ideas, we shall establish the nature of the set of four quantities A_i that may be functions of co-ordinates. Several explanations have been given in literature on this subject, some of them referring to formula (A.5.4), but without a precise conclusion [12].

However, we consider that this analysis may be carried out as follows. We shall refer to Section 3.10 where the transition from the system of reference K to the system of reference K', both being inertial systems, is examined. We shall take into consideration that $V = -A_0$ in (3.115) and (3.116 a), of Chapter 3, as in (A.5.6 c). Rewriting those relations using the symbols above, and replacing A_i by $c\,A_i$ for $i \neq 0$, there follows:

$$A_0' = \alpha \left(A_0 + \frac{v}{c} A_1 \right); \quad A_1' = \alpha \left(\frac{v}{c} A_0 + A_1 \right); \quad A_2' = A_2; \quad A_3' = A_3; \qquad \text{(A.5.7 a, ..., d)}$$

and:

$$x'^0 = \alpha \left(x^0 - \frac{v}{c} x^1 \right); \quad x'^1 = \alpha \left(-\frac{v}{c} x^0 + x^1 \right); \quad x'^2 = x^2; \quad x'^3 = x^3. \qquad \text{(A.5.7 e, ..., h)}$$

By comparing the two systems of relations, it follows that the set A_0, A_1, A_2, A_3 and the set x^0, x^1, x^2, x^3 change in different manners. Therefore, the latter set being a contravariant four-vector, the former will be a covariant four-vector. Then, we can renounce the replacement. According to Linear Algebra relationships, if the set A_0, A_1, A_2, A_3 is a covariant tensor of rank 1, each set obtained from the previous one by multiplying each element (component) by any constant factor, will also be a covariant four-vector, i.e., hence a tensor of rank 1. Therefore the set A_i, $\forall\, i \in [0,3]$, is a covariant four-vector.

In general, in any multidimensional continuum (space), in the present case, a four-dimensional one, any set of four quantities, that is a geometrical object with four components (elements), which can be functions of co-ordinates, may constitute a tensor of rank 1, provided the transformation formulae when passing from a reference frame to another, are like the transformation formulae of the co-ordinates, or opposite. In the former case the tensor will be called contravariant tensor, and in the latter one, the tensor will be called covariant tensor.

From the mathematical point of view, we can choose any of those transformation formulae. It should be added that regardless of the chosen formulae, the length of the tensor will be the same, only the components will differ. From the physical point of view, it is useful to choose a transformation, which is in good agreement with the physical conditions, and requires the smallest volume of computations. Moreover, it is necessary that the nature of tensors entering an equation should be the same, otherwise the equation being not valid.

Having these possibilities in choosing the nature of the geometrical object above, we shall consider it as a covariant tensor of rank 1, which is a covariant four-vector.

We shall make a first verification in order to examine the previous choice. Let us consider relation (A.5.1 b), and calculate the flux of vector $\boldsymbol{B}$ through any simply

connected surface bounded by a closed curve Γ in a three-dimensional continuum (space). Further on, we shall use an integral form. According to Stokes theorem, we shall replace the calculation of the flux, of the right-hand side of the theorem relationship, by the calculation of the circulation along that curve, which will be given by the integral of the left-hand side of the relation.

We shall assume that the flux is a scalar. Then, if the covariant vector component A_i, of the circulation element $A_i\,\mathrm{d}l_i$, multiplied with the contravariant component $\mathrm{d}l_i$, remarking that the line-element can be written $\mathrm{d}l^i$, should yield a scalar, hence in accordance with the physical meaning of the considered case. We can add that, according to relation (A.5.6 b), for calculating the magnetic flux, the component A_0 does not occur.

The velocity of light *in vacuo*, denoted by c is assumed to be constant. According to relations (A.5.6 a) and (A.5.6 b), we shall consider two tensors of rank 2, namely F_{ij} and G_{ij} by which the field state quantities will be expressed in a new general form as follows:

$$F_{ij} = c_i \frac{\partial A_j}{\partial x^i} - c_j \frac{\partial A_i}{\partial x^j}; \quad A_0 = -V; \quad x^0 = ct; \quad \forall i, j \in [0, 3]; \quad i \neq j;$$

$$c_i = 1 \wedge c_j = c, \quad \forall i \neq 0 \wedge j = 0; \qquad c_i = c_j = c; \quad \forall i \wedge j \neq 0.$$

(A.5.8 a, ..., e)

Therefore, we can write:

$$E_i = F_{i0} = \frac{\partial A_0}{\partial x^i} - c \frac{\partial A_i}{\partial x^0}; \quad A_0 = -V; \quad x^0 = ct;$$

$$B_k = B_{ij} = \frac{1}{c}\left(c\frac{\partial A_j}{\partial x^i} - c\frac{\partial A_i}{\partial x^j} \right) = \frac{1}{c} F_{ij}.$$

(A.5.8 f, ..., i)

Introducing the axis coefficients of the Galilean reference frame, e_{ii}, we can write:

$$A_i = e_{ii} A^i; \quad e_{00} = 1, \; e_{ii} = -1, \; \forall i \in [1, 3]; \quad e_{ij} = 0, \; \forall \in i \neq j;$$

(A.5.9 a, ..., e)

$$A_i A^i = \left(A_0\right)^2 - \left(A_1\right)^2 - \left(A_2\right)^2 - \left(A_3\right)^2;$$

and:

$$F_{ij} = e_{ii}\, e_{jj}\, F^{ij}, \; \forall i, j \in [0, 3]; \quad F_{ij} = -F_{ji};$$

$$G_{i0} = c\varepsilon_0 F_{i0}; \quad G_{ij} = \frac{1}{\mu_0} \cdot \frac{1}{c} F_{ij}; \; \forall i, j \in [1, 3];$$

(A.5.10 a, ..., f)

$$D^i = \frac{1}{c} G^{0i}; \quad H^{ij} = G^{ij}; \; \forall i, j \in [1, 3].$$

It is useful to mention that expression (A.5.8 a) is antisymmetric with respect to indices i and j.

We shall verify whether F_{ij} represents a tensor of rank 2. By examining relations (A.5.8 a), we can see that F_{ij} is expressed by two relations, which represent a difference of two terms, each of them containing derivatives of vector A_i.

From the expression (A.6.44), of Appendix 6, for a Galilean system of reference, it follows that the expression (A.5.8 a), after the scaling, represents the difference of two covariant derivatives, hence of two tensors, therefore it is also a tensor, namely covariant and of rank 2, in a four-dimensional continuum.

A.5.3. THE FIELD STATE QUANTITIES FOR NON-POLARIZED MEDIA

Using the tensor functions above, we can write the following expressions, of the usual three-dimensional vectors, along the axes denoted here by subscripts i and k:

$$E_i = F_{i0}; \quad H_k = G_{ij} = G^{ij}; \quad \forall i,j,k \in [1, 3]; \qquad \text{(A.5.11 a, b)}$$

and further:

$$D_i = \varepsilon_0 E_i = \frac{1}{c} G_{i0}; \quad B_k = B_{ij} = \frac{1}{c} F_{ij}; \quad \forall i,j,k \in [1, 3], \qquad \text{(A.5.12 a, b)}$$

here E_i, D_i, H_k, B_k do not represent the components of tensors, but the components of the usual three-dimensional vectors.

It is interesting to be noted that according to the tensor quotient law, the electric displacement (electric induction) D^i, of (A.5.10 e) is a contravariant tensor of rank 1, as it results from the local form of the electric flux law. Indeed the divergence of this vector must be a scalar, and hence the vector must be contravariant.

This result can also be verified by the procedure we have proposed in Section A.5.2, expressing the electric flux through a closed surface, and applying the Gauss-Ostrogradski theorem. In this case, under the integral sign of the left-hand side, there occurs the electric displacement D^i considered as a contravariant vector with three components, which represent a contravariant tensor of rank 1, multiplied by the product $d x^j d x^k$ that represents a contravariant tensor of rank 2. Therefore, in the left-hand side there occurs a contravariant tensor of rank 3. Under the integral sign of the right-hand side, there occurs the divergence of the contravariant vector of component D^i, which is a scalar, and is multiplied by the volume product $d x^i d x^j d x^k$ that represents a contravariant tensor of rank 3. Hence, concerning the type of the vector (tensor of rank 1) D^i, the result above is confirmed.

For the vector electric induction to be contravariant, it is necessary that the covariant vector (tensor of rank 1) E_i be multiplied by a contravariant tensor of the second rank. We can write:

$$D^i = \delta^{ij} \varepsilon_0 E_j, \quad \forall i, j \in [1, 3], \quad \delta^{ij} = \left| e^{ij} \right|.$$
(A.5.13 a, b)

It is to be noted that the tensor D^i is defined using a four-dimensional space, but it has only three components, because the components (elements) of tensor F_{ij} are zero for the components of which the two indices are equal to each other, hence for the time co-ordinate, i.e., for $i = 0$.

The relations (A.5.10 a, e), (A.5.8 c) yield the following tensor relations:

$$D^i = -D_i; \quad E_i = -E^i; \quad \forall i \in [1, 3].$$
(A.5.14 a, b)

The expressions of E_i, and B_k, from relation (A.5.11 a) and (A.5.12 b), respectively, are valid for vacuum as well as for polarized media, while the expressions of D_i and H_k, from the (A.5.12 a) and (A.5.11 b), are valid only for vacuum.

A.5.4. THE FIELD AND SUBSTANCE STATE QUANTITIES FOR POLARIZED MEDIA

For a polarized medium, the substance state quantities of electric and magnetic polarization, respectively, have to be introduced by symbols P and M_J, where we have used the index J in order to avoid any confusion with index j above. In this Section, we shall consider only the temporary polarization. In addition, we shall indicate in this Section, the components of the usual three-dimensional vectors by index k denoting one of the three axes. In this case, the respective quantities can be introduced by the following relations:

$$P_k = P^i; \quad M_k = M^{ij}; \quad H_k = H^{ij};$$
$$P_k = \varepsilon_0 \chi_{e,kj} E_j; \quad M_k = \chi_{m,kj} H_j; \quad M_{J,k} = \mu_0 M_k; \quad \forall i, j, k \in [1, 3],$$
(A.5.15 a, …, f)

where in relations from a to c, in the left-hand side, the usual vector components are written, but in the right-hand side, the tensor components are written; while in relations from d to f, in both sides, only usual vector components are written. In this manner, the components of tensors may be expressed more easily. It follows:

$$D^i = \varepsilon_0 \delta^{ij} E_j + \varepsilon_0 \chi_e^{ij} E_j; \quad B_k = B_{ij} = \mu_0 H_k + M_{J,k}; \quad \forall i, j, k \in [1, 3],$$
(A.5.16 a, b)

where E_j may represent the usual vector component as well as the tensor component, and B_{ij}, and B_k represent the tensor component, and the corresponding vector component, respectively.

This subject has been thoroughly analysed in work [6, p. 156, 268]. However, further on, we have used, to some extent, another way, in order to allow for including, apart from the temporary polarization, also the permanent polarization.

A.5.5. THE MAXWELL EQUATIONS IN TENSOR FORM USING THE INTRODUCED SYMBOLS

A.5.5.1. The Maxwell Equations for Empty Space

Using the symbols introduced above, we can write the Maxwell equations for empty space (vacuum) as follows. We shall consider two sets of equations in the order used by H. A. Lorentz, which differs from that of J. C. Maxwell. For the first set, we shall use the tensor G^{ij} and for the second one, the tensor F_{ij}.

The equations of the first group, using the symbols of the *List of symbols*, are:

$$\frac{\partial G^{ij}}{\partial x^j} = J^i + \frac{\partial G^{0i}}{\partial x^0}, \quad \forall i,j \in [0,3], \quad i \neq j; \quad J^0 = c\rho_v. \tag{A.5.17}$$

The relations (A.5.17) yield the local (differential) form of the: a) electric flux law (after having simplified the factor c of the numerator); b) magnetic circuital law, for each axis; as $i = 0$ or $i \neq 0$, respectively.

The equations of the second set are:

$$\frac{\partial F_{ij}}{\partial x^k} + \frac{\partial F_{jk}}{\partial x^i} + \frac{\partial F_{ki}}{\partial x^j} = 0, \quad \forall i,j,k \in [0,3], \quad i \neq j \neq k. \tag{A.5.18}$$

The relation (A.5.18) yields, after having simplified the factor c of the numerator, the local (differential) form of the: a) magnetic flux law; b) law of electromagnetic induction for each axis; as $i, j, k \neq 0$ or $k = 0$, respectively.

We can write an example for the first set:

$$\frac{\partial G^{1j}}{\partial x^j} = \frac{\partial G^{10}}{\partial x^0} + \frac{\partial G^{11}}{\partial x^1} + \frac{\partial G^{12}}{\partial x^2} + \frac{\partial G^{13}}{\partial x^3} = J^1,$$

$$-\frac{\partial D_1}{\partial t} + \frac{\partial H_3}{\partial x_2} - \frac{\partial H_2}{\partial x_3} = J_1,$$

(A.5.19 a, b)

where in the second relation the vector notation has been used. The sign minus before the first term of the last equation occurs because of relation (A.5.10 e), and before the third term, because of the inversion of the indices. Therefore, the equation of the magnetic circuital law, for the first axis, has been obtained.

We can also write an example for the second set when $i = 2$, $j = 3$, $k = 0$:

$$\frac{\partial F_{23}}{\partial x^0} + \frac{\partial F_{30}}{\partial x^2} + \frac{\partial F_{02}}{\partial x^3} = 0,$$

$$\frac{\partial B_1}{\partial t} + \frac{\partial E_3}{\partial x_2} - \frac{\partial E_2}{\partial x_3} = 0,$$

(A.5.20 a, b)

where in the second relation the vector notation has been used. The last equation has been obtained after having simplified the factor c of the numerator.

Therefore, the local (differential) form of the local (differential) equation of the law of electromagnetic induction, for the first axis, has been obtained.

A.5.5.2. The Maxwell Equations for Polarized Media

In order to consider the equations for polarized media, it is necessary to introduce the polarization tensors of the substance.

For the first set of equations, we shall complete relations of (A.5.17), according to relations (A.5.15 a, b). In order to bring the equation of the first set into a covariant form, with respect to any change of co-ordinates, we shall introduce, in the equation above, the symbols:

$$G^{0j} := G^{0j} + c\,P^j = c\,\varepsilon_0\,\delta^{jj}\,E_j + c\,P^j, \quad \forall\,j \in [1,3];$$

(A.5.21)

$$G^{i0} := -c\,\varepsilon_0\,\delta^{ii}\,E_i - c\,P^i, \quad \forall\,i \in [1,3];$$

(A.5.22)

$$G^{ij} := G^{ij} - M^{ij}, \quad \forall\,i,j \in [1,3];$$

(A.5.23)

$$J^0 = c\,\rho_v; \quad P^0 = 0; \quad M^{0j} = 0; \quad \forall\,j \in [1,3].$$

(A.5.24 a, b, c)

In all definition formulae above, the terms of the form G^{ij} from the right-hand side are those given by relations (A.5.10 c, d). Consequently, equation (A.5.17) becomes:

$$\frac{\partial G^{ij}}{\partial x^j} = J^i, \quad \forall\,i,j \in [0,3], \quad i \neq j; \quad J^0 = c\,\rho_v; \quad P^0 = 0; \quad M^{0j} = 0.$$

(A.5.25 a, ..., d)

The equations of (A.5.25) correspond to equations of (A.5.17) above.

For the second set of equations, we shall obtain the same relations as for non-polarized media because the occurring quantities are not influenced by considering the polarization:

$$\frac{\partial F_{ij}}{\partial x^k} + \frac{\partial F_{jk}}{\partial x^i} + \frac{\partial F_{ki}}{\partial x^j} = 0, \quad \forall\,i,j \in [0,3], \quad i \neq j \neq k.$$

(A.5.26)

The equation (A.5.26) corresponds to equation (A.5.18) above.

A.5.6. THE EXPRESSION OF THE ELECTROMAGNETIC FIELD EQUATIONS IN ANY SYSTEM OF REFERENCE

When passing from a Galilean system of reference to any other system of reference, we shall use formula (A.5.25 a) above and formula (A.6.74) by which the derivatives have to be replaced by the covariant derivatives. Taking into account that the tensor G^{ij} is antisymmetric in indices i and j, we can use formula (A.6.76), and there follows:

$$\nabla_j G^{ij} = \frac{1}{\sqrt{-g}} \cdot \frac{\partial}{\partial x^j} \left(\sqrt{-g}\, G^{ij} \right) = J^i ,$$ (A.5.27)

where g is the expression of the determinant formed by the elements g_{ij} of the fundamental (or metric) tensor. Therefore, we have got the general form of the first set of equations of the electromagnetic field.

When passing from a Galilean system of reference to any other system of reference, equation (A.5.26) may be written, like equation (A.5.18), hence replacing the derivatives by covariant derivatives, in the form:

$$\nabla_k F_{ij} + \nabla_i F_{jk} + \nabla_j F_{ki} = 0, \quad \forall i, j \in [0, 3], \quad i \neq j \neq k .$$ (A.5.28)

By expanding the expression of the covariant derivative of the tensors, one can remark that the terms containing the Christoffel symbols reduce each other, and therefore equation (A.5.28) remains unchanged.

Remark. Some general considerations have to be added. The electric field strength is a covariant vector, because, multiplied by the co-ordinate variation, a scalar is obtained. The electric displacement is a contravariant vector because its divergence is a scalar (law of the electric flux). The magnetic field strength is a contravariant tensor of rank 2, because its curl, like the density of the electric current, is a contravariant vector (law of the magnetic circuital law in local form). The magnetic induction is a covariant tensor of rank 2, because its derivative with respect to time is equal to the curl of the electric field strength changed in sign (law of the electromagnetic induction), which represents a covariant tensor of rank 2. All previous conclusions result as well from the integral form of the laws, as shown in the example of Section A.5.2. It is useful to add that the preceding observations concern the three-dimensional continuum (space).

It is to be noted that the replacement of the tensor of the second rank of the magnetic induction or of the magnetic field strength, as already mentioned in literature [6], represents an approximation by renouncing certain properties of the tensor, and replacing it by a polar or an axial vector. The replacement of the axial tensor by a polar one is compensated by the system of co-ordinates that may be right-handed or left-handed.

List of Symbols Used in Appendix 5

A_i −   component of the four-vector potential;

c −   velocity of light in empty space, supposed to be constant;

D_i −   component of the electric displacement, considered as a usual three-dimensional vector;

D^i −   contravariant component of the electric displacement;

e_{ii} −   axis coefficient, for the axis i of the Galilean reference frame, i.e., element of the fundamental tensor of this frame;

E_i −   covariant component of the electric field strength, as well as component of the electric field strength along axis i as a usual three-dimensional vector;

E^i −   contravariant component of the electric field strength;

F_{ij} −   component of the covariant tensor of rank 2 yielding B_{ij} ;

F_{i0} −   component of a covariant tensor, deriving from the previous one, and yielding the component E_i of the electric field strength, considered as a usual three-dimensional vector;

g −   value of the determinant of the fundamental tensor;

G^{ij} −   contravariant tensor of rank 2, and yielding H^{ij} ;

G_{ij} −   covariant tensor of rank 2, and yielding H_k ;

G_{i0},   −   covariant and contravariant components, deriving from the previous one, and yielding the component D^i ;

G^{i0}

H_k −   component of the magnetic field strength along axis k , considered as a usual three-dimensional vector;

J^i, J_i −   component of a contravariant four-vector representing the density of the conduction electric current, along axis i , and of the usual three-dimensional vector, respectively;

M^{ij} −   component of a contravariant tensor of rank 2, yielding M_k ;

M_k −   magnetisation along axis k, while the magnetic polarization is M_{Jk} , considered as a usual three-dimensional vector;

P^i −   contravariant component of the electric polarization;

P_k −   electric polarization along axis k , considered as a usual three-dimensional vector;

V −   electric potential;

x^i −   co-ordinate along axis i ;

δ_{ij}, δ^{ij} −   symbols equal to unity for equal indices, and equal to zero for different ones (Kronecker symbols);

ε −   electric permittivity, *in vacuo* it is ε_0 ;

μ −   magnetic permeability, *in vacuo* it is μ_0 ;

ρ_v −   volume density of the electric free charge;

χ −   electric and magnetic susceptibility, according to the index e or m.

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APPENDIX 6

TENSOR CALCULUS

In this Appendix, we shall recall certain definitions and several largely used relations. In addition, certain more important proofs will be given. The bibliographic mentions [1]-[14] specific for this Appendix are at the end of this Appendix. Also, a List of symbols is given at the end of this Appendix. We shall have in view the works [8], [9] that present certain advantages for the applications.

A.6.1. TENSOR ALGEBRA

The Vector Calculus aims to renounce the systems of co-ordinates, by using vectors, which are oriented straight-line segments. The various operations over vectors can be performed without resorting to co-ordinates. In the case of more complicated objects, when the number of co-ordinates is greater than number three, the use of vectors does not suffice. For this reason, it is necessary to use systems of co-ordinates (systems of reference, reference frames) [3, p. 25], [14, p. 6], and tensors instead of vectors.

A.6.1.1. Systems of Co-ordinates

The simplest system of co-ordinates, in the three-dimensional space, is the Cartesian system of co-ordinates. The considered system is orthogonal and will be chosen right-handed. In general, any system of co-ordinates may be used, according to the scope. In the Theory of Relativity, taking in view that each event is characterized by its position in space and moment in time, a four-dimensional system of co-ordinates is to be used, and the corresponding space is called four-dimensional space, four-dimensional continuum or space-time continuum. In general, an *n*-dimensional continuum may be considered. The co-ordinates in this continuum are also called Gaussian co-ordinates.

When passing from a system of co-ordinates to another, the co-ordinates of the first will be denoted by x^i, and those of the second by x'^i, hence adding the sign prime. The quantities referring to the second system will also be indicated by the sign prime. The index *i* written as superscript can take every value of the numbers denoting the co-ordinates. In accordance with several well-known works, in the case of the Theory of Relativity, we shall count the axes of co-ordinates starting with zero (for time). For any different case, we shall make mention. The relationship between the co-ordinates of the second system will be expressed in terms of those of the first one. The following relation will be used:

$$x'^i = x^i\left(x^0, x^1, x^2, x^3\right).$$
(A.6.1)

The transformation function (A.6.1) must be continuous, and its derivatives of the first three orders should also be continuous. In addition, the Jacobian determinant must be different from zero, for the inverse transformation to be possible.

Sometimes, it may occur that the number of axes of co-ordinates of the first system, say a Cartesian one, is greater than that of the second system.

According to the previous assumptions, the total differential of the function given by (A.6.1) is:

$$\mathrm{d}\,x'^{i} = \frac{\partial x'^{i}}{\partial x^{0}}\mathrm{d}\,x^{0} + \frac{\partial x'^{i}}{\partial x^{1}}\mathrm{d}\,x^{1} + \frac{\partial x'^{i}}{\partial x^{2}}\mathrm{d}\,x^{2} + \frac{\partial x'^{i}}{\partial x^{3}}\mathrm{d}\,x^{3} = \frac{\partial x'^{i}}{\partial x^{j}}\mathrm{d}\,x^{j}\,, \tag{A.6.2 a}$$

$$\forall\,i,j \in [0,3].$$

As a result, one may write:

$$\mathrm{d}\,x'^{i} = a_{j}^{i}\,\mathrm{d}\,x^{j}\,, \quad \forall\,i,j \in [0,3]. \tag{A.6.2 b}$$

A.6.1.2.   Tensors of Rank 0 and of Rank 1. Summation Convention. Covariant and Contravariant Tensors. Affine transformations.

In order to fix the ideas, we consider the case where n co-ordinates are necessary. The corresponding space is called an n-dimensional space or an n-dimensional continuum. We are going to recall that a set containing, for each point of the space, only a scalar, is called a tensor of rank (also called order) zero. A set containing, for each point of the space, n quantities may constitute a vector also called a tensor of rank 1. Any set of n quantities may represent a tensor of rank (order) 1, also called vector, but for to be a tensor (vector), the set elements must change, when passing (switching) from the first system of co-ordinates to the second one, according to formulae below:

$$A'^{i} = a_{j}^{i}\,A^{j}\,; \quad A'_{i} = a_{i}^{j}\,A_{j}\,; \quad \forall\,i,j \in [0,n-1], \tag{A.6.3 a, b}$$

or

$$A'^{i} = a_{j}^{i}\,A^{j}\,; \quad A'_{i} = a_{i}^{j}\,A_{j}\,; \quad \forall\,i,j \in [1,n], \tag{A.6.4 a, b}$$

depending on the number attributed to the first axis. We shall use various formulae.

A set of quantities satisfying a relation of the form (A.6.3 b) is represented by the differentials of the co-ordinates as in relations (A.6.2 b).

In order to simplify the relations, certain authors denote the co-ordinates of the first system of reference, in the case in which it is a Cartesian system, by y^{i} and those of the second system by x^{i}, hence renouncing the sign prime, what we also shall do, when the case occurs.

In relations (A.6.2) – (A.6.4), and further on, the *summation convention* is used, namely, if in a term, an index occurs twice, as subscript and superscript, respectively, or if two factors of the same term have the same index, regardless of their position (subscript

or superscript), it implies the summation of the terms obtained for all values that the respective index can take.

The coefficients of the last relations (A.6.3), (A.6.4), may be expressed as follows:

$$a_j^i = \frac{\partial x'^i}{\partial x^j}; \quad a_i^j = \frac{\partial x^j}{\partial x'^i}; \quad \forall \det\left[a_j^i\right] \neq 0. \tag{A.6.5 a, b}$$

Among the various possible transformations, one can mention the affine transformation and the orthogonal transformation, due to their respective properties. The affine transformation preserves co-linearity and ratios of distances. Therefore, in an affine transformation all points lying on a line (e.g., a straight-line) remain on the same line after transformation. Also, the ratio between the lengths of two portions of a straight-line segment keeps the same value after transformation. The affine transformations are linear transformations. An affine transformation may be composed of: translation, scaling, rotation, shearing and reflection. However the angle between lines may change. The orthogonal transformations, also called orthonormal transformations, preserve the length of vectors and the angles between them. It may be composed only by rotation or rotation and reflection.

For an orthogonal transformation, the coefficients appearing in (A.6.5) must satisfy the relation:

$$\det\left[a_j^i\right] = \pm 1, \tag{A.6.6}$$

where the transformation for which the plus or minus sign occurs is called a proper or improper transformation, respectively.

In any relation with tensors correctly written, when suppressing the indices denoted by the same letters, and occurring as subscript and superscript, respectively, in each of the two sides of the relation, the remaining indices in each side will be the same.

According to the general convention, if the components of a tensor change according to formulae of type (A.6.3 a), we deal with a *contravariant tensor* (with respect to this index), whereas if we consider formulae of type (A.6.3 b), we now deal with a *covariant* tensor. Each index with respect to which the tensor is contravariant is written as *superscript*, and each one with respect to which the tensor is covariant is a *subscript*. For this reason, the co-ordinates are denoted by using superscripts (except some authors who prefer to keep for co-ordinates the index as subscript, in order to avoid the usage of parentheses in the case of exponents (powers), but they mention this fact as unusual).

In the case of the translation, when passing from a Cartesian system of co-ordinates to another one of the same type, translated and rotated with respect to the former one, the coefficients of relations (A.6.5 a) and (A.6.5 b), are equal. Consequently, then no difference appears between the covariant and contravariant tensors. It is an orthogonal transformation. In the case of the Cartesian systems of the Special Theory of Relativity, which represent Galilean systems of reference, the Lorentz transformation is orthogonal, but it is to be noted that the field state quantities do not change like the co-ordinates or inversely, because they do not represent four-dimensional vectors.

A.6.1.3. Operations with Tensors. Tensors of Rank 2.

In order to fix the ideas, further on, we shall consider a four-dimensional space, and precise some important operations.

1° Addition. The sum of two tensors of rank 1 may be seen in the example:

$$C_i = A_i + B_i ; \quad \forall i \in [0, 3],$$

(A.6.7)

where the values taken by the index i are also given, so that one finally obtains 4 relations. The relation above is not casual and corresponds to the addition of the components of two vectors. As a result the sum of two tensors is also a tensor.

2° Multiplication. The product of two tensors of rank 1 may be seen in the next example:

$$C_{ij} = A_i B_j ; \quad \forall i, j \in [0, 3].$$

(A.6.8)

The following components (elements) will be obtained for the indices $i = 0$ and $i = 1$:

$$C_{00} = A_0 B_0 ; \quad C_{01} = A_0 B_1 ; \quad C_{02} = A_0 B_2 ; \quad C_{03} = A_0 B_3 ;$$
$$C_{10} = A_1 B_0 ; \quad C_{11} = A_1 B_1 ; \quad C_{12} = A_1 B_2 ; \quad C_{13} = A_1 B_3 ; \quad \forall i, j \in [0, 3].$$

(A.6.9)

Similarly, we can write the components for the other values of the index i. Finally, 16 scalar values will be obtained. In this case, it is necessary to use in the left-hand side two indices for obtaining a suggestive representation of all obtained elements. Therefore, we shall obtain a set of 4^2 quantities, which may be a tensor of rank (order) 2, and the symbol of the *general term* (that may also be called *component* or *element*) of the tensor will have two indices.

In this case, the expression of the general term of the tensor, when passing from the first to the second system of reference is:

$$C'_{ij} = A'_i B'_j = \frac{\partial x^k}{\partial x'^i} A_k \frac{\partial x^l}{\partial x'^j} B_l = \frac{\partial x^k}{\partial x'^i} \cdot \frac{\partial x^l}{\partial x'^j} A_k B_l = \frac{\partial x^k}{\partial x'^i} \cdot \frac{\partial x^l}{\partial x'^j} C_{kl} ,$$

(A.6.10)

which, as further shown, is a twice covariant tensor.

There follows that the values of the components of the tensor will change when passing from a reference system to another one, according to the transformation relation, as for the co-ordinates if the respective index is *contravariant*, and inversely if the index is *covariant*. As shown in the last relation, if the letter of a certain index appearing twice, may lead to a confusion, it is convenient to replace it by another letter, as we did, because if the index takes the same values, the result will be not affected.

From the previous example, there follows that the product of two tensors, in general, depends on the order of factors.

3° Tensors of rank 2 (Tensors of the second order). Tensors of higher rank.

According to relation (A.6.10), a set containing n^2 quantities (first condition) may be a tensor of rank 2 in an n-dimensional continuum, whether when passing from the first system of reference to the second, the general term will change according with the character of the respective index contravariant or covariant (second condition), like to the co-ordinates or conversely. The form of the general term of a rank 2-tensor shows one of the following forms: T_{ij}, T^{ij}, $T_i^{\,j}$, namely twice covariant, twice contravariant, mixed once covariant and once contravariant. If the tensor is written as a matrix, the letters i and j denote the row and column, respectively. Other symbols have been proposed in [7].

Any tensor C_{ij} is called *symmetric* or *antisymmetric*, in its indices i and j, as the relation $C_{ij} = C_{ji}$ or $C_{ij} = -C_{ji}$, respectively, is fulfilled [5, p. 69]. An antisymmetric tensor can be built up by the relation:

$$C_{ij} = A_i B_j - A_j B_i, \quad \forall\, i,j \in [1,3].$$
(A.6.11)

By examining this tensor, we can see that it has 9 components, and only 6 are different from zero, the terms with the same double index being zero. From these 6 terms, three of them differ from the other three only in sign. Considering three components, we can see that they just represent the components of the vector product of two three-dimensional vectors.

Any tensor of rank 2, say T_{ij} may be considered as constituted of two tensors: one symmetric S_{ij}, and another antisymmetric A_{ij}, as follows:

$$S_{ij} = \frac{1}{2}\left(T_{ij} + T_{ji}\right); \quad A_{ij} = \frac{1}{2}\left(T_{ij} - T_{ji}\right); \quad T_{ij} = S_{ij} + A_{ij}.$$
(A.6.12)

Similarly, we can write the general term of a tensor several times covariant and several times contravariant, for instance T_{ijk}^{uv}, which represents the general term of a tensor of rank 5, three times covariant and twice contravariant.

4° Equality of two tensors.
Two tensors, A_{ij} and B_{ij} are equal to each other if each element of one is equal to the corresponding element of the other:

$$A_{ij} = B_{ij}, \quad \forall\, i,j \in \mathbf{N}.$$
(A.6.13)

This property of the tensors is invariant, because if this property is fulfilled in a system of reference, it holds in any other system of reference. Consequently, if the components (elements) of a tensor are zero in one system of reference, they will also be zero in any other system of reference.

5° Contraction of a tensor.
The operation results from the following example. Let A_i and B^j be two tensors, their product will be $C_i^{\,j} = A_i B^j$. If one puts $i = j$, the result will be:

$$A_i B^j = A_j B^i = A_1 B^1 + A_2 B^2 + A_3 B^3, \quad \forall i = j \in [1, 3], \tag{A.6.14}$$

which just represents the scalar product of vectors A_i and B^j. The contracted result will be, in this case, a scalar. If the tensors of the product were of higher rank, then the result, after the contraction on an index, will lead to a tensor of a rank smaller of two unities. This operation is called contraction on an index, in the previous case on index i.

6° Tensors of higher ranks. *The rank or order of a tensor is equal to the number of indices*. All operations explained above are valid regardless of the rank of the considered tensors.

It is useful to add that if a set of quantities multiplied with a *tensor* of rank p yields a *tensor* of rank $p + q$, then, the set will be a *tensor* of rank q. This rule is called the *quotient law* [5, p. 71], or more generally, the *theorem of tensor classification* [10, p. 376].

7° Covariant equation. Another remark concerns the equations. In Physics, an equation must keep the same form when passing from a Cartesian system of co-ordinates to another one. This property is expressed saying that the *equation is covariant* with respect to the transformation of the co-ordinates. The property is called *covariance of the equation* with respect to the transformation of co-ordinates. This property may be extended to any system of co-ordinates. We should add that in this case, the meaning of the word covariant is different from the meaning concerning the type of a tensor.

A.6.2. TENSOR ANALYSIS

A.6.2.1. The Metrics of a Space

The metrics of a space represents, in fact, the expression of the distance between two points very near to each other. The expression may have different forms. In the case of the three-dimensional space (Euclidean Geometry), in a Cartesian system of reference, the expression will be:

$$(d s)^2 = \left(d x^1\right)^2 + \left(d x^2\right)^2 + \left(d x^3\right)^2. \tag{A.6.15}$$

Also in the case of a Cartesian system of co-ordinates, but with four dimensions, in the Theory of Relativity, the following metric relation is used:

$$(d s)^2 = e_{ij} \, d x^i \, d x^j, \ \forall i \in [0, 3]; e_{ij} = 0, \ \forall i \neq j; \ e_{00} = 1; \ e_{ii} = -1, \ \forall i \in [1, 3]. \tag{A.6.16}$$

A system of reference the metrics coefficients of which are given by relations (A.6.16) is called Galilean system of reference.

A.6.2.2. The Fundamental Tensor. General Systems of Co-ordinates.

We shall consider anew two systems of co-ordinates, the first being a Cartesian system, like above, and the second any curvilinear system. In this case, we shall denote, as mentioned after formula (A.6.4), the co-ordinates of the first system by y^i, and those of the second by x^i. We want to express the transformation of the metric relation in the second system. The metric relation is assumed invariant when changing the system of co-ordinates. Indeed, the distance between two points must be the same, regardless of the system of co-ordinates. Therefore, we shall equate the metric relation expressed in the two systems. We here consider the case with four co-ordinates used in the Special and General Theory of Relativity. The following quadratic form may be written as follows:

$$(\mathrm{d}s)^2 = e_{ii}\left(\mathrm{d}y^i\right)^2 = e_{ii}\left(a_k^i\,\mathrm{d}x^k\right)^2 = e_{ii}\left(\frac{\partial y^i}{\partial x^k}\,\mathrm{d}x^k\right)^2, \quad \forall\, i,k \in [0,3]. \tag{A.6.17}$$

Expanding the terms of the preceding equation, we shall get:

$$(\mathrm{d}s)^2 = e_{ii}\left(\frac{\partial y^i}{\partial x^k}\cdot\frac{\partial y^i}{\partial x^l}\right)\mathrm{d}x^k\,\mathrm{d}x^l, \quad \forall\, i,k,l \in [0,3]. \tag{A.6.18}$$

The relation (A.6.18) may be written in one of the forms:

$$(\mathrm{d}s)^2 = g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j, \quad \forall\, i,j \in [0,3];$$
$$(\mathrm{d}s)^2 = g_{00}\,\mathrm{d}x^0\,\mathrm{d}x^0 + 2\,g_{0\alpha}\,\mathrm{d}x^0\,\mathrm{d}x^\alpha + g_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta, \quad \forall\, \alpha,\beta \in [1,3]; \tag{A.6.19 a, b}$$

which is a quadratic form, called *fundamental form of the metrics*, and contains the following quantities:

$$g_{ij} = g_{ji} = e_{aa}\frac{\partial y^a}{\partial x^i}\cdot\frac{\partial y^a}{\partial x^j} = \sum_{a=0}^{a=N}\frac{\partial y_a}{\partial x^i}\cdot\frac{\partial y^a}{\partial x^j} =, \quad \forall\, a \in [0,N]; \tag{A.6.20}$$

where $N+1$ represents the number of axes of co-ordinates of the first system. If the numbering of axes began with 1, then the index zero would be replaced by 1, and N will be the number of axes of the first system.

The set of elements g_{ij} represents a tensor called the *fundamental tensor*. Indeed, it fulfils the first condition from 3° of A.6.1.3, containing 16 components (elements), thus n^2 elements in a space with $n = 4$ dimensions. Concerning the second condition from 3° of A.6.1.3, we shall remark what follows. In relation (A.6.19), the product $\mathrm{d}x^i\,\mathrm{d}x^j$, which is the product of two contravariant tensors of rank 1, represents a contravariant

tensor of rank 2. The product of this tensor with the tensor g_{ij} yields a scalar, namely the left-hand side of the equation, therefore the set g_{ij} represents a covariant tensor of rank 2.

In the case of a Cartesian system of co-ordinates, these quantities become:

$$g_{00} = 1; \quad g_{ii} = -1, \quad i \in [1, 3]; \quad g_{ij} = 0, \quad \forall i \neq j; \tag{A.6.20 a}$$

where $g_{ij} = e_{ij}$, and then the quadratic form (A.6.19) takes the form of the expression in (A.6.17) after the first equal sign, and is called normal or quasi-Euclidean form of the metrics of the considered space. If no confusion occurs it may be written x^i instead of y^i. The quantities e_{ij} may also be called axis coefficients.

If a quadratic form has the coefficients g_{ij} different from those of formulae (A.6.20 a), it is called a Riemannian form of the metric relation of the space defined by the co-ordinates x^i. If the fundamental quadratic form of any space cannot be reduced to the form having the components of formulae (A.6.20 a), that space is considered, by analogy with the theory of surfaces, as having a curvature [10], [11].

A.6.2.3.   Relations between Covariant and Contravariant Vectors

Consider as given any covariant vector A_i. We are now looking for the relation between a covariant vector and a contravariant one, in any n-dimensional continuum (space). It is possible to obtain a contravariant vector from the covariant one, by using the following relation:

$$A_i = g_{ij} A^j, \quad i, j \in [0, n]. \tag{A.6.21}$$

The components of the contravariant vector can be obtained by solving the system of equations (A.6.21). One obtains:

$$A^j = \frac{D^{ij}}{g} A_i, \quad g^{ji} = \frac{D^{ij}}{g}, \quad i, j \in [0, n]; \quad g = \det[g_{ij}], \tag{A.6.22}$$

where the quantities of the form D^{ji} are cofactors (minors) corresponding to the element (j, i) of the determinant D of the quadratic fundamental form made of elements g_{ij}.

Similarly, we can write:

$$A^j = g^{ji} A_i, \quad i, j \in [0, n]. \tag{A.6.23}$$

The operations represented by relation (A.6.21) and (A.6.23), respectively, may be referred to as *lowering* and *raising* an index, respectively [5, p. 173].

By replacing expression (A.6.22) into (A.6.23), it results that:

$$\frac{D^{ij}}{g} A_i = g^{ji} A_i; \quad g^{ji} = \frac{D^{ij}}{g}; \quad i, j \in [0, n], \tag{A.6.24}$$

and, if changing the order of indices, the tensor g^{ij} being symmetric in i and j we get:

$$g^{ij} = g^{ji} = \frac{D^{ji}}{g}. \tag{A.6.25}$$

If considered as matrices, g^{ij} and e^{ij} are the inverse of g_{ij} and e_{ij}. Hence $e^{ij} = e_{ij}$.

Similarly, one can transform the character of a tensor, say a contravariant tensor of rank 2, denoted T^{ij}, by lowering one index. For this purpose, we multiply the tensor with the element g_{ku} of the fundamental tensor, and we shall contract on indices i and u, by putting $i = u$, as follows:

$$T^{ij} g_{ku} = T_{k\cdot}^{\cdot j}, \tag{A.6.26}$$

and a half-height dot was placed instead of the index which vanished, in order to show the places of indices on which the tensor was contracted. These dots are used in cases in which, for any reason, this mention is required or useful. We obtained a mixed tensor, once covariant and once contravariant. The same transformation may appear in the form:

$$T^{ij} g_{ki} = T_{k\cdot}^{\cdot j}. \tag{A.6.27}$$

In the case of a Galilean (also called quasi-Euclidean) system of reference, several simple relationships can be obtained, as follows:

$$T^{ij} = e^{iq} e^{js} T_{qs} = e^{ii} e^{jj} T_{ij};$$
$$T_i^j = e_{si} T^{sj} = e_{ii} T^{ij}; \tag{A.6.27 a-e}$$
$$e_{00} = 1, \ e_{ii} = -1, \ \forall i \in [1, 3]; \ e_{ij} = 0, \ \forall \in i \neq j,$$

where, for simplicity we omitted the signs dot.

A.6.2.4. Relations between the Components of the Fundamental Tensor

1° First relation. Consider the term below:

$$g_i^{\ j} = g_{ik} g^{jk}. \tag{A.6.28}$$

This relation may be obtained from:

$$A_i = g_{ik} A^k; \quad A^k = g^{jk} A_j; \quad A_i = g_{ik} g^{jk} A_j = g_i^{\ j} A_j. \tag{A.6.29}$$

In order to get $A_i = A_j$, it follows that in relation (A.6.29) the coefficient of the last right-hand side should be:

$$g_i^j = 0, \quad \forall i \neq j; \quad g_i^j = 1, \quad \forall i = j. \tag{A.6.30}$$

2° Second relation. Consider the product below, where the indices j and k do not vary. One will get:

$$g_{ij}\, g^{ik} = g_{1j} g^{1k} + g_{2j} g^{2k} + \ldots + g_{nj} g^{nk} = g_j^k. \tag{A.6.31}$$

3° Third relation. We shall consider the product above, but we shall put $j = k$. Then, we can start from the result above, and j can take successively n values, and for each value, that product can be different from zero, and thus equal to unity, only if the two indices are equal to each other, hence n times. One immediately obtains:

$$g_{ij}\, g^{ij} = g_j^j = n. \tag{A.6.32}$$

4° Kronecker symbols. Related to the relations above, the symbols $\delta_{ij}, \delta^{ij}, \delta_i^j$, also called Kronecker symbols, have also to be considered. Each of this symbols has the property to have the value zero if $i \neq j$, and the value 1 if $i = j$. Therefore, we can write $\delta_{ij} = \left| e_{ij} \right|$. The sets g_{ij}, e_{ij} and δ_{ij} are tensors of rank 2, twice covariant.

A.6.3. THE LENGTH OF A VECTOR IN ANY SYSTEM OF REFERENCE

In order to express the length of a vector A^i at any point, in any system of reference, one can refer to relation (A.6.19), which expresses the square of the element of length. The components of the considered vector being along the tangents of the lines of co-ordinates supposed translated at that point, the use of the mentioned relation is justified. Therefore, the length will be obtained from the following relation:

$$A^2 = g_{ij} A^i A^j = A_j A^j = A_i A^i, \tag{A.6.33}$$

which is like the square of the modulus of a vector in the three-dimensional space.

The angle between two vectors A^i and B^j can be expressed, on similar bases, as follows:

$$\cos\left(A^i, B^j\right) = \frac{A_i B^i}{\sqrt{A_i A^i}\,\sqrt{B_i B^i}}. \tag{A.6.34}$$

A.6.4. COVARIANT DERIVATIVE OF A COVARIANT VECTOR

For the sake of clarity, we shall present a simple deduction of the covariant derivative of a covariant vector, avoiding several extensions used in literature, like the parallel transport of a vector, gradient of an alternative vector, and other considerations. Nevertheless, we shall obtain the same results necessary for calculations.

Let us consider two systems of reference denoted by K_G (Galilean system) with $N+1$ dimensions (the first dimension being denoted by zero and the last by N), and by K, any inertial or non-inertial system of reference with four dimensions. In the former one, the co-ordinates will be denoted by y^i and in the latter one, by x^i. Also, let us consider one covariant four-vector, denoted in the former system of reference frame by Y_i and in the latter by A_i. The relation between the vector components is:

$$A_i = Y_a \frac{\partial y^a}{\partial x^i}, \quad \forall\, a \in [0, N], \forall\, i \in [0, 3]. \tag{A.6.35}$$

The derivative of vector A_i with respect to x^k will be:

$$\frac{\partial A_i}{\partial x^k} = \frac{\partial Y_a}{\partial x^k} \cdot \frac{\partial y^a}{\partial x^i} + Y_a \frac{\partial^2 y^a}{\partial x^i \partial x^k}, \quad \forall\, i,k \in [0, 3], \forall\, a \in [0, N]. \tag{A.6.36}$$

The existence of the two terms of the last relation corresponds to two reasons: a. The variation of the vector with the position of the observation point in space; b. The variation of the vector components with the type of the system of reference.

Using the fundamental tensors e_{aa} and g_{uv} of the two systems of reference, we have the relations:

$$Y_a = e_{aa} Y^a = e_{aa} A^l \frac{\partial y^a}{\partial x^l} = e_{aa} g^{vl} A_v \frac{\partial y^a}{\partial x^l}, \quad \forall\, v,l \in [0, 3], \forall\, a \in [0, N]. \tag{A.6.37}$$

By substituting the expression (A.6.37) into relation (A.6.36), we shall obtain:

$$\frac{\partial A_i}{\partial x^k} = \frac{\partial Y_a}{\partial x^k} \cdot \frac{\partial y^a}{\partial x^i} + e_{aa}\, g^{vl} A_v \frac{\partial y^a}{\partial x^l} \cdot \frac{\partial^2 y^a}{\partial x^i \partial x^k}. \tag{A.6.38}$$

We shall denote:

$$\nabla_k A_i = \frac{\partial A_i}{\partial x^k} - e_{aa}\, g^{vl} A_v \frac{\partial y^a}{\partial x^l} \cdot \frac{\partial^2 y^a}{\partial x^i \partial x^k} = \frac{\partial Y_a}{\partial x^k} \cdot \frac{\partial y^a}{\partial x^i},$$
$$\forall\, i,k,l,v \in [0, 3], \quad \forall\, a \in [0, N], \tag{A.6.39}$$

and:

$$\nabla_k A_i = \frac{\partial Y_a}{\partial x^k} \cdot \frac{\partial y^a}{\partial x^i} = \frac{\partial Y_a}{\partial y^b} \cdot \frac{\partial y^b}{\partial x^k} \cdot \frac{\partial y^a}{\partial x^i}, \quad \forall i,k \in [0,3], \forall a,b \in [0,N]. \tag{A.6.40}$$

When passing from the reference frame K_G to the reference frame K, the first factor of the last term of equation (A.6.40) changes like a tensor (A.6.3 a, b), therefore the term of the left-hand side is also a tensor twice covariant. It is called the covariant derivative of a covariant vector. One may say that it corresponds to a gradient calculation.

We recall the usual symbol of any element of the fundamental tensor (A.6.20):

$$g_{ij} = e_{aa} \frac{\partial y^a}{\partial x^i} \cdot \frac{\partial y^a}{\partial x^j} = \sum_{a=0}^{a=N} e_{aa} \frac{\partial y^a}{\partial x^i} \cdot \frac{\partial y^a}{\partial x^j}, \tag{A.6.41}$$

where $N+1$ represents the number of dimensions of the former system of reference denoted by K_G. According to (A.6.39), one denotes:

$$\Gamma_{l,ik} = e_{aa} \frac{\partial y^a}{\partial x^l} \cdot \frac{\partial^2 y^a}{\partial x^i \partial x^k}, \quad \forall i,k,l \in [0,3], \forall a \in [0,N], \tag{A.6.42}$$

and also, the known relation:

$$\Gamma_{ik}^v = g^{vl}\Gamma_{l,ik} = \frac{1}{2} g^{vl} \left(\frac{\partial g_{il}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^l} \right), \quad \forall i,k,l,v \in [0,3], \tag{A.6.43}$$

where $\Gamma_{l,ik}$ and Γ_{ik}^v are the Christoffel symbols of the first and second kind, respectively. The last relation, (A.6.43), resulted by calculating the expression (A.6.42) using the expression (A.6.41) and calculate for $\frac{\partial g_{il}}{\partial x_k}, \frac{\partial g_{lk}}{\partial x_i}, \frac{\partial g_{ki}}{\partial x_l}$, the following relations below:

$g_{il} = \frac{\partial y^a}{\partial x_i} \cdot \frac{\partial y^a}{\partial x_l}$, $g_{lk} = \frac{\partial y^a}{\partial x_l} \cdot \frac{\partial y^a}{\partial x_k}$, $g_{ki} = \frac{\partial y^a}{\partial x_k} \cdot \frac{\partial y^a}{\partial x_i}$. After eliminating four like terms of the obtained set of six terms, by a summation and a subtraction, we have obtained a sum composed of two identical terms of the form (A.6.42).

Therefore:

$$\nabla_k A_i = \frac{\partial A_i}{\partial x^k} - g^{vl}\Gamma_{l,ik} A_v, \tag{A.6.44}$$

which represents the covariant derivative of a covariant vector.

In the case in which the derivatives of the components of the fundamental tensor are zero, like in the case of a Galilean system of reference, the additional term of expression

(A.6.44) is zero, and the covariant derivative has the same expression as the usual derivative of a vector.

Sometimes, it may be useful to use the contravariant derivative denoted:

$$\nabla^k = g^{kp} \nabla_p \; ; \quad \nabla^k A_i = g^{kp} \nabla_p A_i \,. \tag{A.6.44 a}$$

The proof of relations (A.6.38) – (A.6.40) is a direct one, and avoids many complex considerations, as it can be seen by comparing to the usual proofs presented in works [1, p. 86], [2, p. 314], [3, p. 222], [5, p. 171], [7, p. 70, 94], [10, p. 395, 401], [11, p. 264, 229, 165, 276].

A.6.5. COVARIANT DERIVATIVE OF A CONTRAVARIANT VECTOR

In this sub-section, we shall present the following deduction, different from the usual ones. We introduce the covariant component into relation (A.6.44) in terms of the components of the contravariant one of the same vector. It follows:

$$\nabla_k \left(g_{ij} A^j \right) = \frac{\partial}{\partial x^k} \left(g_{ij} A^j \right) - g^{vl} \Gamma_{l,ik} \left(g_{vj} A^j \right). \tag{A.6.45}$$

According to relation (A.6.43), after multiplying it with g_{vl} and considering two sets of indices of the Christoffel symbol of the first kind, after summing up, we can obtain the relation:

$$\frac{\partial g_{ij}}{\partial x^k} = \Gamma_{i,jk} + \Gamma_{j,ik} \,. \tag{A.6.46}$$

It results that:

$$\nabla_k \left(g_{ij} A^j \right) = \frac{\partial g_{ij}}{\partial x^k} A^j + g_{ij} \frac{\partial A^j}{\partial x^k} - g^{vl} \Gamma_{l,ik} g_{vj} A^j \,, \quad \forall\, i,j,k,l,v \in \left[0, 3\right]. \tag{A.6.47}$$

From relations (A.6.46) and (A.6.47), it follows:

$$\nabla_k \left(g_{ij} A^j \right) = \Gamma_{i,jk} A^j + \Gamma_{j,ik} A^j + g_{ij} \frac{\partial A^j}{\partial x^k} - g^{vl} \Gamma_{l,ik} g_{vj} A^j \,,$$
$$\forall\, i,j,k,l,v \in \left[0, 3\right]. \tag{A.6.48}$$

Further on, we shall use the well-known s*ummation convention*: if an index occurs twice in the same term, as subscript and superscript, respectively, the summation has to be extended over all the values of that index.

In the last term of relation (A.6.48), according to the summation convention, for any value of the index v, the product $g^{vl} g_{vj}$, is zero, except the case $j = l$, in which it is equal to unity. This results by using the expression $g^{ij} = D^{ji} / D$, where D is the determinant having the elements g_{ij}, and D^{ji} is the cofactor (minor) of the element g_{ji}, or directly, by using formula (A.6.30). But then, since $l = j$, the sum of the second term and of the last term of the right-hand side vanishes. Therefore:

$$\nabla_k \left(g_{ij} A^j \right) = \Gamma_{i,lk} A^l + g_{ij} \frac{\partial A^j}{\partial x^k}, \quad \forall i, j, k, l \in [0, 3], \tag{A.6.49}$$

where we have replaced j by l, both taking successively the same values.

Multiplying both sides of relation (A.6.49) by g^{ir}, where r represents an integer fixed value, comprised between zero and 3 inclusive, we get:

$$g^{ir} \left[\nabla_k \left(g_{ij} A^j \right) \right] = g^{ir} \Gamma_{i,lk} A^l + g^{ir} \left[g_{ij} \frac{\partial A^j}{\partial x^k} \right], \quad \forall i, j, k, l \in [0, 3]. \tag{A.6.50}$$

Considering the summation convention, and making the same remark as before, which have led to relation (A.6.49), we obtain:

$$g^{ir} \nabla_k \left(g_{ij} A^j \right) = \Gamma_{lk}^r A^l + \frac{\partial A^r}{\partial x^k}, \quad \forall i, j, k, l, r \in [0, 3]. \tag{A.6.51}$$

Finally, we denote:

$$\nabla_k A^r = g^{ir} \nabla_k \left(g_{ij} A^j \right) = \frac{\partial A^r}{\partial x^k} + \Gamma_{lk}^r A^l, \quad \forall i, j, k, l, r \in [0, 3], \tag{A.6.52}$$

which is called the covariant derivative of a contravariant vector.

A.6.5.1. Geodesic Lines

Let us replace, in relation (A.6.52), the vector A^r by $\dfrac{\mathrm{d} x^r}{\mathrm{d} s}$, where $x^r(s)$ and $\mathrm{d}s$ denote the co-ordinates of each point and the element of the arc s of a curve, respectively, equate the relation to zero, and multiply the equation with $\dfrac{\mathrm{d} x^k}{\mathrm{d} s}$. We shall obtain the equation of a geodesic curve also called geodesic line.

Accordingly, we consider equation (A.6.52) in the form:

$$\frac{\partial}{\partial x^k}\left(\frac{\mathrm{d}x^r}{\mathrm{d}s}\right) + \Gamma^r_{lk}\frac{\mathrm{d}x^l}{\mathrm{d}s} = 0, \quad \forall k,l,r \in [0,3]. \tag{A.6.53}$$

Since $x^r(s)$ is exclusively depending on the variable s, we may write:

$$\frac{\partial}{\partial x^k}\left(\frac{\mathrm{d}x^r}{\mathrm{d}s}\right) = \frac{\mathrm{d}}{\mathrm{d}x^k}\left(\frac{\mathrm{d}x^r}{\mathrm{d}s}\right) = \left[\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}x^r}{\mathrm{d}s}\right)\right]\frac{\mathrm{d}s}{\mathrm{d}x^k}. \tag{A.6.54}$$

After replacing (6.54) into (6.53), and multiplying with $\dfrac{\mathrm{d}x^k}{\mathrm{d}s}$, we shall obtain:

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}x^r}{\mathrm{d}s}\right) + \Gamma^r_{lk}\frac{\mathrm{d}x^l}{\mathrm{d}s}\cdot\frac{\mathrm{d}x^k}{\mathrm{d}s} = 0. \tag{A.6.55}$$

Finally, we obtain:

$$\frac{\mathrm{d}^2 x^r}{\mathrm{d}s^2} + \Gamma^r_{lk}\frac{\mathrm{d}x^k}{\mathrm{d}s}\cdot\frac{\mathrm{d}x^l}{\mathrm{d}s} = 0, \quad \forall k,l,r \in [0,3], \tag{A.6.56}$$

which represents the equation of a geodesic line.

In the case of a Galilean system of reference, the system of co-ordinates being Cartesian, the Christoffel symbols are zero, and the last equation represents a straight line. For this reason, in any system of reference, the equation (A.6.56) is considered to give the shortest line that is no more a straight line.

A.6.6. COVARIANT DERIVATIVE OF CONTRAVARIANT AND COVARIANT TENSORS OF RANK 2

1° Case of a tensor twice contravariant

The expression of the covariant derivative of a contravariant vector as shown in Section A.6.5, relation (A.6.52), is:

$$\nabla_k A^i = \frac{\partial A^i}{\partial x^k} + \Gamma^i_{lk}A^l, \quad \forall i,k,l \in [0,3], \tag{A.6.57}$$

where the symbol ∇_k denotes the covariant derivative with respect to the co-ordinate k.

The covariant derivative of a tensor of rank 2 is presented using a novel procedure. Let $G^{ij} = G^i G^j$ be the element of a contravariant tensor of rank 2. In order to obtain the expression of the covariant derivative, we apply the previous formula, and multiplying both sides with the same quantity, we obtain:

$$\frac{\partial G^i}{\partial x^k} G^j = \left(\nabla_k G^i\right) G^j - \Gamma^i_{lk} G^l G^j, \quad \forall\, i,j,k,l \in [0,3].$$ (A.6.58)

Similarly, using the same formula, we shall obtain:

$$G^i \frac{\partial G^j}{\partial x^k} = G^i \left(\nabla_k G^j\right) - G^i \Gamma^j_{lk} G^l, \quad \forall\, i,j,k,l \in [0,3].$$ (A.6.59)

Finally, by summing up, we may write:

$$\left(\nabla_k G^i\right) G^j + G^i \left(\nabla_k G^j\right) = \nabla_k G^{ij} = \frac{\partial G^{ij}}{\partial x^k} + \Gamma^i_{lk} G^{lj} + \Gamma^j_{lk} G^{il},$$ (A.6.60)

which is the expression of the covariant derivative of the contravariant tensor G^{ij}.

2° Case of a tensor twice covariant

The expression of the covariant derivative of a covariant vector as shown in Section A.6.4, relation (A.6.44), is:

$$\nabla_k A_i = \frac{\partial A_i}{\partial x^k} - g^{vl} \Gamma_{l,ik}\, A_v.$$ (A.6.61)

Let $G_{ij} = G_i G_j$ be the element of a covariant tensor of rank 2. Proceeding like above, we shall obtain:

$$\left(\nabla_k G_i\right) G_j + G_t \left(\nabla_k G_j\right) = \nabla_k G_{ij} = \frac{\partial G_{ij}}{\partial x^k} - \Gamma^l_{ik} G_{lj} - \Gamma^l_{jk} G_{il}.$$ (A.6.62)

3° Case of a mixed tensor once covariant and once contravariant

Let $G_i^j = G_i G^j$ be the element of the considered tensor. Performing calculations like above, we shall obtain:

$$\left(\nabla_k G_i\right) G^j + G_i \left(\nabla_k G^j\right) = \nabla_k G_i^j = \frac{\partial G_i^j}{\partial x^k} - \Gamma^l_{ik} G_l^j + \Gamma^j_{lk} G_i^l,$$ (A.6.63)

and:

$$\nabla_k G_i^j = \frac{\partial G_i^j}{\partial x^k} - \Gamma^l_{ik} G_l^j + \Gamma^j_{lk} G_i^l.$$ (A.6.64)

4° Case of the fundamental covariant tensor

Let g_{ij} be the element (component) of the considered tensor. Performing calculations like above, we shall obtain:

$$\nabla_k g_{ij} = \frac{\partial g_{ij}}{\partial x^k} - \Gamma_{ik}^l g_{lj} - \Gamma_{jk}^l g_{il},$$ (A.6.65)

and according to (6.43):

$$\Gamma_{ik}^l = g^{lv} \Gamma_{v,ik}, \quad \Gamma_{jk}^l = g^{lv} \Gamma_{v,jk}.$$ (A.6.65 a)

By replacing the factors Γ_{ik}^l and Γ_{jk}^l into (6.65), we shall obtain:

$$\nabla_k g_{ij} = \frac{\partial g_{ij}}{\partial x^k} - \Gamma_{v,ik} g_j^v - \Gamma_{v,jk} g_i^v, \quad \forall i, j, k, v \in [0, 3].$$ (A.6.66)

Taking into account (A.6.30), we shall obtain:

$$\nabla_k g_{ij} = \frac{\partial g_{ij}}{\partial x^k} - \Gamma_{j,ik} - \Gamma_{i,jk}, \quad \forall i, j, k \in [0, 3].$$ (A.6.67)

According to relations (A.6.46), (A.6.67) and also (A.6.32), it results that:

$$\nabla_k g_{ij} = 0; \quad \nabla_k g^{ij} = 0.$$ (A.6.68)

A.6.7. THE DIVERGENCE OF TENSORS OF RANKS 1 AND 2

We shall start, in this Section, from the covariant derivative of a vector. The most general case will be examined in the next Chapter. The divergence of a vector in a Cartesian system of co-ordinates is:

$$\nabla_k A^i = \frac{\partial A^i}{\partial x^k}, \quad \forall k = i; \quad i \in [0, 3].$$ (A.6.69)

Indeed, relation (A.6.57) holds for any value of any value of indices i and k, among which, for instance $i = 2$, $k = 2$, and so on, and summing up side by side, we get the result. Therefore, the expression of the divergence of a vector in any system of co-ordinates is obtained starting from the relation (A.6.57), contracted on indices i, k:

$$\nabla_i A^i = \frac{\partial A^i}{\partial x^i} + \Gamma_{li}^i A^l,$$ (A.6.70)

in which, in the case of a Cartesian system of co-ordinates the second term of the right-hand side vanishes. According to relation (A.6.43), where we introduce the indices for the present case as follows: $v = i$; $l = v$; $i = l$; taking into account the contraction on indices i, k, we shall obtain:

$$\Gamma_{lk}^{i} = g^{iv}\,\Gamma_{v,lk} = \frac{1}{2}\left(\frac{\partial g_{lv}}{\partial x^{k}} + \frac{\partial g_{kv}}{\partial x^{l}} - \frac{\partial g_{lk}}{\partial x^{v}}\right)g^{iv}, \quad i = k. \tag{A.6.71}$$

We now want to calculate:

$$\frac{\partial g_{lv}}{\partial x^{k}}\,g^{iv} - \frac{\partial g_{lk}}{\partial x^{v}}\,g^{iv}, \quad i = k. \tag{A.6.72}$$

Since the indices $k = i$ and v can take all values from zero to 3, we can replace k by v, and v by k in the first term, and the previous expression becomes:

$$\frac{\partial g_{lk}}{\partial x^{v}}\,g^{vk} - \frac{\partial g_{lk}}{\partial x^{v}}\,g^{kv}, \tag{A.6.73}$$

the value of which is zero. Therefore:

$$\Gamma_{lk}^{i} = g^{iv}\,\Gamma_{v,lk} = \frac{1}{2}\,g^{kv}\,\frac{\partial g_{kv}}{\partial x^{l}}, \quad i = k. \tag{A.6.74}$$

Let us use the following symbols: D – the determinant formed by the elements g_{ij}; D^{ij} – the cofactor of the element g_{ij}; g – the value of D. Since the elements g^{ij} are those of the inverse matrix corresponding to the determinant, we express those elements by the corresponding formulae. Taking into account the expression of the derivative of a determinant, and that of the symmetry property of determinant D, we obtain:

$$g^{kv} = \frac{D^{kv}}{D}; \quad D = g_{kv}D^{kv}; \quad \frac{\partial}{\partial x^{l}}D = \frac{\partial g_{kv}}{\partial x^{l}}D^{kv}; \quad \forall k, v \in [0, 3], \tag{A.6.75 a}$$

and also:

$$g^{kv}\frac{\partial g_{kv}}{\partial x^{l}} = \frac{D^{kv}}{D}\cdot\frac{\partial g_{kv}}{\partial x^{l}} = \frac{1}{D}\cdot\frac{\partial D}{\partial x^{l}}, \quad i = k, \quad g = D; \quad \forall k, v \in [0, 3]. \tag{A.6.75 b}$$

Therefore, from relations (A.6.74) and (A.6.75 b), it follows:

$$\Gamma^i_{lk} = g^{iv}\, \Gamma_{v,lk} = \frac{1}{2}\, g^{kv}\, \frac{\partial g_{kv}}{\partial x^l} = \frac{1}{2}\cdot\frac{1}{g}\cdot\frac{\partial g}{\partial x^l}, \quad i=k, \tag{A.6.76}$$

where, for simplicity, we have replaced D by g.

From relations (A.6.70) and (A.6.76), it follows:

$$\nabla_k A^i = \frac{\partial A^i}{\partial x^k} + g^{iv}\, \Gamma_{v,lk}\, A^l = \frac{\partial A^i}{\partial x^i} + \frac{1}{2}\cdot\frac{1}{g}\cdot\frac{\partial g}{\partial x^l}\, A^l, \quad i=k. \tag{A.6.77}$$

Taking into account that in relation (A.6.73), the indices $i=k$ and l take all values from zero to 3, we replace l by $i=k$ in the last term. It follows:

$$\nabla_i A^i = \frac{\partial A^i}{\partial x^i} + \frac{1}{2}\cdot\frac{1}{g}\cdot\frac{\partial g}{\partial x^i}\, A^i. \tag{A.6.78}$$

The last relation may be transformed using (A.6.79 a), in the final form (A.6.79 a, b):

$$\frac{\partial}{\partial x}\left(\sqrt{g}\, A\right) = \frac{1}{2}\cdot\frac{1}{\sqrt{g}}\, A + \sqrt{g}\, \frac{\partial A}{\partial x};$$

$$\tag{A.6.79 a, b}$$

$$\mathrm{Div}\, A^i = \nabla_i A^i = \frac{1}{\sqrt{\pm g}}\cdot\frac{\partial}{\partial x^i}\left(\sqrt{\pm g}\, A^i\right),$$

where the plus and minus signs from denominator and numerator correspond to each other, the sign minus being taken if g is negative. The symbol Div, instead of div, has been used in order to emphasize the general system of reference.

The expression of the divergence of a tensor of rank 2 may be written similarly, starting from formula (A.6.60), contracted on indices j and k:

$$\nabla_j G^{ij} = \frac{\partial G^{ij}}{\partial x^k} + \Gamma^i_{lk}\, G^{lj} + \Gamma^j_{lk}\, G^{il}, \quad \forall i,j \in [0,3], \quad k=j. \tag{A.6.80}$$

In the case in which the tensor G^{lj} is antisymmetric in the superscript indices j and l, whereas the factor Γ^i_{lk} is always symmetric in the subscript indices l and $k=j$, the first term of the right-hand side of (A.6.80) vanishes. It results that:

$$\nabla_j G^{ij} = \frac{\partial G^{ij}}{\partial x^j} + \Gamma^j_{lj}\, G^{il}, \quad \forall i,j,l \in [0,3]. \tag{A.6.81}$$

Similarly as for relations (A.6.70) – (A.6.78), we get:

$$\text{Div}\, G^{ij} = \nabla_j G^{ij} = \frac{1}{\sqrt{\pm g}} \cdot \frac{\partial}{\partial x^j} \left(\sqrt{\pm g}\, G^{ij} \right), \tag{A.6.82}$$

where the plus and minus signs are taken as for relation (A.6.79).

A.6.8. CURVATURE TENSOR OF THE SPACE-TIME CONTINUUM

We shall present a deduction of the curvature tensor. Let A^i be any contravariant vector, each component of which is continuous and its derivatives of the first two orders exist. Also, let us assume that each element of the fundamental tensor is continuous and its derivatives of the first three orders exist.

A.6.8.1. The Expression of the Curvature Tensor

Let us consider the covariant derivatives:

$$\nabla_r A^i = \frac{\partial A^i}{\partial x^r} + \Gamma^i_{ur} A^u \quad \forall\, u \in [0,\,3], \tag{A.6.83}$$

where the left-hand side of each relation is a tensor of rank 2, once covariant and once contravariant. We wish to calculate $\nabla_s \left(\nabla_r A^i \right)$. For this purpose, we shall use relation (A.6.63) by putting: $i = r$, $j = i$, $k = s$, $l = q$, $G^j_i = G^i_r$, where G^i_r is the right-hand side of (A.6.83). After rearranging the terms, there follows:

$$\nabla_s \left(\nabla_r A^i \right) = \frac{\partial^2 A^i}{\partial x^s \partial x^r} + \frac{\partial \Gamma^i_{ur}}{\partial x^s} A^u + \Gamma^i_{ur} \frac{\partial A^u}{\partial x^s} + \Gamma^i_{qs} \left(\frac{\partial A^q}{\partial x^r} + \Gamma^q_{ur} A^u \right) - $$
$$ - \Gamma^q_{rs} \left(\frac{\partial A^i}{\partial x^q} + \Gamma^i_{uq} A^u \right), \tag{A.6.84 a}$$

and

$$\nabla_r \left(\nabla_s A^i \right) = \frac{\partial^2 A^i}{\partial x^r \partial x^s} + \frac{\partial \Gamma^i_{us}}{\partial x^r} A^u + \Gamma^i_{us} \frac{\partial A^u}{\partial x^r} + \Gamma^i_{qr} \left(\frac{\partial A^q}{\partial x^s} + \Gamma^q_{us} A^u \right) - $$
$$ - \Gamma^q_{sr} \left(\frac{\partial A^i}{\partial x^q} + \Gamma^i_{uq} A^u \right). \tag{A.6.84 b}$$

Now, in order to obtain a simple expression, we subtract the last two equations side by side, and we further take into consideration that the terms in which certain indices are denoted by different letters, but take the same values from zero to 3, are similar terms. There follows:

$$\left(\nabla_s\nabla_r - \nabla_r\nabla_s\right)A^i = \left(\frac{\partial\Gamma^i_{ur}}{\partial x^s} - \frac{\partial\Gamma^i_{us}}{\partial x^r}\right)A^u + \left(\Gamma^i_{qs}\Gamma^q_{ur} - \Gamma^i_{qr}\Gamma^q_{us}\right)A^u . \qquad (A.6.85)$$

The expression (A.6.85) may be written:

$$\left(\nabla_s\nabla_r - \nabla_r\nabla_s\right)A^i = R^i_{rsu}A^u . \qquad (A.6.86)$$

The factor R^i_{rsu} of expression (A.6.86), is also called the Riemann curvature tensor, in which we have separated, by parentheses, two terms:

$$R^i_{rsu} = \left(\frac{\partial\Gamma^i_{ur}}{\partial x^s} - \frac{\partial\Gamma^i_{us}}{\partial x^r}\right) + \left(\Gamma^i_{qs}\Gamma^q_{ur} - \Gamma^i_{qr}\Gamma^q_{us}\right), \qquad (A.6.87)$$

which is a tensor of rank 4, three times covariant and once contravariant. By contracting with respect to indices i and r, we obtain:

$$R_{su} = \left(\frac{\partial\Gamma^r_{ur}}{\partial x^s} - \frac{\partial\Gamma^r_{us}}{\partial x^r}\right) + \left(\Gamma^r_{qs}\Gamma^q_{ur} - \Gamma^r_{qr}\Gamma^q_{us}\right), \qquad (A.6.88)$$

which is a covariant tensor of rank 2, also called Ricci tensor.

A.6.8.2. Geodesic Co-ordinates

The geodesic co-ordinates are those in which Christoffel symbols are zero, but their derivatives with respect to co-ordinates may be different from zero. Therefore, the covariant derivative of expression (A.6.87) concerns only the first parenthesis and is:

$$\nabla_k R^i_{rsu} = \frac{\partial^2\Gamma^i_{ur}}{\partial x^k\partial x^s} - \frac{\partial^2\Gamma^i_{us}}{\partial x^k\partial x^r} . \qquad (A.6.89)$$

Using relation (A.6.89), and changing the indices, there follows:

$$\nabla_k R^i_{rsu} + \nabla_s R^i_{kru} + \nabla_r R^i_{sku} = 0 , \qquad (A.6.90)$$

which is the Bianchi identity.

Besides the relations above, the geodesic co-ordinate system has been used in certain applications [3, p. 232]. We recall, more detailed than usually, that a geodesic reference system, generally, can be considered in a small space around any chosen point. The elements (components) of the fundamental tensor are assumed to be functions of point, the first derivatives of which have, at the chosen point, the value zero, but their derivatives of higher orders may exist. If a tensor is zero at that point, it will keep the value zero, at that point, in any other system of reference.

The corresponding expressions of the co-ordinates when passing from any co-ordinate system, to a geodesic system, can be obtained from the formulae of transformation of the Christoffel symbols when passing from a system to another one. Also, for this scope, we can use the expressions of the covariant derivative of a covariant vector, when passing from the former system to the latter.

It is to be mentioned that at any point there are an infinity of geodesic reference systems [5, p. 188], and it is possible to chose the more convenient for the purpose we are looking for.

List of Symbols Used in Appendix 6

a_i^j –   transformation coefficient of the co-ordinates, when passing from one system of reference to another one;

A_i –   component of the four-vector potential;

A_i, A^i –   component (element) of a covariant and contravariant vector (tensor of rank 1), respectively;

B_i, B^i –   component of a covariant and contravariant vector (tensor of rank 1), respectively;

c –   velocity of light in empty space, supposed to be constant;

C_{ij} –   component of a covariant tensor of rank 2 and the contravariant tensor C^{ij};

ds –   distance between two points very near to each other, in any continuum (space);

D –   determinant formed by the elements (components) of the fundamental tensor;

D^{ji} –   cofactor (minor) corresponding to element g_{ji} of determinant D of g_{ij};

e_{ii} –   axis coefficient, for the axis i of the Galilean reference frame;

e_{ij} –   component (element) of the fundamental tensor of a Galilean system of reference;

g –   value of the determinant of the fundamental tensor;

g_{ij} –   covariant element (component) of the covariant fundamental tensor in any system of reference;

g^{ij} –   element (component) of the contravariant fundamental tensor in any system of reference;

G_{ij} –   component of the covariant tensor of rank 2, the contravariant tensor is G^{ij};

K –   any system of reference, a Galilean system of reference is K_G;

N –   total number of co-ordinate axes of a Galilean system of reference, if the number of the last co-ordinate axis is N when the first is 1, and it is $N + 1$ when the first is 0;

T_{ij} –   component of a covariant tensor of rank 2, the contravariant tensor is T^{ij};

x^i –   co-ordinate along axis i in any system of reference;

y^i –   co-ordinate along axis i in a Galilean system of reference;

Y_i, Y^i –   covariant and contravariant vector (tensor of rank 1), respectively, in a Galilean system of reference;

δ_{ij} –   symbol equal to unity for equal indices, and equal to zero for different ones (Kronecker symbol);

$\Gamma_{l,ik}$ –   Christoffel symbol of the first kind;

Γ_{ik}^{v} –   Christoffel symbol of the second kind;

∇_{k} –   covariant derivative with respect to co-ordinate k;

∇^{k} –   contravariant derivative with respect to co-ordinate k.

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APPENDIX 7

THE VARIOUS DIVERGENCE TYPES OF A TENSOR AND OF THE CURVATURE TENSOR IN THE GENERAL THEORY OF RELATIVITY

In Appendix 6, we have presented the calculation of the divergence of various types of tensors that often occurs. Further on, we shall examine the general expressions of the differential operator divergence. The bibliographic mentions [1]-[12], specific for this Appendix, are at the end of this Appendix.

A.7.1. DIVERGENCE OF A TENSOR

The calculation of the divergence of a tensor occurs in many problems of Physical Mathematics. We consider that the notion of divergence of a tensor has not been enough defined and analysed in the usual literature. We shall try to establish a consistent definition of a tensor including the calculation procedure, and to apply it to the calculation of the divergence of the curvature tensor of the four-dimensional continuum space-time.

Several studies on the divergence of tensors have been carried out for a long time, [1]-[10]. In paper [10], a deep study on the differential operations on tensors has been performed without using the tensor calculus, but instead the general method concerning the differential quadratic forms, from the classical mathematical analysis, has been utilized. The methods used in the other papers will be examined further on.

In order to facilitate the understanding, and have a uniform system of symbols, the paper includes three Sub-sections: the first one containing several tensor formulae frequently used in the paper; the second one presenting a relatively simple deduction of the Riemann curvature; and the third one containing a very short deduction of the Bianchi identity.

A.7.2. THE DIVERGENCE OF COVARIANT AND CONTRAVARIANT TENSORS

The calculation of the divergence of a tensor, as known, is based on the calculation of the derivative of that tensor.

The divergence of a vector in a Cartesian system of co-ordinates is:

$$\nabla_k A^i = \frac{\partial A^i}{\partial x^k}, \quad \forall k = i; \quad i \in [0, 3].$$

(A.7.1)

We recall the symbols of the covariant and contravariant derivatives, namely ∇_k and ∇^k, respectively. Correspondingly, we may use the denominations of covariant and contravariant divergence, respectively.

The expression of the covariant derivative of a contravariant tensor of rank 1, i.e. a contravariant vector, is:

$$\nabla_k A^i = \frac{\partial A^i}{\partial x^k} + \Gamma^i_{lk} A^l, \quad \forall i, l, k \in [0, 3], \tag{A.7.2}$$

where $\Gamma_{v,lk}$, and $\Gamma^i_{lk} = g^{iv} \Gamma_{v,lk}$ are the Christoffel symbols of the first and second kind respectively. The expression of the divergence of a vector in any system of co-ordinates is obtained starting from the relation (A.7.2), contracted in indices i, k:

$$\nabla_i A^i = \frac{\partial A^i}{\partial x^i} + \Gamma^i_{li} A^l, \quad \forall i, l \in [0.3], \tag{A.7.3}$$

and represents a tensor of rank zero, i.e., a scalar. Relation (A.7.3) may also be written:

$$\text{Div} \, A^i = \nabla_i A^i = \frac{1}{\sqrt{\pm g}} \cdot \frac{\partial}{\partial x^i} \left(\sqrt{\pm g} \, A^i \right), \quad \forall i \in [0, 3], \tag{A.7.3 a}$$

where g represents the value of the determinant of the fundamental tensor, and the plus and minus signs from denominator and numerator correspond to each other, the sign minus being taken if g is negative. The symbol Div, instead of div, has been used in order to emphasize the general system of reference.

The covariant derivative of a covariant tensor of rank 1, i.e., a covariant vector, is given by the following relation, and its divergence results by contracting the expression in indices i and k:

$$\nabla_k A_i = \frac{\partial A_i}{\partial x^k} - g^{vl} \Gamma_{l,ik} A_v, \quad k = i. \tag{A.7.4}$$

The contravariant derivative of the same tensor is given by the following relation, and its divergence results by contracting the expression in indices i and k:

$$\nabla^k A_i = g^{pk} \nabla_p A_i = g^{pk} \left(\frac{\partial A_i}{\partial x^p} - g^{vl} \Gamma_{l,pi} A_v \right), \quad k = i. \tag{A.7.5}$$

The expression of the divergence of a contravariant tensor of rank 2 may be written similarly, starting from the formula of the covariant derivative of a twice contravariant tensor G^{ij}, contracted in indices j and k:

$$\nabla_j G^{ij} = \frac{\partial G^{ij}}{\partial x^k} + \Gamma_{lk}^i \, G^{lj} + \Gamma_{lk}^j \, G^{il}, \quad \forall \, i, j \in [0, 3], \quad k = j.$$
(A.7.6)

Let us consider a covariant tensor of rank 2, the divergence can be expressed in several manners, as below:

$$\nabla^k G_{kj} = g^{ks} \nabla_s G_{kj} = \nabla_s g^{ks} G_{kj} = \nabla_s G_j^s.$$
(A.7.6 a)

Remarks on the calculation of the divergence of a tensor.
From the relations above, we may realize the following properties:
  a. The divergence of a given contravariant tensor results from the expression of the covariant derivative of that tensor, and due to the contraction, the divergence will be a tensor of a rank smaller with two unities with respect to that of the mentioned expression. In this case, the contraction is performed in each pair of concerned indices, one subscript and the other superscript. It may be called covariant divergence.
  b. The divergence of a given covariant tensor results like for the preceding case, *a*, but in this case, *b*, the contraction must be performed for each pair of concerned indices placed in the same position, both subscript. It may also be called covariant divergence.
  c. For keeping the same manner of contraction as in case *a*, the divergence of a given covariant tensor results from the expression of the contravariant derivative of that tensor. It may be called contravariant divergence. The contraction will be performed as for case *a*. The divergences of a tensor, calculated by procedures *b* and *c*, give results which differ by a factor representing the fundamental tensor.
  d. A convenient manner for simplifying the calculation of the divergence of a tensor, in a geodesic system of co-ordinates, in the case in which the fundamental tensor occurs as a factor in any term, is the transport of the fundamental tensor before or behind the differential operator. However, attention must be paid in order to avoid the transport if in the respective term including the differential operator, besides the differential operator, there are tensors with the same index as the differential operator. If the respective index appears three times or more, the transport may produce errors.
Each of the procedures from *b* and *c* may be advantageous in certain applications.
The calculation of the divergence in the case of a tensor of any rank greater than unity, say of rank 2, is not a unique result as in the case of a contravariant tensor of rank 1, because the calculation has to be carried out with respect to the pair of indices formed by that of the divergence operator and by one of the tensor. Also, by choosing various pairs of indices we may obtain different results for the divergence.

A.7.3.   APPLICATION TO THE CASE OF THE RIEMANN CURVATURE TENSOR AND EINSTEIN TENSOR

The equations of the classical General Theory of Relativity contain the Einstein tensor developed starting from the Riemann tensor of the four-dimensional continuum space-time.

In the work [1], Einstein has directly given the tensor considered to satisfy the three required conditions the value zero of the divergence included. Let us consider the Einstein tensor:

$$R_{ij} + a\, g_{ij} R, \tag{A.7.7}$$

where R_{ij} is the Riemann tensor of rank 2, also called Ricci tensor, R is given by the relation:

$$R = g^{ij} R_{ij} = R_i^i, \tag{A.7.8}$$

and a is a constant. Also, there is added that for the third condition to be satisfied, the constant a should be equal to $\dfrac{-1}{2}$. The same expression is given in many works, among which [3, p. 248], [4, p. 201], [5, p. 99].

7.4.   REMARKS CONCERNING THE CALCULATION OF EXPRESSIONS INVOLVING THE RIEMANN TENSOR

In order to clarify the calculation of the tensor divergence, avoiding certain errors, we shall consider an example concerning the mixed Riemann tensor of rank four and its transformation into a covariant tensor of the same rank. Although well known, we shall remake a simple deduction in order to emphasize a very important property accepted but never explicitly mentioned in the known works. Let us consider the mixed expression of the tensor below and transform it into a covariant one. For the next calculations, we shall take into account relations (A.6.28) – (A.6.32). Therefore, we shall start with (A.6.87):

$$R_{rsu}^{\cdots i} = \frac{\partial \Gamma_{ur}^i}{\partial x^s} - \frac{\partial \Gamma_{us}^i}{\partial x^r} + \Gamma_{qs}^i \Gamma_{ur}^q - \Gamma_{qr}^i \Gamma_{us}^q. \tag{A.7.9}$$

Let us look to express into a covariant form the tensor of the right-hand side of the formula below:

$$R_{rsup} = g_{ip} R_{rsu}^{\cdots i}. \tag{A.7.10}$$

There follows:

$$g_{ip} R_{rsu}^{....i} = g_{ip} \left(\frac{\partial \Gamma_{ur}^i}{\partial x^s} - \frac{\partial \Gamma_{us}^i}{\partial x^r} \right) + g_{ip} \left(\Gamma_{qs}^i \Gamma_{ur}^q - \Gamma_{qr}^i \Gamma_{us}^q \right) =$$

$$= \frac{\partial \left(g_{ip} g^{iw} \Gamma_{w,ur} \right)}{\partial x^s} - \frac{\partial \left(g_{ip} g^{iw} \Gamma_{w,us} \right)}{\partial x^r} - \Gamma_{ur}^i \frac{\partial g_{ip}}{\partial x^s} + \Gamma_{us}^i \frac{\partial g_{ip}}{\partial x^r} + \tag{A.7.11}$$

$$+ g_{ip} \left(\Gamma_{qs}^i \Gamma_{ur}^q - \Gamma_{qr}^i \Gamma_{us}^q \right).$$

We shall calculate the parentheses, having in view that $g_{ip} g^{iw} = g_p^w$, and p being fixed, we have $g_p^w = g_p^p = 1$. We shall obtain:

$$g_{ip} R_{rsu}^{....i} = \frac{\partial \Gamma_{p,ur}}{\partial x^s} - \frac{\partial \Gamma_{p,us}}{\partial x^r} - \Gamma_{ur}^i \left(\Gamma_{p,is} + \Gamma_{i,ps} \right) + \Gamma_{us}^i \left(\Gamma_{p,ir} + \Gamma_{i,pr} \right) +$$

$$+ g_{ip} g^{iw} \left(\Gamma_{w,qs} \Gamma_{ur}^q - \Gamma_{w,qr} \Gamma_{us}^q \right), \tag{A.7.12}$$

and, with the same remark as after (A.7.11), there follows:

$$g_{ip} R_{rsu}^{....i} = \frac{\partial \Gamma_{p,ur}}{\partial x^s} - \frac{\partial \Gamma_{p,us}}{\partial x^r} - \Gamma_{ur}^i \left(\Gamma_{p,is} + \Gamma_{i,ps} \right) + \Gamma_{us}^i \left(\Gamma_{p,ir} + \Gamma_{i,pr} \right) +$$

$$+ g_p^w \left(\Gamma_{w,qs} \Gamma_{ur}^q - \Gamma_{w,qr} \Gamma_{us}^q \right), \tag{A.7.13}$$

or:

$$g_{ip} R_{rsu}^{....i} = \frac{\partial \Gamma_{p,ur}}{\partial x^s} - \frac{\partial \Gamma_{p,us}}{\partial x^r} - \Gamma_{ur}^q \left(\Gamma_{p,qs} + \Gamma_{q,ps} \right) + \Gamma_{us}^q \left(\Gamma_{p,qr} + \Gamma_{q,pr} \right) +$$

$$+ g_p^p \left(\Gamma_{p,qs} \Gamma_{ur}^q - \Gamma_{p,qr} \Gamma_{us}^q \right). \tag{A.7.14}$$

Replacing the Christoffel symbols, we obtain:

$$R_{rsup} = \frac{\partial}{\partial x^s} \frac{1}{2} \left(\frac{\partial g_{pr}}{\partial x^u} + \frac{\partial g_{pu}}{\partial x^r} - \frac{\partial g_{ur}}{\partial x^p} \right) - \frac{\partial}{\partial x^r} \frac{1}{2} \left(\frac{\partial g_{ps}}{\partial x^u} + \frac{\partial g_{pu}}{\partial x^s} - \frac{\partial g_{us}}{\partial x^p} \right) -$$

$$- \Gamma_{ur}^q \Gamma_{q,ps} + \Gamma_{us}^q \Gamma_{q,pr}. \tag{A.7.15}$$

After reducing the like terms, we obtain:

$$R_{rsup} = \frac{1}{2} \left(\frac{\partial^2 g_{pr}}{\partial x^s \partial x^u} - \frac{\partial^2 g_{ur}}{\partial x^s \partial x^p} - \frac{\partial^2 g_{ps}}{\partial x^r \partial x^u} + \frac{\partial^2 g_{us}}{\partial x^r \partial x^p} \right) -$$

$$- \Gamma_{ur}^q \Gamma_{q,ps} + \Gamma_{us}^q \Gamma_{q,pr}. \tag{A.7.16}$$

The relation (A.7.16) contains the components of the fundamental tensor g_{ij} in the system of reference K. These quantities may be obtained by using formula (A.6.20), starting from a quasi-Euclidean system of reference K_G.

Some general relations, deduced from (A.6.3 a, b) extended for tensors of rank 2, and (A.6.41), could facilitate the understanding:

$$g_{ij} = \sum_{a=0}^{N} e_{aa} \frac{\partial y^a}{\partial x^i} \cdot \frac{\partial y^a}{\partial x^j}; \quad g^{ij} = \frac{1}{\sum_{a=0}^{a=N} e_{aa} \frac{\partial y^a}{\partial x^i} \cdot \frac{\partial y^a}{\partial x^j}};$$

$$e_{ab} = \sum_{a,b=0}^{N} g_{ij} \frac{\partial x^i}{\partial y^a} \cdot \frac{\partial x^j}{\partial y^b}; \quad e^{ab} = \frac{1}{\sum_{a,b=0}^{N} g_{ij} \frac{\partial x^i}{\partial y^a} \cdot \frac{\partial x^j}{\partial y^b}}; \quad \text{(A.7.16 a-d)}$$

$$\forall i, j \in [0, 3], a, b \in [0, N];$$

where we have deduced the formulae of the contravariant components from the inverse matrix of the covariant components, having advantages relatively to those of literature.

According to formula (A.7.16 a) above, the passage from the quantities y^a to the components g_{ij} is always possible, obtaining say 10 components in terms of say $(N+1) \geq 4$ other quantities. Conversely, for passing from the components g_{ij} to components e_{ab} we have to resort to the formula (A.7.16 c), say 10 equations. But in this case, supplementary conditions, namely $e_{ab} = 0, \forall a \neq b$ hence other 6 other equations, must be fulfilled (although previously not imposed) the respective reference frame K_G being quasi-Euclidean. Finally, we shall have 4 unknowns and 16 equations. There follows that the number of equations to be fulfilled is greater than the number of unknowns, the solution being, in general, impossible. It follows that the correspondence between K_G and K, is not biunivocal, the obtained expression of the curvature being valid only for the reference frame K.

7.5. CALCULATION OF THE DIVERGENCE GIVEN BY THE AUTHORS USING THE BIANCHI IDENTITY

The majority of the known authors have used the Bianchi identity for this purpose [1]-[5], [8]. In this case, they start from a tensor of rank four, in order to establish the property of a tensor of rank 2, what seems to be an indirect and more complicated way. Further, a set of transformations bring the expression from a vector of rank four to a vector of rank 2, namely a mixed vector.

The proofs, aiming the calculation of the divergence of the curvature tensor are based on Bianchi identity and on the usage of the geodesic co-ordinates, like in works [1]-[5].

According to relation (A.6.90) and the inversion of indices r and s, in the last two terms, the following identity may be written:

$$\nabla_k R_{rsu}^i - \nabla_s R_{rku}^i - \nabla_r R_{ksu}^i = 0.$$ (A.7.17)

By putting $i = r$, and contracting with respect to index i, there follows:

$$\nabla_k R_{su} - \nabla_s R_{ku} - \nabla_i R_{ksu}^i = 0.$$ (A.7.18)

Multiplying the last relation by g^{su}, and taking into account relation (A.6.32), one will obtain:

$$\nabla_k R_s^s - \nabla_s R_k^s - \nabla_i R_k^i = 0,$$ (A.7.19)

and replacing index s by i in the second term of the left-hand side of relation (A.7.19), both taking all values from zero to 3, one obtains:

$$-\nabla_k R_s^s + 2\nabla_i R_k^i = 0.$$ (A.7.20)

From here on, we shall modify the known procedure [4, p. 201], avoiding the Kronecker symbol. Starting from relation (A.7.20) and changing the first index of the first term, we get:

$$-\nabla_i g_k^i R_s^s + 2\nabla_i R_k^i = 0,$$ (A.7.21)

and taking into account (A.7.8), we obtain

$$\nabla_i \left(R_k^i - \frac{1}{2} g_k^i R \right) = 0,$$ (A.7.22)

where the term within parentheses corresponds to that of Einstein tensor, but in a mixed form.

7.6. THE PROPOSED NEW PROOF

Unlike to the other methods we shall consider the tensor we are looking for as not known. Let us start from the general formula of the Riemann curvature in geodesic co-ordinates expressed as a tensor of rank 2:

$$R_{su} = \left(\frac{\partial \Gamma_{ur}^r}{\partial x^s} - \frac{\partial \Gamma_{us}^r}{\partial x^r} \right), \quad \forall r \in [0, 3],$$ (A.7.23)

and s and u are fixed values chosen from the set $[0, 3]$. Then, from formula (A.7.23), also using geodesic co-ordinates, we get:

$$\nabla_k R_{su} = \nabla_k \left(\frac{\partial \Gamma_{ur}^r}{\partial x^s} - \frac{\partial \Gamma_{us}^r}{\partial x^r} \right) = \frac{\partial}{\partial x^k} \left(\frac{\partial \Gamma_{ur}^r}{\partial x^s} - \frac{\partial \Gamma_{us}^r}{\partial x^r} \right) = \frac{\partial}{\partial x^s} \cdot \frac{\partial \Gamma_{ur}^r}{\partial x^k} - \frac{\partial}{\partial x^r} \cdot \frac{\partial \Gamma_{us}^r}{\partial x^k} \qquad \text{(A.7.24)}$$
$$\forall r \in [0, 3],$$

and k, s, u are fixed values chosen from the set $[0, 3]$. Therefore, according to formula (A.7.23) or (A.7.24), by adding and subtracting the same term, we can write:

$$\nabla_k R_{su} = \frac{\partial}{\partial x^s} \left(\frac{\partial \Gamma_{ur}^r}{\partial x^k} - \frac{\partial \Gamma_{uk}^r}{\partial x^r} \right) + \frac{\partial}{\partial x^r} \left(\frac{\partial \Gamma_{uk}^r}{\partial x^s} - \frac{\partial \Gamma_{us}^r}{\partial x^k} \right),$$
$$\forall r \in [0, 3], \qquad \text{(A.7.25)}$$

and k, s, u are fixed values, like in the preceding relation. Consequently:

$$\nabla_k R_{su} = \frac{\partial}{\partial x^s} \left(\frac{\partial \Gamma_{ur}^r}{\partial x^k} - \frac{\partial \Gamma_{uk}^r}{\partial x^r} \right) + \frac{\partial}{\partial x^w} \left(\frac{\partial \Gamma_{uk}^w}{\partial x^s} - \frac{\partial \Gamma_{us}^w}{\partial x^k} \right), \quad \forall r \in [0, 3] \qquad \text{(A.7.26)}$$

or, with the symbols similar to those of (A.7.24), we have:

$$\nabla_k R_{su} = \nabla_s R_{ku} + \nabla_w R_{ksu}^w, \quad \forall w \in [0, 3]. \qquad \text{(A.7.27)}$$

After rearranging the terms, we obtain:

$$\nabla_k R_{su} - \nabla_s R_{ku} - \nabla_w R_{ksu}^w = 0, \quad \forall w \in [0, 3]. \qquad \text{(A.7.28)}$$

By multiplying both sides with g^{su}, it follows:

$$-\nabla_k R_s^s + \nabla_s R_k^s + \nabla_w R_k^w = 0, \quad \forall w \in [0, 3]. \qquad \text{(A.7.29)}$$

Therefore:

$$-\nabla_k R_s^s + 2 \nabla_s R_k^s = 0. \qquad \text{(A.7.30)}$$

Let us use the contravariant derivative:

$$-\nabla^p g_{kp} R_s^s + 2 \nabla^p g_{sp} R_k^s = 0. \qquad \text{(A.7.31)}$$

Taking into account relation (A.7.8), relation (A.7.31) yields:

$$\nabla^p \left(R_{kp} - \frac{1}{2} g_{kp} R \right) = 0, \quad \forall\, p \in [0,\, 3].$$

(A.7.32)

Therefore, we have just deduced the divergence of the Einstein tensor, by a so-called inductive way, without resorting to the Bianchi identity. Due to its form, relation (A.7.32) may be called contravariant divergence.

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APPENDIX 8

ELECTROMAGNETIC ENERGY-MOMENTUM TENSOR FOR NON-HOMOGENEOUS MEDIA IN THE THEORY OF RELATIVITY

This Appendix is devoted to an analysis of the energy momentum tensor for electromagnetic field. The bibliographic mentions [1]-[18], specific for this Appendix, are at the end of this Appendix. Also, a list of symbols is given at the end of this Appendix.

In Electrodynamics and in the Theory of Relativity, the energy-momentum tensor has a very important role as mentioned in references [1]-[9] at the end of this Appendix. Besides the widely accepted fact that this tensor allows a compact way of writing the conservation laws of linear momentum and energy in Electromagnetism, it permits to calculate the energy and stress, in any reference frame in terms of another reference frame, and especially in terms of the reference frame in which the substance is at rest.

The developments of the principles of the concerned mathematical methods, started from a relatively long time, are still examined nowadays [10]-[14].

Many works have been devoted to this subject. However, in the most treated case of empty space as well as in the case of a space filled with substance, the transition from a reference frame to another in motion has not been carefully analysed. In this Appendix, an approach, especially based on [15]-[18], to the analysis of the tensor will be presented, namely, the construction of the tensor, the case of non-homogeneous electrically and magnetically polarized substances, and the transition from a reference frame to another one, with the involved consideration on the Theory of Relativity.

In this Appendix, we shall recall certain definitions and several largely used relations. In addition, certain more important proofs will be given. All bibliographic mentions concern the References [1]-[18] of the end of this Appendix. Also, a list of symbols is given at the end of this Appendix. We shall have in view the works [8], [10], [18] that present certain advantages for the applications we have had in view.

A.8.1. VOLUME DENSITY OF THE ELECTROMAGNETIC FORCE

In the works concerning the Theory of Relativity the analysis of electromagnetic forces is achieved from the Lorentz formula of the force, e.g., [5, p. 133]. In the present section, we shall start from the general formula of the electromagnetic force acting on a substance submitted to an electromagnetic field. It is derived from the principle of conservation of energy and the Theory of Relativity, through certain approximations [8, p. 157]. The reasoning has led to the following formulae, both also deduced in various other manners and accepted by several authors:

$$\boldsymbol{f} = \rho_v \boldsymbol{E} - \frac{1}{2} \boldsymbol{E}^2 \operatorname{grad} \varepsilon - \frac{1}{2} \boldsymbol{H}^2 \operatorname{grad} \mu + \boldsymbol{J} \times \boldsymbol{B}, \tag{A.8.1 a}$$

and

$$f = \rho_v E - \frac{1}{2} E^2 \operatorname{grad} \varepsilon - \frac{1}{2} H^2 \operatorname{grad} \mu + J \times B + \frac{\partial}{\partial t}(D \times B), \qquad \text{(A.8.1 b)}$$

where the symbols are the usual ones. In this case, the quantities ε and μ are considered as constant locally, but strongly depending on the point of the substance, hence varying in space. We shall denote the three axes of a Cartesian system of co-ordinates, by the indices i, j, k. In the further analysis, we shall consider formula (A.8.1 a), and we shall mention the modification occurring due to the supplementary term, if using formula (A.8.1 b). Relations (A.8.1 a) and (A.8.1 b) are considered as having, along each axis, three and four terms (components), respectively:

$$f_k = (f_k)_1 + (f_k)_2 + (f_k)_3 + (f_k)_4 , \qquad \text{(A.8.2)}$$

where the index k indicates the axis. The four terms are given by the following expressions:

$$(f_k)_1 = \rho_v E_k ; \quad (f_k)_2 = [J \times B]_k ;$$

$$(f_k)_3 = \left[-\frac{1}{2} E^2 \operatorname{grad} \varepsilon - \frac{1}{2} H^2 \operatorname{grad} \mu \right]_k ; \qquad \text{(A.8.3 a, …, d)}$$

$$(f_k)_4 = \left[\frac{\partial}{\partial t}(D \times B) \right]_k .$$

Henceforth, we shall write the expressions of the electromagnetic field state quantities by using the scalar and vector potentials V and A, in the well-known form:

$$E_i = -\frac{\partial V}{\partial x^i} - \frac{\partial A_i}{\partial t}; \quad B_k = \frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j}; \quad \forall i, j, k \in [1, 3]. \qquad \text{(A.8.4 a, b)}$$

The relations (A.8.4 a, b) may be written in a new form, using the relations (A.5.8 a-d) – (A.5.10 a-e) from Appendix 5, as follows:

$$F_{ij} = c_i \frac{\partial A_j}{\partial x^i} - c_j \frac{\partial A_i}{\partial x^j}, \quad \forall i, j \in [0, 3];$$

$$c_i = 1, c_j = c, \quad \forall i \neq 0 \wedge j = 0; \quad c_i = c_j = c, \quad i \wedge j \neq 0; \quad \forall i \wedge j \in [0, 3];$$

$$E_i = F_{i0} = \frac{\partial A_0}{\partial x^i} - \frac{\partial A_i}{\partial x^0}, \quad \forall i \in [1, 3]; \qquad \text{(A.8.5 a-e)}$$

$$B_k = B_{ij} = \frac{1}{c} F_{ij} = \frac{1}{c} \left(\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j} \right),$$

$$A_0 = -V, \quad x_0 = ct, \quad \forall i, j, k \in [1, 3].$$

It is useful to mention that the expression (A.8.5 a), like the explanation for (A.5.8 a), of Appendix 5 represents a tensor.

Further, we shall have in view the two groups (sets) of equations of the electromagnetic field (in the order used by H.A. Lorentz, which differs from that of J.C. Maxwell) in a four-dimensional continuum space-time, where the symbols are those of [10], [11]. For the sake of facility, we shall recall these symbols, firstly in the case of empty space (vacuum). The equations of the first group (set) are given by the relationships:

$$\frac{\partial G^{ij}}{\partial x^j} = J^i + \frac{\partial G^{0i}}{\partial x^0}, \quad \forall i,j \in [0,3], \quad J^0 = c\rho_v, \qquad \text{(A.8.6 a, b)}$$

and

$$E_i = F_{i0}; \quad F_{ij} = e_{ii}e_{jj}F^{ij}; \quad F_{ij} = -F_{ji};$$

$$G^{0j} = c\varepsilon_0 F^{0j}; \quad G_{i0} = c\varepsilon_0 F_{i0}; \quad \forall i,j \in [0,3]; \quad c^2 = \frac{1}{\varepsilon_0\mu_0}; \qquad \text{(A.8.7 a-f)}$$

also

$$B_{ij} = \frac{1}{c}F_{ij}; \quad G^{ij} = \frac{1}{\mu_0}\cdot\frac{1}{c}F^{ij};$$

$$D^i = \frac{1}{c}G^{0i} = \varepsilon_0\delta^{ii}E_i; \quad D_k = D^k; \qquad \text{(A.8.8 a-d)}$$

$$\forall i,j,k \in [1,3];$$

$$H^{ij} = G^{ij}; \quad H_k = H^{ij}; \quad B_k = B_{ij}; \quad \forall i,j,k \in [1,3]; \qquad \text{(A.8.8 e-g)}$$

where the subscript index k in the relations (A.8.8 d) and (A.8.8 f, g) refers to the usual three-dimensional vectors, whereas indices i and j refer, as previously, to tensor components. All situations in which the index k has this role will be mentioned. It is to be noted that the components of the form F_{ii} and G^{ii} vanish.

Introducing the axis coefficients of the Galilean reference frame, e_{ii}, [11], as shown in Sub-section A.6.2.1, of Appendix 6, we can write:

$$A_i = e_{ii}A^i = e_{is}A^s; \quad e_{00} = 1; \quad e_{ii} = -1, \quad \forall i \in [1,3];$$

$$e_{ij} = 0, \quad \forall i \neq j; \qquad \text{(A.8.9 a-d)}$$

$$A_i A^i = (A_0)^2 - (A_1)^2 - (A_2)^2 - (A_3)^2.$$

In the case of isotropic, linear, non-homogeneous media, using relation (A.5.21) from Appendix 5 and (3.20) from Chapter 3, respectively, we may write:

$$G^{0j} = c\varepsilon_0(1 + \chi_e)E_j = \varepsilon_0\varepsilon_r F_{j0} = \varepsilon F^{0j}. \qquad \text{(A.8.10)}$$

Similarly, using relations (A.5.23), from Appendix 5, and (3.101 b), from Chapter 3, we may write:

$$G^{ij} = \frac{1}{\mu_0\left(1+\chi_m\right)} \cdot \frac{1}{c} F^{ij} = \frac{1}{\mu_0\mu_r} \cdot \frac{1}{c} F^{ij}; \quad \forall\, i,\, j \in [0,\, 3]. \tag{A.8.11}$$

The equations of the second group (set) are given by the relationship:

$$\frac{\partial F_{ij}}{\partial x^k} + \frac{\partial F_{jk}}{\partial x^i} + \frac{\partial F_{ki}}{\partial x^j} = 0, \quad \forall\, i,\, j,\, k \in [0,\, 3], \quad i \neq j \neq k. \tag{A.8.12}$$

In order to emphasize the tensors F_{ij} and G_{ij}, equations (A.8.3 a-d) can be written in the form below, convenient for passing from three to four dimensions. Relations (A.8.6), (A.8.7 a-f), (A.8.8 a-g), (A.8.10), (A.8.11) have been considered. For instance, relation (A.8.13 c) has been written taking into account relations (A.8.10) and (A.8.11) respectively. Hence:

$$\left(f_k\right)_1 = \frac{1}{c} \cdot \frac{\partial G^{0j}}{\partial x^j} F_{k0}; \quad \left(f_k\right)_2 = \frac{1}{c} \cdot \left(\frac{\partial G^{uj}}{\partial x^j} + \frac{\partial G^{u0}}{\partial x^0}\right) F_{ku};$$

$$\left(f_k\right)_3 = -\frac{1}{2} F_{u0}\, F^{0u}\, \frac{\partial \varepsilon}{\partial x^k} - \frac{1}{2} G_{uv} G^{uv}\, \frac{\partial \mu}{\partial x^k};$$

$$\left(f_k\right)_4 = \frac{1}{c} \cdot \frac{\partial}{\partial x^0}\left(G^{0u}\, F_{ku}\right); \quad \forall\, j,\, u,\, v \in [1,\, 3]; \quad u < v; \quad k \in [1,\, 3], \tag{A.8.13 a-d}$$

and also, we have:

$$G^{0j} = c\,\varepsilon_0 \varepsilon_r\, F^{0j}; \quad G^{uv} = \frac{1}{\mu_0\mu_r} \cdot \frac{1}{c} F^{uv};$$

$$c^2 = \frac{1}{\varepsilon_0\mu_0}; \quad \varepsilon = \varepsilon_0 \varepsilon_r, \quad \mu = \mu_0\mu_r; \tag{A.8.14 a-e}$$

$$\forall\, j \in [1,\, 3]; \quad u,\, v \in [1,\, 3].$$

Summing up, side by side, relations (A.8.13 a) and (A.8.13 b), we shall get the following more compact expression:

$$\left(f_k\right)_{12} = \frac{1}{c} \cdot \frac{\partial G^{0j}}{\partial x^j} F_{k0} + \frac{1}{c} \cdot \frac{\partial G^{uj}}{\partial x^j} F_{ku}, \quad \forall\, k \in [0,\, 3];\, j \in [0,\, 3];\, u \in [1,\, 3]; \tag{A.8.15}$$

relation which, as mentioned, is extended for the four-dimensional continuum since indices j and k may take four values. Then, summing up, there follows:

$$\left(f_k\right)_{12} = \frac{1}{c} \cdot \frac{\partial G^{ij}}{\partial x^j} F_{ki}, \quad \forall\, i,\, j \in [0,\, 3]. \tag{A.8.16}$$

A.8.2. EXPRESSION OF THE FORCE COMPONENTS AND OF THE ENERGY-MOMENTUM TENSOR

We shall now consider the case of a linear isotropic electric and magnetic polarization of the considered medium, with the relative permittivity ε_r and the relative permeability μ_r, point functions. In order to express the force component as the derivative of an expression, we shall write relation (A.8.16), with the same conditions of indices, as:

$$c(f_k)_{12} = \frac{\partial G^{ij}}{\partial x^j} F_{ki} = \frac{\partial}{\partial x^j}\left(G^{ij} F_{ki}\right) - G^{ij}\frac{\partial F_{ki}}{\partial x^j}, \quad \forall\, i, j \in [0, 3], \quad \forall\, k \in [0, 3]. \tag{A.8.17}$$

Now, we shall modify the second term of the right-hand side as follows:

$$G^{ij}\frac{\partial F_{ki}}{\partial x^j} = G^{ij}\frac{\partial F_{ki}}{\partial x^j}, \quad \forall\, i, j \in [0, 3], \quad \forall\, k \in [0, 3],$$

$$G^{ij}\frac{\partial F_{ki}}{\partial x^j} = G^{ji}\frac{\partial F_{kj}}{\partial x^i} = -G^{ij}\frac{\partial F_{kj}}{\partial x^i} = G^{ij}\frac{\partial F_{jk}}{\partial x^i}. \tag{A.8.18 a, b}$$

In the following calculations, we shall use the relation:

$$G^{rs}\frac{\partial}{\partial x^k}\left(\mu\, G_{rs}\right) = \frac{1}{2}G^{pq}\frac{\partial}{\partial x^k}\left(\mu\, G_{pq}\right),$$

$$\forall\, r, s \in [0, 3], r < s, \quad \forall\, p, q \in [0, 3]. \tag{A.8.19}$$

Summing up the left-hand and the last right-hand sides of the two expressions (A.8.18 a, b), and taking into account (A.8.12) and (A.8.19), we shall get:

$$2G^{ij}\frac{\partial F_{ki}}{\partial x^j} = G^{ij}\left(\frac{\partial F_{ki}}{\partial x^j} + \frac{\partial F_{jk}}{\partial x^i}\right) = -2G^{ij}\frac{\partial F_{ij}}{\partial x^k}, \quad \forall\, i, j \in [0, 3],\ i < j,\ \forall\, k \in [0, 3]. \tag{A.8.20}$$

Replacing (A.8.20) into (A.8.17), and taking into account (A.8.19) we shall obtain:

$$c(f_k)_{12} = \frac{\partial G^{ij}}{\partial x^j} F_{ki} = \frac{\partial}{\partial x^j}\left(G^{ij} F_{ki}\right) + G^{ij}\frac{\partial F_{ij}}{\partial x^k}, \quad \forall\, i, j, k \in [0, 3],\ i < j. \tag{A.8.21}$$

By expanding the last term of (A.8.21), there follows:

$$c(f_k)_{12} = \frac{\partial}{\partial x^j}\left(G^{ij} F_{ki}\right) + G^{0v}\frac{\partial F_{0v}}{\partial x^k} + G^{uv}\frac{\partial}{\partial x^k}\left(\mu\, c\, G_{uv}\right),$$

$$\forall\, i, j, k \in [0, 3];\ u, v \in [1, 3];\ u < v. \tag{A.8.22}$$

Initially we shall assume index k different from zero.

Replacing the symbols of (A.8.14 a, b) into (A.8.21), after having divided both sides with c, we shall get:

$$(f_k)_{12} = \frac{1}{c} \cdot \frac{\partial}{\partial x^j} (G^{ij} F_{ki}) - \varepsilon_0 \varepsilon_r F^{0v} \frac{\partial F_{v0}}{\partial x^k} + G^{uv} \frac{\partial}{\partial x^k} (\mu G_{uv});$$

$$\forall k \in [0, 3]; \quad i, j \in [0, 3]; \quad u, v \in [1, 3]; \quad u < v. \tag{A.8.23}$$

We are now going to calculate the components of f_k which, according to the types of the included electromagnetic field state quantities, can be of the following types: electric, magnetic, mixed.

In order to facilitate the understanding of the formulae, we shall successively use the tensor notation and the vector notation. We shall use for indices numbers, instead of letters, because it is easier to perform the computation and to avoid the use of the summation convention when not allowed. Then, the indices may be subscripts. We shall adopt $k = 3$. We shall not write the terms of the form F_{uu} and G^{uu}, being zero.

We shall express the electric component considering expression (A.8.23). We shall take into account the relation:

$$-\varepsilon F \frac{\partial F}{\partial x} = -\frac{\partial}{\partial x} \left(\frac{1}{2} \varepsilon F^2 \right) + \frac{1}{2} F^2 \frac{\partial \varepsilon}{\partial x}. \tag{A.8.24}$$

The electric component will be obtained from the expanded relation (A.8.23), having in view (A.8.24), and that indices u and v take the same values:

$$(f_k)_{\text{elec}} = \frac{1}{c} \cdot \frac{\partial}{\partial x^v} (G^{0v} F_{k0}) - \frac{1}{2} \frac{\partial}{\partial x^k} (\varepsilon F^{0v} F_{v0}) + \frac{1}{2} F^{0u} F_{u0} \frac{\partial \varepsilon}{\partial x^k} =$$

$$= \frac{1}{c} \cdot \frac{\partial}{\partial x^1} (G^{01} F_{30}) + \frac{1}{c} \cdot \frac{\partial}{\partial x^2} (G^{02} F_{30}) + \frac{1}{c} \cdot \frac{\partial}{\partial x^3} (G^{03} F_{30}) - \tag{A.8.25 a}$$

$$- \frac{1}{2} \cdot \frac{1}{c} \cdot \frac{\partial}{\partial x^k} (G^{0v} F_{v0}) + \frac{1}{2} \cdot F^{0u} F_{u0} \frac{\partial \varepsilon}{\partial x^k}, \qquad \forall u, v \in [1, 3],$$

and therefore:

$$(f_k)_{\text{elec}} = \frac{\partial}{\partial x_1} (\varepsilon E_1 E_3) + \frac{\partial}{\partial x_2} (\varepsilon E_2 E_3) + \frac{\partial}{\partial x_3} (\varepsilon E_3 E_3) -$$

$$- \frac{1}{2} \frac{\partial}{\partial x_3} (\varepsilon E^2) + \frac{1}{2} E^2 \frac{\partial \varepsilon}{\partial x_3}. \tag{A.8.25 b}$$

In tensor form, we have:

$$\left(f_k\right)_{\text{elec}} = \frac{1}{c} \cdot \frac{\partial}{\partial x^v}\left(G^{0v}F_{k0}\right) - \frac{1}{2} \cdot \frac{1}{c} \cdot \frac{\partial}{\partial x^k}\left(G^{0v}F_{v0}\right) + \frac{1}{2} \cdot \frac{1}{c}\left(G^{0v}F_{v0}\right)\frac{\partial \varepsilon}{\partial x^k},$$

$$\forall v \in [1,\,3].$$

(A.8.25 c)

Then, we shall express the magnetic component considering also expression (A.8.22). We need to calculate expressions of the form:

$$G^{uv}\frac{\partial}{\partial x}\left(\mu G_{uv}\right) = \frac{1}{\mu}B\frac{\partial B}{\partial x}; \quad G^{uv} = G = \frac{1}{\mu}B.$$

(A.8.26 a)

By differentiating, we obtain:

$$\frac{\partial}{\partial x}\left(\frac{1}{\mu}B^2\right) = \frac{1}{\mu} \cdot \frac{\partial}{\partial x}\left(B^2\right) - \frac{1}{\mu^2}B^2\frac{\partial \mu}{\partial x};$$

$$\frac{\partial}{\partial x}\left(\mu G^2\right) = 2G\frac{\partial}{\partial x}\left(\mu G\right) - G^2\frac{\partial \mu}{\partial x}.$$

(A.8.26 b)

Therefore, replacing the expression (A.8.26 b), into the considered term of (A.8.23), and having in view relations (A.8.20), we shall obtain for $k = 3$:

$$\left(f_k\right)_{\text{mag}} = \frac{\partial}{\partial x^v}\left(\mu G^{uv}G_{ku}\right) + \frac{1}{2} \cdot \frac{\partial}{\partial x^k}\left(\mu G^{uv}G_{uv}\right) + \frac{1}{2}\left(G^{uv}G_{uv}\right)\frac{\partial \mu}{\partial x^k} =$$

(A.8.27a)

$$= \frac{\partial}{\partial x^1}\left(\mu G^{u1}G_{3u}\right) + \frac{\partial}{\partial x^2}\left(\mu G^{u2}G_{3u}\right) + \frac{\partial}{\partial x^3}\left(\mu G^{u3}G_{3u}\right) +$$

$$+ \frac{1}{2} \cdot \frac{\partial}{\partial x^k}\left(\mu G^{uv}G_{uv}\right) + \frac{1}{2}\left(G^{uv}G_{uv}\right)\frac{\partial \mu}{\partial x^k} = \frac{\partial}{\partial x_1}\left(\mu H_3 H_1\right) +$$

$$+ \frac{\partial}{\partial x_2}\left(\mu H_3 H_2\right) - \frac{\partial}{\partial x_3}\left[\mu\left(H_2 H_2 + H_1 H_1\right)\right] + \frac{1}{2} \cdot \frac{\partial}{\partial x_3}\left(\mu H^2\right) + \frac{1}{2}\left(H^2\right)\frac{\partial \mu}{\partial x_3},$$

$$\forall u, v \in [1,\,3], \text{ and } u < v \text{ in the products of the form } G^{uv}G_{uv}.$$

In case of vector notation for the electric and magnetic components, the vector notation has also been used for the co-ordinates x. Rearranging the terms there follows:

$$(f_k)_{\text{mag}} = \frac{\partial}{\partial x_1}(\mu H_3 H_1) + \frac{\partial}{\partial x_2}(\mu H_3 H_2) - \frac{\partial}{\partial x_3}\Big[\mu(H_2^2 + H_1^2) - \mu H^2\Big] -$$

$$- \frac{\partial}{\partial x_3}(\mu H^2) + \frac{1}{2}\cdot\frac{\partial}{\partial x_3}(\mu H^2) + \frac{1}{2} H^2 \frac{\partial\mu}{\partial x_3} =$$

$$= \frac{\partial}{\partial x_1}(\mu H_3 H_1) + \frac{\partial}{\partial x_2}(\mu H_3 H_2) + \frac{\partial}{\partial x_3}(\mu H_3 H_3) - \frac{\partial}{\partial x_3}\Big[\mu(H_1^2 + H_2^2 + H_3^2)\Big] -$$

$$- \frac{1}{2}\cdot\frac{\partial}{\partial x_3}(\mu H^2) + \frac{1}{2} H^2 \frac{\partial\mu}{\partial x_3},$$

(A.8.27 b)

and

$$(f_k)_{\text{mag}} = \frac{\partial}{\partial x_1}(\mu H_3 H_1) + \frac{\partial}{\partial x_2}(\mu H_3 H_2) +$$

$$+ \frac{\partial}{\partial x_3}(\mu H_3 H_3) - \frac{1}{2}\cdot\frac{\partial}{\partial x_3}(\mu H^2) + \frac{1}{2} H^2 \frac{\partial\mu}{\partial x_3}$$

(A.8.27 c)

or in tensor form, taking into account the tensor form of the derivative of μH^2, we have:

$$(f_k)_{\text{mag}} = \frac{\partial}{\partial x^v}(\mu G^{uv} G_{ku}) + \frac{1}{2}\cdot\frac{\partial}{\partial x^k}(\mu G^{uv} G_{uv}) + \frac{1}{2}(G^{uv} G_{uv})\frac{\partial\mu}{\partial x^k},$$

(A.8.27 d)

$$\forall u, v \in [1, 3], \quad u < v.$$

Returning to previous letter indices, and summing side by side relations (A.8.25 c), (A.8.27 b) or (A.8.27 c), we shall get the sum of electric and magnetic terms calculated above:

$$(f_k)_{\text{elec+mag}} = \frac{\partial}{\partial x_v}(\varepsilon E_k E_v) - \frac{1}{2}\cdot\frac{\partial}{\partial x_k}(\varepsilon E^2) + \frac{1}{2} E^2 \frac{\partial\varepsilon}{\partial x_k} + \frac{\partial}{\partial x_v}(\mu H_k H_v) -$$

$$- \frac{1}{2}\frac{\partial}{\partial x_k}(\mu H^2) + \frac{1}{2} H^2 \frac{\partial\mu}{\partial x_k}.$$

(A.8.28 a)

Summing up relations (8.25 c) and (8.27 d), side by side, we get in tensor form:

$$(f_k)_{\text{elec+mag}} = \frac{\partial}{\partial x^v}(\varepsilon F^{0v} F_{k0}) - \frac{\partial}{\partial x^k}\Big(\frac{1}{2}\varepsilon F^{0u} F_{u0}\Big) + \frac{1}{2} F^{0u} F_{u0}\frac{\partial\varepsilon}{\partial x^k} +$$

$$+ \frac{\partial}{\partial x^v}(\mu G^{uv} G_{ku}) - \frac{\partial}{\partial x^k}\Big(\frac{1}{2}\mu G^{uv} G_{uv}\Big) + \frac{1}{2}(G^{uv} G_{uv})\frac{\partial\mu}{\partial x^k}.$$

(A.8.28 b)

$$\forall u, v \in [1, 3], \quad u < v.$$

We shall now express the mixed components, considering the first term of expression (A.8.22), and the expression (A.8.13 d). The first mixed component is given by:

$$(f_k)_{\text{mix1}} = \frac{1}{c} \cdot \frac{\partial}{\partial x^0} \left(G^{u0} F_{ku} \right) = \frac{1}{c} \left[\frac{\partial}{\partial x^0} \left(G^{10} F_{31} \right) + \frac{\partial}{\partial x^0} \left(G^{20} F_{32} \right) \right] =$$

$$= \frac{1}{c} \left[\frac{\partial}{\partial x_0} \left(-cD^1 cB_2 \right) + \frac{\partial}{\partial x_0} \left(cD^2 cB_1 \right) \right],$$

(A.8.29 a)

and

$$(f_k)_{\text{mix1}} = -\frac{\partial}{\partial t} \left(D_1 B_2 - D_2 B_1 \right) = -\frac{\partial}{\partial t} \left(\boldsymbol{D} \times \boldsymbol{B} \right)_3, \quad \forall u \in [1, 3].$$

(A.8.29 b)

The second mixed component is similarly given by:

$$(f_k)_{\text{mix2}} = \frac{1}{c} \cdot \frac{\partial}{\partial x^0} \left(G^{0u} F_{ku} \right) = \frac{\partial}{\partial t} \left(\boldsymbol{D} \times \boldsymbol{B} \right)_3,$$

$$\forall u \in [1, 3].$$

(A.8.30 a)

Returning to previous letter indices, we get:

$$(f_k)_{\text{mix2}} = \frac{\partial}{\partial t} \left(D^i B_j - D^j B_i \right).$$

(A.8.30 b)

Adding up relations (A.8.28 b) and (A.8.29 a), side by side, we obtain:

$$(f_k)_{\text{elec+mag+mix1}} = \frac{1}{c} \left[\frac{\partial}{\partial x^v} \left(G^{0v} F_{k0} \right) - \frac{1}{2} \cdot \frac{\partial}{\partial x^k} \left(G^{0v} F_{v0} \right) \right] + \frac{1}{2} \cdot \frac{1}{c} \left(G^{0v} F_{v0} \right) \frac{\partial \varepsilon}{\partial x^k} +$$

$$+ \frac{1}{c} \cdot \frac{\partial}{\partial x^v} \left(G^{uv} F_{ku} \right) + \frac{1}{2} \cdot \frac{1}{c} \cdot \frac{\partial}{\partial x^k} \left(G^{uv} F_{uv} \right) + \frac{1}{2} \left(G^{uv} G_{uv} \right) \frac{\partial \mu}{\partial x^k} +$$

(A.8.31)

$$+ \frac{1}{c} \cdot \frac{\partial}{\partial x^0} \left(G^{u0} F_{ku} \right), \quad \forall u, v \in [1, 3].$$

By summing up, side by side relations (A.8.31) and (A.8.29 b), we get:

$$(f_k)_{\substack{\text{elec+mag} \\ +\text{mix1+mix2}}} = \frac{1}{c} \left[\frac{\partial}{\partial x^v} \left(G^{0v} F_{k0} \right) - \frac{1}{2} \cdot \frac{\partial}{\partial x^k} \left(G^{0v} F_{v0} \right) \right] + \frac{1}{2} \cdot \frac{1}{c} \left(G^{0v} F_{v0} \right) \frac{\partial \varepsilon}{\partial x^k} +$$

$$+ \frac{1}{c} \cdot \frac{\partial}{\partial x^v} \left(G^{uv} F_{ku} \right) + \frac{1}{2} \cdot \frac{1}{c} \cdot \frac{\partial}{\partial x^k} \left(G^{uv} F_{uv} \right) + \frac{1}{2} \left(G^{uv} G_{uv} \right) \frac{\partial \mu}{\partial x^k},$$

(A.8.32)

$$\forall u, v \in [1, 3],$$

and after summing up with relation (A.8.13 c), side by side, there follows:

$$\left(f_k\right)_{\text{total}} = \frac{1}{c}\left[\frac{\partial}{\partial x^v}\left(G^{0v}F_{k0}\right) + \frac{1}{2}\cdot\frac{\partial}{\partial x^k}\left(G^{0u}F_{u0}\right)\right] +$$

$$+ \frac{1}{c}\left[\frac{\partial}{\partial x^v}\left(G^{uv}F_{ku}\right) + \frac{1}{2}\cdot\frac{\partial}{\partial x^k}\left(G^{uv}F_{uv}\right)\right] + \frac{1}{c}\cdot\frac{\partial}{\partial x^0}\left(G^{v0}F_{kv}\right),$$

$$\forall u, v \in [1, 3], \ u < v. \tag{A.8.33}$$

If we started from formula (A.8.1 b), we should have also added in the right-hand side of relation (A.8.33) the expression (A.8.30 b), and then, the final relation would differ.

The force expression may be written in a compact and general form as follows:

$$f_k = \frac{1}{c}\cdot\frac{\partial}{\partial x^j}\left(G^{ij}F_{ki}\right) + \frac{1}{2}\cdot\frac{1}{c}\cdot\frac{\partial}{\partial x^k}\left(G^{uv}F_{uv}\right),$$

$$\forall i, j, k \in [0, 3], \quad \forall u, v \in [0, 3], u < v, \tag{A.8.34 a}$$

or in a more compact form, as follows:

$$f_k = \frac{1}{c}\cdot\frac{\partial}{\partial x^j}\left(G^{ij}F_{ki} + \frac{1}{2}\delta_k^j\, G^{uv}F_{uv}\right),$$

$$\forall i, j, k \in [0, 3], \quad \forall u, v \in [0, 3], \ u < v. \tag{A.8.34 b}$$

Finally, the component of the volume density of the force along the k-axis can be expressed as:

$$f_k = \frac{\partial}{\partial x^j}W_k^j, \tag{A.8.34 c}$$

where the expression:

$$W_k^j = \frac{1}{c}\left(G^{ij}F_{ki}\right) + \delta_k^j\frac{1}{2}\cdot\frac{1}{c}\left(G^{ij}F_{ij}\right),$$

$$\forall i, j, k \in [0, 3], \quad \forall u, v \in [0, 3], u < v. \tag{A.8.35}$$

represents *the energy-momentum tensor*, also called *tensor of energy and quantity of motion*. It is possible to express the last relation in other forms, taking into account the following relation:

$$W_s^j\, e^{sk} = W^{jk}, \quad e_{sk} = 0, \quad \forall s \neq k;$$

$$e_{00} = 1; \quad e_{jj} = -1, \quad \forall j \in [1, 3]. \tag{A.8.36 a}$$

Also, we get:

$$W_{jk} = e_{jj}e_{kk}\, W^{jk}. \tag{A.8.36 b}$$

The expressions F_{ij} and G^{ij}, according to the mention for relation (A.8.5 a), represent tensors of the second rank. The expression of the form W_{ij} being the sum of the products of two tensors will also be a tensor, and for this reason it is so called.

Remarks

1° If the media were not assumed isotropic and had not linear electric and magnetic polarization, the previous transformations of relations (A.8.24) and (A.8.26 b), respectively, would be no longer possible.

2° Having established the expression of the tensor in one reference frame, we can obtain its expression in any other one. The calculation is to be performed by using the group of co-ordinate transformations, for instance the Lorentz transformations. We consider useful to make the following remark. The Lorentz transformation group has been established for empty space (vacuum), and the involved light velocity is that *in vacuo*. In the present case, we consider that polarization exists, and in this case, also all transformations of the quantities are like those established by Minkowski. But a doubt appears, namely if the transformations are still valid because in any media the velocity of light is different. *For this reason, the Lorentz transformation group may be considered as an assumption that is so better the smaller will be the space regions filled with substance.*

It is to be noted that we have established a new form of the tensors used for defining the field state quantities, which facilitate the analysis.

A.8.3. EXPRESSION OF THE ENERGY-MOMENTUM TENSOR

1° Component W_0^0. Using formula (A.8.35), and after performing the calculation, passing from tensor notation to vector notation, we have got:

$$W_0^0 = \frac{1}{c}\left(G^{i0}\,F_i + \frac{1}{2}\delta_0^0\,G^{0v}F_{0v} \right) + \frac{1}{2}\delta_0^0\,\frac{1}{c}G^{uv}F_{uv}, \quad \forall i, u, v \in [1, 3],\ u < v\,;$$

(A.8.37 a)

$$\frac{1}{c}G^{i0}\,F_{0i} = D_iE_i\,; \quad \frac{1}{2}\cdot\frac{1}{c}\delta_0^0\,G^{0v}F_{0v} = -\frac{1}{2}E_vD_v\,; \quad \frac{1}{2}\cdot\frac{1}{c}\delta_0^0\,G^{uv}F_{uv} = \frac{1}{2}H_iB_i\,,$$

and summing up the calculated terms, we obtain:

$$W_0^0 = D_iE_i - \frac{1}{2}E_iD_i + \frac{1}{2}H_iB_i = \frac{1}{2}\left(\varepsilon E^2 + \mu H^2\right), \quad \forall i \in [1, 3], \qquad \text{(A.8.37 b)}$$

which represents the volume density of the electromagnetic energy, and E_i, D_i are considered as three-dimensional vector components.

2° Component W_k^j for both cases $k \neq j$ and $k = j$. We use, as above, formula (A.8.35), and after performing the calculation, we shall pass from tensor notation to vector notation.

In the first case, remarking that j and k are different, we should keep only the first term of expression (A.8.35). We get:

$$W_k^j = \frac{1}{c} G^{0j} F_{k0} + G^{ij} F_{ki} = \varepsilon_0 \varepsilon_r F_{j0} F_{k0} + G^{ij} F_{ki};$$

$$\varepsilon_0 \varepsilon_r F_{j0} F_{k0} = E_j D_k = E_k D_j; \quad G^{ij} F_{ki} = H_k B_j = H_j B_k;$$

$$W_k^j = -W^{jk} = E_j D_k + H_j B_k;$$

$$\forall\, i,\, j,\, k \in [1,\, 3].$$

(A.8.38 a)

In the second case, for more clarity, instead of letter indices, we shall use number indices, considering a certain case, namely for $j = k = 2$. There follows:

$$W_2^2 = \frac{1}{c}\left(G^{02} F_{20} + G^{12} F_{21} + G^{32} F_{23}\right) + \frac{1}{2} \cdot \frac{1}{c}\left(G^{0v} F_{0v} + G^{uv} F_{uv}\right);$$

$$\forall\, u,\, v \in [1,\, 3];$$

(A.8.38 b)

$$W_2^2 =$$

$$= \frac{1}{c}\left(G^{02} F_{20} + G^{12} F_{21} + G^{32} F_{23}\right) = D_2 E_2 + H_3 \left(-B_3\right) + \left(-H_1\right) B_1;$$

$$\frac{1}{2} \cdot \frac{1}{c}\left(G^{0v} F_{v0}\right) = \frac{1}{2}\left(D_1 E_1 + D_2 E_2 + D_3 E_3\right);$$

$$G^{uv} F_{uv} = H_1 B_1 + H_2 B_2 + H_3 B_3, \quad \forall\, u,\, v \in [1,\, 3];$$

$$\frac{1}{2} \cdot \frac{1}{c}\left(G^{uv} F_{uv}\right) = \frac{1}{2}\left(H_1 B_1 + H_2 B_2 + H_3 B_3,\right), \quad \forall\, u,\, v \in [1,\, 3];$$

$$W_2^2 = -W^{22} =$$

$$= E_2 D_2 - H_3 B_3 - H_1 B_1 + H_2 B_2 - H_2 B_2 -$$

$$-\frac{1}{2}(E_1 D_1 + E_2 D_2 + E_3 D_3) + \frac{1}{2}(H_1 B_1 + H_2 B_2 + H_3 B_3);$$

(A.8.38 c)

and, grouping the terms, we obtain:

$$W_2^2 = -W^{22} = E_2 D_2 + H_2 B_2 - \frac{1}{2}\left(\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{H} \cdot \boldsymbol{B}\right),$$

(A.8.38 d)

and the general form, as expected, is:

$$W_j^j = -W^{jj} = E_j D_j + H_j B_j - \frac{1}{2}\left(\boldsymbol{E} \cdot \boldsymbol{D} + \boldsymbol{H} \cdot \boldsymbol{B}\right),$$

$$\forall\, j \in [1,\, 3].$$

(A.8.38 e)

The results above, expressed by relations (A.8.38 a) and (A.8.38 e), represent the Maxwell stress tensors.

3° Component W_0^j**.** As previously, we shall use formula (A.8.35), and after performing the calculation, we pass from tensor notation to vector notation. We begin with one example for $j = 2$, $k = 0$, and then express the general form. There follows:

$$W_0^2 = \frac{1}{c} G^{i2} F_{0i} = \frac{1}{c} G^{12} F_{01} + \frac{1}{c} G^{32} F_{03} = \frac{1}{c} H^{12}(-E_1) + \frac{1}{c}(-H^{23})(-E_3);$$

$$W_0^2 = \frac{1}{c}(E_3 H_1 - E_1 H_3); \qquad (A.8.39)$$

$$W_0^j = W^{j0} = \frac{1}{c}(E_k H_i - E_i H_k), \quad \forall i, k \in [1, 3];$$

and the general form is:

$$W_0^j = W^{j0} = \frac{1}{c}(E_k H_i - E_i H_k) = \frac{1}{c}(\boldsymbol{E} \times \boldsymbol{H})_j, \qquad (A.8.40)$$

which apart from the denominator c, represents the j-component of Poynting vector, i.e., the rate of the radiated flux of energy per unit of surface and unit of time.

4° The force along the time axis. We shall use formula (A.8.34 a) or (A.8.34 b), putting $k = 0$, and after performing the calculations, we shall pass from tensor notation to vector notation. There follows:

$$f_0 = \frac{1}{c} \cdot \frac{\partial}{\partial x^j}(G^{ij} F_{0i}) + \frac{1}{2} \cdot \frac{1}{c} \cdot \frac{\partial}{\partial x^0}(G^{uv} F_{uv}),$$

$$\forall i, j \in [0, 3], \quad \forall u, v \in [0, 3], \quad u < v, \qquad (A.8.41)$$

and:

$$c f_0 = \frac{\partial}{\partial x^0}(G^{i0} F_{0i}) + \frac{\partial}{\partial x^1}(G^{i1} F_{0i}) + \frac{\partial}{\partial x^2}(G^{i2} F_{02}) + \frac{\partial}{\partial x^3}(G^{i3} F_{0i}) +$$

$$+ \frac{1}{2} \cdot \frac{\partial}{\partial x^k}(G^{uv} F_{uv}), \qquad (A.8.42)$$

$$\forall i \in [0, 3], \quad \forall u, v \in [0, 3], u < v, \quad k = 0.$$

Considering the first and second parentheses, we shall get:

$$G^{i0} F_{0i} = c \varepsilon_0 \varepsilon_r F^{i0} F_{0i} = -c \varepsilon_0 \varepsilon_r(-F_{i0} F_{i0}) = \boldsymbol{E} \cdot \boldsymbol{D}, \qquad (A.8.43 \text{ a})$$

and

$$G^{i1} F_{0i} = G^{21} F_{02} + G^{31} F_{03} = -H^{12}(-E_2) + H^{31}(-E_3) =$$
$$= E_2 H_3 - E_3 H_2 = (\boldsymbol{E} \times \boldsymbol{H})_1 ,$$

(A.8.43 b)

where, for $i = 2$, $G^{i1} = G^{21} = H^{21}$, the vector $\boldsymbol{H}$ being contravariant, and according to relation (A.5.15 b), from Appendix 5, $H^{21} = -H^{12} = -H_3$, since $H_1 = H^{23}$. Calculating the first two parentheses, we get:

$$\frac{\partial}{\partial x^0}\left(G^{i0} F_{0i}\right) = \frac{\partial}{\partial x^0}(\boldsymbol{E} \cdot \boldsymbol{D});$$

$$\frac{\partial}{\partial x^1}\left(G^{i1} F_{0i}\right) = \frac{\partial}{\partial x^1}(\boldsymbol{E} \times \boldsymbol{H})_1 .$$

(A.8.43 c)

Handling similarly the next two parentheses and summing up all terms, there follows:

$$f_0 = \frac{1}{c}\operatorname{div}(\boldsymbol{E} \times \boldsymbol{H}) + \frac{\partial}{\partial x^0}(\boldsymbol{E} \cdot \boldsymbol{D}) + \frac{1}{2}\cdot\frac{1}{c}\cdot\frac{\partial}{\partial x^0}\left(G^{uv} F_{uv}\right),$$
$$\forall\, u, v \in [0, 3],\ u < v.$$

(A.8.44)

For the last parenthesis we shall obtain:

$$G^{uv} F_{uv} = G^{0r} F_{0r} + G^{rs} F_{rs} = c\left(-D_r E_r + H_q B_q\right) =$$
$$= c\left(-\boldsymbol{D} \cdot \boldsymbol{E} + \boldsymbol{H} \cdot \boldsymbol{B}\right),$$
$$\forall\, u, v \in [0, 3],\ u < v,\quad r, s, q \in [1, 3],\ r < s.$$

(A.8.45)

Replacing (A.8.45) into (A.8.44), we get:

$$f_0 = \frac{1}{c}\operatorname{div}(\boldsymbol{E} \times \boldsymbol{H}) + \frac{\partial}{\partial x^0}(\boldsymbol{E} \cdot \boldsymbol{D}) - \frac{1}{2}\cdot\frac{\partial}{\partial x^0}(\boldsymbol{E} \cdot \boldsymbol{D}) + \frac{1}{2}\cdot\frac{\partial}{\partial x^0}(\boldsymbol{H} \cdot \boldsymbol{B}),$$

(A.8.45 a)

hence

$$f_0 = \frac{1}{c}\left[\operatorname{div}(\boldsymbol{E} \times \boldsymbol{H}) + \frac{1}{2}\cdot\frac{\partial}{\partial t}(\boldsymbol{E} \cdot \boldsymbol{D})\right] + \frac{1}{c}\cdot\frac{1}{2}\cdot\frac{\partial}{\partial t}(\boldsymbol{H} \cdot \boldsymbol{B}).$$

(A.8.45 b)

The calculation of the derivatives yields the relation:

$$f_0 = \frac{1}{c}\left[\operatorname{div}(\boldsymbol{E} \times \boldsymbol{H}) + \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{H} \cdot \frac{\partial \boldsymbol{B}}{\partial t}\right],$$

(A.8.46)

where we shall replace the derivatives with respect to time, with the known Maxwell relations, as follows:

$$f_0 = \frac{1}{c}\left[\operatorname{div}(\boldsymbol{E} \times \boldsymbol{H}) + \boldsymbol{E} \cdot (\operatorname{curl}\boldsymbol{H} - \boldsymbol{J}) - \boldsymbol{H} \cdot \operatorname{curl}\boldsymbol{E}\right]. \tag{A.8.47}$$

Grouping the terms, we obtain:

$$f_0 = \frac{1}{c}\left[\operatorname{div}(\boldsymbol{E} \times \boldsymbol{H}) - \operatorname{div}(\boldsymbol{E} \times \boldsymbol{H}) - \boldsymbol{J} \cdot \boldsymbol{E}\right] = -\frac{1}{c}\boldsymbol{J} \cdot \boldsymbol{E}, \tag{A.8.48}$$

which represents the component of the force along the time axis. The same result can be also obtained starting from formula (A.8.16) and by putting $k = 0$.

The set f_k represents a four-vector, according to formulae (A.8.34 c), and can also result from (A.8.1 a) and (A.8.48), indeed the product of the four force components and the four-vector velocity yields a scalar.

A.8.4. THE EQUATIONS OF THE ELECTROMAGNETIC FIELD IN THE THEORY OF RELATIVITY

In this Appendix, we shall present the equations of the electromagnetic field in the Special and General Theory of Relativity in a form permitting to deduce the energy-momentum tensor. This subject has been thoroughly treated in Chapter 3 and in Appendix 5. We shall only recall some important results.

From the formulae of the general theory of the electromagnetic field (3.107 a), (3. 37), (3.200), (3.86), (3.18), (3.99), of Chapter 3, in the usual vector form, we have:

$$\begin{aligned} &\operatorname{curl}\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}; \quad \boldsymbol{B} = \operatorname{curl}\boldsymbol{A}; \\ &\boldsymbol{E} = -\operatorname{grad}V - \frac{\partial \boldsymbol{A}}{\partial t}; \\ &\operatorname{curl}\boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}; \quad \boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}; \\ &\boldsymbol{B} = \mu_0 \boldsymbol{H} + \mu_0 \boldsymbol{M}. \end{aligned} \tag{A.8.49 a-f}$$

We rewrite the equations (A.8.49 d) considering the general case of non-homogeneous media. We have:

$$\operatorname{curl}(\mu_0 \boldsymbol{H} + \mu_0 \boldsymbol{M}) = \mu_0 \boldsymbol{J} + \varepsilon_0 \mu_0 \frac{\partial \boldsymbol{E}}{\partial t} + \mu_0 \frac{\partial \boldsymbol{P}}{\partial t} + \mu_0 \operatorname{curl}\boldsymbol{M}; \tag{A.8.50 a}$$

$$\operatorname{curl}\boldsymbol{B} = \operatorname{curl}\operatorname{curl}\boldsymbol{A} = \mu_0 \boldsymbol{J} + \varepsilon_0 \mu_0 \frac{\partial}{\partial t}\left(-\operatorname{grad}V - \frac{\partial \boldsymbol{A}}{\partial t}\right) + \mu_0\left(\frac{\partial \boldsymbol{P}}{\partial t} + \operatorname{curl}\boldsymbol{M}\right); \tag{A.8.50 b}$$

and by expanding the double curl, there follows:

$$\operatorname{grad}\operatorname{div}\boldsymbol{A} - \nabla^2\boldsymbol{A} = \mu_0\boldsymbol{J} - \varepsilon_0\mu_0\left(\operatorname{grad}\frac{\partial V}{\partial t} + \frac{\partial^2\boldsymbol{A}}{\partial t^2}\right) + \mu_0\left(\frac{\partial\boldsymbol{P}}{\partial t} + \operatorname{curl}\boldsymbol{M}\right). \tag{A.8.51}$$

By rearranging the terms, we get:

$$\nabla^2\boldsymbol{A} - \varepsilon_0\mu_0\frac{\partial^2\boldsymbol{A}}{\partial t^2} = -\mu_0\left(\boldsymbol{J} + \frac{\partial\boldsymbol{P}}{\partial t} + \operatorname{curl}\boldsymbol{M}\right) + \operatorname{grad}\left(\operatorname{div}\boldsymbol{A} + \varepsilon_0\mu_0\frac{\partial V}{\partial t}\right). \tag{A.8.52}$$

Since only the curl of vector $\boldsymbol{A}$ is imposed, the divergence can be chosen by using the L.V. Lorenz (do not confuse with H.A. Lorentz) gauge condition:

$$\operatorname{div}\boldsymbol{A} + \varepsilon_0\mu_0\frac{\partial V}{\partial t} = 0. \tag{A.8.53}$$

The last relation may be written in the form:

$$\frac{\partial A_i}{\partial x^i} + \frac{1}{c^2}\cdot c\frac{\partial V}{\partial x^0} = 0; \quad x^0 = ct;$$

$$c^2 = \frac{1}{\varepsilon_0\mu_0}. \tag{A.8.54 a, b, c}$$

The components along any axis of a Cartesian system of co-ordinates will be:

$$E_i = -\frac{\partial V}{\partial x^i} - \frac{\partial A_i}{\partial t}; \quad B_i = \frac{\partial A_k}{\partial x^j} - \frac{\partial A_j}{\partial x^k};$$

$$\forall i, j, k \in [1, 3]. \tag{A.8 55 a, b}$$

The relations (A.8.49 b, c) may be written using a set of four quantities A_i as follows:

$$E_i = \frac{\partial A_0}{\partial x^i} - \frac{\partial A_i}{\partial t}, \quad B_i = \frac{\partial A_k}{\partial x^j} - \frac{\partial A_j}{\partial x^k}, \tag{A.8.56 a, b, c}$$

$$A_0 = -V, \quad \forall i, j, k \in [1, 3].$$

A new, more general and convenient form may be the following antisymmetric form with respect to indices i and j:

$$F_{ij} = c_i\frac{\partial A_j}{\partial x^i} - c_j\frac{\partial A_i}{\partial x^j}, \quad A_0 = -V, \tag{A.8.57 a, b, c}$$

$$c_i = 1 \wedge c_j = c, \quad \forall i \neq 0 \wedge j = 0, \quad c_i = c_j = c, \quad \forall i \wedge j \neq 0.$$

Therefore:

$$E_i = F_{i0} = \frac{\partial A_0}{\partial x^i} - c\frac{\partial A_i}{\partial x^0}, \quad A_0 = -V ;$$

$$B_{ij} = \frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j}, \quad \forall i, j \in [1, 3].$$

(A.8.58 a, …, d)

It follows that the quantities E_i and B_{ij} can be expressed as follows:

$$E_i = F_{i0} ; \quad B_{ij} = \frac{1}{c}F_{ij} ; \quad \forall i, j \in [1, 3].$$

(A.8.59 a, b)

For expressing the other field state quantities, for the case of empty space (vacuum), we may use formulae (A.5.10 a-f) of Appendix 5.

For expressing the other field state quantities, for the case of a polarized medium, formulae (A.5.15 a-f), of Appendix 5, may be used.

A.8.5. CONCLUSION

The aim of this work has been to establish the expression of the energy-momentum tensor within the frame of the Theory of Relativity, starting from the general formula of the electromagnetic force acting on a substance submitted to an electromagnetic field. The case of linear non-homogeneous media has been examined.

This subject has not been treated in the known papers or works published so far. Meanwhile, the analysis carried out has shown that no all-general known formulae are in agreement with the tensor energy-momentum expression when passing from a system of reference to another one. If the media were not assumed as isotropic and had not linear electric and magnetic polarization, the deduction carried out for obtaining the tensor would not be possible.

The expression of the tensor established in one system of reference can be obtained in any other system of reference owing to the group of Lorentz transformation and the Minkowski transformation formulae using this group. However, a doubt appears because the velocity of light in any media is different, and the Lorentz transformation has been established for this case.

List of Symbols Used in Appendix 8

A_i – component of the four-vector potential;

A – electrodynamic vector potential;

B_{ij} – twice covariant tensor component of magnetic induction, yielding B_k ;

B_k – component of the magnetic induction along axis k, considered as a usual three-dimensional vector;

c – velocity of light in empty space, supposed to be constant;

D_i – component of the electric displacement, considered as a usual three-dimensional vector;

D^i − contravariant component of the electric displacement yielding D_k or D_i considered as a usual three-dimensional vector;

$\boldsymbol{D}$ − electric displacement (electric induction) vector;

E_i − covariant component of the electric field strength, as well as component of the electric field strength along axis i, as a usual three-dimensional vector;

E^i − contravariant component of the electric field strength;

$\boldsymbol{E}$ − electric field strength vector;

e_{ii} − axis coefficient, for the axis i of the Galilean reference frame;

F_{ij} − component of the covariant tensor of rank 2, yielding B_{ij} for i and j non-zero;

F_{i0} − component of the previous covariant tensor, and yielding the component E_i of the electric field strength, considered as a usual three-dimensional vector;

f_k − four-vector component of the volume density of the electromagnetic (mechanical) force;

G^{ij} − contravariant tensor of rank 2, yielding H^{ij};

G_{i0}, G^{0i} − components of the covariant and contravariant tensors, and yielding the component D_i;

H_k − component of the magnetic field strength along axis k, considered as an usual three-dimensional vector;

$\boldsymbol{H}$ − magnetic field strength vector;

J^i − component of a contravariant four-vector, for i non-zero, density of the conduction electric current;

$\boldsymbol{J}$ − conduction electric current density vector;

V − electric potential;

x^i − co-ordinate along axis i;

δ_{ii} − symbol equal to unity for equal indices, and equal to zero for different ones (Kronecker symbol);

ε − electric permittivity, *in vacuo* it is ε_0;

μ_0 − magnetic permeability, *in vacuo* it is μ_0

ρ_v − volume density of the free electric charge.

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APPENDIX 9

DERIVING THE FORMULA OF THE SAGNAC EFFECT
BY USING THE GENERAL THEORY OF RELATIVITY

This Appendix is devoted to the analysis of the Sagnac effect from the point of view of the Theory of General Relativity. The bibliographic mentions [1]-[8], specific for this Appendix, are at the end of this Appendix.

In [1], we have shown that the experiment carried out by G. Sagnac in 1913, and more recently, remade several times using apparatus of high precision, is a very curious one. For a long time this phenomenon was considered as being in contradiction with Theory of Relativity. In a subsequent paper [2] we have shown that although the phenomenon is not in good agreement with the Special Theory of Relativity, it is in good agreement with the General Theory of Relativity. In the present Appendix we shall develop the mentioned proof. All bibliographic mentions concern the References [1]-[8] of the end of this Appendix.

A.9.1. THE EXPERIMENT OF G. SAGNAC

In paper [1], we have shown that the experiment carried out by Georges Sagnac in 1913, and more recently, remade several times using apparatus of high precision, is very interesting. The experiment consists in achieving the interference of two light beams travelling in inverse directions along the same way. The light source, the interferometer and the reflecting mirrors which ensure the desired paths (ways) for the light beams (namely a polygonal trajectory), photographic plate, that is, all apparatus are fixed on a disc, outside which nothing related to the experiment occurs. The light beams travelling around the same way, but in opposite directions, are reflected from the interferometer to a photographic plate. The disc can rotate with any angular velocity ω. A thorough description of the apparatus used by Sagnac may be found in [1].

The experiment has shown that the time for a light beam to travel around a way parallel to the disc surface differed, according to whether the travelling direction was with or against the rotation sense of the disc. Hence, the light beams had different velocities with respect to a reference frame fixed to the disc. The result, referred to as *Sagnac effect*, is not in concordance with the Special Theory of Relativity, in which, in all inertial reference frames, the velocity of light beams emitted by any source of light in empty space, measured with physically identical gauges and clocks, is assumed as having the same value.

As mentioned in paper [1], several authors stated that the Special Theory of Relativity can be applied in the case of a reference frame describing a movement with respect to a reference frame considered at rest, not only if the former has a rectilinear movement, but also if it has a movement along any polygonal or a circular line. According to the

experimental results, as shown in the same paper, this statement appears as no valid. This situation results from the circumstance that the Special Theory of Relativity concerns only inertial reference frames, a condition not satisfied in the present case. As mentioned in paper [1], an attempt to justify the Sagnac formula by the aid of the General Theory of Relativity has been made in paper [3]. For this purpose, several assumptions have been adopted, among which that the photon rest mass is different from zero, and the composition of velocities is performed according to the formula of the Special Theory of Relativity. It is useful to add that the Sagnac effect is not only of a theoretical importance, but it is used in several applications in measurements, instrumentation and metrology.

For the reasons above, in the present paper, we shall analyse the Sagnac effect by using the General Theory of Relativity, and deduce a formula of the Sagnac type, avoiding the above restrictions.

A.9.2. RECALL OF CERTAIN FOUNDATIONS OF THE GENERAL THEORY OF RELATIVITY

A.9.2.1. General Notions

We shall recall several notions of the theory [4]-[8], necessary in the further analysis. The position of a point in space can be defined by using three families (sets) of curves, which intersect forming a three-dimensional net [6]. Any point will be defined by the intersection of three curves of those families. The *space co-ordinates* will be denoted by certain quantities proportional to the distances along these curves, and denoted: x^1, x^2, x^3. In the framework of the General Theory of Relativity, no limitation exists in the choice of the systems of co-ordinates. The distance between two infinitesimally neighbouring points, in any system of co-ordinates, results from the relation:

$$\mathrm{d}s^2 = (\mathrm{d}s)^2 = g_{ii}(\mathrm{d}x^i)^2 + 2g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j = g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j , \qquad\qquad (A.9.1)$$

where the usual rule, of summing up the terms in which an index appears twice, has to be applied. The running index, in a three-dimensional space, will take all values from 1 to 3. Also, $g_{ij} = g_{ji}$. In the term with the coefficient 2, we have $i < j$.

In the case in which we want to define the position of a point in a three-dimensional space, as well as the moment at which that point is considered, a supplementary co-ordinate proportional to time has to be added. This co-ordinate will be denoted by $x^0 = ct$, where t denotes the time and c the velocity of light *in vacuo*, assumed to be constant. Several other symbols are also used, according to the author.

In the same case, of the three-dimensional net of above, a clock is imagined as existing at any node of the net. The time recorded by each of these clocks is assumed to show a small difference with respect to that recorded by the neighbouring ones. In this way, a four-dimensional space of a different type is defined [6, p. 364].

Using the symbols above, an event, e.g., the emission of a light beam from a point, can be marked by a point with four co-ordinates, indicated by one of the four index numbers, from 0 to 3. Therefore, the point, referred to as a *world point* or a *point of the universe*, is

defined in a *four-dimensional space*. The distance between two such points, also referred to as the *space-time interval* between two events, results by suitably modifying expression (A.9.1) above, and the running index will take all values from 0 to 3. Therefore:

$$\mathrm{d}s^2 = (\mathrm{d}s)^2 = g_{ii}\left(\mathrm{d}x^i\right)^2 + 2g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j = g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j =$$
$$= g_{00}\left(\mathrm{d}x^0\right)^2 + g_{\alpha\alpha}\left(\mathrm{d}x^\alpha\right)^2 + 2g_{0\alpha}\,\mathrm{d}x^0\,\mathrm{d}x^\alpha + 2g_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta,\quad \forall\,\alpha \neq \beta,$$

(A.9.2)

where the indices (subscript or superscript) i and j take all values from 0 to 3, whereas, the indices α and β take all values from 1 to 3, and $g_{ij} = g_{ji}$, and also $g_{\alpha\beta} = g_{\beta\alpha}$. The value of the space-time interval, $\mathrm{d}s$, may be zero or not. For instance, if it refers to the propagation of the front of a light signal, it is zero; if it refers to events which took place successively, at the same point in the three-dimensional space, it is different from zero.

In a Galilean system of reference (reference frame), including a Cartesian system of co-ordinates, the coefficients of expression (A.9.2) are the following:

$$g_{00} = 1;\quad g_{ii} = -1, \forall\,i \neq 0;\quad g_{ij} = 0, \forall\,i \neq j.$$

(A.9.2 a)

The space-time interval must keep the same value regardless of the used system of co-ordinates since it is an *invariant*. The trajectory of a point within this space is referred to as a *world line* or *line of the universe*.

In the General Theory of Relativity, the moving reference frame may describe any movement with respect to the fixed one. The passage from a fixed reference frame to another one having an accelerated motion will result in the transformation of a rectilinear uniform motion into a curved, accelerated one [6, p. 342].

The time recorded by a clock carried by any moving body is called *proper time* of this body. Initially, the clocks imagined as attributed to any points of the space are supposed to be not synchronized in any manner. In the General Theory of Relativity, no relation exists between the time recorded by the clocks at different point of the space. However, under certain conditions these can be synchronized [7, p. 310, 313, 384], as it will be further shown. The clocks may be of any type, mechanical as well as atomic. Some precision, not always mentioned, is necessary [5, p. 78, 360]. It is necessary to have into consideration, clocks with a small sensitivity to accelerations. No analysis, without considering minutely the structure of the clock, could predict the exact behaviour of the clock to accelerations. That is why, the proper time is considered to be recorded by ideal clocks, i.e., of very high precision, like certain atomic clocks.

Let us consider two reference frames *K* and *K'*, the former assumed to be immobile (at rest), and the latter in motion. The proper time of a body in motion is always shorter that the time in the immobile reference frame. Indeed, in the reference frame *K'*, where the considered clock is placed, we have $\mathrm{d}x'^\alpha = 0$, all space co-ordinates being unchanged. In this case, from relation (A.9.2), applied for a Cartesian system of co-ordinates, putting $\mathrm{d}s' = \mathrm{d}s$, and introducing the values of coefficients g_{ij} given above, it follows that:

$$\mathrm{d}\,t' = \left[1 - \sum_{\alpha=1}^{3} \left(\mathrm{d}\,x^{\alpha}\right)^2 / \left(c\,\mathrm{d}\,t\right)^2\right]^{\frac{1}{2}} \mathrm{d}\,t, \qquad\qquad (A.9.3)$$

and hence $\mathrm{d}\,t' < \mathrm{d}\,t$.

As already shown, it is to be noted, that in this theory, the length in three-dimensional space cannot be determined simply by putting $\mathrm{d}\,x^0 = 0$, if the quantities $g_{\alpha\beta}$ depend on x^0 at any point of the space. For this reason, in the General Theory of Relativity, the concept of distance has not, always, the known meaning.

In the General Theory of Relativity, there is a very important difference between the electromagnetic field and the gravitational field. The first appears as *an external manifestation* superposed to a *four-space continuum*, in other words, the Maxwell equations are considered, but they are expressed in a four-space continuum that is the *Galilean space-time continuum*. The second appears considered as *an inherent manifestation* of the four-space continuum that is the *Riemannian space-time continuum*. The corresponding co-ordinates are also called Gaussian co-ordinates, [6, p. 364].

In the case of a material point, in a reference frame, in absence of any other body till a distance great enough, the point will describe with respect to the reference frame a curve expressed by a straight line. Therefore, the respective continuum is considered as a quasi-Euclidean continuum (space-time) and gravitation does not exist. In the case of the presence of other bodies, the point will describe any curve, but no more a straight line. Therefore, the respective continuum is considered as a non-Euclidean continuum (space-time), and a curved space-time continuum and gravitation exists.

A.9.3. CALCULATION OF DISTANCE AND DURATION BY USING THE METRICS OF THE FOUR-DIMENSIONAL SPACE-TIME CONTINUUM

In the General Theory of Relativity, in any gravitational field, the proper time recorded by a clock attached to a moving body is related to the time co-ordinate x^0, in a certain manner depending on the position of the points in space. We shall mention two problems that can occur: **1°** The calculation of the duration, i.e., the time interval, between events which take place at a point with given space co-ordinates. **2°** The duration (time interval) between two events which take place at two points. Each of these cases requires a different computational formula.

Case 1

In the case in which the time interval between two events, which take place at the same point, has to be calculated, it is possible to use relation (A.6.19 b), and to introduce $\mathrm{d}x^{\alpha} = 0$, where the superscript index takes all values from 1 to 3. It follows that:

$$\Delta t = \frac{1}{c}\, g_{00}^{1/2}\, \Delta x^0\,, \qquad\qquad (A.9.4)$$

because $\mathrm{d}\,s = c\,\Delta t$.

Case 2

In order to examine this case, it is necessary to define the synchronization of clocks. We shall present two definitions [5, p. 59], [7, p. 313].

A first definition will be given as follows. We shall consider two points in space, at any distance r from each other. Let us consider that a light beam (signal) is emitted from point A at the moment t_1. The light beam will reach the point B, where it will be simultaneously reflected back to point A that it reaches at moment t_2. The clock of point B will be considered synchronized with the clock of point A if at the moment at which the light beam reaches point B, the clock of this point will record the time $t_0 = \dfrac{1}{2}(t_1 + t_2)$.

Another definition will be explained as follows. We shall consider two points infinitesimally neighbouring each other. The clocks attached to these points will record certain times. Let us consider that a light beam is emitted from the point A at any moment and then it reaches the point B, where it is reflected back to point A. Further on, the following symbols will be used. The space co-ordinates of points A and B will be $x^\alpha + \Delta x^\alpha$ and x^α, respectively. The time co-ordinate at which the light beam reaches and leaves, simultaneously, the point B will be x^0.

At the moment of time co-ordinate x^0, at which the light beam reaches and leaves, simultaneously, the point B, the space-time interval, at that point, has to be zero (for both direction of the light beam), because both events (arrival and departure of the light beam) occur at the same moment and at the same point. Therefore, we have $\mathrm{d}s = 0$, and:

$$g_{00}\left(\Delta x^0\right)^2 + 2\,g_{0\alpha}\,\Delta x^0\,\Delta x^\alpha + g_{\alpha\beta}\,\Delta x^\alpha\,\Delta x^\beta = 0\,, \tag{A.9.5}$$

where the unknown is Δx_0, and the Greek indices take all values from 1 to 3. The roots of this equation are given by the relation:

$$\left(\Delta x^0\right)_{1,\,2} = \frac{1}{g_{00}}\left\{-g_{0\alpha}\,\Delta x^\alpha \mp \left[\left(g_{0\alpha}\,\Delta x^\alpha\right)^2 - g_{00}\,g_{\alpha\beta}\,\Delta x^\alpha\,\Delta x^\beta\right]^{\frac{1}{2}}\right\}\,, \tag{A.9.6}$$

where both solutions are acceptable. However, it is justified to choose the former (with the sign minus) for the arriving beam, and the latter, for the leaving beam, respectively. Indeed, the latter value represents the duration of the light travel, after reflection, hence will give a later (ulterior) moment which should be greater.

With the symbols above, the moments at which the light beam is emitted from point A and returns to it correspond to the time co-ordinates $x^0 + \left(\Delta x^0\right)_1$ and $x^0 + \left(\Delta x^0\right)_2$, respectively.

The clock of point A is considered synchronized with the clock of point B, if at the moment at which the light beam reaches point B, the clock of point A will record the time of the middle of the interval between the emission and the return of the light beam from

and to the point A. This time will be $x_0 + \Delta x_0$, where Δx_0 is the mean value of the two quantities $(\Delta x_0)_1$ and $(\Delta x_0)_2$. The mean value will be given by the relation:

$$\Delta x^0 = \frac{-g_{0\alpha} \Delta x^\alpha}{g_{00}}.$$ (A.9.7)

The procedure by which the two clocks are brought to record the mentioned times is referred to as *synchronization*. The procedure can be extended from a point to another infinitesimally neighbouring point. The synchronization of clocks can be achieved, in principle, by light signal tests, and practically by emission and receiving (and recording) of electromagnetic signals, what can be done by a radar station. The total time co-ordinate difference can be obtained by integrating along any curve in three-dimensional space. The proper time is obtained by dividing the result by the velocity c of light *in vacuo*.

Since the path travelled by the light beam is the double of the distance between the two points, it follows that the corresponding space length, i.e., in three-dimensional space, between the two points, in the examined case, results from the relation:

$$\Delta l = \frac{1}{2}\left[(\Delta x^0)_2 - (\Delta x^0)_1\right] = c\,\Delta t = \frac{1}{g_{00}}\left[(g_{0\alpha} \Delta x^\alpha)^2 - g_{00}g_{\alpha\beta} \Delta x^\alpha \Delta x^\beta\right]^{\frac{1}{2}}.$$ (A.9.8)

By considering relations (A.9.7) and (A.9.8), it follows that the time recorded by a synchronized clock, at any point of the reference frame, will have two components: $\Delta l / c$ and $\Delta x^0 / c$.

It is to be noted, that in this theory, the length in three-dimensional space cannot be determined simply by putting $\mathrm{d}\,x^0 = 0$, if the quantities $g_{\alpha\beta}$ depend on x^0 at any point of the space. For this reason, in the General Theory of Relativity, the concept of distance has not, always, the known meaning.

Using the preceding formulae expressing the time, and the expression of the metrics of a four-dimensional continuum, we have shown, in [2], that, as below, the equations of the General Theory of Relativity are in good agreement with the Sagnac effect.

A.9.4.   REFERENCE FRAME IN UNIFORM ROTATION MOTION AND THE SAGNAC EFFECT

Let us consider a fixed (immobile) space reference frame with three co-ordinates. At the same time, we shall consider a reference frame rotating about one of its axes, with respect to the previous one. That is just the configuration that appears in the experiments of the Sagnac type [1], [5, p. 190]. This type of problem has been partially analysed in certain papers, among which [6, p. 346] and [7, p. 336], however, without complete calculations, and with no reference to the Sagnac effect.

A light beam moving in the trigonometric (i.e., counter-clockwise) sense, along the circumference of a circle of radius r, will be considered in the plane $z = 0$.

In both mentioned reference frames, a cylindrical system of co-ordinates will be used for determining the position of any point in the three-dimensional space. Therefore the

space co-ordinates, in the fixed reference frame, are: r_f, φ_f, z_f, and in the rotating reference frame: r, φ, z. The moving reference frame is rotating counter-clockwise, and the angle is also counted counter-clockwise (trigonometric sense). The time is considered to be measured by a clock fixed at a point of the configuration vertical axis, the linear velocity being there zero. The time relations between these systems of co-ordinates are:

$$r_f = r; \quad \varphi_f = \varphi + \omega t; \quad z_f = z, \tag{A.9.9}$$

where ω is the constant angular velocity of the moving reference frame.

If the sense of rotation of the light beam is changed, the situation will be the same as if the light beam keeps its initial sense and the angular velocity ω is of opposite sense.

The expression of the space-time interval in the fixed reference frame is:

$$\mathrm{d}s^2 = c^2 \left(\mathrm{d}t\right)^2 - \left(\mathrm{d}r_f\right)^2 - r_f^2 \left(\mathrm{d}\varphi_f\right)^2 - \left(\mathrm{d}z_f\right)^2. \tag{A.9.10}$$

The expression of the space-time interval in the rotating reference frame is:

$$\left(\mathrm{d}s\right)^2 = c^2 \left(\mathrm{d}t\right)^2 - \left(\mathrm{d}r\right)^2 - r^2 \left(\mathrm{d}\varphi\right)^2 - 2\omega r^2 \,\mathrm{d}\varphi \,\,\mathrm{d}t - \omega^2 r^2 \left(\mathrm{d}t\right)^2 - \left(\mathrm{d}z\right)^2. \tag{A.9.11}$$

Therefore:

$$\left(\mathrm{d}s\right)^2 = \left(c^2 - \omega^2 r^2\right)\left(\mathrm{d}t\right)^2 - \left(\mathrm{d}r\right)^2 - 2\omega r^2 \left(\mathrm{d}\varphi\right)\left(\mathrm{d}t\right) - r^2 \left(\mathrm{d}\varphi\right)^2 - \left(\mathrm{d}z\right)^2, \tag{A.9.12}$$

and:

$$\left(\mathrm{d}s\right)^2 = \frac{1}{c^2}\left(c^2 - \omega^2 r^2\right)\left(\mathrm{d}x^0\right)^2 - \left(\mathrm{d}r\right)^2 - \frac{2}{c}\omega r^2 \left(\mathrm{d}\varphi\right)\left(\mathrm{d}x^0\right) - r^2 \left(\mathrm{d}\varphi\right)^2 - \left(\mathrm{d}z\right)^2. \tag{A.9.13}$$

The 10 coefficients in the expression of $\left(\mathrm{d}s\right)^2$ are:

$$g_{00} = \frac{1}{c^2}\left(c^2 - \omega^2 r^2\right); \quad g_{11} = -1; \quad g_{22} = -r^2; \quad g_{33} = -1;$$

$$g_{01} = 0; \quad g_{02} = -\frac{1}{c}\omega r^2; \quad g_{03} = 0; \quad g_{12} = 0; \quad g_{13} = 0; \quad g_{23} = 0. \tag{A.9.14}$$

The duration for the point of the light beam front to describe a closed path will be calculated from the expression obtained by integrating both sides of formulae (A.9.7) and (A.9.8), and adding up the results. It follows that in this case:

$$\left(\Delta l\right) + \left(\Delta x^0\right)_{\text{total}} =$$

$$= \int\limits_{\substack{x^2=0 \\ x^1=\text{const}}}^{x^2=2\pi} \frac{1}{g_{00}}\left[\left(-g_{00}\,g_{11}\right)\left(dx^1\right)^2 + \left(g_{02}^2 - g_{00}\,g_{22}\right)\left(dx^2\right)^2\right]^{\frac{1}{2}} +$$

$$+ \int\limits_{x_2=0}^{x_2=2\pi} \frac{-g_{02}}{g_{00}}\,dx^2\,.$$

(A.9.15)

After performing the calculations, we get:

$$\left(\Delta l\right) + \left(\Delta x^0\right)_{\text{total}} = \int\limits_0^{2\pi} \frac{r}{1 - \dfrac{\omega^2\,r^2}{c^2}}\,d\varphi + \frac{1}{c}\int\limits_0^{2\pi} \frac{\omega\,r^2}{1 - \dfrac{\omega^2\,r^2}{c^2}}\,d\varphi =$$

$$= 2\,\pi\,\alpha_0^2\,r + \frac{1}{c}\cdot\alpha_0^2\,2\,\pi\,\omega\,r^2 = 2\,\pi\,\alpha_0^2\,r + 2\,\frac{1}{c}\,\alpha_0^2\,\omega\,S_{\text{C}}\,,$$

(A.9.16)

where S_{C} denotes the area of the circle surface, and α_0 will be given in (A.9.17). Therefore, if a light beam travels along the whole circumference of a circle, the time recorded at the end of the circle by the clock of that point, will be:

$$t_{\text{circle}} = \frac{1}{c}\,\alpha_0^2\left(2\,\pi\,r + 2\,\frac{1}{c}\,\omega\,S_{\text{C}}\right);$$

$$\alpha_0 = \frac{1}{\sqrt{1 - \dfrac{v^2}{c^2}}}; \qquad v = \omega\,r\,.$$

(A.9.17)

It can be added that formula (A.9.17) may be derived directly, by solving equation (A.9.5), obtained from relation (A.9.2) considered for the front of the wave (light beam), and hence, taking $ds = 0$. The equation has to be solved with respect to Δx^0, and, after performing all calculations, we shall obtain just the last result.

It is interesting to add, that the coefficient α_0^2 has, in the considered case, the expression of the factor used for calculating the period of any oscillations in an acceleration field [6, p. 383].

From the last relation, it follows that the time taken by a light beam to travel along the whole circumference of the circle, in a reference frame turning with the angular velocity ω, depends on the magnitude of this velocity and on its sense. In the case of the Sagnac effect, two light beams travelling along the circumference in opposite senses are considered. From the last formula, it follows that the difference of the times taken by the two light beams, is given by the relation:

$$t_2 - t_1 = t_S = \alpha_0^2 \, \frac{4 \, S_C \, \omega}{c^2}, \tag{A.9.18}$$

where, like above, S_C denotes the area of the circle surface.

It is to be noted that the last formula slightly differs from formula (10) of paper [1] established without resorting to the Special or General Theory of Relativity.

It is interesting to mention that in the Special Theory of Relativity, the velocity of light *in vacuo* has the known value c, but in the General Theory of Relativity, the velocity of light and bodies can take any value [6, p. 389]. It should be added that it is possible to choose a reference frame so that, in the vicinity of a certain point, the reference frame is a Galilean one. In this reference frame, and within a limited domain, the velocity of light has the known value c.

A.9.5. CONCLUSION

The analysis carried out above, based on the General Theory of Relativity, has permitted to derive the Sagnac formula directly from the invariant expression of the space-time interval.

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APPENDIX 10

DERIVING A GENERAL FORMULA OF THE MAGNETIC FIELD STRENGTH OF A SOLENOID

In this Appendix, we shall present a derivation of the general formula of the magnetic field strength of a solenoid. The bibliographic mentions [1]-[16], specific for this Appendix, are at the end of this Appendix. Also, a list of symbols is given at the end of this Appendix. The derivation was given by the author in previous papers presented in references.

A.10.1. THE ANALYSE OF THE MAGNETIC FIELD STRENGTH OF A SOLENOID

Coils having the form of a solenoid are widely used in many electromagnetic applications [1]-[11]. The calculation of the magnetic field strength produced by solenoids carrying an electric current are of interest in many cases aiming at practical purposes as well as theoretical ones. For this reason, several thorough studies have been devoted to this subject. Most of them concern the case of solenoids having the form of a right cylinder with circular cross-section, considered in vacuum medium. Among the various studies dealing with this subject, we can note [2], [3], [6]. In work [2], very detailed analyses for various solenoids are presented. At the same time, in the same work, various methods for the calculation of magnetic fields are presented: the Biot-Savart-Laplace formula [2, p. 35, 99], magnetic systems of dipoles associated to electric currents and magnetized bodies [2, pp. 14-21, 99-101, 137]. Analytical solutions for the case of circular cylindrical solenoids are presented. In those solutions, of high accuracy, elliptic functions of the first, second, and third kind are used. In work [2], and paper [5], fictitious magnetic charges have been used. As mentioned already by Maxwell [1], it is an artificial method. In the mentioned papers, especially the case of a solenoid having a circular cross-section has been taken into consideration. In certain treatises and textbooks, like [2], [4], [7], the expression of the magnetic field strength at a point of the solenoid axis is given in closed form.

In the present paper, we shall give a new derivation of a general expression of the magnetic field strength of a solenoid. We shall start, also, from the Biot-Savart-Laplace formula, and we shall use only the relations from Vector Analysis, in order to obtain a general expression, and at the same time, to highlight the components of the resultant magnetic field strength without resorting to the general theory of expressing a vector starting from the values of its divergence and curl, respectively.

The SI International System of units of measure has been used.

A.10.2. AN EXPRESSION OF THE MAGNETIC FIELD STRENGTH PRODUCED BY A SOLENOID STARTING FROM THE BIOT-SAVART-LAPLACE FORMULA

In order to fix the ideas, we shall consider, as shown in Fig. A.10.1, a solenoid with the symbols given in the figure. For the simplicity of the figure, we shall consider a solenoid of a circular or elliptic cross-section. This circumstance will not restrict the generality of the solution because all the relations will have a general use. The cylinder bases are considered to be perpendicular to its generatrix.

The current sheet round the cylinder will be considered infinitely thin and constant with time. The medium is assumed as vacuum. Under the preceding assumptions, the magnetic field strength produced by the electric current carried by the solenoid can be calculated by the Biot-Savart-Laplace formula, (2.48), as follows:

$$\boldsymbol{H} = \int_{\Gamma} \frac{i}{4\pi} \cdot \frac{\mathrm{d}\boldsymbol{l} \times \boldsymbol{r}}{r^3}, \tag{A.10.1}$$

with the usual symbols.

The following symbols are used: Γ – helical curve, a very closed one, considered as the mean curve of a very thin and narrow strip, which forms the lateral surface of the cylinder; i – intensity of the electric current carried by the path corresponding to that curve; $\boldsymbol{r}$ – position vector having its origin at any source point denoted by P, and its arrow end at the field (observation) point denoted by N; J_{0L} – linear current density of the current sheet; $\mathrm{d}S = \mathrm{d}h\,\mathrm{d}l$ – area of an element of the lateral surface of the cylinder; Δh and $\mathrm{d}h$ – breadth of the strip, and element of cylinder height; Δl and $\mathrm{d}l$ – element of length along the circumference.

With the symbols above, the link between the vector representing an element of the lateral surface of the cylinder, and the current carried by the strip represented by curve Γ, can be expressed as follows:
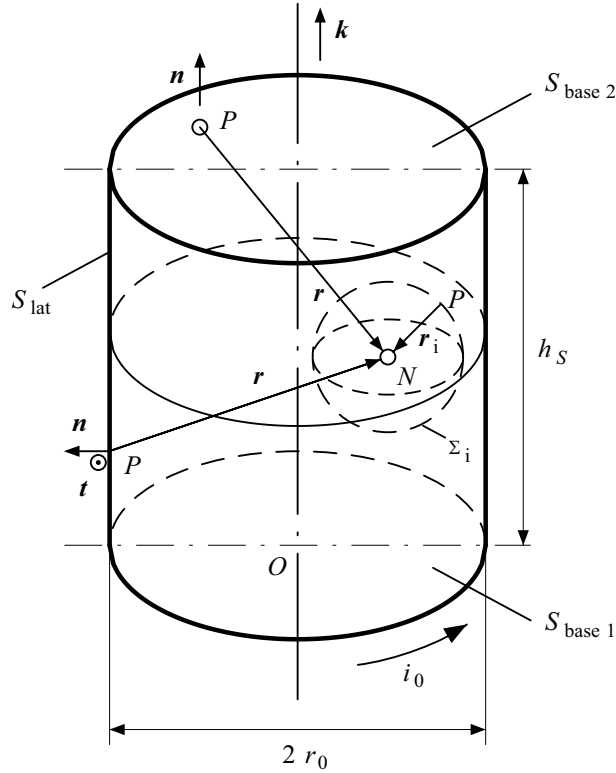
Fig. A.10.1. Solenoid of finite length carrying a current sheet: P – source point, running point, in three positions; N – field point; S_{lat} – lateral surface of the solenoid cylinder; $S_{base\ 1}$, $S_{base\ 2}$, – surfaces of both bases; Σ_i – small sphere surface surrounding the field point. The field point is inside the cylinder but it may be also outside.

$$\Delta S = (\Delta h)(\Delta l); \quad \Delta \boldsymbol{l} = \boldsymbol{t}\,\Delta l = (\boldsymbol{k} \times \boldsymbol{n})\Delta l; \quad i = J_{0L}\,\Delta h; \qquad \text{(A.10.2)}$$

$$\mathrm{d}\boldsymbol{S} = \boldsymbol{n}\,\mathrm{d}S. \qquad \text{(A.10.2 a)}$$

The relations can be written considering relation (A.10.2 a), and $\mathrm{d}S$ would be shifted at the end of the expression, keeping at the starting place only vector $\boldsymbol{n}$, but for conciseness reason, we have used in most cases only the vector $\mathrm{d}\boldsymbol{S}$.

With the symbols above, relations (A.10.1) and (A.10.2), yield:

$$\boldsymbol{H} = \int\limits_{S_{lat}} \frac{J_{0L}\left[(\boldsymbol{k} \times \boldsymbol{n}) \times \boldsymbol{r}\right]\mathrm{d}S}{4\pi r^3}. \qquad \text{(A.10.3)}$$

The last integral, changing the order of factors and modifying the surface over which it is extended, can be written as follows:

$$\boldsymbol{H} = -J_{0L} \frac{1}{4\pi} \int\limits_{\Sigma_{\mathrm{cyl}}} \frac{[\boldsymbol{r} \times (\boldsymbol{k} \times \boldsymbol{n})]\,\mathrm{d}S}{r^3},\tag{A.10.4}$$

where Σ_{cyl} represents the closed surface of the cylinder, i.e., the lateral surface, Σ_{lat}, unified with the surface of each of both bases, Σ_{bases}, and $\boldsymbol{n}$ denotes the unit normal at any point of that surface. In the last formula, $\mathrm{d}S$ represents the surface element around any point of the cylinder surface, because, on the bases of the cylinder, the numerator of the integrand is zero, hence the results given by relations (A.10.3) and (A.10.4) will not differ from each other.

Let us consider the integrand of the integral (A.10.4) as composed of two factors: $\dfrac{\boldsymbol{r}}{r^3}$ and $(\boldsymbol{k} \times \boldsymbol{n})$. Then, taking into account the expansion of any term of the form $\mathrm{curl}[\boldsymbol{a}\,\varphi(r)]$, where $\boldsymbol{a}$ is a constant vector, the integral can be written using (A.1.71):

$$\boldsymbol{H} = J_{0L} \frac{1}{4\pi} \int\limits_{\Sigma_{\mathrm{cyl}}} \mathrm{curl}\left(\frac{\boldsymbol{k} \times \boldsymbol{n}}{r}\right)\mathrm{d}S = J_{0L} \frac{1}{4\pi} \int\limits_{\Sigma_{\mathrm{cyl}}} \mathrm{curl}\left(\boldsymbol{k} \times \frac{\mathrm{d}\boldsymbol{S}}{r}\right),\tag{A.10.5}$$

where the curl is calculated at the field point (observation point) N, which is not singular, whereas the surface element is at the source point P.

A.10.3.   CALCULATION OF THE INTEGRAL BY USING VECTOR ANALYSIS TRANSFORMATIONS

We want to calculate the integral:

$$\boldsymbol{I} = \int\limits_{\Sigma_{\mathrm{cyl}}} \mathrm{curl}\left(\frac{\boldsymbol{k} \times \mathrm{d}\boldsymbol{S}}{r}\right),\tag{A.10.6}$$

$$\mathrm{d}\boldsymbol{S} = \boldsymbol{n}\,\mathrm{d}S.\tag{A.10.6 a}$$

Having in view the form of relation (A.10.6), and several known examples, one could suppose that the calculation of expression (A.10.6) would be easier performed if one uses the vector potential. However, this procedure does not lead, in this case, to a favourable result. That is why we shall use a new procedure, more general, and simpler than those we have previously presented in [9] and [10].

Unlike to the case of relation (A.10.4), in relation (A.10.6), we shall consider the vector quantity the curl of which has to be calculated, as composed of two factors: $\boldsymbol{k}$ and $\dfrac{\boldsymbol{n}}{r}$. Taking into account that the first factor is constant; we shall obtain using (A.1.76):

$$\operatorname{curl}\left(\frac{\boldsymbol{k}\times\boldsymbol{n}}{r}\right)=\boldsymbol{k}\operatorname{div}\frac{\boldsymbol{n}}{r}-(\boldsymbol{k}\cdot\nabla)\frac{\boldsymbol{n}}{r}. \tag{A.10.7}$$

The right-hand side contains a term that appears also in the expression of the gradient of the scalar product, one vector of which, namely $\boldsymbol{k}$, is constant, and by (A.1.74) we get:

$$\operatorname{grad}\left(\boldsymbol{k}\cdot\frac{\boldsymbol{n}}{r}\right)=(\boldsymbol{k}\cdot\nabla)\frac{\boldsymbol{n}}{r}+\boldsymbol{k}\times\operatorname{curl}\left(\frac{\boldsymbol{n}}{r}\right)=(\boldsymbol{k}\cdot\nabla)\frac{\boldsymbol{n}}{r}+\boldsymbol{k}\times\left(\boldsymbol{n}\times\frac{\boldsymbol{r}}{r^3}\right). \tag{A.10.8}$$

Therefore:

$$\operatorname{curl}\left(\boldsymbol{k}\times\frac{\boldsymbol{n}}{r}\right)=-\operatorname{grad}\left(\boldsymbol{k}\cdot\frac{\boldsymbol{n}}{r}\right)+\boldsymbol{k}\operatorname{div}\left(\frac{\boldsymbol{n}}{r}\right)+\boldsymbol{k}\times\left(\boldsymbol{n}\times\frac{\boldsymbol{r}}{r^3}\right). \tag{A.10.9}$$

From relations (A.10.6) and (A.10.9), it follows that:

$$\boldsymbol{I}=\int\limits_{\Sigma_{\text{cyl}}}\left[-\operatorname{grad}\left(\boldsymbol{k}\cdot\frac{\mathrm{d}\boldsymbol{S}}{r}\right)+\boldsymbol{k}\operatorname{div}\frac{\mathrm{d}\boldsymbol{S}}{r}+\boldsymbol{k}\times\left(\mathrm{d}\boldsymbol{S}\times\frac{\boldsymbol{r}}{r^3}\right)\right],$$
$$\Sigma_{\text{cyl}}=S_{\text{lat}}\cup S_{\text{bases}}. \tag{A.10.10}$$

We shall calculate the component produced by each of the three terms contained by the integral.

The first term yields according to (A.1.169):

$$\boldsymbol{I}_1=-\int\limits_{\Sigma_{\text{cyl}}}\operatorname{grad}\left(\boldsymbol{k}\cdot\frac{\mathrm{d}\boldsymbol{S}}{r}\right)=\int\limits_{\Sigma_{\text{cyl}}}\frac{\boldsymbol{r}}{r^3}(\boldsymbol{k}\cdot\mathrm{d}\boldsymbol{S})=\int\limits_{S_{\text{lat}}}\frac{\boldsymbol{r}}{r^3}(\boldsymbol{k}\cdot\mathrm{d}\boldsymbol{S})+\int\limits_{S_{\text{bases}}}\frac{\boldsymbol{r}}{r^3}(\boldsymbol{k}\cdot\mathrm{d}\boldsymbol{S}). \tag{A.10.11}$$

In the integral extended over the lateral surface of the cylinder, the scalar product $\boldsymbol{k}\cdot\mathrm{d}\boldsymbol{S}$ is zero because the two vectors are, at any point of this surface, perpendicular to each other. Hence:

$$\boldsymbol{I}_1=\int\limits_{S_{\text{bases}}}\frac{\boldsymbol{r}}{r^3}(\boldsymbol{k}\cdot\mathrm{d}\boldsymbol{S}). \tag{A.10.12}$$

The second term yields according to (A.1.70):

$$\boldsymbol{I}_2=\int\limits_{\Sigma_{\text{cyl}}}\boldsymbol{k}\operatorname{div}\left(\frac{\mathrm{d}\boldsymbol{S}}{r}\right)=-\boldsymbol{k}\int\limits_{\Sigma_{\text{cyl}}}\frac{\boldsymbol{r}\cdot\mathrm{d}\boldsymbol{S}}{r^3}. \tag{A.10.13}$$

In order to simplify the calculation, we shall apply the theorem of Gauss-Ostrogradski for obtaining the transformation of a surface integral into a volume one. For applying the usual formulae containing differential operators, taking into account that the point N is fixed, we shall consider, when performing the calculation, the vector $\boldsymbol{r}'=-\boldsymbol{r}$, and after

the calculation has been performed, we shall return to vector $\boldsymbol{r}$. However, for conciseness reason, we shall not write this intermediary transformation.

In order to ensure the applicability of the mentioned theorem, we shall first separate, inside the cylinder, the singular point, that is the point corresponding to $\boldsymbol{r} = 0$, by surrounding it with a sphere, having the centre at that point, and small enough, so that it does not touch the cylinder. Then, we have to calculate the integral:

$$-\int_{\Sigma} \frac{\boldsymbol{r} \cdot \mathrm{d}\boldsymbol{S}}{r^3} = -\int_{\Sigma_{\text{cyl}}} \frac{\boldsymbol{r} \cdot \mathrm{d}\boldsymbol{S}}{r^3} - \int_{\Sigma_{\text{i}}} \frac{\boldsymbol{r} \cdot \mathrm{d}\boldsymbol{S}}{r^3}, \quad \Sigma = \Sigma_{\text{cyl}} \cup \Sigma_{\text{i}},$$

(A.10.14)

where Σ_{i} represents the surface of the sphere, which surrounds the singular point above.

The above mentioned theorem yields:

$$-\int_{\Sigma} \frac{\boldsymbol{r} \cdot \mathrm{d}\boldsymbol{S}}{r^3} = -\int_{V_{\Sigma}} \operatorname{div} \frac{\boldsymbol{r}}{r^3} \, \mathrm{d}v = 0, \quad \Sigma = \Sigma_{\text{cyl}} \cup \Sigma_{\text{i}}.$$

(A.10.15)

For the sphere surface, the unit normal is oriented, as in general, outwards the considered volume, that is, towards the sphere centre, as $\boldsymbol{r}_{\text{i}}$ in A.10.1. Therefore, relations (A.10.14) and (A.10.15) yield:

$$-\int_{\Sigma_{\text{cyl}}} \frac{\boldsymbol{r} \cdot \mathrm{d}\boldsymbol{S}}{r^3} = \int_{\Sigma_{\text{i}}} \frac{\boldsymbol{r} \cdot \mathrm{d}\boldsymbol{S}}{r^3}.$$

(A.10.16)

The relations (A.10.13) and (A.10.16) yield:

$$\boldsymbol{I}_2 = \boldsymbol{k} \int_{\Sigma_i} \frac{\boldsymbol{r} \cdot \mathrm{d}\boldsymbol{S}}{r^3} = \boldsymbol{k} \int_{\Sigma_i} \frac{\boldsymbol{r}_{\text{i}} \cdot \boldsymbol{r}_{\text{i}}}{r_{\text{i}}^4} \, \mathrm{d}S = \boldsymbol{k} \, 4\pi.$$

(A.10.17)

If the point N is outside the cylinder, the term $\boldsymbol{I}_2$ is zero.

The third term yields:

$$\boldsymbol{I}_3 = \int_{\Sigma_{\text{cyl}}} \boldsymbol{k} \times \left(\mathrm{d}\boldsymbol{S} \times \frac{\boldsymbol{r}}{r^3} \right).$$

(A.10.18)

It will be calculated similarly as the second one. Using the same definition of Σ as in relation (A.10.14), we have:

$$\int_{\Sigma} k \times \left(dS \times \frac{r}{r^3} \right) = \int_{\Sigma_{cyl}} k \times \left(dS \times \frac{r}{r^3} \right) + \int_{\Sigma_i} k \times \left(dS \times \frac{r}{r^3} \right). \tag{A.10.19}$$

It follows that we have to calculate:

$$\int_{\Sigma} dS \times \frac{r}{r^3} = \int_{\Sigma_{cyl}} dS \times \frac{r}{r^3} + \int_{\Sigma_i} dS \times \frac{r}{r^3}. \tag{A.10.20}$$

We have, using (A.1.92):

$$\int_{\Sigma} dS \times \frac{r}{r^3} = \int_{V_\Sigma} \text{curl}\left(\frac{r}{r^3} \right) dv = 0 \tag{A.10.21}$$

and

$$\int_{\Sigma_i} dS \times \frac{r}{r^3} = \int_{\Sigma_i} \frac{r_i}{r_i} \times \frac{r_i}{r^3} dS = 0. \tag{A.10.22}$$

Therefore:

$$I_3 = 0. \tag{A.10.23}$$

It results that:

$$I = I_1 + I_2 + I_3. \tag{A.10.24}$$

We obtain:

$$I = k\, 4\pi + \int_{S_{bases}} \frac{r}{r^3} (k \cdot dS). \tag{A.10.25}$$

A.10.4. COMPUTING THE EXPRESSION OF THE MAGNETIC FIELD STRENGTH

From relations (A.10.5), (A.10.6), (A.10.25), it results for the case of the solenoid an expression with separated components:

$$H = k\, J_{0L} + \frac{1}{4\pi} J_{0L} \int_{S_{bases}} \frac{r}{r^3} k \cdot d S. \tag{A.10.26}$$

Therefore:

$$\boldsymbol{H}(N) = m\,\boldsymbol{k}\,J_{0L} + \frac{1}{4\pi}J_{0L}\int\limits_{S_{\text{bases}}}\frac{\boldsymbol{r}}{r^3}\boldsymbol{k}\cdot\mathrm{d}\boldsymbol{S};$$

$$J_{0L} = \frac{i_0}{h_S};\quad m=1,\quad \forall\,N\in V_{\text{cyl}}\backslash\Sigma_{\text{cyl}};\quad m=0,\quad \forall\,N\notin V_{\text{cyl}},$$

(A.10.27)

where the following symbols have been used: V_{cyl} – volume domain occupied by the cylinder; Σ_{cyl} – cylinder surface; S_{bases} – surface of each cylinder base; i_0 – total current sheet; J_{0L} – linear current density; $\boldsymbol{k}$ – unit vector along the cylinder generatrix; $\boldsymbol{r}$ – position vector of observation (field) point N with respect to the origin (source) point P. Therefore: if the point N is outside the cylinder, the first term of the right-hand side is zero, and if it is inside, it is different from zero.

The last formula emphasizes that, generally, the expression for any observation (field) point inside the cylinder has two components, the latter of Coulombian type and the former of non-Coulombian type. If the observation point is outside the cylinder, the former term vanishes, while the latter subsists.

It is interesting to emphasize that the last relation permits to establish a known general relation. If we change the symbols and denote the second term by $\boldsymbol{H}(N)$ the Coulombian component, and by $\boldsymbol{M}$ the value of the first non-Coulombian component, we can write:

$$\frac{1}{\mu_0}\boldsymbol{B}(N) = \boldsymbol{M}(N) + \boldsymbol{H}(N),$$

(A.10.28)

where, the quantity $\boldsymbol{H}$ has the character of the magnetic field strength, and the quantity $\boldsymbol{M}$ has the character of the magnetization. In general:

$$\boldsymbol{B} = \mu_0\,\boldsymbol{H} + \mu_0\,\boldsymbol{M},$$

(A.10.29)

or introducing the magnetic polarization:

$$\boldsymbol{B} = \mu_0\,\boldsymbol{H} + \boldsymbol{M}_j.$$

(A.10.30)

If any other magnetic field, produced by other causes for instance electric currents, exists it has to be included into the vector $\boldsymbol{H}$. Therefore, in such cases, the magnetic field strength may be of non-Coulombian type.

It follows that formula (A.10.27) allows for deducing the relation between the magnetic field strength, polarization and induction, in the general case.

If any substance contains a large number of elementary solenoids, the relations above are valid for the quantities $\boldsymbol{H}, \boldsymbol{M}, \boldsymbol{B}$ of every solenoid and also for the average of these quantities over any physically infinitesimal interval of space and time. We have, hence, found anew, simpler and in a more general way a result of paper [9] and [10].

A.10.5. CALCULATION OF THE MAGNETIC FIELD STRENGTH PRODUCED BY A CIRCULAR CROSS-SECTION SOLENOID BY USING DIRECTLY THE BIOT-SAVART-LAPLACE FORMULA

In order to verify the accuracy of formulae above, we shall apply formula (A.10.3) for calculating the magnetic field strength of a right circular cylinder. We shall consider the configuration of Fig. A.10.1, and use formula (A.10.3). It will be recalled that P is the source point, whereas N is the observation (field) point. For applying the formula, we shall consider a slice of the cylinder, of a very small height Δz. Let x_N, y_N, z_N be the co-ordinates of point N, and, z_{b1} and z_{b2}, the z-co-ordinates of the two bases of the cylinder. The slice will be taken at any co-ordinate z. For simplicity, without restricting the generality, we shall take $y_N = 0$.

The projection of the considered slice on the plane xOy is shown in Fig. A.10.2.

The product $\Delta l \times r$ will be expressed by using the polar co-ordinates ρ and θ where, in this case, $\rho = r_0$. It follows:
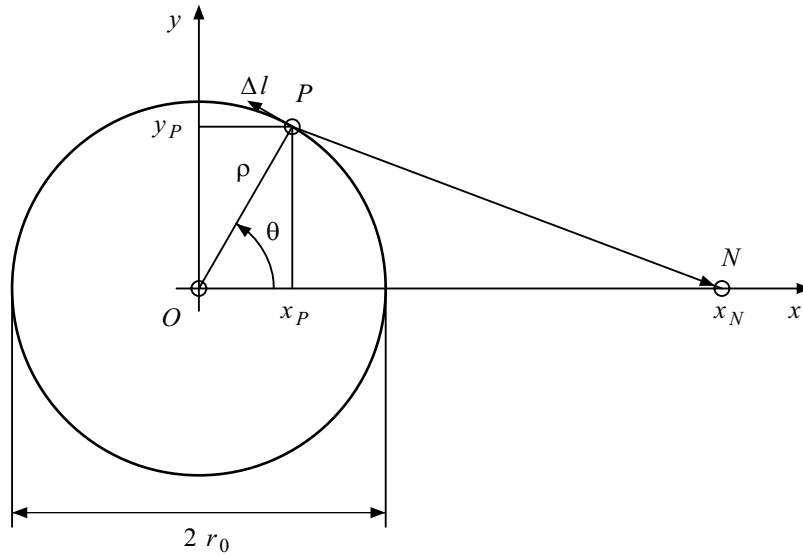
Fig. A.10.2. The projection of the circumference of a cross-section
of the solenoid cylinder on the xOy - plane: The representation of the projection of
the position vector PN using polar co-ordinates. The segment OP belongs to the
plane of the circle.

$$\Delta l = i\left(-r_0\,\Delta\theta\right)\sin\theta + j\left(r_0\,\Delta\theta\right)\cos\theta + k\,0, \tag{A.10.31}$$

$$r = i\left(x_N - r_0\cos\theta\right) + j\left(-r_0\sin\theta\right) + k\left(z_N - z\right), \tag{A.10.32}$$

$$\Delta \boldsymbol{l} \times \boldsymbol{r} = \boldsymbol{i}\left(z_N - z\right)r_0 \cos\theta\, \Delta\theta + \boldsymbol{j}\left(z_N - z\right)r_0 \sin\theta\, \Delta\theta + \boldsymbol{k}\left(r_0 - x_N \cos\theta\right)r_0\, \Delta\theta. \qquad \text{(A.10.33)}$$

The utilization of formula (A.10.31) results in calculating a double integral with respect to variables z and θ, for each of the three axes of Cartesian co-ordinates.

Taking into account that, for reasons of symmetry, H_y is zero, it results that:

$$H_x = 2f \int_0^\pi I_x(\theta)\,\mathrm{d}\theta; \quad H_y = 0; \quad H_z = 2f \int_0^\pi I_z(\theta)\,\mathrm{d}\theta; \quad f = \frac{1}{4\pi} J_{0L}\, r_0, \qquad \text{(A.10.34 a,..,d)}$$

where, in order to shorten the explanations, we shall use the following symbols:

$$I_x(\theta) = \int_{z_{b1}}^{z_{b2}} \frac{\left(z_N - z\right)\cos\theta}{r^3}\,\mathrm{d}z; \quad I_z(\theta) = \int_{z_{b1}}^{z_{b2}} \frac{r_0 - x_N \cos\theta}{r^3}\,\mathrm{d}z; \qquad \text{(A.10.35)}$$

and

$$r = \left[x_N^2 + r_0^2 - 2x_N\, r_0 \cos\theta + \left(z_N - z\right)^2\right]^{\frac{1}{2}}. \qquad \text{(A.10.36)}$$

We shall denote:

$$F = F(z) = r^2 = a + c\left(z_N - z\right)^2. \qquad \text{(A.10.37)}$$

For the calculation of expressions (A.10.35 a, b), we can use one of the procedures of Sub-Section A.10.6.1. For this purpose, we shall denote:

$$a = x_N^2 - 2x_N\, r_0 \cos\theta + r_0^2; \quad b = 0; \quad c = 1;$$
$$D = 4ac = 4a; \quad x = z_N - z; \quad \mathrm{d}x = -\mathrm{d}\left(z_N - z\right). \qquad \text{(A.10.38 a-f)}$$

We get:

$$I_x(\theta) = \left[\frac{1}{\sqrt{F}}\right]_{z_{b1}}^{z_{b2}} \cos\theta, \qquad \text{(A.10.39)}$$

$$I_z(\theta) = -\left[\frac{4\left(z_N - z\right)}{D\sqrt{F}}\right]_{z_{b1}}^{z_{b2}} \left(r_0 - x_N \cos\theta\right). \qquad \text{(A.10.40)}$$

Then, we shall calculate the integrals (A.10.34 a, c) above, with respect to the variable θ. This calculation has been carried out by a numerical method, using the Gauss integration method, by a procedure with 32-point ordinates.

A.10.6. CALCULATION OF THE MAGNETIC FIELD STRENGTH PRODUCED BY A CIRCULAR CROSS-SECTION SOLENOID BY USING THE FORMULA WITH SEPARATE COMPONENTS

In order to verify the accuracy of formulae above, we shall apply formula (A.10.27) for calculating the magnetic field strength of a right circular cylinder. We shall consider the configuration of Fig. A.10.1, and use formula (A.10.27). We shall consider, the configuration of Fig. A.10.3. It will be recalled that, as previously, P is the source point, whereas N is the observation (field) point.
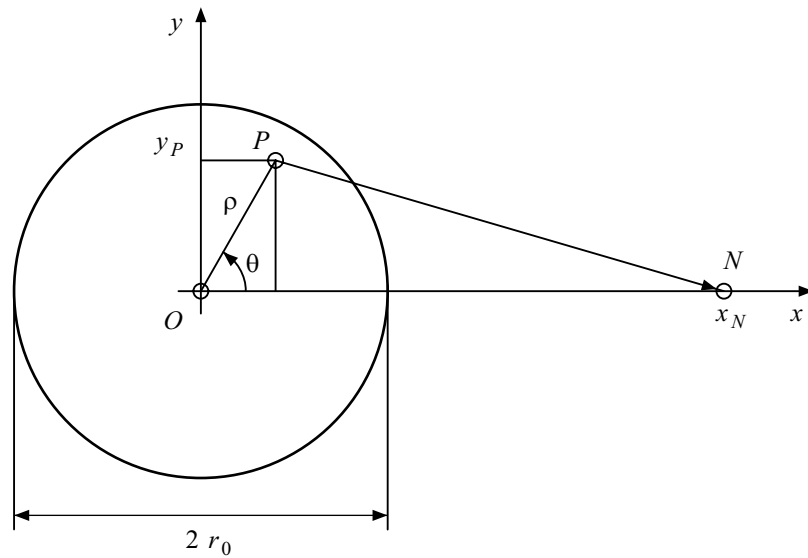
Fig. A.10.3. The projection of a thin slice of the end of the solenoid cylinder on the xOy - plane: The representation of the projection of the position vector PN using polar co-ordinates. The segment OP belongs to the plane of the circle.

Let, as above, x_N, y_N, z_N be the co-ordinates of point N, and, z_{b1} and z_{b2}, be the z-co-ordinates of the two bases of the cylinder. For simplicity, without restricting the generality, we shall take $y_N = 0$. For describing the position of a point on each of the two bases of the solenoid cylinder, polar co-ordinates ρ and θ will be used, as in Fig. A.10.3. The vector oriented from point P to point N will be:

$$\boldsymbol{r} = \boldsymbol{i}\left(x_N - \rho\cos\theta\right) + \boldsymbol{j}\left(-\rho\sin\theta\right) + \boldsymbol{k}\left(z_N - z\right). \tag{A.10.41}$$

The calculation by formula (A.10.27) results in calculating a double integral with respect to variables ρ and θ, for each of the three axes of Cartesian co-ordinates.

Taking into account that, for reasons of symmetry, H_y is zero, it results that:

$$H_x = 2f \int_0^\pi I_x(\theta) \mathrm{d}\theta; \quad H_y = 0; \quad H_z = 2f \int_0^\pi I_z(\theta) \mathrm{d}\theta; \quad f = \frac{1}{4\pi} J_{0L}; \qquad \text{(A.10.42 a, b, c, d)}$$

where, in order to shorten the explanations, we have used the following symbols:

$$I_x(\theta, z) = \int_0^{r_0} \frac{(x_N - \rho\cos\theta)}{r^3} \rho \, \mathrm{d}\rho; \quad I_z(\theta, z) = \int_0^{r_0} \frac{z_N - z}{r^3} \rho \, \mathrm{d}\rho; \qquad \text{(A.10.43 a, b)}$$

$$I_x(\theta) = I_x(\theta, z_{b2}) - I_x(\theta, z_{b1});$$
$$I_z(\theta) = I_z(\theta, z_{b2}) - I_z(\theta, z_{b1});$$

$$\left[\frac{x_N - \rho\cos\theta}{r^3} \rho \right]_{z_{b1}}^{z_{b2}}; \quad \left[\frac{z_N - z}{r^3} \rho \right]_{z_{b1}}^{z_{b2}};$$

(A.10.44 a, b, c, d)

and the order of the computing operations can be: (A.10.43), (A.10.44), (A.10.42) or (A.10.44), (A.10.43), (A.10.42). We shall denote:

$$r = \left[x_N^2 + (z_N - z)^2 - 2 x_N \rho\cos\theta + \rho^2 \right]^{\frac{1}{2}}, \qquad \text{(A.10.45)}$$

and

$$F = F(\rho) = r^2 = a + b\rho + c\rho^2. \qquad \text{(A.10.46)}$$

For the calculation of expressions (A.10.44 a, b), we shall use one of the procedures of Section A.10.6.1. For this purpose, we shall denote:

$$a = x_N^2 + (z_N - z)^2; \quad b = -2 x_N \cos\theta; \quad c = 1; \qquad \text{(A.10.47 a-e)}$$
$$D = 4ac - b^2; \quad x = \rho.$$

The Coulombian component of the magnetic field strength, as mentioned, is given by the integral of the right-hand side of relation (A.10.27). We obtain:

$$I_x(\theta) = x_N \left[-\frac{2(2a + b\rho)}{D\sqrt{F}} \right]_0^{r_0} -$$

$$-\left[\frac{(-D + b^2)\rho + 2ab}{cD\sqrt{F}} + \ln(2\sqrt{F} + 2\rho + b) \right]_0^{r_0} (-\cos\theta), \qquad \text{(A.10.48 a, b)}$$

$$I_z(\theta) = (z_N - z) \left[-\frac{2(2a + b\rho)}{D\sqrt{F}} \right]_0^{r_0}.$$

Further on, it is necessary to calculate the integrals (A.10.42 a, c) with respect to the variable θ. This calculation has been carried out, as previously, by a numerical method, using the Gauss integration method, by a procedure with 32 ordinates.

A.10.6.1. Calculation of the Integrals of Sections A.10.5 and A.10.6
Calculation of Integrals Using Tables

The integrals occurring above may be calculated either by Computer Symbolic Calculus, using softwares, like Derive, MathCAD, Maple, Mathematica. The author prepared computer programs in Maple which lead to relatively complicate expressions. However, it is possible to use tables of integrals, like [12], [13], which, in this case, completed with some arrangements by hand, lead to symmetrical and much simpler expressions. The functions below and the corresponding indefinite integrals can be used:

$$F = a + bx + cx^2. \qquad \text{(A.10.I)}$$

$$D = 4ac - b^2. \qquad \text{(A10.II)}$$

$$\int \frac{x\,\mathrm{d}x}{F^{\frac{3}{2}}} = -\frac{2(2a + bx)}{DF^{\frac{1}{2}}}. \qquad \text{(A.10.III)}$$

$$\int \frac{\mathrm{d}x}{F^{\frac{3}{2}}} = \frac{2(2cx + b)}{DF^{\frac{1}{2}}}. \qquad \text{(A.10.IV)}$$

$$\int \frac{\mathrm{d}x}{F^{\frac{1}{2}}} = \frac{1}{\sqrt{c}} \ln(2\sqrt{cF} + 2cx + b); \qquad \forall c > 0. \qquad \text{(A.10.V)}$$

$$\int \frac{x^2 \, \mathrm{d}x}{F^{\frac{3}{2}}} = \frac{2\,a\,b + \left(-D + b^2\right)x}{c\,D\,F^{\frac{1}{2}}} + \frac{1}{c\sqrt{c}} \ln\left(2\sqrt{cF} + 2\,c\,x + b\right). \tag{A.10.VI}$$

A.10.7. COMPUTER PROGRAMS AND NUMERICAL RESULTS

For the methods developed in the present paper, we have developed computer programs in Fortran.F77 language. For integration by the Gauss method, the subroutine DQG32 of the IBM Library has been used. The program has been written in double precision, that is, each word has two bytes, i.e., 16 bits, what ensures results of high precision even at points in the vicinity of singular points. The obtained results are given in Table 1. The programs are called SOLENBSL.F77 based on the Biot-Savart-Laplace formula, and SOLENCNC.F77 based on the separated components formula.

If the two types of formulae ensure a good accuracy, they have to give results very close to each other. We have compared the results for various points placed at various positions with respect to the cylinder. The greatest relative deviation has been of the order of magnitude of 10^{-5}.

Table 1

Results obtained by numerical computation of the magnetic field strength by three methods:

1. Biot-Savart-Laplace formula. 2. Formula with the separated Coulombian components.

3. Formula in closed form. Data of the solenoid: $z_{b1} = -1$, $z_{b2} = 1$, $r_0 = 1$, $J_{0L} = 1$.

a. Co-ordinates of the observation point: $x_N = 0$, $y_N = 0$. Component H_z.

Value of z_N and method	-1.01	-1 (-1.0000001)	-0.5	0	0.5	1 (1.0000001)
1	0.4016784 E+00	0.4472135 E+00	0.6396319 E+00	0.7071068 E+00	0.6396319 E+00	0.4472135 E+00
2	0.4016784 E+00	0.4472135 E+00	0.6396319 E+00	0.7071068 E+00	0.6396319 E+00	0.4472135 E+00
3	0.4016784 E+00	0.4472136 E+00	0.6396319 E+00	0.7071068 E+00	0.6396319 E+00	0.4472135 E+00

b. Co-ordinates of the observation point: $x_N = 2$, $y_N = 0$. Component H_z.

Value of z_N and method	-1.5	-1	-0.5	0	0.5	1
1	-0.458168 E-03	-0.224792 E-01	-0.440966 E+00	-0.518527 E+00	-0.440966 E-01	-0.224792 E-01
2	-0.458168 E-03	-0.224972 E-01	-0.440966 E+00	-0.518527 E+00	-0.440966 E-01	-0.224792 E-01

c. Co-ordinates of the observation point: $x_N = 2$, $y_N = 0$. Component H_x.

Value of z_N and method	-1.5	-1	-0.5	0	0.5	1
1	-0.466422 E-01	-0.489889 E-01	-0.305329 E-01	0.000000 E+00	0.305329 E-01	0.489889 E-01
2	-0.466422	-0.489889	-0.305329	0.000000	0.305329	0.489889

The obtained results show a small difference between the two methods, these are due to the rounding error, different for each formula. It has to be noted that the errors are very small even in the neighbourhood of singular points. Afterwards has remade the computation using the Maple 12 software, for the case in which we are not interested in the simplicity of the form of formulae. We could realize certain programming facilities and greatest relative deviation has been of the order of magnitude of less than 10^{-7}. In the table above, there are given the last results, for this reason no deviation appears, which appeared in the foregoing papers of the author. The comparison of the results obtained using by formulae in closed form and numerical procedures, respectively, has shown that, at least, the first seven digits of the mantissa (or significant), the results being written in the form of the table above, coincide.

A.10.8. CALCULATION OF THE MAGNETIC FIELD STRENGTH OF A SOLENOID BY USING FICTITIOUS MAGNETIC CHARGES

We shall consider two space domains of the same form and sizes, both of finite length, containing a cylindrical solenoid and a uniformly magnetically polarized cylinder.

The cylindrical solenoid produces a magnetic field as well as the cylindrical uniformly magnetically polarized body occupying the same domain. According to paper [9], a cylindrical solenoid may be decomposed in a set of elementary solenoids, the neighbouring current sheets of which being of similar directions, their effect will be the same as that of the given solenoid. At the same time, according to the same paper, the cylindrical uniformly magnetized cylinder may be decomposed in a set of elementary magnetic dipoles. If one magnetic dipole is equivalent to one elementary solenoid, then the magnetic field produced by each of them in their neighbourhood will be the same. According to the same paper, at a microscopic scale, the magnetic field strength of elementary dipole and elementary solenoid have different values. Then, if we consider a single cylindrical solenoid and a single magnetized cylinder, the magnetic field strength inside the considered domains will be also different, regardless of the cylinder size, hence microscopically as well as macroscopically. But, the magnetic induction will be the same in both cases. Therefore, we shall consider the magnetic induction of the two domains.

In the case of the magnetically polarized cylinder of finite length, the magnetic field strength may be obtained by using the principle of superposition, like in paper [9]. The considered magnetically polarized cylinder will be considered as belonging to an infinitely long cylinder. We can consider that the infinitely long cylinder consists of three parts: the considered cylinder of finite length, and other two semi-infinite cylinders, each of them situated on one part of the cylinder of finite length, in order to built-up the whole infinitely long cylinder.

The magnetic induction (flux density) at any point inside or outside the cylinder of finite length will have two components:

1. A component, say of type *a*, produced by the infinitely long cylinder.

   Because the effect of magnetic dipoles cancels each other, the volume density of the fictitious magnetic charge will be zero, and the magnetic field strength produced by

the infinite cylinder will be zero, inside as well outside the cylinder, but the magnetic induction will be different from zero inside, and zero outside.

2. A component, say of type d, produced by the two semi-infinite cylinders, each of them situated on one side of the cylinder of finite length, taken with changed sign, because it has to be subtracted from the previous one.

Because the effect of magnetic dipoles cancels each other, the volume density of the fictitious magnetic charge will be zero, and only the fictitious magnetic charge of the base of the semi-infinite cylinder will contribute to the production of a magnetic field.

The component of type a will be:

$$\boldsymbol{B}_{\mathrm{a}}(N) = \boldsymbol{M}_j \, , \qquad\qquad \text{(A.10.49 a)}$$

inside, and

$$\boldsymbol{B}_{\mathrm{a}}(N) = 0 \, , \qquad\qquad \text{(A.10.49 b)}$$

outside the infinite cylinder.

The magnetic induction produced at any point inside as well as outside the cylinder of finite length will be given by the known relation (like in Electrostatics):

$$\boldsymbol{B}_{\mathrm{d}}(N) = \frac{1}{4\pi} \int\limits_{S_{\mathrm{bases}}} \frac{\boldsymbol{r}}{r^3} \boldsymbol{M}_j \cdot \mathrm{d}\boldsymbol{S} \, , \qquad\qquad \text{(A.10.50)}$$

where not minus sign has to be introduced, because the integral limits above refer to the cylinder of finite length, and the normals to the bases of this cylinder are opposite to those on the bases of the semi-infinite length.

By summing up the formulae (A.10.49) and (A.10.50), dividing by the permeability of vacuum both sides, and replacing, the magnetic polarization, according to formulae (14), (18) of [9], we have just formula (A.10.27) above.

This latter deduction has not the generality of the former one, but, in turn, it has the advantage of being suggestive.

A.10.9. CONCLUSION

The presented new method of establishing the magnetic field strength of a solenoid calculating the magnetic field strength produced by a right cylindrical solenoid of finite length and with a cross-section of any form has the advantage of using only the Vector Analysis formulae, without resorting to the method of fictitious magnetic charges or to general formulae of the field theory. The method leads to an expression with separated components. The obtained expression permits the calculation of the magnetic field strength produced by various form of the solenoid, is easier to be applied than other largely used expressions and ensures a high accuracy. Also, it permits to obtain directly the relation between induction (flux density), field strength and polarization of a magnetic field.

Certain results concerning the usage of various softwares have been obtained, what has not been known in other publications.

List of Symbols Used in Appendix 10

$\boldsymbol{B}$	–	macroscopic magnetic induction;
$\mathrm{d}\boldsymbol{l}$	–	line element vector;
$\mathrm{d}\boldsymbol{S}$	–	Surface element vector;
$\boldsymbol{H}$	–	macroscopic magnetic field strength;
h_S	–	length of a cylinder that can be a solenoid or a uniformly magnetized body;
$\boldsymbol{I}$	–	symbol of any vector integral with a certain suffix;
$\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$	–	unit vectors of a three-orthogonal right-handed system of co-ordinates;
i	–	electric current intensity;
i_0	–	total electric current intensity of a solenoid;
J_{0L}	–	linear density of the current sheet of a solenoid;
$\boldsymbol{k}$	–	unit vector of the generatrix of a right cylinder;
$\boldsymbol{M}$	–	magnetization vector;
$\boldsymbol{M}_j$	–	$\boldsymbol{M}_j = \mu_0 \boldsymbol{M}$, magnetic polarization vector;
$\boldsymbol{n}$	–	unit vector of the normal to a surface;
$\boldsymbol{r}$	–	position vector of any observation (field) point P relatively to the origin or source point N;
r_i	–	radius of the interior sphere with the centre at the observation point;
r_0	–	radius of the cylinder;
x, y, z	–	co-ordinates of any source point P;
$x_N,$	–	co-ordinates of any observation point N;
y_N, z_N		
$z_{\mathrm{b1}},$	–	co-ordinate along the Oz-axis of the two bases of the cylinder;
z_{b2}		
S	–	open surface that can have a suffix indicating it;
Γ	–	curve, in most cases, a closed one;
μ	–	magnetic permeability, *in vacuo* it is μ_0;
θ	–	angle in cylindrical co-ordinates;
Σ	–	closed surface that can contain a suffix indicating it.

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SUBJECT INDEX

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