Chapter 6

Work, Kinetic Energy and Potential Energy

6.1 The Important Stuff

6.1.1 Kinetic Energy

For an object with mass m and speed v, the **kinetic energy** is defined as

$$K = \tfrac{1}{2}mv^2 \tag{6.1}$$

Kinetic energy is a scalar (it has magnitude but no direction); it is always a positive number; and it has SI units of $\mathrm{kg \cdot m^2/s^2}$. This new combination of the basic SI units is known as the **joule**:

$$1\,\mathrm{joule} = 1\,\mathrm{J} = 1\,\tfrac{\mathrm{kg \cdot m^2}}{\mathrm{s^2}} \tag{6.2}$$

As we will see, the joule is also the unit of work W and potential energy U. Other energy units often seen are:

$$1\,\mathrm{erg} = 1\,\tfrac{\mathrm{g \cdot cm^2}}{\mathrm{s^2}} = 10^{-7}\,\mathrm{J} \qquad 1\,\mathrm{eV} = 1.60 \times 10^{-19}\,\mathrm{J}$$

6.1.2 Work

When an object moves while a force is being exerted on it, then **work** is being done on the object by the force.

If an object moves through a displacement $\mathbf{d}$ while a *constant* force $\mathbf{F}$ is acting on it, the force does an amount of work equal to

$$W = \mathbf{F} \cdot \mathbf{d} = Fd\cos\phi \tag{6.3}$$

where ϕ is the angle between $\mathbf{d}$ and $\mathbf{F}$.

Work is also a scalar and has units of $1\,\mathrm{N \cdot m}$. But we can see that this is the same as the joule, defined in Eq. 6.2.

Work can be negative; this happens when the angle between force and displacement is larger than 90°. It can also be *zero*; this happens if $\phi = 90°$. To do work, the force must have a component along (or opposite to) the direction of the motion.

If several different (constant) forces act on a mass while it moves though a displacement **d**, then we can talk about the **net work** done by the forces,

$$W_{\rm net} = \mathbf{F}_1 \cdot \mathbf{d} + \mathbf{F}_1 \cdot \mathbf{d} + \mathbf{F}_1 \cdot \mathbf{d} + \ldots \tag{6.4}$$

$$= \left(\sum \mathbf{F}\right) \cdot \mathbf{d} \tag{6.5}$$

$$= \mathbf{F}_{\rm net} \cdot \mathbf{d} \tag{6.6}$$

If the force which acts on the object is *not* constant while the object moves then we must perform an integral (a sum) to find the work done.

Suppose the object moves along a straight line (say, along the x axis, from x_i to x_f) while a force whose x component is $F_x(x)$ acts on it. (That is, we know the force F_x as a function of x.) Then the work done is

$$W = \int_{x_i}^{x_f} F_x(x)\, dx \tag{6.7}$$

Finally, we can give the most general expression for the work done by a force. If an object moves from $\mathbf{r}_i = x_i\mathbf{i} + y_i\mathbf{j} + z_i\mathbf{k}$ to $\mathbf{r}_f = x_f\mathbf{i} + y_f\mathbf{j} + z_f\mathbf{k}$ while a force $\mathbf{F}(\mathbf{r})$ acts on it the work done is:

$$W = \int_{x_i}^{x_f} F_x(\mathbf{r})\, dx + \int_{y_i}^{y_f} F_y(\mathbf{r})\, dy + \int_{z_i}^{z_f} F_z(\mathbf{r})\, dz \tag{6.8}$$

where the integrals are calculated along the path of the object's motion. This expression can be abbreviated as

$$W = \int_{\mathbf{r}_i}^{\mathbf{r}_f} \mathbf{F} \cdot d\mathbf{r} \ . \tag{6.9}$$

This is rather abstract! But most of the problems where we need to calculate the work done by a force will just involve Eqs. 6.3 or 6.7

We're familiar with the force of gravity; gravity does work on objects which move vertically. One can show that if the height of an object has changed by an amount Δy then gravity has done an amount of work equal to

$$W_{\rm grav} = -mg\Delta y \tag{6.10}$$

regardless of the horizontal displacement. Note the minus sign here; if the object increases in height it has moved *oppositely* to the force of gravity.

6.1.3 Spring Force

The most famous example of a force whose value depends on position is the **spring force**, which describes the force exerted on an object by the end of an **ideal spring**. An ideal spring will pull inward on the object attached to its end with a force proportional to the amount by which it is stretched; it will push outward on the object attached to its with a force proportional to amount by which it is compressed.

If we describe the motion of the end of the spring with the coordinate x and put the origin of the x axis at the place where the spring exerts no force (the equilibrium position) then the spring force is given by

$$F_x = -kx \tag{6.11}$$

Here k is **force constant**, a number which is different for each ideal spring and is a measure of its "stiffness". It has units of $\mathrm{N/m} = \mathrm{kg/s^2}$. This equation is usually referred to as **Hooke's law**. It gives a decent description of the behavior of real springs, just as long as they can oscillate about their equilibrium positions and they are not stretched by *too* much!

When we calculate the work done by a spring on the object attached to its end as the object moves from x_i to x_f we get:

$$W_{\rm spring} = \tfrac{1}{2}kx_i^2 - \tfrac{1}{2}kx_f^2 \tag{6.12}$$

6.1.4 The Work–Kinetic Energy Theorem

One can show that as a particle moves from point $\mathbf{r}_i$ to $\mathbf{r}_f$, the change in kinetic energy of the object is equal to the net work done on it:

$$\Delta K = K_f - K_i = W_{\rm net} \tag{6.13}$$

6.1.5 Power

In certain applications we are interested in the *rate* at which work is done by a force. If an amount of work W is done in a time Δt, then we say that the **average power** $\overline{P}$ due to the force is

$$\overline{P} = \frac{W}{\Delta t} \tag{6.14}$$

In the limit in which both W and Δt are very small then we have the instantaneous power P, written as:

$$P = \frac{dW}{dt} \tag{6.15}$$

The unit of power is the **watt**, defined by:

$$1\,\mathrm{watt} = 1\,\mathrm{W} = 1\,\tfrac{\mathrm{J}}{\mathrm{s}} = 1\,\tfrac{\mathrm{kg\cdot m^2}}{\mathrm{s^3}} \tag{6.16}$$

The watt is related to a quaint old unit of power called the horsepower:

$$1\,\mathrm{horsepower} = 1\,\mathrm{hp} = 550\,\tfrac{\mathrm{ft\cdot lb}}{\mathrm{s}} = 746\,\mathrm{W}$$

One can show that if a force $\mathbf{F}$ acts on a particle moving with velocity $\mathbf{v}$ then the instantaneous rate at which work is being done on the particle is

$$P = \mathbf{F}\cdot\mathbf{v} = Fv\cos\phi \tag{6.17}$$

where ϕ is the angle between the directions of $\mathbf{F}$ and $\mathbf{v}$.

6.1.6 Conservative Forces

The work done on an object by the force of gravity does not depend on the path taken to get from one position to another. The same is true for the spring force. In both cases we just need to know the initial and final coordinates to be able to find W, the work done by that force.

This situation also occurs with the general law for the force of gravity (Eq. 5.4.) as well as with the electrical force which we learn about in the second semester!

This is a different situation from the friction forces studied in Chapter 5. Friction forces do work on moving masses, but to figure out how much work, we need to know *how* the mass got from one place to another.

If the net work done by a force does not depend on the path taken between two points, we say that the force is a **conservative force**. For such forces it is also true that the net work done on a particle moving on around any closed path is zero.

6.1.7 Potential Energy

For a conservative force it is possible to find a function of position called the **potential energy**, which we will write as $U(\mathbf{r})$, from which we can find the work done by the force.

Suppose a particle moves from $\mathbf{r}_i$ to $\mathbf{r}_f$. Then the work done on the particle by a conservative force is related to the corresponding potential energy function by:

$$W_{\mathbf{r}_i \to \mathbf{r}_f} = -\Delta U = U(\mathbf{r}_i) - U(\mathbf{r}_f) \tag{6.18}$$

The potential energy $U(\mathbf{r})$ also has units of joules in the SI system.

When our physics problems involve forces for which we *can* have a potential energy function, we usually think about the *change in potential energy* of the objects rather than the work done by these forces. However for non–conservative forces, we *must* directly calculate their work (or else deduce it from the data given in our problems).

We have encountered two conservative forces so far in our study. The simplest is the force of gravity near the surface of the earth, namely $-mg\mathbf{j}$ for a mass m, where the y axis points upward. For this force we can show that the potential energy function is

$$U_{\text{grav}} = mgy \tag{6.19}$$

In using this equation, it is *arbitrary* where we put the origin of the y axis (i.e. what we call "zero height"). But once we make the choice for the origin we must *stick* with it.

The other conservative force is the spring force. A spring of force constant k which is extended from its equilibrium position by an amount x has a potential energy given by

$$U_{\text{spring}} = \tfrac{1}{2}kx^2 \tag{6.20}$$

6.1.8 Conservation of Mechanical Energy

If we separate the forces in the world into conservative and non-conservative forces, then the work–kinetic energy theorem says

$$W = W_{\rm cons} + W_{\rm non-cons} = \Delta K$$

But from Eq. 6.18, the work done by *conservative* forces can be written as a change in potential energy as:

$$W_{\rm cons} = -\Delta U$$

where U is the sum of *all* types of potential energy. With this replacement, we find:

$$-\Delta U + W_{\rm non-cons} = \Delta K$$

Rearranging this gives the general theorem of the **Conservation of Mechanical Energy**:

$$\Delta K + \Delta U = W_{\rm non-cons} \tag{6.21}$$

We define the **total energy** E of the system as the sum of the kinetic and potential energies of all the objects:

$$E = K + U \tag{6.22}$$

Then Eq. 6.21 can be written

$$\Delta E = \Delta K + \Delta U = W_{\rm non-cons} \tag{6.23}$$

In words, this equation says that the total mechanical energy changes by the amount of work done by the non–conservative forces.

Many of our physics problems are about situations where all the forces acting on the moving objects are conservative; loosely speaking, this means that there is no friction, or else there is negligible friction.

If so, then the work done by non–conservative forces is zero, and Eq. 6.23 takes on a simpler form:

$$\Delta E = \Delta K + \Delta U = 0 \tag{6.24}$$

We can write this equation as:

$$K_i + U_i = K_f + U_f \qquad \text{or} \qquad E_i = E_f$$

In other words, for those cases where we can ignore friction–type forces, if we add up all the kinds of energy for the particle's *initial* position, it is equal to the sum of all the kinds of energy for the particle's *final* position. In such a case, the amount of mechanical energy stays the same...it is conserved.

Energy conservation is useful in problems where we only need to know about positions or speeds but not *time* for the motion.

6.1.9 Work Done by Non–Conservative Forces

When the system does have friction forces then we must go back to Eq. 6.23. The change in total mechanical energy equals the work done by the non–conservative forces:

$$\Delta E = E_f - E_i = W_{\mathrm{non-cons}}$$

(In the case of sliding friction with the rule $f_k = \mu_k N$ it *is* possible to compute the work done by the non–conservative force.)

6.1.10 Relationship Between Conservative Forces and Potential Energy (Optional?)

Eqs. 6.9 (the general expression for work W) and 6.18 give us a relation between the force $\mathbf{F}$ on a particle (as a function of position, $\mathbf{r}$) and the change in potential energy as the particle moves from $\mathbf{r}_i$ to $\mathbf{r}_f$:

$$\int_{\mathbf{r}_i}^{\mathbf{r}_f} \mathbf{F} \cdot d\mathbf{r} = -\Delta U \tag{6.25}$$

Very loosely speaking, potential energy is the (negative) of the integral of $\mathbf{F}(\mathbf{r})$. Eq. 6.25 can be rewritten to show that (loosely speaking!) the force $\mathbf{F}(\mathbf{r})$ is the (minus) derivative of $U(\mathbf{r})$. More precisely, the components of $\mathbf{F}$ can be gotten by taking *partial derivatives* of U with respect to the Cartesian coordinates:

$$F_x = -\frac{\partial U}{\partial x} \qquad F_y = -\frac{\partial U}{\partial y} \qquad F_z = -\frac{\partial U}{\partial z} \tag{6.26}$$

In case you haven't come across partial derivatives in your mathematics education yet: They come up when we have functions of several variables (like a function of x, y and z); if we are taking a partial derivative with respect to x, we treat y and z as constants.

As you may have already learned, the three parts of Eq. 6.26 can be compactly written as

$$\mathbf{F} = -\nabla U$$

which can be expressed in words as "$\mathbf{F}$ is the negative gradient of U".

6.1.11 Other Kinds of Energy

This chapter covers the mechanical energy of particles; later, we consider extended objects which can rotate, and they will also have *rotational kinetic energy*. Real objects also have temperature so that they have *thermal energy*. When we take into account *all* types of energy we find that total energy is *completely* conserved…we never lose any! But here we are counting only the *mechanical* energy and if (in real objects!) friction is present some of it can be lost to become thermal energy.

6.2 Worked Examples

6.2.1 Kinetic Energy

---

1. If a Saturn V rocket with an Apollo spacecraft attached has a combined mass of $2.9 \times 10^5\,\mathrm{kg}$ and is to reach a speed of $11.2\,\frac{\mathrm{km}}{\mathrm{s}}$, how much kinetic energy will it then have? [HRW5 7-1]

(Convert some units first.) The speed of the rocket will be

$$v = (11.2\,\tfrac{\mathrm{km}}{\mathrm{s}})\left(\frac{10^3\,\mathrm{m}}{1\,\mathrm{km}}\right) = 1.12 \times 10^4\,\tfrac{\mathrm{m}}{\mathrm{s}}\ .$$

We know its mass: $m = 2.9 \times 10^5\,\mathrm{kg}$. Using the definition of kinetic energy, we have

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}(2.9 \times 10^5\,\mathrm{kg})(1.12 \times 10^4\,\tfrac{\mathrm{m}}{\mathrm{s}})^2 = 1.8 \times 10^{13}\,\mathrm{J}$$

The rocket will have $1.8 \times 10^{13}\,\mathrm{J}$ of kinetic energy.

---

2. If an electron (mass $m = 9.11 \times 10^{-31}\,\mathrm{kg}$) in copper near the lowest possible temperature has a kinetic energy of $6.7\times10^{-19}\,\mathrm{J}$, what is the speed of the electron? [HRW5 7-2]

Use the definition of kinetic energy, $K = \frac{1}{2}mv^2$ and the given values of K and m, and solve for v. We find:

$$v^2 = \frac{2K}{m} = \frac{2(6.7 \times 10^{-19}\,\mathrm{J})}{(9.11 \times 10^{-31}\,\mathrm{kg})} = 1.47 \times 10^{12}\,\tfrac{\mathrm{m}^2}{\mathrm{s}^2}$$

which gives:

$$v = 1.21 \times 10^6\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

The speed of the electron is $1.21 \times 10^6\,\frac{\mathrm{m}}{\mathrm{s}}$.

6.2.2 Work

---

3. A floating ice block is pushed through a displacement of $\mathbf{d} = (15\,\mathrm{m})\mathbf{i} - (12\,\mathrm{m})\mathbf{j}$ along a straight embankment by rushing water, which exerts a force $\mathbf{F} = (210\,\mathrm{N})\mathbf{i} - (150\,\mathrm{N})\mathbf{j}$ on the block. How much work does the force do on the block during the displacement? [HRW5 7-11]

Here we have the simple case of a straight–line displacement $\mathbf{d}$ and a *constant* force $\mathbf{F}$. Then the work done by the force is $W = \mathbf{F}\cdot\mathbf{d}$. We are given all the components, so we can compute the dot product using the components of $\mathbf{F}$ and $\mathbf{d}$:

$$W = \mathbf{F}\cdot\mathbf{d} = F_x d_x + F_y d_y = (210\,\mathrm{N})((15\,\mathrm{m}) + (-150\,\mathrm{N})(-12\,\mathrm{m}) = 4950\,\mathrm{J}$$

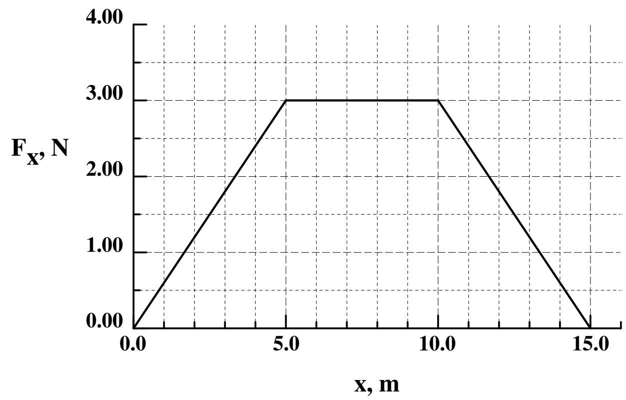

Figure 6.1: Force F_x, which depends on position x; see Example 4.

The force does 4950 J of work.

---

4. A particle is subject to a force F_x that varies with position as in Fig. 6.1. Find the work done by the force on the body as it moves (a) from $x = 0$ to $x = 5.0\,\mathrm{m}$, (b) from $x = 5.0\,\mathrm{m}$ to $x = 10\,\mathrm{m}$ and (c) from $x = 10\,\mathrm{m}$ to $x = 15\,\mathrm{m}$. (d) What is the total work done by the force over the distance $x = 0$ to $x = 15\,\mathrm{m}$? [Ser4 7-23]

(a) Here the force is *not* the same all through the object's motion, so we can't use the simple formula $W = F_x x$. We must use the more general expression for the work done when a particle moves along a straight line,

$$W = \int_{x_i}^{x_f} F_x \, dx \ .$$

Of course, this is just the "area under the curve" of F_x vs. x from x_i to x_f.

In part (a) we want this "area" evaluated from $x = 0$ to $x = 5.0\,\mathrm{m}$. From the figure, we see that this is just *half* of a rectangle of base 5.0 m and height 3.0 N. So the work done is

$$W = \tfrac{1}{2}(3.0\,\mathrm{N})(5.0\,\mathrm{m}) = 7.5\,\mathrm{J} \ .$$

(Of course, when we evaluate the "area", we just keep the *units* which go along with the base and the height; here they were meters and newtons, the product of which is a *joule*.)

So the work done by the force for this displacement is 7.5 J.

(b) The region under the curve from $x = 5.0\,\mathrm{m}$ to $x = 10.0\,\mathrm{m}$ is a *full* rectangle of base 5.0 m and height 3.0 N. The work done for this movement of the particle is

$$W = (3.0\,\mathrm{N})(5.0\,\mathrm{m}) = 15.\,\mathrm{J}$$

(c) For the movement from $x = 10.0\,\mathrm{m}$ to $x = 15.0\,\mathrm{m}$ the region under the curve is a *half* rectangle of base 5.0 m and height 3.0 N. The work done is

$$W = \tfrac{1}{2}(3.0\,\mathrm{N})(5.0\,\mathrm{m}) = 7.5\,\mathrm{J} \ .$$

(d) The total work done over the distance $x = 0$ to $x = 15.0\,\text{m}$ is the sum of the three separate "areas",

$$W_{\text{total}} = 7.5\,\text{J} + 15.\,\text{J} + 7.5\,\text{J} = 30.\,\text{J}$$

---

5. What work is done by a force $\mathbf{F} = (2x\,\text{N})\mathbf{i} + (3\,\text{N})\mathbf{j}$, with x in meters, that moves a particle from a position $\mathbf{r}_i = (2\,\text{m})\mathbf{i} + (3\,\text{m})\mathbf{j}$ to a position $\mathbf{r}_f = -(4\,\text{m})\mathbf{i} - (3\,\text{m})\mathbf{j}$? [HRW5 7-31]

We use the general definition of work (for a two–dimensional problem),

$$W = \int_{x_i}^{x_f} F_x(\mathbf{r})\,dx + \int_{y_i}^{y_f} F_y(\mathbf{r})\,dy$$

With $F_x = 2x$ and $F_y = 3$ [we mean that F in newtons when x is in meters; work W will come out with units of *joules*!], we find:

$$\begin{aligned} W &= \int_{2\,\text{m}}^{-4\,\text{m}} 2x\,dx + \int_{3\,\text{m}}^{-3\,\text{m}} 3\,dy \\ &= x^2\Big|_{2\,\text{m}}^{-4\,\text{m}} + 3x\Big|_{3\,\text{m}}^{-3\,\text{m}} \\ &= [(16) - (4)]\,\text{J} + [(-9) - (9)]\,\text{J} \\ &= -6\,\text{J} \end{aligned}$$

6.2.3 Spring Force

---

6. An archer pulls her bow string back $0.400\,\text{m}$ by exerting a force that increases from zero to $230\,\text{N}$. (a) What is the equivalent spring constant of the bow? (b) How much work is done in pulling the bow? [Ser4 7-25]

(a) While a bow string is not literally *spring*, it may behave like one in that it exerts a force on the thing attached to it (like a hand!) that is *proportional to the distance of pull* from the equilibrium position. The correspondence is illustrated in Fig. 6.2.

We are told that when the string has been pulled back by 0.400 m, the string exerts a restoring force of 230 N. The magnitude of the string's force is equal to the force constant k times the magnitude of the displacement; this gives us:

$$|F_{\text{string}}| = 230\,\text{N} = k(0.400\,\text{m})$$

Solving for k,

$$k = \frac{(230\,\text{N})}{(0.400\,\text{m})} = 575\,\tfrac{\text{N}}{\text{m}}$$

The (equivalent) spring constant of the bow is $575\,\frac{\text{N}}{\text{m}}$.

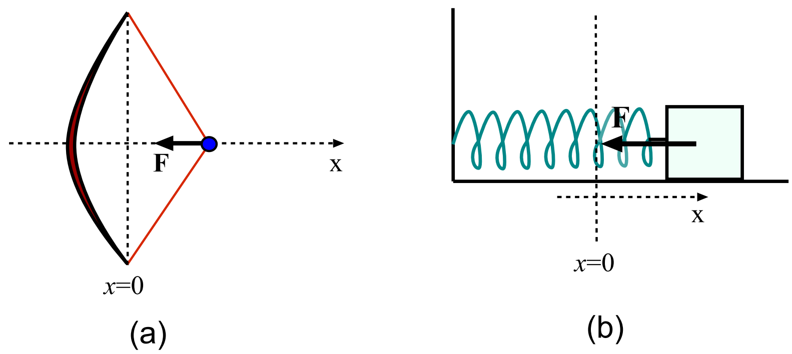

Figure 6.2: The force of a bow string (a) on the object pulling it back can be modelled as as ideal spring (b) exerting a restoring force on the mass attached to its end.

(b) Still treating the bow string as if it were an ideal spring, we note that in pulling the string from a displacement of $x = 0$ to $x = 0.400\,\mathrm{m}$ the *string* does as amount of work *on the hand* given by Eq. 6.12:

$$\begin{aligned} W_{\rm string} &= \tfrac{1}{2}kx_i^2 - \tfrac{1}{2}kx_f^2 \\ &= 0 - \tfrac{1}{2}(575\,\tfrac{\mathrm{N}}{\mathrm{m}})(0.400\,\mathrm{m})^2 \\ &= -46.0\,\mathrm{J} \end{aligned}$$

Is this answer to the question? Not quite... we were really asked for the work done *by the hand on the bow string*. But at all times during the pulling, the hand exerted an equal and opposite force on the string. The force had the opposite direction, so the work that it did has the opposite sign. The work done (by the hand) in pulling the bow is $+46.0\,\mathrm{J}$.

6.2.4 The Work–Kinetic Energy Theorem

---

7. A $40\,\mathrm{kg}$ **box initially at rest is pushed** $5.0\,\mathrm{m}$ **along a rough horizontal floor with a constant applied horizontal force of** $130\,\mathrm{N}$**. If the coefficient of friction between the box and floor is** 0.30**, find (a) the work done by the applied force, (b) the energy lost due to friction, (c) the change in kinetic energy of the box, and (d) the final speed of the box.** [Ser4 7-37]

(a) The motion of the box and the forces which do work on it are shown in Fig. 6.3(a). The (constant) applied force points in the same direction as the displacement. Our formula for the work done by a constant force gives

$$W_{\rm app} = Fd\cos\phi = (130\,\mathrm{N})(5.0\,\mathrm{m})\cos 0^\circ = 6.5\times 10^2\,\mathrm{J}$$

The applied force does 6.5×10^2 J of work.

(b) Fig. 6.3(b) shows *all* the forces acting on the box.

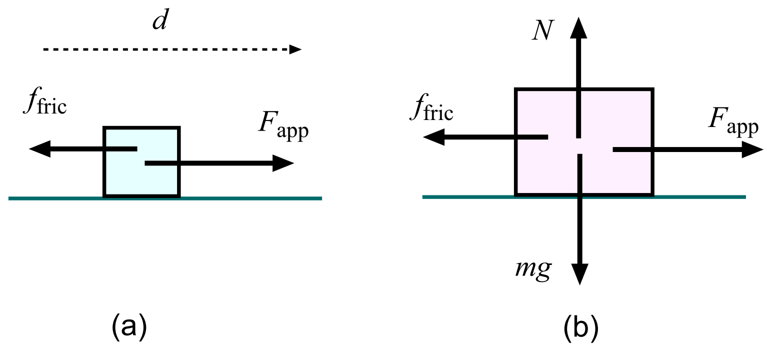

Figure 6.3: (a) Applied force and friction force both do work on the box. (b) Diagram showing *all* the forces acting on the box.

The vertical forces acting on the box are gravity (mg, downward) and the floor's normal force (N, upward). It follows that $N = mg$ and so the magnitude of the friction force is

$$f_{\text{fric}} = \mu N = \mu mg = (0.30)(40\,\text{kg})(9.80\,\tfrac{\text{m}}{\text{s}^2}) = 1.2 \times 10^2\,\text{N}$$

The friction force is directed opposite the direction of motion ($\phi = 180^\circ$) and so the work that it does is

$$\begin{aligned} W_{\text{fric}} &= Fd\cos\phi \\ &= f_{\text{fric}}\,d\cos 180^\circ = (1.2 \times 10^2\,\text{N})(5.0\,\text{m})(-1) = -5.9 \times 10^2\,\text{J} \end{aligned}$$

or we might say that 5.9×10^2 J is *lost* to friction.

(c) Since the normal force and gravity do no work on the box as it moves, the net work done is

$$W_{\text{net}} = W_{\text{app}} + W_{\text{fric}} = 6.5 \times 10^2\,\text{J} - 5.9 \times 10^2\,\text{J} = 62\,\text{J} .$$

By the work–Kinetic Energy Theorem, this is equal to the change in kinetic energy of the box:

$$\Delta K = K_f - K_i = W_{\text{net}} = 62\,\text{J} .$$

(d) Here, the initial kinetic energy K_i was *zero* because the box was initially at rest. So we have $K_f = 62$ J. From the definition of kinetic energy, $K = \frac{1}{2}mv^2$, we get the final speed of the box:

$$v_f^2 = \frac{2K_f}{m} = \frac{2(62\,\text{J})}{(40\,\text{kg})} = 3.1\,\tfrac{\text{m}^2}{\text{s}^2}$$

so that

$$v_f = 1.8\,\tfrac{\text{m}}{\text{s}}$$

---

8. A crate of mass $10.0\,\text{kg}$ is pulled up a rough incline with an initial speed of $1.50\,\frac{\text{m}}{\text{s}}$. The pulling force is $100\,\text{N}$ parallel to the incline, which makes an angle of 20.0° with the horizontal. The coefficient of kinetic friction is 0.400, and the crate

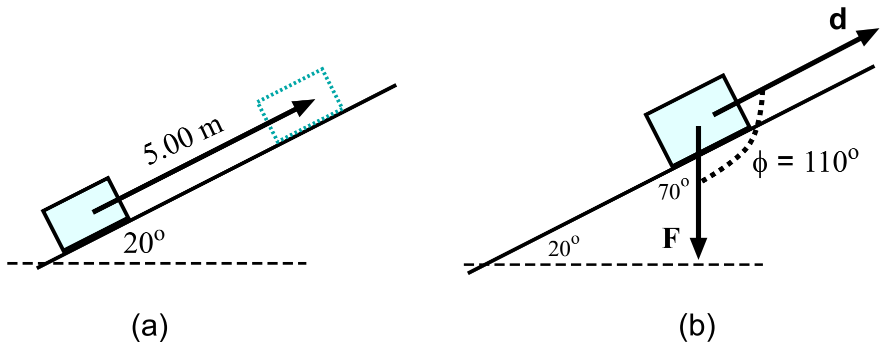

Figure 6.4: (a) Block moves 5.00 m up plane while acted upon by gravity, friction and an applied force. (b) Directions of the displacement and the force of gravity.

is pulled 5.00 m. (a) How much work is done by gravity? (b) How much energy is lost due to friction? (c) How much work is done by the 100 N force? (d) What is the change in kinetic energy of the crate? (e) What is the speed of the crate after being pulled 5.00 m? [Ser4 7-47]

(a) We can calculate the work done by gravity in two ways. First, we can use the definition: $W = \mathbf{F}\cdot\mathbf{d}$. The magnitude of the gravity force is

$$F_{\rm grav} = mg = (10.0\,{\rm kg}(9.80\,\tfrac{\rm m}{\rm s^2}) = 98.0\,{\rm N}$$

and the displacement has magnitude 5.00 m. We see from geometry (see Fig. 6.4(b)) that the angle between the force and displacement vectors is 110°. Then the work done by gravity is

$$W_{\rm grav} = Fd\cos\phi = (98.0\,{\rm N})(5.00\,{\rm m})\cos\,110^\circ = -168\,{\rm J}\ .$$

Another way to work the problem is to plug the right values into Eq. 6.10. From simple geometry we see that the change in height of the crate was

$$\Delta y = (5.00\,{\rm m})\sin\,20^\circ = +1.71\,{\rm m}$$

Then the work done by gravity was

$$W_{\rm grav} = -mg\Delta y = -(10.0\,{\rm kg})(9.80\,\tfrac{\rm m}{\rm s^2})(1.71\,{\rm m}) = -168\,{\rm J}$$

(b) To find the work done by friction, we need to know the force of friction. The forces on the block are shown in Fig. 6.5(a). As we have seen before, the normal force between the slope and the block is $mg\cos\theta$ (with $\theta = 20^\circ$) so as to cancel the normal component of the force of gravity. Then the force of kinetic friction on the block points down the slope (opposite the motion) and has magnitude

$$\begin{aligned} f_{\rm k} &= \mu_{\rm k}N = \mu mg\cos\theta \\ &= (0.400)(10.0\,{\rm kg})(9.80\,\tfrac{\rm m}{\rm s^2})\cos\,20^\circ = 36.8\,{\rm N} \end{aligned}$$

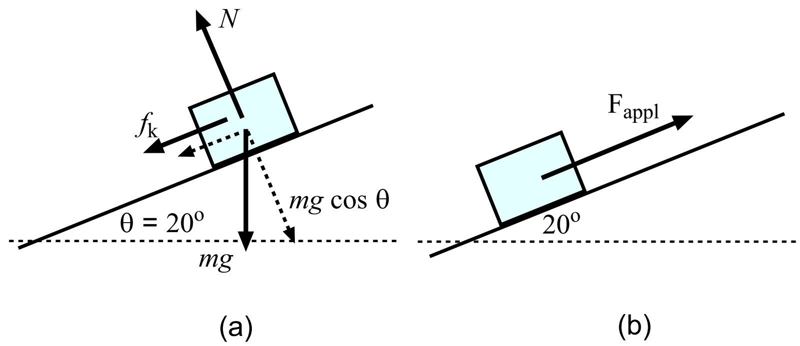

Figure 6.5: (a) Gravity and friction forces which act on the block. (b) The applied force of 100 N is along the direction of the motion.

This force points exactly opposite the direction of the displacement **d**, so the work done by friction is

$$W_{\rm fric} = f_{\rm k} d \cos 180^\circ = (36.8\,{\rm N})(5.00\,{\rm m})(-1) = -184\,{\rm J}$$

(c) The 100 N applied force pulls in the direction up the slope, which is *along* the direction of the displacement **d**. So the work that is does is

$$W_{\rm appl} = F d \cos 0^\circ = (100\,{\rm N})(5.00\,{\rm m})(1) = 500.\,{\rm J}$$

(d) We have now found the work done by each of the forces acting on the crate as it moved: Gravity, friction and the applied force. (We should note the the normal force of the surface *also* acted on the crate, but being perpendicular to the motion, it did no work.) The net work done was:

$$\begin{aligned} W_{\rm net} &= W_{\rm grav} + W_{\rm fric} + W_{\rm appl} \\ &= -168\,{\rm J} - 184\,{\rm J} + 500.\,{\rm J} = 148\,{\rm J} \end{aligned}$$

From the work–energy theorem, this is equal to the change in kinetic energy of the box: $\Delta K = W_{\rm net} = 148\,{\rm J}$.

(e) The initial kinetic energy of the crate was

$$K_i = \tfrac{1}{2}(10.0\,{\rm kg})(1.50\,\tfrac{\rm m}{\rm s})^2 = 11.2\,{\rm J}$$

If the final speed of the crate is v, then the change in kinetic energy was:

$$\Delta K = K_f - K_i = \tfrac{1}{2}mv^2 - 11.2\,{\rm J}\ .$$

Using our answer from part (d), we get:

$$\Delta K = \tfrac{1}{2}mv^2 - 11.2\,{\rm J} = 148\,{\rm J} \qquad \Longrightarrow \qquad v^2 = \frac{2(159\,{\rm J})}{m}$$

So then:

$$v^2 = \frac{2(159\,{\rm J})}{(10.0\,{\rm kg})} = 31.8\,\tfrac{\rm m^2}{\rm s^2} \qquad \Longrightarrow \qquad v = 5.64\,\tfrac{\rm m}{\rm s}\ .$$

The final speed of the crate is $5.64\,\frac{\rm m}{\rm s}$.

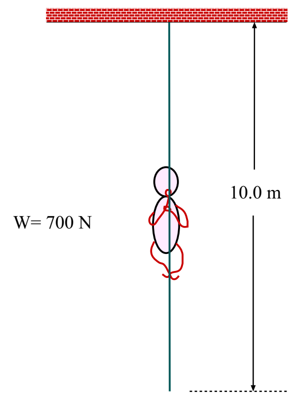

Figure 6.6: Marine climbs rope in Example 9. You don't like my drawing? Tell it to the Marines!

6.2.5 Power

---

9. A 700 N **marine in basic training climbs a** 10.0 m **vertical rope at a constant speed of** 8.00 s**. What is his power output?** [Ser4 7-53]

Marine is shown in Fig. 6.6. The speed of the marine up the rope is

$$v = \frac{d}{t} = \frac{10.0\,\mathrm{m}}{8.00\,\mathrm{s}} = 1.25\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

The forces acting on the marine are gravity (700 N, downward) and the force of the rope which must be 700 N upward since he moves at constant velocity. Since he moves in the same direction as the rope's force, the rope does work on the marine at a rate equal to

$$P = \frac{dW}{dt} = \mathbf{F}\cdot\mathbf{v} = Fv = (700\,\mathrm{N})(1.25\,\tfrac{\mathrm{m}}{\mathrm{s}}) = 875\,\mathrm{W}\ .$$

(It may be hard to think of a stationary rope doing work on anybody, but that is what is happening here.)

This number represents a rate of change in the potential energy of the marine; the energy comes from someplace. *He* is losing (chemical) energy at a rate of 875 W.

---

10. Water flows over a section of Niagara Falls at a rate of $1.2\times10^{6}\,\mathrm{kg/s}$ **and falls** 50 m**. How many** 60 W **bulbs can be lit with this power?** [Ser4 7-54]

Whoa! Waterfalls? Bulbs? What's going on here??

If a certain mass m of water *drops* by a height h (that is, $\Delta y = -h$), then from Eq. 6.10, gravity does an amount of work equal to mgh. If this change in height occurs over a time interval Δt then the rate at which gravity does work is $mgh/\Delta t$.

For Niagara Falls, if we consider the amount of water that falls in one second, then a mass $m = 1.2\times10^{6}$ kg falls through 50 m and the work done by gravity is

$$W_{\mathrm{grav}} = mgh = (1.2\times10^{6}\,\mathrm{kg})(9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(50\,\mathrm{m}) = 5.88\times10^{8}\,\mathrm{J}\ .$$

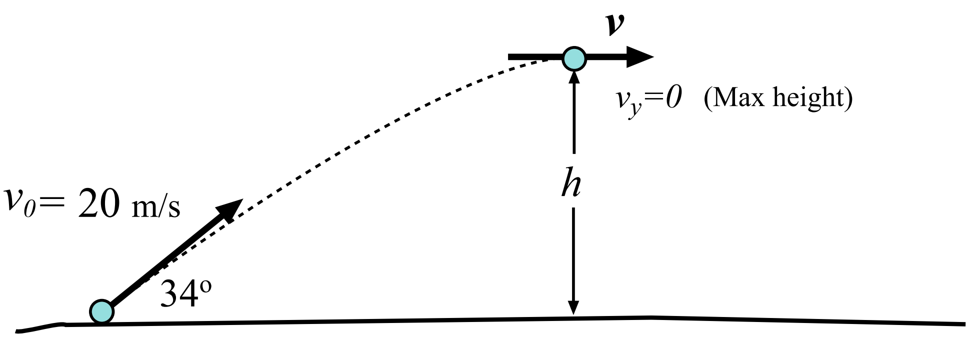

Figure 6.7: Snowball is launched at angle of 34° in Example 11.

This occurs every second, so gravity does work at a rate of

$$P_{\rm grav} = \frac{mgh}{\Delta t} = \frac{5.88 \times 10^8\,{\rm J}}{1\,{\rm s}} = 5.88 \times 10^8\,{\rm W}$$

As we see later, this is also the rate at which the water *loses potential energy*. This energy can be converted to other forms, such as the electrical energy to make a light bulb function. In this highly idealistic example, all of the energy is converted to electrical energy.

A 60 W light bulb uses energy at a rate of $60\,\frac{\rm J}{\rm s} = 60\,{\rm W}$. We see that Niagara Falls puts out energy at a rate much bigger than this! Assuming *all* of it goes to the bulbs, then dividing the *total* energy consumption rate by the rate for *one* bulb tells us that

$$N = \frac{5.88 \times 10^8\,{\rm W}}{60\,{\rm W}} = 9.8 \times 10^6$$

bulbs can be lit.

6.2.6 Conservation of Mechanical Energy

---

11. A $1.50\,{\rm kg}$ snowball is shot upward at an angle of 34.0° to the horizontal with an initial speed of $20.0\,\frac{\rm m}{\rm s}$. (a) What is its initial kinetic energy? (b) By how much does the gravitational potential energy of the snowball–Earth system change as the snowball moves from the launch point to the point of maximum height? (c) What is that maximum height? [HRW5 8-31]

(a) Since the initial speed of the snowball is $20.0\,\frac{\rm m}{\rm s}$, we have its initial kinetic energy:

$$K_i = \tfrac{1}{2}mv_0^2 = \tfrac{1}{2}(1.50\,{\rm kg})(20.0\,\tfrac{\rm m}{\rm s})^2 = 300.\,{\rm J}$$

(b) We need to remember that since this projectile was not fired straight up, it will still have *some* kinetic energy when it gets to maximum height! That means we have to think a little harder before applying energy principles to answer this question.

At maximum height, we know that the y component of the snowball's velocity is zero. The x component is *not* zero.

But we do know that since a projectile has no horizontal acceleration, the x component will remain *constant*; it will keep its initial value of

$$v_{0x} = v_0 \cos\theta_0 = (20.0\,\tfrac{\mathrm{m}}{\mathrm{s}})\cos 34^\circ = 16.6\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

so the speed of the snowball at maximum height is $16.6\,\frac{\mathrm{m}}{\mathrm{s}}$. At maximum height, (the final position) the kinetic energy is

$$K_f = \tfrac{1}{2}mv_f^2 = \tfrac{1}{2}(1.50\,\mathrm{kg})(16.6\,\tfrac{\mathrm{m}}{\mathrm{s}})^2 = 206.\,\mathrm{J}$$

In this problem there are only conservative forces (namely, gravity). The mechanical energy is conserved:

$$K_i + U_i = K_f + U_f$$

We already found the initial kinetic energy of the snowball: $K_i = 300.\,\mathrm{J}$. Using $U_{\mathrm{grav}} = mgy$ (with $y = 0$ at ground level), the initial potential energy is $U_i = 0$. Then we can find the final potential energy of the snowball:

$$\begin{aligned} U_f &= K_i + U_i - K_f \\ &= 300.\,\mathrm{J} + 0 - 206.\,\mathrm{J} \\ &= 94.\,\mathrm{J} \end{aligned}$$

The final gravitational potential energy of the snowball–earth system (a long–winded way of saying what U is!) is then $94.\,\mathrm{J}$. (Since its original value was zero, this is the answer to part (b).)

(c) If we call the maximum height of the snowball h, then we have

$$U_f = mgh$$

Solve for h:

$$h = \frac{U_f}{mg} = \frac{(94.\,\mathrm{J})}{(1.5\,\mathrm{kg})(9.80\,\frac{\mathrm{m}}{\mathrm{s}^2})} = 6.38\,\mathrm{m}$$

The maximum height of the snowball is $6.38\,\mathrm{m}$.

---

12. A pendulum consists of a $2.0\,\mathrm{kg}$ stone on a $4.0\,\mathrm{m}$ string of negligible mass. The stone has a speed of $8.0\,\frac{\mathrm{m}}{\mathrm{s}}$ when it passes its lowest point. (a) What is the speed when the string is at 60° to the vertical? (b) What is the greatest angle with the vertical that the string will reach during the stone's motion? (c) If the potential energy of the pendulum–Earth system is taken to be zero at the stone's lowest point, what is the total mechanical energy of the system? [HRW5 8-32]

(a) The condition of the pendulum when the stone passes the lowest point is shown in Fig. 6.8(a). Throughout the problem we will measure the height y of the stone from the

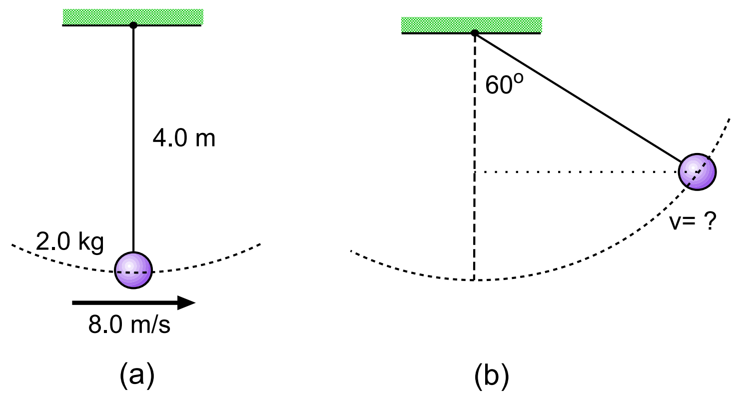

Figure 6.8: (a) Pendulum in Example 12 swings through lowest point. (b) Pendulum has swung 60° past lowest point.

bottom of its swing. Then at the bottom of the swing the stone has zero potential energy, while its kinetic energy is

$$K_i = \tfrac{1}{2}mv_0^2 = \tfrac{1}{2}(2.0\,\mathrm{kg})(8.0\,\tfrac{\mathrm{m}}{\mathrm{s}})^2 = 64\,\mathrm{J}$$

When the stone has swung up by 60° (as in Fig. 6.8(b)) it has some potential energy. To figure out how much, we need to calculate the height of the stone *above the lowest point of the swing*. By simple geometry, the stone's position is

$$(4.0\,\mathrm{m})\cos 60^\circ = 2.0\,\mathrm{m}$$

down from the top of the string, so it must be

$$4.0\,\mathrm{m} - 2.0\,\mathrm{m} = 2.0\,\mathrm{m}$$

up from the lowest point. So its potential energy at this point is

$$U_f = mgy = (2.0\,\mathrm{kg})(9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(2.0\,\mathrm{m}) = 39.2\,\mathrm{J}$$

It will also have a kinetic energy $K_f = \frac{1}{2}mv_f^2$, where v_f is the final speed.

Now in this system there are only a *conservative force* acting on the particle of interest, i.e. the stone. (We should note that the string tension also acts on the stone, but since it always pulls perpendicularly to the motion of the stone, it does no work.) So the total mechanical energy of the stone is conserved:

$$K_i + U_i = K_f + U_f$$

We can substitute the values found above to get:

$$64.0\,\mathrm{J} + 0 = \tfrac{1}{2}(2.0\,\mathrm{kg})v_f^2 + 39.2\,\mathrm{J}$$

which we can solve for v_f:

$$(1.0\,\mathrm{kg})v_f^2 = 64.0\,\mathrm{J} - 39.2\,\mathrm{J} = 24.8\,\mathrm{J} \qquad \Longrightarrow \qquad v_f^2 = 24.8\,\tfrac{\mathrm{m}^2}{\mathrm{s}^2}$$

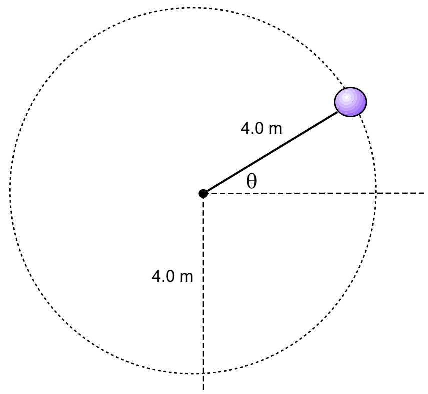

Figure 6.9: Stone reaches its highest position in the swing, which we specify by some angle θ measured above the horizontal.

and then:

$$v_f = 5.0\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

The speed of the stone at the 60° position will be $5.0\,\tfrac{\mathrm{m}}{\mathrm{s}}$.

(b) Clearly, since the stone is still in motion at an angle of 60°, it will keep moving to greater angles and larger heights above the bottom position. For all we know, it may keep rising until it gets to some angle θ above the position where the string is horizontal, as shown in Fig. 6.9. We *do* assume that the string will stay straight until this point, but that is a reasonable assumption.

Now at this point of maximum height, the speed of the mass is instantaneously zero. So in *this* final position, the kinetic energy is $K_f = 0$. Its height above the starting position is

$$y = 4.0\,\mathrm{m} + (4.0\,\mathrm{m})\sin\theta = (4.0\,\mathrm{m})(1 + \sin\theta) \tag{6.27}$$

so that its potential energy there is

$$U_f = mgy_f = (2.0\,\mathrm{kg})(9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(4.0\,\mathrm{m})(1+\sin\theta) = (78.4\,\mathrm{J})(1+\sin\theta)$$

We use the conservation of mechanical energy (from the position at the bottom of the swing) to find θ: $K_i + U_i = K_f + U_f$, so:

$$U_f = K_i + U_i - K_f \qquad \Longrightarrow \qquad (78.4\,\mathrm{J})(1+\sin\theta) = 64\,\mathrm{J} + 0 - 0$$

This gives us:

$$1 + \sin\theta = \frac{78.4\,\mathrm{J}}{64\,\mathrm{J}} = 1.225 \qquad \Longrightarrow \qquad \sin\theta = 0.225$$

and finally

$$\theta = 13^\circ$$

We do get a sensible answer of θ so we were right in writing down Eq. 6.27. Actually this equation would also have been correct if θ were negative and the pendulum reached its highest point with the string below the horizontal.

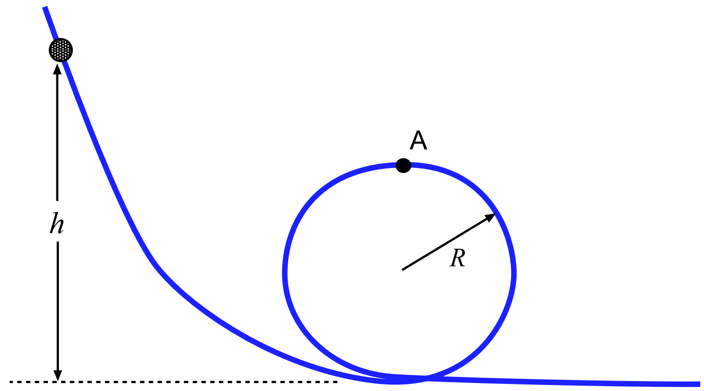

Figure 6.10: Bead slides on track in Example 13.

---

13. A bead slides without friction on a loop–the–loop track (see Fig. 6.10). If the bead is released from a height $h = 3.50R$, what is its speed at point A? How large is the normal force on it if its mass is $5.00\,\mathrm{g}$? [Ser4 8-11]

In this problem, there are no friction forces acting on the particle (the bead). Gravity acts on it and gravity is a conservative force. The track will exert a *normal* forces on the bead, but this force does no work. So the total energy of the bead —kinetic plus (gravitational) potential energy— will be conserved.

At the initial position, when the bead is released, the bead has no speed; $K_i = 0$. But it is at a height h above the bottom of the track. If we agree to measure height from the bottom of the track, then the initial potential energy of the bead is

$$U_i = mgh$$

where $m = 5.00\,\mathrm{g}$ is the mass of the bead.

At the final position (A), the bead has *both* kinetic and potential energy. If the bead's speed at A is v, then its final kinetic energy is $K_f = \frac{1}{2}mv^2$. At position A its height is $2R$ (it is a full diameter above the "ground level" of the track) so its potential energy is

$$U_f = mg(2R) = 2mgR \ .$$

The total energy of the bead is conserved: $K_i + U_i = K_f + U_f$. This gives us:

$$0 + mgh = \tfrac{1}{2}mv^2 + 2mgR \ ,$$

where we want to solve for v (the speed at A). The mass m cancels out, giving:

$$gh = \tfrac{1}{2}v^2 = 2gR \qquad \Longrightarrow \qquad \tfrac{1}{2}v^2 = gh - 2gR = g(h - 2R)$$

and then

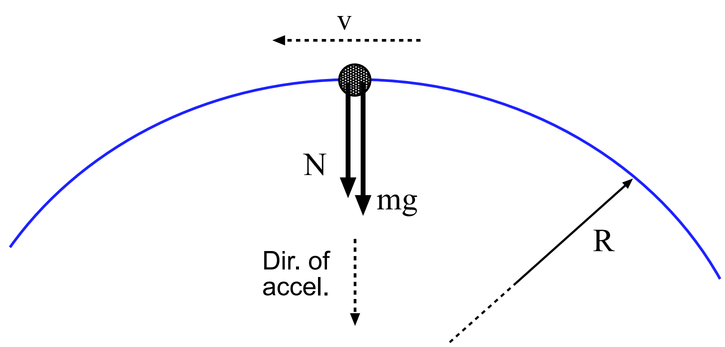

Figure 6.11: Forces acting on the bead when it is at point A (the top of the loop).

$$v^2 = 2g(h - 2R) = 2g(3.50R - 2R) = 2g(1.5R) = 3.0\,gR \tag{6.28}$$

and finally

$$v = \sqrt{3.0\,gR}\ .$$

Since we don't have a numerical value for R, that's as far as we can go.

In the next part of the problem, we think about the *forces* acting on the bead at point A. These are diagrammed in Fig. 6.11. Gravity pulls down on the bead with a force mg. There is also a normal force from the track which I have *drawn* as having a downward component N. But it is possible for the track to be pushing *upward* on the bead; if we get a negative value for N we'll know that the track was pushing *up*.

At the top of the track the bead is moving on a circular path of radius R, with speed v. So it is accelerating *toward the center of the circle*, namely downward. We know that the downward forces must add up to give the centripetal force mv^2/R:

$$mg + N = \frac{mv^2}{R} \qquad \Longrightarrow \qquad n = \frac{mv^2}{R} - mg = m\left(\frac{v^2}{R} - g\right)\ .$$

But we can use our result from Eq. 6.28 to substitute for v^2. This gives:

$$N = m\left(\frac{3.0\,gR}{R} - g\right) = m(2g) = 2mg$$

Plug in the numbers:

$$N = 2(5.00 \times 10^{-3}\,\mathrm{kg})(9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2}) = 9.80 \times 10^{-2}\,\mathrm{N}$$

At point A the track is pushing *downward* with a force of $9.80 \times 10^{-2}\,\mathrm{N}$.

---

14. Two children are playing a game in which they try to hit a small box on the floor with a marble fired from a spring –loaded gun that is mounted on the table. The target box is $2.20\,\mathrm{m}$ horizontally from the edge of the table; see Fig. 6.12.

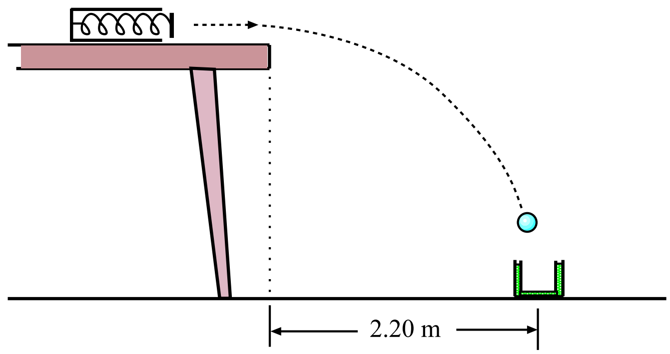

Figure 6.12: Spring propels marble off table and hits (or misses) box on the floor.

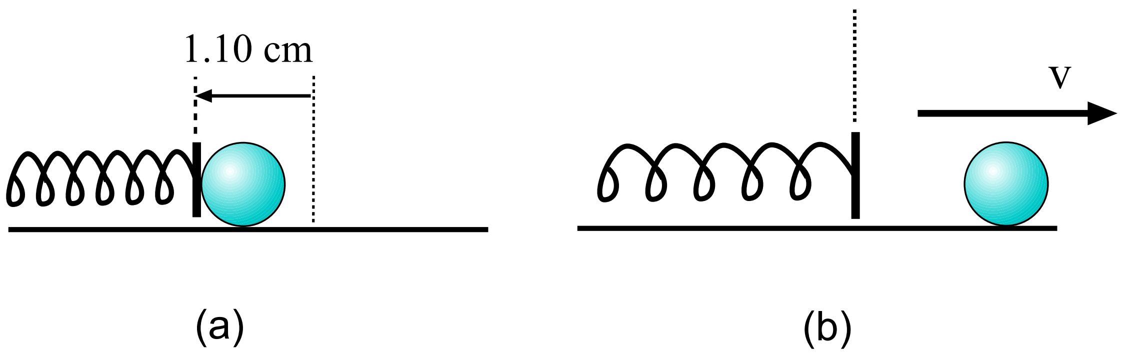

Figure 6.13: Marble propelled by the spring–gun: (a) Spring is compressed, and system has potential energy. (b) Spring is released and system has kinetic energy of the marble.

Bobby compresses the spring 1.10 cm**, but the center of the marble falls** 27.0 cm **short of the center of the box. How far should Rhoda compress the spring to score a direct hit?** [HRW5 8-36]

Let's put the origin of our coordinate system (for the motion of the marble) at the edge of the table. With this choice of coordinates, the object of the game is to insure that the x coordinate of the marble is 2.20 m when it reaches the level of the floor.

There are many things we are not told in this problem! We don't know the spring constant for the gun, or the mass of the marble. We don't know the height of the table above the floor, either!

When the gun propels the marble, the spring is initially compressed and the marble is motionless (see Fig. 6.13(a).) The energy of the system here is the energy stored in the spring, $E_i = \frac{1}{2}kx^2$, where k is the force constant of the spring and x is the amount of compression of the spring.) When the spring has returned to its natural length and has given the marble a speed v, then the energy of the system is $E_f = \frac{1}{2}mv^2$. If we can neglect friction then mechanical energy is conserved during the firing, so that $E_f = E_i$, which gives us:

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}kx^2 \qquad \Longrightarrow \qquad v = \sqrt{\frac{k}{m}x^2} = x\sqrt{\frac{k}{m}}$$

We will let x and v be the compression and initial marble speed for Bobby's attempt. Then we have:

$$v = (1.10 \times 10^{-2}\,\mathrm{m})\sqrt{\frac{k}{m}} \tag{6.29}$$

The marble's trip from the edge of the table to the floor is (by now!) a fairly simple kinematics problem. If the time the marble spends in the air is t and the height of the table is h then the equation for vertical motion tells us:

$$h = \tfrac{1}{2}gt^2 \ .$$

(This is true because the marble's initial velocity is all *horizontal*. We do know that on Bobby's try, the marble's x coordinate at impact was

$$x = 2.20\,\mathrm{m} - 0.27\,\mathrm{m} = 1.93\,\mathrm{m}$$

and since the horizontal velocity of the marble is v, we have:

$$vt = 1.93\,\mathrm{m} \ . \tag{6.30}$$

There are too many unknowns to solve for k, v, h and t...but let's go on.

Let's suppose that Rhoda compresses the spring by an amount x' so that the marble is given a speed v'. As before, we have

$$\tfrac{1}{2}mv'^2 = \tfrac{1}{2}kx'^2$$

(it's the same spring and marble so that k and m are the same) and this gives:

$$v' = x'\sqrt{\frac{k}{m}} \ . \tag{6.31}$$

Now when Rhoda's shot goes off the table and through the air, then if its time of flight is t' then the equation for vertical motion gives us:

$$h = \tfrac{1}{2}gt'^2 \ .$$

This is the same equation as for t, so that the times of flight for both shots is the same: $t' = t$. Since the x coordinate of the marble for Rhoda's shot will be $x = 2.20\,\mathrm{m}$, the equation for horizontal motion gives us

$$v't = 2.20\,\mathrm{m} \tag{6.32}$$

What can we do with these equations? If we divide Eq. 6.32 by Eq. 6.30 we get:

$$\frac{v't}{vt} = \frac{v'}{v} = \frac{2.20}{1.93} = 1.14$$

If we divide Eq. 6.31 by Eq. 6.29 we get:

$$\frac{v'}{v} = \frac{x'\sqrt{\frac{k}{m}}}{(1.10 \times 10^{-2}\,\mathrm{m})\sqrt{\frac{k}{m}}} = \frac{x'}{(1.10 \times 10^{-2}\,\mathrm{m})} \ .$$

With these last two results, we can solve for x'. Combining these equations gives:

$$1.14 = \frac{x'}{(1.10 \times 10^{-2}\,\mathrm{m})} \qquad \Longrightarrow \qquad x' = 1.14(1.10 \times 10^{-2}\,\mathrm{m}) = 1.25\,\mathrm{cm}$$

Rhoda should compress the spring by 1.25 cm in order to score a direct hit.

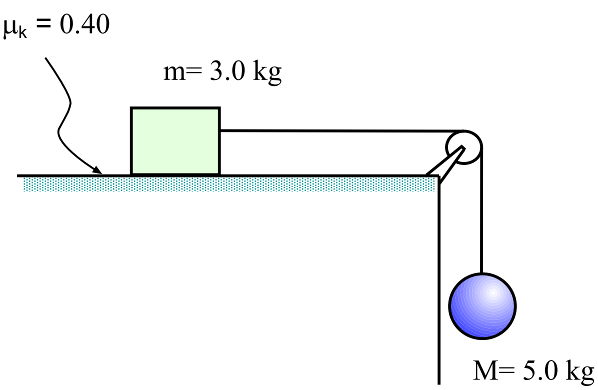

Figure 6.14: Moving masses in Example 15. There is friction between the surface and the 3.0 kg mass.

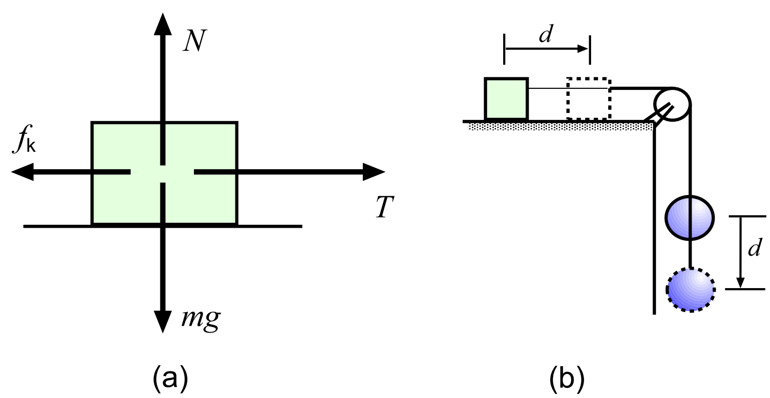

Figure 6.15: (a) Forces acting on m. (b) Masses m and M travel a distance $d = 1.5\,\mathrm{m}$ as they increase in speed from 0 to v.

6.2.7 Work Done by Non–Conservative Forces

---

15. The coefficient of friction between the $3.0\,\mathrm{kg}$ mass and surface in Fig. 6.14 is 0.40. The system starts from rest. What is the speed of the $5.0\,\mathrm{kg}$ mass when it has fallen $1.5\,\mathrm{m}$? [Ser4 8-25]

When the system starts to move, both masses accelerate; because the masses are connected by a string, *they always have the same speed*. The block (m) slides on the rough surface, and friction does work on it. Since its height does not change, its potential energy does not change, but its kinetic energy increases. The hanging mass (M) drops freely; its potential energy decreases but its kinetic energy increases.

We want to use energy principles to work this problem; since there *is* friction present, we need to calculate the work done by friction.

The forces acting on m are shown in Fig. 6.15(a). The normal force N must be equal to mg, so the force of kinetic friction on m has magnitude $\mu_k N = \mu_k mg$. This force *opposes* the motion as m moves a distance $d = 1.5\,\mathrm{m}$, so the work done by friction is

$$W_{\mathrm{fric}} = f_k d \cos\phi = (\mu_k m g)(d)(-1) = -(0.40)(3.0\,\mathrm{kg})(9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(1.5\,\mathrm{m}) = -17.6\,\mathrm{J}\ .$$

Mass m's initial speed is zero, and its final speed is v. So its change in kinetic energy is

$$\Delta K = \tfrac{1}{2}(3.0\,\mathrm{kg})v^2 - 0 = (1.5\,\mathrm{kg})v^2$$

As we noted, m has no change in potential energy during the motion.

Mass M's change in kinetic energy is

$$\Delta K = \tfrac{1}{2}(5.0\,\mathrm{kg})v^2 - 0 = (2.5\,\mathrm{kg})v^2$$

and since it has a *change in height* given by $-d$, its change in (gravitational) potential energy is

$$\Delta U = Mg\Delta y = (5.0\,\mathrm{kg})(9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(-1.5\,\mathrm{m}) = -73.5\,\mathrm{J}$$

Adding up the changes for both masses, the total change in mechanical energy of this system is

$$\begin{aligned} \Delta E &= (1.5\,\mathrm{kg})v^2 + (2.5\,\mathrm{kg})v^2 - 73.5\,\mathrm{J} \\ &= (4.0\,\mathrm{kg})v^2 - 73.5\,\mathrm{J} \end{aligned}$$

Now use $\Delta E = W_{\mathrm{fric}}$ and get:

$$(4.0\,\mathrm{kg})v^2 - 73.5\,\mathrm{J} = -17.6\,\mathrm{J}$$

Solve for v:

$$(4.0\,\mathrm{kg})v^2 = 55.9\,\mathrm{J} \qquad \Longrightarrow \qquad v^2 = \frac{55.9\,\mathrm{J}}{4.0\,\mathrm{kg}} = 14.0\,\tfrac{\mathrm{m}^2}{\mathrm{s}^2}$$

which gives

$$v = 3.74\,\tfrac{\mathrm{m}}{\mathrm{s}}$$

The final speed of the 5.0 kg mass (in fact of both masses) is $3.74\,\frac{\mathrm{m}}{\mathrm{s}}$.

---

16. A $10.0\,\mathrm{kg}$ block is released from point A in Fig. 6.16. The track is frictionless except for the portion BC, of length $6.00\,\mathrm{m}$. The block travels down the track, hits a spring of force constant $k = 2250\,\mathrm{N/m}$, and compresses it $0.300\,\mathrm{m}$ from its equilibrium position before coming to rest momentarily. Determine the coefficient of kinetic friction between surface BC and block. [Ser4 8-35]

We know that we *must* use energy methods to solve this problem, since the path of the sliding mass is curvy.

The forces which act on the mass as it descends and goes on to squish the spring are: gravity, the spring force and the force of kinetic friction as it slides over the rough part. Gravity and the spring force are conservative forces, so we will keep track of them with the potential energy associated with these forces. Friction is a non-conservative force, but in

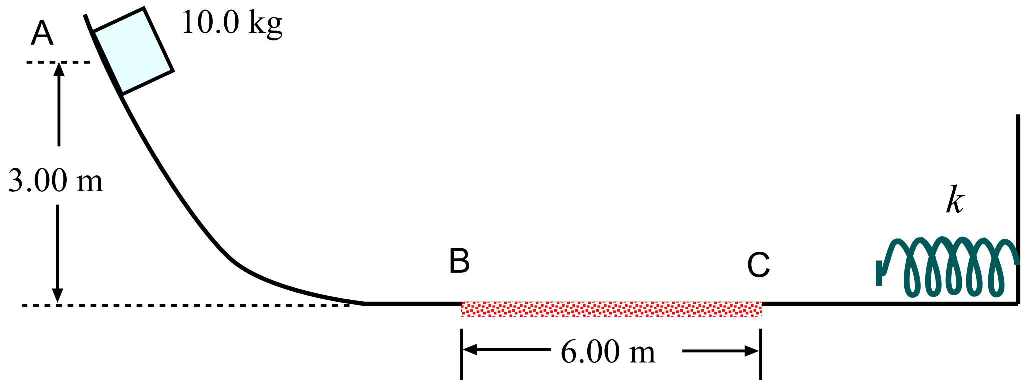

Figure 6.16: System for Example 16

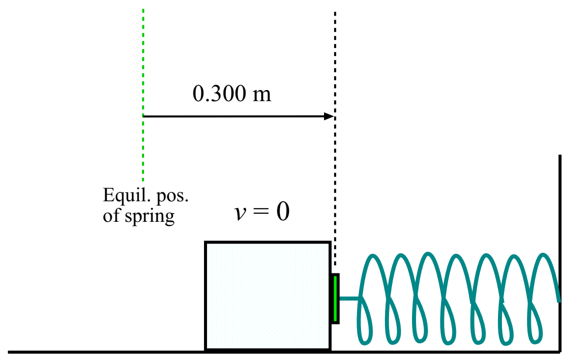

Figure 6.17: After sliding down the slope and going over the rough part, the mass has maximally squished the spring by an amount $x = 0.300\,\mathrm{m}$.

this case we can calculate the work that it does. Then, we can use the energy conservation principle,

$$\Delta K + \Delta U = W_{\rm non-cons} \tag{6.33}$$

to find the unknown quantity in this problem, namely $\mu_{\rm k}$ for the rough surface. We *can* get the answer from this equation because we have numbers for all the quantities except for $W_{\rm non-cons} = W_{\rm friction}$ which depends on the coefficient of friction.

The block is *released* at point A so its initial speed (and hence, kinetic energy) is zero: $K_i = 0$. If we measure height upwards from the level part of the track, then the initial potential energy for the mass (all of it *gravitational*) is

$$U_i = mgh = (10.0\,\mathrm{kg})(9.80\,\tfrac{\mathrm{m}}{\mathrm{s}^2})(3.00\,\mathrm{m}) = 2.94 \times 10^2\,\mathrm{J}$$

Next, for the "final" position of the mass, consider the time at which it has maximally compressed the spring and it is (instantaneously) at rest. (This is shown in Fig. 6.17.) We don't need to think about what the mass was doing in between these two points; we don't care about the speed of the mass during its slide.

At this final point, the mass is again at rest, so its kinetic energy is zero: $K_f = 0$. Being at zero height, it has no gravitational potential energy but now since there is a compressed spring, there is stored (potential) energy in the *spring*. This energy is given by:

$$U_{\rm spring} = \tfrac{1}{2}kx^2 = \tfrac{1}{2}(2250\,\tfrac{\rm N}{\rm m})(0.300\,{\rm m})^2 = 1.01\times 10^2\,{\rm J}$$

so the final potential energy of the system is $U_f = 1.01\times 10^2$ J.

The total mechanical energy of the system changes because there is a non–conservative force (friction) which does work. As the mass (m) slides over the rough part, the vertical forces are gravity (mg, downward) and the upward normal force of the surface, N. As there is no vertical motion, $N = mg$. The magnitude of the force of kinetic friction is

$$f_{\rm k} = \mu_{\rm k}N = \mu_{\rm k}mg = \mu_{\rm k}(10.0\,{\rm kg})(9.80\,\tfrac{\rm m}{\rm s^2}) = \mu_{\rm k}(98.0\,{\rm N})$$

As the block moves 6.00 m this force points *opposite* (180° from) the direction of motion. So the work done by friction is

$$W_{\rm fric} = f_{\rm k}d\cos\phi = \mu_{\rm k}(98.0\,{\rm N})(6.00\,{\rm m})\cos 180^\circ = -\mu_{\rm k}(5.88\times 10^2\,{\rm J})$$

We now have everything we need to substitute into the energy balance condition, Eq. 6.33. We get:

$$(0-0) + (1.01\times 10^2\,{\rm J} - 2.94\times 10^2\,{\rm J}) = -\mu_{\rm k}(5.88\times 10^2\,{\rm J})\ .$$

The physics is done. We do algebra to solve for $\mu_{\rm k}$:

$$-1.93\times 10^2\,{\rm J} = -\mu_{\rm k}(5.88\times 10^2\,{\rm J}) \qquad\Longrightarrow\qquad \mu_{\rm k} = 0.328$$

The coefficient of kinetic friction for the rough surface and block is 0.328.

6.2.8 Relationship Between Conservative Forces and Potential Energy (Optional?)

---

17. A potential energy function for a two–dimensional force is of the form $U = 3x^3y - 7x$. Find the force that acts at the point (x, y). [Ser4 8-39]

(We presume that the expression for U will give us U in *joules* when x and y are in meters!)

We use Eq. 6.26 to get F_x and F_y:

$$\begin{aligned} F_x &= -\frac{\partial U}{\partial x} \\ &= -\frac{\partial}{\partial x}(3x^3y - 7x) \\ &= -(9x^2y - 7) = -9x^2y + 7 \end{aligned}$$

and:

$$\begin{aligned} F_y &= -\frac{\partial U}{\partial y} \\ &= -\frac{\partial}{\partial y}(3x^3y - 7x) \\ &= -(3x^3) = -3x^3 \end{aligned}$$

Then in unit vector form, **F** is:

$$\mathbf{F} = (-9x^2y + 7)\mathbf{i} + (-3x^3)\mathbf{j}$$

where, if x and y are in meters then **F** is in newtons. Got to watch those units!