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Solutions Manual For

MATHEMATICAL METHODS

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The Complex Number System.

The set 'C' $= R \times R = \{(a,b) / a,b \in R\}$ is called the set of complex number if the following conditions are satisfied.

i) $(a,b) + (c,d) = (a+c,\ b+d)$ (Addition)

ii) $(a,b) \cdot (c,d) = (ac-bd,\ ad+bc)$ (Multiplication)

iii) $K(a,b) = (Ka, Kb)$ where $K \in R$ (Scalar Multiple)

iv) $(a,b) = (c,d)$ iff $a=c,\ b=d$ (Equality)

Note. $\boxed{(a,b) = a+bi}$ $a =$ Real Part, $b =$ Imag Part

$i = (0,1)$

$(a,b) = (a,0)+(0,b)$
$= a(1,0)+b(0,1)$
$= a(1)+b(i)$
$(a,b) = a+bi$

$i = (0,1)$
$i \cdot i = (0,1)(0,1)$
$i^2 = (0\cdot0-1\cdot1,\ 0\cdot1+1\cdot0)$
$i^2 = (-1,0)$
$i^2 = -1+0i$
$i^2 = -1$

where $(a,b) \in C$ & $a+bi \in C$
$\because (a,b) = a(1,0)+b(0,1)$
$(a,b) = a\cdot1 + b\,i = a+bi$

Multiplicative Identity in C. $(1,0) = 1 = 1+0i$

Additive Identity in C. $(0,0) = 0 = 0+0i$

Additive Inverse of (a,b) is $(-a,-b)$

Multiplicative Inverse of $(a,b) \in C$

Let $z = (a,b) = a+ib$

$z^{-1} = (a+ib)^{-1} = \dfrac{1}{a+ib}$

$= \dfrac{1}{a+ib} \times \dfrac{(a-ib)}{(a-ib)}$

$z^{-1} = \dfrac{a-ib}{a^2+b^2} = \dfrac{a}{a^2+b^2} - \dfrac{ib}{a^2+b^2}$

$z^{-1} = \left(\dfrac{a}{a^2+b^2}, \dfrac{-b}{a^2+b^2}\right)$

(To verify $zz^{-1} = (1,0) = 1$)

Modulus of $(a,b) \in C$

If $z = (a,b)$ then $|z| = \sqrt{a^2+b^2}$

Conjugate of $(a,b) \in C$

If $z = (a,b) = a+bi$
then $\bar{z} = \overline{a+bi} = a-bi$

To Prove $z\cdot\bar{z} = |z|^2 = (\operatorname{Re} z)^2+(\operatorname{Im} z)^2$

LHS $z\cdot\bar{z}$
$= (a+bi)(a-bi)$
$= a^2+b^2 = (\operatorname{Re} z)^2+(\operatorname{Im} z)^2$

$\boxed{z\bar{z} = |z|^2}$

Also $z^2 = (a+bi)(a+bi)$
$z^2 = a^2-b^2+2abi$
$|z^2| = \sqrt{(a^2-b^2)^2+(2ab)^2}$
$= \sqrt{a^4+b^4-2a^2b^2+4a^2b^2}$
$= \sqrt{a^4+b^4+2a^2b^2}$
$= \sqrt{(a^2+b^2)^2} = a^2+b^2 = |z|^2$
$\therefore \boxed{|z^2| = |z|^2}$

Imp

Th Let z_1, z_2 be complex numbers. Show that

(i) $\overline{z_1+z_2} = \overline{z_1}+\overline{z_2}$

(ii) $\overline{z_1 z_2} = \overline{z_1}\cdot\overline{z_2}$

(iii) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$

Sol (i) Let $z_1 = a+bi \quad \overline{z_1} = a-bi$

$z_2 = c+di, \quad \overline{z_2} = c-di$

LHS $z_1+z_2 = a+bi+c+di$

$z_1+z_2 = (a+c)+(b+d)i$

$\overline{z_1+z_2} = (a+c)-(b+d)i$ —— ①

RHS $\overline{z_1}+\overline{z_2} = a-bi+c-di$

$= (a+c)-(b+d)i$ —— ⑪

① = ⑪ $\Rightarrow$ $\overline{z_1+z_2} = \overline{z_1}+\overline{z_2}$

(ii)

LHS $z_1z_2 = (a+bi)(c+di)$

$= ac+adi+bci+bd(i^2)$

$= ac+adi+bci+bd(-1)$

$z_1z_2 = (ac-bd)+(ad+bc)i$

$\overline{z_1z_2} = (ac-bd)-i(ad+bc)$ —— ①

RHS $\overline{z_1}\,\overline{z_2} = (a-bi)(c-di)$

$= ac-adi-bci+bd(i)^2$

$\overline{z_1}\,\overline{z_2} = (ac-bd)-i(ad+bc)$ —— ⑪

① = ② $\Rightarrow$ $\overline{z_1z_2} = \overline{z_1}\,\overline{z_2}$

(iii) LHS $\dfrac{z_1}{z_2} = \dfrac{a+bi}{c+di}$

$\dfrac{z_1}{z_2} = \dfrac{a+bi}{c+di}\times\dfrac{c-di}{c-di}$

$= \dfrac{ac-adi+bci-bd(i^2)}{c^2-(id)^2}$

$= \dfrac{ac-adi+bci-bd(-1)}{c^2+d^2}$

$\dfrac{z_1}{z_2} = \dfrac{(ac+bd)+i(bc-ad)}{c^2+d^2}$

$\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{(ac+bd)-i(bc-ad)i}{c^2+d^2}$ —— ①

RHS $\dfrac{\overline{z_1}}{\overline{z_2}} = \dfrac{a-bi}{c-di}$

$= \dfrac{a-bi}{c-di}\times\dfrac{c+di}{c+di}$

$= \dfrac{ac+adi-bci-bd(i^2)}{c^2-(id)^2}$

$= \dfrac{ac+adi-bci+bd}{c^2+d^2}$

$\dfrac{\overline{z_1}}{\overline{z_2}} = \dfrac{(ac+bd)-i(bc-ad)}{c^2+d^2}$ —— ⑪

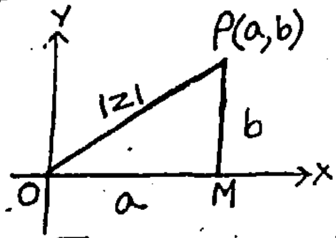
① = ⑪ $\Rightarrow \overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$

Imp

Th Show that the modulus $|z|$ of a complex number $z = a+bi$ is the distance of a point from origin, or length of $\overline{OP}$.

Sol We know to each complex number $z = a+bi$ there corresponds a pt $P(a,b)$, in the cartesian plane and vice versa.

But the point (a,b) in the plane is represented as



$|\overline{OM}| = a$

$|\overline{PM}| = b$

$|\overline{OP}| = |z|$

By pythagoras Th.

$|\overline{OP}|^2 = |\overline{OM}|^2 + |\overline{PM}|^2$

$|OP|^2 = a^2+b^2$

$|OP| = \sqrt{a^2+b^2}$

$|z| = \sqrt{a^2+b^2}$ Distance of (a,b) from $(0,0)$

Note 1)

$z = a+ib$

$\overline{z} = a-ib$

$\overline{\overline{z}} = a+ib \quad \therefore \boxed{\overline{\overline{z}} = z}$

Note 2)

$z = a+ib \Rightarrow |z| = \sqrt{a^2+b^2}$

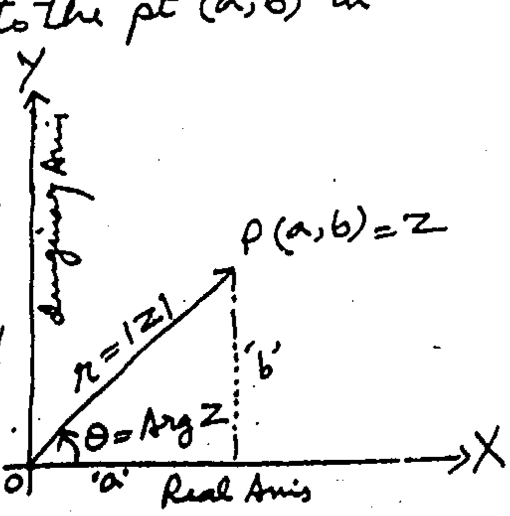
$\overline{z} = a-ib \Rightarrow |\overline{z}| = \sqrt{a^2+b^2}$

$-z = -a-ib \Rightarrow |-z| = \sqrt{a^2+b^2}$

$\therefore |z| = |-z| = |\overline{z}|$

Complex Plane

A complex number $z = a + bi$ corresponds to the pt (a, b) in the XY-plane and vice versa.



The XY-plane in which a complex number z is represented by a vector $\overline{OP}$ is called Complex Plane or Z-Plane, X-axis is called Real Axis and Y-axis is called Imaginary Axis. & figure so obtained is called Argand Diagram.

The inclination 'θ' of a complex vector $\overline{OP}$ with positive direction of x-axis is called Argument or Amplitude of z, written as $\arg z$.

$$\arg z = \theta = \tan^{-1} \frac{b}{a}$$

$\sin\theta = \frac{b}{|z|}$

$\cos\theta = \frac{a}{|z|}$

If value of θ is such that $-\pi < \theta \leq \pi$ then θ is called Principal argument of z i.e $\operatorname{Arg} z$

Arg of 0 is not defined.

Imp Note

i) When $z = (a, b)$ is in Ist Quad then angle is 'θ'

ii) When $z = (a, b)$ is in 2nd Quad then angle is $(\pi - \theta)$

iii) When $z = (a, b)$ is in 3rd Quad then angle is $-(\pi - \theta)$

iv) When $z = (a, b)$ is in 4th Quad then angle is $-\theta$'

It is because $-\pi < \theta \leq \pi$ i.e value of θ is not greater than π.

Properties

i) $|z| = |-z| = |\bar{z}|$

ii) $|z|^2 = |z^2| = z\bar{z}$

iii) $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$

iv) $|z_1 z_2| = |z_1||z_2|$

v) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$

vi) $|\operatorname{Re} z| \leq |z|$

vii) $|\operatorname{Im} z| \leq |z|$

viii) $z\bar{z} = (\operatorname{Re} z)^2 + (\operatorname{Im} z)^2$

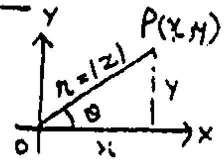
ix) $|z_1 - z_2| = |z_2 - z_1|$

x) $|z_1 - z_2| \geq |z_1| - |z_2|$

xi) $|z_1 + z_2| \leq |z_1| + |z_2|$

Polar Form of Complex Number

From fig



$x = r\cos\theta, \; y = r\sin\theta$

$$z = x + iy \quad \text{——— (I)}$$

$$= r\cos\theta + i\, r\sin\theta$$

$$= r(\cos\theta + i\sin\theta)$$

$$z = r \operatorname{cis} \theta \quad \text{——— (II)}$$

(I) is Cartesian form of complex number z.

(II) is Polar form of complex number z.

Imp

Th For all $z_1, z_2 \in C$

$$\boxed{|z_1 z_2| = |z_1||z_2|}$$

Proof Let $z_1 = a+ib$, $z_2 = c+id$, then

$$z_1 z_2 = (a+ib)(c+id)$$

$$z_1 z_2 = ac + iad + ibc + i^2 bd$$

$$z_1 z_2 = (ac-bd) + i(ad+bc)$$

$$|z_1 z_2| = \sqrt{(ac-bd)^2 + (ad+bc)^2}$$

$$= \sqrt{a^2c^2 + b^2d^2 - \cancel{2acbd} + a^2d^2 + b^2c^2 + \cancel{2adbc}}$$

$$= \sqrt{a^2(c^2+d^2) + b^2(d^2+c^2)}$$

$$= \sqrt{(a^2+b^2)(c^2+d^2)}$$

$$= \sqrt{a^2+b^2}\ \sqrt{c^2+d^2}$$

$$|z_1 z_2| = |z_1||z_2| \quad \text{proved}$$

Th Prove that for $z_1, z_2 \in C$

$$\boxed{|z_1| - |z_2| \leq |z_1 + z_2| \leq |z_1| + |z_2|}$$

Proof Let $|z_1+z_2|^2$

$$= (z_1+z_2)(\overline{z_1+z_2}) \qquad \because |z|^2 = z\bar{z}$$

$$= (z_1+z_2)(\bar{z}_1 + \bar{z}_2) \qquad \because \overline{z_1+z_2} = \bar{z}_1 + \bar{z}_2$$

$$= z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2 \qquad \because z_1\bar{z}_2 + z_2\bar{z}_1 = 2Re\, z_1\bar{z}_2$$

$$= |z_1|^2 + 2Re(z_1\bar{z}_2) + |z_2|^2 \qquad \because |Re\, z| \leq |z|$$

$$\leq |z_1|^2 + 2|z_1\bar{z}_2| + |z_2|^2 \qquad \because |z_1z_2| = |z_1||z_2|$$

$$= |z_1|^2 + 2|z_1||\bar{z}_2| + |z_2|^2 \qquad |\bar{z}| = |z|$$

$$= |z_1|^2 + 2|z_1||z_2| + |z_2|^2$$

$$|z_1+z_2|^2 \leq (|z_1|+|z_2|)^2$$

$$\boxed{|z_1+z_2| \leq |z_1| + |z_2|} \quad \text{——— (I)}$$

Now $|z_1| = |z_1 + z_2 - z_2|$ (+ − z_2)

$$\leq |z_1+z_2| + |-z_2|$$

$$|z_1| = |z_1+z_2| + |z_2|$$

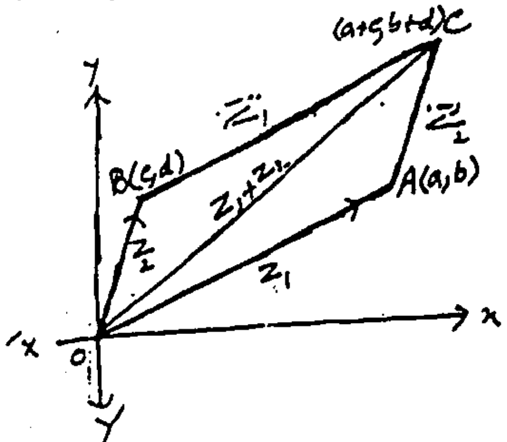
$$\boxed{|z_1| - |z_2| \leq |z_1+z_2|} \quad \text{——— (II)}$$

Combining (I) & (II) $|z_1| - |z_2| \leq |z_1+z_2| \leq |z_1| + |z_2|$

Imp 2nd Method (Also see on Page 10)

For any two complex numbers

$$|z_1| - |z_2| \leq |z_1+z_2| \leq |z_1| + |z_2|$$



Let $|z_1| = |\overline{OA}|$ $|z_1+z_2| = |\overline{OC}|$

$|z_2| = |\overline{OB}|$

In ΔOAC $|\overline{OA}| + |\overline{AC}| > |\overline{OC}|$ ——— (I)

$|z_1| + |z_2| > |z_1+z_2|$

For Collinear pts

$$|\overline{OA}| + |\overline{AC}| = |\overline{OC}|$$

$$|z_1| + |z_2| = |z_1+z_2| \text{ ——— (II)}$$

Combining (I) & (II)

$$|z_1| + |z_2| \geq |z_1+z_2| \quad \text{proved}$$

Now $|z_1| = |z_1 + z_2 - z_2|$

$$|z_1| \leq |z_1+z_2| + |-z_2|$$

$$|z_1| \leq |z_1+z_2| + |z_2|$$

$$|z_1| - |z_2| \leq |z_1+z_2| \text{ ——— (III)}$$

Combining (II) + (III) we get the result.

To Prove $|z_1| - |z_2| \leq |z_1 - z_2|$

Proof Since $|z_1+z_2| \leq |z_1| + |z_2|$

Replace z_1 by $z_1 - z_2$

$$\therefore |z_1 - z_2 + z_2| \leq |z_1 - z_2| + |z_2|$$

$$\Rightarrow |z_1| \leq |z_1-z_2| + |z_2|$$

$$|z_1| - |z_2| \leq |z_1 - z_2|$$

To Prove $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$

Proof $\because z\bar{z} = |z|^2$

$$\Rightarrow \frac{1}{z\bar{z}} = \frac{1}{|z|^2}$$

$$\Rightarrow \frac{1}{z} = \frac{\bar{z}}{|z|^2} \quad \text{proved.}$$

EXERCISE 1.1

EXPRESS each of the following complex numbers in the polar form. (Problem 1—6):

Q.1 Let $z = x + iy = -\sqrt{3} + i \Rightarrow x = -\sqrt{3},\ y = 1$, $\quad r = |z| = \sqrt{(-\sqrt{3})^2 + 1^2} = \sqrt{3+1} = 2$

$\because \cos\theta = \frac{x}{r} = \frac{-\sqrt{3}}{2} \Rightarrow \theta = \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

& $\sin\theta = \frac{y}{r} = \frac{1}{2} \Rightarrow \theta = \sin^{-1}\left(\frac{1}{2}\right)$ $\Bigg\} \Rightarrow \theta = \frac{5\pi}{6}$

(x, is -ive & y, is +ive, So θ lies in 2nd Quad. Since 2nd Quadrant, $\therefore \theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$)

Hence $z = r(\cos\theta + i\sin\theta)$

$= 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

$= 2\ \text{cis}\,\frac{5\pi}{6}$ Ans

$\boxed{\text{2nd Method}}$

$z = x + iy = -\sqrt{3} + i \Rightarrow x = -\sqrt{3},\ y = 1$

$\tan\theta = \frac{y}{x} = \frac{1}{-\sqrt{3}}$

$\theta = \tan^{-1}\left(\frac{1}{-\sqrt{3}}\right) = \frac{5\pi}{6}$

($\because$ x -ive, y +ive, $\therefore$ θ lies in 2nd Quad. So Principal Arg $= \pi - \theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$)

Hence $z = r(\cos\theta + i\sin\theta)$

$= 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

Q.2 Let $z = x + iy = -i = 0 + (-i) \Rightarrow x = 0,\ y = -1$

$\Rightarrow r = |z| = \sqrt{0^2 + (-1)^2} = 1$

$\because \cos\theta = \frac{x}{r} = \frac{0}{1} = 0 \Rightarrow \theta = \cos^{-1}(0)$

& $\sin\theta = \frac{y}{r} = -\frac{1}{1} = -1 \Rightarrow \theta = \sin^{-1}(-1)$ $\Bigg\}\ \theta = -\frac{\pi}{2}$

($\because$ x +ive & y -ive $\therefore$ 4th Quad. So Principal Arg $z = -\theta = -\frac{\pi}{2}$)

Hence $z = r(\cos\theta + i\sin\theta)$

$= 1\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$

or $z = \left(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2}\right)$

$\boxed{\text{2nd Method}}$

$z = x + iy = -i$

$\Rightarrow x = 0,\ y = -1$

$\tan\theta = \frac{y}{x} = \frac{-1}{0} = \infty$

$\theta = \tan^{-1}(\infty) = -\frac{\pi}{2}$

($\because$ x +ive, y -ive $\therefore$ 4th Quad. So Principal Arg $= -\theta = -\frac{\pi}{2}$)

Hence $z = r(\cos\theta + i\sin\theta)$

$= 1\left(\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right)$

$z = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2}$

Q.3 Let $z = x + iy = -1 - \sqrt{3}i \Rightarrow x = -1,\ y = -\sqrt{3}$

$\Rightarrow r = |z| = \sqrt{(-1)^2 + (-\sqrt{3})^2}$

$\Rightarrow r = \sqrt{1+3} = \sqrt{4} = 2$

$\because \cos\theta = \frac{-1}{2} \Rightarrow \theta = \cos^{-1}\left(-\frac{1}{2}\right)$

& $\sin\theta = \frac{-\sqrt{3}}{2} \Rightarrow \theta = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$ $\Bigg\}\ \theta = -\frac{2\pi}{3}$

(3) $z = x + iy = -1 - \sqrt{3}i \Rightarrow x = -1,\ y = -\sqrt{3}$

$\tan\theta = \frac{y}{x} = \frac{-\sqrt{3}}{-1}$ $\quad r = \sqrt{1+3} = \sqrt{4},\ r = 2$

$\theta = \tan^{-1}\left(\frac{-\sqrt{3}}{-1}\right)$

$\theta = -\frac{2\pi}{3}$

$\because$ x -ive, y -ive III Quad

$\therefore -\left(\pi - \frac{\pi}{3}\right) = -\frac{2\pi}{3}$