CARAVAN'S

MATHEMATICAL TECHNIQUES

For
B.A. / B.Sc.

by

Dr. Karamat H. Dar
Irfan-ul-Haq   M. Ashraf Jajja

Caravan Book House-Lahore

Muhammad Saad

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Pdf

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PREFACE TO 3RD EDITION

Al- "Hamdo-lillah", the 3rd edition is out for the readers. The delay in this edition is apologized, but the authors feel happy in presenting this edition with the following mentionable features:

(a) The content has been improved by introducing the following topics:

(i) Orthogonal trajectories in chapter 7.

(ii) Solution of the differential equations with undetermined coefficients in chapter 8.

(iii) The detailed discussion on permutation groups in chapter 2.

(b) The proofs of some of the results have been revised in the light of the suggestions received from our colleagues, friends and the readers.

(c) In order to elaborate some of the concepts more, new solved examples have been added in the content and a few unsolved, problems have been inserted in some of the exercises.

(d) The authors do not claim mistake-free content but they claim of putting hard work to reduce the errors and insert the omissions in order to improve upon it.

(e) Unnecessary matter has been deleted after careful examination and discussion with our colleagues and friends.

We feel it our duty to acknowledge the cooperation of our colleagues, friends and the readers in general who rendered their sincere cooperation and gave their moral support to us. The names of Mr. Mohammad Akram and Prof. Arif Javed in this regard are mentionable in particular.

The authors owe thanks to the Caravan Book House (Publishers) and Mr. Shahbaz Muzammil whose efforts have made it possible to present it to the readers in this form.

At the last and not the least at any standard, we thank, in anticipation to all those who will communicate their suggestions to the authors / publishers, for the improvement of this book.

Authors

Dated at Lahore the 3rd Sept., 1998

Preface to the Second Edition

Although, the plan and most of the features of the original manuscript have been retained in this edition but some addition's deletions and refinements have been made to increase clarity and achieve satisfaction to the liking of the readers. Notable changes are made through reorganization of exercise 4.1 and insertion of some additional problems in exercise 4.5. A few solved examples have been added wherever thought necessary.

We are indebted and deeply grateful to all our colleagues and well wishers who contributed to the improvement of the present edition either by writing us their words of encouragement, by constructive criticism or by helpful remarks and suggestions. In this regard we feel thankful, in particular to Sh. Mohammad Hafeez, Dr. Shamim Arif, M. Munir Abdi, Nazir Ahmad Cheema, Abdul Hameed, Mohammad Alam, Mohammad Saeed, Mohammad Mudassir, Abdul Karim, Mohammad Mushtaq, Dr. Sabir H. Shah, Abdul Rauf, Abdul Wahab, Obaid Ullah, Miss Shama Javed, Miss Abida, Mohammad Rafaq Siddqui, Abdur Raheem, Mohammad Nawaz Shina and Mohammad Imtiaz.

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Sept., 1, 1992

Authors

Preface to the First Edition

Our long experience of teaching mathematics developed us with the feeling of necessity of bringing some competition in the quantity and quality of good text books at the degree level. Probably no book is ever written without a lot of help and encouragement. In putting forward a humble attempt in the shape of this book, the motivation and help is provided by our sincere colleagues and teachers of the degree classes. This book comprises the introduction of some of the essential mathematical techniques for an easy understanding of the various topics. It provides the reader a foundation for further study in a broad variety of mathematical disciplines. The content being restricted to the current syllabus recommended for the B.A./B.Sc. degree classes of various local universities, has been presented in a simple manner without ignoring the mathematical vigour of the concepts. The subject matter is supported by liberal typical worked out examples followed by well-arranged exercises selected from standard books. In fact, the students are introduced, in this book, to the process of abstraction as a way of making it easier to conceptualize and cope with the problems. Every care has been taken to avoid printing and other mistakes but perfection cannot be claimed. So any intimation of errors and suggestions for improvement will be duly appreciated and acknowledged.

The authors owe an immense gratitude to their worthy teachers and respectable authors of the books from which they sought help during the process of learning mathematics at any stage.

The authors are grateful to M/s The Caravan Book House for taking up the responsibility of publishing this work.

Thanks are also due to the Design Dynamic Group who extended their cooperation in its publishable shape.

Authors

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1

COMPLEX NUMBER

1.1 INTRODUCTION

We assume that the reader is already familiar with the concept of complex number while considering the solution of $x^2 + 1 = 0$. There is no real number satisfying this equation. We then introduce the imaginary number $i = \sqrt{-1}$ by solving this equation getting i and –i as its roots. In order to solve $ax^2 + bx + c = 0$ and in general, the equation

$$x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots.. + a_n = 0,$$

we introduce the complex number x + iy which is the sum of a real number x and an imaginary number iy.

This introduction of complex numbers is not satisfactory, for it makes them appear as objects not existing in reality, that is, imaginary in the literal sense of the word.

So we introduce the complex number by the following axiomatic approach:

1.1.1 THE COMPLEX NUMBER SYSTEM

The complex number system is the set $\mathbb{C} = R \times R = \{(a,b) : a, b \in R\}$ whose elements satisfy the following properties;

(i) $(a, b) = (l, m) \Leftrightarrow a = l$ and $b = m$

(ii) $(a, b) + (l, m) = (a + l, b + m)$

(iii) If r is any real number

$$r(a, b) = (ra, rb)$$

(iv) $(a, b) \cdot (l, m) = (al - bm, am + bl)$

The elements of R x R are complex numbers. In a complex number

$z = (a, b) \in \mathbb{C}$, a is called the real part of z and b its imaginary part.

In symbols,

$$a = \text{Re}(z), \quad b = \text{Im}(z)$$

1.1.2 ALGEBRAIC ASPECTS

It can be easily verified that the above mentioned operations of addition and multiplication obey all the rules of algebra. i.e. The commutative, associative and distributive laws. The reader has already verified them in intermediate classes. For revision purposes we enlist them below;

1. COMMUTATIVE LAWS

$\forall z_1, z_2 \in \mathbb{C}$

(i) $z_1 + z_2 = z_2 + z_1$

(ii) $z_1 \cdot z_2 = z_2 \cdot z_1$

2. ASSOCIATIVE LAWS

$\forall z_1, z_2, z_3 \in \mathbb{C}$

(i) $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$

(ii) $(z_1 z_2)z_3 = z_1(z_2 z_3)$

3. DISTRIBUTIVE LAW

$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

Note: unlike real numbers, no two complex numbers are comparable by the ordering $<$ or $>$. So the statment, that a complex number $z_1 > z_2$ or $z_1 < z_2$ is meaningless for any pair z_1, z_2 of complex numbers.

The complex numbers (0, 0) and (1, 0) assume the role of additive and multiplicative identities respectively. The inverse operations of addition and multiplications — namely, subtraction and division — can always be performed within $\mathbb{C}$, except for division by (0, 0). This is evidenced by the following formulas;

If $z_1 = (a_1, b_1)$ and $z_2 = (a_2, b_2)$ and $-z_2 = (-a_2, -b_2)$, then

(i) $z_1 - z_2 = z_1 + (-z_2)$

$= (a_1, b_1) + (-a_2, -b_2)$

$= (a_1 - a_2, b_1 - b_2)$

(ii) $\frac{z_1}{z_2} = \left(\frac{a_1a_2 + b_1b_2}{a^2_2 + b^2_2}, \frac{a_2b_1 - a_1b_2}{a^2_2 + b^2_2}\right), \quad z_2 \neq 0$

which can be easily verified.

This can also be verified that $\mathbb{C}$ is closed with respect to addition, subtraction, multiplication and division, except for division by (0, 0); In other words $\mathbb{C}$ forms the structure of a field

$\mathbb{C}$ contains a subset $\mathbb{C}_o = \{(a, 0)/a \in R\}$

$$\text{We write } (a, 0) \pm (b, 0) = (a \pm b, 0)$$

$$(a, 0)\,(b, 0) = (ab, 0)$$

$$\frac{(a, 0)}{(b, 0)} = \left(\frac{a}{b}, 0\right), \quad b \neq 0$$

so by $(a, 0) \longleftrightarrow a \quad \forall\, a \in R$

the set $\mathbb{C}_0$ is in one-to-one correspondence with the real set R.

Moreover, $(a, b) = (a, 0) + (b, 0)\,(0, 1)$

but $(0, 1)\,(0, 1) = (-1, 0) \longleftrightarrow -1 = i^2$

writing $(a, 0) = a$

$(b, 0) = b$

$(0, 1) = i$

$(a, b) = a + ib$ or $a + bi$

1.1.3 GEOMETRIC REPRESENTATION OF COMPLEX NUMBERS

The one-to-one (1 – 1) correspondence between $\mathbb{C}$ and R x R = R^2 (say) suggests a geometric interpretation of the complex numbers. All one has to do is to let a point P(x, y) in a rectangular co-ordinate system represent the complex number z = x + iy.

In that way every complex number is mapped uniquely onto a specified point of the xy – plane and, vice versa, every point of that plane corresponds to one and only one complex number. The plane involved is called the complex, or Gaussian plane and is denoted by $\mathbb{C}$ or z-plane.

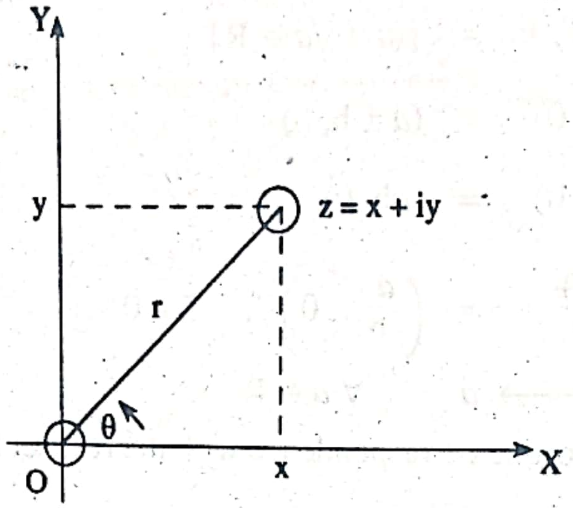
In the $\mathbb{C}$-plane, the x-axis of the co-ordinate system is called the real axis, whereas the y-axis is called the imaginary axis.

1.1.4. POLAR REPRESENTATION OF COMPLEX NUMBERS

It is convenient to have representation of z in the polar co-ordinates by introducing (r, θ) as the pair of polar co-ordinates of the point (x, y) in the z-plane.

Thus, $x = r\cos\theta$, and $y = r\sin\theta$

from which we obtain $r = \sqrt{x^2 + y^2} > 0$, $\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$



The positive quantity r is called the *modulus* or *absolute value of* z and is denoted by $r = |z|$, while θ is called the *argument* or *amplitude of* z and is denoted by

$$\theta = \arg(z) = \arg(x + iy) = \tan^{-1}\left(\frac{y}{x}\right)$$

Argument of z is not single-valued function, since to any $z \neq 0$ there correspond infinitely many values of arg (z). We shall say that arg (z) is a multi-valued function. i.e. If θ is one of the values of arg (z), we have

$$\arg(z) = \theta + 2k\pi, \quad k = 0, \pm 1, \pm 2, \ldots$$

If, $-\pi < \theta \leq \pi$ then θ is called the principal argument of z and is denoted by Arg (z). Therefore, if θ is *principal argument* then we write

$$\theta = \text{Arg}(z)$$

So, from the above figure,

$$z = x + iy = r\cos\theta + i\, r\sin\theta$$
$$= r(\cos\theta + i\sin\theta)$$

If we use the abbreviation

$$\cos\theta + i\sin\theta = \text{cis}\,\theta$$

$$z = r\,\text{cis}\,\theta \Rightarrow |z| = r, \quad |\text{cis}\,\theta| = 1$$

The complex number $x - iy$ is called the *conjugate* of $z = x + iy$.