The History Of Trigonometry

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Trigonometry is often a source of difficulty and grief for students at both the high school and collegiate level. The usefulness of trigonometry in the real world, in surveying, land measure, mensuration, and navigation, insured the importance of the subject in colonial times (Allen 71). High school students in modern times have their first experience in trigonometry in 10th or 11th grade using the concepts of sine, cosine, and tangent, to measure the angles of a right plane triangle in courses like pre-calculus or trigonometry. The emphasis in the study of plane trigonometry in the high school is changing from the study of measurement to the study of functions. The problem is that even when simply using these concepts to measure angles of a triangle, many students do not see the connections to real life, or understand where these concepts came from. Rather than incorporating trigonometric ideas with algebra and geometry, its predecessors, it is often introduced as a separate entity in the mathematical world of knowledge. As a result of having no connection to previously acquired mathematical learning, students are baffled by the intricacies of the subject and question the purpose it fulfills, not only in mathematics, but in everyday life as well. Perhaps by providing a brief insight into where these concepts came from, how they were discovered, and how their uses in the past relate to how they are currently used and taught may provide students with the extra understanding they need to put these concepts to use, whether it be as measuring devices, or as functions. With this better insight, one would be able to see the value in studying trigonometry as a component of mathematics, instead of a detached unit from the subject.

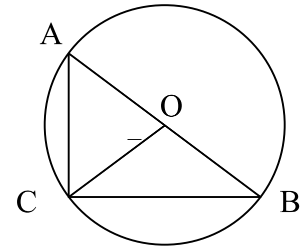
The term "trigonometry," although not of native Greek origin, comes from the Greek word *trigonon*, meaning "triangle," and the Greek word *-metria*, meaning "measurement." As the name implies, trigonometry ultimately developed from the study of right triangles by applying the relationships between the measures of its sides and angles to the study of similar triangles (Gullberg 458). However, the word "trigonometry" did not exist upon the birth of the subject, but was later introduced by the German mathematician and astronomer, Bartholomaeus Pitiscus in the title of his work, *Trigonometria sive de solutione triangularum tractatus brevis et perspicius…*, published in 1595. It was then revised in 1600 and published again as *Trigonometria sive de dimensione triangulae.*

As far as the origin of the subject is concerned, trigonometry and the development of trigonometric functions have a rich, diverse history. Trigonometry is not the work of one man or a nation. In fact, the ancient Egyptians and Babylonians had developed theorems on ratios of the sides of similar triangles (Boyer 158), before trigonometry was ever formalized as a subdivision of mathematics. These two groups had no clear usage of trigonometric functions but were able to use them unknowingly to their advantage. Egyptians used trigonometry to their benefit in land surveying and the building of pyramids. Babylonian astronomers related trigonometric functions to arcs of circles and the lengths of chords subtending their arcs (Gullberg 458). The ancestral beginnings of trigonometry are thought to be the first numerical sequences correlating shadow lengths with the time of day. The shadow tables are the ancestors of cotangent and tangent; hence, these functions were later derived from these early discoveries. Shadow tables consisted of simple sequences of numbers, which primitively applied, "the fact that the shadow of a vertical stick (gnomon)—or of a person, for that matter—is long in the early morning, shortens to a minimum at noon, then becomes longer and longer as the afternoon wears on" (Kennedy 335). The shadow tables related a particular hour of time with a particular length of the gnomon shadow, and were used as early as 1500 B.C. by the Egyptians. Similar tables were developed later by other civilizations, including the Indians and Greeks. The results of the shadow tables varied among the different civilizations, since shadow lengths were dependent on the position of the sun in relation to the place on Earth of each civilization that the shadow observations were taking place. It is important to note that these civilizations were inadvertently corresponding shadow lengths to time as a function; this attests that, at least 3000 years ago, humans inherently used the notion of a function, before even knowing or understanding what a function was.

In retrospect, shadow tables were a great development in the creation of trigonometry, but it was really the Greeks, however, who first developed trigonometry into an ordered science. "Trigonometry, perhaps more than any other branch of mathematics, developed as the result of a continual and fertile interplay of supply and demand: the supply of applicable mathematical theories and techniques available at any given time and the demands of a single applied science, astronomy" (Kennedy 333). The Greeks took over from the Babylonians as astronomers and studied the relationship between angles in a circle and lengths of chords to develop theories of planetary position and motion (Mankiewicz 18). By studying Babylonian planetary theory and astronomy, the Greeks inherited the sexagesimal number system, which is based on the number sixty (rather than our current decimal number system based on ten). "Although sixty may appear to be a large value to have as a base, it does convey certain advantages. Sixty is the smallest number that can be wholly divided by two, three, four, five, and six, and of course, it can also be divided by ten, fifteen, twenty, and thirty" (maxmon.com/1900bc.htm).

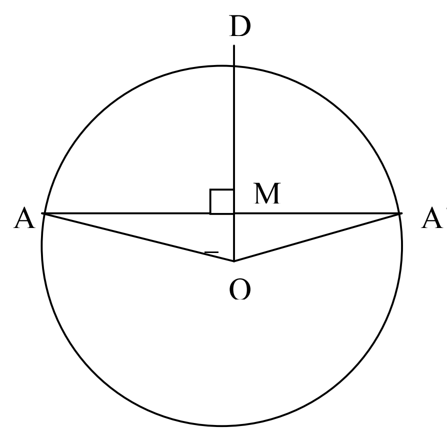
Other than that, little is known and much is lost in prehistory. "Knowledge of the subject did not grow steadily. It progressed instead by a series of discontinuous jumps" (Kennedy 335). From the primitive landmarks of shadow tables and the Greeks' gain and expansion of astronomical knowledge from the Babylonians, there was a gap in the improvement of trigonometry until the time of Hipparchus. The earliest contributions to sine, cosine and tangent came during the Alexandrian Period (300BC and 30BC). Hipparchus, a highly credited Greek astronomer who came to be known as "the father of trigonometry," had a great influence in the developments of trigonometry and is the first person whose use of trigonometry is documented (Heath 257). Trigonometric tables were created for computations related to the scientific field known as astronomy. The stars were thought to have been fixed on a sphere of great size. Only the planets moved on the sphere and therefore, to understand these positions on the sphere, mathematicians used spherical geometry (alepho.clarku.edu), which is the geometry of circles, angles, and figures on the surface of a sphere. Hipparchus was the first person to create such a trigonometric table of ratios. He did so by considering every triangle as being inscribed in a circle of large fixed radius. The advantage of choosing a large fixed radius is that fractions can be avoided because when the radius is chosen large enough, when divisions are made, these parts become whole numbers. Each side of the triangle then became a chord, defined as a straight line drawn between two points on a circle (alepho.clarku.edu).

In order to find the various parts of the triangle he needed to find the length of the chord as a function of the central angle (Maor 24). For example, in the diagram to the right, triangle ACB is inscribed in circle O. So the sides of the triangle become chord AC, chord CB and chord AB. Hipparchus would have sought to find the length of the chord, say AC, as a function of the central angle _. In essence, he deduced a trigonometric formula for the length of a chord sketched from one point on the circumference of a circle to another (Motz 26). This could therefore be used to help understand the positioning of the planets on the sphere. It is not clear really when the development of the 360 degree circle came into usage but it seems to result from Hipparchus' table of chords. Though Hipparchus is attributed as the father of trigonometry all of his work is lost except one but we gain knowledge of his work through Ptolemy.

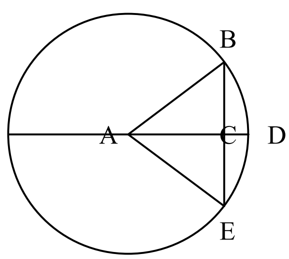


Menelaus was the next major influence on trigonometry, since he made many contributions to spherical trigonometry. He wrote a six-book treatise on chords and a three-book work, known as *Sphaerica*, which touched on the Greek development of trigonometry. In Book I of *Sphaerica*, he developed a basis for spherical triangles, which are comparable to Euclid's plane triangles. In Book II, he described the application of spherical geometry to astronomical phenomena. Book III contained the theorem of Menelaus, which became part of spherical trigonometry in the Greek form. This theorem came to play a major role in spherical trigonometry and astronomy. It was also believed that Menelaus devised a second table of chords that was based on Hipparchus', however these were also lost (Smith 615). Though Menelaus's developments were crucial, it was Ptolemy, who developed the most influential work known as *The Mathematical Syntaxis*, which was a work of thirteen books. This book later became known as "Almagest," an Arabic word meaning "greatest" because it was competing with a lesser work written by Aristarchus and was seen as superior (Boyer 164). This book was a composition of both astronomy and trigonometry and derived much of the work of Hipparchus and Menelaus. Since much of the work of Hipparchus and Menelaus was lost, it is difficult to grasp a clear portrayal of the exact developments of trigonometry. However, the *Almagest* has been passed down through many generations, and as a result, Ptolemy is often regarded as the most influential person in the development of trigonometry.

Ptolemy noted that Menelaus started by dividing a circle into 360^o, and the diameter into 120 parts. He did this because 3 x 120 = 360, using the previous application of 3 for π. Then each part is divided into sixty parts, each of these again into sixty parts, and so on. It is believed that this system of parts was based on the Babylonian sexagesimal or base 60 numeration system which was the only system available at the time for handling fractions (Maor 26). This system was based on 60 so that the number of degrees corresponding to the circumference of a circle would be the same as the number of days in a year, which the Babylonians



believed to be 360 days (Ball 243). Menelaus created a circle, and then formed an arc AD drawing lines OD and OA such that they meet at the central angle _. In the diagram above, O indicates the center point of the circle. Then a line is drawn from A, perpendicular to OD, that meets the circumference of the circle at A'. The point where AA' meets OD is called M. Since OD is the radius of the circle and AA' cuts this radius perpendicularly, Menelaus knew that AM = _ AA'. If the line OA' is drawn, the angle AOA' is now an angle of 2_. So instead of writing AM = _ AA', AM was denoted as _ the chord created by an angle 2_ or, AM = _ crd · 2_, where crd is the abbreviation for chord. It can be shown that this expression that Menelaus came up with, _ crd · 2_, is equivalent to the modern function, sin_. Implementing the current right triangle definition for sin_ (opposite side ÷ hypotenuse) and the diagram above, sin_ = AM/AO. Looking at Menelaus' construction, AM = _ crd · 2_ and the equivalences created above, we have _ AA' = AM = _ crd · 2_. So, AM/AO = AA'/2AO. Therefore, sin_ is equivalent to _ crd · 2_ (Heath 265).

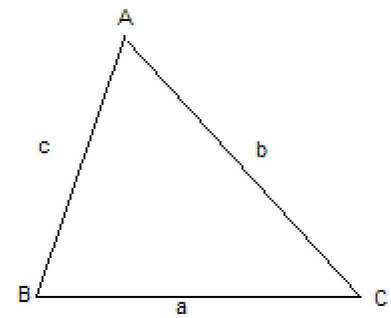


In layman's terms, a sine is half of a chord, or the sine of an angle is half the chord of twice the angle (alepho.clarku.edu). For example, look at <BAD in the figure to the left, since A is the origin, AB is the radius. Let C be the point where the perpendicular line from B hits line *AD.* Then, these mathematicians are saying that the sine of *<BAD* is defined to be the length of line BC. Nowadays, this is equivalent to saying that the angle is equal to the y component of point B, because sin_ = y. In terms of chords, if we double <BAD we get <BAE*,* so the chord of <BAE is BE. Therefore using this definition of sine, sine of angle BAD, which equals BC, is half the sine of angle BAE, which is BE. (alepho.clarku.edu). As is understood from this explanation, the Greeks' concept of a chord is directly related to our current concept of sine, the Greeks simply used a different terminology.

Since the Greeks used the chord of an arc as their function, they had no special use for the chord of the complement. However, when studies moved from spherical triangles to simple right-angled triangles, it became handy to think about the complement angle (Smith 619). This is how an idea similar to our modern cosine function was discovered. Ptolemy took Menelaus' construction _ crd · 2_ and said that the complement angle could be written as _ crd · (180^o-2_), since 180^o was half the circumference of the circle. Since today, cos_ = sin(90^o-_), it can be shown that cos_ = _ crd · (180^o-2_), using a similar argument as the one shown above (Heath 265). From these two expressions, one of the greatest identities known today was created. That is, $(\_ \text{ crd} \cdot 2\_)^2 + \{\_ \text{ crd} \cdot (180^o\text{-}2\_)\}^2 = 1$ which is exactly $\sin^2\_ + \cos^2\_ = 1$ (Heath 259). Although all of these men used roughly the same formulas to determine a similar table, it is believed that Ptolemy's work was more accurate then both Hipparchus' and Menelaus', because he worked with half chords and came up with a much more detailed table of chords. Using his table, Ptolemy believed that one could solve any planar triangle, if given at least one side of the triangle (Maor 27). Although we use sine and cosine today as a pair, as can be seen by this history, their discovery did not occur as such.

Ptolemy's *Almagest* also contained theorems corresponding to the present day law of sines, which states:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$



Compound angle and half angle identities were discussed as well. In his work, Ptolemy founded formulas for the chord of difference and an equivalent for our modern day half-angle formulas. Because of Ptolemy's discoveries, given a chord of an arc in a circle, the chord of half an arc can be determined as well. Ptolemy also discovered chords of sum and difference, chords of half an arc, and chords of half degree, from which he then built up his tables to the nearest second of chords of arcs from half degree.

It was in his work, the *Almagest*, in which a true distinction was made between plane and spherical trigonometry. Plane trigonometry is the branch of trigonometry which applies its principles to plane triangles; this includes the kind of trigonometric problems that one comes across in pre-calculus courses. Spherical trigonometry, on the other hand, is the branch of trigonometry in which its principles are applied to spherical triangles, which are triangles on the surface of the sphere. This trigonometry is only discussed in more advanced mathematics classes, such as a college-level geometry course. Ptolemy began with spherical trigonometry, for he worked with spherical triangles in many of his theorems and proofs. However, when calculating the chords of arcs, he unintentionally developed a theory for plane trigonometry. "Trigonometry was created for use in astronomy; and because spherical trigonometry was for this purpose the more useful tool, it was the first to be developed. The use of plane trigonometry...is foreign to Greek mathematicians" (Kline 126). Therefore, it is apparent that spherical trigonometry was developed out of necessity for the interest and application of astronomers. In fact, spherical trigonometry was the most prevalent branch of trigonometry until the 1450s, even though Ptolemy did introduce a basis for plane trigonometry in the *Almagest* in 150 A.D.

Next in the history of trigonometry came the age of the *Siddhantas*. This was a book thought to be established in the late fourth or early fifth century A.D. However, the origin of the information contained in this work is undecided. The Hindu scholars insisted on originality of this source, but many western writers were inclined to believe that this work stemmed from Greek influence (Boyer 209). There do appear to be similarities in this work and that of the trigonometry and astronomy of Ptolemy. "The *Siddhantas* is a compendium of astronomy made up of cryptic rules in Sanskrit verse with little explanation and no proofs…the extant version has been revised so frequently that it is difficult to say which sections are in their original form" (Kennedy 346). Even if the Hindus were influenced by the Greeks and gained knowledge from their works, the Hindu trigonometry definitely took on a new form. The trigonometry of Ptolemy was based on the functional relationship between chords of a circle and the central angles they subtend. The writers of the *Siddhantas* changed this to a study of the relationship between half of a chord of a circle and half of the angle subtended at the center by the whole chord. From this stemmed the predecessor of the modern trigonometric function known as the sine of the angle. So, the chief contribution of India and mainly the *Siddhantas* is the more formal introduction of the sine function to the history of mathematics. Though this great change took place in India, some scholars believe the transformation of trigonometry occurred in Alexandria in the post-Ptolemaic period. Whether or not this is deemed true, it is still through the Hindus that the half chord was derived, as well as the development of the word "sine" (Boyer 209).

As it was just mentioned, Indian mathematicians had their hands in trigonometry, and they created their own version of the sine table. Arya-Bhata, born in 476, was a great Indian mathematician and astronomer (Ball 147). He composed a book called *Aryabhathiya*, which contained most of the essential ideas we associate with sine and cosine. His most outstanding contribution to the topic, which distinguishes him from the other mathematicians of this time, was his work on sine differences (Clark 19). His definition of sine was literally "half chord" and was abbreviated *jya* or *jiva*, which simply meant, "chord" (Smith 615). Sines were given in minutes, at intervals of 225 minutes. This measurement was not of the sines themselves, but instead, it was the measurement of the differences between the sines. His method of calculating them was as follows. The first sine was equal to 225. The second sine was defined as any particular sine being worked with in order to calculate the sine that directly follows (Clark 29). It was found using the following pattern:

$$(225 - \text{the previous sine}) + \frac{(225 + \text{the previous sine})}{225}$$

This total was then subtracted from 225 to obtain the sine table. For example,

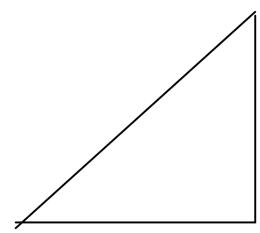
Second sine:
$225 – 225 = 0$
$225 / 225 = 1$
$0 + 1 = 1$
$225 – 1 = \underline{224}$

Third sine:
$225 – 224 = 1$
$(225 + 224) / 225 \cong 2$
$225 – 2 = \underline{222}$
(Clark 29).

Arya-Bhata had concluded that if one divided a quarter of the circumference of a circle (essentially one quadrant of the unit circle) into as many equal parts as desired, with the resulting triangles and quadrilaterals one would have, on the radius, the same amount of sines of equal arcs. Doing this, he was able to form a table of natural sines corresponding to the angles in the first quadrant (Ball 147). Although much of his work had the right idea, many of Arya-Bhata's calculations were inaccurate. Later, in 1150AD, an Indian mathematician known as Bhaskara gave a more accurate method of constructing a table of sines, which considered sines in every degree (Smith 615). Although the Indian mathematicians made attempts at creating a table to help with astronomy, their table of sines was not as accurate as that of the Greeks.

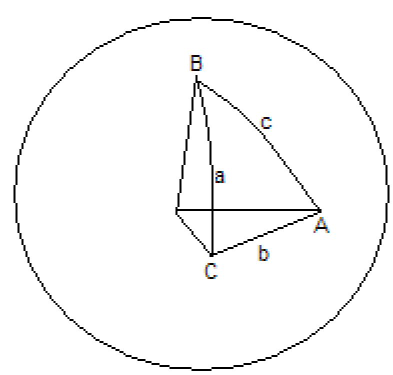
In Arabia, there was a competition of two types of trigonometry, that of the Greeks, dealing with the geometry of chords, and that of the Hindus, involving their table of sines. There was a conflict, but the Hindus prevailed. The Arabs adopted the Hindu line of thinking, resulting in most Arab trigonometry being based on the sine function (Boyer 237). In the period from the ninth century to the fifteenth century the new sine function and the old shadow functions, such as tangent, cotangent, and secant, were tabulated as sexagesimals. As a result of this development, the first real trigonometry emerged, and the object of study then became the spherical or plane triangle, with focus on its sides or angles (Kennedy 334). The sine and cosine functions were developed in the context of astronomy, but the tangent and cotangent functions were derived from practical measurements of heights and distances.

Later on, a Hindu by the name of Ahmed Ibn Abdallah became interested in shadows in relation to the sundial. He was fascinated with the shadows created when the sun hit an object, and he wanted to find a way to determine the height of the object hit by the sun using the angle of the suns rays and information about the shadow. In the diagram shown, he named the straight shadow "umbra recta" and the turned shadow "umbra versa" (Smith 620). As time moved on, these shadows became known as the horizontal and vertical shadows, and using these shadows, Abdallah, in 860 AD, created a table, which today corresponds to our table of tangents and cotangents (Smith 620). This table was used to measure heights.



Abu'l-Wefa is believed to have helped introduced the concept of the tangent function. He also may have had something to do with the development of secant and cosecant. He was an algebraist as well as a trigonometer. His trigonometry took on a more systematic form in which he proved theorems for double and half angle formulas. The law of sines, is also attributed to Abu'l-Wefa, even though it was first introduced by Ptolemy. This is in part due to the fact that Abu'l-Wefa presented a straightforward formulation of the law of sines for spherical triangles, which states

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C},$$



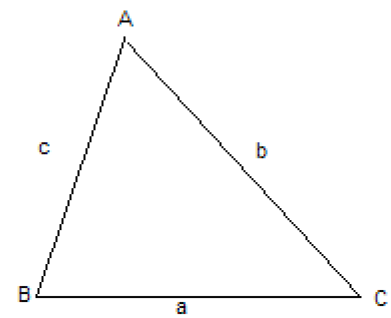
where A, B, and C are surface angles of the spherical triangle and a, b, and c are the central angles of the spherical triangle. He developed a new sine table, using eight decimal places, and he contributed a table of tangents. The table of tangents made use of all six of the common trigonometric functions (Boyer 238). Strangely, unlike the break in time between the discovery of sine and cosine, tangent and cotangent developed side by side. Even though there was no difficulty in calculating the tangent and many other trigonometric functions of an angle using solely the Greek table of sines, the Indian mathematicians were the first to give a real-life application to tangent, relating it to shadows and heights. So, by 1151 AD, the ideas of the six trigonometric functions existed, they were just not named as we know them today.

It is from the Arabic influence that trigonometry reached Europe. Western Europe favored Arabic mathematics over Greek geometry. Arabic arithmetic and algebra were on a more elementary level than Greek geometry had been during the time of the Roman Empire. Romans did not display much interest in Greek trigonometry or any facets of Greek math. Therefore, Arabic math appealed to them since it was easier for them to comprehend. Leonardo Fibonacci was one mathematician who became acquainted with trigonometry during his extensive travels in Arab countries. He then presented the knowledge he gained in *Practica geometriae* in 1220 AD (Gullberg 464).

The first revelation of trigonometry as a science separate from astronomy is credited to the Persian, Nasir Eddin. He helped to differentiate plane trigonometry and spherical trigonometry. Other than that, little development occurred from the time of the 1200's to the 1500's, aside for the developments of the Germans in the late 15th and early 16th century. Germany was becoming a prosperous nation at the time and was engaged in much trade. Their interests also developed in navigation, calendar formation, and astronomy. This interest in astronomy precipitated a general interest and need for trigonometry (Kline 238). Included in this movement around the time of 1464, the German astronomer and mathematician, Regiomontanus (also known as Iohannes Molitoris) formulated a work known as *De Triangulis Omnimodis*, a compilation of the trigonometry of that time. When it was finally printed in 1533, it became an important medium of spreading the knowledge of trigonometry throughout Europe (Gullberg 464). The first book began with fifty propositions on the solutions of triangles using the properties of right triangles. Although the word "sine" was derived from the Arabs, Regiomontanus read the term in an Arabic manuscript in Vienna and was the first to use it in Europe. The second book began with a proof of the law of sines and then included problems involving how to determine sides, angles, and areas of plane triangles. The third book contained theorems found on Greek spherics before the use of trigonometry, and the fourth was based on spherical trigonometry.

In the sixteenth century the greatest progress in mathematics was algebra; however, trigonometry was not far behind. Nicholas Copernicus was a revolutionary astronomer who could also be deemed as a trigonometer. He studied law, medicine and astronomy. He completed a treatise, known as *De revolutionibus orbium coelestium,* the year he died in 1543. This work contained information on trigonometry and it was similar to that of Regiomontanus, although it is not clear if they were connected or not. While this was a great achievement, Copernicus' student, Rheticus, an Indian mathematician, who lived during the years 1514-1576, went further and combined the work of both these men and published a two-volume work, *Opus palatinum de triangulus*. Trigonometry really began to expand and formalize at this point. The functions with respect to the arc of a circle were disregarded. Outstandingly, even though most work had previously been done using spherical triangles, he knew that the ratios sine, cosine, tangent, and cotangent could all be defined with the use of the ratios of the sides of a right-angled triangle, instead of using the circle (Ball 226). All six trigonometric functions came into full use, and Rheticus had calculated elaborate tables of them (Boyer 292). He began tables of tangents and secants but never had time to finish them.

Also around this time surfaced Francois Viete who practiced law and spent his leisure time devoted to mathematics. He contributed to arithmetic, algebra, geometry and trigonometry. He came to be known as "the father of the generalized analytic approach to trigonometry" (Boyer 307). He thought of trigonometry as an independent branch of mathematics, and he worked without direct reference to chords in a circle. He also made tables for all six trigonometric functions for angles to the nearest minute. Viete was also one of the first to use the formula for the law of tangents, which states the following:



$$\frac{a-b}{a+b}=\frac{\tan[\frac{1}{2}(A-B)]}{\tan[\frac{1}{2}(A+B)]}.$$

Trigonometric identities of many forms were also appearing at this time in Europe. As a result, there was a reduced emphasis on the computation of solutions of triangles and what developed was a greater focus on analytic functional relationships (Boyer 308).

Though Viete was one of the first mathematicians to focus on analytical trigonometry, that branch of trigonometry which focuses on the relations and properties of the trigonometric functions, this new trigonometry became more prevalent around the time of 1635 with the work of Roberval and Torricelli. They developed the first sketch of half an arch of a sine curve. This became an important development, since it assisted in the progression of trigonometry from a computational emphasis, which dominated most of the thought at this time, to a functional approach. This development formed the basis of the European contribution of trigonometry. From the influence of oriental scientists, the Europeans focused on the computation of tables and the discovery of functional relations between parts of triangles. However, Europe developed appropriate symbols, which replaced the verbal rules and ordinary language in which the subject was usually presented. Previously, trigonometry was expressed in lengthy passages of confusing words, but the Europeans introduced such symbols as sin, cos, tan, etc. to simplify the subject and make it more concise. "But the invention of the infinitesimal Calculus, following hard after, foreshadowed the speedy end of trigonometry as an independent and growing branch of mathematics: for with the discovery of and exploitation of the complex domain the whole mass of theory was subsumed into analysis" (Kennedy 334).

Prior to the analytic approach, the main usage of trigonometry was to measure geometric figures, but the transition of its influence from geometry to calculus began with the discovery of infinite series representations for the trigonometric functions. Trigonometric series became useful in the theory of astronomy, around the time of the eighteenth century. Since astronomical phenomena are periodic, it was useful to have trigonometric series because they are periodic functions as well. Use of trigonometric series was introduced to determine the positions of the planets and interpolation, which is a mathematical procedure that estimates the values of a function at positions between given values (Kline 454).

Euler also had some input in the development of trigonometry. Around 1729, he began to take an interest in interpolation. In using this method he was able to develop a trigonometric series representation of the function (Kline 454). The idea of function became an integral part of trigonometry and analysis, credited to Euler in his work, *Introductio*. In this work, strict analysis of trigonometric functions was established. Sine was no longer a line segment; rather, it transformed into a number or ratio, the ordinate point on a unit circle (Boyer 443). First introduced in the *Introductio* as well were the common names of our trigonometric functions today, namely sin, cos, tan, csc, sec, and cot. His versions are very similar to our modern day English versions. However, today this branch of mathematics far extends the uses of the eighteenth century. It far surpasses the study of triangles in modern mathematics because today it is used in such fields as cartography, surveying, astronomy, and navigation (Gullberg 458).

Yet, even after one rummages through the history of trigonometry, finding information about its beginnings and uses, many questions still remain. Only one thing is clear: “The earlier lengths are uncertain conjectures pieced out with an occasional fact” (Kennedy 335). The history does provide a basis for the initial beginnings of trigonometry. Certainly, it was derived from the field of astronomy and from the Egyptian and Babylonian intelligences. Its later development and progression can be attributed to the Greeks, Hindus, and Arabs. These influences eventually led to its expansion throughout Europe. However, many of the great works of the past, such as those contributions from Hipparchus and Menelaus, are either only known through mention in the works of later mathematicians or are completely lost. Due to this, one is deprived of really knowing all of the intentions of the primary developers of trigonometry, and as a result, one may still feel lost or unfulfilled. “The ever-accelerating development of trigonometry provides a ready illustration of the fact that knowledge tends to accumulate at a rate proportional to the quantity that is already at hand; broadly speaking, its growth is exponential with respect to time” (Kennedy 334). So much was being discovered at the onset of the development of trigonometry, and consequently, it was impossible to record every little bit of data and information that was introduced at that time. Karpinski is noted as saying, “Few of the widely circulated general works in the history of mathematics place proper emphasis upon the significance of trigonometrical ideas in the evolution of mathematical ideas” (Allen 10). Therefore, the early history of trigonometry is very vague, with respect to the details of why certain trigonometric principles were developed and initiated in the first place.

Despite the shortcomings of its history, trigonometry is still a very important component of mathematics. The history marks the progression of its uses in astronomy and geometry to its advancements as trigonometric functions and series in calculus. The movement from spherical trigonometry to plane trigonometry allowed for further developments and uses in everyday life. Plane trigonometry spanned from such necessities as surveying during the eighteenth century. Since then there has not been much progression in the field of trigonometry but it is considered an important component of calculus and geometry. As Wentworth and Smith point out,

> "…the history of trigonometry is, from one standpoint, the history of the world: [1] on the use of the shadow in the life of the people, as seen in the telling time and the seasons; [2] in the growth of mathematical symbols along with that of measures and their use; [3] in the relation of algebra to trigonometry and of each to surveying and to astronomy; and [4] in the growth of trigonometry in relation to such important sciences as physics and mechanics and to the recent remarkable advances in our studies of the universe in which we seem to play such an infinitesimal part." (Allen 8)

In society today, trigonometry is used in physics to aide in the understanding of space, engineering and chemistry. Within mathematics it is typically seen in mainly in calculus, but also in linear algebra and statistics. (aleph0.clarku.edu) By exploring the history and beginnings of trigonometry, one can see its importance and necessity in mathematics.

Being given a brief history of trigonometry, one might want to understand how these concepts are related to the concepts that are taught in a trigonometry course. Elementary trigonometry has, as central ideas, the set of six lines, ratios or functions, the sine, cosine, tangent, cotangent, secant and cosecant. Defined as lines, these six concepts may be viewed as segments and be thought of, with reference to the unit circle, as functions of arc length or central angle measures. Defined as trigonometric ratios, the six represent different measures of sides of a right triangle, taken two at a time. Finally, defined as trigonometric functions of angles, the six may be displayed as a winding or wrapping function on the unit circle (Allen 7). In the early nineteenth century, Algebra and Geometry became college admission requirements, rather than college level studies, but trigonometry still lacked a strong presence. By the 1890's, Elementary Trigonometry was made a component of college freshman mathematics in both the United States and Canada, and could also be allowed as a high school senior offering, for interested students (Allen 32). It was first taught as entirely geometrical, in terms of trigonometric line, and later ratio definitions became the common practice. Unfortunately as time went on, the general consensus, as stated by Minnick, was that

> "The majority of teachers follow the textbook page by page. They lack either the ability or preparation to select new material or vary the organization. For such teachers, the textbook is the only source of material and it virtually serves as a series of lesson plans, since the pages are slavishly followed." (Allen 43)

So, students who are presently experiencing difficulty with concepts in trigonometry may be experiencing the same difficulties as past learners, being that the subject was not taught with the correct preparation and organization in earlier years. Reliance on the textbook lead to poorly emphasized concepts and problems in the students' understanding of the material. Since the study of trigonometry became more of a college-prep course, rather than a first-year college offering, the United States began working to fix this problem, to better prepare elementary trigonometers. Unification of secondary school mathematics resulted in a right-triangle unit in junior high school, angle trigonometry as an upper level elective, and functional trigonometry as a senior (Allen 72). In 1903, course content in specific subject areas was outlined in detail to meet college entrance requirements. Definitions of requirements for trigonometry were as follows:

> "Definitions and relations of the six trigonometric functions as ratios; circular measurement of angles
>
> Proofs of principal formulas, in particular for the sine, cosine, and tangent of the sum and difference of two angles, of the double angle and the half angle, the product expressions for the sum or the difference of two sines or of two cosines, etc.; the transformation of trigonometric expressions by means of these formulas
>
> Solution of trigonometric equations of a simple character
>
> Theory and use of logarithms (without the introduction of work involving infinite series)
>
> The solution of right and oblique triangles, and practical applications, including the solution of right spherical triangles"
>
> (Allen 91)

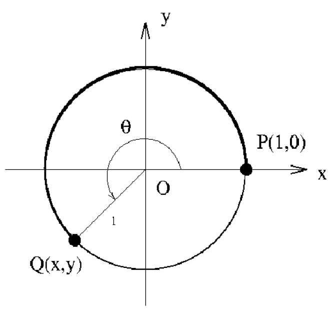
These definitions continued to serve as "College Board" mathematics requirements until 1923, when they were replaced and when junior high schools were developed. With junior high school years now available, it was hoped that an algebraic approach to trigonometry in grade nine would be beneficial and lay the groundwork for the ratio definitions that followed in grade ten (Allen 108). This ninth grade unit was to be called "Numerical Trigonometry" and it was described as,

> "The use of sine, cosine, and tangent in solving right triangles
>
> The use of the four-place tables of natural trigonometric functions is assumed, but the teacher may find it useful to include some preliminary work with three-place tables
>
> It is important that the pupil should acquire facility in simple interpolation; in general, emphasis should be laid on

carrying the computation to the limit of accuracy permitted
by the table." (Allen 123)

The new trigonometry (tenth grade) syllabus gave greater detail then the one from 1903, and included topics like, reduction formula, circular measure, "fundamental formulas," derived identities, and a complete omission of Spherical Trigonometry (Allen 124). During 1940-1957, the cold war presented mathematical needs in society. For example, it was believed that students had entered the armed forces without adequate understanding of maps, graphs, logarithms and basic concepts of trigonometry (Allen 150). From here on in, trigonometry and its place and function in junior high and high school mathematics took a huge turn. The course description and syllabus were advanced, and curricular recommendations were applied to classroom materials and into educational practice (Allen 179). When Sputnik was created, there was a need to create abler students to compete with other nations. In the 1950's, The Program for College Preparatory Mathematics advocated the teaching of Trigonometry concepts in junior high school, fundamental trigonometry in grade 11, based on coordinates, vectors, and complex numbers, and circular functions in grade 12 (Allen 185).

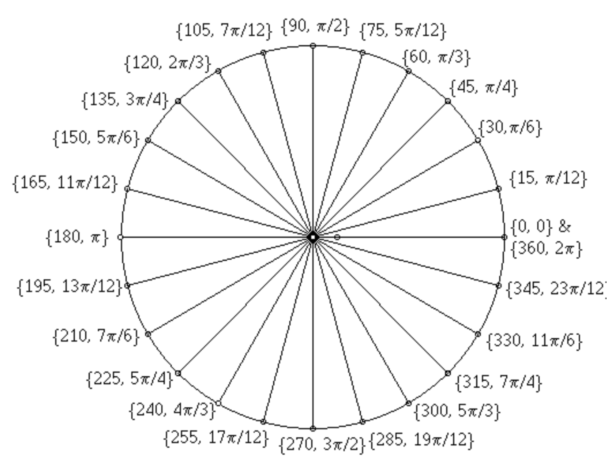
Picture from: www.sosmath.com



Most modern trigonometry textbooks begin with a chapter on angles, degrees and radians, using an x-y coordinate plane, with examples like the diagram given to the left. All measurements begin at the x-axis and, by rotating counter clockwise, angle measures can be determined. Radian, a topic that was not discovered till many years after the discovery of the concepts of sine, cosine and tangent, is strangely now taught before these topics, almost to segue into them. Students are then asked to consider the unit circle, defined as a circle with a radius of 1 unit and center at the origin of a rectangular coordinate system (Blitzer 437). The unit circle is taught using important degree measures (30, 60, 45, 90 degrees, etc) and then students are asked to put them into radian form as well. This leaves them with the given diagram as a tool. The textbooks then incorporate this tool, the labeled unit circle, into the next lesson to help students learn sine, cosine and tangent. Sine, cosine, and tangent are first introduced as functions, expressed as sin(t), cos(t), and tan(t). Their initial definitions are as follows:

$\sin(t) = y$ $\cos(t) = x$ $\tan(t) = y/x$

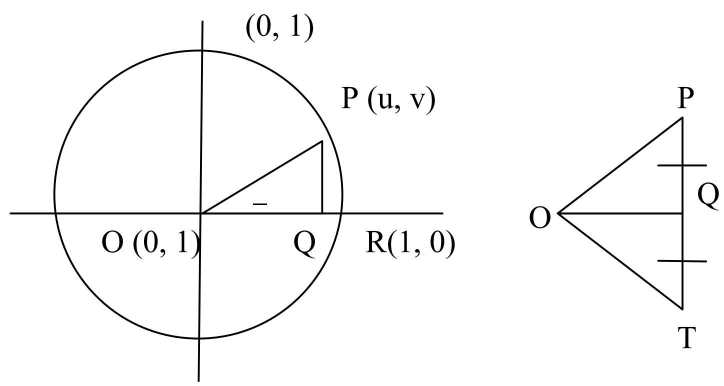
Therefore, their function values are expressed in terms of coordinates of a point on the unit circle, sine being the y-coordinate, cosine the x-coordinate, and



Picture from: http://math.rice.edu

tangent the ratio of the two (Blitzer 439). Textbook examples and problems ask students to plug and chug values directly into these definitions. For example, in Robert Blitzer's textbook, *Precalculus*, problems such as, "Find the values when t=π/2," are given, where students just plug this value into each function to find the answer (440). Unfortunately, although students are able to successfully complete problems of this type, it is questionable whether students actually understand the three concepts, or if they are thoughtlessly mimicking the solving method. Next, students are given an explanation of how to use these functions in the context of right triangles. This is where a better understanding of the functions can be gained, if taught properly. An example of how many high school students are introduced to this topic is as follows:

Given a circle with arc PR, create a right triangle OPQ by connecting point P to the origin and dropping a perpendicular line form point p to the x-axis. A matching triangle can be formed directly below triangle OPQ, sharing side OQ, to form an equilateral triangle OPT. If given the measure of angle _, say 30^o, a student is taught how to find the length of side, or chord, PQ. Since PO = OT = TP, we know PQ = _ TP = _ PO.



If PO, the radius of the circle, is equal to 1, then v is the coordinate that corresponds to side PQ and therefore v = _. So, the Pythagorean Theorem can be used to find the point P (Willoughby 328). Defining sine and cosine is then done by explaining that sin_ is the vertical coordinate, while cos_ is the horizontal coordinate, and tan_ becomes the ratio sin_/cos_. Therefore, in the above diagram, point P becomes (cos_, sin_). If we compare this diagram with the one used by the Greek mathematicians, in the discovery of the chord/sine tables, we should take note of the resemblance. The concept of the unit circle, and the triangle we form to create it, directly correlate to the concepts the ancient Greeks used to determine their table of chords. So our unit circle mimics the Greeks' thinking.

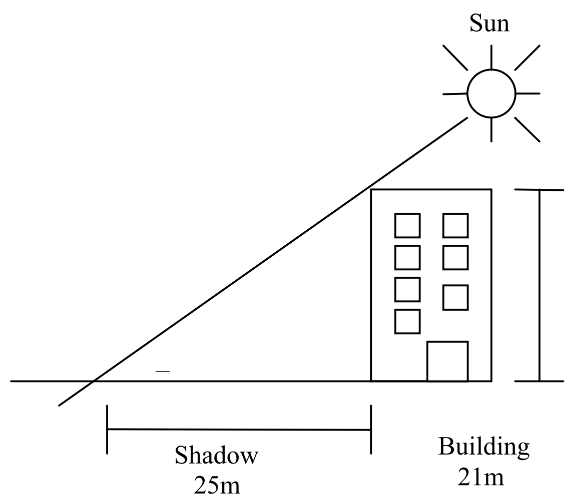
From this diagram, students are taught the relationships between the three functions using a mnemonic device, like the popular phrase "SOHCAHTOA", where

SOH: sin_ = $\frac{\text{opposite side}}{\text{hypotenuse}}$ CAH: cos_ = $\frac{\text{adjacent side}}{\text{hypotenuse}}$

TOA: $\tan\_ = \frac{\sin\_}{\cos\_} = \frac{O}{H} \div \frac{A}{H} = \frac{O}{A} = \frac{\text{opposite side}}{\text{adjacent side}}$

Students then learn how to use the Pythagorean Theorem and these ratios to solve for lengths and angles of many different triangles. The connection between how sine and cosine were discovered is not directly seen in the current teaching. However, ideas in the teaching of tangent do seem to directly correspond to its Indian discovery and early Egyptian and Babylonian uses. Blitzer poses the following textbook problem in chapter 4, directly after teaching students about trig functions related to right angles:

> "6. A building that is 21 meters tall casts a shadow that is 25 meters long. Find the angle of elevation of the sun to the nearest degree." (460)



This problem is almost an exact replica of the thought processes that Indians used to try to determine the heights of objects that cast a certain shadow, in the umbra recta and umbra versa diagram. The angle _ can be found by finding the ratio of the y length to the x length, giving us the equation tan_ = y/x = 21m/25m. This application problem is a great way to introduce the history of the subject and to help students better understand where these concepts originated. The methods for teaching sine and cosine do not have this direct correlation between how they were discovered in Greek times, and how they are currently taught, but tangent provides us with an almost perfect glimpse into the past and the reasons these functions were needed and discovered.

As can be seen, the history of sine, cosine, and tangent, as well as the general history of trigonometry, is ultimately entangled with astronomy. Most mathematicians who studied astronomy created or used trigonometry in some form to perform their calculations. The creation of trigonometry took nearly a thousand years, and it was being developed and expanded before anyone even recognized it as a subject worth studying. Although much of the work done in these times was not entirely accurate, the concepts that came out of their research are indispensable and are still with us today. Without the study of trigonometry, little would have been accomplished in the fields of navigation, land measure, mensuration, and surveying. There is a great necessity for students to be introduced to this topic, and although it is a great source of frustration for many, trigonometry is an essential part of the mathematics core curriculum. Even though not much has developed with trigonometry since the eighteenth century, perhaps the future will hold some bright minds that will make remarkable discoveries. Only time will tell if the richness of the history of trigonometry will grow even greater.

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