

# INTRODUCTION TO GEOMETRY

Lecture notes by Răzvan Gelca



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# Chapter 1

## Absolute Geometry

### 1.1 The axioms

#### 1.1.1 Properties of incidence

Lines and points are primary notions, they are not defined. A point can belong to a line or not.

- I1. Given two points, there is one and only one line containing those points.
- I2. Any line has at least two points.
- I3. There exist three non-collinear points in the plane.

When a line contains a point, we also say that the line passes through that point. Points are denoted by capital letters of the Roman alphabet. Given two distinct points  $A$  and  $B$ , the unique line passing through  $A$  and  $B$  is denoted by  $AB$ . Sometimes lines will be denoted by lower case letters of the Roman alphabet. Several points belonging to the same line are called collinear. If  $C$  belongs to  $AB$  we write  $C \in AB$ . We identify a line with the set of all the points belonging to it.

**Example 1.1.1.** *The Euclidean plane.*

**Example 1.1.2.** *The plane consists of three noncollinear points  $A, B, C$ , and the lines are the sets  $\{A, B\}$ ,  $\{A, C\}$ ,  $\{B, C\}$ .*

**Problem 1.1.1.** *Show that if  $A, B, C$  are three non-collinear points, then the lines  $AB$ ,  $BC$ , and  $AC$  are pairwise distinct.*

*Proof.* We argue by contradiction (*reductio ad absurdum*). Assume that  $AB = BC$ . We know that  $C \in BC$ , and hence  $C \in AB$  as well. This implies that  $A, B, C$  are collinear, a contradiction. So our initial assumption was false, which implies that  $AB \neq BC$ . QED  $\square$

#### 1.1.2 Properties of ordering

The notion of *between* is not defined, but it is given through its properties. It applies to three points  $A, B, C$ , with  $A \neq C$ , by saying that “ $B$  is between  $A$  and  $C$ ”.

- O1. If  $B$  is between  $A$  and  $C$  then  $A, B, C$  are collinear and  $B$  is also between  $C$  and  $A$ .
- O2. If  $B$  is between  $A$  and  $C$ , then  $A$  is not between  $B$  and  $C$ .

- O3. If  $A, B, C$  are collinear and distinct such that  $A$  is not between  $B$  and  $C$  and  $C$  is not between  $A$  and  $B$ , then  $B$  is between  $A$  and  $C$ .
- O4. If  $B$  is between  $A$  and  $C$  and  $C$  is between  $B$  and  $D$ , then  $B$  and  $C$  are between  $A$  and  $D$ .
- O5. If  $A$  and  $B$  are distinct points, there is  $C$  such that  $B$  is between  $A$  and  $C$ .
- O6. If  $A$  and  $B$  are distinct points, there is  $C$  between  $A$  and  $B$ .

**Theorem 1.1.1.** Every line contains infinitely many points.

*Proof.* Let us consider a line. By I2 it contains two points  $A$  and  $B$ . Using O5 we can find a point  $M_1$  such that  $B$  is between  $A$  and  $M_1$ . From O1 and O2 we deduce that  $M_1 \neq A$  and  $M_1 \neq B$ .

Using O5 we can find  $M_2$  such that  $M_1$  is between  $B$  and  $M_2$ . Then by O4,  $M_1$  and  $B$  are between  $A$  and  $M_2$ . Consequently  $M_2$  is different from  $A, B, M_1$ .

Next using O5 we can find  $M_3$  such that  $M_2$  is between  $M_1$  and  $M_3$ . By O4,  $M_1$  and  $M_2$  are between  $B$  and  $M_3$ , and so by the same O4,  $B, M_1, M_2$  are between  $A$  and  $M_3$ . So  $M_3$  is different from  $A, B, M_1, M_2$ .

Inductively we construct the sequence of points  $M_1, M_2, M_3, \dots$ , such that for each  $k > 1$ ,  $B, M_1, M_2, \dots, M_{k-1}$  are between  $A$  and  $M_k$ . These points are distinct, so the line has infinitely many points.  $\square$

**Definition.** Given two points  $A, B$ , the (closed) *segment*  $|AB|$  consists of  $A, B$  and all points between  $A$  and  $B$ .

**Definition.** Given two points  $A, B$ , the *ray*  $|AB$  is the set of all points  $M$  such that  $A$  is not between  $M$  and  $B$ .

**Definition.** Given three noncollinear points  $A, B, C$ , the *half-plane* containing  $C$  and bounded by  $AB$  is the set of all points  $M$  such that there does not exist  $N$  on  $AB$  with  $N$  between  $C$  and  $M$ .

**Definition.** Given three noncollinear points  $A, B, C$ , the triangle  $\triangle ABC$  is the union of the segments  $|AB|$ ,  $|BC|$ , and  $|AC|$ .

We introduce one more ordering axiom.

- O7. (Pasch' axiom) If a line does not pass through any of the noncollinear points  $A, B, C$  and intersects the segment  $|AB|$ , then it intersects one and only one of the segments  $|AC|$  and  $|BC|$ .

**Definition.** Given the points  $A_1, A_2, \dots, A_n$ , the union of the segments  $|A_1A_2|, |A_2A_3|, \dots, |A_nA_1|$  is called a *polygon* ( $\pi\omicron\lambda\upsilon\gamma\omicron\upsilon\omicron\nu$ ). If  $n = 3$ , the polygon is a triangle, if  $n = 4$  it is a quadrilateral. The points  $A_1, A_2, \dots, A_n$  are called *vertices*, while the segments  $|A_1A_2|, |A_2A_3|, \dots, |A_nA_1|$  are called *sides*.

If nonconsecutive sides intersect, the polygon is called *skew*. By default, we assume that polygons are nonskew. If any two consecutive vertices determine a line such that all other vertices are in the same half-plane determined by that line, then the polygon is called *convex*.

In a polygon  $A_1A_2 \dots A_n$ , if  $A_j$  and  $A_k$  are *not* consecutive vertices, then the segment  $|A_jA_k|$  is called a *diagonal*.

**Definition.** We call *angle* the union of two rays with the same origin.

If the rays are  $|AB$  and  $|AC$  we denote the angle they form by  $\angle BAC$  or  $\widehat{BAC}$ .

Adjacent, supplementary, and opposite angles are defined as in Figure 1.1.

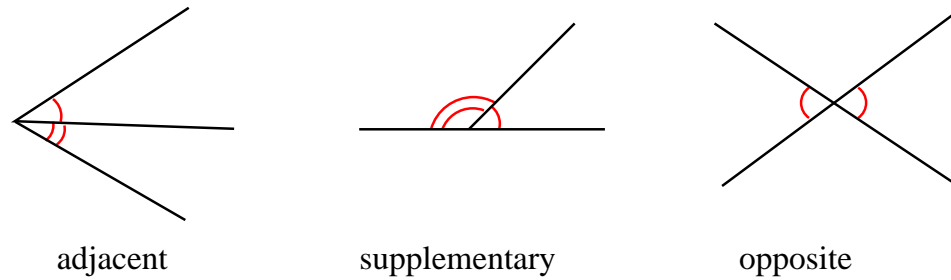


Figure 1.1: Types of angles

### 1.1.3 Congruence

“Congruence” is the notion of equality in Euclidean geometry, in the same way as “isomorphic” is the notion of equality in group theory. The congruence of segments and angles is again a primary notion, defined by properties, but intuitively two segments or angles are congruent if one can be overlaid on top of the other. Congruence is denoted by  $\equiv$ .

- C1. Given a ray with origin  $O$  and a segment  $|AB|$ , there exists one and only one point  $M$  on the ray such that  $|OM| \equiv |AB|$ .
- C2.  $|AB| \equiv |AB|$  and  $|AB| \equiv |BA|$ .  
 If  $|AB| \equiv |A'B'|$  then  $|A'B'| \equiv |AB|$ .  
 If  $|AB| \equiv |A'B'|$  and  $|A'B'| \equiv |A''B''|$  then  $|AB| \equiv |A''B''|$ .
- C3. If  $B$  is between  $A$  and  $C$  and  $B'$  is between  $A'$  and  $C'$ , and if  $|AB| \equiv |A'B'|$  and  $|BC| \equiv |B'C'|$ , then  $|AC| \equiv |A'C'|$ .
- C4. Given an angle  $\angle AOB$  and a ray  $|O'A'$  and given any of the half-planes bounded by  $O'A'$ , there is a unique ray  $OB'$  contained in this half-plane such that  $\angle AOB \equiv \angle A'O'B'$ .
- C5.  $\angle AOB \equiv \angle AOB$ ,  $\angle AOB \equiv \angle BOA$ .  
 If  $\angle AOB \equiv \angle A'O'B'$  then  $\angle A'O'B' \equiv \angle AOB$ .  
 If  $\angle AOB \equiv \angle A'O'B'$  and  $\angle A'O'B' \equiv \angle A''O''B''$ , then  $\angle AOB \equiv \angle A''O''B''$ .
- C6. Given two triangles  $\triangle ABC$  and  $\triangle A'B'C'$  such that  $\angle BAC \equiv \angle B'A'C'$ ,  $|AB| \equiv |A'B'|$ , and  $|AC| \equiv |A'C'|$  then  $\angle ABC \equiv \angle A'B'C'$ .

Note that there is no analogue of C3 for angles. The property is actually true, but it is a theorem provable from these axioms (Theorem 1.2.6).

**Definition.** Let  $|AB|$  and  $|CD|$  be two segments and  $P, Q, R$  three points such that  $Q$  is between  $P$  and  $R$ . If  $|AB| \equiv |PQ|$  and  $|CD| \equiv |QR|$ , we say that  $|PR| \equiv |AB| + |CD|$ . We say that  $|AB| > |CD|$  if  $|AB| \equiv |CD| + |EF|$  for some segment  $|EF|$ . Also  $|CD| \equiv |AB| - |EF|$ .

If  $|AB| \equiv |A'B'| + |A'B'| + \dots + |A'B'|$ , we write  $|AB| = n|A'B'|$  or  $|A'B'| = \frac{1}{n}|AB|$ . If  $|AB| \equiv n|A'B'|$  and  $|A''B''| \equiv m|A'B'|$ , we write  $|A''B''| \equiv \frac{m}{n}|AB|$ . This can also be written as

$$\frac{|A''B''|}{|AB|} = \frac{m}{n}.$$

If  $M$  is between  $A$  and  $B$ , and  $|AB| \equiv 2|AM|$ , then  $M$  is called the midpoint of  $|AB|$ .

## 1.2 Congruence of triangles

### 1.2.1 Theorems of congruence of triangles

**Definition.** One says that  $\triangle ABC \equiv \triangle A'B'C'$  if  $\angle BAC \equiv \angle B'A'C'$ ,  $\angle ABC \equiv \angle A'B'C'$ ,  $\angle ACB \equiv \angle A'C'B'$  and  $|AB| \equiv |A'B'|$ ,  $|BC| \equiv |B'C'|$  and  $|AC| \equiv |A'C'|$ .

**Theorem 1.2.1.** (SAS) If in  $\triangle ABC$  and  $\triangle A'B'C'$ ,  $|AB| \equiv |A'B'|$ ,  $\angle BAC \equiv \angle B'A'C'$ , and  $|AC| \equiv |A'C'|$ , then  $\triangle ABC \equiv \triangle A'B'C'$ .

*Proof.* By C6 we have  $\angle ABC \equiv \angle A'B'C'$  and  $\angle ACB \equiv \angle A'C'B'$ . We are left to show that  $|BC| \equiv |B'C'|$ .

We refer to Figure 1.2. On the ray  $|BC$ , choose  $C''$  such that  $|BC''| \equiv |B'C'|$  (which is possible by C1). We want to show that  $C'' = C$ . Applying C6 to triangles  $\triangle BAC''$  and  $\triangle B'A'C'$  ( $\angle ABC'' \equiv \angle A'B'C'$ ,  $|BC''| \equiv |B'C'|$ ,  $|AB| \equiv |A'B'|$ ), we obtain that  $\angle BAC'' \equiv \angle B'A'C'$ . The later is congruent to  $\angle BAC$  by hypothesis. Axiom C4 implies that  $|AC''| = |AC|$ . But  $C, C'' \in BC$  and since the line  $AC$  and  $BC$  cannot have more than one point in common, by I1, it follows that  $C = C''$ . Therefore  $|BC| \equiv |B'C'|$ , and the theorem is proved.  $\square$

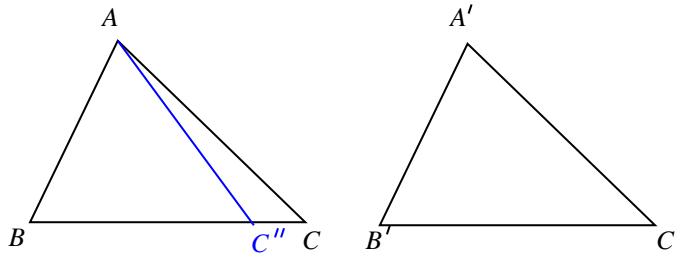


Figure 1.2: Theorem SAS

**Theorem 1.2.2.** (ASA) If in  $\triangle ABC$  and  $\triangle A'B'C'$ ,  $\angle ABC \equiv \angle A'B'C'$ ,  $|BC| \equiv |B'C'|$ , and  $\angle ACB \equiv \angle A'C'B'$ , then  $\triangle ABC \equiv \triangle A'B'C'$ .

*Proof.* We argue on Figure 1.3. On  $|BA$  choose  $A''$  such that  $|BA''| \equiv |B'A'|$ . Since  $|BC| \equiv |B'C'|$ ,  $\angle A''BC \equiv \angle A'B'C'$  and  $|BA''| \equiv |B'A'|$ , by applying Theorem SAS to the triangles  $\triangle A''BC$  and  $\triangle A'B'C'$  we deduce that  $\triangle A''BC \equiv \triangle A'B'C'$ . Hence  $\angle BCA'' \equiv \angle B'C'A'$ . But by hypothesis,  $\angle BCA \equiv \angle B'C'A'$ . From C4 we obtain  $|CA''| = |CA|$ , and since  $CA$  and  $BA$  can have at most one point in common,  $A = A''$ . But then we have  $\triangle ABC \equiv \triangle A'B'C'$ , as desired.  $\square$

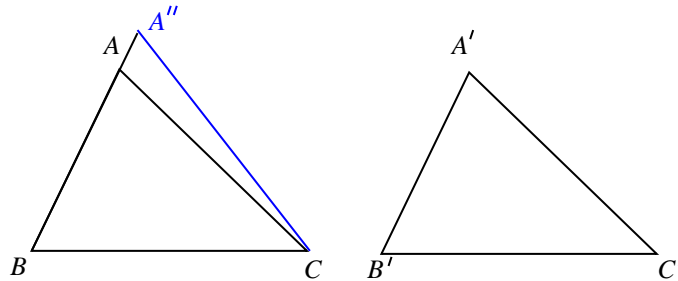


Figure 1.3: Theorem ASA

**Theorem 1.2.3.** In  $\triangle ABC$ ,  $|AB| \equiv |AC|$  if and only if  $\angle ABC \equiv \angle ACB$ .



*Proof.*  $\Rightarrow$  Let us prove first the direct implication. If  $|AB| \equiv |AC|$ , by applying Theorem SAS to triangles  $\triangle ABC$  and  $\triangle ACB$  ( $|AB| \equiv |AC|$ ,  $\angle BAC \equiv \angle CAB$ ,  $|AC| \equiv |AB|$ ), we obtain that these triangles are congruent. It follows that  $\angle ABC \equiv \angle ACB$ .

$\Leftarrow$  Now let us prove the converse. If  $\angle ABC \equiv \angle ACB$ , then by applying Theorem ASA to triangles  $\triangle ABC$  and  $\triangle ACB$  ( $\angle ABC \equiv \angle ACB$ ,  $|BC| \equiv |CB|$ ,  $\angle ACB \equiv \angle ABC$ ) we deduce that these triangles are congruent. Consequently  $|AB| \equiv |AC|$ . The theorem is proved.  $\square$

**Definition.** A triangle with two congruent sides (or equivalently two congruent angles) is called *isosceles*. A triangle with all sides congruent (or equivalently all angles congruent) is called *equilateral*.

**Theorem 1.2.4.** Given  $\angle AOB, \angle BOC, \angle A'O'B', \angle B'O'C'$  such that  $\angle AOB$  is the supplement of  $\angle BOC$ ,  $\angle A'O'B'$  is the supplement of  $\angle B'O'C'$  and  $\angle AOB \equiv \angle A'O'B'$ , then  $\angle BOC \equiv \angle B'O'C'$ .

Said in plain words, “angles with congruent supplements are congruent”.

*Proof.* Using the axiom C1, we can actually choose the points  $A, B, A', B'$  such that  $|OA| \equiv |O'A'|$ ,  $|OB| \equiv |O'B'|$ , and  $|OC| \equiv |O'C'|$ , as in Figure 1.4.

Since  $|OA| \equiv |O'A'|$ ,  $\angle AOB \equiv \angle A'O'B'$ , and  $|OB| \equiv |O'B'|$ , by Theorem SAS  $\triangle OAB \equiv \triangle O'A'B'$ . Hence  $|AB| \equiv |A'B'|$  and  $\angle OAB \equiv \angle O'A'B'$ .

Applying Theorem SAS to triangles  $\triangle ABC$  and  $\triangle A'B'C'$ , in which  $|AC| \equiv |A'C'|$ ,  $\angle BAC \equiv \angle B'A'C'$ , and  $|AB| \equiv |A'B'|$ , we deduce that these triangles are congruent, hence  $|BC| \equiv |B'C'|$  and  $\angle OBC \equiv \angle O'B'C'$ .

By Theorem SAS,  $\triangle O'B'C' \equiv \triangle OBC$  ( $|BC| \equiv |B'C'|$ ,  $\angle ACB \equiv \angle A'C'B'$ ,  $|AC| \equiv |A'C'|$ ). It follows that  $\angle BOC \equiv \angle B'O'C'$ , as desired.

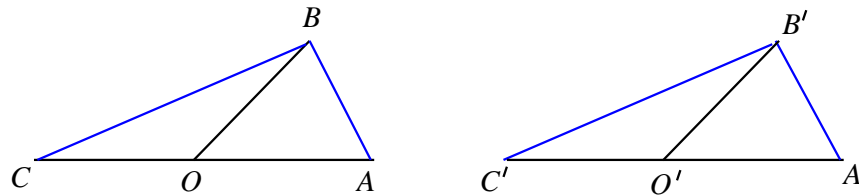


Figure 1.4: Angles with congruent supplements are congruent.

$\square$

**Theorem 1.2.5.** Given the points  $O, A, A', B, B'$  such that  $O \in |AA'|$  and  $O \in |BB'|$ , we then have  $\angle AOB \equiv \angle A'O'B'$ .

In other words “opposite angles are congruent”.

*Proof.* The two angles have the common supplement  $\angle AOB'$ . By Theorem 1.2.4 they are congruent.  $\square$

**Theorem 1.2.6.** In the configuration from Figure 1.5,  $\angle AOB \equiv \angle A'O'B'$  and  $\angle BOC \equiv \angle B'O'C'$ . Then  $\angle AOC \equiv \angle A'O'C'$ .

*Proof.* Choose  $A, B, C, A', B', C'$  such that  $|OA| \equiv |O'A'|$ ,  $|OB| \equiv |O'B'|$ ,  $A, B, C$  are collinear, and  $A', B', C'$  are collinear (see Figure 1.6).

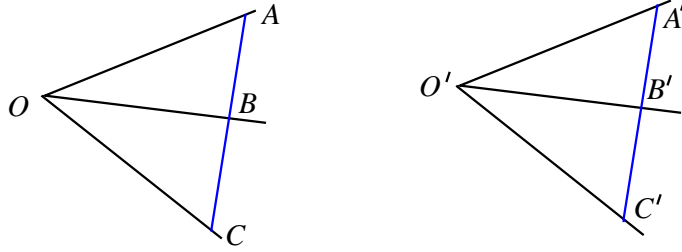


Figure 1.5: Sums of congruent angles are congruent.

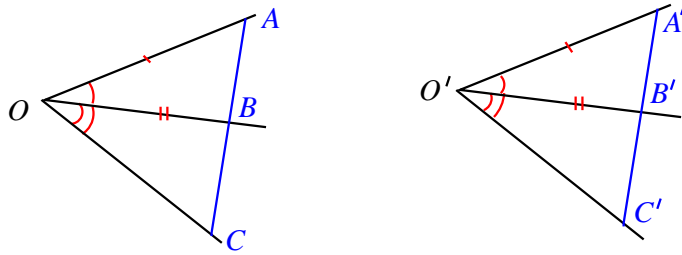


Figure 1.6: Proof that some of congruent angles are congruent.

Since  $|OA| \equiv |O'A'|$ ,  $\angle AOB \equiv \angle A'O'B'$ , and  $|OB| \equiv |O'B'|$ , by Theorem SAS,  $\triangle OAB \equiv \triangle O'A'B'$ . It follows that  $\angle OBA \equiv \angle O'B'A'$ . Because  $\angle OBC$  and  $\angle O'B'C'$  have congruent supplements, by Theorem 1.2.4 they are congruent.

Hence by Theorem ASA,  $\triangle OBC \equiv \triangle O'B'C'$ , because  $\angle BOC \equiv \angle B'O'C'$ ,  $|OB| \equiv |O'B'|$ ,  $\angle OBC \equiv \angle O'B'C'$ . From this we obtain that  $|AB| \equiv |A'B'|$  and  $|BC| \equiv |B'C'|$ . Adding these and using C3 we obtain  $|AC| \equiv |A'C'|$ .

Returning to the congruent triangles  $\triangle OAB$  and  $\triangle O'A'B'$ , we have  $\angle OAB \equiv \angle O'A'B'$ . Hence in triangles  $\triangle AOC$  and  $\triangle A'O'C'$ ,  $|OA| \equiv |O'A'|$ ,  $\angle OAC \equiv \angle O'A'C'$ ,  $|AC| \equiv |A'C'|$ , and so  $\triangle AOC \equiv \triangle A'O'C'$ . From this congruence it follows that  $\angle AOC \equiv \angle A'O'C'$ , and we are done.  $\square$

Now we can introduce some notation.

If  $\angle AOB = \angle A'O'B' + \angle A'O'B' + \cdots + \angle A'O'B'$ , with  $n$  terms on the right, we say that  $\angle AOB = n\angle A'O'B'$ .

**Definition.** If  $\angle AOB \equiv \angle A'O'B' + \angle B'O'C'$ , we say that  $\angle AOB > \angle A'O'B'$ . Also  $\angle A'O'B' \equiv \angle AOB - \angle B'O'C'$ .

**Definition.** An angle congruent to its supplement is called a right angle.

A straight angle is twice a right angle.

**Definition.** Let  $AB$  and  $BC$  be two lines intersecting at  $B$ . The lines are called orthogonal (or perpendicular) if  $\angle ABC$  is right.

**Proposition 1.2.1.** There exist right angles.

*Proof.* Consider an isosceles triangle  $\triangle ABC$ , and let  $M$  be the midpoint of  $|AB|$ . Then Theorem SAS implies that  $\triangle ABM \equiv \triangle ACM$ , so  $\angle AMB \equiv \angle AMC$ . This shows that  $\angle AMC$  is right.  $\square$

**Theorem 1.2.7.** (SSS) If in  $\triangle ABC$  and  $\triangle A'B'C'$ ,  $|AB| \equiv |A'B'|$ ,  $|BC| \equiv |B'C'|$ , and  $|AC| \equiv |A'C'|$ , then  $\triangle ABC \equiv \triangle A'B'C'$ .

*Proof.* In the half-plane bounded by  $BC$  which does not contain  $A$ , choose a point  $A''$  such that  $|BA''| \equiv |B'A'|$  and  $\angle CBA'' \equiv \angle C'B'A'$ , as shown in Figure 1.7. We can make this choice by Axioms C1 and C4. By Theorem SAS,  $\triangle BCA'' \equiv \triangle B'C'A'$ , because  $|BC| \equiv |B'C'|$ ,  $\angle CBA'' \equiv \angle C'B'A'$  and  $|BA''| \equiv |B'A'|$ .

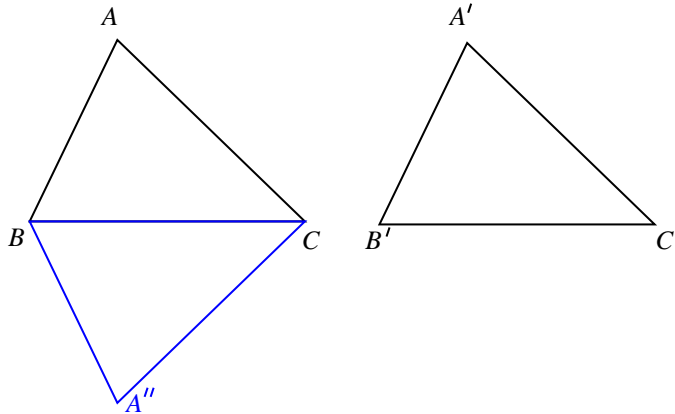


Figure 1.7: Theorem SSS

Since  $A, A''$  do not lie in the same half-plane, the segment  $AA''$  intersects  $BC$ . Let  $O$  be this intersection. We distinguish the following cases:  $O = B$ ,  $O = C$ ,  $O \in |BC|$ ,  $C \in |OB|$ ,  $B \in |OC|$ . The first two and the last two cases are similar. So we need to consider only three cases, described in Figure 1.8.

*Case 1.*  $O = B$ .

Then  $\triangle CAA''$  is isosceles ( $|CA''| \equiv |C'A'| \equiv |CA|$ ), hence  $\angle BAC \equiv \angle BA''C$ . Theorem SAS implies that  $\triangle ABC \equiv \triangle A'B'C'$ , as desired.

*Case 2.*  $O \in |BC|$ . From the isosceles triangle  $\triangle BAA''$  we obtain  $\angle BAO \equiv \angle BA''O$ . From the isosceles triangle  $\triangle CAA''$  we obtain  $\angle CAO \equiv \angle CA''O$ . Using Theorem 1.2.6, we obtain

$$\angle BAC \equiv \angle BAO + \angle OAC \equiv \angle BA''O + \angle OA''C \equiv \angle BA''C.$$

Now we have  $|AB| \equiv |A''B|$ ,  $\angle BAC \equiv \angle BA''C$ , and  $|AC| \equiv |A''C|$ , so by Theorem SAS,  $\triangle ABC \equiv \triangle A''BC$ . Consequently  $\triangle ABC \equiv \triangle A'B'C'$ , so this case is solved, as well.

*Case 3.*  $B \in |OC|$ . Like before, in the isosceles triangle  $\triangle BAA''$ ,  $\angle BAO \equiv \angle BA''O$ . In the isosceles triangle  $\triangle CAA''$ ,  $\angle CAO \equiv \angle CA''O$ . Hence by Theorem 1.2.6,

$$\angle BAC \equiv \angle CAO - \angle BAO \equiv \angle CA''O - \angle BA''O \equiv \angle BA''C.$$

Again by Theorem SAS,  $\triangle ABC \equiv \triangle A''BC \equiv \triangle A'B'C'$ . The theorem is proved.  $\square$

### 1.2.2 Problems

- 1.\* Prove that a line determines exactly two half planes.
2. Let  $\triangle ABC \equiv \triangle A'B'C'$  and let  $M$  and  $M'$  be the midpoints of  $|BC|$  respectively  $|B'C'|$ . Prove that  $\angle BAM \equiv \angle B'A'M'$ .

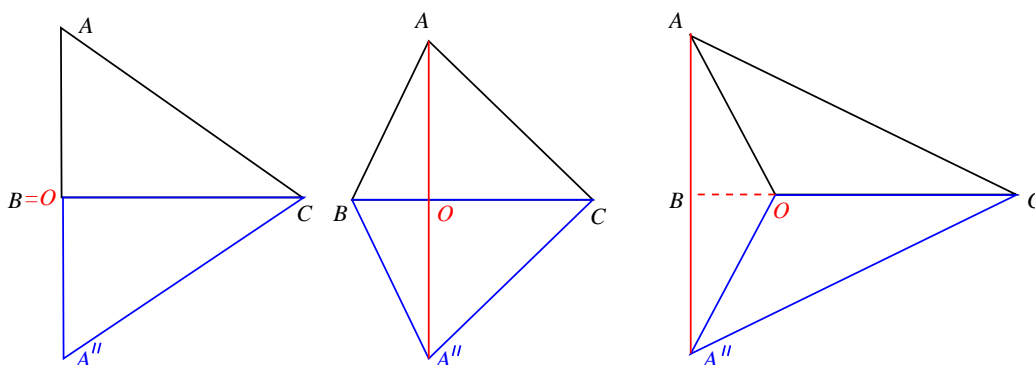


Figure 1.8: The 3 cases for the proof of Theorem SSS

3. Consider the triangle  $\triangle ABD$ . Suppose there is a point  $C$  such that  $\angle ACB \equiv \angle ACD$  and  $\angle BAC \equiv \angle DAC \equiv \frac{1}{2}\angle BAD$ . Prove that the triangles  $\triangle ABD$  and  $\triangle CBD$  are isosceles.
4. Let  $\triangle ABC$  be an isosceles triangle, with  $|AB| \equiv |AC|$ . Prove that the medians  $|BN|$  and  $|CP|$  are congruent.
5. In Figure 1.5 assume that  $\angle AOB \equiv \angle A'O'B'$  and  $\angle AOC \equiv \angle A'O'C'$ . Prove that  $\angle BOC \equiv \angle B'O'C'$ .
- 6.\* Let  $|AB|$  be a segment. Prove that there exists  $M \in |AB|$  such that  $|AM| \equiv |BM|$ .
- 7.\* Prove that there is a triangle  $\triangle ABC$  such that if  $M$  is the midpoint of  $|BC|$  then  $AM$  is not orthogonal to  $BC$ .

### 1.3 Inequalities in a triangle

#### 1.3.1 The results

**Theorem 1.3.1.** (The exterior angle theorem) In a triangle  $\triangle ABC$ , the supplement of  $\angle B$  is greater than  $\angle A$ .

*Proof.* We argue on Figure 1.9. Choose  $E$  such that  $B$  is between  $C$  and  $E$ , so that the supplement of  $\angle B = \angle ABE$ . Assume by way of contradiction that  $\angle ABE \leq \angle A$ , and choose the point  $C' \in |BC|$  such that  $\angle C'AB \equiv \angle ABE$ .

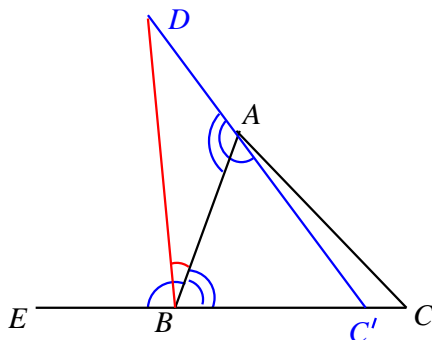


Figure 1.9: The exterior angle theorem

Let  $D \in AC'$  such that  $A$  is between  $C'$  and  $D$ , and  $|BC'| \equiv |AD|$ . Then  $\angle DAB \equiv \angle ABC'$ , because they have the same supplement. This combined with  $|AB| \equiv |AB|$  and  $|BC'| \equiv |AD|$  implies that  $\triangle ABC' \equiv \triangle BAD$  (by Theorem SAS). Hence

$$\angle ABD \equiv \angle BAC' \equiv \angle ABE.$$

By Axiom C4,  $|BD$  coincides with  $|BE$ . So the lines  $AC'$  and  $BC'$  have two points of intersection, namely  $C'$  and  $D$ , contradicting Axiom I1. Hence our assumption was false, and the theorem is proved.  $\square$

**Proposition 1.3.1.** If a line forms congruent alternate interior angles with two other lines, then those lines do not intersect.

*Proof.* If the two lines intersected, then the two angles would be one interior and one exterior to a triangle (see Figure 1.10).

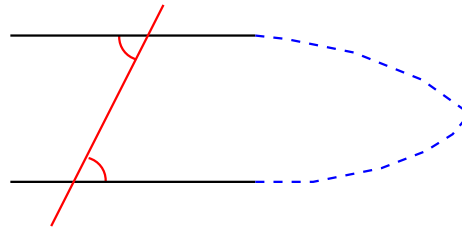


Figure 1.10: Parallel lines

But the Exterior Angle Theorem shows that this is impossible.  $\square$

**Definition.** Two lines that do not intersect are called parallel.

“Parallel lines exist!”

**Theorem 1.3.2.** In a triangle the larger side is opposite to the larger angle.

*Proof.* Rephrasing the statement, in  $\triangle ABC$ ,  $|AB| < |AC|$  if and only if  $\angle C < \angle B$ .

$\Rightarrow$  If  $|AB| < |AC|$ , choose  $D \in |AC|$  such that  $|AD| \equiv |AB|$ , as shown in Figure 1.11. The triangle  $\triangle ABD$  is isosceles, hence  $\angle ABD \equiv \angle ADB$ . Using the Exterior Angle Theorem, we can write

$$\angle ABC > \angle ADB > \angle ACB.$$

Thus  $\angle C < \angle B$ .

$\Leftarrow$  If  $\angle C < \angle B$ , then  $|AB| \equiv |AC|$  contradicts the isosceles triangle theorem, and  $|AB| > |AC|$  contradicts what we just proved. We can only have  $|AB| < |AC|$ .  $\square$

**Theorem 1.3.3.** (The triangle inequality) In a triangle the sum of two sides is greater than the third.

*Proof.* Construct  $C'$  such that  $B \in |CC'|$  and  $|BC'| \equiv |AB|$  (see Figure 1.12). The triangle  $\triangle BAC'$  is isosceles, so

$$\angle BC'A \equiv \angle BAC' < \angle CAC'.$$

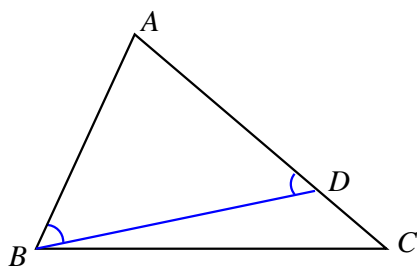


Figure 1.11: Larger side opposes larger angle.

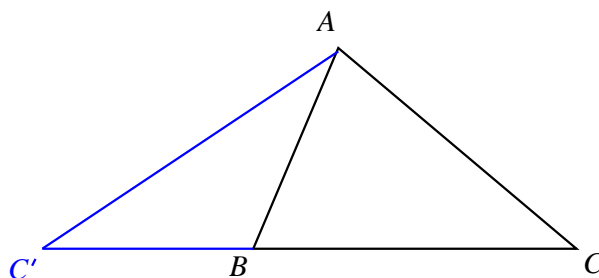


Figure 1.12: The triangle inequality

By Theorem 1.3.2,

$$|CC'| > |AC|.$$

Because  $|CC'| \equiv |AB| + |BC|$ , we obtain

$$|AB| + |BC| > |AC|.$$

The other two inequalities are obtained the same way. □

**Theorem 1.3.4.** If in  $\triangle ABC$ ,  $|AB| \leq |AC|$  and  $D \in |BC|$ , then  $|AD| < |AC|$ .

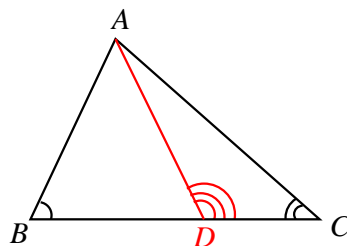


Figure 1.13: The secant is shorter than the longer side.

*Proof.* We argue on Figure 1.13. By Theorem 1.3.2

$$\angle B \geq \angle C.$$

By the exterior angle theorem

$$\angle ADC > \angle ABC.$$

Hence

$$\angle ADC > \angle C,$$

and so by Theorem 1.3.2  $|AD| < |AC|$ , as desired.  $\square$

### 1.3.2 Problems

1. Let  $ABCD$  be a quadrilateral. Prove that  $|AB| + |BC| + |CD| \geq |AD|$ .
2. Let  $M$  be a point in the interior of triangle  $\triangle ABC$ . Prove that  $|AM|$  is shorter than the longest side of the triangle.
3. Show that a triangle can have at most one obtuse angle (you cannot use the fact that the sum of the angles of a triangle is  $180^\circ$  since this is not true in absolute geometry).
4. Let  $\triangle ABC$  be a right triangle, with  $\angle A$  being the right angle. Prove that  $|AB| + |AC| < 2|BC|$ .
- 5.\* Let  $\triangle ABC$  be an equilateral triangle, and  $M \in |BC|$ ,  $N \in |AC|$  and  $P \in |AB|$  such that  $AM$  perpendicular to  $BC$ ,  $BN$  perpendicular to  $AC$  and  $CP$  perpendicular to  $AB$ . Show that  $M$ ,  $N$  and  $P$  are the midpoints of the sides, and that  $|AM|$ ,  $|BN|$  and  $|CP|$  are the angle bisectors of the triangle.
6. Let  $\triangle ABC$  be a triangle, and let  $M$  be the midpoint of  $|BC|$ . Prove that  $|AM| \leq \frac{1}{2}(|AB| + |AC|)$ . Conclude that the sum of the medians of a triangle is less than the sum of the sides.

## 1.4 Right angles

### 1.4.1 Properties of right angles

Recall that a right angle is an angle that is congruent to its supplement.

*Remark 1.4.1.* The supplement and the opposite of a right angle are right angles.

**Definition.** Two intersecting lines are called orthogonal if the four angles they determine are right. If  $AB$  and  $CD$  are the two lines, we write  $AB \perp CD$ .

**Proposition 1.4.1.** Any angle congruent to a right angle is a right angle.

*Proof.* Suppose that  $\angle AOB \equiv \angle A'O'B'$ , and  $\angle A'O'B'$  is right. By Theorem 1.2.4, the supplement of  $\angle AOB$  is congruent to the supplement of  $\angle A'O'B'$ . But the latter is congruent to  $\angle A'O'B'$ , hence to  $\angle AOB$ . So  $\angle AOB$  is congruent to its supplement, so it is right.  $\square$

**Theorem 1.4.1.** Any two right angles are congruent.

*Remark 1.4.2.* Euclid lists this as an axiom, but we work with a more modern system of axioms, in which this statement can be proved.

*Proof.* Arguing by contradiction, let us assume there exist two right angles  $\angle ABC$  and  $\angle A'B'C'$  such that  $\angle ABC > \angle A'B'C'$ . Let  $\angle DBA$  be the supplement of  $\angle ABC$ .

Choose  $A''$  in the half-plane bounded by  $BC$  which contains  $A$  such that  $\angle A''BC \equiv \angle A'B'C'$  (see Figure 1.14). Then

$$\angle A''BC < \angle ABC,$$

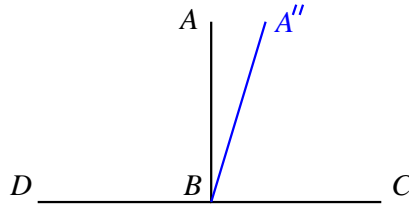


Figure 1.14: Right angles are congruent.

and so their supplements should satisfy

$$\angle DBA < \angle DBA''.$$

But since  $\angle ABC$  and  $\angle A''BC$  are right,

$$\angle A''BC \equiv \angle DBA'' > \angle DBA \equiv \angle ABC.$$

This contradicts  $\angle A''BC < \angle ABC$ . So our initial assumption was false, proving that any two right angles are congruent.  $\square$

**Definition.** An angle that is smaller than a right angle is called *acute*. An angle that is greater than a right angle is called *obtuse*.

**Definition.** If two angles add up to a right angle, they are called *complementary*. Each is the *complement* of the other.

### 1.4.2 Theorems of congruence for right triangles

In all five theorems below, the triangles are  $\triangle ABC$  and  $\triangle A'B'C'$ , with the right angles  $\angle A$  respectively  $\angle A'$ .

**Theorem 1.4.2.** If  $|AB| \equiv |A'B'|$  and  $|AC| \equiv |A'C'|$ , then  $\triangle ABC \equiv \triangle A'B'C'$ .

*Proof.* This is an easy application of Theorem SAS.  $\square$

**Theorem 1.4.3.** If  $|AB| \equiv |A'B'|$  and  $\angle B \equiv \angle B'$ , then  $\triangle ABC \equiv \triangle A'B'C'$ .

*Proof.* This is an easy application of Theorem ASA.  $\square$

**Theorem 1.4.4.** If  $|AB| \equiv |A'B'|$  and  $|BC| \equiv |B'C'|$ , then  $\triangle ABC \equiv \triangle A'B'C'$ .

*Proof.* We give three proofs to this result, the second and the third suggested by students.

I. We argue on Figure 1.15. Choose  $C'' \in |AC|$  such that  $|AC''| \equiv |A'C'|$ . Let us assume that  $C \neq C''$ . By Theorem SAS,  $\triangle AC''B \equiv \triangle A'C'B'$ . Hence  $|BC''| \equiv |B'C'| \equiv |BC|$ . It follows that  $\triangle BC''C$  is isosceles.

But the angles of a right triangle are acute, by the Exterior Angle Theorem applied to the right angle. Thus one of the two congruent angles of  $\triangle BC''C$  is acute, and the other has acute supplement, hence is obtuse. This is a contradiction, which proves that our assumption was false. It follows that  $A = A''$ , and hence  $\triangle ABC \equiv \triangle A'B'C'$ .

II. There is another way to end this proof. In the beginning, we may assume  $C'' \in |AC|$ , or else switch the triangles. Once we have that  $\triangle BCC''$  is isosceles, we notice that  $|BC''| < \max(|BC|, |AB|)$  by Theorem 1.3.4. But the “larger side opposes the larger angle” shows that



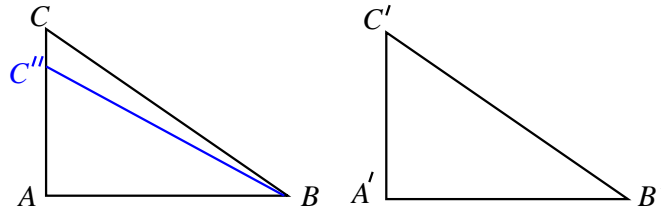


Figure 1.15: Congruence of right triangles.

in a right triangle the hypotenuse is the longest side. So  $\max(|BC|, |AB|) = |BC|$ . Hence  $|BC''| < |BC|$ , a contradiction. The conclusion follows.

III. Construct the points  $D$  and  $D'$  such that  $A$  is the midpoint of  $|BD|$ , and  $A'$  is the midpoint of  $|B'D'|$ . Then triangles  $\triangle CAB$  and  $\triangle CAD$  are congruent, by Theorem SAS, so  $|BC| \equiv |DC|$ . Similarly  $\triangle C'A'B' \equiv \triangle C'A'D'$ , so  $|B'C'| \equiv |D'C'|$ . Then triangles  $\triangle ABD$  and  $\triangle A'B'D'$  are congruent by Theorem SSS, because  $|DC| \equiv |BC| \equiv |B'C'| \equiv |D'C'|$  and  $|BD| \equiv 2|AB| \equiv 2|A'B'| \equiv |B'D'|$ . We obtain that  $\angle B \equiv \angle B'$ , and then Theorem SAS implies  $\triangle ABC \equiv \triangle A'B'C'$ , as desired.  $\square$

**Theorem 1.4.5.** If  $|AB| \equiv |A'B'|$  and  $\angle C \equiv \angle C'$ , then  $\triangle ABC \equiv \triangle A'B'C'$ .

*Proof.* We argue again on Figure 1.15. Choose  $C'' \in |AC$  such that  $|AC''| \equiv |A'C'|$ . Then  $\triangle ABC'' \equiv \triangle A'B'C'$ , by Theorem SAS. Hence  $\angle BC''A \equiv \angle B'C'A' \equiv \angle BCA$ . This would contradict the Exterior Angle Theorem, unless  $C'' \equiv C$ . We conclude that  $\triangle ABC \equiv \triangle A'B'C'$ .  $\square$

**Theorem 1.4.6.** If  $|BC| \equiv |B'C'|$  and  $\angle B \equiv \angle B'$ , then  $\triangle ABC \equiv \triangle A'B'C'$ .

*Proof.* Choose  $A'' \in |BA$  such that  $|BA''| \equiv |B'A'|$ , as shown in Figure 1.16. Suppose that  $A'' \neq A$ . Then Theorem SAS implies that  $\triangle BCA'' \equiv \triangle B'C'A'$ , so  $\angle CA''A$  is right. It follows that in  $\triangle CAA''$ , there is an exterior angle and an interior angle not adjacent to it, both of which are right. This contradicts the Exterior Angle Theorem. Hence  $A'' = A$  and  $\triangle ABC \equiv \triangle A'B'C'$ .  $\square$

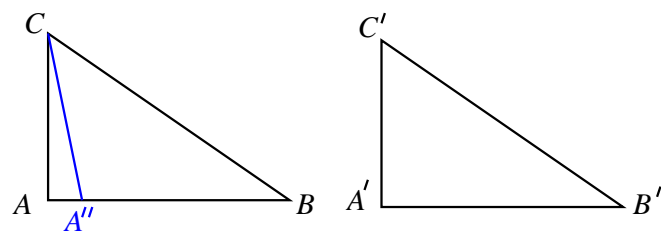


Figure 1.16: Congruence of right triangles.

We can summarize these results as follows.

**Theorem 1.4.7.** (The Theorem of Congruence of Right Triangles) If in two right triangles, two pairs of corresponding elements are congruent, one of which is a pair of sides, then the triangles are congruent.

**Definition.** The *bisector* of an angle  $\angle AOB$  consists of those points  $M$  with the property that  $\angle MOA \equiv \angle MOB \equiv \frac{1}{2}\angle AOB$ .

**Theorem 1.4.8.** In a triangle the three angle bisectors intersect at one point.

*Proof.* We argue on Figure 1.17. Let  $I$  be the point of intersection of the bisectors from  $\angle A$  and  $\angle B$ . By Exercise 2, there are  $D \in BC$ ,  $E \in AC$ , and  $F \in AB$  such that  $ID \perp BC$ ,  $IE \perp AC$ , and  $IF \perp AB$ .

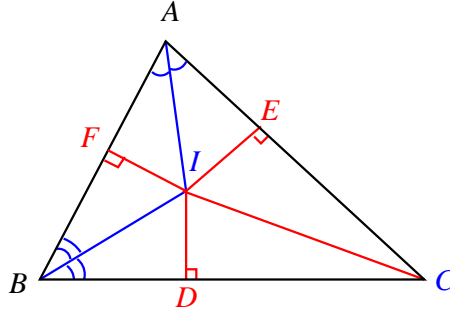


Figure 1.17: The angle bisectors intersect.

By Theorem 1.4.6,  $\triangle BID \cong \triangle BIF$ , so  $|ID| \equiv |IF|$ . Also by Theorem 1.4.6,  $\triangle AIF \cong \triangle AIE$ , and so  $|IF| \equiv |IE|$ . Hence  $|ID| \equiv |IE|$ .

In triangles  $\triangle CID$  and  $\triangle CIE$ ,  $|ID| \equiv |IE|$ ,  $|IC| \equiv |IC|$ , so by Theorem 1.4.4 they are congruent. It follows that  $\angle ICE \equiv \angle ICD$ . Hence  $|CI|$  is the angle bisector of  $\angle C$ . We conclude that the angle bisectors intersect at one point.  $\square$

### 1.4.3 Problems

1. Let  $AB$  be a line and  $C$  a point that does not belong to it. Show that there is a point  $D \in AB$  such that  $CD \perp AB$ .
- 2.\* Let  $l$  be a line and let  $C$  be a point that does not belong to  $l$ . Prove that on  $l$  there exists two distinct points  $A$  and  $B$  such that  $|AC| \equiv |BC|$ .
3. Let  $ABC$  be an isosceles triangle with  $|AB| \equiv |AC|$ . Let  $E \in AC$  and  $F \in AB$  be such that  $BE \perp AC$  and  $CF \perp AB$ . Prove that  $|BE| \equiv |CF|$ . (Hint: The angle  $\angle BAC$  might be acute or obtuse).

## 1.5 The axioms of continuity

We conclude the discussion of absolute geometry by adding two axioms that allow us to establish a one-to-one correspondence between the points of a line and the real numbers that preserves the ordering.

- R1. (Archimedes) If  $A$  and  $B$  are two points of a ray  $|OX$ , then there is a finite set of points  $\{A_1, A_2, \dots, A_k\}$  on  $|OX$  such that

$$A \in |OA_1|, A_1 \in |OA_2|, \dots, A_{k-1} \in |OA_k|,$$

$$|OA| \equiv |AA_1| \equiv |A_1A_2| \equiv \dots \equiv |A_{k-1}A_k|,$$

and  $B \in |OA_k|$ .

- R2. (Cantor-Dedekind) Given a line and two sequences of points  $A_1, A_2, A_3, \dots, B_1, B_2, B_3, \dots$  on this line such that for every  $j$  the segment  $|A_{j+1}B_{j+1}|$  is contained in the segment  $|A_jB_j|$ , then there exists a point  $P$  contained in all of these segments.

## Chapter 2

# Euclidean Geometry

### 2.1 Euclid's fifth postulate

#### 2.1.1 Parallel lines

E1. (Euclid's fifth postulate) Given a line  $l$  and a point  $A$  that does not belong to  $l$ , there is a unique line  $l'$  passing through  $A$  such that  $l$  and  $l'$  are parallel.

Notation:  $l \parallel l'$ .

**Theorem 2.1.1.** Given two lines  $l_1$  and  $l_2$  that are intersected by a third line  $l$  as shown in Figure 2.1, then  $l_1 \parallel l_2$  if and only if  $\angle\alpha \equiv \angle\beta$ , where  $\angle\alpha$  and  $\angle\beta$  are a pair of alternate interior angles.

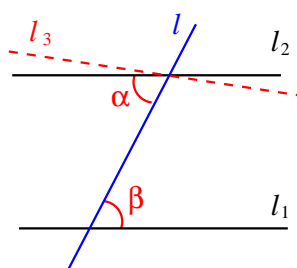


Figure 2.1: Characterization of parallel lines

*Proof.*  $\Rightarrow$  We start with  $l_1 \parallel l_2$ . Construct a line  $l_3$  through the intersection of  $l_2$  and  $l$  such that  $l_3$  and  $l_1$  form congruent alternate interior angles with  $l$ . Then  $l_3$  and  $l_1$  are parallel, by Proposition 1.3.1, so by Euclid's fifth postulate,  $l_3 = l_2$ . Hence  $\angle\alpha \equiv \angle\beta$ .

$\Leftarrow$  The converse statement follows from Proposition 1.3.1. □

**Corollary 2.1.1.** Two pairs of parallel lines form congruent angles.

**Theorem 2.1.2.** The sum of the angles of a triangles is congruent to a straight angle.

*Proof.* Let  $\triangle ABC$  be a triangle. On  $AB$  choose a point  $D$  such that  $A$  is between  $B$  and  $D$ . Take the only line through  $A$  that is parallel to  $BC$ , as shown in Figure 2.2. This line divides  $\angle CAD$  into two angles. Let these angles be  $\angle CAE$  and  $\angle EAD$ . Then by Theorem 2.1.1,

$$\angle CAE \equiv \angle ACB \text{ and } \angle EAD \equiv \angle ABC.$$

The conclusion follows, since the angles  $\angle BAC$ ,  $\angle CAE$  and  $\angle EAD$  add up to a straight angle.  $\square$

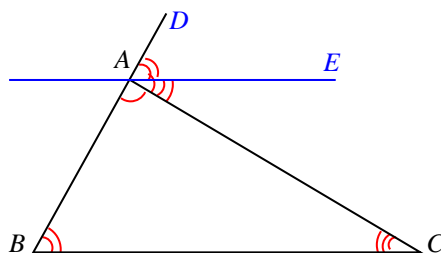


Figure 2.2: The sum of the angles of a triangle is a straight angle

**Definition.** A *trapezoid* is a quadrilateral that has a pair of opposite sides that are parallel.

**Definition.** A *parallelogram* is a quadrilateral whose opposite sides are parallel.

**Proposition 2.1.1.** A quadrilateral is a parallelogram if and only if one of the following properties holds:

- (i) Two opposite sides are parallel and congruent.
- (ii) The two pairs of opposite sides are congruent.
- (iii) The diagonals intersect at their midpoint.

**Definition.** A parallelogram whose angles are right is called a *rectangle*. A parallelogram whose sides are congruent is called a *rhombus*. A quadrilateral that is both a rectangle and a rhombus is called a *square*.

## 2.1.2 Problems

1. Show that if  $l \parallel l'$  and  $l' \parallel l''$ , then  $l \parallel l''$ .
2. Prove Proposition 2.1.1.
3. Show that if a parallelogram has one right angle, then all of its angles are right.
4. What is the sum of the angles of a polygon with  $n$  sides?

## 2.2 Similarity

### 2.2.1 The Theorems of Thales

Before we state Thales' Theorems, we introduce the following notation. We say that  $|AB|/|CD| = x$ , where  $x$  is a positive real number, if  $|AB| \equiv x|CD|$ . It is easy to see how this works if  $x$  is rational; the axioms of continuity imply that for any choice of a segment  $|CD|$  and any positive real number  $x$ , there is a segment  $|AB|$  such that  $|AB| \equiv x|CD|$ .

**Theorem 2.2.1.** (Thales) Let  $l$  and  $l'$  be two distinct lines in the plane, and  $A, A', A'' \in l$ ,  $B, B', B'' \in l'$  such that  $AB \parallel A'B'$ ,  $B' \in |BB''|$ , and  $|AA'| \equiv |A'A''|$ . Then  $A'B' \parallel A''B''$  if and only if  $B'$  is the midpoint of the segment  $|BB''|$ .

*Proof.*  $\Rightarrow$  Construct  $M \in |A'B'|$  and  $M' \in |A''B''|$  such that  $BM \parallel AA'$  and  $B'M' \parallel A'A''$  (see Figure 2.3). Applying repeatedly Theorem 2.1.1 we deduce that

$$\angle BMB' \equiv \angle AA'B' \equiv \angle AA''B'' \equiv \angle B'M'B'',$$

and

$$\angle MBB' \equiv \angle M'B'B''.$$

Also, by Proposition 2.1.1,

$$|BM| \equiv |AA'| \equiv |A'A''| \equiv |B'M'|.$$

Hence by Theorem ASA,  $\triangle B'BM \equiv \triangle B''B'M'$ . It follows that  $|BB'| \equiv |B'B''|$ , so  $B'$  is the midpoint of  $|BB''|$ .

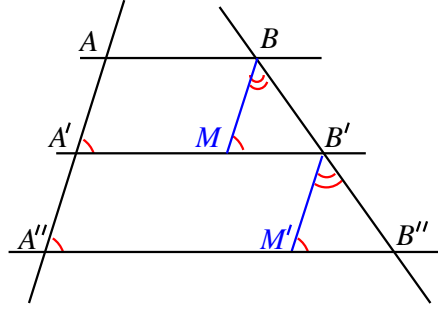


Figure 2.3: Proof of Thales' Theorem, the particular case

$\Leftarrow$  Choose  $B_1 \in l'$  such that  $BB_1 \parallel A'B'$ . Then by what we just proved,  $|B'B_1| \equiv |BB'|$ . It follows that  $B_1$  is on the ray  $|B'B''|$ , and  $|B'B_1| \equiv |B'B''|$ . This can only happen if  $B_1 = B''$ , and we are done.  $\square$

**Theorem 2.2.2.** (Thales) Let  $l, l'$  be two distinct lines in the plane,  $A, A', A'' \in l$  and  $B, B', B'' \in l'$  such that  $A' \in |AA''|$ ,  $B' \in |BB''|$  and  $A'B' \parallel AB$ . Then  $A''B'' \parallel A'B'$  if and only if

$$\frac{|AA'|}{|A'A''|} = \frac{|BB'|}{|B'B''|}.$$

*Proof.* Case 1.  $|AA'|/|A'A''| = 1/n$ ,  $n$  a positive integer. We argue on Figure 2.4.

$\Rightarrow$  Choose  $A_1, A_2, \dots, A_{n-1} \in |A'A''|$  such that

$$|A'A_1| \equiv |A_1A_2| \equiv \dots \equiv |A_{n-1}A''| \equiv |AA'|.$$

Choose also  $B_1, B_2, \dots, B_{n-1} \in |B'B''|$  such that

$$A_1B_1 \parallel A_2B_2 \parallel \dots \parallel A_{n-1}B_{n-1} \parallel A'B'$$

Applying the previous theorem successively, we obtain

$$|B'B_1| \equiv |BB'|, |B_1B_2| \equiv |B'B_1|, \dots, |B_{n-1}B''| \equiv |B_{n-2}B_{n-1}|.$$

It follows that  $|B'B''| \equiv n|BB'|$ , as desired.

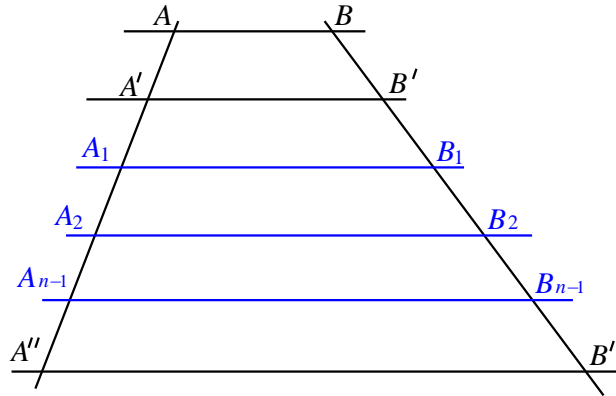


Figure 2.4: Proof of Thales' Theorem

$\Leftarrow$  We use the same figure but this time we choose the  $B_i$ 's such that

$$|B'B_1| \equiv |B_1B_2| \equiv \cdots \equiv |B_{n-1}B''| \equiv |BB'|.$$

Applying again successively the previous theorem we deduce that

$$A_1B_1 \parallel A'B', A_2B_2 \parallel A_1B_1, \dots, A''B'' \parallel A_{n-1}B_{n-1}.$$

We obtain that  $A''B'' \parallel A'B'$ , which proves this case.

Case 2.  $|AA'|/|A'A''| = m/n$ ,  $m, n$  positive integers,  $m > 1$ . We argue on Figure 2.5.

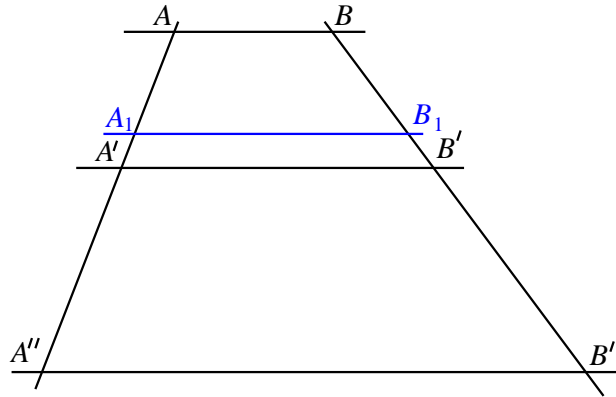


Figure 2.5: Proof of Thales' Theorem

$\Rightarrow$  Choose  $A_1B_1$  such that

$$\frac{|A'A_1|}{|AA_1|} = \frac{1}{m-1} \text{ and } A_1B_1 \parallel A'B'.$$

By Case 1 of this theorem,

$$\frac{|B_1B'|}{|B_1B|} = \frac{1}{m-1} \text{ and } \frac{|B_1B'|}{|B'B''|} = \frac{1}{n}.$$

An algebraic computation shows that

$$\frac{|BB'|}{|B'B''|} = \frac{m}{n} = \frac{|AA'|}{|A'A''|},$$

as desired.

$\Leftarrow$  Choose now  $B_1$  such that

$$\frac{|A'A_1|}{|AA_1|} = \frac{|B_1B'|}{|BB_1|} = \frac{1}{m-1}.$$

Then on the one hand, by Case 1 applied "bottom to top",  $A_1B_1 \parallel A'B'$ . On the other hand

$$\frac{|A_1A'|}{|A'A''|} = \frac{|B_1B'|}{|B'B''|} = \frac{1}{n}.$$

Applying again Case 1, we deduce that  $A''B'' \parallel A'B'$ .

**Case 3.** If  $|AA'|/|A'A''| = x$ , with  $x$  a real number, approximate  $x$  by rational numbers, then pass to the limit. In the process we use the axioms of continuity. Note that if  $A_nB_n \parallel AB$  for all  $n$ , and  $A_n \rightarrow A_*$ ,  $B_n \rightarrow B_*$ , then  $A_*B_* \parallel AB$ .  $\square$

**Corollary 2.2.1.** Let  $\angle AOA'$  be an angle,  $M \in |OA$  and  $M' \in |OA'$ . Then

$$AA' \parallel MM' \Leftrightarrow \frac{|OM|}{|OA|} = \frac{|OM'|}{|OA'|}.$$

*Proof.* In Thales' theorem, choose  $A = B$ .  $\square$

### 2.2.2 Similar triangles

**Definition.** We say that  $\triangle ABC$  is similar to  $\triangle A'B'C'$ , and write  $\triangle ABC \sim \triangle A'B'C'$ , if

$$\angle A \equiv \angle A', \quad \angle B \equiv \angle B', \quad \angle C \equiv \angle C',$$

and

$$\frac{|AB|}{|A'B'|} = \frac{|BC|}{|B'C'|} = \frac{|AC|}{|A'C'|}.$$

**Theorem 2.2.3.** If in  $\triangle ABC$  and  $\triangle A'B'C'$  we have  $\angle A \equiv \angle A'$ ,  $\angle B \equiv \angle B'$  and  $\angle C \equiv \angle C'$ , then  $\triangle ABC \sim \triangle A'B'C'$ .

*Proof.* Let  $B'' \in |AB$  and  $C'' \in |AC$  such that  $|AB''| \equiv |A'B'|$  and  $|AC''| \equiv |A'C'|$  (Figure 2.6). Then by Theorem SAS,  $\triangle AB''C'' \equiv \triangle A'B'C'$ . Hence  $\angle AB''C'' \equiv \angle A'B'C' \equiv \angle ABC$ . By Theorem 2.1.1,  $B''C'' \parallel BC$ . Applying Thales' Theorem, we deduce that

$$\frac{|AB''|}{|B''B|} = \frac{|AC''|}{|C''C|}.$$

A little algebra gives

$$\frac{|A'B'|}{|AB|} = \frac{|A'C'|}{|AC|}. \quad (2.2.1)$$

Repeating the argument at vertex  $B$ , we deduce that

$$\frac{|A'B'|}{|AB|} = \frac{|B'C'|}{|BC|}. \quad (2.2.2)$$

Combining (2.2.1) and (2.2.2), we obtain

$$\frac{|AB|}{|A'B'|} = \frac{|AC|}{|A'C'|} = \frac{|BC|}{|B'C'|},$$

which together with the congruence of angles shows that  $\triangle ABC \sim \triangle A'B'C'$ .  $\square$

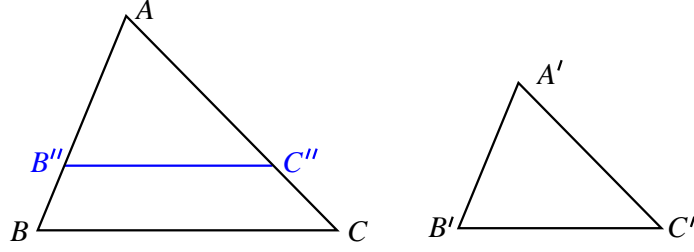


Figure 2.6: Similarity of triangles

Note that because of Theorem 2.1.2, in order for the triangles to be similar it suffices to check that two pairs of angles are respectively congruent.

**Theorem 2.2.4.** If in  $\Delta ABC$  and  $\Delta A'B'C'$

$$\angle A \equiv \angle A' \text{ and } \frac{|AB|}{|A'B'|} = \frac{|AC|}{|A'C'|},$$

then  $\Delta ABC \sim \Delta A'B'C'$ .

*Proof.* Let  $B'' \in |AB$  and  $C'' \in |AC$  such that  $|AB''| \equiv |A'B'|$  and  $|AC''| \equiv |A'C'|$  (Figure 2.6). Then by Theorem SAS,  $\Delta AB''C'' \equiv \Delta A'B'C'$ . We have

$$\frac{|AB''|}{|AB|} = \frac{|A'B'|}{|AB|} = \frac{|A'C'|}{|AC|} = \frac{|AC''|}{|AC|},$$

so by Thales' theorem  $B''C'' \parallel BC$ . This implies that

$$\begin{aligned} \angle A'B'C' &\equiv \angle AB''C'' \equiv \angle ABC \\ \angle A'C'B' &\equiv \angle AC''B'' \equiv \angle ACB. \end{aligned}$$

Because the triangles  $\Delta ABC$  and  $\Delta A'B'C'$  have the angles respectively congruent, by Theorem 2.2.3, they are similar.  $\square$

**Theorem 2.2.5.** If in  $\Delta ABC$  and  $\Delta A'B'C'$ ,

$$\frac{|AB|}{|A'B'|} = \frac{|AC|}{|A'C'|} = \frac{|BC|}{|B'C'|},$$

then  $\Delta ABC \sim \Delta A'B'C'$ .

*Proof.* Choose  $B'' \in |AB$  and  $C'' \in |AC$  such that  $|AB''| \equiv |A'B'|$  and  $|AC''| \equiv |A'C'|$  (Figure 2.6). Then

$$\frac{|AB''|}{|AB|} = \frac{|AC''|}{|AC|},$$

so by Theorem 2.2.4,  $\Delta ABC \sim \Delta AB''C''$ . Then

$$\frac{|B''C''|}{|BC|} = \frac{|AB''|}{|AB|} = \frac{|B'C'|}{|BC|},$$

which implies  $|B''C''| \equiv |B'C'|$ . We can apply Theorem SSS to conclude that  $\Delta AB''C'' \equiv \Delta A'B'C'$ . As the first of these two triangles is similar to  $\Delta ABC$ , so is the second.  $\square$



### 2.2.3 Problems

1. Let  $\triangle ABC$  be a triangle and let  $M, N, P$  be the midpoints of  $|BC|$ ,  $|AC|$ , respectively  $|AB|$ . Let also  $M', N', P'$  be the midpoints of  $|NP|$ ,  $|MP|$ ,  $|MN|$ . Prove that the triangles  $\triangle ABC$  and  $\triangle M'N'P'$  are similar.
2. Let  $\triangle ABC$  be an equilateral triangle. On the rays  $|AB|$ ,  $|BC|$ , and  $|CA|$  choose the points  $M, N, P$  respectively, such that  $|AM| \equiv |BN| \equiv |CP|$ . Prove that  $\triangle MNP$  is equilateral.
3. Let  $\triangle ABC$  be an equilateral triangle and let  $M \in AC$  and  $N \in BC$  be such that  $|AM|/|MC| = 1/2$  and  $|BN|/|NC| = 1/3$ . Let  $P$  be the intersection of  $AN$  and  $BM$ . Find  $|BP|/|PM|$ .
4. Let  $\triangle ABC$  be a triangle and let  $|AD|$  and  $|BE|$  be its altitudes from  $A$  and  $B$ . Prove that  $\triangle CEB \sim \triangle CDA$  and that  $\triangle CED \sim \triangle CBA$ .

## 2.3 The four important points in a triangle

### 2.3.1 The incenter

Recall that Theorem 1.17 shows that the three angle bisectors of a triangle intersect at one point.

**Definition.** The point of intersection of the three angle bisectors of a triangle is called the incenter of the triangle.

The incenter is denoted by  $I$ .

### 2.3.2 The centroid

**Definition.** In a triangle, the segments that join the vertices with the midpoints of the opposite sides are called medians.

**Theorem 2.3.1.** In a triangle the three medians intersect at a point, called the *centroid* of the triangle. The centroid divides each median in the ration 2 : 1.

*Proof.* (The physical proof) Place three equal masses at the vertices  $A, B, C$  of the triangle. Combine the masses at  $B$  and  $C$  to a mass twice as large placed at the midpoint  $M$  of  $BC$ . Since  $M$  is the center of mass of the system formed by  $B$  and  $C$ , the old and the new system of masses have the same center of mass. The second system has its center of mass on  $|AM|$ , dividing  $|AM|$  in the ration 2 : 1.

Now combine the masses at  $A$  and  $C$ , respectively  $A$  and  $B$ , to conclude that the center of mass of the system lies on the medians from  $B$  and  $C$  as well, and divides these medians in the ration 2 : 1.  $\square$

*Proof.* (The mathematical proof) Let  $M$  be the midpoint of  $|BC|$  and  $G \in |AM|$  such that  $|AG|/|GM| = 2$ . Construct  $A' \in |GM|$  such that  $|MG| \equiv |MA'|$  (Figure 2.8). The diagonals of  $BGCA'$  intersect at their midpoint, so by Lemma 2.1.1,  $BGCA'$  is a parallelogram. It follows that  $BG \parallel A'C$  and  $CG \parallel A'B$ .

Let  $N$  be the intersection of  $BG$  and  $AC$ , and  $P$  the intersection of  $CG$  and  $AB$ . By Thales' Theorem

$$\frac{|AN|}{|NC|} = \frac{|AG|}{|GA'|} = \frac{|AP|}{|PB|}.$$

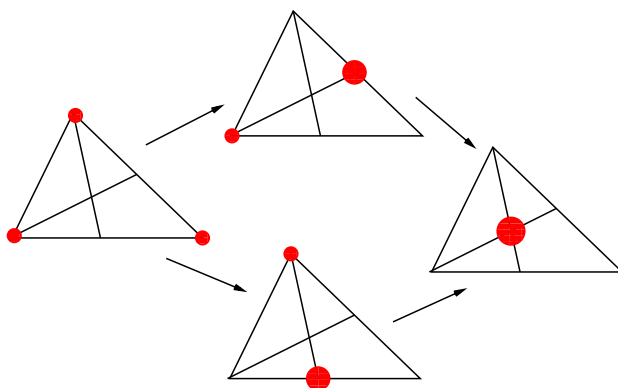


Figure 2.7: The medians of a triangle intersect at one point (physical proof)

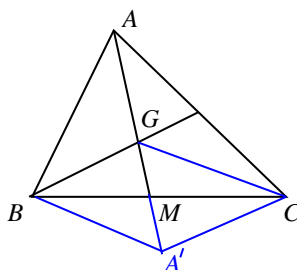


Figure 2.8: The medians of a triangle intersect at one point

It follows that  $N$  and  $P$  are the midpoints of the sides. So the three medians intersect at one point. Of course, as  $|AG|/|GM| = 2$ , the same must be true about  $|BG|/|GN|$  and  $|CG|/|GP|$ , by just repeating the construction using the vertices  $B$ , respectively  $C$  instead of  $A$ . The theorem is proved.  $\square$

The centroid is denoted by  $G$ .

### 2.3.3 The orthocenter

**Definition.** In a triangle  $\triangle ABC$ , the *altitude* from  $A$  is the segment  $|AD|$  with  $D \in |BC|$  that is perpendicular to  $|BC|$ .

Note that the altitude from  $A$  is unique. A triangle has three altitudes, one for each vertex.

**Theorem 2.3.2.** In a triangle, the three altitudes intersect at one point, called the *orthocenter* of the triangle.

*Proof.* Let us consider first the case where  $\triangle ABC$  is acute. We argue on Figure 2.9. Let  $|AD|$ ,  $|BE|$ ,  $|CF|$  be the three altitudes. Because  $\angle C \equiv \angle C$  and  $\angle ADC \equiv \angle BEC$ , being right angles, it follows from Theorem 2.2.3 that  $\triangle ADC \sim \triangle BEC$ . Hence

$$\frac{|CE|}{|CD|} = \frac{|BC|}{|AC|}.$$

Using Theorem 2.2.4, we deduce that  $\triangle CED \sim \triangle CBA$ . It follows that  $\angle EDC \equiv \angle BAC$ .

A similar argument shows that  $\triangle BDF \sim \triangle BAC$ , so  $\angle BDF \equiv \angle BAC$ . It follows that  $\angle EDC \equiv \angle BDF$  so  $\angle FDA \equiv \angle ADE$ , because they have congruent complements. We thus

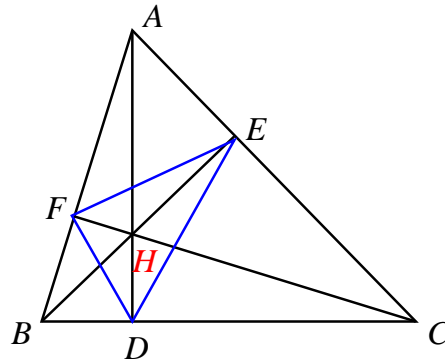


Figure 2.9: The altitudes of a triangle intersect at one point

found that  $|DA|$  is the angle bisector of  $\angle FDE$ . Similarly  $|EB|$  and  $|FC|$  are angle bisectors in  $\triangle DEF$ . We conclude that  $|AD|$ ,  $|BE|$  and  $|CF|$  intersect at the incenter of  $\triangle DEF$ . The theorem is proved.

Now assume that  $\angle BAC$  is obtuse. Let  $H$  be the intersection of  $BE$  and  $CF$ . Then  $BA$  and  $CA$  are altitudes in the acute triangle  $\triangle HBC$ , so  $AH$  is also an altitude. This implies that  $AH$  is perpendicular to  $BC$ , and so  $AH = AD$ . Consequently  $AD$  passes through  $H$  and we are done.  $\square$

The orthocenter is denoted by  $H$ .

**Definition.** The points  $D$ ,  $E$ ,  $F$  are called the feet of the altitudes. Triangle  $\triangle DEF$  is called the *orthic triangle* of  $\triangle ABC$ .

### 2.3.4 The circumcenter

**Definition.** Given two points in the plane, the *perpendicular bisector* of the segment  $|AB|$  is the locus of the points  $P$  in the plane such that  $|AP| \equiv |BP|$

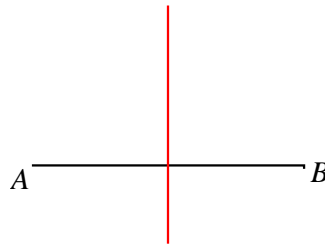


Figure 2.10: Perpendicular bisector of a segment

**Proposition 2.3.1.** The perpendicular bisector of a segment is the line perpendicular to the segment passing through its midpoint.

*Proof.* Let  $|AB|$  be the segment, and  $M$  its midpoint. If  $P$  is a point on the perpendicular bisector, then  $\triangle PAB$  is isosceles ( $|PA| \equiv |PB|$ ), so  $\angle PAB \equiv \angle PBA$ . Given that also  $|AM| \equiv |BM|$ , we have that  $\triangle PMA \equiv \triangle PMB$ , by Theorem SAS. Thus  $\angle PMA \equiv \angle PMB$ , so both are right, showing that  $M$  is on the line perpendicular to  $AB$  passing through  $M$ .

Conversely, if  $P$  belongs to the line through  $M$  that is perpendicular to  $AB$ , then in the right triangles  $\triangle PMA$  and  $\triangle PMB$ ,  $|PM| \equiv |PM|$ , and  $|AM| \equiv |BM|$ , so the triangles are congruent. It follows that  $|PA| \equiv |PB|$ , so  $P$  belongs to the perpendicular bisector.  $\square$

**Theorem 2.3.3.** In a triangle, the perpendicular bisectors of the sides intersect at one point, called the circumcenter of the triangle.

*Proof.* Let the triangle be  $\triangle ABC$ , and let  $M, N, P$  be the midpoints of  $|BC|, |AC|,$  and  $|AB|$ , respectively. Consider the perpendicular bisectors of  $|AB|$  and  $|AC|$ . Because they are perpendicular to lines that are not parallel, they are not parallel themselves, so they intersect at a point  $O$  (see Figure 2.11). Then  $|OA| \equiv |OB|$ , and  $|OA| \equiv |OC|$ , which implies  $|OB| \equiv |OC|$ . It follows that  $O$  is on the perpendicular bisector of  $|BC|$  as well, so the three perpendicular bisectors intersect at one point. □

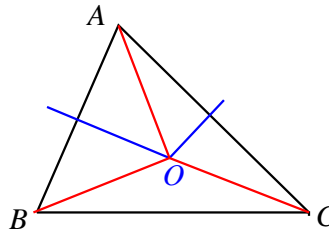


Figure 2.11: The perpendicular bisectors of the sides of a triangle intersect.

It is standard to denote the circumcenter by  $O$ .

### 2.3.5 The Euler line

**Theorem 2.3.4.** (L. Euler) In a triangle the circumcenter, centroid, and orthocenter are collinear. Moreover, the centroid lies between the circumcenter and the orthocenter, and divides the segment formed by the circumcenter and the orthocenter in the ratio 1 : 2.

*Proof.* Let  $M$  and  $N$  be the midpoints of  $|BC|$  respectively  $|AC|$ . Then by Thales' Theorem  $MN \parallel AB$ . Consequently,  $\triangle CMN$  and  $\triangle CBA$  have congruent angles, so they are similar. It follows that  $|MN|/|AB| = |CM|/|CB| = 1/2$ .

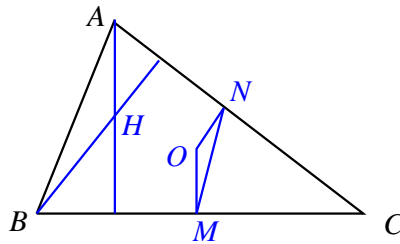


Figure 2.12: Proof of Euler's Theorem

On the other hand  $OM \parallel AH$  because both are perpendicular to  $BC$ , and  $ON \parallel BH$  because both are perpendicular to  $AC$  (see Figure 2.12). This combined with  $MN \parallel AB$  implies that triangles  $\triangle HBA$  and  $\triangle ONM$  have parallel sides, hence they have congruent angles. It follows that  $\triangle OMN \sim \triangle HAB$ . We thus have

$$\frac{|OM|}{|AH|} = \frac{|ON|}{|BH|} = \frac{|MN|}{|AB|} = \frac{1}{2}. \quad (2.3.1)$$

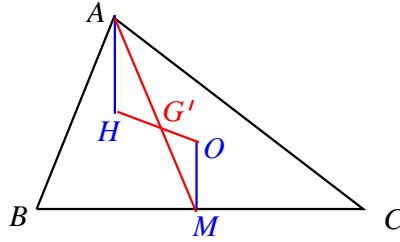


Figure 2.13: Proof of Euler's Theorem

Next, let  $G'$  be the intersection of the median  $|AM|$  with  $|OH|$  (Figure 2.13). Because  $OM \parallel AH$ , we have that  $\angle HAG' \equiv \angle OMG'$ ,  $\angle AHG' \equiv \angle MOG'$ . Also  $\angle AG'H \equiv \angle OG'M$ , being opposite angles. Hence  $\triangle G'AH \sim \triangle G'MO$ . We obtain

$$\frac{|G'H|}{|G'O|} = \frac{|AG'|}{|G'M|} = \frac{|OM|}{|AH|} = \frac{1}{2}.$$

Comparing this to (2.3.1), we deduce that  $G = G'$ , the centroid, and

$$\frac{|OG|}{|GH|} = \frac{1}{2}.$$

The theorem is proved. □

### 2.3.6 Problems

1. Show that in an equilateral triangle, the incenter, centroid, orthocenter, and circumcenter coincide.
2. Where does the circumcenter of a right triangle lie?
3. Show that in an isosceles triangle the Euler line passes through one of the vertices. Show conversely, that if in a triangle the Euler line passes through one of the vertices, then the triangle is isosceles.

## 2.4 Quadrilaterals

### 2.4.1 The centroid of a quadrilateral

The following results apply to all quadrilaterals, including the skew ones.

**Lemma 2.4.1.** The midpoints of the four sides of a quadrilateral form a parallelogram.

*Proof.* Let the quadrilateral be  $ABCD$ , and let  $M, N, P, Q$  be the midpoints of  $|AB|$ ,  $|BC|$ ,  $|CD|$  and  $|DA|$  respectively. Because

$$\frac{|AQ|}{|AD|} = \frac{|AM|}{|AB|} = \frac{1}{2},$$

by Thales' Theorem  $QM \parallel BD$ . Similarly  $PN \parallel BD$ . Therefore  $QM \parallel PN$ .

A similar argument shows that  $QP \parallel PN$ . The lemma is proved. □

**Theorem 2.4.1.** The two segments joining the midpoints of the opposite sides of a quadrilateral have the same midpoint.

*Proof.* (The physical proof) Let the quadrilateral be  $ABCD$ . Place at each vertex a weight of one pound. Then the system consisting of  $A$  and  $B$  has the center of mass at the midpoint of  $|AB|$  and a combined weight of 2 pounds, and the system consisting of  $C$  and  $D$  has the center of mass at the midpoint of  $|CD|$  and a total weight of 2 pounds. The center of mass of the system  $ABCD$  is the midpoint of the segment that joins the midpoints of  $|AB|$  and  $|CD|$ . A similar argument shows that the center of mass of this system is the segment that joins the midpoints of  $|AD|$  and  $|BC|$ , and it is also the midpoint of the segment that joins the midpoints of  $|AC|$  and  $|BD|$ .  $\square$

*Proof.* (The mathematical proof) Let the quadrilateral be  $ABCD$ , and let  $M, N, P, Q$  be the midpoints of  $|AB|, |BC|, |CD|, |DA|$ , respectively. By Lemma 2.4.1,  $MNPQ$  is a parallelogram. By Proposition 2.1.1, the diagonals  $|MP|$  and  $|NQ|$  intersect at their midpoint.  $\square$

**Definition.** The common midpoint of the two segments that join the midpoints of opposite sides in a quadrilateral is called the centroid of the quadrilateral.

## 2.4.2 Problems

1. Show that the segment joining the midpoints of the diagonals of a quadrilateral passes through the centroid and is divided by it into two equal parts.
2. Let  $ABCD$  be a quadrilateral. Show that the segments joining  $A$  with the centroid of  $\triangle BCD$ ,  $B$  with the centroid of  $\triangle CDA$ ,  $C$  with the centroid of  $\triangle DAB$ , and  $D$  with the centroid of  $\triangle ABC$  intersect at one point.

## 2.5 Measurements

### 2.5.1 Measuring segments and angles

Euclidean geometry can be scaled, so there is no a priori unit of length for segments. To measure segments we start by fixing a segment  $|OX|$  and declare it to have length 1.

We define the length of a segment  $|AB|$  to be

$$\|AB\| = \frac{|AB|}{|OX|}.$$

Note that congruent segments have equal lengths, and that length is additive, meaning that the length of the sum of two segments is the sum of the lengths of the segments. The axioms of continuity imply that any positive real number can be the length of a segment.

We want to measure angles so that congruent angles have equal measures and the measure of the sum of two angles is equal to the sum of the measures of the angles.

There are two standard ways of measuring angles. One was introduced by the Babylonians, in which a straight angle was declared to have  $180^\circ$ . Then a right angle has  $90^\circ$ , and the angles of an equilateral triangle have all  $60^\circ$ .

There is a modern way of measuring angles, in which the straight angle is declared to have  $\pi$  radians (where  $\pi$  is the length of a semicircle of radius 1- we will talk later about this). Then the right angle has  $\pi/2$  radians, and the angles of an equilateral triangle have  $\pi/3$  radians.

### 2.5.2 Areas

To be able to talk about areas we need a unit of length. Thus we start by declaring a certain segment  $|OX|$  to be of length 1.

**Definition.** The *interior of a triangle* consists of all points  $P$  with the property that there are two points  $M$  and  $N$  on the sides of the triangle such that  $P$  is between  $M$  and  $N$ .

**Definition.** A polygonal surface is the union of several triangles and their interiors.

We define a function

$$\Sigma \mapsto A(\Sigma),$$

called *area*, which associates to each finite union of polygonal surfaces a number, such that the following four properties are satisfied:

- A1. The area of the square whose sides have length 1 is equal to 1.
- A2.  $A(\Sigma) > 0$  for all unions of polygonal surfaces  $\Sigma$ .
- A3. Congruent triangles have equal areas.
- A4. If  $\Sigma_1$  and  $\Sigma_2$  are disjoint, or if they share only a finite union of segments (on the boundary), then

$$A(\Sigma_1 \cup \Sigma_2) = A(\Sigma_1) + A(\Sigma_2).$$

As a corollary of A2 and A4, if  $\Sigma \subset \Sigma'$ , then  $A(\Sigma) \leq A(\Sigma')$ .

**Theorem 2.5.1.** The area of a rectangle  $ABCD$  is equal to  $\|AB\| \cdot \|BC\|$ .

*Proof.* Case 1. The side-lengths of the rectangle are integer numbers.

Let the sides have lengths  $m$  respectively  $n$ . Divide the rectangle into unit squares, then count the squares. There are  $mn$  squares, and by condition A4, their total area is  $mn$ .

Case 2. The side-lengths of the rectangle are rational numbers.

Let the side-lengths be  $m_1/n$  and  $m_2/n$ , with  $m_1, m_2, n$  integers (use the common denominator!). Divide the unit square into  $n^2$  congruent squares. By A4, the area of each square is  $1/n^2$ . Next, divide the rectangle into  $m_1 \times m_2$  squares, each of size  $\frac{1}{n} \times \frac{1}{n}$ . Using A4 again, we conclude that the total area is

$$m_1 m_2 \cdot \frac{1}{n^2} = \frac{m_1}{n} \cdot \frac{m_2}{n},$$

as desired.

Case 3. The side-lengths are arbitrary numbers.

Note that A4 implies that the area is an increasing function, namely that if  $\Sigma_1 \subset \Sigma_2$ , then  $A(\Sigma_1) < A(\Sigma_2)$  (because  $\Sigma_2$  is the union of  $\Sigma_1$  and the "piece" inbetween). We can approximate the rectangle from above and below by rectangles with rational side-lengths, and pass to the limit to obtain the conclusion.  $\square$

**Theorem 2.5.2.** The area of a triangle is equal to half the product of the lengths of a side and of the altitude from the opposite angle.

*Proof.* We argue on Figure 2.14. The idea is to double the triangle to a rectangle. For that, let  $\triangle ABC$  be the triangle and consider the altitude  $|AD|$ . Let  $E$  and  $F$  be such that  $FB$  and  $EC$  are both perpendicular to  $BC$ , and  $EF$  is parallel to  $BC$ . Then  $\angle FAB \equiv \angle DBA$ , and because  $|AB| \equiv |AB|$ , by Theorem 1.4.6  $\triangle AFB \equiv \triangle BDA$ . For a similar reason  $\triangle AEC \equiv \triangle CDA$ . Using A3 and A4, we conclude that

$$\begin{aligned} A(\triangle ABC) &= \frac{1}{2}[A(\triangle ABD) + A(\triangle AFB) + A(\triangle ADC) + A(\triangle AEC)] = \frac{1}{2}A(BCEF) \\ &= \frac{1}{2}\|AD\| \cdot \|BC\|. \end{aligned}$$

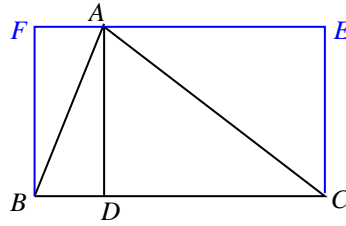


Figure 2.14: The area of a triangle

□

**Corollary 2.5.1.** The area of a right triangle is equal to the half the product of the lengths of the sides adjacent to the right angle.

**Theorem 2.5.3.** Let  $ABCD$  be a trapezoid,  $AD \parallel BC$ . Let  $M$  be a point on  $BC$  such that  $AM$  is perpendicular to  $BC$ . Then

$$A(ABCD) = \frac{\|AM\|}{2}(\|AD\| + \|BC\|).$$

*Proof.* Divide the trapezoid into the triangles  $\triangle ACD$  and  $\triangle ABC$  as shown in Figure 2.15, then apply Theorem 2.5.2 to these triangles. □

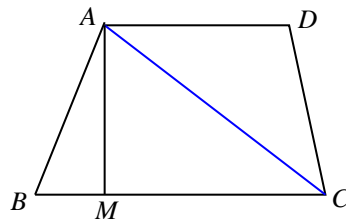


Figure 2.15: The area of a trapezoid

**Corollary 2.5.2.** The area of a parallelogram is equal to the product of the length of the base and the height.

### 2.5.3 The Pythagorean Theorem

**Theorem 2.5.4.** Let  $\triangle ABC$  be a right triangle, with  $\angle A$  right. Then

$$\|AB\|^2 + \|AC\|^2 = \|BC\|^2.$$



### 2.5.4 Problems

1. What is the area of a regular polygon of side-length 1?
2. Prove the Pythagorean theorem.
3. Prove that the area of a parallelogram with side-lengths 2 and 3 does not exceed 6.

## 2.6 The circle

**Definition.** The circle of center  $O$  and radius  $|AB|$  is the locus of points  $M$  such that  $|OM| \equiv |AB|$ .

The radius is usually specified by a segment, or by a length. In the latter case it is denoted by one letter.

Given two circles of centers  $O_1$  and  $O_2$  and radii  $R_1$  and  $R_2$ , their relative position can be:

1. one interior to the other, if  $\|O_1O_2\| < |R_1 - R_2|$ ;
2. interior tangent, if  $\|O_1O_2\| = |R_1 - R_2|$ ;
3. intersecting, if  $|R_1 - R_2| < \|O_1O_2\| < R_1 + R_2$ ;
4. exterior tangent, if  $\|O_1O_2\| = R_1 + R_2$ ;
5. exterior to each other, if  $\|O_1O_2\| > R_1 + R_2$ .

These five situations are shown in Figure 2.16.

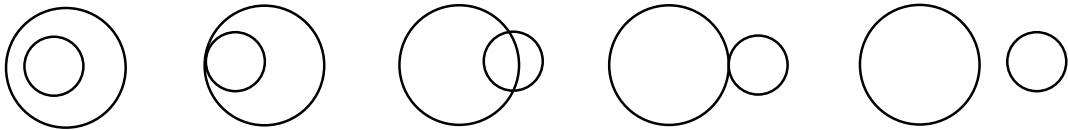


Figure 2.16: The relative position of two circles

The segment determined by two points on the circle is called *chord*. If the center of the circle belongs to the chord, the chord is called *diameter*. A line that intersects a circle at exactly one point is called *tangent*.

### 2.6.1 Measuring angles using arcs

**Definition.** Given a circle of center  $O$  and two points  $A$  and  $B$  on this circle, the arc  $\widehat{AB}$  is the set of points on the circle that lie inside the angle  $\angle AOB$  together with  $A$  and  $B$ .

**Definition.** We define the measure of the arc  $\widehat{AB}$  by the equality  $m(\widehat{AB}) = m(\angle AOB)$ .

**Theorem 2.6.1.** Let  $A, B, C$  be points on a circle. Then

$$m(\angle BAC) = \frac{1}{2}m(\widehat{BC}),$$

where  $\widehat{BC}$  is the arc of the circle that lies inside the angle.

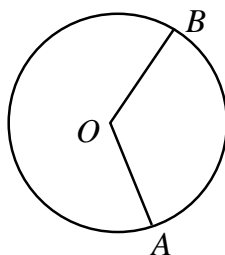


Figure 2.17: Measuring angles by arcs

*Proof.* Case 1.  $O \in |AC|$  (Figure 2.18). The triangle  $\triangle OAB$  is isosceles, so  $\angle OAB \equiv \angle OBA$ . By the Exterior Angle Theorem,  $\angle BOC \equiv \angle OAB + \angle OBA$ . It follows that

$$m(\angle BAC) = \frac{1}{2}m(\angle BOC) = \frac{1}{2}m(\widehat{BC}).$$

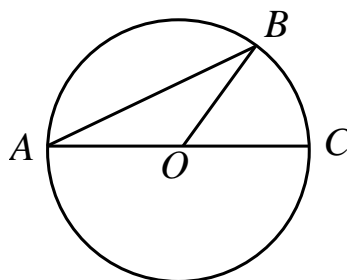


Figure 2.18: Inscribed angles, case 1

Case 2.  $O$  is in the interior of the angle  $\angle BAC$  (Figure 2.19). Let  $M$  be on the circle such that  $O \in |AM|$ . Then

$$\begin{aligned} m(\angle BAC) &= m(\angle BAM) + m(\angle MAC) = \frac{1}{2}m(\widehat{BM}) + \frac{1}{2}m(\widehat{MC}) \\ &= \frac{1}{2}m(\widehat{BC}). \end{aligned}$$

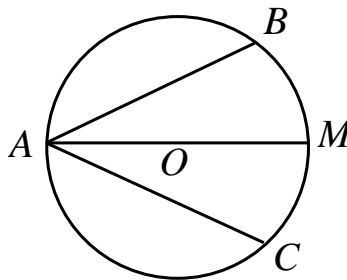


Figure 2.19: Inscribed angles, case 2

Case 3.  $O$  is outside the angle  $BAC$  (Figure 2.20). Let  $M$  be on the circle such that  $O \in |AM|$ .

Assume without loss of generality that  $C$  is in the interior of the angle  $\angle BAM$ . Then

$$\begin{aligned} m(\angle BAC) &= m(\angle BAM) - m(\angle CAM) = \frac{1}{2}m(\widehat{BM}) - \frac{1}{2}m(\widehat{CM}) \\ &= \frac{1}{2}m(\widehat{BC}). \end{aligned}$$

The theorem is proved.

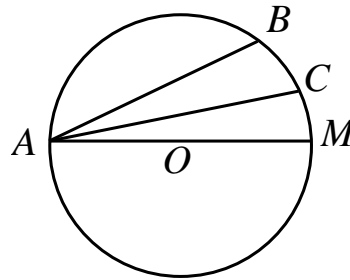


Figure 2.20: Inscribed angles, case 3

□

**Theorem 2.6.2.** Assume that the chords  $|AB|$  and  $|CD|$  of a circle intersect at  $M$ . Then

$$m(\angle BMD) = \frac{1}{2}m(\widehat{BD}) + \frac{1}{2}m(\widehat{AC}),$$

where the two arcs lie inside the angle and its opposite.

*Proof.* We argue on Figure 2.21. By the euclidean version of the Exterior Angle Theorem,  $\angle BMD \equiv \angle MAD + \angle MDA$ . It follows that

$$m(\angle BMD) = m(\angle MAD) + m(\angle MDA) = \frac{1}{2}m(\widehat{AC}) + \frac{1}{2}m(\widehat{BD}).$$

We are done.

□

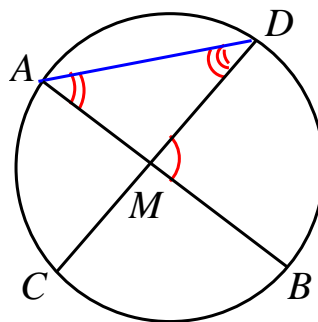


Figure 2.21:

**Theorem 2.6.3.** Assume that the lines of support of the chords  $AB$  and  $CD$  of a circle intersect outside the circle at  $M$  such that  $A$  is between  $M$  and  $B$  and  $C$  is between  $M$  and  $D$ . Then

$$m(\angle AMC) = \frac{1}{2}m(\widehat{BD}) - \frac{1}{2}m(\widehat{AC}),$$

where the arcs are defined as to lie inside the angle.

*Proof.* We argue on Figure 2.22. Again we apply the Exterior Angle Theorem to conclude that  $\angle BCD \equiv \angle MBC + \angle BMC$ . We have

$$m(\angle AMC) = m(\angle BCD) - m(\angle MBC) = \frac{1}{2}m(\widehat{BD}) - \frac{1}{2}m(\widehat{AC}),$$

as desired.  $\square$

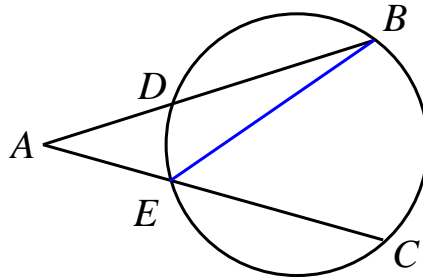


Figure 2.22:

**Proposition 2.6.1.** Let  $\angle BAC$  be an angle such that  $|AC|$  is a chord in a circle and  $AB$  is tangent to the circle. Then

$$m(\angle BAC) = \frac{1}{2}m(\widehat{AC}),$$

where the arc is taken as to lie inside the angle.

*Proof.* Consider the case where  $|AB|$  is a chord, then rotate the chord until it becomes tangent.  $\square$

**Corollary 2.6.1.** The tangent is perpendicular to the radius at the point of contact.

## 2.6.2 The circumcircle and the incircle of a triangle

**Theorem 2.6.4.** Given a triangle  $\triangle ABC$ , there is a unique circle containing the vertices  $A, B, C$ .

*Proof.* Let us recall the proof of Theorem 2.3.3. There, we showed that the three perpendicular bisectors of the sides  $|AB|$ ,  $|BC|$ , and  $|CA|$  intersect at a point  $O$  which lie at equal distance from the vertices. Hence  $O$  is the center of a circle passing through  $A, B, C$ .

If there were a second circle containing  $A, B, C$ , then using the characterization of the perpendicular bisector of a segment given by Proposition 2.3.1, we deduce that the center of that circle is also at the intersection of the perpendicular bisectors of the sides, hence it equals  $O$ . And so the circle is the same, which proves uniqueness.  $\square$

**Definition.** The unique circle passing through the vertices of a triangle is called the circumcircle.

**Theorem 2.6.5.** Given a triangle  $\triangle ABC$ , there is a unique circle tangent to the segments  $|AB|$ ,  $|BC|$ ,  $|AC|$ .

*Proof.* In Theorem 1.17 we showed that the angle bisectors of a triangle intersect at one point by showing that this point, denoted by  $I$ , is at equal distance from the sides. Let  $D, E, F$  be the projections of  $I$  onto  $BC, AC$ , and  $AB$ . Because in triangle  $\triangle IBC$ ,  $\angle IBC$  and  $\angle ICB$  are acute,  $D$  is between  $B$  and  $C$ . Similarly we deduce that the other projections are on the sides.

Because  $|ID| \equiv |IE| \equiv |IF|$ ,  $I$  is the circumcenter of  $\triangle DEF$ . And because  $IE$  is perpendicular to  $BC$ ,  $BC$  is tangent to the circle. For the same reason  $AC$  and  $AB$  are tangent, which shows that the circumcircle of  $\triangle DEF$  has the desired property.

Uniqueness follows from the fact that if another circle is tangent to the sides, and  $I'$  is its center, then  $I'$  is equally distanced from the sides, and hence it lies on the angle bisector of each angle of  $\triangle ABC$ . It follows that  $I' = I$ , and consequently the circle is unique.  $\square$

**Definition.** The unique circle tangent to the three sides of the triangle is called the *incircle*.

*Remark 2.6.1.* If we require the circle to be only tangent to the lines  $AB$ ,  $AC$ , and  $BC$ , there are *four* such circles, the other three are called the excircles.

### 2.6.3 Cyclic quadrilaterals

**Theorem 2.6.6.** A quadrilateral is cyclic if and only if two opposite angles add up to a straight angle.

*Proof.* Let the quadrilateral be  $ABCD$  (see Figure 2.23). Assume that it is cyclic. Then

$$m(\angle DAC) + m(\angle DCB) = \frac{1}{2}m(\widehat{DCB}) + \frac{1}{2}m(\widehat{DAB}) = \frac{1}{2}2\pi = \pi.$$

Hence  $\angle DAC$  and  $\angle DCB$  add up to a straight angle.

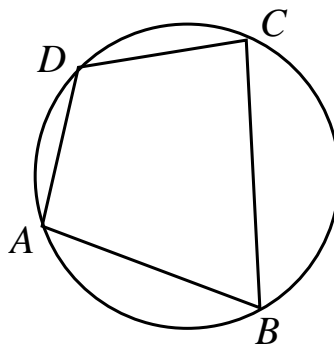


Figure 2.23:

Conversely, assume that  $m(\angle DAB) + m(\angle DCB) = \pi$ . Arguing by contradiction, assume that  $ABCD$  is not cyclic.

Choose  $C' \in BC$  such that  $A, B, C', D$  lie on a circle. We only discuss the case where  $C' \in |BC|$ , the other two cases ( $B \in |CC'|$  and  $C \in |BC'|$ ) being analogous (see Figure 2.24). Then  $\angle DCB \equiv \angle DC'B$ , since both have  $\angle DAB$  as their supplement. But this contradicts the Exterior Angle Theorem in  $\triangle DCC'$ . Hence the conclusion.  $\square$

**Theorem 2.6.7.** A quadrilateral is cyclic if and only if the angle formed by one side with a diagonal is congruent to the angle formed by the opposite side with the other diagonal.

*Proof.* If  $ABCD$  is cyclic then, as can be seen in Figure 2.25,

$$m(\angle BAC) = \frac{1}{2}m(\widehat{BC}) = m(\angle BDC).$$

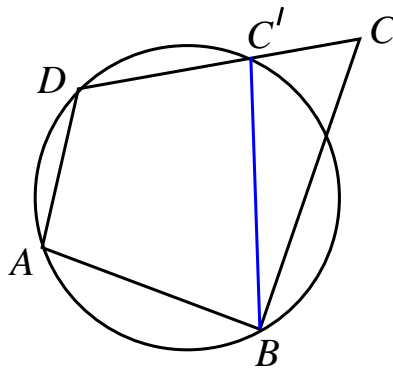


Figure 2.24:

Conversely, if  $\angle BAC \equiv \angle BDC$  but  $ABCD$  is not cyclic, let  $A'$  be the second intersection of  $AB$  with the circumcircle of  $\triangle BCD$ . We discuss only the case where  $A' \in |AB|$  (Figure 2.26), the other cases being similar. We have

$$\angle BAC \equiv \angle BDC \equiv \angle BA'C$$

which contradicts the exterior angle theorem for  $\triangle DAA'$ . Thus our assumption was false, showing that  $ABCD$  is cyclic.

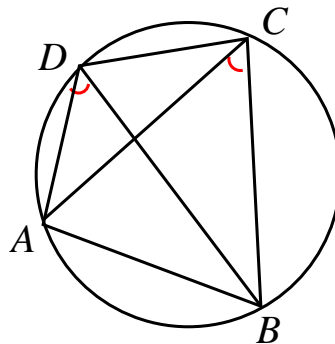


Figure 2.25:

□

**Corollary 2.6.2.** A quadrilateral with two opposite right angles is cyclic.

**Corollary 2.6.3.** The rectangle is the only cyclic parallelogram.

**Corollary 2.6.4.** The isosceles trapezoid is the only cyclic trapezoid.

**Theorem 2.6.8.** (Simson's line) Let  $\triangle ABC$  be a triangle and let  $P$  be a point on the circumcircle of  $\triangle ABC$ . Then the projections of  $P$  onto the lines  $AB$ ,  $AC$ , and  $BC$  are collinear.

*Proof.* Without loss of generality, let us assume that  $P$  is on the arc  $\widehat{AC}$ . Let  $L, M, N$  be the projections onto  $BC$ ,  $AC$ , respectively  $AB$ . Let us also assume that  $L$  is between  $B$  and  $C$ ,  $M$  between  $A$  and  $C$ , and  $A$  between  $B$  and  $N$ , as in Figure 2.27, the proofs for the other possible configurations being similar.

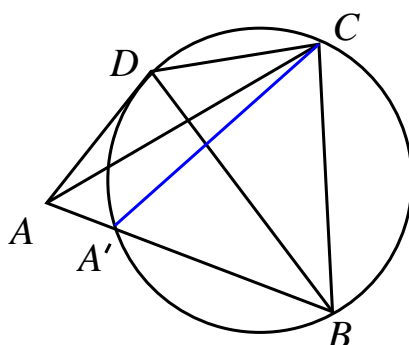


Figure 2.26:

The quadrilateral  $AMPN$  is cyclic. Hence

$$\angle PNM \equiv \angle PAM = \angle PAC.$$

The quadrilateral  $PNBL$  is cyclic, so

$$\angle PNL \equiv \angle PBL = \angle PBC.$$

The quadrilateral  $ABCP$  is cyclic, hence

$$\angle PAC \equiv \angle PBC.$$

It follows that

$$\angle PNM \equiv \angle PNL,$$

so  $|NM| = |NL|$ , which means that the points  $L, M, N$  are collinear.

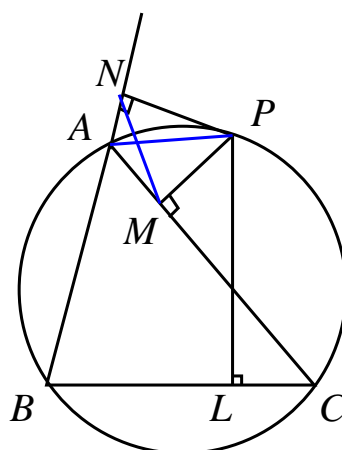


Figure 2.27: Simson's line

□

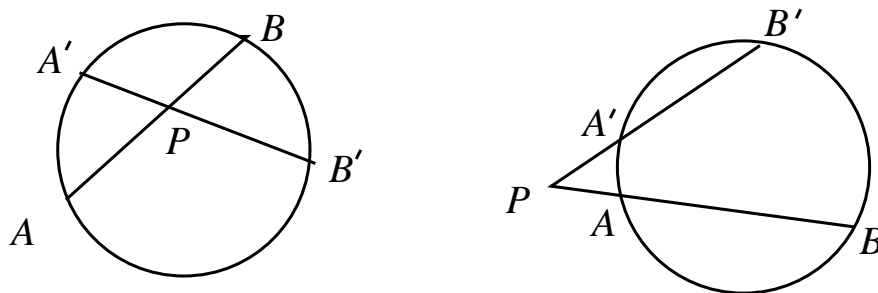


Figure 2.28: Power of a point

### 2.6.4 Power of a point with respect to a circle

**Theorem 2.6.9.** If  $P$  is a point in the plane of a circle, and  $AB$  and  $A'B'$  are two lines passing through  $P$  such that  $A, B, A', B'$  are on the circle, then

$$\|PA\| \cdot \|PB\| = \|PA'\| \cdot \|PB'\|.$$

*Proof.* If  $P$  is on the circle, then the product is zero in both cases.

If  $P$  is outside the circle, suppose that  $A$  is between  $P$  and  $B$  and that  $A'$  is between  $P'$  and  $B'$  (Figure 2.29). Then  $ABB'A'$  is cyclic, hence  $\angle A'AB$  and  $\angle BB'A'$  add up to a straight angle. It follows that  $\angle PAA' \equiv \angle PB'B$ . This combined with the fact that  $\angle APA' = \angle BPB'$  implies that  $\triangle PAA' \sim \triangle PB'B$ . Thus

$$\frac{|PA'|}{|PB|} = \frac{|PB'|}{|PA|}.$$

Multiplying out the denominators we obtain the equality from the statement.

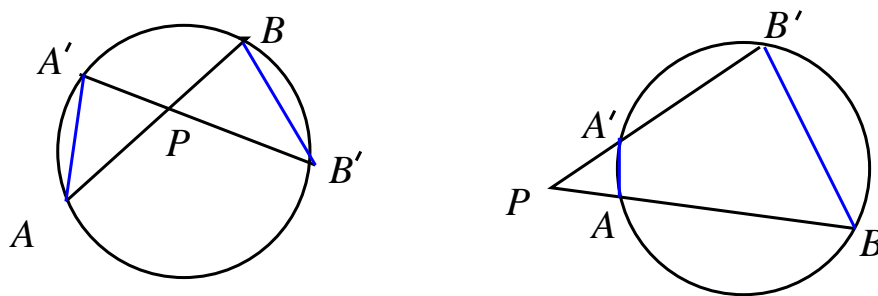


Figure 2.29: Proof of power of a point property

The case where  $P$  is inside the circle is similar (look carefully at Figure 2.29). □

**Definition.** Given a point  $P$  and a circle, let  $AB$  be a line through  $P$ , with  $A$  and  $B$  on the circle. The power of  $P$  with respect to the circle is equal to  $\|PA\| \cdot \|PB\|$  if  $P$  is outside of the circle, to 0 if  $P$  is on the circle, and to  $-\|PA\| \cdot \|PB\|$  if  $P$  is inside the circle.

**Proposition 2.6.2.** Given a point  $P$  and a circle of center  $O$  and radius  $R$ , the power of  $P$  with respect to the circle is equal to  $\|PO\|^2 - R^2$ .



*Proof.* The property is true if  $P$  is on the circle. Let  $A$  and  $B$  be the intersections of line  $PO$  with the circle. If  $P$  is outside of the circle, then

$$\|PA\| \cdot \|PB\| = (\|PO\| + R)(\|PO\| - R) = \|PO\|^2 - R^2.$$

If  $P$  is inside the circle, then

$$-\|PA\| \cdot \|PB\| = -(R + \|PO\|)(R - \|PO\|) = -(R^2 - \|PO\|^2) = \|PO\|^2 - R^2.$$

□

**Theorem 2.6.10.** The locus of points that have equal powers with respect to two circles is a line.

*Proof.* Assume the circles have centers and radii  $O_1, O_2$  respectively  $R_1, R_2$ . Let  $P$  be a point on the locus, and consider  $Q$  on  $O_1O_2$  such that  $PQ$  is orthogonal to  $O_1O_2$  (see Figure 2.30).

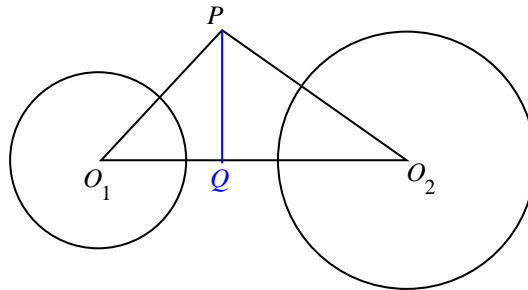


Figure 2.30: The radical axis

By the Pythagorean theorem,

$$\|PO_1\|^2 = \|PQ\|^2 + \|QO_1\|^2 \text{ and } \|PO_2\|^2 = \|PQ\|^2 + \|QO_2\|^2.$$

Subtracting we obtain

$$\|QO_1\|^2 - \|QO_2\|^2 = \|PO_1\|^2 - \|PO_2\|^2,$$

and the latter is equal to  $R_1^2 - R_2^2$ , by Proposition 2.6.2. This completely determines the position of  $Q$  on the line  $O_1O_2$ , and consequently  $P$  must belong to a line that passes through this point  $Q$  and is perpendicular to  $O_1O_2$ . Conversely, if  $P$  belongs to this line, then the same application of the Pythagorean theorem implies that  $P$  has equal powers with respect to the two circles. □

**Definition.** The set of points that have equal powers with respect to two circles is called the *radical axis* of the two circles.

**Theorem 2.6.11.** Given three circles with noncollinear centers, there is a unique point in the plane, called *radical center*, that has equal powers with respect the three circles.

*Proof.* Let the circles be  $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ . Let  $P$  be the intersection of the radical axis of  $\mathcal{C}_1$  and  $\mathcal{C}_2$  with the radical axis of  $\mathcal{C}_1$  and  $\mathcal{C}_3$ . Then  $P$  has equal power with respect to  $\mathcal{C}_1$  and  $\mathcal{C}_2$ , and equal power with respect to  $\mathcal{C}_1$  and  $\mathcal{C}_3$ . Consequently, it has equal power with respect to all three circles. □

**Corollary 2.6.5.** Given three circles  $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ , whose centers are noncollinear, the radical axes of the pairs  $\{\mathcal{C}_1, \mathcal{C}_2\}, \{\mathcal{C}_1, \mathcal{C}_3\}, \{\mathcal{C}_2, \mathcal{C}_3\}$ , intersect at the radical center of the three circles.

## 2.6.5 Problems

1. Given two circles that are exterior tangent at  $M$ , consider a line  $AB$  that passes through  $M$  such that  $A$  is on the first circle and  $B$  is on the second circle. Prove that the measure of one of the arcs determined by  $A$  and  $M$  is equal to the measure of one of the arcs determined by  $B$  and  $M$ .
2. Let  $\triangle ABC$  be a triangle, and let  $|AA'|$ ,  $|BB'|$ ,  $|CC'|$  be diameters in the circumcircle. Prove that  $\triangle ABC \cong \triangle A'B'C'$ .
3. Let  $ABCD$  be a quadrilateral with the property that  $AC$  and  $BD$  are orthogonal. Let  $M, N, P, Q$  be the midpoints of the sides  $|AB|$ ,  $|BC|$ ,  $|CD|$ , and  $|DA|$  respectively. Prove that  $MNPQ$  is cyclic.
4. Let  $\triangle ABC$  be an acute triangle, and let  $D \in |BC|$  and  $E \in |AC|$  be the feet of the altitudes. Prove that  $\angle ABE \cong \angle ADE$ .
5. In the triangle  $\triangle ABC$ ,  $\angle A = 60^\circ$  and the angle bisectors  $|BB'|$  and  $|CC'|$  intersect at  $I$ . Prove that  $|IB'| \cong |IC'|$ .
6. Let  $\angle AOB$  be a right angle,  $M$  and  $N$  points on the rays  $|OA$  respectively  $|OB$  and let  $MNPQ$  be a square such that  $MN$  separates points  $O$  and  $P$ . Find the locus of the center of the square when  $M$  and  $N$  vary.
- 7.\* (Euler's circle) Show that in a triangle the feet of the altitudes, the midpoints of the sides, and the midpoints of the segments connecting the orthocenter to the vertices are on a circle.
8. Assume that the circles  $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$  have noncollinear centers and assume that  $\mathcal{C}_1$  and  $\mathcal{C}_2$  intersect at  $A$  and  $B$ ,  $\mathcal{C}_1$  and  $\mathcal{C}_3$  intersect at  $C$  and  $D$ , and  $\mathcal{C}_2$  and  $\mathcal{C}_3$  intersect at  $E$  and  $F$ . Prove that  $AB$ ,  $CD$ , and  $EF$  intersect at one point.
9. Let  $P$  be a point inside a circle such that there exist three chords through  $P$  of equal length. Prove that  $P$  is the center of the circle.
- 10.\* (Țițeica's Five Lei Coin Problem) Let  $\mathcal{C}_1, \mathcal{C}_2$ , and  $\mathcal{C}_3$  be three circles of equal radii that pass through a common point  $P$  and intersect pairwise at  $A, B, C$ . Prove that the circumcircle of  $\triangle ABC$  has the same radius as  $\mathcal{C}_1, \mathcal{C}_2$ , and  $\mathcal{C}_3$ .

# Chapter 3

## Geometric Transformations

### 3.1 Isometries

#### 3.1.1 Translations

A vector is an oriented segment. More precisely, a vector is an equivalence class of oriented segments, all parallel, congruent, and pointing in the same direction. A vector is denoted by a lower case letter with an arrow on top of it, or by two upper case letters, the endpoints, with an arrow on top.

**Definition.** Given a vector  $\vec{v}$ , the translation of a point  $A$  by  $\vec{v}$  is a point  $A'$  such that  $\vec{AA'} = \vec{v}$ .

**Theorem 3.1.1.** 1. The translation of a segment is a segment parallel and congruent to it.  
2. The translation of an angle is an angle congruent to it.  
3. The translation of a triangle, is a triangle congruent to it.

*Proof.* For the proof of 1, let  $|AB|$  be a segment, and let  $A'$  and  $B'$  be the translates of  $A$  and  $B$ . Then  $|AA'|$  and  $|BB'|$  are parallel and congruent, so  $ABB'A'$  is a parallelogram. It follows that  $|AB|$  and  $|A'B'|$  are parallel and congruent.

That the translation of a triangle is a triangle congruent to it follows from 1 and Theorem SSS. Finally, we get 2 by including the angle in a triangle.  $\square$

The addition of vectors is defined using the parallelogram rule. This turns the set of vectors into a group, with the identity element being the zero vector, and the negative of a vector being the vector of same length pointing in the opposite direction.

The set of translations endowed with composition is a group which is isomorphic to the group of vectors.

**Example 3.1.1.** *Two cities lie on the opposite sides of a river, at some distance from the river. The river has non-negligible width. Construct a road and a bridge that minimize the distance traveled between the cities. (Note: The bridge should be perpendicular to the river.)*

*Proof.* Let the river be defined by two parallel lines  $l$  and  $l'$ , with city  $A$  on the shore  $l$  and the city  $B$  on the shore  $l'$ . Let  $\vec{v}$  be a vector orthogonal to  $l$  and  $l'$ , of length equal to the distance between  $l$  and  $l'$  and pointing from  $l$  to  $l'$ . Let  $A'$  be the image of  $A$  through the translation. Consider the line  $A'B$  and let  $M'$  be the intersection of  $l'$  and  $A'B$ . Of course,  $M'$  is the image through the translation of a point  $M$  on  $l$ . We claim that the bridge should be  $|MM'|$ .

Indeed, for another location of the bridge, say  $|NN'|$ , the total distance

$$|AN| + |NN'| + |N'B| \equiv |A'N'| + |MN| + |N'B| < |A'B| + |MN| = |AM| + |MM'| + |M'B|,$$

where the inequality is follows from the triangle inequality. The problem is solved.  $\square$

### 3.1.2 Reflections

**Definition.** Given a line  $l$  in the plane, the *reflection* of a point  $A$  over the line  $l$  is a point  $A'$  such that  $AA' \perp l$ ,  $A$  and  $A'$  are in different half-planes determined by  $l$  and the distances from  $A$  and  $A'$  to  $l$  are equal.

**Theorem 3.1.2.** 1. The reflection of a segment over a line is a segment congruent to it. 2. The reflection of an angle over a line is an angle congruent to it. 3. The reflection of a triangle over a line is a triangle congruent to it.

*Proof.* 1. Let  $l$  be the line over which we reflect. Consider a segment  $|AB|$  and let  $A'$  and  $B'$  be the reflections of  $A$  and  $B$  over  $l$ . Let  $AA'$  intersect  $l$  at  $M$  and  $BB'$  intersect  $l$  at  $N$ . Then  $|AM| \equiv |A'M|$  and  $|MN| \equiv |MN|$ , so the right triangle  $\triangle AMN$  and  $\triangle A'MN$  are congruent. It follows that  $|AN| \equiv |A'N|$  and  $\angle ANM \equiv \angle A'NM$ . Hence  $\angle ANB \equiv \angle A'NB'$ , and Theorem SAS implies  $\triangle ABN \equiv \triangle A'B'N$ . It follows that  $|AB| \equiv |A'B'|$ .

By the same argument, for every  $P \in |AB|$ , with  $P'$  its image under the reflection,  $|PA| \equiv |P'A|$  and  $|PB| \equiv |P'B|$ , and hence  $|A'B'| \equiv |A'P'| + |P'B'|$ . This implies that  $P' \in |A'B'|$ . It is not hard to see that every point on  $|A'B'|$  is the image of a point on  $|AB|$ . This proves the first part.

3. This follows from 1, by using Theorem SSS.

2. By 1, the reflection of an angle is an angle, namely the two rays that determine the angle are mapped to two rays. To show that the reflected angle is congruent to the original one, place the original angle in a triangle, then reflect the triangle and use 3.  $\square$

**Proposition 3.1.1.** If  $\sigma$  is the reflection with respect to a line  $l$ , then  $\sigma^2$  is the identity map. Consequently, the two-element set formed by a reflection and the identity map is a group.

*Remark 3.1.1.* This is a geometric realization of the group  $\mathbb{Z}_2$ .

### 3.1.3 Rotations

**Definition.** The rotation about  $O$  by angle  $\alpha$  maps a point  $A$  to a point  $A'$  such that  $m(\angle AOA') = \alpha$  and  $|OA| \equiv |OA'|$ .

We agree to measure  $\alpha$  in degrees. One distinguishes between positive (counter-clockwise) rotations, and negative (clockwise) rotations.

**Theorem 3.1.3.** 1. The rotation of a segment is a segment congruent to the original one. 2. The rotation of an angle is an angle congruent to the original one. 3. The rotation of a triangle is a triangle congruent to the original one.

*Proof.* Exercise.  $\square$

**Proposition 3.1.2.** 1. The inverse of the counter-clockwise rotation about  $O$  by  $\alpha$  is the counter-clockwise rotation by  $360^\circ - \alpha$  about the same point. 2. The composition of the rotation of angle  $\alpha_1$  about  $O$  and the rotation of angle  $\alpha_2$  about  $O$  is the rotation of angle  $\alpha_1 + \alpha_2$  about  $O$ .

Using addition of angles, and composition of rotations we can define rotations for every real value of  $\alpha$ .

**Corollary 3.1.1.** The set of all rotations about a point is a group.

**Theorem 3.1.4.** (D. Pompeiu) Given an equilateral triangle  $\Delta ABC$  and a point  $P$  that does not lie on the circumcircle of  $\Delta ABC$ , one can construct a triangle of side-lengths  $\|PA\|, \|PB\|, \|PC\|$ .

*Proof.* Rotate  $\Delta ABC$  about  $C$  by  $60^\circ$ . Then  $|AP'| \equiv |B'P'| \equiv |BP|$ . Also  $|CP| \equiv |CP'|$ , which means that  $\Delta CPP'$  is an isosceles triangle with one angle of  $60^\circ$ . It follows that  $\Delta CPP'$  is equilateral (prove it!). Then in  $\Delta APP'$ ,  $|AP|, |AP'| \equiv |BP|$ , and  $|PP'| \equiv |CP|$ , so this triangle has the desired property.  $\square$

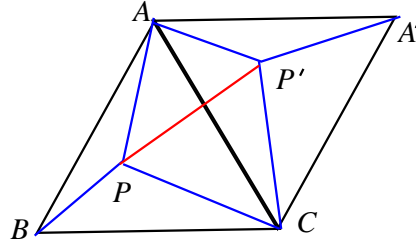


Figure 3.1: Proof of Pompeiu's Theorem

**Example 3.1.2.** Given a closed polygonal line, show that it contains 3 points which form an equilateral triangle.

Solution: To prove this fact, rotate the polygonal line by  $60^\circ$  around a point  $A$  on a side. Let  $B$  be a point where the new line intersects the old. Point  $C$  which is the preimage of  $B$  together with  $A$  and  $B$  form an equilateral triangle.

### 3.1.4 Isometries

**Definition.** A transformation of the plane which preserves lengths is called *isometry*.

Rotations, reflections, and translations are isometries.

Remark: An isometry is a one-to-one transformation of the plane. The next result will also prove that it is onto.

**Theorem 3.1.5.** Every isometry of the plane is of the form  $r \circ t$  or  $s \circ r \circ t$ , where  $s$  is a reflection,  $r$  is a rotation, and  $t$  is a translation.

*Proof.* Let  $f$  be the isometry. Consider a segment  $|AB|$ , and let  $A' = f(A)$  and  $B' = f(B)$ . If  $M$  is on  $|AB|$ , and  $M' = f(M)$ , then  $|A'M'| + |M'B'| \equiv |AM| + |MB| \equiv |AB| \equiv |A'B'|$ , so  $M'$  is on  $|A'B'|$ , and moreover, any  $M'$  on  $|A'B'|$  can be obtained as the image of a point  $M$ . This shows that  $f(|AB|) = |A'B'|$ . In fact by slightly modifying the argument we deduce that  $f(AB) = A'B'$ . Let  $r$  and  $t$  be a rotation and a translation such that  $r \circ t(|AB|) = |A'B'|$ . Then  $rt$  restricted to  $AB$  is equal to  $f$ .

Consider a point  $P$  which does not belong to  $AB$ . Then because  $|AP| \equiv |A'P'|$  and  $|BP| \equiv |B'P'|$ , there are only two locations that  $P' = f(P)$  can have, namely either  $P' = r \circ t(P)$ , or  $P' = s \circ r \circ t(P)$ , where  $s$  is the reflection over  $AB$ .

Now let us show that an isometry is completely determined by the image of 3 non-collinear points. Let  $A, B, P$  be the three noncollinear points mapped to  $A', B', P'$ . We saw above that the restriction of the isometry is completely determined when restricted to  $AB, AP$ , and  $BP$ . Let  $Q$  be a point that does not lie on any of these lines. Then  $Q$  is mapped to one of two points lying on

one side or the other of  $AB$ . If  $Q$  is not separated by  $AB$  from  $P$ , then  $|PQ|$  does not intersect  $A'B'$ , so  $|P'Q'|$  does not intersect  $AB$  (or else the isometry wouldn't be one-to-one). This shows that there is a unique choice for the location of  $Q'$ . Similarly, if  $|PQ|$  intersects  $AB$ , then  $|P'Q'|$  intersects  $A'B'$ , so again the location of  $Q'$  is unique. This proves our claim.

We conclude that the isometry can only be either  $rot$  or  $sorot$ , and the theorem is proved.  $\square$

### 3.1.5 Problems

1. What is the composition of two reflections over parallel lines?
2. What is the composition of two reflections over non-parallel lines?
3. Show that every isometry is a composition of reflections.
4. Given a polygon, and a point  $P$  in its interior, show that there are two points  $A$  and  $B$  on the polygon such that  $P$  is the midpoint of  $|AB|$ .
5. Two towns are on the same side of the river, at some distance from the river. They want to build a common water pump that would supply both with water. In what location should the pump be built in order to minimize the total length of the two pipes that connect it to the cities?

## 3.2 Homothety and Inversion

### 3.2.1 Homothety

Let  $O$  be a point in the plane and  $r$  a real number.

**Definition.** The homothety of center  $O$  and ratio  $r$  sends a point  $P$  to a point  $P'$  such that  $\vec{OP'} = r\vec{OP}$ .

**Theorem 3.2.1.** Homothety maps a segment to a segment parallel to it whose length is  $|r|$  times the length of the original segment.

**Corollary 3.2.1.** Homothety maps an angle to an angle congruent to it. Homothety maps a triangle to a triangle similar to it, with similarity ratio  $|r|$ .

### 3.2.2 Inversion

**Definition.** Given a circle of center  $O$  and radius  $r > 0$ , the inverse of a point  $P \neq O$  with respect to this circle is a point  $P'$  on the ray  $|OP$  such that

$$\|OP\| \cdot \|OP'\| = r^2.$$

By abuse of language, we map  $O$  to the "point at infinity". With this convention, the square of the inversion is the identity map.

**Theorem 3.2.2.** Inversion maps circles through  $O$  into lines not passing through  $O$ , and lines not passing through  $O$  into circles through  $O$ . Inversion maps lines through  $O$  into themselves, and circles that do not pass through  $O$  into circles that do not pass through  $O$ .

**Theorem 3.2.3.** Inversion distorts distances according to the following formula

$$\|A'B'\| = \frac{r^2}{\|OA\| \cdot \|OB\|} \|AB\|.$$

# Chapter 4

## Non-Euclidean Geometry

### 4.1 The negation of Euclid's fifth postulate

**Postulate:** There is a line  $l$ , and a point  $A$  that does not belong to  $l$ , such that through  $A$  pass two lines that do not intersect  $l$ .

**Theorem 4.1.1.** For the line  $l$  and the point  $A$  from the above postulate, there exist infinitely many lines passing through  $A$  that do not intersect  $l$ .

*Proof.* Let  $l_1$  and  $l_2$  be the lines through  $A$  that do not intersect  $l$ . There is a ray of  $l_2$  that is separated from  $l$  by  $l_1$ . Pick  $P$  on this ray and  $Q$  on  $l$ . Let  $R$  be the intersection of  $|PQ|$  with  $l_1$ . Choose one of the infinitely many points of the segment  $|PR|$ , call this point  $M$ .

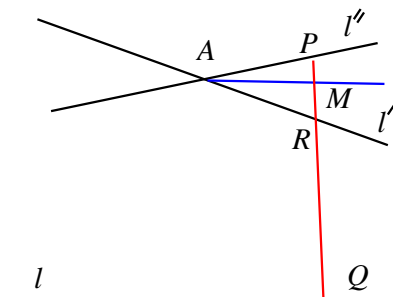


Figure 4.1: Proof that there are infinitely many parallels

Assume that  $AM$  intersects  $l$  at some point  $S$ . Then  $l_1$  intersects side  $|MQ|$  of  $\triangle MQS$ , but does not intersect side  $|QS|$ , and it cannot intersect  $|MS|$  since it already intersected  $MA$  at  $A$ . This contradicts the Axiom of Pasch, proving that  $AM$  does not intersect  $l$ . Since each point  $M \in |PR|$  determines a different line, there are infinitely many lines that pass through  $A$  and don't intersect  $l$ .  $\square$

### 4.2 Euclid's fifth postulate and the sum of the angles of a triangle

**Theorem 4.2.1.** The sum of the angles of a triangle is less than or equal to a straight angle.

*Proof.* We will need two results.

**Lemma 4.2.1.** The sum of two interior angles of a triangle is less than a flat angle.

*Proof.* Let  $\triangle ABC$  be a triangle. By the Exterior Angle Theorem

$$\angle A < \text{supplement of } \angle C.$$

Adding  $\angle C$  to both sides we obtain that  $\angle A + \angle C$  is strictly less than a straight angle.  $\square$

**Lemma 4.2.2.** For any triangle, there exists a triangle with the same sum of angles and with one of the angles as small as we want.

*Proof.* In  $\triangle ABC$ , let  $M$  be the midpoint of  $|AC|$ . Either  $\angle ABM$  or  $\angle MBC$  is less than or congruent to  $\angle B/2$ . Say  $\angle MBC \leq \angle B/2$ . Let  $C'$  be such that  $M$  is the midpoint of  $|BC'|$  (Figure 4.2). Consider the triangle  $\triangle ABC'$ . By Theorem SAS,  $\triangle MAC' \equiv \triangle MBC$ . We have

$$\begin{aligned} \angle BAC' + \angle AC'B + \angle ABC' &\equiv \angle BAC + \angle CAC' + \angle AC'B + \angle ABC' \\ &\equiv \angle BAC + \angle ACB + \angle ABC' + \angle C'BC \equiv \angle BAC + \angle ACB + \angle CBA. \end{aligned}$$

Triangle  $\triangle ABC'$  has the same sum of angles as  $\triangle ABC$  and  $\angle C' \leq \angle B/2$ . Repeating the construction  $n$  times we find a triangle with the same sum of angles as  $\triangle ABC$  and with one angle less than or congruent to  $\angle B/2^n$ .  $\square$

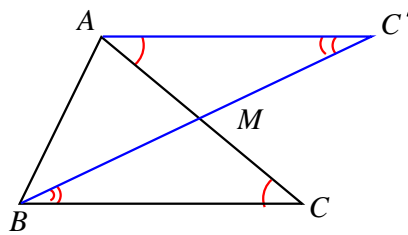


Figure 4.2: Sum of angles of a triangle

Let us return to the proof of the theorem. Assume that for some triangle  $\triangle ABC$ ,

$$\angle A + \angle B + \angle C \equiv \text{straight angle} + \angle\alpha, \quad \angle\alpha > 0.$$

Construct a triangle  $\triangle A'B'C'$  with the same sum of angles and with  $\angle C' < \angle\alpha/2$ . Then, by the first lemma,  $\angle A' + \angle B'$  is less than a straight angle. Adding  $\angle C'$  we obtain  $\angle A' + \angle B' + \angle C'$  less than a straight angle plus  $\angle\alpha/2$ . This contradicts the fact that  $\angle A' + \angle B' + \angle C'$  equal to a straight angle plus  $\angle\alpha$ . Hence the conclusion.  $\square$

**Theorem 4.2.2.** If the sum of the angles of a certain triangle is *strictly* less than a straight angle, then the same is true for any triangle.

*Proof.* Let  $M$  be a point on  $|BC|$ . If the sum of the angles of  $\triangle ABC$  is less than a straight angle, (recall Theorem 4.2.1) then the same must be true for one of the triangles  $\triangle ABM$  and  $\triangle AMC$ , otherwise by adding we would obtain equality for  $\triangle ABC$ . Dividing further we can obtain an arbitrarily small triangle with the sum of angles less than a straight angle.

Consider now some triangle  $\triangle A'B'C'$  in the plane. Place inside a small triangle with the sum of the angles less than a straight angle. Divide  $\triangle A'B'C'$  into triangles one of which is the one with sum of angles less than a straight angle. By adding the angles of the triangles in the decomposition and removing the straight angles that are formed we deduce that the sum of the angles of  $\triangle A'B'C'$  is less than a straight angle. The theorem is proved.  $\square$



**Corollary 4.2.1.** Either in every triangle the sum of the angles is a straight angle, or in every triangle the sum of the angles is strictly less than a straight angle.

**Proposition 4.2.1.** If the fifth postulate holds for one point and one line, then there exists a triangle with the sum of the angles equal to a straight angle.

*Proof.* Let  $A$  and  $l$  be the point and line for which the postulate holds. Choose  $B$  and  $C$  on  $l$ . Pick  $l'$  such that the alternate angles formed by  $AC$  with  $l$  and  $l'$  are congruent. The Exterior Angle Theorem implies that  $l$  and  $l'$  do not intersect.

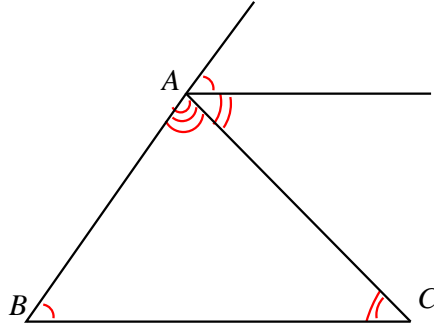


Figure 4.3: Fifth postulate and the sum of angles of a triangle

Pick a line  $l''$  so that the alternate angles formed by  $AB$  with  $l$  and  $l''$  are congruent. Again  $l$  and  $l''$  do not intersect. Uniqueness implies  $l' = l''$ . Examining the Figure 4.3 we see that  $\angle A + \angle B + \angle C$  is congruent to a straight angle.  $\square$

**Theorem 4.2.3.** If the sum of the angles of every triangle is congruent to a straight angle, then Euclid's fifth postulate holds.

*Proof.* Let  $l$  be a line and  $A$  a point,  $A \notin l$ . Let  $AB$  be the perpendicular to  $l$ ,  $B \in l$ . The line  $l'$  perpendicular to  $AB$  at  $A$  does not intersect  $l$ . We will show that any other line through  $A$  intersects  $l$ . We argue on Figure 4.4.

Let  $l''$  be a line passing through  $A$  and  $\angle\alpha$  the acute angle this line makes with  $|AB|$ . Consider the point  $B_1, B_2, \dots, B_n, \dots$  on  $l$  that lie on the same side of  $AB$  as  $\angle\alpha$ , and such that

$$|BB_1| \equiv |AB|, \quad |B_1B_2| \equiv |AB_1|, \quad \dots, \quad |B_{n-1}B_n| \equiv |AB_{n-1}|.$$

In  $\triangle ABB_1$ ,

$$m(\angle A) = m(\angle B_1) = \frac{\pi}{4}.$$

Using the fact that the sum of the angles of a triangle is  $\pi$ , we further compute, in  $\triangle AB_1B_2$ ,

$$m(\angle A) = m(\angle B_2) = \frac{\pi}{8}, \dots$$

in  $\triangle AB_{n-1}B_n$ ,

$$m(\angle A) = m(\angle B_n) = \frac{\pi}{2^{n+1}}.$$

For sufficiently large  $n$ ,

$$m(\angle BAB_n) = \frac{\pi}{4} + \frac{\pi}{8} + \dots + \frac{\pi}{2^{n+1}}$$

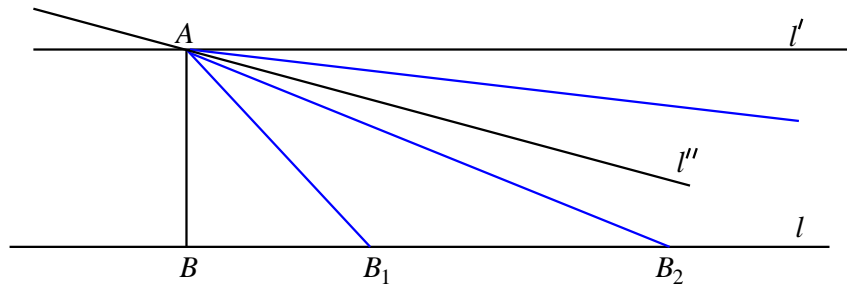


Figure 4.4: Euclid's fifth postulate and the sum of the angle of triangles

will be very close to  $\pi/2$ , hence greater than  $\angle\alpha$ . Then  $l''$  will run inside  $\angle BAB_n$ , so it will have to cross  $|BB_n|$ . Hence  $l''$  intersects  $l$ . □

**Theorem 4.2.4.** If Euclid's fifth postulate fails for a point and a line, then it fails for every other point and every other line not containing the point.

*Proof.* If fifth postulate holds for a point and a line, then the sum of the angles of some triangle is a straight angle. But then the sum of the angles of any other triangle is congruent to a straight angle. In that case the fifth postulate must hold for every point and every line. □

**Corollary 4.2.2.** Either Euclid's fifth postulate holds, or for every line  $l$  and every point  $A$ ,  $A \notin l$ , there are infinitely many lines passing through  $A$  which do not intersect  $l$ .

### 4.3 The area of a triangle in non-euclidean geometry

Since in non-euclidean geometry the area of a triangle is strictly less than  $\pi$ , we can associate to each triangle  $\Delta ABC$  the positive number

$$\epsilon(\Delta ABC) = \pi - m(\angle A) - m(\angle B) - m(\angle C).$$

**Proposition 4.3.1.** If the triangular surface  $T$  is the union of the triangular surfaces  $T_1$  and  $T_2$ , then

$$\epsilon(T) = \epsilon(T_1) + \epsilon(T_2).$$

Because of this we can make the following definition.

**Definition.** The area of a triangle  $\Delta ABC$  is equal to  $\epsilon(\Delta ABC)$ .

As a corollary, we have that the area of a polygon with  $n$  sides is  $(n - 2)\pi$  minus the sum of the angles of the polygon. This satisfies the four properties that the area should satisfy.