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FUNCTIONAL ANALYSIS

BS 4-Years
M.Sc. Mathematics

Z.R. Bhatti

FUNCTIONAL ANALYSIS

3rd Edition

For

BS 4-Years

M.Sc. Mathematics

According to New Syllabus Approved by the

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1-1 Definitions and Examples

In this section we shall discuss basic definitions and some examples of metric spaces.

Metric Space: Let X be a nonempty set. A function $d: X \times X \to R$ is said to be a *metric* on X, if for all $x,y,z \in X$, it satisfies the following axioms:

M_1): $d(x,y) \geq 0$ (Non-negativity)

M_2): $d(x,y) = 0 \Leftrightarrow x = y$ (Reflexive property)

M_3): $d(x,y) = d(y,x)$ (Symmetric property)

M_4): $d(x,z) \leq d(x,y) + d(y,z)$ (Triangle inequality)

The pair (X,d) is called the *metric space*. The set X is called the *underlying set* or the *ground set*. The elements of X are called the *points* of the metric space (X,d). Instead of (X,d), we may write X for a metric space if there is no danger of confusion.

We shall see in the next examples that the given definition of d will play an important role in satisfying the above four axioms of metric space. However, in some concepts and results, the value of d will not play any role. In such cases the metric space (X,d) is denoted by X. In fact, mathematicians do not bother about specifying the metric. If it is said that X is a metric space then one should himself understand that there is a metric defined on the set X.

Next we discuss the geometrical meanings of the metric. If we concentrate on the four axioms of the metric space, we realise that these axioms are exactly the properties of distance.

For example, the distance between two points is never negative, and the first axiom of metric space also shows that the value of metric d is always nonnegative.

Similarly, the distance of two points is zero if these points are same, and if the points are same then the distance between theses points will be zero. This is exactly which is expressed in the second axiom of metric.

The third axiom of metric expresses the fact that the distance between two points is same whether it is measured from first point to second point or from second point to first point.

The fourth and final axiom of metric is the justification for three points. If the tree points are collinear then the distance between two points which lie at the ends is equal to the sum of distances of endpoints from the intermediate point. On the other hand, if the points are non-collinear then a triangle is formed and the length of one side of the triangle is always lesser than the sum of the lengths of the other two sides. Fourth axiom of metric space describes this property of distance without specifying whether the points are collinear or non-collinear.

This geometric interpretation of metric concludes that the metric on a set is exactly the distance function, that is, it gives the distance between the points of metric space under consideration.

Thus, $d(x,y)$ is the distance between the points x and y. The fourth axiom of the metric is called the *triangle inequality* and is motivated by the elementary geometry as shown in Fig.1.1.

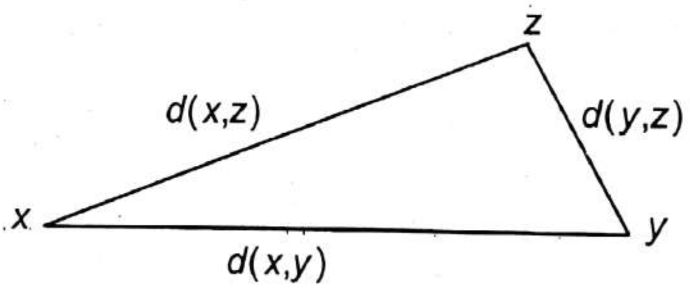

Figure 1.1: The Triangle Inequality

Remark: If $x_1, x_2, \ldots, x_n$ are n distinct points of the metric space (X,d), then fourth axiom can be generalised as follows:

$$\begin{aligned} d(x_1,x_n) &\leq d(x_1,x_2)+d(x_2,x_n) \\ &\leq d(x_1,x_2)+d(x_2,x_3)+d(x_3,x_n) \\ &\leq d(x_1,x_2)+d(x_2,x_3)+d(x_3,x_4)+\ldots+d(x_{n-1},x_n) \end{aligned}$$

$$\Rightarrow d(x_1,x_n) \leq d(x_1,x_2)+d(x_2,x_3)+d(x_3,x_4)+\ldots+d(x_{n-1},x_n)$$