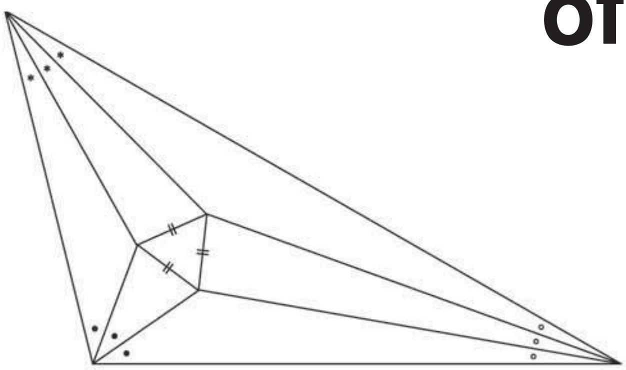
Mathematics, the language of watchmaking

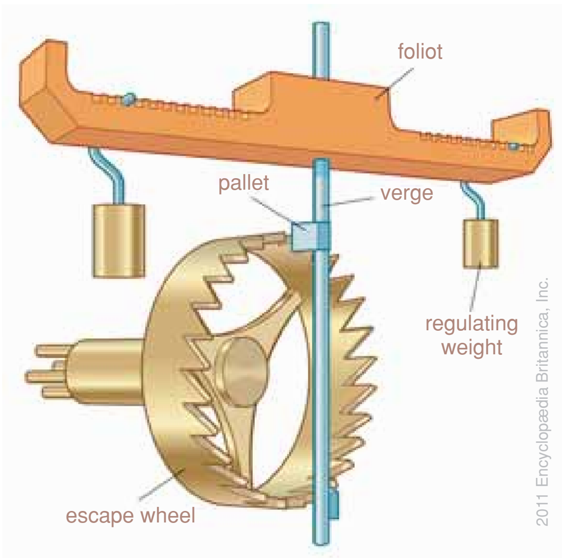


Morley's theorem (1898) states that if you trisect the angles of any triangle and extend the trisecting lines until they meet, the small triangle formed in the centre will always be equilateral.

Ilan Vardi[1]

I am often asked to explain mathematics; is it just about numbers and equations? The best answer that I've found is that mathematics uses numbers and equations like a language. However what distinguishes it from other subjects of thought – philosophy, for example – is that in maths complete understanding is sought, mostly by discovering the order in things. That is why we cannot have real maths without formal proofs and why mathematicians study very simple forms to make profound discoveries. One good example is the triangle, the simplest geometric shape that has been studied since antiquity. Nevertheless the world had to wait 2,000 years for Morley's theorem, one of the few mathematical results that can be expressed in a diagram.

Horology is of interest to a mathematician because it enables a complete understanding of how a watch or clock works. His job is to impose a sequence, just as a conductor controls an orchestra or a computer's real-time clock controls data processing. Comprehension of a watch can be compared to a violin where science can only confirm the preferences of its maker. Another example is the bicycle that has defied mathematical analysis for more than a century. Bicycle wheels roll without sliding which leads to a mathematical model that has no clear solution[2]. The watchmaker also needs



Foliot clock.

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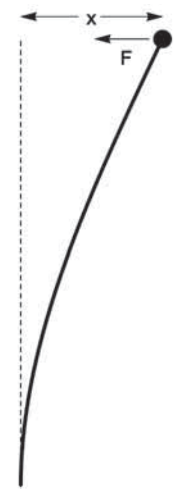


total understanding to make his watch work, so mathematicians and watchmakers always have something to discuss.

Since they first appeared at the end of the Middle Ages, mechanical clocks and later watches have always been worked by the energy of a falling weight or an unwinding spring, the rates of which are controlled by a regulator. The original regulator was the foliot, a centrally pivoted arm that is swung back and forth by a gear train driven by a spring or weight. The inertia of the foliot slowed down the gear train, and with trial-and-error adjustment the clock might keep time to within a quarter of an hour a day. There was little room for improvement in spring-powered timepieces because the braking effect of the foliot is directly linked to the torque of the spring. When the spring is at full tension, the foliot swings faster and the watch gains; conversely the watch goes slow when the spring runs down. These time keepers really did nothing more than indicate the varying output of the mainspring.

A craft becomes a science. Until the 17th century, clocks and watches were designed and built by artisans who managed, by dint of mechanical adjustment, to improve the precision to several minutes a day. The great leap forward in horology was the introduction of an oscillator with its own recoil energy. In such an oscillator the period of each oscillation is independent of the energy source. Isochronism – equal periods – means that the oscil-

The isochronism of an oscillator became easy to describe once Isaac Newton's laws of motion were known in 1687. An oscillator like a blade spring obeys Hooke's Law: the force of a spring, *F,* is proportional to the distance it is tensed, *x*. Following Newton's laws it could be expressed by a differential equation

$$m\frac{d^2x}{dt^2} = -kx$$

where *m* is the mass and *k,* the stiffness of the spring, is a constant; *t* is the time and d^2x/dt^2, the acceleration. Using differential calculus, which we also owe to Isaac Newton, we obtain the explicit answer for the position of the blade

$$x(t) = A\,\sin(2\pi f t + \varphi)$$

where *A* is the amplitude, *f,* the frequency and ϕ is a phase that describes the initial position of the oscillator. You can also derive a formula for the frequency

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

This formula shows that the frequency is only dependent on the mass and elasticity; it is thus independent of the amplitude, *A,* demonstrating the much sought-after isochronism.

[2] History of bicycle steer and dynamics equations, *bicycle.tudelft.nl/schwab/Bicycle/BicycleHistoryReview*

Portrait of Christiaan Huygens, painted by Caspar Netscher in 1671. Collection of the Historical Museum of the Hague. On permanent loan to the Boerhaave Museum in Leiden, the Netherlands.

The return force of a pendulum is due to gravity and does not obey Hooke's Law. A pendulum is thus not a perfect oscillator. The equation for the return force is more like

$$L\,\frac{d^2\theta}{dt^2} = -g\sin\theta$$

where θ is the angle of the pendulum, L is its length, g acceleration due to gravity; t is the time and $d^2\theta/dt^2$, the angular acceleration. Knowing that sin θ is close to θ when θ is small, one can abstract the difference and write

$$L\,\frac{d^2\theta}{dt^2} = -g\theta$$

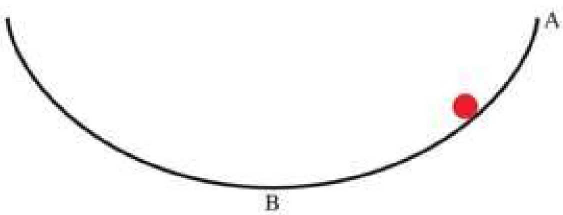
This is the equation of a true frequency oscillator

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{L}}$$

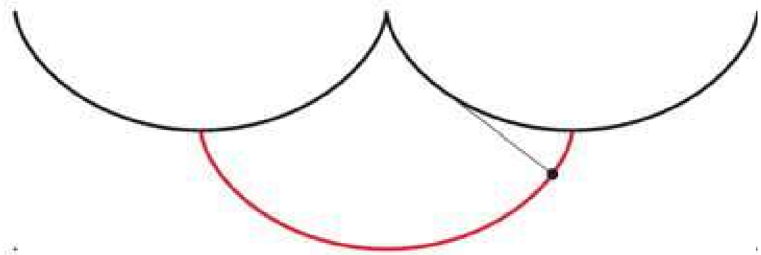
However this approximation is insufficient for a proper precision clock. Increasing the amplitude of a standard pendulum by 2 to 2.5 degrees causes a loss of 3.7 seconds a day.

lator makes its own time. Isochronism is the basic principle of time measurement. We owe it to Galileo, who discovered it by timing a swinging chandelier in a church by the rate of his pulse. He concluded that the period of the oscillation was independent of its amplitude, and so defined isochronism.
Replacing the foliot by the pendulum increased precision to within 15 seconds a day, almost a 100-fold improvement. It could be said that time had been conquered, because clocks had become more regular than the sun, which can deviate by as much as 30 seconds a day. The accurate clock triggered a cultural revolution, since time measurement was no longer based on natural phenomena, but on an artificial and technical construct.
This revolution was built on the laws of physics to formulate the problem, and of mathematics to solve it. And that is how horology became a science.

The horological mathematician. Christiaan Huygens (1629-1695) is perhaps the embodiment of the horological mathematician. The physicist, Galileo, had discovered isochronism by observing a pendulum, but Huygens the mathematician went much further and deeper. On December 25, 1656, he found out how to regulate a clock with a pendulum to make it substantially more accurate. His calculations immediately showed that Galileo was wrong and that pendulums were not isochronous.
The mathematical Huygens was not satisfied by this and found the full answer by devising an



A ball rolling along a cycloid.



The involute of a cycloid is also a cycloid.

isochronous pendulum. This is his solution in three stages:

1 The tautochrone curve. Huygens sought to find the curve down which a ball will roll in a given time, irrespective of where it starts on the curve, as shown in the above left-hand diagram. He showed that the curve was a cycloid. It followed that the oscillations of ball on a cycloid are isochronous.
2 Involutes and evolutes. Huygens proposed the notion of the involute – the curve traced by the end of a taut string unwinding from a circle or other curve (the evolute).
3 The involute of a cycloid is another cycloid.

Verifying these results is an easy task with the knowledge of today, but it should be noted that Huygens had arrived at them in 1657, meaning 30 years before Isaac Newton's laws of motion and differential calculus in 1687. Huygens' methods are well explained in an excellent book by Léopold Defossez[3].

The result was that a pendulum on a cord that unwinds off a cycloid, as depicted on the above right-hand diagram, is isochronous. Huygens' idea was implemented by the clockmaker, Salomon Coster, in 1657 with an immediate improvement in timekeeping precision.

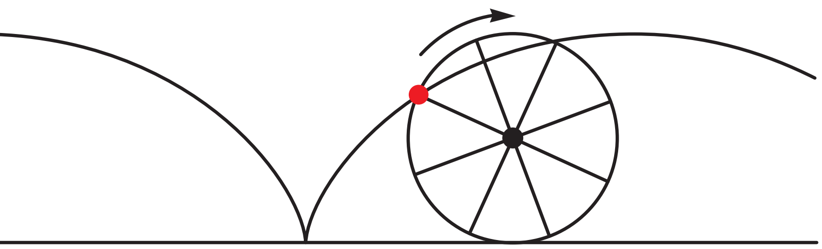
This story is not well known to mathematicians, but it does illustrate how abstract concepts are thought up to solve concrete problems, and that once the theory is established their origins are often forgotten. Huygens' solution is typical of the mathematician's work: a sound and elegant theory, but of not much practical use. Indeed, the cycloidal pendulum did not give very satisfactory results and was abandoned in favour of a rigid pendulum beating at low amplitude. But Huygens did not stop at the pendulum. In 1675 he introduced the sprung balance, which is theoretically isochronous according to Hooke's Law (Robert Hooke should always be mentioned as the co-inventor of the balance spring). The sprung balance continues to regulate the wristwatches of today.

Other eminent mathematicians took an interest in horology. George Biddle Airy (1801-1892) drew up the specifications of London's "Big Ben" clock and William Thomson (1824-1907), better known as Lord Kelvin of the temperature scale, built an astronomical clock with a unique system of maintaining power.

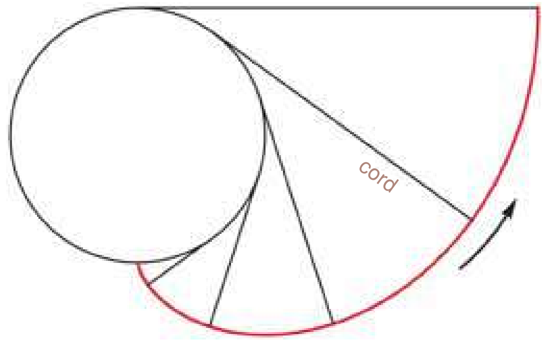
We can't all be Huygens, yet advances in mechanical horology are still possible. Our laboratory, for instance, last year announced the first mechanical oscillator to regulate a timepiece without an escapement[5].

Horological theory. Mathematics continues to play a role in the watchmaking industry. The theory of time measurement is embodied in physics, the technical expression of which is the language of mathematics. The teaching of horological theory

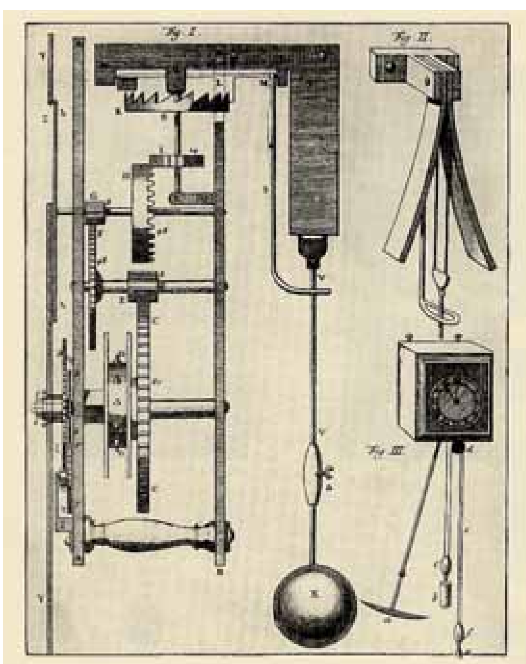
[3] *Léopold Defossez,* Les savants du XVIIe siècle et la mesure du temps, *published by the Journal Suisse d'Horlogerie, Lausanne 1946.*



A cycloid is the curve described by a point on the rim of a wheel rolling without slipping on a straight line.



The involute of a curve is the line drawn by the end of a string unwinding from the curve. Gear teeth are often profiled according to the involute of a circle.



Clock designed by Huygens with cycloidal banking.[4]

and its application date back to Jules Grossmann (1829-1907), director of the horological school at Le Locle. He realised that teaching people to adjust watches was difficult because it was a knack that was not easily transmitted. He thus taught the theory of time measurement to save hours of fiddling about with the adjustment. It was the equivalent of today's computer-aided design that enables watchmakers to test concepts without having to make a string of prototypes.

With his son, Hermann, he wrote the first Swiss book describing the theoretical basis of timekeeping. His successor at the Le Locle school, Léopold Defossez, wrote the definitive book on the subject, *Théorie Générale de l'Horlogerie,* published in 1950 and since used throughout the Swiss watch industry. Today the textbooks for the technical schools are *The Theory of Horology* of the Joux valley's technical school and the *Traité de construction horlogère* by the professors of the HE-ARC technical university. •

[4] *Christian Huygens,* Horologium Oscillatorium, *translated from the Latin by Ian Bruce, www.17centurymaths.com/contents/huygenscontents.html*

[5] *S. Henein, I. Vardi, L. Rubbert, R. Bitterli, N. Ferrier, S. Fifanski, D. Lengacher,* IsoSpring : vers la montre sans échappement, actes de la Journée d'Etude de la SSC 2014, 49–58.