Geometry

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Geometry

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NOTICE

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I. Module 5: Geometry

II. Prerequisite Courses Or Knowledge

High school geometry.

III. Time

The total time for this module is 120 study hours.

IV. Material

You are highly recommended to use the interactive computer software in the CD-ROM–*Resources* included in your study material package. The software *Geo-Gebra* or *WinGeom* will help you to explore the geometry world in an interesting and dynamic way and with lesser expenses of paper and time!

When starting each Unit you should visit, at least one time, the on-line material in the Internet as indicated in *Relevant Readings*. Almost all links in *Relevant Readings* have contents in off-line in the CD-ROM–*Resources.* This includes free and open e-books and the software mentioned above.

V. Module Rationale

The Module on Geometry starts by looking at the historical development of knowledge that the humankind gather along centuries and became later, about 300 BC, the mathematical subject called "Euclidian Geometry" because of the grate work of Euclid. The inductive-deductive reasoning which characterizes this subject will be developed through investigation of your own conjectures on geometric objects and properties. You will explore geometry by using basic mechanical instruments (compass and straightedge) and computer software.

As you progress you will treat the Euclidian geometry using a referential system to locate points. The orthogonal Cartesian system of coordinates that you already know from secondary school is the most common referential system you will use in both two and three dimensions. You will also learn some other systems of coordinates that will empower you to do research in geometry and in other mathematical modules as well.

Going deeper in analysing the axiomatic construction of Euclidean geometry you will learn new geometrical structures, generally designated as Non-Euclidian geometry. So, summarily speaking, this Module is about Euclidean geometry treated in both syntactical and analytical ways and encloses an introduction to Non-Euclidean Geometry, handled synthetically only.

VI. Overview

The module is divided in four Units. Each sub-Unit constitutes a Learning Activity (LA). The total time of the Module is 120 hours, distributed as in table below:

Module 5: Geometry	Time
Revising Euclidean Geometry	24
Plane Analytic Geometry	34
Solid Analytic Geometry	42
Non-Euclidean Geometry	20
Total time	120 hours

6.1 Outline: Syllabus

Each sub-Unit constitutes a Learning Activity (LA)

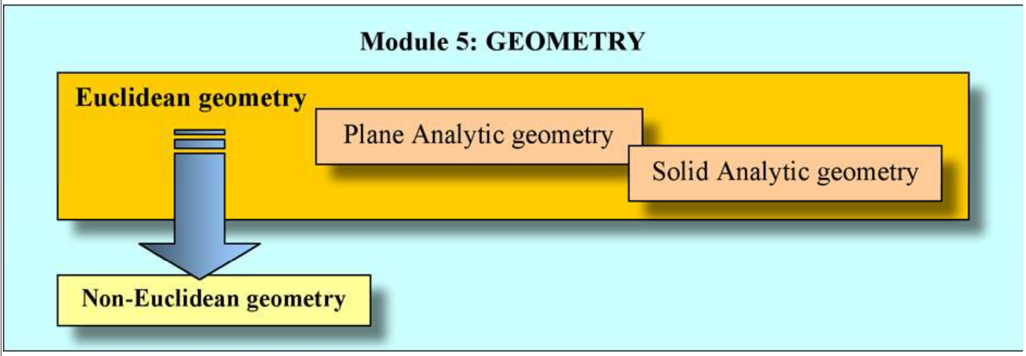
Unit 1:	Revising Euclidean Geometry	Time
LA1.1	On history of Euclidean geometry	1
LA1.2	Axiomatic development of Euclidean geometry	3
LA1.3	Euclidean plane isometry	6
LA1.4	Triangles	5
LA1.5	Circles	3
LA1.6	Geometric similarity	4
Miscellaneous exercises		2
Total time		24 hours

Unit 2:	Plane Analytic Geometry	Time
LA2.1	Vector space in two dimensions	6
LA2.2	The straight line	8
LA2.3	Transformation of coordinates in the plane	4
LA2.4	Conic sections	12
Miscellaneous exercises		4
Total time		34 hours

Unit 3:	Solid Analytic Geometry	Time
LA3.1	Vectors in three dimensions	8
LA3.2	Planes and lines	10
LA3.3	Transformations of coordinates in 3D-space	6
LA3.4	Quadric surfaces	14
Miscellaneous exercises		4
Total time		42 hours

Unit 4:	Non-Euclidean Geometry	Time
LA4.1	Introduction to non-Euclidian geometry	6
LA4.2	Affine transformations	4
LA4.3	Projective transformations	8
Miscellaneous exercises		2
Total time		20 hours

6.2 Graphic Organiser



Note: Euclidean geometry is the central topic of Module 5. The first Unit indeed is dedicated on revising Euclidean geometry in *syntactical* approach. Afterwards, using Linear Algebra (Module 4) with the orthogonal Cartesian system of coordinates, Euclidean geometry will be treated in *analytical* approach covering two Units: Plane Analytic Geometry and Solid Analytic Geometry. After these Units, a construction of new geometrical structures, generally designated as Non-Euclidean geometry, will be introduced.

VII. Teaching And Learning Activities

Pre-assessment

UNIT 1: Revising Euclidean Geometry

Pre-assessment (60 minutes)

Before you come to the Learning Activities on Euclidean geometry you will need to refresh you knowledge on some topics of school geometry. The **Pre-assessment** consists of five questions in a multiple-choice format. Each question has four answers and you should choose one response only, the one you consider to be completely correct. The sign X should be placed in the box of your considered answer.

Question 1
The word "geometry" means:

A The same as "geography"

B "Earth measurement", where geo stays for "earth" and metry for "measurement"

C "Earth matrix", where geo stays for "earth" and metry stays for "matrix"

D Study of a landscape.

Question 2

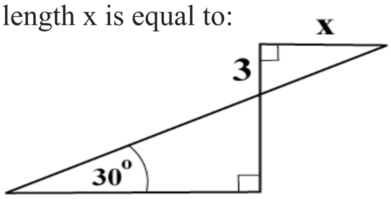
Visual reasoning is peculiar in working out questions on Euclidean geometry.

Which shape is the next in the following pattern : 

A  **B**  **C**  **D** 

Question 3

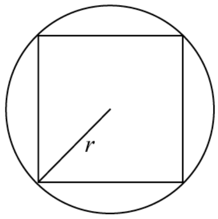
Consider the figure on left side. The length x is equal to:



A 3.5 **B** 4 **C** $3\sqrt{3}$ **D** 5

Question 4

Consider a square inscribed in a circle of radius *r*. The ratio of the area of circle to the area of square is equal to:



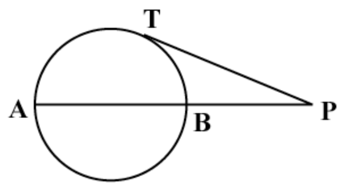
A	$2r$	**B**	$\frac{1}{2}\pi$	**C**	$\pi\sqrt{r}$	**D**	r^2

Question 5

At the figure on the right, the segment *PT* is tangent to the circle at *T*.

If $\overline{AB} = 4$ and $\overline{BP} = 3$,

them $\overline{PT}^2$ is equal to:



A	12	**B**	25	**C**	28	**D**	21

Answer key

Question	1	2	3	4	5
Answer	B	A	C	B	D

UNIT 2: Plane Analytic Geometry

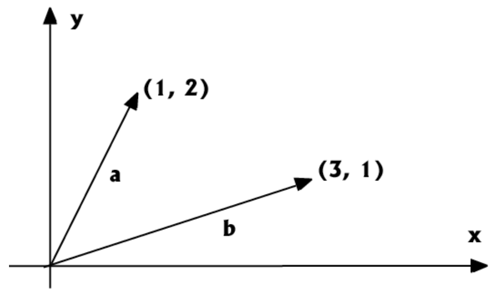
Pre-assessment (60 minutes)

Before you come to the Learning Activities on Plane Analytic Geometry you will need to refresh you knowledge on some topics of school geometry. The **Pre-assessment** consists of five questions in a multiple-choice format. Each question has four answers and you should choose one response only, the one you consider to be completely correct. The sign X should be placed in the box of your considered answer.

Question 1

Considering the vectors *a* and b at the figure on the left, the co-ordinates of the vector *a+b* are:

A (2, 3)
B (-2, -1)
C (4, -1)
D (4, 3)



Question 2

The distance d between two known points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ is given by

A. $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

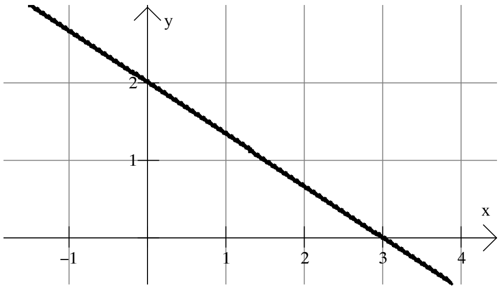
B. $d = (x_2 - y_2) - (x_1 - y_1)$

C. $d = (x_2 - x_1)^2 + (y_2 - y_1)$

D. $d = \sqrt{(x_2 - x_1) + (y_2 - y_1)}$

Question 3

Considering the graph below, the equation of the straight line is:



A $3y + 2x + 6 = 0$

B $x - \frac{2}{3}y - 3 = 0$

C $2x + 3y - 6 = 0$

D $y - \frac{2}{3}x + 2 = 0$

Question 4

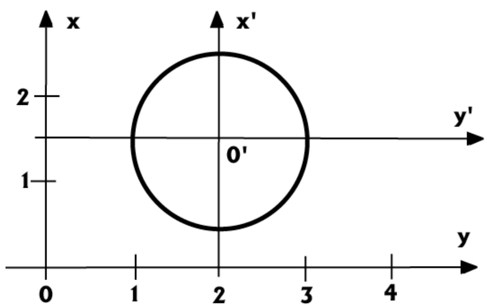
Consider the circle of radius ***r*** and centre ***O'***.
Its equation can be given as:

A $(x-3)^2 + (y-2)^2 = 1$

B $(x-2)^2 + (y-\frac{3}{2})^2 = 1$

C $(x-\frac{3}{2})^2 + (y-\frac{1}{2})^2 = 1$

D $(x-2)^2 + (y-3)^2 = 1$



Question 5

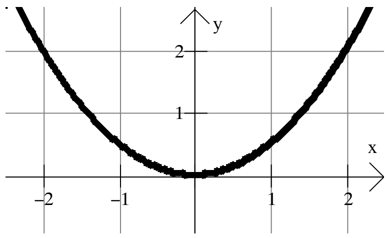
Considering the graph below, the equation of the parabola is:

A $y = 2x^2$

B $y = -x^2$

C $y = \frac{1}{2}x^2$

D $y = x^2$



Answer key

Question	1	2	3	4	5
Answer	D	A	C	B	C

UNIT 3: Solid Analytic Geometry

Pre-assessment (60 minutes)

Before you come to the Learning Activities on Solid Analytic Geometry you will need to refresh you knowledge on some topics of the previous Unit on Plane Analytic Geometry. The **Pre-assessment** consists of five questions in a multiple-choice format. Each question has four answers and you should choose one response only, the one you consider to be completely correct. The sign X should be placed in the box of your considered answer.

Question 1

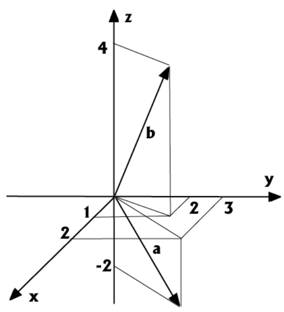
Considering the vectors a and b at the figure on the left, the sum *a+b* is equal to:

A (4, 5, -2)

B (3, -5, -2)

C (4, -1, 2)

D (3, 5, 2)



Question 2

Considering the figure of the previous question, the magnitude of the angle between the vectors *a* and *b* is equal to:

A 120^0

B 90^0

C 245^0

D 100^0

Question 3

The distance d between two known points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is given by

A $d = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (Z_2 - Z_1)^2$

B $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

C $d = (x_1 - y_2) - (y_1 - z_2) + (z_1 - x_2)$

D $d = \sqrt{(x_2 - y_1) + (y_2 - z_1) + (z_2 - x_1)}$

Question 4

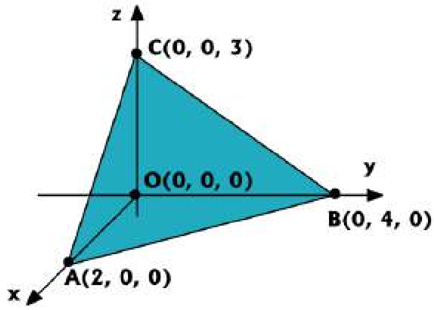
Considering the figure below, the volume of the tetrahedron [OABC] is equal to:

A 24 cubic units

B 18 cubic units

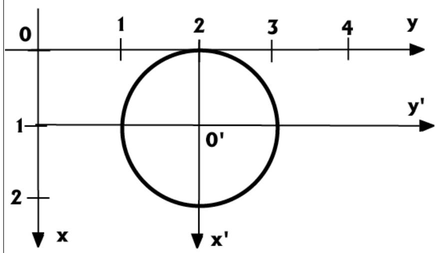
C 8 cubic units

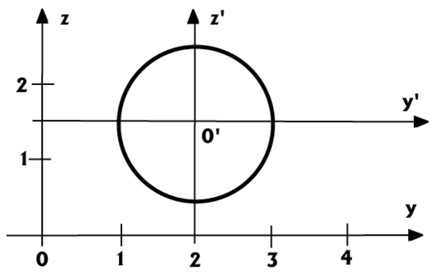
D 10 cubic units



Question 5

The graphs below show the projection of a sphere on plane xy and on plane yz.





The co-ordinates of the centre of the sphere is:

A (1, 2, 1.5)

B (2, 1, 1.5)

C (1, 1.5, 2)

D (1.5, 2, 1)

Answer key

Question	1	2	3	4	5
Answer	D	B	B	C	A

UNIT 4: Non-Euclidean Geometry

Pre-assessment (60 minutes)

Before you come to the Learning Activities on Non-Euclidean Geometry you will need to refresh you knowledge on some topics of the Unit 1 on Euclidean geometry. The **Pre-assessment** consists of five questions in a multiple-choice format. Each question has four answers and you should choose one response only, the one you consider to be completely correct. The sign X should be placed in the box of your considered answer.

Question 1

The following statement:

"Given any line *l* and a point *P* not on *l*, there is a unique line m in the same plane as *P* and *l* which passes through *P* and does not meet *l*."

A Is the definition of parallel lines

B Is an axiom on incident point to line

C Is the 5th postulate of Euclid

D Is the theorem of parallel lines

Question 2

Which of the following statements is true:

A Euclidean geometry was invented by Euclid

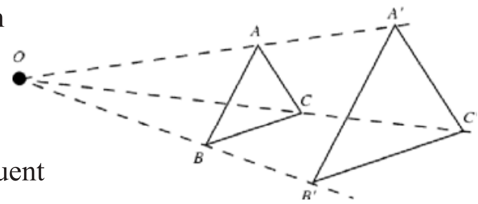
B The work Elements was written by Euclid's father in about 300 BC

C Euclid was born in Roma

D The work Elements is composed by thirteen books

Question 3

Consider a triangle ABC and its image A'B'C' by a central dilatation of centre O. If the centre O tends to infinity, then



A The two triangles tend to be congruent

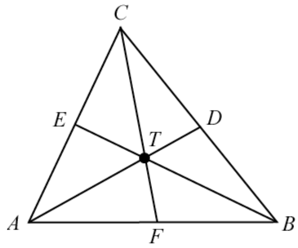
B The triangle ABC tends to be bigger than the triangle A'B'C'

C The triangle A'B'C' tends to be bigger than the triangle ABC

D The size relation of the two triangles still unchanged

Question 4

Consider any triangle ABC. If *T* is the centroid (the centroid of a triangle is the point of intersection of its <u>medians</u> - the lines joining each <u>vertex</u> with the mid-point of the opposite side), then always gilt:



A $\overline{AT} = \frac{2}{3}\overline{AD}$

B $\overline{ET} = \overline{FT}$

C $\overline{BT} = \overline{CT}$

D $\overline{TE} + \overline{TF} + \overline{TD} = \overline{EB}$

Question 5

The inverse matrix of $A = \begin{pmatrix} 3 & 2 \\ 7 & 5 \end{pmatrix}$

A Is $A^{-1} = \begin{pmatrix} 2 & 3 \\ 5 & 7 \end{pmatrix}$

B Is $A^{-1} = \begin{pmatrix} 5 & -2 \\ -7 & 3 \end{pmatrix}$

C Is $A^{-1} = \begin{pmatrix} 5 & -7 \\ -2 & 3 \end{pmatrix}$

D Does not exist!

Answer key

Question	1	2	3	4	5
Answer	C	D	A	A	B

IX. Learning Activities

Unit 1: Revising Euclidean Geometry

This Unit has 6 **Learning Activities (LA)**:

LA1.1: On history of Euclidean geometry
LA1.2: Axiomatic development of Euclidean geometry
LA1.3: Euclidean plane isometry
LA1.4: Triangles
LA1.5: Circles
LA1.6: Geometric similarity

Now you are going to work on

Activity 1.1 On history of Euclidean geometry

Learning Objectives

At the end of this Learning Activity the learner should be able to:

- Explain about the development of *geometric knowledge* along the history of different civilizations.
- Indicate locations, periods, outstanding geometers and their contributions for the development Euclidean geometry

Summary

In this Learning Activity you will learn how geometrical thinking was developed along centuries as a product of interaction of humankind with the Nature. You will know about locations, periods of outstanding geometers and their contributions for the knowledge that became a mathematical subject called *Euclidean* geometry. You will find that apart from the Greeks, Babylonians, Romans, Indians and Chinese, the ancient Egyptians - an African ancient civilization – in about 4000 years ago possessed already interesting mathematical knowledge that embodies the famous work of Euclid - *Elements*.

Rationale

Frequently pupils ask where do theorems come from, or how do the mathematicians discover them? It is import for you as future educator to explain your students about this kind of questions. Knowing how the humankind developed geometry they will become stimulated to investigate geometry from their own. If they can feel themselves as researchers they will become more confident in doing mathematics and you will be very proud of your profession as a teacher.

Key concepts

Euclid's Elements**:** The classic treatise in geometry written by Euclid (about 300 BC). It consists of 465 propositions, divided into 13 «books». (See LA1.1 RR4)

Rind Mathematical Papyrus**:** The Rhind Mathematical Papyrus (RMP) is an ancient mathematical document written on *papyrus* (paper). It is named after Alexander Henry Rhind, a Scottish antiquarian, who purchased the papyrus in 1858 in Luxor, Egypt. (See LA1.1 RR5)

List of relevant readings (RR)

LA1.1 RR1: Euclid's Elements
    http://aleph0.clarku.edu/~djoyce/java/elements/toc.html

LA1.1 RR2: Euclidean Geometry
    http://en.wikipedia.org/wiki/Euclidean_geometry

LA1.1 RR3: The Euclidean geometry of the plane
    http://mathworld.wolfram.com/Euclidean.html

LA1.1 RR4: *Elements* - The classic treatise in geometry written by Eucli
    http://mathworld.wolfram.com/Elements.html

LA1.1 RR5: Ancient Egyptian mathematics
    http://en.wikipedia.org/wiki/Ancient_Egyptian_mathematics

List of relevant resources

Particularly for this starting Learning Activity of Unit 1, apart from the *CD-ROM-Resources* that you will get in your study material package, you will need to visit sites in the Internet, at least once, during the study of the Learning Activity.

Development of the Learning Activity

On history of Euclidean geometry

You may have asked yourself the following questions:

- Who created (or invented) Euclidean geometry?
- What was the merit of Euclid as geometer?
- Did ancient civilizations in Africa developed geometric knowledge?

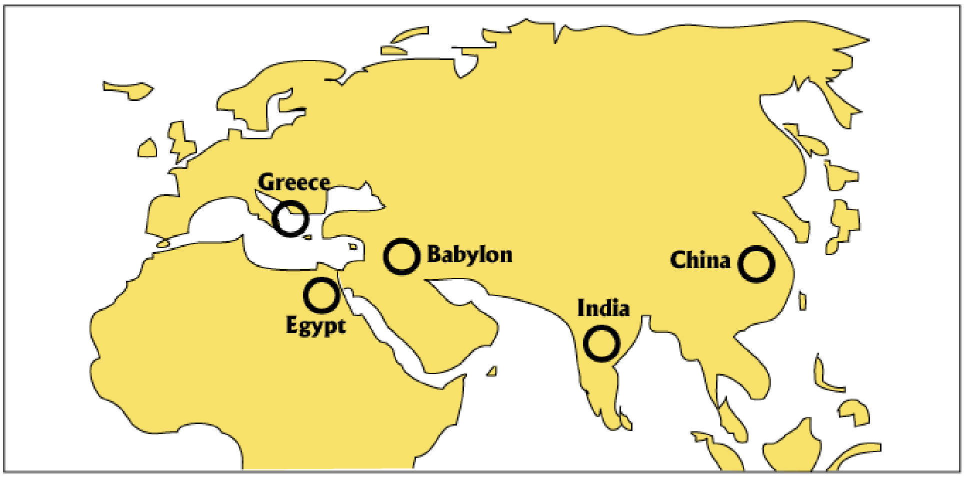


Figure 1.1.1: Ancient centers of development of geometric knowledge

In this activity you are going to learn about development geometric knowledge and be able to respond to questions like those presented above. In fact the Euclidean geometry – that you already know about from secondary school – is a mathematical subject that has been developed by different civilizations all around the world since ancient times.

You will learn about Euclidean geometry and know why it has been fascinating people and influencing the way of thinking in mathematics and in other scientific subjects since about 300 years BC when Euclid wrote his famous work entitled *Elements*.

Task 1

Read about the development of *Euclidean geometry* (LA1.1 RR2 – RR5) along the history of different civilizations. Find out how different civilization along centuries contributed to the construction of knowledge known today as Euclidean geometry and **write** an essay (500-700 words) on this issue.

Figure 1.1.2: Giza pyramids – a magnificent sign of geometric knowledge of the Ancient Egypt

Task 2

Read about Ancient Egyptian mathematics (LA1.1 RR5) and **write** about some of the Egyptian contribution to Euclidean geometry.

Task 3

Read about Ancient Egyptian mathematics (LA1.1 RR5) and **write** about some of the Egyptian contribution to Euclidean geometry.

Figure 1.1.3: A portion of the Rind papyrus

Task 4

When did Euclid write is work "Elements"? **Describe** shortly the content of the 13 books of the Elements.

Unit 1: Revising Euclidean Geometry

This Unit has 6 **Learning Activities (LA)**:

LA1.1: On history of Euclidean geometry
LA1.2: Axiomatic development of Euclidean geometry
LA1.3: Euclidean plane isometry
LA1.4: Triangles
LA1.5: Circles
LA1.6: Geometric similarity

Now you are going to work on

Activity 1.2 Axiomatic development of Euclidean geometry

Learning objectives

At the end of this Learning Activity the learner should be able to:

- Explain the development of Euclidean geometry in a concrete example of applying axioms, theorems and other geometric concepts
- State own conjectures from exploration of solid manipulative aids or computer software.
- Draw geometric figures using basic instruments and/or a computer.
- Proof classic and simple Euclidean theorems.

Summary

In this activity you will learn how geometrical thinking was developed along centuries as product of interaction of humankind with the nature. The essential goal of this activity is to get you developing the *inductive-deductive reasoning* and to stimulate you for further investigation in geometry and in mathematics in general. Therefore, you will have to state *conjectures* by exploring the questions that will be posed to you throughout the activity. Based on your own conjectures you will develop formal *mathematical proofs*. You will also learn some classic and common *theorems* of Euclidean geometry and asked to apply them in concrete problems. You are highly recommended to use geometric tools (at lest a straightedge and compass) wherever is necessary.

Rationale

It is important to reflect on how Euclidean geometry – like any other mathematical structure – was constructed. You will find what characterizes the foundation of geometry and what the innovations of David Hilbert (1862-1943) to the axiomatic of Euclidean geometry are. On the basis of such a reflection, you can better understand what distinguish this geometry from other types of geometry that you will learn later in Unit 4 (the non-Euclidean geometry).

Key concepts

Axiom: A sentence or proposition that is taken for granted as true.

Theorem: A statement that has been proven to be true or proposed as a demonstrable truth.

Conjectures: A conjecture is the same as a hypothesis, a statement drown from some evidence but not proved at all.

Inductive-deductive reasoning: A way of thinking that starts from particular cases to generalizations (*induction*) – and once having general (proved) statements then we got implications for particular case (*deduction*).

Mathematical proof: A thorough demonstration of a proposition using the logic of the *axiomatic* structure of that proposition.

List of relevant readings (RR)

LA1.2 RR1: Euclid's Elements, by *D. E. Joyce* (1997)
http://aleph0.clarku.edu/~djoyce/java/elements/toc.html

LA1.2 RR2: Euclidean Geometry
http://en.wikipedia.org/wiki/Euclidean_geometry

LA1.2 RR3: The Euclidean geometry of the plane
http://mathworld.wolfram.com/Euclidean.html

LA1.2 RR4: *Elements* - The classic treatise in geometry written by Euclid
http://mathworld.wolfram.com/Elements.html

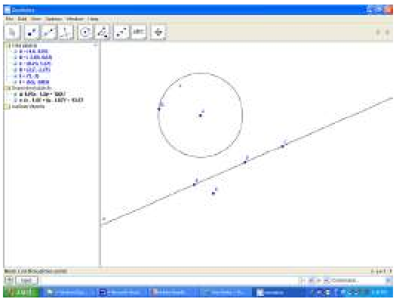
LA1.2 RR5: The Foundation of Geometry, by *D. Hilbert* (ed. 2007)
http://www.gutenberg.org/etext/17384

LA1.2 RR6: Axiom system
http://en.wikipedia.org/wiki/Axiomatic_system

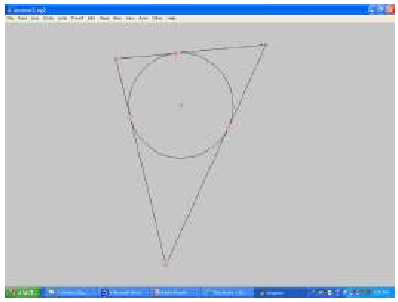
List of relevant resources

WinGeom or ***GeoGebra*** (Geometry software in *CD-ROM-Resources*)

Both *WinGeom* and *GeoGebra* are free open source dynamic software similar to Cabri geometry and are of very high quality. They also draw function graphs and create a dynamic relationship between the graph and the function. You may download *WinGeom* and *GeoGebra* from the site http://freestatistics.altervista.org/math.php



GeoGebra template



WinGeom template

Figure 1.2.2: Screen captures of GeoGebra and of WinGeom templates

Development of the Learning Activity

Axiomatic development of Euclidean geometry

Figure 1.2.1: A building being constructed

You may have seen people building a house. Tray to reflect on the following questions:

- What are the basic materials they use to raise up a wall?
- Do you know some rules – precautions – that the constructors apply in order to be sure that the walls will not fall down?

Even if you do not know the precautions the constructors use to build a house, you are sure that it is necessary to respect certain physical rules.

The construction of our subject – Euclidean geometry – can be compared to the construction of a house. The "stones" or "bricks" of the Euclidean geometry are sentences or propositions that are taken for granted as true and serve as a starting point for deducing other truths. These kid of statements are called *axioms*. There are further basic material which are called *theorems*. A theorem is a statement that has been proven to be true or proposed as a demonstrable truth.

So, with this "material" together with definitions of geometrical concepts you already know from secondary school, let us start building our house (the Euclidean geometry) using stones and bricks (axioms, theorems, definitions) and respecting rules of house construction (logic, deductive-inductive reasoning) putting all these material together, in such a way that walls can raise up firmly.

Keeping constructing attending the rules we will finally get the whole house having bedrooms, kitchen, bathroom, doors, windows, etc. Like stones or bricks of a house, the whole set of Euclidean axioms respect following rules or principles:

- ***Entirety***: *An axiom system is entire if from it all propositions of the subject (in this case the Euclidean Geometry) can be deduced.*
- ***Independence***: *It is not allowed to consider any statement as axiom if it is deduced for the other axiom.*
- ***Non-contradiction***: *It is not allowed that through deductive conclusion from axioms the negative of an axiom or proposition can be stated.*

These are the *principles of axiom system* of Euclidean geometry. Thus, since we are involved in a construction of a geometric theory on the basis of such a set of axioms, our activity will be on the axiomatic development of Euclidean geometry.

Now you have an idea of what your activity is going be about. In fact the Euclidean geometry has been fascinating people and influencing the way of thinking in other scientific subjects since about 300 years BC when Euclid wrote his famous *Elements*.

Task 1

Read about the Foundation of Geometry (LA1.2 RR5, pgs 1-16). From the five groups of axiom, choose two groups and **present** their axioms.

Task 2

Read about the *Euclidean Geometry* (LA1.2 RR2).

Explain the difference of an axiom and a theorem. **Give** three **examples** of both.

Task 3

Read about the *Euclid's Elements* (LA1.2 RR1), particularly the following items:

- Introduction (on *Euclid's Elements*)
- Book I (The fundamentals of geometry)
- Book IV (Constructions for inscribed and circumscribed figures.)

Choose one of the software (*GeoGebra* or *WinGeom*) tray to **illustrate** three propositions of Book I, and five propositions of Book IV.

Task 4

Read about the axiom system of Euclidean geometry (LA1.2 RR5 and LA1.2 RR6) and **explain** its properties.

Unit 1: Revising Euclidean Geometry

This Unit has 6 **Learning Activities (LA)**:

LA1.1: On history of Euclidean geometry
LA1.2: Axiomatic development of Euclidean geometry
LA1.3: Euclidean plane isometry
LA1.4: Triangles
LA1.5: Circles
LA1.6: Geometric similarity

Now you are going to work on

Activity 1.3 Euclidean plane isometry

Learning objectives

At the end of this Learning Activity you should be able to:

- Define all four types of Euclidean plane isometry
- Draw examples of Euclidean plane isometry by using paper and pencil and by using software *GeoGebra* or *WinGeom.*

Summary

In this activity you will revise geometric transformations you already know from the secondary school geometry. The four types of Euclidean plane isometry are: translations, rotations, reflections, and glide reflections. Basically you will be stimulated to revise these transformations by exploring the computer software in your CD-ROM-*Resources*.

Rationale

This Learning Activity helps you to upgrade your knowledge in isometric transformations you learned at secondary school. It is important to recognize and to classify the different types of plane isometry in your geographical and cultural environment.

Key concepts

Euclidean plane isometry: *Euclidean plane isometry* is an isometry of the Euclidean plane, or more informally, a way of transforming the plane that preserves geometrical properties such as length. (LA1.3 RR1)

Translation: In geometry *translation* is moving every point a constant distance in a specified direction. Formally, a translation is denoted by T_v, where v is a vector in R^2. This has the effect of shifting the plane in the direction of v. (LA1.3 RR1)

Rotation: A *rotation* is the movement of an object in a circular motion. In the plane a object rotates around a *center* (or *point*) *of rotation*. A rotation is denoted by $R_{c,\theta}$, where *c* is a point in the plane (the centre of rotation), and θ is the angle of rotation. (LA1.3 RR1)

Reflection: In mathematics, a *reflection* (also spelt *reflexion*) is a map that transforms an object into its mirror image. A reflection or *mirror isometry* is denoted by $F_{c,v}$, where *c* is a point in the plane and *v* is a unit vector in R^2. (*F* is for «flip».) This has the effect of reflecting the point *p* in the line *L* that is perpendicular to *v* and that passes through *c*. The line *L* is called the *reflection axis* or the associated *mirror*. (LA1.3 RR1)

Glide reflection: A *glide reflection* is the combination of a reflection in a line and a translation along that line. A glide reflection is denoted by $G_{c,v,w}$, where *c* is a point in the plane, *v* is a unit vector in R^2, and *w* is a vector perpendicular to *v*. (LA1.3 RR1)

Geometric congruence: Two geometric figures are said to exhibit *geometric congruence* (or «be geometrically *congruent*») iff one can be transformed into the other by an *isometry* (LA1.3 RR3). This relationship is written $A \cong B$.

List of relevant readings (RR)

LA1.3 RR1: Euclidean plane isometry
http://en.wikipedia.org/wiki/Euclidean_plane_isometry

LA1.3 RR2: Theory of symmetry and ornament, by *Slavik V. Jablan* (1995)
http://www.emis.de/monographs/jablan/index.html

LA1.3 RR3
http://mathworld.wolfram.com/GeometricCongruence.html

List of relevant resources

The relevant resources for this learning activity are:

- Geometric basic drawing instruments (straightedge and compass). Further hand drawing instruments may be useful.
- *GeoGebra* or *WinGeom*

Development of the Learning Activities

Euclidean plane isometry

Anywhere in our environment we have the opportunity to appreciate the beauty of natural and artificial objects. There are plants, animals and human made objects, whose beauty is characterized by displaying repeated patterns, keeping forms and size unchanged. It is that characteristic the matter of our learning activity now.

Figure 1.3.1: Examples of isometric transformations in the Nature and in art

Task 1

Read about *Euclidean plane isometry* (LA1.3 RR1). Find in your environment two examples of each type of Euclidean plane isometry and **sketch** them by hand.

Task 2

Investigate the following question: In a glide reflection *G*, if starting by the translation followed by the reflection, do it results the same as starting by the reflection followed by the translation?

Task 3

Draw three examples of each type of Euclidean plane isometry using GeoGebra.

Unit 1: Revising Euclidean Geometry

This Unit has 6 **Learning Activities (LA)**:

LA1.1: On history of Euclidean geometry
LA1.2: Axiomatic development of Euclidean geometry
LA1.3: Euclidean plane isometry
LA1.4: Triangles
LA1.5: Circles
LA1.6: Geometric similarity

Now you are going to work onLearning

Activity 1.4 Triangles

Learning objectives

At the end of this Learning Activity you should be able to:

- Demonstrate some common theorems on triangles
- Construct a triangle when given three of its elements by using straightedge and compass only or by using *GeoGebra* or *WinGeom*.

Summary

In this activity you will demonstrate synthetically some of the most popular theorems on triangles. These are propositions about points, lines and circles associated with a triangle. For triangle constructions, apart from using the classic instruments (compass and straightedge), you will be stimulated to solve construction problems by exploring *GeoGebra* or *WinGeom* - the computer software in your CD-ROM-*Resources*.

Rationale

A triangle is the polygon with the minor number of sides. Every polygon can be split into triangles. Therefore, knowing properties of triangles you will be able to solve problems related to different polygonal shapes. Triangular shapes are very used by engineers because they are the simplest form that one can construct strong physical structures for several technical applications.

Key concepts

Triangle: A *triangle* is one of the basic shapes of geometry: a polygon with three vertices and three sides which are straight line segments. (LA1.4 RR1)

Circumcircle**:** The *circumcircle* (or *circumscribed circle*) of a triangle is the circle which contains all the vertices of the triangle. The centre of this circle is called the *circumcenter.* (LA1.4 RR1)

Incircle**:** The *incircle* or *inscribed circle* of a triangle is the largest circle contained in the triangle; it touches (is tangent to) the three sides. The center of the incircle is called the triangle's *incenter*. (LA1.4 RR1)

Centroid: The centroid of a triangle is the point of intersection of its medians (the lines joining each vertex with the midpoint of the opposite side). (LA1.4 RR1)

Orthocenter: The three altitudes of a triangle intersect in a single point, called the *orthocenter* of the triangle. (LA1.4 RR1)

Nine-point circle**:** The *nine-point circle* (also known as *Feuerbach's circle*, *Euler's circle*, *Terquem's circle*) is a circle that can be constructed for any given triangle. It is so named because it passes through nine significant points, six lying on the triangle itself (unless the triangle is obtuse). They include: the midpoint of each side of the triangle, the foot of each altitude, and the midpoint of the segment of each altitude from its vertex to the orthocenter.

List of relevant readings (RR)

LA1.4 RR1: Triangle
http://en.wikipedia.org/wiki/Triangle

LA1.4 RR2: Triangle
http://mathworld.wolfram.com/Triangle.html

Introduction to the Learning Activities

Triangle

Figure 1.4.1: An example of application of triangular shapes in construction technology

You will start the activities by reminding the special points, lines and circumferences related with a triangle:

Altitude to a side of a triangle is a segment from a vertex perpendicular to the opposite side. The three altitudes of a triangle intersect in a single point, called the *orthocenter* of the triangle.

Median of a triangle is a segment from a vertex to the midpoint of the opposite side. The *centroid* or *baricenter* of a triangle is the point of intersection of its medians.

Angle bisector of a triangle is a segment of ray that bisects an angle und extends to the opposite side. The intersection of the angle bisectors finds the center of the *incircle*.

Perpendicular bisector of a side of a triangle is a line that bisects and is perpendicular to a side. The three perpendicular bisectors of a triangle meet in a single point, the triangle's *circumcenter*; this point is the center of the *circumcircle*, the *circle* passing through all three vertices.

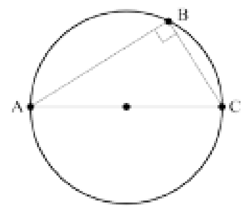
After this review you are going workout those elements related to a triangle by stating conjectures of their properties. This will lead you to learn theorems which will enable you to solve problems on triangles constructions, using compass and straightedge. Using *GeoGebra* or *WinGeom* you also explore interesting propositions about triangles.

Task 1

Read about *Triangles* (LA1.4 RR1 and LA1.4 RR2).

In general, a triangle is defined by its three elements. SAS (Side, Angle, Side), ASA (Angle, Side, Angle), SSS (Side, Side, Side) provide three well known examples. But there is much more. Using a compass and straightedge, **construct** a triangle for each of the cases SAS, ASA, and SSS.

Task 2



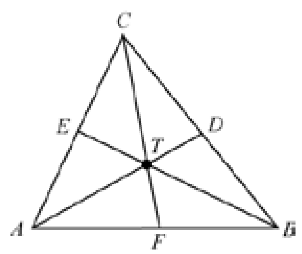
Thales' theorem (named after Thales of Miletus) states that if A, B and C are points on a circle where the line AC is a diameter of the circle, then the angle ABC is a right angle.

Proof the converse of this Thales' theorem.

Task 3

Euler line, named after Leonhard Euler, is a line determined from any triangle that is not equilateral; it passes through the *orthocenter*, the *circumcenter*, the *centroid*, and the center of the *nine-point circle* of the triangle. **Draw** the Euler line of a triangle ABC.

Task 4



Investigate the **Ceva's theorem.** This is a well-known theorem in elementary geometry. Given a triangle *ABC*, and points *D*, *E*, and *F* that lie on lines *BC*, *CA*, and *AB* respectively, the theorem states that lines *AD*, *BE* and *CF* are concurrent if and only if

$$\frac{\overline{AF}}{\overline{FB}} \cdot \frac{\overline{BD}}{\overline{DC}} \cdot \frac{\overline{CE}}{\overline{EA}} = 1$$

Proof this Ceva's theorem.

Task 5

Given a triangle ABC, **draw** its *Feuerbach's circle* using compass and straigh-tedge.

Unit 1: Revising Euclidean Geometry

This Unit has 6 **Learning Activities (LA)**:

LA1.1: On history of Euclidean geometry
LA1.2: Axiomatic development of Euclidean geometry
LA1.3: Euclidean plane isometry
LA1.4: Triangles
LA1.5: Circles
LA1.6: Geometric similarity

Now you are going to work on

Activity 1.5 Circles

Learning objectives

At the end of this Learning Activity you should be able to:

- Demonstrate some common theorems on circles
- Explore properties of circles using *GeoGebra* or *WinGeom*.

Summary

In this activity you will demonstrate synthetically some of the most popular theorems on triangles. Some of those theorems are related to the previous LA (on Triangles).

Constructions problems of circles will be solved by using compass and straightedge as well as using *GeoGebra* or *WinGeom*.

Rationale

In this LA you will learn properties of circle related with segments and angles. Therefore it is important to review some theorems on circumscribed and inscribed triangles.

Key concepts

Circle**:** A *circle* is the set of all points in a plane that are at a fixed distance from a fixed point called the *centre.*

Circumference**:** A *circumference* of a circle is the distance around the circle.

Radius**:** A *radius* of a circle is the line segment joining the centre to a point on the circle.

Note: Since all radii of a given circle have the same length, we may at times use the word *radius* to mean the number that is "the length of the radius".

Chord**:** A *chord* of a circle is a line segment joining to points on a circle.

Diameter**:** A *diameter* is a chord through the centre.

Arc**:** An *arc* is a continuous part of a circle.

List of relevant readings (RR)

LA1.5 RR1: Circle
http://en.wikipedia.org/wiki/Circle

LA1.5 RR2: Circles
http://mathworld.wolfram.com/topics/Circles.html

LA1.5 RR3: Circle theorems
http://www.mathsrevision.net/gcse/pages.php?page=13

List of relevant resources

The relevant resources for this learning activity are:

- Geometric classic drawing instruments: compass and straightedge.
- *GeoGebra* or *WinGeom*.

Introduction to the Learning Activities

Circles

What shape is the set of all points staying at the same distance of a fixed point?

Circle is a very common shape we find in Nature and in artificial environment.

Figure 1.5: Example of application of circular shapes

Task 1

Read about *Circles* (LA1.5 RR1, LA1.5 RR2 and LA1.5 RR3) and **prove** three of the following theorems:

(1) Angles subtended by an arc at the circle and lying in the same segment are equal.

(2) The angle that an arc subtends at the centre of a circle is twice the angle that the arc subtends at points on the remainder of the circumference.

(3) An angle subtended at the circumference by a semicircle is a right angle.

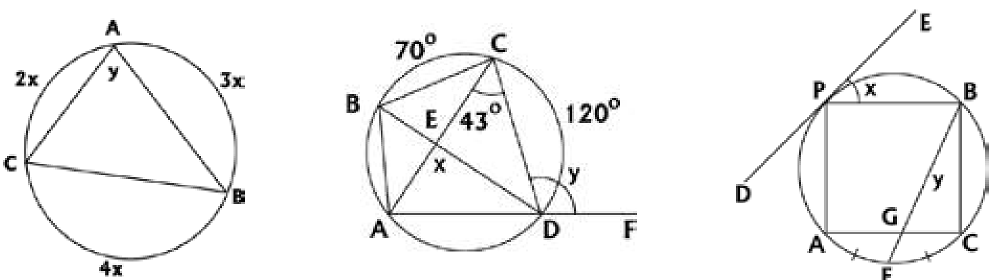
(4) If two tangents are drawn from an external point P to a circle then: (a) the tangents have equal length; (b) the tangents subtend equal angles at the centre of the circle; (c) the line from the point to the centre bisects the angle between the tangents.

(5) If a tangent PA and a secant PBC are drawn from an external point P, then $PA^2=PB.PC$.

(6) If two chords AB and CD intersect at a point Y, then AY.BY=CY.DY.

Task 2

Considering the figures below, find *x* and y.



A . B. C .

Task 3

Solve one of the following construction problem:

(1) Construct a circle with a given radius and tangent to two given intersecting lines.

(2) Construct a circle with a given radius tangent to a given line and tangent to a given circle.

Unit 1: Revising Euclidean Geometry

This Unit has 6 **Learning Activities (LA)**:

LA1.1: On history of Euclidean geometry
LA1.2: Axiomatic development of Euclidean geometry
LA1.3: Euclidean plane isometry
LA1.4: Triangles
LA1.5: Circles
LA1.6: Geometric similarity

Now you are going to work on

Activity 1.6 Geometric similarity

Learning objectives

At the end of this Learning Activity you should be able to:

- Define and construct examples geometric similarities
- Explore geometric similarity by using *GeoGebra* or *WinGeom*.

Summary

In this activity you will learn geometric similarity and relate this transformation with the isometric transformation you learned in LA1.3. Particularly you will learn about similar triangles. Apart from using the classic drawing instruments for geometrical construction you will use also *GeoGebra* or *WinGeom.*

Rationale

Geometric similarity includes isometric transformations as particular cases. Therefore, it is important that you recall the properties of transformations learned in LA1.3. It is also important to review basic proportion principles, particularly those related to similar triangles.

Key concepts

Geometric similarity: A transformation that preserves angles and changes all distances in the same ratio, called the *ratio of magnification*. A similarity can also be defined as a transformation that preserves ratios of distances. (LA1.6 RR2)

Ratio of magnification: A *ratio of magnification* is a constant ratio by which all distances are increased (or decreased) in a *similarity*. A similarity with ratio of magnification equal to 1 is called an *isometry*. (LA1.6 RR2)

Central dilation: A similarity transformation which transforms each line to a parallel line whose length is a fixed multiple of the length of the original line. The simplest dilation is therefore a *translation*, and any dilation that is not merely a translation is called a *central dilation*.

***Directly similar*:** Two figures are *directly similar* when all corresponding angles are equal and described in the same rotational sense. (LA1.6 RR2)

***Inversely similar*:** Two figures are *inversely similar* when all corresponding angles are equal and described in the opposite rotational sense. (LA1.6 RR2)

List of relevant readings (RR)

LA1.6 RR1: Similarity (geometric)
http://en.wikipedia.org/wiki/Similarity_(geometry)

LA1.6 RR2: Geometric similarity
http://mathworld.wolfram.com/SimilarityTransformation.html

List of relevant resources

The relevant resources for this learning activity are:

- Geometric classic drawing instruments: compass and straightedge.
- *GeoGebra* or *WinGeom*.

Introduction to the Learning Activities

Geometric similarity

Figure 1.5: Examples similar geometric shapes

Task 1

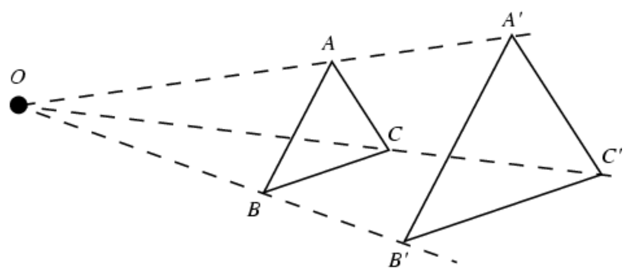
Read about *geometric similarity* (LA1.6 RR1, LA1.6 RR2 and LA1.6 RR3).

Suppose that triangle *ABC* is similar to triangle *DEF* in such a way that the angle at vertex *A* is congruent with the angle at vertex *D*, the angle at *B* is congruent with the angle at *E*, and the angle at *C* is congruent with the angle at *F*. Then, once this is known, **prove** one of the following proportionalities between corresponding sides of the two triangles:

$$\frac{AB}{BC}=\frac{DE}{EF}, \quad \frac{AB}{AC}=\frac{DE}{DF}, \quad \frac{AC}{BC}=\frac{DF}{EF}, \quad \frac{AB}{DE}=\frac{BC}{EF}=\frac{AC}{DF}$$

Task 2

Consider a triangle ABC whose area is equal to 3 square units. If ABC is transformed into A'B'C' by a central dilatation of with ratio of magnification equal to $\frac{3}{2}$ what will the area of A'B'C' be?



Task 3

Find the lengths of the arms of a right triangle whose hypotenuse has length *c* if these arms have a ratio of

(a) 3:4 and *c*=15; (b) 5:12 and *c*=26 (c) 1:2 and *c*=10.

Unit 2: Plane Analytic Geometry

This Unit has 4 **Learning Activities (LA)**:

LA2.1: Vector space (in two dimensions)
LA2.2: The straight line
LA2.3: Transformation of coordinates on plane
LA2.4: Conic sections

Now you are going to work on

Activity 2.1 Vector space (in two dimensions)

Learning objectives

At the end of this Learning Activity you should be able to:

- Define the concept of vector
- Operate with vectors (addition, scalar multiplication, and dot product).

Summary

In this Learning Activity you will learn the concept of vector. It should be observed that a vector is defined by the following aspects: magnitude and direction. The fundamental operations of vectors you will start with in this Unit 2 are addition, multiplication by a scalar, and the dot product. Later on Unit 3, you will learn further different types of products of two vectors.

Rationale

The laws of operation in vector space are similar to the operation laws in scalar algebra. So, it is important to review basic operation principles learned in the module on linear algebra.

Key concepts

Vector space: A *vector space* is a mathematical domain of quantities possessing a *magnitude* and *direction* – ***vector***s – together with a defined set of operation laws. (See formal definition in LA2.1 RR3.)

Vector: A *vector* is a quantity characterized by a *magnitude* (in mathematics a number, in physics a number times a unit) and a *direction*, often represented graphically by an arrow. (LA2.1 RR2)

Scalar multiplication: A *scalar multiplication* is the multiplication of a vector by a scalar.

Norm of a vector: The *length* or *magnitude* of a vector **v** is often called ***norm*** of **v** and it is donated by $\|\mathbf{v}\|$. It follows from the theorem of Pythagoras that the norm of a vector $\mathbf{v} = (v_1,\ v_2)$ in two dimensional space is $\|\mathbf{v}\| = \sqrt{v_1^{\ 2} + v_2^{\ 2}}$.

Dot product: A *dot product* (or *scalar product*, owing to the fact that its value is scalar) of two vectors *A* and *B* is the scalar quantity (donated $\mathbf{a} \times \mathbf{b}$) obtained by multiplying the product of the *norms* of the vectors by the *cosine* of the angle between them. That is $\mathbf{a} \times \mathbf{b} = \|\mathbf{a}\| \times \|\mathbf{b}\| cos(\mathbf{a},\ \mathbf{b})$. The dot product is also called *direct product.*

List of relevant readings (RR)

LA2.1 RR1: **Vector analysis**, a text-book for the use of students of mathematics and Physics, founded upon the lectures of J. Willard Gibbs, c1929 (a PDF file in
http://www.archive.org/details/117714283)

LA2.1 RR2: **Vector (spatial)**
http://en.wikipedia.org/wiki/Vector_(spatial)

LA2.1 RR3: **Vector space**
http://en.wikipedia.org/wiki/Vector_space

LA2.1 RR4: **Vector**
http://mathworld.wolfram.com/Vector.html

List of relevant resources

The relevant resources for this learning activity are:

- Straightedge and angle-measurer
- *GeoGebra* or *WinGeom*.

Introduction to the Learning Activities

Vectors space (in two dimensions)

In mathematics and in physical world we may consider two different kinds of quantities. Some quantities are simply described by its magnitude. On the other hand, there are quantities that in addition to a magnitude a direction should be given. For example, force is given by a magnitude together with a direction to which it is pointing. The wind is also presented by a magnitude together with direction north, south, east, west or an intermediate direction. Such quantities – possessing a ***magnitude*** and a ***direction*** – are called ***vectors***.

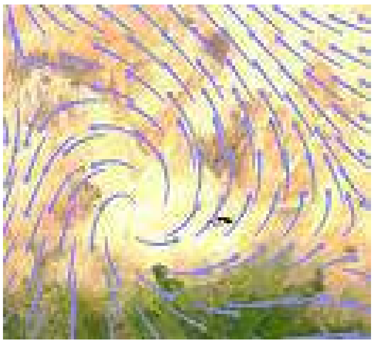


Figure 2.1.7: A vector grid layer

Mathematically, a space where quantities of that kind are studied is called ***vector space*** and it is formally defined as follows:

Let F be a field (such as the real numbers or complex numbers), whose elements will be called *scalars*. A ***vector space over the field*** F is a set V together with two binary operations,

- *Vector addition*: $V \times V \to V$ denoted $\mathbf{v} + \mathbf{w}$, where $\mathbf{v}, \mathbf{w} \in V$, and
- *Scalar multiplication*: $F \times V \to V$ denoted $a\,\mathbf{v}$, where $a \in F$ and $\mathbf{v} \in V$,

satisfying the axioms below. Four require vector addition to be an Abelian group, and two are distributive laws

1. Vector addition is ***associative***:
   For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, we have $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.

2. Vector addition is ***commutative:***
   For all $\mathbf{v}, \mathbf{w} \in V$, we have $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$.

3. Vector addition has an ***identity element:***
   There exists an element $\mathbf{0} \in V$, called the ***zero vector,*** such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in V$.

4. Vector addition has an ***inverse element:***
   For all $\mathbf{v} \in$ V, there exists an element $\mathbf{w} \in V$, called the ***additive inverse*** of $\mathbf{v}$, such that $\mathbf{v} + \mathbf{w} = \mathbf{0}$.

5. Distributivity holds for scalar multiplication over vector addition:
   For all $a \in F$ and $\mathbf{v}, \mathbf{w} \in V$, we have $a\,(\mathbf{v} + \mathbf{w}) = a\,\mathbf{v} + a\,\mathbf{w}$.

6. Distributivity holds for scalar multiplication over field addition:
   For all $a, b \in F$ and $\mathbf{v} \in V$, we have $(a + b)\ \mathbf{v} = a\ \mathbf{v} + b\ \mathbf{v}$.
7. Scalar multiplication is compatible with multiplication in the field of scalars:
   For all $a, b \in F$ and $\mathbf{v} \in V$, we have $a\ (b\ \mathbf{v}) = (ab)\ \mathbf{v}$.
8. Scalar multiplication has an ***identity element:***
   For all $\mathbf{v} \in V$, we have $1\ \mathbf{v} = \mathbf{v}$, where 1 denotes the ***multiplicative identity*** in F.

Formally, these are the axioms for a *module*, so a vector space may be concisely described as *a module over a field.*

Note that the seventh axiom above, stating $a\ (b\ \mathbf{v}) = (ab)\ \mathbf{v}$, is not asserting the *associativity* of an operation, since there are two operations in question, scalar multiplication: $b\ \mathbf{v}$; and field multiplication: ab

Some sources choose to also include two axioms of *closure:*

1. V is *closed* under vector addition:
   If $\mathbf{u}, \mathbf{v} \in V$, then $\mathbf{u} + \mathbf{v} \in V$.
2. V is *closed* under scalar multiplication:
   If $a \in F$, $\mathbf{v} \in V$, then $a\ \mathbf{v} \in V$.

(Source: LA2.1 RR3)

Task 1

Read about the concept of *vector*, its properties and operations in the readings LA2.1 RR1, Chapter I, pages 1-17, 51-52; Chapter II, pages 55-60.

Task 2

Sketch the following vectors **v** with the initial points located at the origin:

(a) $\mathbf{v}_1 = (3, 5)$ (b) $\mathbf{v}_2 = (-2, 7)$ (c) $\mathbf{v}_3 = (-5, -4)$

Task 3

Find the components of the vector having initial point P_1 and terminal point P_2.

(a) $P_1(2,\ 3)$, $P_2(3, -5)$ (b) $P_1(7, -2)$ $P_2(0, 0)$

Task 4

Solve the exercises 1, 3, 5, 11-16, on LA2.1 RR1, Chapter II, pages 113-114.

Unit 2: Plane Analytic Geometry

This Unit has 4 **Learning Activities (LA)**:

LA2.1: Vector space (in two dimensions)
LA2.2: The straight line
LA2.3: Transformation of coordinates on plane
LA2.4: Conic sections

Now you are going to work on

Activity 2.2 The straight line

Learning objectives

At the end of this Learning Activity you should be able to:

- present a working definition of a straight line through the different equation forms;
- transform a given equation form of a straight line to another equation form;
- solve problems involving straight lines and related points, angles, or distances.

Summary

A straight line can be represented by different equation forms, some of them you already know from the secondary school. In this Learning Activity you will learn more equation forms, using the Cartesian and the polar system of coordinates.

Rationale

We may consider a straight line as the simplest "curve" since it is represented by a first degree equation. Later on this Unit you will learn about the very curves whose equations are of second degree. It is important to master the deduction of the different equations of a straight line before you come to the curved lines.

Key concepts

Slope of a straight line: A *slope* (of a straight line) is a way to describe the angle between a straight line and a horizontal straight line. See formal definition of a slope of a straight line in the plane containing the x and y axes on LA2.2 RR1.

Slope-intercept form: The linear form $y = ax + b$ is called the *slope-intercept form* of the equation of a straight line, where a is the slope of the line, and b is the y-intercept.

General form: The linear form $Ax + By + C = 0$ where A, B, C are integers, is called the *general form* of the equation of a straight line.

List of relevant readings (RR)

LA2.2 RR1: **The elements of plane and solid analytic geometry**, Candy, Albert L.; Boston, D.C. Heath & co, (1904)

(a PDF file in http://www.archive.org/details/elemplanesolidan00candrich)

LA2.2 RR2: **Analytic geometry for colleges, universities, and technical schools,** Nichols, E. W. (Edward West), 1858-1927; Hill, Theodore Preston. Early American mathematics books. CU-BANC; Boston [Mass.] : Leach, Shewell & Sanborn, (c1892)

(a PDF file in http://www.archive.org/details/analygeomcoll00nichrich)

LA2.2 RR3: **Line**

http://en.wikipedia.org/wiki/Line_(mathematics)

LA2.2 RR4: **Line**

http://mathworld.wolfram.com/Line.html

List of relevant resources

The relevant resources for this learning activity are:

- Straightedge and angle-measurer
- *GeoGebra* or *WinGeom*.

Introduction to the Learning Activities

The straight line

Figure 2.2.7: Straight lines

Task 1

Read about representing a straight line by different equation forms in LA2.2 RR1, Chapter II, § 31, pages 38-39; and LA2.2 RR2, Chapter III, pages 27-49;

Task 2

Solve the following exercises:

LA2.2 RR1,page 51, Exercises 1-15 (solve 8 exercises only);
page 52, Exercises 1-13 (solve 7 exercises only);
page 54, Exercises 1-11 (solve 6 exercises only);
page 57, Exercises 1-18 (solve 9 exercises only);
pages 63-66, Exercises 1-39 (solve 20 exercises only);

Task 3

Solve the following exercises:

LA2.2 RR2,pages 27-29, Exercises 1-15 (solve 8 exercises only);
pages 30-31, Exercises 1-12 (solve 7 exercises only);
page 34, Exercises 1-7 (solve 4 exercises only);

Unit 2: Plane Analytic Geometry

This Unit has 4 **Learning Activities (LA)**:

LA2.1: Vector space (in two dimensions)
LA2.2: The straight line
LA2.3: Transformation of coordinates in the plane
LA2.4: Conic sections

Now you are going to work on

Activity 2.3 Transformation of coordinates in the plane

Learning objectives

At the end of this Learning Activity you should be able to:

- define the two motions (translation, rotation) of coordinates transformation in the plane;
- find for a given geometric object the new coordinates by a translation and/or rotation of the original axes.

Summary

An equation of a geometric object may be complicated and not easy to analyze as given. By a transformation of coordinates that equation can became simpler. Simplification can be realized by one or two distinct operations: a translation or/ and a rotation of the axes. In this Learning Activity you will learn how to deduce the new coordinates from the original ones.

Rationale

The transformation of coordinates constitutes a device that enables us to simplify an equation of a geometric object to most recognized form – the *canonic* form. So, once you are able to transform coordinates, you can then analyze the properties of that geometric object much easier.

Key concepts

Coordinate transformation: A *coordinate transformation* is a conversion from one system to another, to describe the same space.

Translation of the coordinate axes: A *translation of the coordinate axes* is the operation of moving the coordinate axes in the coordinate plane to a different position so that the new axes are parallel to the old axes, respectively, and similarly directed.

Rotation of the coordinate axes: A *rotation of the coordinate axes* is the operation of moving the coordinate axes by a turn about their origin as a fixed point.

List of relevant readings (RR)

LA2.3 RR1: **The elements of plane and solid analytic geometry**, Candy, Albert L.; Boston, D.C. Heath & co, (1904)
(PDF file in http://www.archive.org/details/elemplanesolidan00candrich)

LA2.3 RR2: **Analytic geometry for colleges, universities, and technical schools,** Nichols, E. W. (Edward West), 1858-1927; Hill, Theodore Preston. Early American mathematics books. CU-BANC; Boston [Mass.]: Leach, Shewell & Sanborn, (c892)
(PDF file in http://www.archive.org/details/analygeomcoll00nichrich)

List of relevant resources

The relevant resources for this learning activity are:

- Straightedge and angle-measurer
- *GeoGebra* or *WinGeom*.

Introduction to the Learning Activities

Transformation of coordinates in the plane

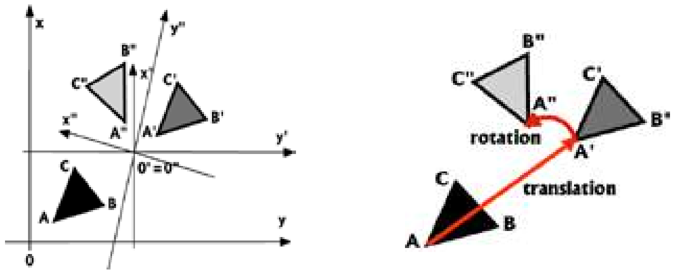


Figure 2.3.7: Example of a transformation of coordinates in the plane (a translation followed by a rotation)

Task 1

Read about transformation of coordinates in the plane in LA2.3 RR1, Chapter IV, pages 67-69; and LA2.3 RR2, Chapter IV, pages 50-58.

Task 2

Solve the following exercises:

LA2.3 RR1,page 69, Exercises 1-24 (solve 15 exercises only).

Task 3

Solve the following exercises:

LA2.3 RR2, page 54, Exercises 1-13 (solve 8 exercises only);

pages 57-58, Exercises 1-24 (solve 15 exercises only).

Unit 2: Plane Analytic Geometry

This Unit has 4 **Learning Activities (LA)**:

LA2.1: Vector space (in two dimensions)
LA2.2: The straight line
LA2.3: Transformation of coordinates in the plane
LA2.4: Conic sections

Now you are going to work on

Activity 2.4 Conic sections

Learning objectives

At the end of this Learning Activity you should be able to:

- define conic sections in terms of locus of points and by presenting their equations;
- solve problems related to conic sections;
- draw conic sections by free hand and using *GeoGebra* or *WinGeom*.

Summary

In this Learning Activity you will define authentic conic sections (ellipse, parabola, and hyperbola) and apply their properties in solving related problems. You will start by treating the general equation of the second degree, and the standard forms to which it can be transformed while on the other hand it will be presented that the locus of such an equation is always a curve that can be obtained by making a plane section of right circular cone.

Rationale

Conic sections have properties with several technical applications. It is important for you to know about those properties, particularly, as you will need to use them in the next Learning Activity when treating surfaces generated by conic sections.

Key concepts

Conic section: A *conic section* (or just *conic*) is a curve that can be obtained by making a plane section of a right circular cone.

Ellipse: An *ellipse* is a conic section generated by the intersection of a right circular cone and a plane which does not intersect the cone's base. An ellipse can also be defined as the locus of points on a plane where the sum of the distances from any point on the curve to two fixed points is constant.

Parabola: A *parabola* is a conic section generated by the intersection of a right circular cone and a plane parallel to a generating straight line of that surface. A parabola can also be defined as locus of points in a plane which are equidistant from a given point (the *focus*) and a given line (the *directrix*).

Hyperbola: A *hyperbola* is the type of conic section defined as the intersection between a right circular conical surface and a plane which cuts through both halves of the cone. It may also be defined as the locus of points where the difference in the distance to two fixed points is constant.

List of relevant readings (RR)

LA2.4 RR1: **The elements of plane and solid analytic geometry**, Candy, Albert L.; Boston, D.C. Heath & co, (1904), pages 113-192.
(a PDF file in http://www.archive.org/details/elemplanesolidan00candrich)

LA2.4 RR2: **Analytic geometry for colleges, universities, and technical schools,** Nichols, E. W. (Edward West), 1858-1927; Hill, Theodore Preston. Early American mathematics books. CU-BANC; Boston [Mass.] : Leach, Shewell & Sanborn, (c1892), pages 59-169.
(a PDF file in http://www.archive.org/details/analygeomcoll00nichrich)

LA2.4 RR3: **Conic section**
http://en.wikipedia.org/wiki/Conic_section

LA2.4 RR4: **Conic section**
http://mathworld.wolfram.com/ConicSection.html

List of relevant resources

The relevant resources for this learning activity are:

- Straightedge and compass.
- *GeoGebra* or *WinGeom*.

Introduction to the Learning Activities

Conic sections

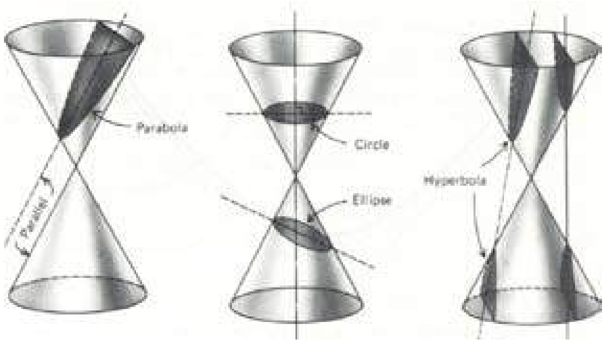


Figure 2.4.7: Conic sections

Task 1

Read about conic sections in LA2.4 RR1, Chapter VII-Chapter XI, pages 113-192; and LA2.4 RR2, Chapter V-Chapter IX, pages 59-187.

Task 2

Solve the following exercises:

LA2.4 RR1, from each group of pages, solve five exercises only. Pages: 137-, 146-, 161-, 175-, 187-.

Task 3

Solve the following exercises:

LA2.4 RR2, from each group of pages, solve five exercises only. Pages: 63-82, 102-105, 136-140, 166-169.

Unit 3: Solid Analytic Geometry

This Unit has 4 **Learning Activities (LA)**:

LA3.1: Vectors in three dimensions
LA3.2: Planes and lines
LA3.3: Transformations of coordinates in 3D-space
LA3.4: Quadric surfaces

Now you are going to work on

Activity 3.1 Vectors in three dimensions

Learning objectives

At the end of this Learning Activity you should be able to:

- Operate with vectors (addition, scalar multiplication, and dot product in three dimension);
- Apply the products of vectors (dot product, cross product, the scalar triple product, and the vector triple product)

Summary

In this Learning Activity you will start by revising the concept of vector and extend it to the three dimensional space. In fact, in terms of space, this Unit is an extension of the previous one. All operations of vectors you learned in Unit 2, will be applied in three dimensions now. A different type of product of two vectors will be introduced – the cross product (or vector product). Furthermore, combinations of dot product and cross product involving three vectors – the scalar triple product and the vector triple product – will be learned.

Rationale

The laws of operation in vector space of three dimensions are the same you already now from the previous Unit. So, the only aspect you should be aware is that you are now working in a three dimensional space. Because of this reason, you are recommended to review the properties of vector space you learned in Unit 2, before you start the present one.

Key concepts

Cross product: A *cross product* (also known as vector product) is a binary operation on vectors in a three-dimensional Euclidean space that results in a vector. It differs from the dot product, which results in a scalar. Its main use lies in the fact that the cross product of two vectors is orthogonal to both of them. (LA3.1 RR4)

Scalar triple product: The *scalar triple product* is defined as the dot product of one of the vectors with the cross product of the other two. It is a scalar (more precisely, it can be either a scalar or a *pseudoscalar*. Geometrically, this product is the (signed) volume of the parallelepiped formed by the three vectors given. It can be evaluated numerically using any one of the following equivalent characterizations: $a \cdot (b \times c) = b \cdot (c \times a) = c \cdot (a \times b)$. (LA3.1 RR4)

Vector triple product: A *vector triple product* is defined as the cross product of one vector with the cross product of the other two. The following relationships hold: $a \times (b \times c) = b(a \cdot c) - c(a \cdot b)$; $(a \times b) \times c = -c \times (a \times b) = -a(b \cdot c) + b(a \cdot c)$

List of relevant readings (RR)

LA3.1 RR1: **Vector analysis**, a text-book for the use of students of mathematics and physics, founded upon the lectures of J. Willard Gibbs, c1929; pages 55-74.
(a PDF file in http://www.archive.org/details/117714283)

LA3.1 RR4: **Cross product, scalar triple product**
http://en.wikipedia.org/wiki/Cross_product
http://en.wikipedia.org/wiki/Scalar_triple_product

LA3.1 RR5: **Cross product, scalar triple product, vector triple product**
http://mathworld.wolfram.com/CrossProduct.html
http://mathworld.wolfram.com/ScalarTripleProduct.html
http://mathworld.wolfram.com/VectorTripleProduct.html

List of relevant resources

The relevant resources for this learning activity are:

- *GeoGebra* or *WinGeom*.

Introduction to the Learning Activities

Vectors in three dimensions

Since you learned in the previous Unit to operate with vectors in two dimensions, in this LA you have extend your knowledge in this area a bit more.

Task 1

Let **u**=(2, -1, 3), **v**=(0, 1, 7), **w**=(1, 4, 5). Compute:

a) **u** x **w**

b) **u** x (**v** x **w**)

c) (**u** x **v**) -2**w**

Task 2

In each case find a vector orthogonal to both **u** and **v**:

a) **u** = (-7, 3, 1), **v** = (2, 0, 4)

b) **u** = (-1, -1, -1), **v** = (2, 0, 2)

Task 3

Given the vertices of the triangle *A*(1, -1, 2), *B*(5, -6, 2), *C*(1, 3, -1), find the altitude from the vertex *B*.

Task 4

In each case find the area of the triangle having vertices *P*, *Q*, and *R*.

a) *P*(1, 5, -2), *Q*(0, 0, 0), *R*(3, 5, 1)

b) *P*(2, 0, -3), *Q*(1, 4, 5), *R*(7, 2, 9)

Task 5

The volume of the tetrahedron ABCD is equal to 5.

If *A*(2, 1, -1), *B*(3, 0, 1), *C*(2, -1, 3), and the vertex *D* belongs to the axis *Y*, find the coordinates of *D*.

Unit 3: Plane Analytic Geometry

This Unit has 4 **Learning Activities (LA)**:

LA3.1: Vectors in three dimensions
LA3.2: Planes and lines
LA3.3: Transformations of coordinates in 3D-space
LA3.4: Quadric surfaces

Now you are going to work on

Activity 3.2 Planes and lines

Learning objectives

At the end of this Learning Activity you should be able to:

- Deduce different forms of equations of a plane and of a line;
- Solve problems relating planes and lines

Summary

In this Learning Activity you will start by deducing different forms of equation of a plane and later on of a line. Having deduced equations of a plane and of a line you will solve problems relating the two geometric objects. Using *GeoGebra* or *WinGeom* you will be able to visualize different positions of a sketch illustrating a given problem so that you can solve it in easier and more comprehensive way.

Rationale

You already learned about straight line in Unit 2 (Plane Analytic Geometry). Now you are going to upgrade you knowledge on this topic by treating it in three dimensional space. Furthermore, you will now learn to drive equations of planes and solve problems related to planes and lines in three dimensions. So, it is important to revise the Unit 2 before you engage in Solid Analytic Geometry.

Key concepts

Planes and straight lines are given by equations in different *forms.* Here are explained briefly some of the forms.

A **plane** containing the point $(x_0\ y_0\ z_0)$ and having a ***normal*** vector (an orthogonal vector to the plane) $\vec{n} = \langle a,\ b,\ c \rangle$ can be represented in ***standard form*** by the equation: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

By regrouping terms, you obtain the ***general form*** of the equation of a plane:

$$Ax + By + Cz + D = 0$$

A **line** parallel to the vector $\vec{v} = \langle a,\ b,\ c \rangle$ and passing through the point $P(x_0,\ y_{0,}\ z_0)$ is represented by the ***parametric equations***:

$$x = x_0 + ta$$

$$y = y_0 + b$$

$$z = z_0 + tc\ .$$

If the direction numbers *a*, *b,* and *c* are all nonzero, you can eliminate the parameter t to obtain the ***symmetric equations (***or ***standard form)*** of a line:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

List of relevant readings (RR)

LA3.2 RR1: **The elements of plane and solid analytic geometry**, Candy, Albert L.; Boston, D.C. Heath & co, (1904), pages 209-216.
(a PDF file in http://www.archive.org/details/elemplanesolidan00candrich)

LA3.2 RR2: **Analytic geometry for colleges, universities, and technical schools,** Nichols, E. W. (Edward West), 1858-1927; Hill, Theodore Preston. Early American mathematics books. CU-BANC; Boston [Mass.] : Leach, Shewell & Sanborn, (c1892), pages 226-258.
(a PDF file in http://www.archive.org/details/analygeomcoll00nichrich)

List of relevant resources

The relevant resources for this learning activity are:

- *GeoGebra* or *WinGeom*.

Introduction to the Learning Activities

Planes and lines

In this Unit you will use vectors to drive equations of planes and lines in three dimensional space, and you will use these equations to solve basic geometric problems. In plane analytic geometry a line can be specified by giving its slope and one of its points. Similarly, a plane in three dimensional space can be determined by giving its inclination and specifying one of its points.

Planes and straight lines are given by equations in different *forms.* In the Relevant Readings for this Unit you find explanations of how to drive all the different forms. Before you solve a problem algebraically, try to make a related sketch; this will help you to interpret more easily the question of the problem.

Task 1

Read about planes and lines in LA3.2 RR1, chapter XIV, pages 209-216; and LA3.2 RR2, chapter II-III, pages 226-252.

Task 2

Solve the following exercises:

LA3.2 RR1, page 216, from exercises 1-12, it is to solve nine exercises only.

Task 3

Solve the following exercises:

LA3.2 RR2, from each group of pages 234-235, and 240-242, it is to solve fifteen exercises only.

Unit 3: Plane Analytic Geometry

This Unit has 4 **Learning Activities (LA)**:

LA3.1: Vectors in three dimensions
LA3.2: Planes and lines
LA3.3: Transformations of coordinates in 3D-space
LA3.4: Quadric surfaces

Now you are going to work on

Activity 3.3 Transformations of coordinates in 3D-space

Learning objectives At the end of this Learning Activity you should be able to:

- Reduce general forms of equations in three dimensions to standard forms by a translation and/or by rotation of coordinate axes.

Summary

Similarly to the transformation of coordinates in the plane you learned in the previous Unit, in this Learning Activity you will now learn to reduce equations from general forms standard forms by a translation and/or by a rotation of coordinate axes in three dimensions. Using hand sketch or the software *Geogebra* or *WinGeom* you can visualize a given problem before you solve it algebraically.

Rationale

You already know about transformation of coordinates in a plane. Now it is important to learn about the same transformations (translation, rotation), but this time, in three dimensional space. These operations are required to solve problems in the next Learning Activity on quadric surfaces.

Key concepts

The key concepts are the same as presented in Learning Activity on transformation of coordinates in the plane (Unit 2, LA3), but now defined in three dimensional space.

Coordinate transformation: A *coordinate transformation* is a conversion from one system to another, to describe the same space.

Translation of the coordinate axes: A *translation of the coordinate axes* is the operation of moving the coordinate axes in the coordinate plane to a different position so that the new axes are parallel to the old axes, respectively, and similarly directed.

Rotation of the coordinate axes: A *rotation of the coordinate axes* is the operation of moving the coordinate axes by a turn about their origin as a fixed point.

List of relevant readings (RR)

LA3.3 RR1: **The elements of plane and solid analytic geometry**, Candy, Albert L.; Boston, D.C. Heath & co, (1904), pages 217-218.
(a PDF file in http://www.archive.org/details/elemplanesolidan00candrich)

LA3.3 RR2: **Analytic geometry for colleges, universities, and technical schools,** Nichols, E. W. (Edward West), 1858-1927; Hill, Theodore Preston. Early American mathematics books. CU-BANC; Boston [Mass.]: Leach, Shewell & Sanborn, (c1892), pages 247-249.
(a PDF file in http://www.archive.org/details/analygeomcoll00nichrich)

List of relevant resources

The relevant resources for this learning activity are:

- *GeoGebra* or *WinGeom*.

Introduction to the Learning Activities

Transformations of coordinates in 3D-space

Task 1

Read about Transformation of coordinates in 3D-space in LA3.3 RR1, pages 217-218; and LA3.3 RR2, pages 247-249.

Task 2

Solve the exercises 1-17, in LA3.3 RR1, page 217.

Unit 3: Plane Analytic Geometry

This Unit has 4 **Learning Activities (LA)**:

LA3.1: Vectors in three dimensions
LA3.2: Planes and lines
LA3.3: Transformations of coordinates in 3D-space
LA3.4: Quadric surfaces

Now you are going to work on

Activity 3.4 Quadric surfaces

Learning objectives

At the end of this Learning Activity you should be able to:

- define quadric surfaces in terms of locus of points and by presenting their equations;
- solve problems related to quadric surfaces;
- draw quadric surfaces by free hand and using *GeoGebra* or *WinGeom*.

Summary

In this Learning Activity you will learn about surfaces strictly related to the conic sections you learned in Unit 2. Quadric surfaces are represented by quadratic equations in three dimensions. Similarly to the study of conic sections, here you will learn to reduce equations of quadric surfaces from general forms standard forms. Also, using hand sketch or the software *Geogebra* or *WinGeom* you can visualize a given problem before you solve it algebraically.

Rationale

The techniques of transformation of quadratic equations used in the Learning Activity on conic sections are extended to three dimensions. So, it is important to revise the Learning Activity on conic sections (Unit 2) and the Learning Activity on transformation of coordinates in 3D-space, in Unit 3.

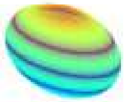
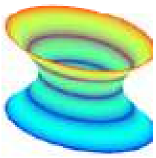
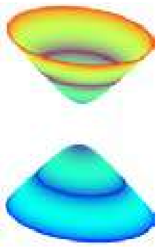
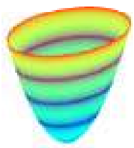
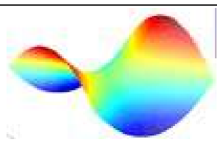
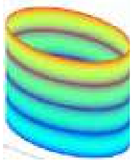
Key concepts

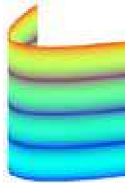
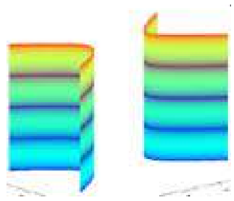
Quadric surfaces: *Quadric surfaces* (or simply *quadrics*) are surfaces represented by any equation that can be put into the general form

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

where $A, \ldots J$ are constants.

There is no way that we can possibly list all of them, but there are some standard equations so here is a list of some of the more common quadric surfaces.

Quadric surface	Sketch	Equation
Ellipsoid		$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$
Cone		$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$
Hyperboloid of one sheet		$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$
Hyperboloid of two sheets		$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$
Paraboloid		$\frac{x^2}{a^2}+\frac{y^2}{b^2}-z=0$
Hyperbolic paraboloid		$\frac{x^2}{a^2}-\frac{y^2}{b^2}-z=0$
Elliptic cylinder		$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Parabolic cylinder		$y^2 + 2px = 0$
Hyperbolic cylinder		$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

List of relevant readings (RR)

LA3.4 RR1: **The elements of plane and solid analytic geometry**, Candy, Albert L.; Boston, D.C. Heath & co, (1904), pages 217-218.
(a PDF file in http://www.archive.org/details/elemplanesolidan00candrich)

LA3.4 RR2: **Analytic geometry for colleges, universities, and technical schools,** Nichols, E. W. (Edward West), 1858-1927; Hill, Theodore Preston. Early American mathematics books. CU-BANC; Boston [Mass.]: Leach, Shewell & Sanborn, (c1892), pages 247-249.
(a PDF file in http://www.archive.org/details/analygeomcoll00nichrich)

List of relevant resources

The relevant resources for this learning activity are:

- *GeoGebra* or *WinGeom*.

Introduction to the Learning Activities

Quadric surfaces

Task 1

Read about "Conicoids" LA3.4 RR1, pages 220-246 and about "A discussion of surfaces of the second order" in LA3.4 RR2, pages 259-275.

Task 2

Solve the exercises 1-10, in LA3.4 RR1, page 246 and the exercises 1-5, in LA3.4 RR2, page 275.

Unit 4: Non-Euclidean Geometry

This Unit has 3 **Learning Activities (LA)**:

LA4.1: Introduction to non-Euclidean geometry
LA4.2: Affine transformations
LA4.3: Projective transformations

Now you are going to work on

Activity 4.1 Introduction to non-Euclidean geometry

Learning objectives

At the end of this Learning Activity you should be able to:

- Describe the foundation of non-Euclidean, relating it to Euclidean geometry;
- Work on basic concepts and theorems of hyperbolic geometry using the disk model and the upper half-plane model of Poincaré.

Summary

The study of the Unit on non-Euclidean geometry will help you to develop an understanding of an axiomatic system through investigating and comparing Euclidean and Non-Euclidean geometries. You will work on some definitions and theorems of hyperbolic geometry, based on the disk model and the upper half-plane model of Poincaré using ruler and compass constructions and exploring *NonEuclid* – an interactive Java software incorporating explanations, exercises.

Rationale

Non-Euclidean geometry is becoming increasingly important in its role in modern science and technology. A study of non-Euclidean geometry makes clear that geometry is not a static subject but it is a current and active field of research.

Key concepts

Non-Euclidean geometry: *non-Euclidean geometry* is any geometry that is different from Euclidean geometry. Each non-Euclidean geometry is a consistent system of definitions, assumptions, and proofs that describe such objects as points, lines and planes. The two most common non-Euclidean geometries are spherical geometry and hyperbolic geometry. (see LA4.1 RR3)

Disk model of Poincaré: the *Poincaré disk model*, also called the conformal disk model, is a model of n-dimensional hyperbolic geometry in which the points of the geometry are in an n-dimensional disk, or unit ball, and the lines of the geometry are segments of circles contained in the disk orthogonal to the boundary of the disk, or else diameters of the disk. (see LA4.1 RR2)

List of relevant readings (RR)

LA4.1 RR1: **Non-Euclidean geometry**
http://en.wikipedia.org/wiki/Non-Euclidean_geometry

LA4.1 RR2: **Poincaré disc model**
http://en.wikipedia.org/wiki/Poincar%C3%A9_disk_model

LA4.1 RR3: **What is Non-Euclidean Geometry**
http://cs.unm.edu/~joel/NonEuclid/NonEuclid.html

List of relevant resources

***NonEuclid*:**

NonEuclid is an interactive Java software for creating ruler and compass constructions in both the Poincaré disk and the upper half-plane models of hyperbolic geometry.

The software package includes explanations, activities, and strategies for incorporating non-Euclidean geometry into high school curriculum.

NonEuclid can be downloaded at:

http://cs.unm.edu/~joel/NonEuclid/NonEuclid.html

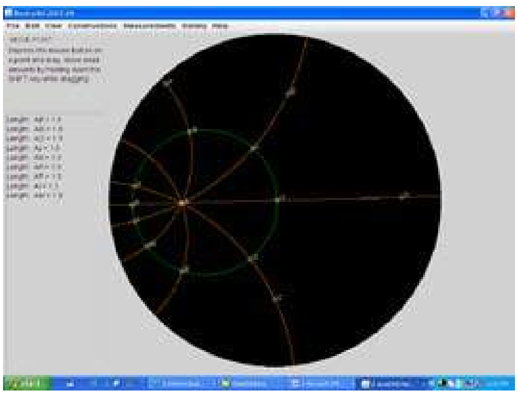


Figure 4.1.1: A screen capture of NonEuclid

Introduction to the Learning Activities

Non-Euclidean geometry

You will now learn a new kind of geometry. Imagine a triangle on a sphere: Is the sum of angles in a spherical triangle equal to 180°? Reflect on this question before you proceed.

On a sphere, the sum of the angles of a triangle is not equal to 180°. A sphere is not a Euclidean space, but locally the laws of the Euclidean geometry are good approximations. In a small triangle on the face of the earth, the sum of the angles is very nearly 180°.

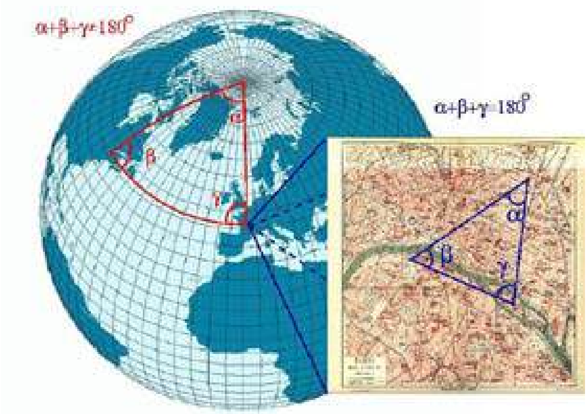


Figure 4.1.2: On a sphere, the sum of the angles of a triangle is not equal to 180°.

Non-Euclidean geometry is any geometry that is different from Euclidean geometry. Each non-Euclidean geometry is a consistent system of definitions, assumptions, and proofs that describe such objects as points, lines and planes. The two most common non-Euclidean geometries are spherical geometry and hyperbolic geometry.

The essential difference between Euclidean geometry and these two non-Euclidean geometries is the nature of parallel lines: In Euclidean geometry, given a point and a line, there is exactly one line through the point that is in the same plane as the given line and never intersects it. In spherical geometry there are no such lines. In hyperbolic geometry there are at least two distinct lines that pass through the point and are parallel to (in the same plane as and do not intersect) the given line. (see LA4.1 RR3)

Task 1

Read about the concept and about the history of foundations of non-Euclidean geometry, in LA4.1 RR1.

Task 2

Using *NonEuclid*, try to **explore properties** in hyperbolic geometry.

(To **get activities** load up ***NonEuclid***, click **Help** and then **What-to-do: Activities**.)

Unit 4: Non-Euclidean Geometry

This Unit has 3 **Learning Activities (LA)**:

LA4.1: Introduction to non-Euclidian geometry
LA4.2: Affine transformations
LA4.3: Projective transformations

Now you are going to work on

Activity 4.2 Affine transf

Learning objectives

At the end of this Learning Activity you should be able to:

- Describe the properties of affine transformations
- Solve construction problems on affine figures (triangles, quadrangles, and conic sections)

Summary

After the introduction of non-Euclidean geometry, the content of present Learning Activity occupies a place intermediate between Euclidean geometry and projective geometry – the content of the next Learning Activity. Affine transformations have some feature in common with Euclidean transformations, but also some very different features. For example, a remarkable difference is that by an affine transformation any triangle can be mapped to any other triangle; that means all triangles are congruent! This result constitutes the essence of the Fundamental Theorem of Affine Geometry.

Affine transformations can be defined in R^n, for any natural $n \geq 2$; here we restrict our attention to the case n=2.

Rationale

In this Unit you will workout properties of geometric figures in manner that will make you clear of the difference between Euclidean transformations and affine transformations. The algebra required to compose affine transformations is similar to the algebra that we used to compose Euclidean transformations. Therefore, is important that you revise operations with matrices in R^2.

Key concepts

***Affine transformation*:** an *affine transformation* or affine map (from the Latin, *affinis*, “connected with”) between two vector spaces (strictly speaking, two affine spaces) consists of a linear transformation followed by a translation: $x \rightarrow Ax + b$. In the finite-dimensional case each affine transformation is given by a matrix *A* and a vector *b*, which can be written as the matrix *A* with an extra column *b*.

(see LA4.3 RR1)

Fundamental theorem of affine geometry: Lat *P*, *Q*, *R* and *P’*, *Q’*, *R’* be two sets of three non-collinear points in R^2. Then: (a) there is an affine transformation *t* which maps *P*, *Q*, and R to *P’*, *Q’*, and *R’*, respectively; (b) the affine transformation *t* is unique.

List of relevant readings (RR)

LA4.2 RR1: **Affine transformation**
http://en.wikipedia.org/wiki/Affine_transformation

LA4.2 RR2: **Affine transformation**
http://mathworld.wolfram.com/AffineTransformation.html

List of relevant resources

- Geometric basic drawing instruments (straightedge and compass).
- *GeoGebra* or *WinGeom*

Introduction to the Learning Activities

Affine transformations

In this Learning Activity you meet the first of the new geometries (non-Euclidean geometries) on plane: the ***affine geometry*** in R^2. This geometry consists of the set R^2 together with a group of transformations, the affine transformations, acting on R^2. Affine transformations can be defined in R^n, for any natural $n \geq 2$; here we restrict our attention to the case n=2.

Figures in affine transformation have the following basic properties:

Affine transformations:

1. map straight lines to straight lines;
2. map parallel straight lines to parallel straight lines;
3. preserve ratios of lengths along a given straight line.

Recalling what you studied in the previous LA, you will see that affine transformations indicate a special class of projective transformations.

Task 1

Read about *affine transformations* in LA4.2 RR1, and LA4.2 RR2.

Task 2

Prove Ceva's and Menelaus's theorems presented in LA4.2 RR1.

Task 3

Determine whether or not each of the following transformations of R^2 is an affine transformation.

A $$t_1(x) = \begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} x + \begin{pmatrix} 4 \\ -2 \end{pmatrix}$$

B $$t_2(x) = \begin{pmatrix} -6 & 5 \\ 3 & 2 \end{pmatrix} x + \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

C $$t_3(x) = \begin{pmatrix} -2 & -1 \\ 8 & 4 \end{pmatrix} x + \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

D $$t_4(x) = \begin{pmatrix} 5 & -3 \\ -2 & 2 \end{pmatrix} x$$

Task 4

The affine transformation t: $R^2 \rightarrow R^2$ is given by

$$t(x) = \begin{pmatrix} 1 & -1 \\ 2 & -3 \end{pmatrix} x + \begin{pmatrix} 2 \\ -4 \end{pmatrix}$$

Determine the image under t of each of the following lines:

a) $y = -2x$

b) $2y = 3x - 1$

Task 5

The triangle ΔABC has vertices $A(-1, 2)$, $B(-3, -1)$, and $C(3, 1)$ and the points

$P(1, \frac{1}{3})$, $Q(1, \frac{3}{2})$ and $R(-\frac{5}{3}, 1)$ lie on BC, CA, and AB.

a) Determine the ratios in which P, Q and R divide the sides of the triangle.

b) Determine whether or not the lines AP, BQ, and CR are concurrent.

Unit 4: Non-Euclidean Geometry

This Unit has 3 **Learning Activities (LA)**:

LA4.1: Introduction to non-Euclidian geometry
LA4.2: Affine transformations
LA4.3: Projective transformations

Now you are going to work on

Activity 4.3 Projective transformations

Learning objectives

At the end of this Learning Activity you should be able to:

- Present definition and properties of basic projective figures (complete quadruples, Desargues' configuration, Pappus' configuration, conic sections)
- Solve problems about projective transformations on the plane.

Summary

In this Learning Activity you will study transformations belonging to the projective geometry – a geometry that includes all the other geometries you studied before. In the words of Arthur Cayley (1821-1895): "*Projective geometry is all geometry*". As an introductory study, this Learning Activity will be restricted to treatment of projective transformations on the plane.

Rationale

Being the projective geometry the "all geometry" it is important to you at this stage to summarise the "genealogy" of the geometric subjects known in our days. Of course, this will also make you aware of existence of different "worlds" of geometry and that new geometrical structures may be created since we establish the invariants of the transformations involved in that "world". .

Key concepts

***Projective transformation*:** A *projective transformation* is a transformation used in projective geometry: it is the composition of a pair of perspective projections. It describes what happens to the perceived positions of observed objects when the point of view of the observer changes. Projective transformations do not preserve sizes or angles but do preserve incidence and cross-ratio: two properties which are important in projective geometry. A projective transformation can also be called **projectivity.**

A projective transformation can be in the (real) one-dimensional projective line RP^1, the two-dimensional projective plane RP^2, and the three-dimensional projective 3-space RP^3. (see LA4.3 RR2)

***The Fundamental Theorem of Projective Geometry*:**

Let *ABCD* and *A'B'C'D'* be two quadrilaterals in RP^2. Then:

(a) there is a projective transformation *t* which maps *A* to *A*', *B* to *B*', and *C* to *C*', and *D* to *D*";

(b) the projective transformation *t* is unique.

***Complete quadrangle*:** In mathematics, specifically projective geometry, a *complete quadrangle* is a projective configuration $(4_3 6_2)$ consisting of four points, no three of which are collinear, and the six lines defined by those four points. This configuration was called a *tetrastigm* by Lachlan (1893), and that term is occasionally still used.

***Complete quadrilateral*:** The projective dual configuration $(6_2 4_3)$ to a complete quadrangle is a *complete quadrilateral*

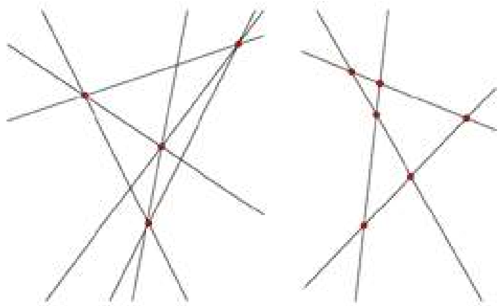


Figure 4.3.1: A *complete quadrangle* (at left) and a *complete quadrilateral* (at right).

(called a *tetragram* by Lachlan), a configuration consisting of 4 lines, no 3 of which pass through a common point, and the 6 points of intersection of those four lines.

Desargues' theorem: In projective geometry, Desargues' theorem, named in honor of Gérard Desargues, states: In projective space, two triangles are in perspective axially if and only if they are in perspective centrally.

The 10 lines and 10 3-line intersections form a 103 configuration sometimes called Desargues' configuration.

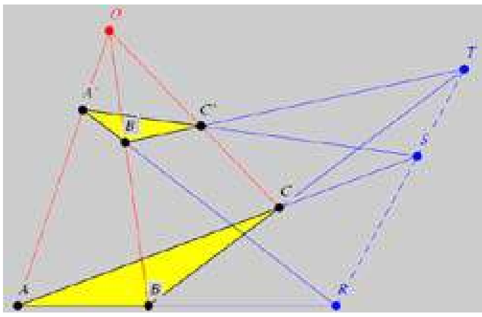


Figure 4.3.2: Desargues' configuration

Pappus configuration**:** *Pappus configuration* consists of a pair ((*A*,*B*,*C*), (*D*,*E*,*F*)) of triplets of points, which pair is located either on a pair of lines or on two sides of a conic section, with a hexagon *AECDBF* defined on the points, and a collinear triplet of intersections of opposite sides (e.g. *AE* and *DB*) of the hexagon. When each triplet of points is collinear, it forms a projective configuration of nine points incident to nine lines, with the notation $(9_3 9_3)$.

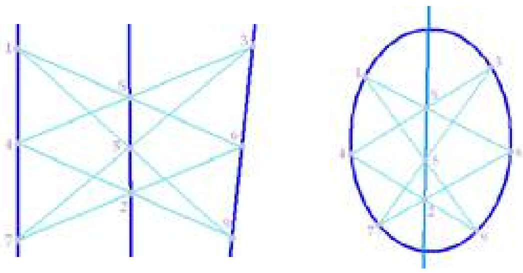


Figure 4.3.3: Pappus' configurations

List of relevant readings (RR)

LA4.3 RR1: **Projective geometry**, Veblen, Oswald, Boston and Young, John W.; Ginn & Co. (1938), pages 1-108;
(PDF file in http://www.archive.org/details/117714799_001)

LA4.3 RR2: **Projective geometry**
http://en.wikipedia.org/wiki/Projective_geometry

LA4.3 RR3: **Projective geometry**
http://mathworld.wolfram.com/ProjectiveGeometry.html

List of relevant resources

- Straightedge (without any additional hand drawing instrument!)
- *GeoGebra or WinGeom*

Introduction to the Learning Activities

Projective transformations

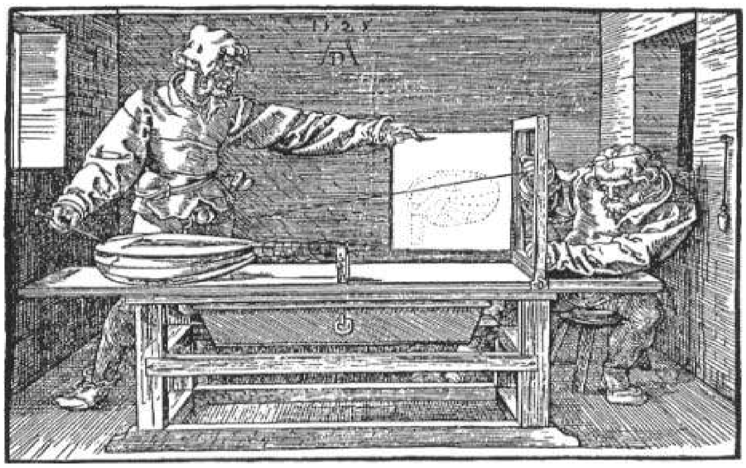


Figure 4.3.4: A perspective machine

"*Projective geometry is all geometry*" (Cayley, 1821-1895).

This statement of the English mathematician Arthur Cayley is due the fact that in projective geometry the points at infinity are not distinguished from the ordinary points and therefore, projective geometry exhibits much more symmetry than the affine geometry.

Projective properties were discovered by artists when attempting over many centuries to paint realistic-looking pictures of scenes composed of objects situated at different distances from the eye. The question raised sound like that: How can three-dimensional scenes be represented on a two-dimensional canvas? Projective geometry explains though its transformations how eyes perceive "the real world" and so explains how can artists achieve realism in their paintings.

Among mathematicians who have developed this geometry we can mentions the French architect Girard Desargues (1591-1661) who invented points at infinity, which leads to a unified approach and deeper insight into geometric structures.

In this Learning Activity you will study the projective transformations on the plane and use them to solve basic projective geometry problems.

Task 1

Read about projective geometry in LA4.3 RR1, on the following topics:

a) Ideal elements in geometry, in introductory chapter, pages 12-14;
b) Principle of duality, chapter 1, pages 26-28;
c) Elementary configurations, chapter 2, pages 34-54;
d) Projectivity of the primitive geometric forms, chapter 3, pages 55-74
e) Harmonic constructions and the Fundamental Theorem of projective geometry, chapter 1, pages 79-102.

Task 2

Which of the following homogeneous coordinates represent the same point of RP^2 as [4, -8, 2]?

A [1, 4, -2] **B** $[\frac{1}{4}, -\frac{1}{2}, \frac{1}{8}]$ **C** $[-\frac{1}{2}, -2, 1]$

D [-2, 4, -1] **E** $[-\frac{1}{8}, -\frac{1}{2}, \frac{1}{4}]$

Task 3

Determine which of the following transformations *t* of RP^2 are projective transformations? For those that are projective transformations, write down a matrix associated with *t*.

a) $t: \lfloor x,\ y,\ z \rfloor \mapsto \lfloor 2x,\ y+3z,\ 1 \rfloor$

b) $t: \lfloor x,\ y,\ z \rfloor \mapsto \lfloor x,\ x-y+3z,\ x+y \rfloor$

c) $t: \lfloor x,\ y,\ z \rfloor \mapsto \lfloor 2y,\ y-4z,\ x \rfloor$

d) $t: \lfloor x,\ y,\ z \rfloor \mapsto \lfloor x+y-z,\ y+3z,\ x+2y+2z \rfloor$

Task 4

For each of the following sets of points *A*, *B*, *C*, *D*, **calculate** the cross-ratio (*ABCD*).

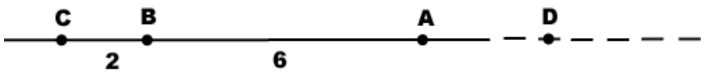
a) *A*[2, 1, 3], *B*[1, 2, 3], *C*[8, 1, 9], *D*[4, -1, 3]

b) *A*[2, 1, 1], *B*[-1, 1, -1], *C*[1, 2, 0], *D*[-1, 4, -2]

c) *A*[-1, 1, 1], *B*[0, 0, 2], *C*[5, -5, 3], *D*[-3, 3, 7]

Task 5

Calculate the cross-ratio (*ABCD*) for the collinear points *A*, *B*, *C*, and *D* illustrated below, where *D* is an ideal point.



X. Key Concepts (Glossary)

Euclid's Elements: The classic treatise in geometry written by Euclid (about 300 BC). It consists of 465 propositions, divided into 13 «books». (See LA1.1 RR4)

Rind Mathematical Papyrus: The Rhind Mathematical Papyrus (RMP) is an ancient mathematical document written on *papyrus* (paper). It is named after Alexander Henry Rhind, a Scottish antiquarian, who purchased the papyrus in 1858 in Luxor, Egypt. (See LA1.1 RR5)

Axiom: A sentence or proposition that is taken for granted as true.

Theorem: A statement that has been proven to be true or proposed as a demonstrable truth.

Conjectures: A conjecture is the same as a hypothesis, a statement drown from some evidence but not proved at all.

Inductive-deductive reasoning: A way of thinking that starts from particular cases to generalizations (*induction*) – and once having general (proved) statements then we got implications for particular case (*deduction*).

Mathematical proof: A thorough demonstration of a proposition using the logic of the *axiomatic* structure of that proposition.

Euclidean plane isometry: *Euclidean plane isometry* is an isometry of the Euclidean plane, or more informally, a way of transforming the plane that preserves geometrical properties such as length. (LA1.3 RR1)

Translation: In geometry *translation* is moving every point a constant distance in a specified direction. Formally, a translation is denoted by T_v, where v is a vector in R^2. This has the effect of shifting the plane in the direction of v. (LA1.3 RR1)

Rotation: A *rotation* is the movement of an object in a circular motion. In the plane a object rotates around a *center* (or *point*) *of rotation*. A rotation is denoted by $R_{c,\square}$, where c is a point in the plane (the centre of rotation), and □ is the angle of rotation. (LA1.3 RR1)

Reflection: In mathematics, a *reflection* (also spelt *reflexion*) is a map that transforms an object into its mirror image. A reflection or *mirror isometry* is denoted by $F_{c,v}$, where c is a point in the plane and v is a unit vector in R^2. (F is for «flip».) This has the effect of reflecting the point p in the line L that is perpendicular to v and that passes through c. The line L is called the *reflection axis* or the associated *mirror*. (LA1.3 RR1)

Glide reflection: A *glide reflection* is the combination of a reflection in a line and a translation along that line. A glide reflection is denoted by $G_{c,v,w}$, where c is a point in the plane, v is a unit vector in R^2, and w is a vector perpendicular to v. (LA1.3 RR1)

Geometric congruence: Two geometric figures are said to exhibit *geometric congruence* (or «be geometrically *congruent*») iff one can be transformed into the other by an *isometry* (LA1.3 RR3). This relationship is written. *A*□*B*

***Triangle*:** A *triangle* is one of the basic shapes of geometry: a polygon with three vertices and three sides which are straight line segments. (LA1.4 RR1)

***Circumcircle*:** The *circumcircle* (or *circumscribed circle*) of a triangle is the circle which contains all the vertices of the triangle. The centre of this circle is called the *circumcenter.* (LA1.4 RR1)

***Incircle*:** The *incircle* or *inscribed circle* of a triangle is the largest circle contained in the triangle; it touches (is tangent to) the three sides. The center of the incircle is called the triangle's *incenter*. (LA1.4 RR1)

Centroid: The centroid of a triangle is the point of intersection of its medians (the lines joining each vertex with the midpoint of the opposite side). (LA1.4 RR1)

Orthocenter: The three altitudes of a triangle intersect in a single point, called the *orthocenter* of the triangle. (LA1.4 RR1)

***Nine-point circle*:** The *nine-point circle* (also known as *Feuerbach's circle*, *Euler's circle*, *Terquem's circle*) is a circle that can be constructed for any given triangle. It is so named because it passes through nine significant points, six lying on the triangle itself (unless the triangle is obtuse). They include: the midpoint of each side of the triangle, the foot of each altitude, and the midpoint of the segment of each altitude from its vertex to the orthocenter.

***Circle*:** A *circle* is the set of all points in a plane that are at a fixed distance from a fixed point called the *centre.*

***Circumference*:** A *circumference* of a circle is the distance around the circle.

***Radius*:** A *radius* of a circle is the line segment joining the centre to a point on the circle.

Note: Since all radii of a given circle have the same length, we may at times use the word *radius* to mean the number that is "the length of the radius".

***Chord*:** A *chord* of a circle is a line segment joining to points on a circle.

***Diameter*:** A *diameter* is a chord through the centre.

***Arc*:** An *arc* is a continuous part of a circle.

Geometric similarity: A transformation that preserves angles and changes all distances in the same ratio, called the *ratio of magnification*. A similarity can also be defined as a transformation that preserves ratios of distances. (LA1.6 RR2)

Ratio of magnification: A *ratio of magnification* is a constant ratio by which all distances are increased (or decreased) in a *similarity*. A similarity with ratio of magnification equal to 1 is called an *isometry.* (LA1.6 RR2)

Central dilation: A similarity transformation which transforms each line to a parallel line whose length is a fixed multiple of the length of the original line. The simplest dilation is therefore a *translation*, and any dilation that is not merely a translation is called a *central dilation*.

***Directly similar*:** Two figures are *directly similar* when all corresponding angles are equal and described in the same rotational sense. (LA1.6 RR2)

***Inversely similar*:** Two figures are *inversely similar* when all corresponding angles are equal and described in the opposite rotational sense. (LA1.6 RR2)

Vector space*:** A *vector space* is a mathematical domain of quantities possessing a *magnitude* and *direction* – ***vectors – together with a defined set of operation laws. (See formal definition in LA2.1 RR3.)

***Vector*:** A *vector* is a quantity characterized by a *magnitude* (in mathematics a number, in physics a number times a unit) and a *direction*, often represented graphically by an arrow. (LA2.1 RR2)

Scalar multiplication: A *scalar multiplication* is the multiplication of a vector by a scalar.

Norm of a vector: The *length* or *magnitude* of a vector **v** is often called ***norm*** of **v** and it is donated by $\|\mathbf{v}\|$. It follows from the theorem of Pythagoras that the norm of a vector $\mathbf{v} = (v_1,\ v_2)$ in two dimensional space is $\|\mathbf{v}\| = \sqrt{v_1^{\ 2} + v_2^{\ 2}}$.

Dot product: A *dot product* (or *scalar product*, owing to the fact that its value is scalar) of two vectors *A* and *B* is the scalar quantity (donated $\mathbf{a} \times \mathbf{b}$) obtained by multiplying the product of the *norms* of the vectors by the *cosine* of the angle between them. That is $\mathbf{a} \times \mathbf{b} = \|\mathbf{a}\| \times \|\mathbf{b}\| cos(\mathbf{a}, \mathbf{b})$. The dot product is also called *direct product.*

Slope of a straight line: A *slope* (of a straight line) is a way to describe the angle between a straight line and a horizontal straight line. See formal definition of a slope of a straight line in the plane containing the *x* and *y* axes on LA2.2 RR1.

Slope-intercept form: The linear form $y = ax + b$ is called the *slope-intercept form* of the equation of a straight line, where *a* is the slope of the line, and *b* is the *y*-intercept.

General form: The linear form $Ax + By + C = 0$ where *A*, *B*, *C* are integers, is called the *general form* of the equation of a straight line.

Coordinate transformation: A *coordinate transformation* is a conversion from one system to another, to describe the same space.

Translation of the coordinate axes: A *translation of the coordinate axes* is the operation of moving the coordinate axes in the coordinate plane to a different position so that the new axes are parallel to the old axes, respectively, and similarly directed.

Rotation of the coordinate axes: A *rotation of the coordinate axes* is the operation of moving the coordinate axes by a turn about their origin as a fixed point.

Conic section: A *conic section* (or just *conic*) is a curve that can be obtained by making a plane section of a right circular cone.

Ellipse: An *ellipse* is a conic section generated by the intersection of a right circular cone and a plane which does not intersect the cone's base. An ellipse can also be defined as the locus of points on a plane where the sum of the distances from any point on the curve to two fixed points is constant.

Parabola: A *parabola* is a conic section generated by the intersection of a right circular cone and a plane parallel to a generating straight line of that surface. A parabola can also be defined as locus of points in a plane which are equidistant from a given point (the *focus*) and a given line (the *directrix*).

Hyperbola: A *hyperbola* is the type of conic section defined as the intersection between a right circular conical surface and a plane which cuts through both halves of the cone. It may also be defined as the locus of points where the difference in the distance to two fixed points is constant.

Cross product: A *cross product* (also known as vector product) is a binary operation on vectors in a three-dimensional Euclidean space that results in a vector. It differs from the dot product, which results in a scalar. Its main use lies in the fact that the cross product of two vectors is orthogonal to both of them. (LA3.1 RR4)

Scalar triple product: The *scalar triple product* is defined as the dot product of one of the vectors with the cross product of the other two. It is a scalar (more precisely, it can be either a scalar or a *pseudoscalar*. Geometrically, this product is the (signed) volume of the parallelepiped formed by the three vectors given. It can be evaluated numerically using any one of the following equivalent characterizations: $a \cdot (b \times c) = b \cdot (c \times a) = c \cdot (a \times b)$. (LA3.1 RR4)

Vector triple product: A *vector triple product* is defined as the cross product of one vector with the cross product of the other two. The following relationships hold: $a \times (b \times c) = b(a \cdot c) - c(a \cdot b)$;

$$(a \times b) \times c = -c \times (a \times b) = -a(b \cdot c) + b(a \cdot c).$$ (LA3.1 RR4)

Planes and straight lines are given by equations in different *forms.* Here are explained briefly some of the forms.

A **plane** containing the point $(x_0\ y_0\ z_0)$ and having a ***normal*** vector (an orthogonal vector to the plane) $\vec{n} = \langle a,\ b,\ c \rangle$ can be represented in ***standard form*** by the equation:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0\ .$$

By regrouping terms, you obtain the ***general form*** of the equation of a plane:

$$Ax + By + Cz + D = 0\ \cdot$$

A **line** parallel to the vector $\vec{v} = \langle a,\ b,\ c \rangle$ and passing through the point $P(x_0,\ y_{0,}\ z_0)$ is represented by the ***parametric equations***:

$x=x_0+ta$

$y=y_0+tb$

$z=z_0+tc.$

If the direction numbers *a*, *b,* and *c* are all nonzero, you can eliminate the parameter t to obtain the ***symmetric equations (***or ***standard form)*** of a line:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

The key concepts are the same as presented in Learning Activity on transformation of coordinates in the plane (Unit 2, LA3), but now defined in three dimensional space.

Coordinate transformation: A *coordinate transformation* is a conversion from one system to another, to describe the same space.

Translation of the coordinate axes: A *translation of the coordinate axes* is the operation of moving the coordinate axes in the coordinate plane to a different position so that the new axes are parallel to the old axes, respectively, and similarly directed.

Rotation of the coordinate axes: A *rotation of the coordinate axes* is the operation of moving the coordinate axes by a turn about their origin as a fixed point.

Quadric surfaces: *Quadric surfaces* (or simply *quadrics*) are surfaces represented by any equation that can be put into the general form

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

where *A*,…*J* are constants.

Source of illustrations

Unit 1, LA1

Figure 1.1.1: Ancient centers of development of geometric knowledge (Source: Drawn by the author using the Adobe Illustrator program)

Figure 1.1.2: Giza pyramids – a magnificent sign of geometric knowledge from the Ancient Egypt
(Source: http://office.microsoft.com/clipart/results.aspx?lc=en-us&Scope=MP&Query=pyramid)

Figure 1.1.3: A portion of the Rind papyrus
(Source: http://en.wikipedia.org/wiki/Rhind_Mathematical_Papyrus)

Unit 1, LA2

Figure 1.2.1: A building being constructed
(Source: Photograph by the author)

Figure 1.2.2: Screen captures of GeoGebra and of WinGeom templates
(Source: Templates captured by the author)

Unit 1, LA3

Figure 1.3.1: Examples of isometric transformations in the Nature and in art
(Source: butterfly: http://keithclan.com/buterfly.gif ; woven basket: photograph by the author; flower: http://i1.treknature.com/photos/3302/foto0010.jpg)

Unit 1, LA4

Figure 1.4.1: An example of application of triangular shape in construction technology
(Source: http://www.cnllanca.cat/images/Grua.jpg)

Unit 1, LA5

Figure 1.5.1: Example of application of circular shapes
Woven plate (Source: Photograph by the author); Bicycle weel (Source: http://home.subnet.at/peter/wp-content/scott_speedster.jpg); Circular clock (Source: http://www.homeclick.com/products/bulova_C4192v1.jpg)

Unit 2, LA1

Figure 2.1.7: A vector grid layer
(Source: http://www.jhlabs.com/maps/doc/vector_grid_layer.html)

Unit 2, LA2

Figure 2.2.7: Straight lines
(Source: http://www.coral-lab.org/~arm1/digital/arlington/arlington-Images/31.jpg ,

http://www.sitemason.com/files/d7HNCg/ncstLRR.JPG , http://www.toweringflat.com/straight%20not.JPG)

Unit 2, LA3

Figure 2.3.7: Example of a transformation of coordinates in the plane
(Source: Drawn by the author using the Adobe Illustrator program)

Unit 2, LA4

Figure 2.4.7: Conic sections
(Source: http://www.andrews.edu/~calkins/math/webtexts/conicsl.jpg)

Unit 4, LA1

Figure 4.1.1: A screen capture of *NonEuclid*

Figure 4.1.2: On a sphere, the sum of the angles of a triangle is not equal to 180°. A sphere is not a Euclidean space
(Source: http://en.wikipedia.org/wiki/Image:Triangle_on_globe.jpg)

Unit 4, LA3

Figure 4.3.1: A complete quadrangle (at left) and a complete quadrilateral (at right).
(Source: http://en.wikipedia.org/wiki/Image:Complete-quads.png)

Figure 4.3.2: Desargues' configuration
(Source: http://mathworld.wolfram.com/DesarguesTheorem.html)

Figure 4.3.3: Pappus' configuration
(Source: http://en.wikipedia.org/wiki/Image:PappusConfiguration.PNG)

Figure 4.3.4: A **perspective machine**
(Source: http://www.dartmouth.edu/~matc/math5.geometry/unit11/durer.gif)

XI. Compulsory Readings

Reading #1

The elements of plane and solid analytic geometry, Candy, Albert L.; Boston, D.C. Heath & co, (1904)

(PDF file in http://www.archive.org/details/elemplanesolidan00candrich)

Abstract/Rationale **:** A historical textbook in geometry. This is referenced throughout the module as a core text.

Reading #2

Analytic geometry for colleges, universities, and technical schools, Nichols, E. W. (Edward West), 1858-1927; Hill, Theodore Preston. Early American mathematics books. CU-BANC; Boston [Mass.]: Leach, Shewell & Sanborn, (c892)

(PDF file in http://www.archive.org/details/analygeomcoll00nichrich)

Abstract/Rationale **:** A historical textbook in geometry. This is referenced throughout the module as a core text.

Reading #3

Projective geometry, Veblen, Oswald, Boston and Young, John W.; Ginn & Co. (1938), pages 1-108;

(PDF file in http://www.archive.org/details/117714799_001)

Abstract/Rationale **:** A historical textbook in geometry. This is referenced throughout the module as a core text.

Reading # 4

Wolfram MathWorld ***(visited 03.11.06)***

Complete reference : http://mathworld.wolfram.com

Abstract : Wolfram MathWorld is a specialised on-line mathematical encyclopedia.

Rationale: Detailed references are provided throughout the text.

Reading # 5

Wikipedia (visited 03.11.06)

Complete reference : http://en.wikipedia.org/wiki

Abstract : Wikipedia is an on-line encyclopedia.

Rationale: Detailed references are provided throughout the text.

XII. Multimedia Resources And Useful Links

Reading # 1

Wolfram MathWorld ***(visited 03.11.06)***

Complete reference : http://mathworld.wolfram.com

Abstract : Wolfram MathWorld is a specialised on-line mathematical encyclopedia.

Rationale: It provides the most detailed references to any mathematical topic. Students should start by using the search facility for the module title. This will find a major article. At any point students should search for key words that they need to understand. The entry should be studied carefully and thoroughly.

Reading # 2

Wikipedia (visited 03.11.06)

Complete reference : http://en.wikipedia.org/wiki

Abstract : Wikipedia is an on-line encyclopedia. It is written by its own readers. It is extremely up-to-date as entries are contunally revised. Also, it has proved to be extremely accurate. The mathematics entries are very detailed.

Rationale: Students should use wikipedia in the same way as MathWorld. However, the entries may be shorter and a little easier to use in the first instance. Thy will, however, not be so detailed.

Reading # 3

MacTutor History of Mathematics (visited 03.11.06)

Complete reference : http://www-history.mcs.standrews.ac.uk/Indexes

Abstract : The MacTutor Archive is the most comprehensive history of mathematics on the internet. The resources are organsied by historical characters and by historical themes.

Rationale: Students should search the MacTutor archive for key words in the topics they are studying (or by the module title itself). It is important to get an overview of where the mathematics being studied fits in to the hostory of mathematics. When the student completes the course and is teaching high school mathematics, the characters in the history of mathematics will bring the subject to life for their students. Particularly, the role of women in the history of mathematics should be studied to help students understand the difficulties women have faced while still making an important contribution.. Equally, the role of the African continent should be studied to share with students in schools: notably the earliest number counting devices (e.g. the Ishango bone) and the role of Egyptian mathematics should be studied.

XIII. Synthesis of the module

After accomplishment of this module you will have learned the basics of Euclidean Geometry both in synthetic and in analytical approach. The first Unit of the Module starts by looking at the historical development of knowledge that the humankind gather along centuries and became later, about 300 BC, the mathematical subject called "Euclidian geometry". Through investigation of your own conjectures on geometric objects and properties you will develop inductive-deductive reasoning - which characterises the way of working in Euclidean geometry.

As you progress in the Units two and three you will treat the Euclidian geometry analytically both in two and in three dimensions. The last unit of the module gives an introduction to non-Euclidean geometries, where you will learn the hierarchy of the groups of transformations in geometry.

It is also expected that you will be able to explore geometric world in a dynamic and interesting way and with lesser expenses of paper and time by using computer software.

The indicated readings of the module on Geometry include exercises and solutions for many of them and help as resources for self study.

XIV. Summative Evaluation

Final Assessment

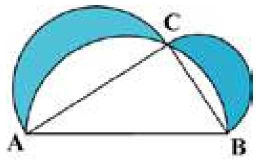
UNIT 1: Revising Euclidean Geometry

Summative assessment (60 minutes)

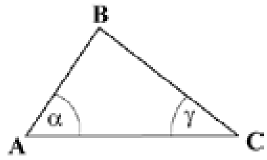
You have finished the Learning Activities on Euclidean geometry Unit. Now it is time to assess what you have learned in the whole Unit 1.

Out of the following five problems solve three only.

1. Right at the beginning of the Unit on Euclidean geometry you revised the history of development of geometrical knowledge in different civilizations along centuries. Write a short review (about 100 words) on how you would introduce your students to Euclidean geometry using its history as starting point.
2. In the figure at the right side, semicircles are drawn with each side of right triangle ABC as diameter. Prove that the sum of the areas of the areas of the two lunar regions is equal to the area of triangular region ABC.



3. Consider the designation of the triangle at the left. Given is the side $a = 4.0\ \text{cm}$, the angle $\alpha = 62°$, and the angle. Find the length of the side AB.



UNIT 2: Plane Analytic Geometry

Summative assessment (60 minutes)

You have finished the Learning Activities on Plane Analytic Geometry. Now it is time to assess what you have learned in the whole Unit 2.

Out of the following seven problems <u>solve four only</u>.

1. Determine the equation, centre, and radius of the circle whose centre lies on the line $3x + 7y + 2 = 0$ and which passes through the two points (6, 2), (8, 0).
2. Prove analytically that any angle inscribed in a semicircle is a right angle.
3. Transform the equation $2x^2 + \sqrt{3}xy + y^2 = 4$ by rotating the coordinate axes through an angle of 30^0. Plot the locus, and show both sets of axes.
4. Find the equation of the parabola whose axis is parallel to the X-axis and which passes through the three points $\left(\frac{3}{2},\ -1\right)$, $\left(0,\ 5\right)$, and $\left(-6,\ -7\right)$.
5. The equation of an ellipse is $x^2 + 4y^2 + 2x - 12y + 5 = 0$. Reduce the equation to the standard form, and determine the coordinates of the centre, vertices, and foci, the lengths of major and minor axis, and each latus rectum, and the eccentricity.
6. The vertices of a hyperbola are the points (0, 3) and (0, -3), and its foci are the points (0, 5) and (0, -5). Find the equation of the hyperbola, the lengths of its transverse and conjugate axes, its eccentricity, and the length of each latus rectum.
7. Discuss the locus of the equation $9x^2 - 4y^2 - 54x + 8y + 113 = 0$.

UNIT 3: Solid Analytic Geometry

Summative assessment (60 minutes)

You have finished the Learning Activities on Solid Analytic Geometry. Now it is time to assess what you have learned in the whole Unit 3.

Out of the following five problems <u>solve three only</u>.

1. Find the equations of the line passing through the point (-3, 2, 1) and perpendicular to the plane 4x+3y-12=0.

2. Find the equation of the plane passing through the point P(2, 1, -3) and parallel to the plane 5x-2y+4z-9=0.

3. Find the acute angle between the plane 5x+4y-z+8=0 and the XY-plane.

4. Transform the equation $x^2 + y^2 - 4z^2 - 2x + 4y + 24z = 31$ of a surface by translating the coordinate axes to the new origin (1, -2, 3). Construct the surface and show both sets of axes.

5. Discuss and construct the surface whose equation is $x^2 + y^2 - 4z = 0$.

UNIT 4: Non-Euclidean Geometry

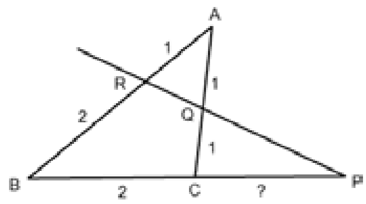
Summative assessment (60 minutes)

You have finished the Learning Activities on Non-Euclidean Geometry. Now it is time to assess what you have learned in the whole Unit 4.

Out of the following five problems solve three only.

1. Write a summary explanation (max. 100 words) with illustration of three key concepts you studied in this Unit on non-Euclidean geometry of plane.

2. In the figure on the right side:
   AR=1, RB=2 ,
   BC=2, CQ=1, QA=1
   Calculate the distance PC
   and justify your solving strategy.



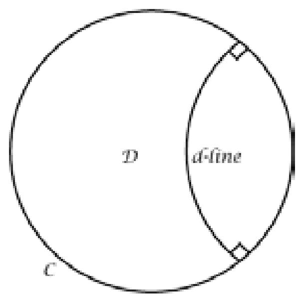
3. Prove one of the following theorems in RP^2:
   a) The Desargues' theorem on two perspective triangles.
   b) The Pappus' theorem on triplets of points.

4. Recalling that in the Poincaré disk model, a d-line is that part of a (Euclidean) generalized circle which approach the boundary line of the disk at a right angle and which lies in interior of the disk.
   Sketch the following parts of generalized circles, and determine which of them d-lines are.

$$d_1 = \{(x, y) : y = 3x\} \cap D$$

$$d_2 = \{(x, y) : 3x + y = 1\} \cap D$$

$$d_3 = \{(x, y) : x^2 + y^2 + 2x + 2y + 1 = 0\} \cap D$$



5. At the beginning of the Unit you learned about the historical foundation of non-Euclidean geometry. Write a short review (about 100 words) on how you would introduce your students to non-Euclidean geometry using its history as starting point.

XV. References

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XVI. Main Author of the Module

Marcos Cherinda was born in Maputo, Mozambique in 1963. Very early, from 1980-1981 he was trained as mathematics and physics teacher for secondary school education at the Faculty of Education of the Eduardo Mondlane University in Maputo. After two years of teaching in the northern Province of Nampula he went to Germany where in 1989 concluded is "Diplom Lehrer" in Mathematics. In 2002 he received his doctorates (Ph.D.) in mathematics education from the University of the Witwatersrand, Johannesburg, South Africa.

Presently he is Assistant Prof. at the Universidade Pedagogica in Mozambique and serves the Research Capacity Building Committee (RCBC) of Southern African Association for Research in Mathematics, Science and Technology Education (SAARMSTE). His special interest is in geometry and in ethnomathematics – a new research area in mathematics (education) in which we works with Professor Paulus Gerdes at the Ethnomathematics Research Centre in Mozambique. He wrote (as co-author with P. Gerdes, 1991) the book Teoremas Famosos da Geometria (Famous Theorems of Geometry) and published several articles about ethnomathematics and mathematics education.