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Fundamentals of Complex Analysis

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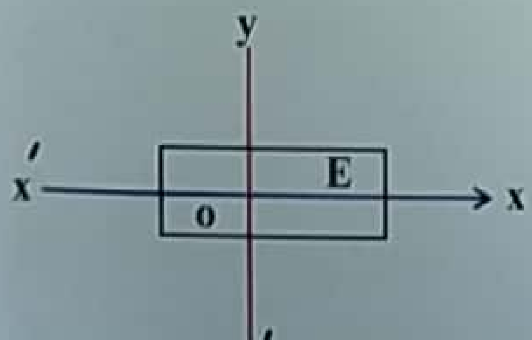

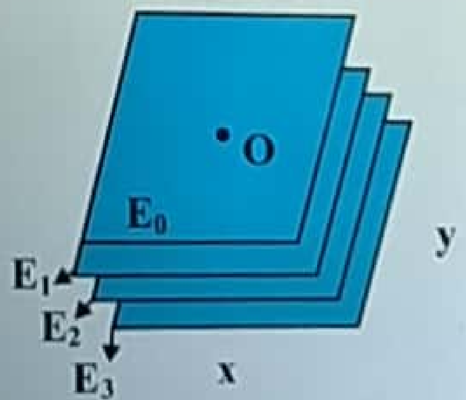

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CHAPTER ZERO

BASIC CONCEPTS AND COMPLEX NUMBERS

We begin by describing some intuitive ideas about sets.

A collection of objects under a rule or a property is called a set. The underlying rule or property will determine, whether the given object belongs to the collection or not. The objects which belong to a set are called its members or elements or occasionally its points.

The theory of sets has been systematically studied only since about 1880, when G. Cantor formulated the basic definitions and axioms.

He remarked that a set must consists of definite , "well defined" elements, that is to say If we have in mind some particular collection then we call it a "set" if, given any object we can decide whether it belongs to the collection we are interested in or whether it does not. This seemingly obvious property is violated by some objects, we can illustrate this by an example due to Bertrand Russell. Suppose we have a set of letters (alphabets), we call it S, then since S is itself an object, either S will belong to the set S (itself) or (the object) S will not belong to (the set) S.

We therefore, see that there are pitfalls in a deep study of set theory. We shall take it as an axiom that sets do exist and they contain well defined objects.

The reader may feel that this is in some sense cheating. Let us look at the problem of language. If we look up a word in a dictionary, its meaning is given using other words.

For instance, take the word "Straight", According to the dictionary. This means "without bend", so we look up "bend" ; a "bend" is defined to be a "curve" and upon looking up "curve", we see that it is "a line no part of which is straight". If we put all these definitions together, we see that we obtain a circular definition , "straight" means "straight".

We adopt the same viewpoint here that a set is what we would intuitively think of as a collection of objects.

Need for Complex Numbers.

The extension to the concept of complex numbers from that of real numbers was first necessitated by the solution of algebraic equations. for example the quadratic equations.

$$x^2 + 4 = 0 \quad \text{or} \quad x^2 + x + 1 = 0 \quad \text{or} \quad x^2 - 2x + 3 = 0$$

do not possess real roots. In order to find the solutions of the above quadratic equations, Euler. (1707 - 1783) was first to introduce the symbol $i = \sqrt{-1}$ i.e. $i^2 = -1$.

Gauss (1777 – 1855) a German mathematician was first to prove in a satisfactory manner that some algebraic equations with real coefficients have complex roots in the form $a \pm i\, b$ (Note that complex and irrational roots always occur in pairs)

Complex number has two parts namely.

(i) Real and (ii) Imaginary.

Real numbers can be further classified as

(a) *Natural Numbers* : as 1, 2, 3, . . . etc

(b) *Positive, Negative Integers* : as 0, $\pm$ 1, $\pm$ 2, . . .

(c) *Rational Numbers* : A number of the form p/q when p and q are integers and q $\neq$ 0 e.g 1/3, 2/5, 2/6, 7/9 etc.

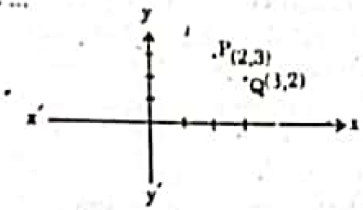
(d) *Irrational Numbers* : A number which can be represented as infinite non-recurring decimals

$$\sqrt{2} = 1.4142 \ldots$$

$$\sqrt{3} = 1.732 \ldots$$

$$\pi = 3.141592 \ldots$$

$$e = 2.716 \ldots$$



Ordered pair : A pair (a, b) such that (a, b) $\neq$ (b , a) unless a = b is called an ordered pair. It can be represented as a point. e.g. P is a point with ordered pair (2, 3) & Q with ordered pair (3, 2)

Complex number : An ordered pair (x , y) which is usually denoted by $Z = (x , y) = x + iy$ is called a complex number, both x and y are real numbers. x is called the real part and y the imaginary part of the complex number Z.

Note : We shall denote the set of all complex numbers by the symbol C.

Operations on Complex Numbers.

Let $Z_1 = (x_1 , y_1) = x_1 + i\, y_1$

$Z_2 = (x_2 , y_2) = x_2 + i\, y_2$.

Addition : The sum of two complex numbers Z_1 & Z_2

is $Z_1 + Z_2 = (x_1 , y_1) + (x_2 , y_2)$

$= (x_1 + x_2 , y_1 + y_2) = (x_1 + x_2) + i\,(y_1 + y_2)$

Subtraction : The difference of two complex numbers

is $Z_1 - Z_2 = (x_1 , y_1) - (x_2 , y_2)$

$= (x_1 - x_2 , y_1 - y_2) = (x_1 - x_2) + i\,(y_1 - y_2)$

Multiplication : The product of two complex numbers

is $Z_1 \times Z_2 = (x_1 , y_1) \times (x_2 , y_2)$

$= (x_1 + i y_1)(x_2 + i y_2) = (x_1 x_2 - y_1 y_2 , x_1 y_2 + x_2 y_1)$

$= (x_1 x_2 - y_1 y_2) + i\,(x_1 y_2 + x_2 y_1)$

Division : The quotient of two complex numbers

is $\dfrac{Z_1}{Z_2} = \dfrac{x_1 + i y_1}{x_2 + i y_2} = \dfrac{(x_1 , y_1)}{(x_2 , y_2)} \dfrac{(x_2 , - y_2)}{(x_2 , - y_2)}$

$\dfrac{Z_1}{Z_2} = \left(\dfrac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} , \dfrac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}\right)$

$= \dfrac{(x_1 x_2 + y_1 y_2) + i\,(x_2 y_1 - x_1 y_2)}{x_2^2 + y_2^2}$

Equality : Two complex numbers are said to be equal if real part of one is equal to real part of the other and imaginary part of one is equal to imaginary part of the other.

i.e. $Z_1 = Z_2 \Rightarrow (x_1, y_1) = (x_2, y_2)$

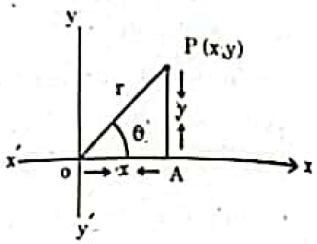
if $x_1 = x_2$ and $y_1 = y_2$

Scalar multiplication : If Z is a complex number and K is a real constant then

$$KZ = K(x, y) = (Kx, Ky) = Kx + iKy$$

Remark : A set of complex numbers forms a field.

Polar Form of a Complex Number



We represent the complex number $Z = x + iy$ by a point P whose Cartesian coordinates are (x, y) referred to rectangular axes ox, oy, usually called the real and imaginary ones respectively. Clearly the polar coordinates of P are (r, θ) where r is the modulus and θ the argument of complex number Z.

From the figure $x = r\cos\theta$, $y = r\sin\theta$

$x^2 + y^2 = r^2$ and $\theta = \tan^{-1}(y/x)$

$r = \sqrt{x^2 + y^2}$ (r is positive)

or $Z = (x, y) = x + iy = r(\cos\theta + i\sin\theta)$.

The plane is called Argand plane or Argand diagram, complex plane or Gaussian plane.

Modulus and argument of a Complex number : As we have discussed previously $|Z| = r = \sqrt{x^2 + y^2}$ is called the modulus of Z. and $\theta = \tan^{-1}(y/x)$ is called the argument or amplitude of Z written as arg Z. or amp(z) for $z \neq 0$

It follows that $|Z| = 0$ if and only if $x = 0$ and $y = 0$ i.e. $|Z|$ is the distance of the point from the origin. The argument θ of a complex number is not unique since if θ be a value of the argument so also is $2n\pi + \theta$, when $n = 0, \pm 1, \pm 2, \ldots$

The value of argument which satisfies the inequality $-\pi \leq \theta \leq \pi$ is called the principal value of the argument.

EXAMPLE 1: Find the locus of Z, where $\arg\left(\frac{Z-1}{Z+1}\right) = \pi/3$

Solution :

Consider $\frac{Z-1}{Z+1} = \frac{x+iy-1}{x+iy+1} = \frac{(x-1)+iy}{(x+1)+iy} \times \frac{(x+1)-iy}{(x+1)-iy}$

$$\frac{Z-1}{Z+1} = \frac{x^2-1+y^2+2iy}{(x+1)^2+y^2}$$

$$\arg\left(\frac{Z-1}{Z+1}\right) = \tan^{-1}\frac{2y}{x^2+y^2-1} = \pi/3$$

$$\therefore \quad \tan(\pi/3) = \sqrt{3} = \frac{2y}{x^2+y^2-1}$$

$$x^2+y^2-1 = \frac{2}{\sqrt{3}}y$$

$$x^2+y^2 - \frac{2}{\sqrt{3}}y - 1 = 0$$

which is the equation of the circle.

EXAMPLE 2 : Find the locus of Z such that

$$\left|\frac{Z-i}{Z+i}\right| \geq 2 \quad \text{or} \quad \left|\frac{Z-i}{Z+i}\right|^2 \geq 4$$

$$(\because |Z|^2 = Z\bar{Z})$$

or $\left(\frac{Z-i}{Z+i}\right)\left(\overline{\frac{Z-i}{Z+i}}\right) \geq 4$

$$\frac{(Z-i)(\bar{Z}+i)}{(Z+i)(\bar{Z}-i)} \geq 4$$

$$\Rightarrow Z\bar{Z} - i(\bar{Z}-Z) + 1 \geq 4Z\bar{Z} + 4i(\bar{Z}-Z) + 4$$

or $3Z\bar{Z} + 5i(\bar{Z}-Z) + 3 \leq 0$

or $3(x^2+y^2) + 5i(-2iy) + 3 \leq 0$

or $3x^2 + 3y^2 + 10y + 3 \leq 0 \qquad (1)$

(1) represents the interior and boundary of the circle.