Solutions Manual of

FUNCTIONAL ANALYSIS

BS 4-Years / M.Sc. Mathematics

Z.R. Bhatti

According to New Syllabus Approved by the University of Punjab, Govt. College University, Lahore, Lahore College University for Women, Kinnaird College, F C College, University of Lahore, UCP, Comsat, FAST, GCU, Faisalabad, Global Institute, Lahore, UMT, University of Sargodha, University of Gujrat, International Islamic University, Islamabad, Peshawar University, University of AJK, Bahauddin Zakariya University, Multan, Quaid-i-Azam University, Islamabad, University of Education, Lahore, Islamia University, Bahawalpur.

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Chapter 1

METRIC SPACES

EXERCISE 1

Short Questions

Q.9 Write answers of the following short questions:

(i) Define a metric.

Sol: Let X be a nonempty set. A function $d: X \times X \to R$ is said to be a metric on X, if for all $x,y,z \in X$, it satisfies the following axioms:

M_1): $d(x,y) \geq 0$ (Non-negativity)

M_2): $d(x,y) = 0 \Leftrightarrow x = y$ (Reflexive property)

M_3): $d(x,y) = d(y,x)$ (Symmetric property)

M_4): $d(x,z) \leq d(x,y) + d(y,z)$ (Triangle inequality)

(ii) For all $x,y \in R$, $d(x,y) = |x-y|$. Verify the triangle inequality.

Sol: $d(x,z) = |x-z|$

$= |x-y+y-z| \leq |x-y| + |y-z|$

$\Rightarrow d(x,z) \leq d(x,y) + d(y,z), \forall x,y,z \in R$

(iii) If $d(x,y) = \dfrac{|x-y|}{1+|x-y|}, \forall x,y \in R$, then prove the triangle inequality.

Sol: $|x-z| = |x-y+y-z| \leq |x-y| + |y-z|$

$\Rightarrow 1+|x-z| \leq 1+|x-y|+|y-z|$ (Adding 1 on both sides)

$$\Rightarrow \frac{1}{1+|x-y|+|y-z|} \leq \frac{1}{1+|x-z|}$$

$$\Rightarrow -\frac{1}{1+|x-z|} \leq -\frac{1}{1+|x-y|+|y-z|}$$

$$\Rightarrow 1-\frac{1}{1+|x-z|}\le 1-\frac{1}{1+|x-y|+|y-z|} \text{ (Adding 1 on both sides)}$$

$$\Rightarrow \frac{1+|x-z|-1}{1+|x-z|}\le \frac{1+|x-y|+|y-z|-1}{1+|x-y|+|y-z|}$$

$$\Rightarrow \frac{|x-z|}{1+|x-z|}\le \frac{|x-y|+|y-z|}{1+|x-y|+|y-z|}$$

$$=\frac{|x-y|}{1+|x-y|+|y-z|}+\frac{|y-z|}{1+|x-y|+|y-z|}$$

$$\le \frac{|x-y|}{1+|x-y|}+\frac{|y-z|}{1+|y-z|}$$

$$\Rightarrow d(x,z)\le d(x,y)+d(y,z), \forall x,y,z\in R$$

(iv) **Let $d(P_1,P_2)=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}$, where $P_1=(x_1,y_1)$, $P_2=(x_2,y_2)$, $x_1,x_2,y_1,y_2\in R$. Show that $d(P_1,P_2)=0\Leftrightarrow P_1=P_2$.**

Sol: $d(P_1,P_2)=0\Leftrightarrow \sqrt{(x_1-x_2)^2+(y_1-y_2)^2}=0$

$$\Leftrightarrow (x_1-x_2)^2+(y_1-y_2)^2=0$$

$$\Leftrightarrow x_1-x_2=0, y_1-y_2=0$$

$$\Leftrightarrow x_1=x_2, y_1=y_2$$

$$\Leftrightarrow (x_1,y_1)=(x_2,y_2)$$

$$\Leftrightarrow P_1=P_2$$

(v) **Show that $d(P_1,P_2)=|x_1-x_2|+|y_1-y_2|$ satisfies the triangle inequality, where $P_1=(x_1,y_1), P_2=(x_2,y_2), x_1,x_2,y_1,y_2\in R$.**

Sol: $d(P_1,P_3)=|x_1-x_3|+|y_1-y_3|$

$$=|x_1-x_2+x_2-x_3|+|y_1-y_2+y_2-y_3|$$

$$\le |x_1-x_2|+|x_2-x_3|+|y_1-y_2|+|y_2-y_3|$$

$$\Rightarrow d(P_1,P_3)\le |x_1-x_2|+|y_1-y_2|+|x_2-x_3|+|y_2-y_3|$$

$$\Rightarrow d(P_1,P_3)\le d(P_1,P_2)+d(P_2,P_3), \forall P_1,P_2,P_3\in R^2$$

(vi) **Let $\underline{x}(x_1,x_2,...,x_n), \underline{y}(y_1,y_2,...,y_n)\in R^n$, then show that $d(\underline{x},\underline{y})=\sum_{i=1}^{n}|x_i-y_i|$ satisfies the triangle inequality.**

Sol: $d(\underline{x},\underline{z})=\sum_{i=1}^{n}|x_i-z_i|=\sum_{i=1}^{n}|x_i-y_i+y_i-z_i|$