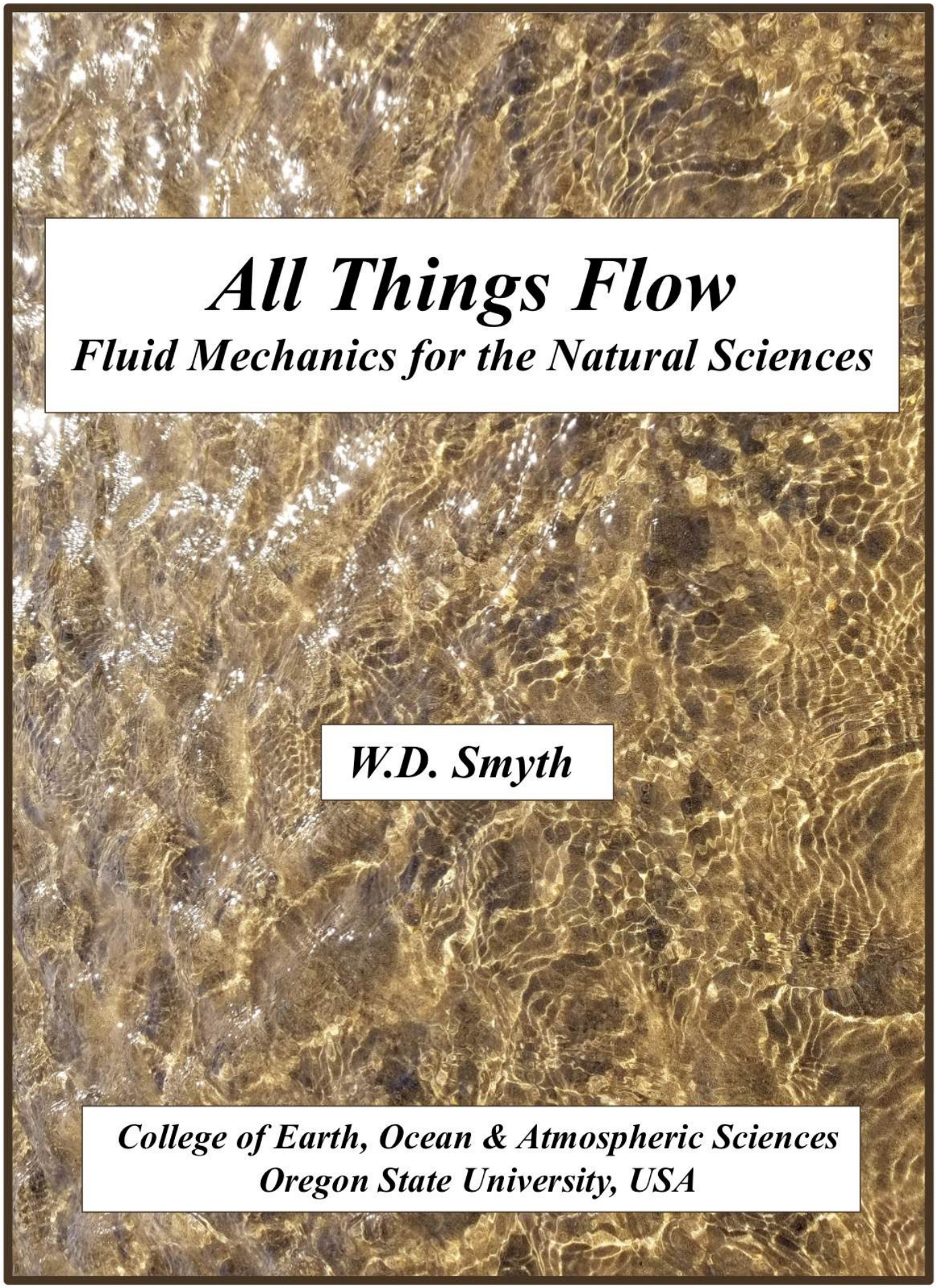

All Things Flow

Fluid Mechanics for the Natural Sciences

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Navigation

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Disclaimer

Every effort has been made to ensure that information contained here is accurate. However, neither the author nor Oregon State University takes legal responsibility for its correctness. The book is provided as is, and all legal responsibility for its use resides solely with the user.

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Thanks

Thanks to the many students who have pointed out errors and other opportunities for improvement. Thanks for the especially incisive inspections by Chih-Lun Liu and Margaret Conley. Thanks to colleagues Alex Kurapov, Jon Nash, Roger Samelson, Jeff Carpenter and Martin Hoecker-Martinez for good ideas and challenging critiques, and to Abby Metzger for formatting advice. Thanks to my Ph.D. advisor Dick Peltier for teaching me how to teach.

Thanks to my wife, Rita Brown, for proofreading the text and designing the cover, to my kids, Nick and Laura, and stepkids, Kevin and Ryan, for inspiration and above all to my folks, Barbara and Bill Smyth, for the second chance.

Photo credits

An unexpected pleasure of this project was learning about the work of the many gifted photographers whose pictures of natural fluid phenomena grace these pages. They join the ranks of the sailors, engineers and whitewater canoeists I have known who understand fluid mechanics on a whole other level.

Cover: An Oregon coastal stream, photo by Ryan Diltz.

Figure 1.1(a): (upper) Vortex photo by Paul Groom Photography, www.paulgroom.com, for Hydro International Ltd, UK, used with permission.
(lower) Tornado photo courtesy of the National Oceanic and Atmospheric Administration (NOAA).

Figure 1.1(b): (upper) Mozambique beach waves photographed by Barry Tuck (former professional photographer, now digital marketing specialist), www.linkedin.com/in/barrytuck/, used with permission.
(lower) Unstable waves in the near-surface boundary layer in the Canadian Arctic by Scott McAuliffe, Gulf of Maine Research Institute, Portland, ME.

Figure 1.1(c): (upper) Kayakers negotiate Itanda Falls on the Nile River, Uganda. This feature is known locally as "The Bad Place" for obvious reasons. Thanks to Sim Davis of Coldhouse Collective for permission to use the photo. Thanks also to Sam Ward of Kayak the Nile, Uganda, kayakthenile.com.
(lower) Sunrise over Cataloochee Ranch, North Carolina (Wikimedia commons).

Figure 4.1: Satellite wind measurements by NOAA; vector field graphic courtesy of Ansley Manke, software specialist.

Figure 5.1: Enhanced photo of Jupiter by National Aeronautics and Space Administration (NASA).

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Figure 8: *The Ninth Wave*, oil painting by Ivan Aivazovsky (Wikimedia commons).

Figure 8.6: (a) Photo by Barry Tuck, https://www.linkedin.com/in/barrytuck/, used with permission. (b) Photo by Aurelie Moulin, Ph.D., Applied Physics Laboratory, University of Washington, Seattle, WA, USA, used with permission.

Figure 9.1: (a) Photo by Jonathan Ball. (b) Photo by Robert Symons, taken 5 March 1950 from a P-38 fighter.

Figure 9.7: River photo by Zachary Collier, Northwest Rafting Company, https://www.nwrafting.com. Used with permission.

Figure 9.9: Colbuco volcano photo by Marcelo Reyes, public domain.

Contents

Chapter 1

Introduction

"No single thing abides, but all things flow." - Heraclitus

1.1 Preface

This book began as lecture notes for an Oregon State University course in fluid mechanics, designed for beginning graduate students in physical oceanography. Because of its fundamental nature, this course is often taken by students outside physical oceanography, e.g., atmospheric science, civil engineering, physics and mathematics.

In later courses, the student will discover esoteric fluid phenomena such as internal waves that propagate through the sky, water phase changes that govern clouds, and planetary rotation effects that control large-scale winds and ocean currents. In contrast, this course concerns phenomena that we have all been familiar with since childhood: flows you see in sinks and bathtubs, in rivers, and at the beach. In this context, we develop the mathematical techniques and scientific reasoning skills needed for higher-level courses and professional research.

Prerequisites are few: basic linear algebra, differential and integral calculus and Newton's laws of motion. As we go along we discover the need for the more advanced tools of tensor analysis.

The science of fluid mechanics is vast. Most books on the topic are concerned with technological applications, e.g., flow through pipes and machinery, that have little relevance in nature. But even among naturally occurring flows we cannot, and should not, try to cover everything. What I have done here is to identify three canonical flow structures that are common in nature (figure 1.1):

- vortices
- waves
- hydraulic jumps

The inner workings of these three phenomena involve all of the basic flow processes, and their study demands a thorough understanding of the theory. The goal, then, is to learn what we need to know to thoroughly understand vortices, waves and hydraulic jumps. Master this, and you will be well prepared to study the much broader range of fluid phenomena found in nature.

The discussion is in three, roughly equal parts:

- Chapters 2-4 are mainly mathematical; we review some advanced aspects of linear algebra and then develop the tools of tensor analysis.
- Chapters 5 and 6 are the crux. The mathematical tools from part 1 are used to develop a theoretical description of flow. The development is thorough, rigorous and (I hope) intuitive.
- In chapters 7-9, we apply the theory to our three common phenomena. Besides exercising the analytical skills we have developed, these examples allow us to test the assumptions that underlie the theory by comparing the results with our everyday experience. In some cases we find that the theory is inadequate, laying the groundwork for further study.

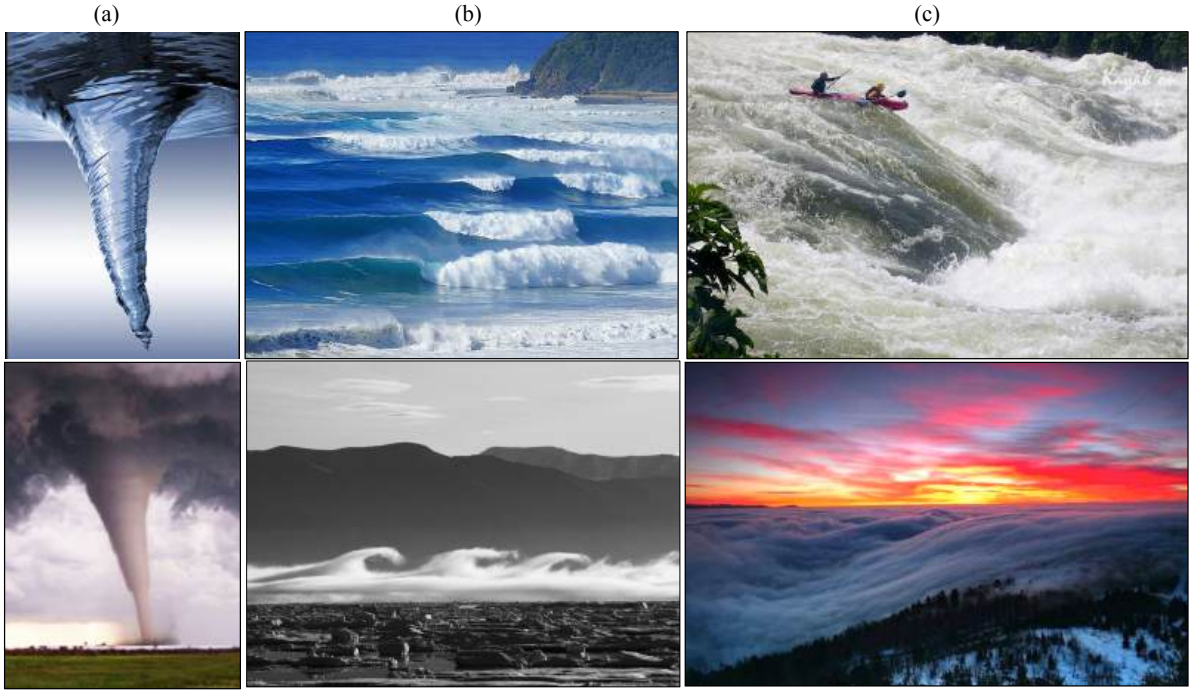


Figure 1.1: Fluid phenomena in water and air: (a) vortices, (b) gravity waves, and (c) hydraulic jumps.

(For photo credits see page ii.)

Homework exercises are included (chapter 12) and are integral to the course. The main text is designed to be covered in 40 hours of lectures. Appendices give auxiliary information and additional topics that can be covered or assigned. It is expected that students will devote an addition 80 hours to homework and independent study. Instructors are invited to contact the author (smythw@oregonstate.edu) for additional materials such as suggested assignments, solutions and possible exam questions.

This book is also intended for self-study, with detailed explanations and frequent exercises to confirm your understanding. If you take this route, feel free to email me with any questions that may arise.

We will not shy away from proofs of the mathematical results we encounter; indeed, I expect the student to demand them. Professionals with graduate degrees are expected not only to know facts, but to understand why they are true and how they came to be known. Be skeptical. To believe something just because a professor said it is to invite error. As a young student, I was taught that the continents do not move and that the planet Mercury always keeps the same face toward the Sun, two statements that we now know are untrue. I absolutely guarantee that, at some point in each student's education, and perhaps in this book, a "fact" will be learned that will turn out to be total hogwash. Be on the lookout.

Smaller errors in logic or mathematics turn up all the time. I have frequently had a math error corrected, or learned a clearer way to explain a difficult concept, thanks to an alert student. That is true in every course but perhaps more so in this one, because every student arrives with an intuitive feel for the fluid phenomena that motivate the analysis. If you have taken this course, thank you. I have learned from you.

There is yet room for improvement - if you spot something wrong or unclear, please let me know and I will fix it.

Bill Smyth, December, 2019
Bill.Smyth@oregonstate.edu

1.2 What is a fluid? The continuum hypothesis

> *"Empty your mind. Be formless, shapeless, like water. When you put water in a bottle, it becomes the bottle; when you put water in a teapot, it becomes the teapot. Water can flow or it can crash. Be water, my friend."* - Bruce Lee

Nearly all materials are either solid or fluid. The distinction depends on what happens if you apply a force that acts to deform (e.g., bend or twist) the material, and then remove the force. A solid will return to its original shape, whereas a fluid will continue to deform. Deformation that continues after the force is removed is called *flow*. Therefore, a fluid is simply a material that flows.

Real fluids are made of molecules far too small for us to observe directly. Those molecules are in constant motion. Besides traveling through space, they spin, wiggle and change their shape. In air at sea level, the mean translational speed is several hundred m/s, close to the speed of sound (not coincidentally). As a result of this motion, molecules occasionally collide. A useful measure of molecule spacing is the average distance (or *mean free path*) between collisions. Typical values are 10^{-9} m in water and 50×10^{-9} m in air.

What we perceive as the "motion" of a fluid is in fact the average motion of many individual molecules. For example, if you hold out your hand to test the wind, the motion you sense is really the average over all of the air molecules striking your hand, i.e., billions of molecules. If you subtract out that average velocity, what's left is the independent motion of each molecule, as well as whatever little dance the molecule may execute as it travels. On the macroscopic level, we experience those residual motions in two ways: as *pressure* and as *heat*.

What we observe as a fluid, then, is indistinguishable from a continuous medium (a *continuum*) which has the properties of *velocity, pressure* and *temperature* at each point in space. Modeling a fluid in this way excuses us from having to account for the behavior of each individual molecule, a dramatic simplification. It also allows us to bring to bear the powerful analytical machinery of the calculus. On the other hand, the continuum view introduces some concepts that don't arise in the familiar physics of solid bodies, e.g., stress, strain and advection. It is for these phenomena that tensor analysis is needed.

Despite the power of the continuum idea, we'll occasionally find it helpful to remember that the fluid is really made up of molecules. A *fluid parcel* is defined as a collection of molecules occupying a simply connected region of space (i.e., a single, continuous blob) that is much bigger than the mean free path. The fluid parcel can move and change its shape, but it is always composed of the same molecules. Another name for a fluid parcel is a *material volume*.

While a fluid parcel is three-dimensional, we can define two-, one- and zero-dimensional analogues. A *material surface* is a two-dimensional surface that is always made up of the same molecules. This is a good model for the surface of a lake or ocean. A *material line* is a one-dimensional curve that bends and twists with the flow but is nonetheless always composed of the same molecules. A *material point*, more commonly called a *fluid particle* is a fluid parcel of infinitesimal size. More specifically, its size is much smaller than the length scales of interest, e.g. the size of a container that bounds the fluid, but still much larger than the mean free path, so that the continuum view makes sense.

Back to Table of Contents.

Chapter 2

Review of elementary linear algebra

We'll first get ourselves into the right frame of mind by reviewing the basic concepts of linear algebra.[1] In the process, we'll introduce notational conventions that will be used in the rest of the book. In particular, we'll define an index notation based on the Einstein summation convention, an extremely convenient device for simplifying lengthy calculations. Later, the student will need to be fluent in this notation.

Most of these definitions and facts should be familiar. Highlighted text indicates concepts that are especially important and/or likely to be new to students with the minimal prerequisite background.

2.1 Scalars and vectors

- A *scalar* a is, in the simplest definition, a single number.
- A *vector* $\vec{v}$ is, in the simplest definition, a list of numbers [2].

$$\text{Column vector } \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_N \end{pmatrix}, \qquad \text{Row vector } \vec{v} = (v_1, v_2, \cdots v_N).$$

  In index notation, v_i is the i^{th} component of $\vec{v}$, for $i = 1, 2, 3, \cdots, N$.
- Vectors can be added by adding corresponding elements: $\vec{w} = \vec{u} + \vec{v}$, or $w_i = u_i + v_i$, for $i = 1, 2, 3, \cdots, N$.
- A vector can be multiplied by a scalar: $\vec{u} = a\vec{v}$, or $u_i = av_i$.
- Dot product (or scalar product or inner product): $\vec{u} \cdot \vec{v} = \sum_{i=1}^{N} u_i v_i$.
- Einstein summation notation: $\vec{u} \cdot \vec{v} = u_i v_i$; summation over the repeated index is implied. The repeated index is called a *dummy index*. The Einstein notation is also called index notation.
- The *magnitude* (or length, or absolute value) of a vector: $|\vec{v}| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_i v_i} = \sqrt{v_i^2}$.
- A *unit vector* has magnitude equal to 1. We will identify unit vectors with a hat rather than a vector symbol, e.g., $\hat{e}$, and $|\hat{e}| = 1$.
- Thinking geometrically, the dot product of $\vec{u}$ and $\vec{v}$ can be written in terms of the magnitudes of the two vectors and the angle between them, θ: $\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$.

[1] *"Be water, my friend."* The quote from Bruce Lee at the beginning of section 1.2 is not entirely facetious. Mathematics is a game of symbol manipulation, like tic-tac-toe or checkers, but for some reason it is extraordinarily effective at representing physical reality. Because of that, we sometimes set aside our intuitive, visceral understanding of our physical environment in order to focus on those symbols. The danger is that we get lost in the game, forgetting that it's just a means to an end. To counteract this tendency, spend time watching water. Meditate on it - its inner dynamics, its ebb and flow. Strive to understand it intuitively, without symbols, numbers or words.

[2] Our definitions of scalars and vectors will become more specific when used in the context of Cartesian coordinates.

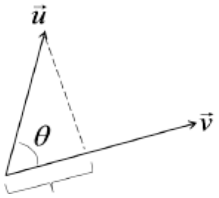


Figure 2.1: Component of $\vec{u}$ in the direction of $\vec{v}$.

- *Orthogonal vectors* are vectors at right angles to each other, i.e., $\theta = \pm\pi/2$, so $\cos\theta = 0$, so $\vec{u}\cdot\vec{v} = 0$.
- A unit vector in the direction of $\vec{v}$ can be defined as $\hat{e}_v = \vec{v}/|\vec{v}|$. $\hat{e}_v$ is parallel to $\vec{v}$, and $|\hat{e}_v| = 1$.
- Projection: The component of $\vec{u}$ in the direction of $\vec{v}$ is given by $\vec{u}\cdot\hat{e}_v = |\vec{u}|\cos\theta$ (see figure 2.1).[3]
- The *cross product* of two vectors gives a third vector: $\vec{u}\times\vec{v} = \vec{w}$.
  - The magnitude $|\vec{w}|$ is given by $|\vec{u}||\vec{v}||\sin\theta|$, where θ is the angle separating the two vectors.
  - The direction of $\vec{w}$ is perpendicular to both $\vec{u}$ and $\vec{v}$ in the sense specified by the right-hand rule: if right-hand fingers curl from $\vec{u}$ to $\vec{v}$, thumb points to $\vec{w}$.
  - The cross product of 3-element vectors $\{u_1,u_2,u_3\}$ and $\{v_1,v_2,v_3\}$ can be expanded as

$$\vec{u}\times\vec{v} = \{u_2v_3 - v_2u_3, -u_1v_3 + v_1u_3, u_1v_2 - v_1u_2\}. \tag{2.1.1}$$

Test your understanding of this section by completing exercise 1.

2.2 Matrices

A *matrix* $\underset{\sim}{A}$ is a two-dimensional table of numbers, e.g.,

$$A_{ij} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix} \tag{2.2.1}$$

A general element of the matrix $\underset{\sim}{A}$ can be written as A_{ij}, where i and j are labels whose values indicate the address. The first index (i, in this case) indicates the *row*; the second (j) indicates the *column*. A matrix can have any size; the size of the example shown in (2.2.1) is denoted 2×3. A matrix can also have more than two dimensions, in which case we call it an *array*.

Below are some basic facts and definitions associated with matrices.

- A vector may be thought of as a matrix with only one column (or only one row).
- Matrices of the same shape can be added by adding the corresponding elements, e.g., $C_{ij} = A_{ij} + B_{ij}$.
- Matrices can also be multiplied: $\underset{\sim}{C} = \underset{\sim}{A}\underset{\sim}{B}$. For example, a pair of 2×2 matrices can be multiplied as follows:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11}+A_{12}B_{21} & A_{11}B_{12}+A_{12}B_{22} \\ A_{21}B_{11}+A_{22}B_{21} & A_{21}B_{12}+A_{22}B_{22} \end{bmatrix}. \tag{2.2.2}$$

- Using index notation, the matrix multiplication $\underset{\sim}{C} = \underset{\sim}{A}\underset{\sim}{B}$ is expressed as

$$C_{ij} = A_{ik}B_{kj}. \tag{2.2.3}$$

  Note that the index k appears twice on the right-hand side. As in our previous definition of the dot product (section 2.1), the repeated index is summed over.

[3] This quantity is more specifically called the *scalar projection*. Some texts also define the *vector projection*, which is the scalar projection times the unit vector $\hat{e}_v$. Here we use only the scalar projection, and we call it "the projection" for simplicity.

- In (2.2.3), k is called a dummy index, while i and j are called free indices. The dummy index is summed over. Note that the dummy index appears only on one side of the equation, whereas each free index appears on both sides. For a matrix equation to be consistent, free indices on the left- and right-hand sides must correspond. Test your understanding of these important distinctions by trying the exercises in (2) and (4).
- Matrix multiplication is not commutative, i.e., $\underset{\sim}{A}\underset{\sim}{B} \neq \underset{\sim}{B}\underset{\sim}{A}$ except in special cases. In index notation, however, $A_{ik}B_{kj}$ and $B_{kj}A_{ik}$ are the same thing. This is because A_{ik}, the i,k element of the matrix A, is just a number, like 3 or 17, and two numbers can be multiplied in either order. The results are summed over the repeated index k regardless of where k appears.
- The standard form for matrix multiplication is such that the dummy indices are *adjacent*, as in (2.2.3).

2.2.1 Matrix-vector multiplication

A matrix and a vector can be multiplied in two ways, both of which are special cases of matrix multiplication.

- Right multiplication :

$$\vec{u} = \underset{\sim}{A}\vec{v}, \quad \text{or} \quad u_i = A_{ij}v_j. \tag{2.2.4}$$

  Note the sum is on the *second*, or right index of $\underset{\sim}{A}$.

- Left multiplication:

$$\vec{u} = \vec{v}\underset{\sim}{A}, \quad \text{or} \quad u_j = v_iA_{ij}.$$

  The sum is now on the *first*, or left, index of $\underset{\sim}{A}$.

As in other cases of matrix multiplication, $\vec{v}\underset{\sim}{A} \neq \underset{\sim}{A}\vec{v}$, but $v_iA_{ij} = A_{ij}v_i$.

2.2.2 Properties of square matrices

A *square matrix* has the same number of elements in each direction, e.g., the matrices appearing in (2.2.2). **From here on, we will restrict our discussion to square matrices.**

- The *main diagonal* of a square matrix is the collection of elements for which both indices are the same. In figure 2.2, for example, the main diagonal is outlined in red.
- The *transpose* of a matrix is indicated by a superscript "T", i.e., the transpose of $\underset{\sim}{A}$ is $\underset{\sim}{A}^T$. To find the transpose, flip the matrix about the main diagonal: $A^T_{ij} = A_{ji}$ as shown in figure 2.2.

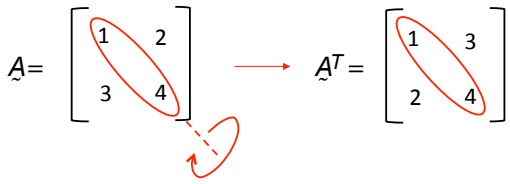


Figure 2.2: Taking the transpose of a 2×2 matrix.

- In a *diagonal* matrix, only the elements on the main diagonal are nonzero, i.e., $A_{ij} = 0$ unless $i = j$.
- The *trace* is the sum of the elements on the main diagonal, i.e., $\mathrm{Tr}(\underset{\sim}{A}) = A_{ii}$. For the example shown in figure 2.2, the trace is equal to 5. Note that $Tr(\underset{\sim}{A}^T) = Tr(\underset{\sim}{A})$.
- For a *symmetric* matrix, $\underset{\sim}{A}^T = \underset{\sim}{A}$ or $A_{ji} = A_{ij}$. We say that a symmetric matrix is *invariant under transposition*. Note that a diagonal matrix is automatically symmetric.
- For an *antisymmetric* matrix, transposition changes the sign: $\underset{\sim}{A}^T = -\underset{\sim}{A}$ or $A_{ji} = -A_{ij}$.

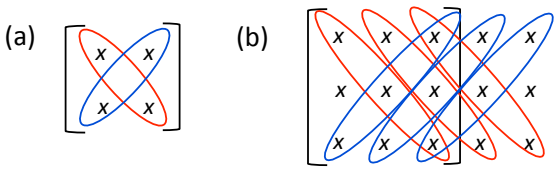


Figure 2.3: Groups of elements to be multiplied when computing the determinant. Products of terms outlined in red are added; products in blue are subtracted. In the 3×3 case, the first two columns are repeated outside the matrix.

- *Identity* matrix:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \qquad \text{e.g.} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{2.2.5}$$

Multiplying a matrix by $\underset{\sim}{\delta}$ leaves the matrix unchanged: $\underset{\sim}{\delta}\underset{\sim}{A} = \underset{\sim}{A}$. The same is true of a vector: $\underset{\sim}{\delta}\vec{v} = \vec{v}$. The identity matrix is often called I.

Test your understanding:

- Is an identity matrix diagonal?
- What is the trace of a 2×2 identity matrix? A 3×3 identity matrix?
- Is an identity matrix symmetric, antisymmetric, both, or neither?

- The *determinant* of $\underset{\sim}{A}$, denoted $\det(\underset{\sim}{A})$, is a scalar property that will be described in detail later (section D.3.3). For the general 2×2 matrix

$$\underset{\sim}{A} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \tag{2.2.6}$$

the determinant is

$$\det(\underset{\sim}{A}) = A_{11}A_{22} - A_{21}A_{12}.$$

The pattern illustrated in figure 2.3a provides a simple mnemonic. For the general 3×3 matrix,

$$\underset{\sim}{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix},$$

the determinant is obtained by multiplying diagonals, then adding the products of diagonals oriented downward to the right (red in figure 2.3b) and subtracting the products of diagonals oriented upward to the right (blue in figure 2.3b):

$$\det \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32} - A_{31}A_{22}A_{13} - A_{32}A_{23}A_{11} - A_{33}A_{21}A_{12}.$$

The determinant can also be evaluated via expansion of cofactors along a row or column. In this example we expand along the top row:

$$\det \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} = A_{11} \times \det \begin{bmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{bmatrix} - A_{12} \times \det \begin{bmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{bmatrix} + A_{13} \times \det \begin{bmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix}$$

- Matrix *inverse*: If $\underset{\sim}{A}\underset{\sim}{B} = \underset{\sim}{\delta}$ and $\underset{\sim}{B}\underset{\sim}{A} = \underset{\sim}{\delta}$ then $\underset{\sim}{B} = \underset{\sim}{A}^{-1}$ and $\underset{\sim}{A} = \underset{\sim}{B}^{-1}$. For the 2×2 matrix (2.2.6), the inverse is

$$\underset{\sim}{A}^{-1} = \frac{1}{\det(\underset{\sim}{A})} \begin{bmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{bmatrix}.$$

Exercise: Multiply this by $\underset{\sim}{A}$ as given in (2.2.6) and confirm that you get the identity matrix. Unfortunately, there is no simple formula for the 3×3 case.

- *Orthogonal* matrix: $\underset{\sim}{A}^{-1} = \underset{\sim}{A}^{T}$.
- *Singular* matrix: $\det(\underset{\sim}{A}) = 0$. A singular matrix has no inverse.

Test your understanding of this section by completing exercises 2, 3, 4, 5 and 6. (Exercise 7 is outside the main discussion and can be done any time.)

2.3 Systems of linear equations

- $\vec{u} = \underset{\sim}{A}\vec{v}$, or $\underset{\sim}{A}\vec{v} = \vec{u}$, represents a set of linear equations that can (usually) be solved for $\vec{v}$. If the matrix is square and $\det(\underset{\sim}{A}) \neq 0$, then $\vec{v} = \underset{\sim}{A}^{-1}\vec{u}$.
- A *homogeneous* set of equations has the form $\underset{\sim}{A}\vec{v} = 0$, i.e., it has $\vec{v} = 0$ as a solution (like homogeneous differential equations). In this case, nonzero solutions for $\vec{v}$ exist only if $\det(\underset{\sim}{A}) = 0$.

2.4 Eigenvalues and eigenvectors

Every $M \times M$ square matrix $\underset{\sim}{A}$ has M *eigenvalue-eigenvector pairs*:

$$\underset{\sim}{A}\vec{v}^{(m)} = \lambda^{(m)}\vec{v}^{(m)} \qquad \text{(no sum on } m\text{)}. \tag{2.4.1}$$

The superscript (m) is the *"mode number"* an index running from 1 to M that labels the eigenvalue-eigenvector pairs. To find $\lambda^{(m)}$ and $\vec{v}^{(m)}$, $m = 1, 2, \ldots, M$, rewrite (2.4.1) as:

$$\underset{\sim}{A}\vec{v}^{(m)} = \lambda^{(m)}\underset{\sim}{\delta}\,\vec{v}^{(m)} \qquad \Rightarrow \qquad \left[\underset{\sim}{A} - \lambda^{(m)}\underset{\sim}{\delta}\right]\vec{v}^{(m)} = 0, \tag{2.4.2}$$

where $\underset{\sim}{\delta}$ is the $M \times M$ identity matrix. This is a homogeneous set of equations, and can therefore only have a nontrivial solution if the determinant is zero:

$$\det(\underset{\sim}{A} - \lambda^{(m)}\underset{\sim}{\delta}) = 0. \tag{2.4.3}$$

Because $\underset{\sim}{A}$ is an $M \times M$ matrix, (2.4.3) can be written as an M^{th} order polynomial for the eigenvalues λ, referred to as the *characteristic polynomial*. It has M solutions for λ. These may be complex, and they are not necessarily distinct. Each particular eigenvalue solution $\lambda^{(m)}$ has an associated non-zero eigenvector $\vec{v}^{(m)}$ that is obtained by substituting the value of $\lambda^{(m)}$ into (2.4.2) and solving the resulting set of M equations for the M components of $\vec{v}^{(m)}$.

An important property of (2.4.2) is that both sides of the equation can be multiplied by an arbitrary constant c and the equation still holds. So, if $\vec{v}^{(m)}$ is an eigenvector of $\underset{\sim}{A}$ with eigenvalue $\lambda^{(m)}$, then $c\vec{v}^{(m)}$ is also an eigenvector of A with the same eigenvalue $\lambda^{(m)}$.

The eigenvectors are therefore defined only to within an arbitrary scalar multiple. In many cases, we choose the eigenvectors to be unit vectors by making the appropriate choice for c.

Interesting tidbits:

- The sum of the eigenvalues equals the trace.
- The product of the eigenvalues equals the determinant.

You will prove these in due course.

Back to Table of Contents.

Chapter 3

Cartesian vectors and tensors

3.1 Measuring space with Cartesian coordinates

A convenient way to measure space is to assign to each point a label consisting of three numbers, one for each dimension. We begin by choosing a single point to serve as the origin. The location of any point can now be quantified by its position vector, the vector extending from the origin to the point in question. We'll name this position vector $\vec{x}$.

Next, choose three basis vectors, each beginning at the origin and extending away for some distance. The only real restriction on this choice is that the vectors must not all lie in the same plane. It will be easiest if we choose the basis vectors to be unit vectors, in which case we'll name them $\hat{e}^{(1)}$, $\hat{e}^{(2)}$ and $\hat{e}^{(3)}$. It's also easiest if we choose the vectors to be mutually orthogonal:

$$\boxed{\hat{e}^{(i)} \cdot \hat{e}^{(j)} = \delta_{ij}.} \tag{3.1.1}$$

Finally, it's easiest if we choose the system to be right-handed, meaning that if you take your right hand and curve the fingers in the direction from $\hat{e}^{(1)}$ to $\hat{e}^{(2)}$, your thumb will point in the direction of $\hat{e}^{(3)}$ (figure 3.1). Interchanging any two basis vectors renders the coordinate system *left-handed*. This "handedness'' property is also referred to as *parity*.

Every position vector $\vec{x}$ can be expressed as a linear combination of the basis vectors:

$$\vec{x} \;=\; x_1\hat{e}^{(1)} + x_2\hat{e}^{(2)} + x_3\hat{e}^{(3)} \;=\; x_i\hat{e}^{(i)}. \tag{3.1.2}$$

The component x_k can be isolated by projecting $\vec{x}$ onto $\hat{e}^{(k)}$:

$$\vec{x}\cdot\hat{e}^{(k)} \;=\; x_i\hat{e}^{(i)}\cdot\hat{e}^{(k)} \;=\; x_i\delta_{ik} \;=\; x_k,$$

therefore

$$\boxed{x_k \;=\; \vec{x}\cdot\hat{e}^{(k)}.} \tag{3.1.3}$$

The components of the position vector are called the Cartesian coordinates of the point.

A note on terminology

A position vector in a Cartesian coordinate system can be expressed as $\{x_1, x_2, x_3\}$ or, equivalently, as $\{x, y, z\}$. The numerical index notation is useful in a mathematical context, e.g., when using the summation convention. In physical applications, it is traditional to use the separate letter labels $\{x, y, z\}$. In the geosciences, for example, we commonly denote the eastward direction as x, northward as y and upward as z. (Note that this coordinate system is right-handed.) Similarly, the basis vectors $\{\hat{e}^{(1)}, \hat{e}^{(2)}, \hat{e}^{(3)}\}$ are written as $\{\hat{e}^{(x)}, \hat{e}^{(y)}, \hat{e}^{(z)}\}$. As we transition from mathematical concepts to real-world applications, we will move freely between these two labeling conventions.

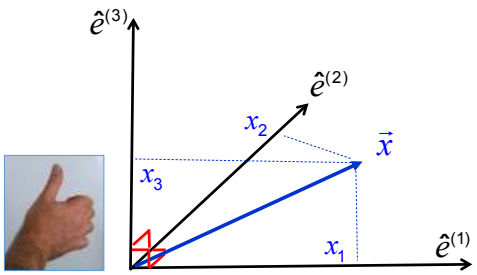


Figure 3.1: A position vector $\vec{x}$ in a right-handed Cartesian coordinate system. The x_i are the components of $\vec{x}$ or, equivalently, the coefficients for the linear combination (3.1.2).

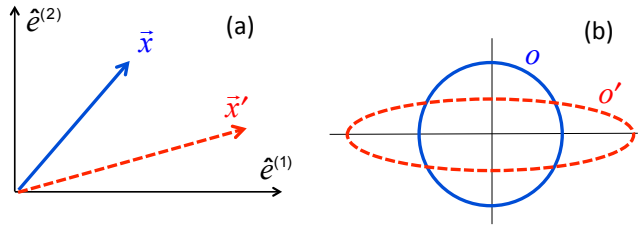


Figure 3.2: Transformation by the matrix (3.1.5). (a) Transformation of an arbitrary vector. (b) Transformation of a unit circle.

Cartesian geometry on the plane

Advanced concepts are often grasped most easily if we restrict our attention to a two-dimensional plane such that one of the three coordinates is constant, and can therefore be ignored. For example, on the plane $x_3 = 0$, the position vector at any point can be written as $\vec{x} = x_1\hat{e}^{(1)} + x_2\hat{e}^{(2)}$, or $\{x, y\}$. Two-dimensional geometry is also a useful first approximation for many natural flows, e.g., large-scale motions in the Earth's atmosphere.

3.1.1 Matrices as geometrical transformations

When we multiply a matrix $\underset{\sim}{A}$ onto a position vector $\vec{x}$, we transform it into another position vector $\vec{x}'$ with different length and direction (except in special cases). In other words, the point the vector points to moves to a new location. Often, but not always, we can reverse the transformation and recover the original vector. The matrix needed to accomplish this reverse transformation is $\underset{\sim}{A}^{-1}$. To fix these ideas, we'll now consider a few very simple, two-dimensional examples.

Example 1: Consider the 2×2 identity matrix $\underset{\sim}{\delta}$. Multiplying a vector by $\underset{\sim}{\delta}$ is a *null transformation*, i.e., $\vec{x}' = \vec{x}$; the vector is transformed into itself. Not very interesting.

Example 2: Now consider a slightly less trivial example: the identity matrix multiplied by a scalar, say, 2:

$$\underset{\sim}{A} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}. \tag{3.1.4}$$

Multiplying $\underset{\sim}{A}$ onto some arbitrary vector $\vec{x}$ has the effect of doubling its length but leaving its direction unchanged: $\vec{x}' = 2\vec{x}$.

Is this transformation *reversible*? Suppose I gave you the transformed vector $\vec{x}'$ and the transformation $\underset{\sim}{A}$ that produced it, then asked you to deduce the original vector $\vec{x}$. This would be very easy; you'd simply divide $\vec{x}'$ by 2 to get $\vec{x}$.

Example 3:

Next we'll look at a more interesting example:

$$\underset{\sim}{A} = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix} \tag{3.1.5}$$

Multiplication of $\underset{\sim}{A}$ onto any vector doubles the vector's first component while halving its second (figure 3.2a):

$$\text{If } \vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \text{then } \vec{x}' = \underset{\sim}{A}\vec{x} = \begin{bmatrix} 2x \\ y/2 \end{bmatrix}. \tag{3.1.6}$$

In general, the transformation changes both the length and the direction of the vector. There are, however, exceptions that you can now confirm for yourself:

- Show that the length of a vector is not changed *if* its y component is $\pm$ twice its x component.
- Identify a particular vector $\vec{x}$ whose direction is not changed in this transformation. Now identify another one.

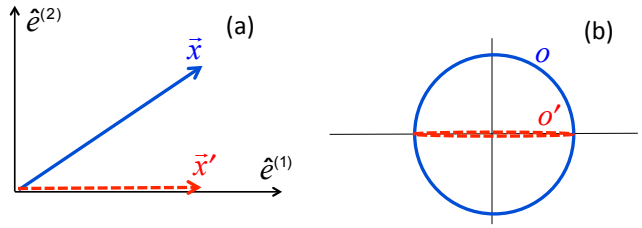


Figure 3.3: Transformation by the matrix (3.1.8). (a) Transformation of an arbitrary vector. (b) Transformation of a unit circle.

- Compute the eigenvectors of $\underset{\sim}{A}$. You should find that they are simply the basis vectors $\hat{e}^{(x)}$ and $\hat{e}^{(y)}$. In general, any multiple of either of these unit vectors is an eigenvector. Note that this class of vectors is also the class of vectors whose direction is unchanged! [1]

Because a matrix transformation can be applied to *any* position vector, it can be thought of as affecting any geometrical shape, or indeed *all of space*. A simple way to depict the general effect of the matrix is to sketch its effect on the unit circle centered at the origin. In this case the circle is transformed into an ellipse (figure 3.2b), showing that the general effect of the matrix is to compress things vertically and expand them horizontally.

Is this transformation reversible? Certainly: just halve the x component and double the y component. As an exercise, write down a matrix that accomplishes this reverse transformation, and show that it is the inverse of $\underset{\sim}{A}$.

Example 4: Now consider

$$\underset{\sim}{A} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{3.1.7}$$

This transformation, applied to a vector, leaves the x component unchanged but changes the y component to zero. The vector is therefore projected onto the x axis (figure 3.3a). Applied to a general shape, the transformation squashes it flat (figure 3.3b). Does this transformation change the length and direction of the vector? Yes, with certain exceptions which the reader can deduce.

Is the transformation reversible? No! All information about vertical structure is lost. Verify that the determinant $|\underset{\sim}{A}|$ is zero, i.e., that the matrix has no inverse.

Test your understanding by completing exercises (8) and (9).

Example 5: As a final example, consider

$$\underset{\sim}{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{3.1.8}$$

The matrix transforms

$$\begin{bmatrix} x \\ y \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} y \\ -x \end{bmatrix}.$$

The length of the vector is unchanged (check), and the transformed vector is orthogonal to the original vector (also check). In other words, the vector has been rotated through a 90^o angle. In fact, it is generally true that an antisymmetric matrix rotates a vector through 90^o. To see this, apply a general antisymmetric matrix $\underset{\sim}{A}$ to an arbitrary vector $\vec{u}$, then dot the result with $\vec{u}$:

$$\begin{aligned} (\underset{\sim}{A}\vec{u}) \cdot \vec{u} = A_{ij}u_j u_i &= -A_{ji}u_j u_i && \text{(using antisymmetry)} \\ &= -A_{ij}u_i u_j && \text{(relabeling } i \leftrightarrow j\text{)} \\ &= -A_{ij}u_j u_i. && \text{(reordering)} \end{aligned}$$

The result is equal to its own negative and must therefore be zero. We conclude that the transformed vector is orthogonal to the original vector. [2]

[1] Generalizing from this example, we could propose that an eigenvector is a vector which, when multiplied by the matrix, maintains its direction. There is a common exception to this, though, and that is when the eigenvalues or eigenvectors are complex. In that case, geometric interpretation of the eigenvectors becomes, well, complex.

[2] Be sure you understand this proof; there'll be many others like it.

3.1.2 Coordinate rotations and the principle of relativity

It is an essential principle of science that the reality you and I experience exists independently of us. If this were not true, science would be a waste of time, because an explanation of your reality would not work on mine.

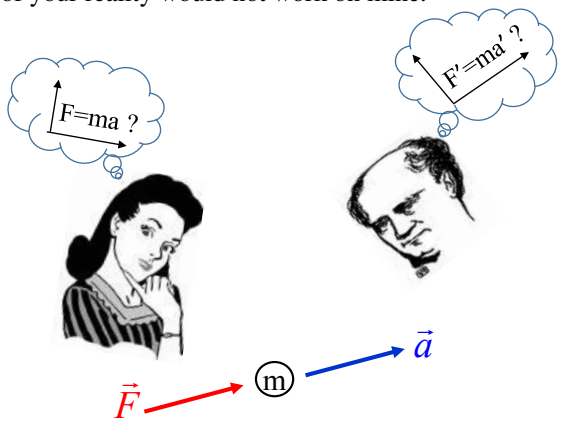


Figure 3.4: Two observers evaluating Newton's second law.

Is there any way to prove this principle? No, but we can test it by comparing our experiences of reality. For example, if you measure a force and get the vector $\vec{F}$, and if I measure the same force, I expect to get the same result. Now, we know from the outset that this will not be quite true, because we observe from different perspectives. For example, if you see $\vec{F}$ pointing to the right, I may see it pointing to the left, depending on where I'm standing. For a meaningful comparison, we must take account of these expected differences in perspective. Let's call the force vector that I measure $\vec{F}'$. The components of this vector will have different values than the ones you measured ($\vec{F}$), but if we can somehow translate from your perspective to mine, the vectors should be the same. We expect this because the force is a physically real entity that exists independently of you and me.

Now imagine that this force acts on a mass m and we each measure the resulting acceleration: you get $\vec{a}$ and I get $\vec{a}'$. As with the force, the components of $\vec{a}'$ will differ from those of $\vec{a}$, but if we correct for the difference in perspective we should find that $\vec{a}'$ and $\vec{a}$ represent the same acceleration.

Now suppose further that we are making these measurements in order to test Newton's second law of motion. If that law is valid in your reference frame, then the force and acceleration you measure should be related by $\vec{F} = m\vec{a}$. If the law is also valid in my reference frame then I should get $\vec{F}' = m\vec{a}'$. This leads us to the relativity principle, attributed to Galileo: The laws of physics are the same for all observers. Just as with other kinds of laws, if a law of physics works for you but not for me, then it is not a very good law.

Our goal here is to deduce physical laws that describe the motion of fluids. The selection of possible *hypotheses* (candidate laws) that we could imagine is infinite. How do we determine which one is valid? To begin with, we can save ourselves a great deal of trouble if we consider only hypotheses that are consistent with the relativity principle. To do this, we must first have a mathematical language for translating between different reference frames. In particular, we need to be able to predict the effect of a coordinate rotation on the components of any vector, or of any other quantity we may want to work with.

3.1.3 The rotation matrix

Suppose that, having defined an orthogonal, right-handed set of basis vectors $\hat{e}^{(i)}, i = 1,2,3$, we switch to a new set, $\hat{e}'^{(i)}$ (figure 3.5). Each of the new basis vectors can be written as a linear combination of the original basis vectors in accordance with (3.1.2). For example, $\hat{e}'^{(1)}$ can be written as

$$\hat{e}'^{(1)} = C_{i1}\hat{e}^{(i)}.$$

This is analogous to (3.1.2), but in this case the coefficients of the linear combination have been written as one column of a 3x3 matrix $\underset{\sim}{C}$. Doing the same with the other two basis vectors ($\hat{e}'^{(2)}$ and $\hat{e}'^{(3)}$) yields the other two columns:

$$\boxed{\hat{e}'^{(j)} = C_{ij}\hat{e}^{(i)}, \quad j = 1,2,3.} \tag{3.1.9}$$

To rephrase (3.1.9), $\underset{\sim}{C}$ is composed of the rotated basis vectors, written as column vectors and set side-by-side:

$$\boxed{\underset{\sim}{C} = [\ \hat{e}'^{(1)}\ \hat{e}'^{(2)}\ \hat{e}'^{(3)}\ ].} \tag{3.1.10}$$

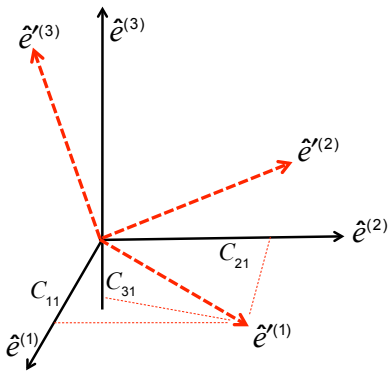


Figure 3.5: Original (black, solid) and rotated (red, dashed) Cartesian coordinate systems. Thin dashed lines show the components of $\hat{e}'^{(1)}$ in the original coordinates, which also make up the first column of the rotation matrix C.

Now suppose that the new basis vectors have been obtained simply by *rotating* the original basis vectors about some axis.[3] In this case, both the lengths of the basis vectors and the angles between them should remain the same. In other words, the new basis vectors, like the old ones, are orthogonal unit vectors: $\hat{e}'^{(i)} \cdot \hat{e}'^{(j)} = \delta_{ij}$. This requirement restricts the forms that $\underset{\sim}{C}$ can take. Substituting (3.1.9), we have:

$$\hat{e}'^{(i)} \cdot \hat{e}'^{(j)} = C_{ki}\hat{e}^{(k)} \cdot C_{lj}\hat{e}^{(l)} = C_{ki}C_{lj}\hat{e}^{(k)} \cdot \hat{e}^{(l)} = C_{ki}C_{lj}\delta_{kl} = C_{li}C_{lj} = C^T_{il}C_{lj} = \delta_{ij}.$$

The final equality is equivalent to

$$\underset{\sim}{C}^T = \underset{\sim}{C}^{-1}, \tag{3.1.11}$$

i.e., $\underset{\sim}{C}$ is an *orthogonal matrix*. [4]

Recall also that the original basis vectors form a right-handed set. Do the rotated basis vectors share this property? In section D.3.3, it is shown that the determinant of an orthogonal matrix equals ± 1. Moreover, if $|\underset{\sim}{C}| = -1$, then $\underset{\sim}{C}$ represents an *improper* rotation: the coordinate system undergoes both a rotation and a parity switch, from right-handed to left-handed. This is not usually what we want. So, if $\underset{\sim}{C}$ is orthogonal and its determinant equals +1, we say that $\underset{\sim}{C}$ represents a proper rotation.

To reverse a coordinate rotation, we simply use the inverse of the rotation matrix, $\underset{\sim}{C}^{-1}$, or $\underset{\sim}{C}^T$:

$$\boxed{\hat{e}^{(j)} = C_{ji}\hat{e}'^{(i)}, \quad j = 1,2,3.} \tag{3.1.12}$$

Comparing (3.1.12) with (3.1.9), we see that, on the right-hand side, the dummy index is in the first position for the forward rotation and in the second position for the reverse rotation.

Example 6: Suppose we want to rotate the coordinate frame around the $\hat{e}^{(1)}$ axis by an angle ϕ. Referring to figure 3.6, we can express the rotated basis vectors using simple trigonometry:

$$\hat{e}'^{(1)} = \hat{e}^{(1)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}; \quad \hat{e}'^{(2)} = \begin{bmatrix} 0 \\ \cos\phi \\ \sin\phi \end{bmatrix}; \quad \hat{e}'^{(3)} = \begin{bmatrix} 0 \\ -\sin\phi \\ \cos\phi \end{bmatrix};$$

Now, in accordance with (3.1.10), we simply place these column vectors side-by-side to form the rotation matrix:

$$\underset{\sim}{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix}. \tag{3.1.13}$$

Looking closely at (3.1.13), convince yourself that the following are properties of $\underset{\sim}{C}$:

- $\underset{\sim}{C}^T\underset{\sim}{C} = \underset{\sim}{\delta}$, i.e., $\underset{\sim}{C}$ is orthogonal.
- $|\underset{\sim}{C}| = 1$, i.e., $\underset{\sim}{C}$ represents a proper rotation.

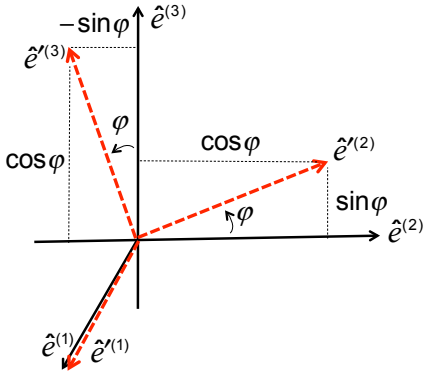


Figure 3.6: Original (black, solid) and rotated (red, dashed) basis vectors after a rotation by an angle φ about $\hat{e}^{(x)}$. Dashed lines show the components of the rotated basis vectors in the original coordinates.

- Changing the sign of ϕ produces the transpose (or, equivalently, the inverse) of $\underset{\sim}{C}$, as you would expect.

Test your understanding by completing exercise 10.

3.2 Vectors and scalars

3.2.1 The position vector in a rotated coordinate system

The position vector $\vec{x}$ that labels a point in space (e.g., the blue vector in figure 3.1) is a physical entity that exists *independently of any observer* and of any coordinate frame used to measure it. If you use the basis vectors $\hat{e}$ and I use $\hat{e}'$, then $\vec{x}$ can be expressed as a linear combination of either set:

$$\vec{x} \;=\; x_i\,\hat{e}^{(i)} \;=\; x'_i\,\hat{e}'^{(i)}. \tag{3.2.1}$$

Now suppose that our coordinate frames are related by a rotation matrix $\underset{\sim}{C}$. If we know the components x_i that you measure and the rotation matrix $\underset{\sim}{C}$, can we predict the components x'_i that I will measure? We'll do this by solving (3.2.1) for the components of the vector $\vec{x}'$. Start with the second equality of (3.2.1):

$$x'_i\,\hat{e}'^{(i)} \;=\; x_i\,\hat{e}^{(i)}.$$

How do we solve this equality for x'_i? If $\hat{e}'^{(i)}$ were a scalar, we'd simply divide it out of both sides, but division by a vector makes no sense. Instead, we dot both sides with $\hat{e}'^{(j)}$:

$$x'_i\,\hat{e}'^{(i)}\cdot\hat{e}'^{(j)} \;=\; x_i\,\hat{e}^{(i)}\cdot\hat{e}'^{(j)}.$$

Now use (3.1.9) on the right-hand side[5],

$$\begin{aligned} x'_i\,\underbrace{\hat{e}'^{(i)}\cdot\hat{e}'^{(j)}}_{=\delta_{ij}} &= x_i\,\underbrace{\hat{e}^{(i)}\cdot\hat{e}^{(k)}}_{=\delta_{ik}}C_{kj} \\ x'_i\,\delta_{ij} &= x_i\,\delta_{ik}C_{kj}. \end{aligned}$$

Finally, multiplying out the deltas, we have

$$\boxed{x'_j \;=\; x_i\,C_{ij}.} \tag{3.2.2}$$

This result is similar to the transformation rule (3.1.9) for the basis vectors, though its meaning is entirely different. In each case, the dummy index is in the first position of the rotation matrix. The relation (3.2.2) can also be written in vector notation:

$$\boxed{\vec{x}' = \vec{x}\,\underset{\sim}{C}, \quad \text{or} \quad \vec{x}' = \underset{\sim}{C}^T\vec{x}.} \tag{3.2.3}$$

Now consider the reverse transformation, i.e., my measurement into yours. Multiplying both sides of (3.2.2) by C_{kj}, we have

$$x'_j\,C_{kj} \;=\; x_i\,C_{ij}C_{kj} \;=\; x_i\,C_{ij}C^T_{jk} \;=\; x_i\,\delta_{ik} \;=\; x_k,$$

[3] The axis needn't be a coordinate axis; any line will do.

[4] We now see why a matrix with this property is called "orthogonal"; orthogonal vectors remain orthogonal after transformation by such a matrix.

[5] Compare with (3.1.9) and note that the dummy index i has been relabeled as k to avoid conflict with the dummy index i that is already in use. Also, the order of the basis vector and the rotation matrix has been reversed, as is always permissible.

and therefore

$$\boxed{x_j = x'_i\, C_{ji}.} \tag{3.2.4}$$

Compare this with (3.2.2). The letters used to label the indices don't matter. The important distinction is that, for the reverse transformation, the dummy index is in the second position whereas for the forward tranformation it is in the first. As with (3.2.2), there is an equivalent expression in vector form:

$$\boxed{\vec{x} = \vec{x}'\, \underset{\sim}{C}^T, \quad \text{or} \quad \vec{x} = \underset{\sim}{C}\vec{x}'.} \tag{3.2.5}$$

3.2.2 Differentiating the position vector

In a Cartesian coordinate system $\{x,y\}$, what is $\partial y/\partial x$? This is just the change in y as we move in the x-direction, i.e., zero. In contrast, $\partial x/\partial x = 1$. In general, for a 3-dimensional system, we can write:

$$\frac{\partial x_j}{\partial x_i} = \delta_{ij}. \tag{3.2.6}$$

Now imagine rotating the system using rotation matrix $\underset{\sim}{C}$. How rapidly does the jth coordinate in the *rotated* system change if we move along the ith coordinate axis in the *original* system? First, expand (3.2.2) as

$$x'_j \;=\; x_1\, C_{1j} \;+\; x_2\, C_{2j} \;+\; x_3\, C_{3j}.$$

We can easily differentiate x'_j with respect to any of the un-primed coordinates. For example:

$$\frac{\partial x_{j'}}{\partial x_1} = C_{1j}.$$

More generally:

$$\boxed{\frac{\partial x_{j'}}{\partial x_i} = C_{ij}.} \tag{3.2.7}$$

3.2.3 Defining Cartesian vectors and scalars

We now establish a more precise definition of a *vector* using the position vector as a prototype.

Definition A vector is a quantity possessed of direction and magnitude independent of the observer. A vector must therefore transform in the same way as the position vector:

$$\boxed{v'_j \;= v_i\, C_{ij}; \quad v_j = v'_i\, C_{ji}.} \tag{3.2.8}$$

Before referring to any quantity as a vector, we should check to see that it satisfies this criterion.

Examples:

- Consider the *velocity* of a moving point:

$$u_i = \frac{d}{dt}x_i. \tag{3.2.9}$$

  In rotated coordinates,

$$u'_j = \frac{d}{dt}x'_j = \frac{d}{dt}(x_i C_{ij}) = \frac{dx_i}{dt}\, C_{ij} = u_i\, C_{ij}.$$

  This confirms that the velocity transforms in the same way as the position vector, and therefore that it qualifies as a vector [6] .

- The wave vector of a plane wave transforms as a vector. Consider a surface gravity wave whose surface elevation is given by $\eta = \eta_0 \cos(\vec{k}\cdot\vec{x} - \omega t)$. The phase function $\vec{k}\cdot\vec{x} - \omega t$ is a scalar; for example, the crest of a wave is a real entity that looks like the crest to every observer. Therefore $\vec{k}\cdot\vec{x}$ must be a scalar. Defining that scalar as D, we can write:

$$D' = k'_i x'_i = k'_i x_j C_{ji} = \underbrace{k'_i C_{ji}}_{k_j} x_j = D,$$

  which requires that $k_j = k'_i C_{ji}$.

[6] The velocity components $\{u_1, u_2, u_3\}$ are commonly replaced by $\{u, v, w\}$.

Counterexample: The phase velocity of a plane wave sounds like a vector. It has three components, and we usually use "velocity'' rather than "speed'' to differentiate the vector from the scalar. However, the definition of phase velocity is inextricably linked to a particular coordinate system. From the inverse relationship between phase velocity and wave vector, $c_i = \omega/k_i$, it should be clear that c_i does not transform according to (3.2.2). If not, try it.

Definition A scalar is a single number that is the same in every reference frame.

Examples:

- Temperature is a scalar. You and I can look from different angles and we will still perceive the same temperature.
- The dot product of two vectors is a scalar. To see this, write the dot product of $\vec{x}$ and $\vec{y}$ as $D = x_i y_i$. Then ask what the value of D would be in a rotated reference frame:

$$\begin{aligned} D' = x_i' y_i' &= \overbrace{x_j C_{ji}}^{x_i'} \overbrace{y_k C_{ki}}^{y_i'} \\ &= x_j y_k \, C_{ji} C_{ki} \\ &= x_j y_k \, C_{ji} C_{ik}^T \\ &= x_j y_k \, \delta_{jk} = x_j y_j = D. \end{aligned}$$

Counterexample: A single element of a vector, while consisting of a single number, is not a scalar. For example, the x-coordinate of a point in space differs, obviously, in different coordinate systems.

3.3 Cartesian tensors

We have seen how to represent a vector in a rotated coordinate system. Can we do the same for a matrix? The basic idea is to identify a mathematical operation that the matrix represents, then require that it represent the *same* operation in the new coordinate system. We'll do this in two ways: first, by seeing the matrix as a geometrical transformation of a vector, and second by seeing it as a recipe for a bilinear product of two vectors.

3.3.1 Derivation #1: preserving geometrical transformations

In section 3.1.1, we saw how a matrix can be regarded as a geometric transformation that acts on any vector or set of vectors (such as those that terminate on the unit circle). Look carefully at figure 3.7a. Originally spherical, the ball is compressed in the $\hat{e}^{(3)}$ direction to half its original radius, and is therefore expanded in the $\hat{e}^{(1)}$ and $\hat{e}^{(2)}$ directions by a factor $\sqrt{2}$ (so that the volume is unchanged). The strain, or change of shape, can be expressed by the matrix transformation $\underset{\sim}{A}$:

$$\underset{\sim}{A} = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

Note that $\det(\underset{\sim}{A}) = 1$.

Now suppose we rotate our coordinate system by 90^o about $\hat{e}^{(1)}$, so that $\hat{e}'^{(3)} = -\hat{e}^{(2)}$ and $\hat{e}'^{(2)} = \hat{e}^{(3)}$ (figure 3.7b). In the new coordinate system, the compression is aligned in the $\hat{e}'^{(2)}$ direction. We therefore expect that the matrix will look like this:

$$\underset{\sim}{A}' = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \tag{3.3.1}$$

The compression represented by $\underset{\sim}{A}'$ is *exactly the same process* as that represented by $\underset{\sim}{A}$, but but the numerical values of the matrix elements are different because the process is measured using a different coordinate system.

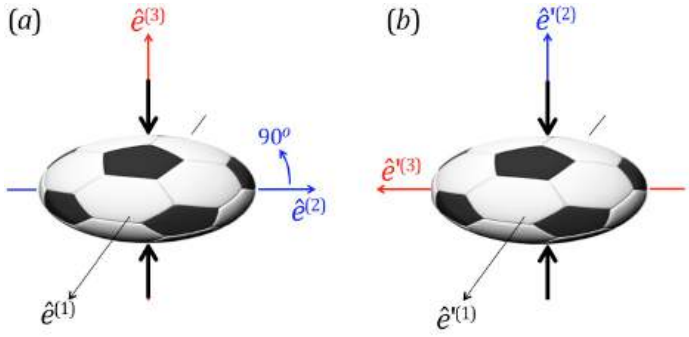


Figure 3.7: Somebody stepped on a soccer ball. (a) original coordinate system; (b) rotated coordinate system. The rotation is the one shown in figure 3.6, with $\varphi = 90^o$.

Can we generalize this result? In other words, given an arbitrary geometrical transformation $\underset{\sim}{A}$, can we find the matrix $\underset{\sim}{A}'$ that represents the same transformation in an arbitrarily rotated coordinate system? Suppose that $\underset{\sim}{A}$ transforms a general vector $\vec{u}$ into some corresponding vector $\vec{v}$:

$$A_{ij}u_j = v_i. \tag{3.3.2}$$

Now we require that the same relationship be valid in an arbitrary rotated coordinate system:

$$A'_{ij}u'_j = v'_i. \tag{3.3.3}$$

What must $\underset{\sim}{A}'$ look like? To find out, we start with (3.3.2) and substitute the *reverse* rotation formula (3.2.4) for $\vec{u}$ and $\vec{v}$:

$$A_{ij}\underbrace{u'_l C_{jl}}_{u_j} = \underbrace{v'_k C_{ik}}_{v_i}. \tag{3.3.4}$$

Now we try to make (3.3.4) look like (3.3.3) by solving for $\vec{v}'$. Naively, you might think that you could simply divide both sides by the factor multiplying v'_k, namely C_{ik}. This doesn't work because, although the right hand side *looks* like a single term, it is really a sum of three terms with $k = 1,2,3$, and C_{ik} has a different value in each. But instead of dividing by $\underset{\sim}{C}$, we can multiply by its inverse. To do this, multiply both sides of (3.3.4) by C_{im}:

$$\begin{aligned} A_{ij}u'_l C_{jl}C_{im} &= v'_k C_{ik}C_{im} \\ A_{ij}u'_l C_{jl}C_{im} &= v'_k \underbrace{C^T_{ki}C_{im}}_{\delta_{km}} \\ A_{ij}C_{im}C_{jl}u'_l &= v'_m. \end{aligned}$$

So we have successfully solved for the elements of $\vec{v}'$.

Now if (3.3.3) holds, we can replace the right-hand side with $A'_{ml}u'_l$:

$$A_{ij}C_{im}C_{jl}u'_l = A'_{ml}u'_l$$

or

$$(A_{ij}C_{im}C_{jl} - A'_{ml})u_l = 0.$$

This relation must be valid *not just for a particular vector $\vec{u}$ but for every vector $\vec{u}$*, and that can only be true if the quantity in parentheses is identically zero. (Make sure you understand that last statement; it will come up again.)

The transformation rule for the matrix $\underset{\sim}{A}$ is therefore (after a minor relabelling of indices):

$$\boxed{A'_{ij} = A_{kl}C_{ki}C_{lj}.} \tag{3.3.5}$$

This transformation law can also be written in matrix form as

$$\boxed{\underset{\sim}{A}' = \underset{\sim}{C}^T\underset{\sim}{A}\underset{\sim}{C}.} \tag{3.3.6}$$

There is also a reverse transformation

$$\boxed{A_{ij} = A'_{kl}C_{ik}C_{jl}.} \tag{3.3.7}$$

As in the transformation of vectors, the dummy index on $\underset{\sim}{C}$ is in the first position for the forward transformation, in the second position for the reverse transformation.

Exercise: Use this law to check (3.3.1).

Definition A *2nd-order tensor* is a matrix that transforms according to (3.3.5).

3.3.2 Tensors in the laws of physics

In the above derivation of the matrix transformation formula (3.3.5), we thought of the vector $\vec{u}$ as a position vector identifying, for example, a point on the surface of a soccer ball. Likewise, $\vec{v}$ is the same point after undergoing the geometrical transformation $\underset{\sim}{A}$, as described by (3.3.2). We then derived (3.3.5) by assuming that the *geometrical relationship* between the vectors $\vec{u}$ and $\vec{v}$, as represented by $\underset{\sim}{A}$, be the same in a rotated coordinate system, i.e. (3.3.4).

Now suppose that the vectors $\vec{u}$ and $\vec{v}$ are not position vectors but are instead some other vector quantities such as velocity or force, and (3.3.2) represents a *physical relationship* between those two vector quantities. If this relationship is to be valid for all observers (as discussed in section 3.1.2), then the matrix $\underset{\sim}{A}$ must transform according to (3.3.5).

A simple example is the rotational form of Newton's second law of motion:

$$\vec{T} = \underset{\sim}{I}\vec{\alpha}. \tag{3.3.8}$$

Here, $\vec{\alpha}$ represents the angular acceleration of a spinning object (the rate at which its spinning motion accelerates). If $\vec{\alpha}$ is multiplied by the matrix $\underset{\sim}{I}$, called the *moment of inertia*, the result is $\vec{T}$, the torque (rotational force) that must be applied to create the angular acceleration. If the process is viewed by two observers using different coordinate systems, their measurements of $\underset{\sim}{I}$ are related by (3.3.5), as is shown in appendix B. If this were not true, (3.3.8) would be useless as a law of physics and would have been discarded long ago.

The number of mathematical relationships that might conceivably exist between physical quantities is infinite, and a theoretical physicist must have some way to identify those relationships that might actually be true before spending time and money testing them in the lab. The requirement that physical laws be the same for all observers, as exemplified in (3.3.5), plays this role. It was used extensively in Einstein's derivation of the theory of relativity, and the modern form of the theory was inspired by that success. It is equally useful in the development of the laws of fluid dynamics as we will see shortly.

3.3.3 Derivation #2: preserving bilinear products

The dot product of two vectors $\vec{u}$ and $\vec{v}$ is an example of a *bilinear product*:

$$D(\vec{u},\vec{v}) = u_1v_1 + u_2v_2 + u_3v_3. \tag{3.3.9}$$

We call it "bilinear" because it is linear in each argument separately, e.g., $D(\vec{a}+\vec{b},\vec{c}) = D(\vec{a},\vec{c}) + D(\vec{b},\vec{c})$.

With a little imagination we can invent many other bilinear products, such as

$$u_1v_2 - u_2v_3 + 2u_3v_2. \tag{3.3.10}$$

Such products can be written in the compact form

$$A_{ij}u_iv_j.$$

For the dot product (3.3.9), the matrix $\underset{\sim}{A}$ is just $\underset{\sim}{\delta}$. In the second example (3.3.10), the matrix is

$$\underset{\sim}{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 2 & 0 \end{bmatrix}.$$

Exercise: Write out $A_{ij}u_iv_j$ and verify that the result is (3.3.10).

So, every matrix can be thought of as the recipe for a bilinear product. Now suppose we rotate to a new coordinate system. How would the matrix $\underset{\sim}{A}$ have to change so that the bilinear product it represents remains the same (i.e., is a scalar)? The following must be true:

$$A'_{ij}u'_iv'_j = A_{kl}u_kv_l.$$

As in section 3.3.1, we substitute for $\vec{u}$ and $\vec{v}$ using the reverse rotation formula (3.2.8):

$$A'_{ij}u'_iv'_j = A_{kl}u'_iC_{ki}v'_jC_{lj}.$$

Collecting like terms, we can rewrite this as

$$(A'_{ij} - A_{kl}C_{ki}C_{lj})u'_iv'_j = 0.$$

If this is to be true for *every* pair of vectors $\vec{u}$ and $\vec{v}$, the quantity in parentheses must vanish. This leads us again to the tensor transformation law (3.3.5).

Examples:

- A dyad is a matrix made up of the components of two vectors: $A_{ij} = u_iv_j$. Is the dyad a tensor?

  $$A'_{ij} = u'_iv'_j = u_kC_{ki}v_lC_{lj} = u_kv_lC_{ki}C_{lj} = A_{kl}C_{ki}C_{lj}.$$

  The answer is yes, the dyad transforms according to (3.3.5) and therefore qualifies as a tensor.
- The identity matrix is the same in every coordinate system. Does it qualify as a tensor? Let $\underset{\sim}{C}$ represent an arbitrary rotation matrix and then write the rotated identity matrix using the fact that $\underset{\sim}{C}$ is orthogonal:

  $$\delta'_{ij} = C^T_{ik}C_{kj}.$$

  With a bit of index-juggling, we can show that this equality is equivalent to (3.3.5):

  $$\delta'_{ij} = C^T_{ik}C_{kj} = C_{ki}C_{kj} = \delta_{kl}C_{ki}C_{lj}.$$

  We conclude that $\underset{\sim}{\delta}$ transforms as a tensor.

Test your understanding by completing exercises 11 and 12.

3.3.4 Higher-order tensors

A matrix that transforms according to (3.3.5) is called a "2nd-order'' tensor because it has two indices. By analogy, we can think of a vector as a 1st-order tensor, and a scalar as a 0th-order tensor. Each obeys a transformation law involving a product of rotation matrices whose number equals the order:

$$\begin{aligned} &\text{order 0:} && T' = T; \\ &\text{order 1:} && u'_p = u_i\,C_{ip}; \\ &\text{order 2:} && A'_{pq} = A_{ij}\,C_{ip}C_{jq}. \end{aligned}$$

We can now imagine 3rd and 4th order tensors that transform as

$$\begin{aligned} &\text{order 3:} && G'_{pqr} = G_{ijk}\,C_{ip}C_{jq}C_{kr}; \\ &\text{order 4:} && K'_{pqrs} = K_{ijkl}\,C_{ip}C_{jq}C_{kr}C_{ls}; \end{aligned}$$

The 3rd-order tensor is a three-dimensional array that expresses a relationship among three vectors, or one vector and one 2nd-order tensor. The 4th-order tensor may express a relationship among four vectors, two 2nd-order tensors or a vector and a 3rd-order tensor. We will see examples of both of these higher-order tensor types.

Test your understanding of tensors by completing exercises 13, 14, 15 and 16.

3.3.5 Symmetry and antisymmetry in higher-order tensors

We're familiar with the properties that define symmetric and antisymmetric 2nd-order tensors: $A_{ij} = A_{ji}$ and $A_{ij} = -A_{ji}$, respectively. For higher-order tensors, these properties become a bit more involved. For example, suppose a 3rd-order tensor has the property

$$G_{ijk} = G_{jik}.$$

This tensor is symmetric with respect to its 1st and 2nd indices. The tensor could also be symmetric with respect to its 1st and 3rd, or 2nd and 3rd indices.

The tensor could also be *antisymmetric* with respect to one or more pairs of indices, e.g.

$$G_{ijk} = -G_{jik}.$$

A tensor that is antisymmetric with respect to *all* pairs of indices is called "completely antisymmetric". The same nomenclature applies to 4th and higher-order tensors.

Symmetry and antisymmetry are *intrinsic* properties of a tensor, in the sense that they are true in all coordinate systems.[7] You will show this in exercise 19.

3.3.6 Isotropy

Definition: An isotropic tensor is one whose elements have the same values in all coordinate systems, i.e., it is invariant under rotations.

Every scalar is isotropic but no vector is. How about 2nd order tensors? As is shown in appendix C, the only isotropic 2nd-order tensors are those proportional to the identity matrix.

Isotropic tensors are of particular importance in defining the basic operations of linear algebra. For example, how many ways can you think of to multiply two vectors? If you were awake in high school then you already know two - the dot product and the cross product - but in fact it would be easy to invent more. The reason we use these two products in particular is that both are based on isotropic tensors.

We saw in section 3.3.3 that the dot product is an example of a bilinear product, and that bilinear products in general are represented by 2nd-order tensors. In the case of the dot product, the tensor is $\underset{\sim}{\delta}$, i.e. $\vec{u}\cdot\vec{v} = \delta_{ij}u_iv_j$. Now, *every* 2nd-order tensor defines a bilinear product that could potentially be an alternative to the dot product. But as it turns out (appendix C), the identity tensor $\underset{\sim}{\delta}$ is the *only* 2nd-order tensor that is isotropic[8]. A bilinear product based on any other tensor would have to be computed differently in each reference frame, and would therefore be useless as a general algebraic tool.[9]

How about the cross product? We can imagine many ways to multiply two vectors such that the product is another vector. Simply define three separate bilinear combinations and use one for each component of the resulting vector. For example, the cross product is defined by the following three bilinear combinations:

$$\begin{aligned} w_1 &= u_2v_3 - u_3v_2 \\ w_2 &= -u_1v_3 + u_3v_1 \\ w_3 &= u_1v_2 - u_2v_1. \end{aligned} \tag{3.3.11}$$

Any such formula can be written using a 3rd-order tensor:

$$w_i = A_{ijk}u_jv_k.$$

And conversely, every 3rd-order tensor defines a vector bilinear product. But again, if we want the formula to be independent of the reference frame, then the tensor must be isotropic. As with 2nd-order tensors, there is only one choice for a 3rd-order isotropic tensor, and that is the choice that defines the cross product. We will discuss that tensor in the following section.

Isotropic tensors are also useful for describing the physical properties of isotropic materials, i.e., materials whose inner structure does not have a preferred direction. Both water and air are isotropic to a good approximation. A counterexample is wood, which has a preferred direction set by its grain. In an upcoming chapter we will see that the relationship between stress and strain in an isotropic material is described by an isotropic 4th-order tensor.

Table 3.1 summarizes the various orders of tensors, their rotation rules, and the subset that are isotropic. The isotropic forms are derived in appendices C and D.

[7] A counterexample to this is *diagonality*: a tensor can be diagonal in one coordinate system and not in others, e.g., exercise 10, or section 5.3.4. So diagonality is *not* an intrinsic property.

[8] up to a multiplicative constant

[9] This is true in Cartesian coordinates. In curved coordinate systems, e.g. Appendix I, the dot product is computed using a more general object called the *metric tensor* which we won't go into here.

order	common name	rotation rule	isotropic
0	scalar	$T' = T$	all
1	vector	$v'_i = v_j C_{ji}$	none
2	matrix	$A'_{ij} = A_{kl} C_{ki} C_{lj}$	$a\ \delta_{ij}$
3		$A'_{ijk} = A_{lmn} C_{li} C_{mj} C_{kn}$	completely antisymmetric
4		$A'_{ijkl} = A_{mnpq} C_{mi} C_{nj} C_{pk} C_{ql}$	$a\ \delta_{ij}a_{kl} + b\ \delta_{ik}\delta_{jl} + c\ \delta_{il}\delta_{jk}$

Table 3.1: Summary of tensor properties. The variables a, b and c represent arbitrary scalars. The rotation rules are for *forward* rotations, hence the dummy index is on the left. To reverse the rotation, place the dummy index on the right.

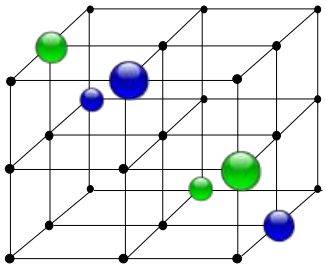


Figure 3.8: Schematic of the Levi-Civita alternating tensor. Blue = 1, green = -1, black = 0.

3.3.7 The Levi-Civita tensor: properties and applications

In this section we'll describe the so-called Levi-Cevita alternating tensor[10] (also called the "antisymmetric tensor" and the "permutation tensor"). This tensor holds the key to understanding many areas of linear algebra, and has application throughout mathematics, physics, engineering and many other fields. In this section we will only summarize the most useful properties of the Levi-Civita tensor. For a more complete explanation including proofs, the student is encouraged to examine Appendix D.

The alternating tensor is defined as follows:

$$\varepsilon_{ijk} = \begin{cases} 1, & \text{if } ijk = 123, 312, 231, \\ -1, & \text{if } ijk = 213, 321, 132, \\ 0, & \text{otherwise.} \end{cases} \tag{3.3.12}$$

Illustrated in figure 3.8, the array is mostly zeros, with three 1s and three -1s arranged antisymmetrically. The essential property of the alternating tensor is that it is *completely antisymmetric*, meaning that the interchange of any two indices results in a sign change. Two other properties follow from this one:

1. Any element with two equal indices must be zero.

2. Cyclic permutations of the indices have no effect. What is a cyclic permutation? The idea is illustrated in figure 3.9. For example, suppose we start with ε_{ijk}, then move the k back to the first, or leftmost position, shifting i and j to the right to make room, resulting in ε_{kij}. That's a cyclic permutation.

Examine (3.3.12) and convince yourself that these two statements are true. See Appendix D for a proof that they *have to* be true.

[10] Tullio Levi-Civita (1873-1941) was an Italian mathematician. Among other things, he consulted with Einstein while the latter was developing the general theory of relativity.

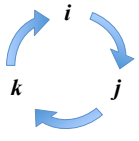


Figure 3.9: Cyclic permutations of the indices i, j and k.

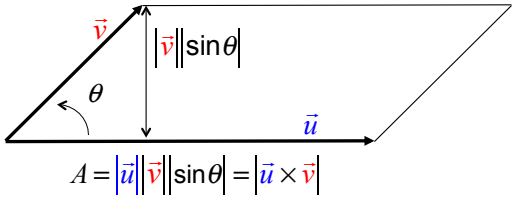


Figure 3.10: The area of the parallelogram bounded by two vectors $\vec{u}$ and $\vec{v}$ is the magnitude of their cross product.

The $\varepsilon - \delta$ relation

The alternating tensor is related to the 2nd-order identity tensor in a very useful way:

$$\boxed{\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}.} \tag{3.3.13}$$

For us, the main value of (3.3.13) will be in deriving vector identities, as we will see below. Test your understanding of the $\varepsilon - \delta$ relation by completing exercise 17.

The cross product

The cross product is defined in terms of $\underset{\sim}{\varepsilon}$ by

$$\boxed{z_k = \varepsilon_{ijk}u_i v_j.} \tag{3.3.14}$$

You'll sometimes see (3.3.14) written with the free index in the first position: $z_k = \varepsilon_{kij}u_i v_j$. The two expressions are equivalent because kij is a cyclic permutation of ijk (see property 2 of $\underset{\sim}{\varepsilon}$ listed above).

We'll now list some essential properties of the cross product.

1. The cross product is anticommutative:

$$\begin{aligned}
(\vec{v}\times\vec{u})_k &= \varepsilon_{ijk}v_i u_j \\
&= \varepsilon_{ijk}u_j v_i \qquad \text{(reordering)} \\
&= \varepsilon_{jik}u_i v_j \qquad \text{(relabeling } i \text{ and } j\text{)} \\
&= -\varepsilon_{ijk}u_i v_j \qquad \text{(using antisymmetry)} \\
&= -(\vec{u}\times\vec{v})_k.
\end{aligned}$$

2. The cross product is perpendicular to both $\vec{u}$ and $\vec{v}$. This is left for the reader to prove (exercise 18).

3. The magnitude of the cross product is

$$|\vec{z}| = |\vec{u}||\vec{v}||\sin\theta|, \tag{3.3.15}$$

where θ is the angle between $\vec{u}$ and $\vec{v}$. This is easily proven by writing the squared magnitude of $\vec{z}$ as

$$z_k z_k = \varepsilon_{ijk}u_i v_j\ \varepsilon_{klm}u_l v_m$$

then applying the $\underset{\sim}{\varepsilon} - \underset{\sim}{\delta}$ relation. A geometric interpretation of (3.3.15) is that the magnitude of the cross product is equal to the area of the parallelogram bounded by $\vec{u}$ and $\vec{v}$ (figure 3.10).
Test your understanding of the cross product by completing exercise 18.

The determinant

The determinant of a 3x3 matrix $\underset{\sim}{A}$ can be written using $\underset{\sim}{\varepsilon}$:

$$\det(\underset{\sim}{A}) = \varepsilon_{ijk}A_{i1}A_{j2}A_{k3}, \tag{3.3.16}$$

where the columns are treated as vectors.[11] The many useful properties of the determinant all follow from this definition (Appendix D).

Back to Table of Contents.

[11] This definition does not require that the columns actually transform as vectors; in general they do not.

Chapter 4

Tensor calculus

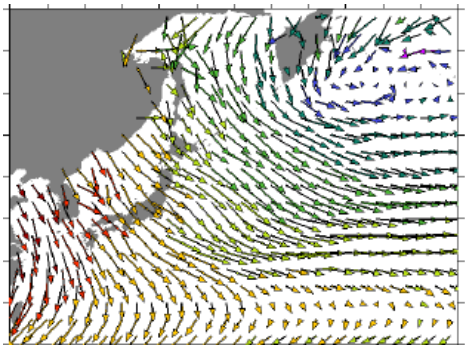


Figure 4.1: Vector field representation of the wind over the northwest Pacific ocean. The curl of this vector field is a dominant influence on ocean currents. Its divergence can tell us about vertical motion and precipitation (Ansley Manke, NOAA).

A field is (for our purposes) a quantity that varies in space and can therefore be differentiated with respect to position. Scalars, vectors and tensors can all be fields (e.g., figure 4.1). The various derivative operations that can be applied to fields are fundamental tools in the study of flow. I assume that the reader is comfortable with the calculus as applied to functions of a single variable and has some familiarity with partial derivatives. With this knowledge, it is straightforward to apply the calculus to scalar, vector and tensor fields.

4.1 Vector calculus operations

In multidimensional calculus, the role of the derivative is taken by the vector differential operator $\vec{\nabla}$, pronounced "del'', or sometimes "nabla'':

$$\nabla_i = \frac{\partial}{\partial x_i}.$$

This operator can be written in several different but equivalent ways:

$$\vec{\nabla} = \{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\} = \hat{e}^{(i)}\frac{\partial}{\partial x_i} = \hat{e}^{(i)}\nabla_i.$$

4.1.1 Gradient of a scalar

Let $\phi(\vec{x})$ be a scalar field and $\vec{x}$ the position vector in a Cartesian coordinate system. Application of $\vec{\nabla}$ yields the gradient:

$$\vec{G} = \vec{\nabla}\phi = \{\frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2}, \frac{\partial\phi}{\partial x_3}\} = \hat{e}^{(i)}\frac{\partial\phi}{\partial x_i}.$$

The gradient has three components and appears to be a vector, but we should check. Identifying a vector is more complicated when spatial derivatives are involved.

$$\begin{aligned} G_i' &= \frac{\partial \phi'}{\partial x_i'} \\ &= \frac{\partial \phi}{\partial x_i'} \qquad (\phi \text{ is a scalar}) \\ &= \frac{\partial \phi}{\partial x_j}\frac{\partial x_j}{\partial x_i'} \qquad (\text{applying the chain rule and summing over } j) \\ &= G_j \frac{\partial x_j}{\partial x_i'}. \end{aligned} \tag{4.1.1}$$

The partial derivative in the final term is the Jacobian matrix for the transformation. Using the reverse transformation rule (3.2.4),

$$\frac{\partial x_j}{\partial x_i'} = \frac{\partial x_k' C_{jk}}{\partial x_i'} = C_{jk}\frac{\partial x_k'}{\partial x_i'} = C_{jk}\delta_{ki} = C_{ji}. \tag{4.1.2}$$

This useful result pertains to every orthogonal coordinate system, so we'll highlight for later reference:

$$\boxed{\frac{\partial x_j}{\partial x_i'} = C_{ji}.} \tag{4.1.3}$$

Now, combining (4.1.1) and (4.1.2), we have $G_i' = G_j C_{ji}$, i.e., the gradient of a scalar transforms as a vector.

4.1.2 Divergence

The divergence normally results from applying $\vec{\nabla}$ to a vector:

$$\vec{\nabla}\cdot\vec{u} = \frac{\partial u_i}{\partial x_i}.$$

The result is a scalar. A vector whose divergence is zero is called solenoidal. The divergence may also be applied to either dimension of a matrix or a 2nd order tensor. For example,

$$\frac{\partial A_{ij}}{\partial x_j}\hat{e}^{(i)}.$$

If $\underset{\sim}{A}$ is a tensor, the result is a vector.

4.1.3 Curl

The curl is applied to a vector field $\vec{u}$ by taking the cross product with $\vec{\nabla}$:

$$\vec{\nabla}\times\vec{u} = \varepsilon_{ijk}\nabla_i u_j \hat{e}^{(k)}.$$

The result is another vector field. It can be expanded as

$$\vec{u}\times\vec{v} = \hat{e}^{(1)}(u_2v_3 - u_3v_2) - \hat{e}^{(2)}(u_1v_3 - u_3v_1) + \hat{e}^{(3)}(u_1v_2 - u_2v_1). \tag{4.1.4}$$

See Appendix D, section D.3.3 for details.

4.1.4 Laplacian

The Laplacian results from successive applications of $\vec{\nabla}$:

$$\vec{\nabla}\cdot\vec{\nabla} = \nabla^2 = \nabla_i\nabla_i = \frac{\partial^2}{\partial x_i^2}.$$

The Laplacian may be applied to either a scalar or a vector:

$$\nabla^2\phi = \vec{\nabla}\cdot\vec{\nabla}\phi = \frac{\partial^2\phi}{\partial x_i^2},$$

or

$$\nabla^2\vec{u} = \{\nabla^2 u_1, \nabla^2 u_2, \nabla^2 u_3\} = \hat{e}^{(i)}\nabla^2 u_i.$$

4.1.5 Advective derivative

The advective derivative is an operation unique to fluid mechanics. Besides $\vec{\nabla}$, it requires a vector field $\vec{u}(\vec{x})$, which is often (though not always) chosen to be the velocity of the flow. When this choice is made, we use the more common term material derivative (discussed in section 5.1.1). The operation is

$$[\vec{u}\cdot\vec{\nabla}] = u_i\frac{\partial}{\partial x_i}.$$

The advective derivative can be applied to a scalar, resulting in another scalar:

$$[\vec{u}\cdot\vec{\nabla}]\phi = \vec{u}\cdot\vec{\nabla}\phi = u_i\frac{\partial\phi}{\partial x_i},$$

or to a vector, resulting in another vector:

$$[\vec{u}\cdot\vec{\nabla}]\vec{v} = u_i\frac{\partial}{\partial x_i}(v_j\hat{e}^{(j)}) = \hat{e}^{(j)}u_i\frac{\partial v_j}{\partial x_i}.$$

4.1.6 Vector identities

Many equations hold for only certain values of a variable, and one may want to solve the equation to find those values. In contrast, *identities* are equations that hold for all values of a certain class of variables, e.g., all vectors that vary continuously in space. The identity

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{u}) = 0$$

tells us that the divergence of the curl of a vector is zero, and this is true for every continuously-varying vector $\vec{u}$. Such identities are tremendously useful in vector calculus. For example, if a term includes the divergence of the curl of a vector, you can throw it out regardless of what the vector is.

Appendix E lists 21 of the most useful vector identities. All of these can (and should) be proved using the methods we have covered so far. For example:

- Proof of identity #15: $\vec{\nabla}\times\vec{\nabla}\phi = 0$. We start with the kth component of $\vec{\nabla}\times\vec{\nabla}\phi$:

$$\begin{aligned}
[\vec{\nabla}\times\vec{\nabla}\phi]_k = \varepsilon_{ijk}\frac{\partial}{\partial x_i}\frac{\partial\phi}{\partial x_j} &= \varepsilon_{ijk}\frac{\partial^2\phi}{\partial x_i\partial x_j} \\
&= \varepsilon_{ijk}\frac{\partial^2\phi}{\partial x_j\partial x_i} && \text{(reverse order of differentiation)} \\
&= \varepsilon_{jik}\frac{\partial^2\phi}{\partial x_i\partial x_j} && \text{(relabel } i \text{ and } j \text{ as each other)} \\
&= -\varepsilon_{ijk}\frac{\partial^2\phi}{\partial x_i\partial x_j} && \text{(use antisymmetry of } \underset{\sim}{\varepsilon}\text{)} \\
&= -[\vec{\nabla}\times\vec{\nabla}\phi]_k.
\end{aligned}$$

  We have shown that $[\vec{\nabla}\times\vec{\nabla}\phi]_k$ is equal to its own additive inverse, and therefore can have no other value but zero.

- Proof of identity #21. It's easier if we rearrange the identity like this: $(\vec{\nabla}\times\vec{u})\times\vec{u} \equiv [\vec{u}\cdot\vec{\nabla}]\vec{u} - \frac{1}{2}\vec{\nabla}(\vec{u}\cdot\vec{u})$. Now define

$$\vec{\omega} = \vec{\nabla}\times\vec{u}, \quad \text{or} \quad \omega_k = \varepsilon_{ijk}\frac{\partial}{\partial x_i}u_j \tag{4.1.5}$$

  and

$$\vec{F} = (\vec{\nabla}\times\vec{u})\times\vec{u}, \quad \text{or} \quad F_m = \varepsilon_{klm}\omega_k u_l. \tag{4.1.6}$$

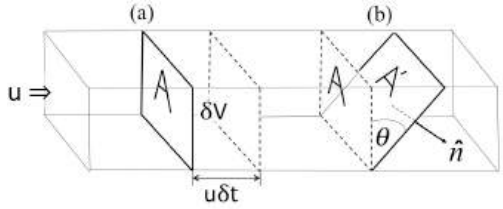


Figure 4.2: Volume flux through a rectangular channel. (a) A rectilinear cross-section showing the volume transport in a time δt. (a) A tilted cross-section with unit normal $\hat{n}$.

We now substitute (4.1.5) into (4.1.6) and use the $\varepsilon - \delta$ relation:

$$\begin{aligned} F_m &= \varepsilon_{klm}\left(\varepsilon_{ijk}\frac{\partial}{\partial x_i}u_j\right)u_l \\ &= \varepsilon_{ijk}\varepsilon_{klm}\frac{\partial u_j}{\partial x_i}u_l \\ &= (\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})\frac{\partial u_j}{\partial x_i}u_l \\ &= \delta_{il}\delta_{jm}\frac{\partial u_j}{\partial x_i}u_l - \delta_{im}\delta_{jl}\frac{\partial u_j}{\partial x_i}u_l \\ &= \frac{\partial u_m}{\partial x_l}u_l - \frac{\partial u_l}{\partial x_m}u_l. \end{aligned}$$

(In that last step we had two choices: we summed over the dummy indices i and j, but we could just as well have chosen to sum over i and l or j and l.) Now it's just a matter of recognizing the remaining two terms on the right-hand side as the terms in the identity we want to prove.

$$\begin{aligned} F_m &= u_l\frac{\partial}{\partial x_l}u_m - \frac{1}{2}\frac{\partial}{\partial x_m}(u_l u_l) \\ &= [\vec{u}\cdot\vec{\nabla}]\,u_m - \frac{1}{2}\frac{\partial}{\partial x_m}(\vec{u}\cdot\vec{u}), \end{aligned}$$

or $\vec{F} = [\vec{u}\cdot\vec{\nabla}]\vec{u} - \frac{1}{2}\vec{\nabla}(\vec{u}\cdot\vec{u})$, and the identity is proven.

Identity-proving is not only an excellent diversion for a rainy day; it will also, like pushups for a quarterback or scales for a musician, prepare you for great things. You are therefore encouraged to see the list of vector identities in appendix E as a fine pile of puzzles awaiting your attention. Exercise 20 lists a few that you should definitely try.

4.2 Flux and divergence

The flux of a quantity is the rate at which it is transported across a surface, expressed as transport per unit surface area. A simple example is the volume flux, which we denote as Q.

4.2.1 Volume flux through a rectilinear surface

Consider the simple, rectilinear channel in figure 4.2. The flow velocity $\vec{u}$ is assumed to be uniform with magnitude $|\vec{u}| = U$, and the cross-sectional area is A. After a time δt, the flow through the cross-section marked (a) has travelled a distance $U\delta t$ and occupies a volume $\delta V = AU\delta t$. The volume flux is then

$$Q = \frac{\delta V}{\delta t} = AU.$$

Examples:

- A river 100m wide and 2m deep has cross-sectional area 200m^2. If we take the velocity to be 1ms^{-1}, then we estimate the volume flux as $200\text{m}^2 \times 1\text{ms}^{-1} = 200\text{m}^3\text{s}^{-1}$.
- The total volume flux of all of Earth's rivers is $\sim 10^6\text{m}^3\text{s}^{-1}$.

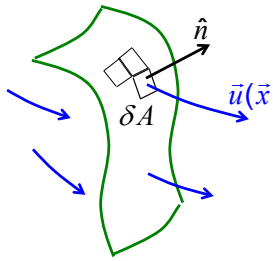


Figure 4.3: Volume flux through an arbitrary surface tiled by surface elements with area δA and unit normal $\hat{n}$.

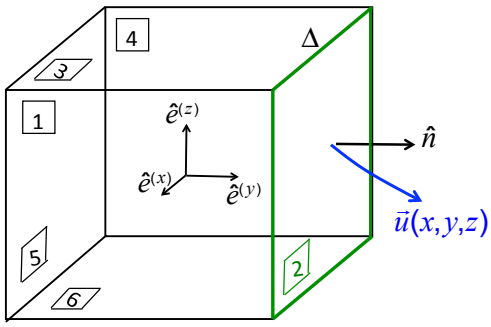


Figure 4.4: Definition sketch for the volume flux through a cube.

- The Gulf Stream, a large ocean current that flows north along the east coast of the U.S., is typically 100km wide and 1000m deep, so the cross-sectional area is $10^8 \mathrm{m}^2$. A typical velocity is $1 \mathrm{ms}^{-1}$, so the corresponding volume flux is $Q = 10^8 \mathrm{m}^3\mathrm{s}^{-1}$.

Oceanographers measure volume flux in units of *Sverdrups*[1]: 1Sv = $10^6 \mathrm{m}^3\mathrm{s}^{-1}$. The world's rivers therefore carry about 1Sv., while the Gulf Stream carries 100Sv.

Suppose now that the surface through which we calculate the volume flux is tilted at an angle θ from the vertical (marked (b) in figure 4.2). The volume flux is, of course, the same as that through the vertical section. The area of the vertical section is $A' \cos\theta$. We can therefore define the volume flux through a surface tilted at an arbitrary angle θ from the vertical as $Q = UA' \cos\theta$. We can also express this flux in terms of the unit vector $\hat{n}$, drawn normal to the surface A'. Note that the product $U \cos\theta$ is equal to $\vec{u} \cdot \hat{n}$.

4.2.2 Volume flux through a curved surface

A curved surface can be thought of as being tiled by small, flat, surface elements with area δA and unit normal $\hat{n}$. The tiling matches the surface exactly as the tile size shrinks to zero. The volume flux through each tile is $\delta Q = \vec{u} \cdot \hat{n} \delta A$, just as in the case of the tilted surface in section 4.2.1. If we now sum over all of the tiles and take the limit as $\delta A \to 0$, we obtain the general expression

$$\boxed{Q = \int_A \vec{u} \cdot \hat{n} dA.} \tag{4.2.1}$$

4.2.3 Volume flux through an arbitrary closed surface: the divergence theorem

Flux through an infinitesimal cube

Consider a general velocity field $\vec{u}(\vec{x}) = \{u(\vec{x}), v(\vec{x}), w(\vec{x})\}$, and somewhere within it a small, imaginary cube with edge dimension Δ (figure 4.4). We would like to know the net volume flux *out of the cube*. We define a Cartesian coordinate system aligned with the cube as shown. We will now compute the outward volume flux across each of the faces, numbered 1-6 in the figure.

We begin with face #2, highlighted in green. On this face $y = \Delta/2$, and the outward unit normal is $\hat{n} = \hat{e}^{(y)}$. The volume flux may be written as

$$Q^{[2]} = \int_{[2]} \vec{u} \cdot \hat{n} \, dA = \int_{-\Delta/2}^{\Delta/2} dx \int_{-\Delta/2}^{\Delta/2} dz \, v(x, \Delta/2, z). \tag{4.2.2}$$

[1] Harald Sverdrup (1888-1957) was a Norwegian oceanographer and meteorologist. He was the scientific director for the Amundsen expedition to the North pole, and was later director of the Scripps Institute of Oceanography in San Diego. He discovered the fundamental balance between wind and the Earth's rotation that governs the large-scale ocean currents.

We now approximate the spatial variation of v by means of a first-order Taylor series expansion about the origin:

$$v(x,y,z) \;=\; v^0 + v^0_x\, x + v^0_y\, y \;+\; v^0_z\, z + \ldots \tag{4.2.3}$$

Here, subscripts indicate partial derivatives (for brevity) and the superscript "0" specifies evaluation at the origin. The dots at the end represent higher-order terms that will vanish later when we take the limit $\Delta \to 0$; from here on we ignore these. Replacing the integrand in (4.2.2) with (4.2.3), we have

$$\begin{aligned} Q^{[2]} = \int_{-\Delta/2}^{\Delta/2} dx \int_{-\Delta/2}^{\Delta/2} dz \quad & \Big[\quad v^0 \;+\; v^0_x\, x \;+\; v^0_y\, \frac{\Delta}{2} \;+\; v^0_z\, z \,\Big] \\ = \quad & \Big[\quad \Delta^2 v^0 \;+\; 0 \;+\; \Delta^2 v^0_y\, \frac{\Delta}{2} \;+\; 0 \,\Big] \\ = \quad & \Big(\quad v^0 \;+\; v^0_y\, \frac{\Delta}{2} \;\Big)\, \Delta^2. \end{aligned}$$

Now we repeat the process for the opposite face, #5. The only differences are that the uniform value of y becomes $-\Delta/2$ and the outward normal becomes $-\hat{e}^{(y)}$. The calculation therefore gives

$$Q^{[5]} = -\big(v^0 - v^0_y\, \frac{\Delta}{2}\big)\, \Delta^2.$$

Summing the fluxes from faces 2 and 5 gives

$$Q^{[2]} + Q^{[5]} = 2 \times v^0_y\, \frac{\Delta}{2}\, \Delta^2 = v^0_y \Delta^3.$$

Note that, if the velocity v were uniform, this net outward flux would be zero, i.e., what comes in one face goes out the other. The net flux is nonzero only when the velocities through the two faces differ. We can now repeat this process for each of the other two opposite pairs of faces:

$$Q^{[1]} + Q^{[4]} = u^0_x \Delta^3, \quad \text{and} \quad Q^{[3]} + Q^{[6]} = v^0_z \Delta^3.$$

Adding these results, we have the net outflow:

$$Q = (u^0_x + v^0_y + w^0_z)\Delta^3.$$

At this stage we take the limit as $\Delta \to 0$ so that the higher-order terms that we have neglected vanish. Because our cube could have been placed anywhere in the velocity field, this result is true at every point and we don't need drop the superscript "0". The infinitesimal volume flux δQ from this small cube therefore expresses the divergence of the velocity field:

$$\delta Q = \vec{\nabla} \cdot \vec{u}\, \delta V, \tag{4.2.4}$$

where δV is the limit of the volume Δ^3.

In many situations, the flows into and out of a small volume balance, and therefore $\vec{\nabla} \cdot \vec{u} = 0$. There are two exceptions:

1. There is a volume source, e.g., fluid is being pumped into the cube through a hose.
2. The fluid expands or contracts, e.g., as a result of heating or cooling.

Summing the cubes

Suppose we now want to know the net outflow from two adjacent cubes. We would sum the flows through each face as before. Note, however, that the volume fluxes through the two adjacent faces exactly cancel. (The velocities are the same and the unit normals are opposite.) By (4.2.4), this net outflow equals the divergence evaluated at the center of each cube multiplied by the volume δV and summed over the two cubes.

We can generalize this to any assemblage of adjacent cubes: the net outflow is the sum of the outflows through the *exterior faces only*, because the flows through the interior faces cancel. Moreover, this is equal to the sum of the divergences in each cube times δV.

The divergence theorem

An arbitrary volume can be approximated with arbitrary precision as an assemblage of small cubes. The foregoing results regarding the flux from a small cube, in the limit as $\delta V \to 0$, give us the divergence theorem (also called Gauss' theorem [2]):

[2] Carl Friedrich Gauss (1777-1855) was a German mathematician and physicist. He is considered one of the greatest scientists in history, and it would be an insult to try to describe his accomplishments in a footnote. However, he did not actually discover the theorem that bears his name - it was used by Lagrange fifty years before Gauss found it.

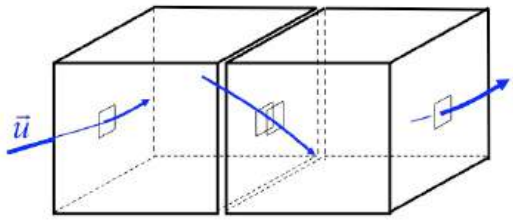


Figure 4.5: Volume flux through two cubes, showing that fluxes through adjacent interior surfaces cancel.

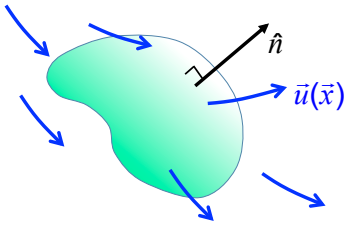


Figure 4.6: An arbitrary, simply-connected volume, with outward normal $\hat{n}$, embedded in an arbitrary flow field $\vec{u}(\vec{x})$.

Theorem: Within a given flow field $\vec{u}(\vec{x})$, imagine volume of space V bounded by an arbitrary closed surface A. At each point on the surface, define the outward-pointing unit normal $\hat{n}$. Then the net volume flux out the surface is given by the integral of its divergence throughout the volume:

$$Q = \oint_A \vec{u}\cdot\hat{n}\, dA = \int_V \vec{\nabla}\cdot\vec{u}\, dV, \tag{4.2.5}$$

or, in index notation:

$$Q = \oint_A u_i n_i\, dA = \int_V \frac{\partial u_i}{\partial x_i}\, dV. \tag{4.2.6}$$

In physical terms, the divergence theorem tells us that the flux out of a volume equals the sum of the sources minus the sinks within the volume.

The divergence theorem can be generalized considerably. First, $\vec{u}$ does not have to be the flow velocity; the theorem holds for any vector field. Second, the theorem can be applied to higher-dimensional objects. Suppose, for example, that we take three separate vectors and concatenate them to form the columns of a matrix $\underset{\sim}{A}(\vec{x}) = \{\vec{u}^{(1)}, \vec{u}^{(2)}, \vec{u}^{(3)}\}$, or $A_{ij} = u_i^{(j)}$. We can then concatenate (4.2.6) and find

$$\oint_A A_{ij} n_i\, dA = \int_V \frac{\partial A_{ij}}{\partial x_i}\, dV \quad \text{for } j = 1,2,3. \tag{4.2.7}$$

One could also let the three vectors be the rows of $\underset{\sim}{A}$, in which case the dummy index in (4.2.7) would be the second index of $\underset{\sim}{A}$ instead of the first. [3]

4.3 Vorticity and circulation

Vorticity and circulation are two related measures of the tendency of a flow to rotate. The vorticity is the curl of the velocity field:

$$\vec{\omega} = \vec{\nabla}\times\vec{u},$$

or in index form

$$\omega_k = \varepsilon_{ijk}\frac{\partial}{\partial x_i}u_j.$$

Using our previous formula (4.1.4) for the curl, we can write the vorticity as

$$\vec{\omega} = \hat{e}^{(x)}(w_y - v_z) + \hat{e}^{(y)}(u_z - w_x) + \hat{e}^{(z)}(v_x - u_y), \tag{4.3.1}$$

where the subscripts denote partial differentiation.

[3] The derivation does not rely on $\underset{\sim}{A}$ having the transformation properties of a Cartesian tensor. Indeed, if its columns transform as vectors, then it will not. Conversely, $\underset{\sim}{A}$ may transform as a second-order tensor in which case its columns $\vec{u}^{(1)}$ will not transform as vectors. The theorem works regardless.

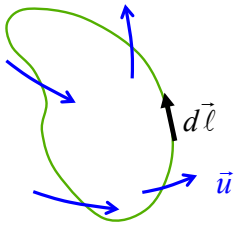


Figure 4.7: Definition sketch for the circulation. An arbitrary closed curve, with line element $d\vec{\ell}$, is embedded in an arbitrary flow field $\vec{u}(\vec{x})$.

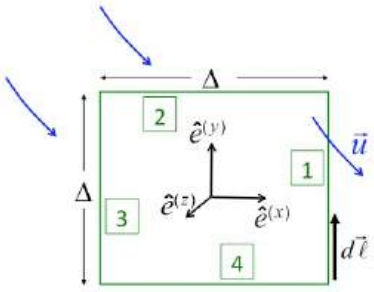


Figure 4.8: Circulation (4.3.2) around a $\Delta \times \Delta$ square on the $\hat{e}^{(x)} - \hat{e}^{(y)}$ plane with line element $d\vec{\ell}$.

The circulation is a line integral around an arbitrarily-chosen closed curve within the space occupied by the fluid. Its value depends on the curve, velocity of the fluid on the curve, and the direction (e.g., clockwise or counterclockwise) in which the curve is traversed. The circulation is defined as

$$\boxed{\Gamma = \oint \vec{u} \cdot d\vec{\ell}.} \tag{4.3.2}$$

4.3.1 Stokes' theorem

Stokes' theorem[4] makes the connection between the circulation around a curve and the vorticity within the curve. To derive Stokes' theorem, we first evaluate the circulation around a small square (figure 4.8). The flow field can be three-dimensional: $\vec{u} = \{u, v, w\}$, but the coordinates are aligned so that the square lies in the $\hat{e}^{(x)} - \hat{e}^{(y)}$ plane. We now calculate the line integral of $\vec{u} \cdot d\vec{\ell}$ along each of the four edges.

On edge #1, $\vec{\ell} = \hat{e}^{(y)}$, so $\vec{u} \cdot d\vec{\ell} = v\, dy$, and $x = \Delta/2$. Expanding the spatial variation of the velocity field using a first-order Taylor series, the line integral becomes

$$\Gamma^{[1]} = \int_{-\Delta/2}^{\Delta/2} dy \left[v^0 + v_x^0 \frac{\Delta}{2} + v_y^0\, y + v_z^0\, 0 \right] \tag{4.3.3}$$

$$= \left[\Delta v^0 + \Delta\, v_x^0 \frac{\Delta}{2} + 0 + 0 \right] \tag{4.3.4}$$

$$= \Delta \left(v^0 + v_x^0 \frac{\Delta}{2} \right). \tag{4.3.5}$$

On edge #3, $\vec{\ell} = -\hat{e}^{(y)}$, so $\vec{u} \cdot d\vec{\ell} = -v\, dy$, and $x = -\Delta/2$. Therefore

$$\Gamma^{[3]} = -\Delta \left(v^0 - v_x^0 \frac{\Delta}{2} \right),$$

and

$$\Gamma^{[1]} + \Gamma^{[3]} = 2\Delta v_x^0 \frac{\Delta}{2} = \Delta^2 v_x^0.$$

Similarly,

$$\Gamma^{[2]} + \Gamma^{[4]} = -\Delta^2 u_y^0.$$

Adding, we get the net circulation around the square:

$$\Gamma = \Delta^2 (v_x^0 - u_y^0).$$

[4] Sir George Stokes (1819-1903) was an Irish-born mathematician, physicist, politician and theologian. He was the Lucasian professor of mathematics at Cambridge University, the position held previously by Isaac Newton and today by Stephen Hawking. Sir George will join us again in chapter 6 when we discuss the Navier-Stokes equations.

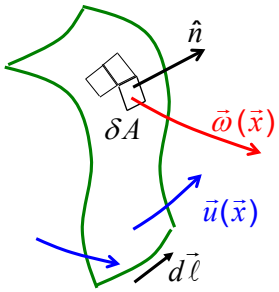


Figure 4.9: An arbitrary closed curve, with line element $d\vec{\ell}$, is embedded in an arbitrary flow field $\vec{u}(\vec{x})$ with vorticity $\vec{\omega}(\vec{x})$. A surface bounded by the curve (but otherwise arbitrary) is approximated by a collection of infinitesimal tiles with area δA and unit normal $\hat{n}$.

Referring back to (4.3.1), we see that the quantity in parentheses is the z-component of the vorticity, so taking the limit $\Delta \to 0$ we have

$$\delta\Gamma = \omega^{(z)}\delta A,$$

where δA is the limit of Δ^2 and $\omega^{(z)} = \vec{\omega}\cdot\hat{e}^{(z)}$.

We now generalize this result to the case of an arbitrary surface. We imagine the surface as an assemblage of square tiles. The circulation around each tile is given by (4.3.1). As we found in the case of the volume flux (figure 4.5), the line integrals along adjacent edges cancel. As a result, the net circulation is just the sum of the line integrals along the exterior edges. Summing over all of those edges in the limit as the tile size goes to zero, we have Stokes' theorem:

Theorem: Let Γ be the circulation around an arbitrary closed curve ℓ, taken in the direction specified by the line element $d\vec{\ell}$. Moreover, let A be any surface bounded by that closed curve. The unit normal to the surface, $\hat{n}$, is directed in accordance with the right-hand rule (with the fingers pointed along $d\vec{\ell}$). The circulation is then given by

$$\boxed{\Gamma = \oint \vec{u}\cdot d\vec{\ell} = \int_A \vec{\omega}\cdot\hat{n}\, dA.} \tag{4.3.6}$$

Back to Table of Contents.

Chapter 5

Fluid kinematics

> *"Flood time will repay you just to sit and watch. The river seems to be holding itself up before you like a page open to be read. There is no knowing how the currents move. They shift, and boil, and eddy. They are swifter in some places than in others. To think of a "place" on a flowing surface soon baffles your mind, for the places are ever changing and moving. The current, in all its various motions and speeds, flows along, and that flowing may be stirred again at the surface by the wind in all its various motions. Who can think of it?"* - Wendell Berry in "Jayber Crow"

The study of motion can be divided into two parts. *Kinematics* concerns the description of motion, while *dynamics* inquires into its causes. In elementary mechanics we are concerned with the motion of solid bodies, e.g., orbiting planets, billiard balls and the apple that supposedly fell on Newton's head. In these cases the motion is simple to describe, so we don't pay much attention to kinematics. In contrast, fluid motion can be extremely complicated. The task of describing such motion, a precondition to understanding its causes, is not trivial and is the subject of this chapter.

5.1 Lagrangian and Eulerian descriptions

We can observe a flow in two ways, first by focusing on the motion of a specific fluid parcel (see section 1.2), second by stepping back and looking at the pattern as a whole. These are called the Lagrangian and Eulerian descriptions of flow, respectively. Here we will seek to understand the distinction more fully, to become fluent with both points of view, and to translate between them.

Definitions:

1. Lagrangian information concerns the nature and behavior of fluid parcels.
2. Eulerian information concerns fields, i.e., properties like velocity, pressure and temperature that vary in time and space.

Figure 5.1: Cloud patterns reveal complex motions in Jupiter's atmosphere. The large vortex at the right is the Great Red Spot. Image courtesy of NASA.

Here are some examples:

1. Statements made in a weather forecast
   - "A cold air mass is moving in from the North." (Lagrangian)
   - "Here (your city), the temperature will decrease." (Eulerian)
2. Ocean observations
   - Moorings fixed in space (Eulerian)
   - Drifters that move with the current (Lagrangian)

The Lagrangian perspective is a natural way to describe the motion of solid objects. For example, suppose an apple falls from a tree. Newton taught us to describe the height and velocity *of the apple* as functions of time. This is a Lagrangian description. To try to describe this event in terms of Eulerian fields would be very awkward. You might define $A(z,t)$ as the "appliness": $A = 1$ at points in space and time occupied by the apple and 0 everywhere else, then try to derive a differential equation for A. Good luck.

The Eulerian perspective, while useless for solid objects, is natural for fluids. As we will see, it makes the math easier by providing partial differential equations for fields like velocity $\vec{u}(\vec{x},t)$ and temperature $T(\vec{x},t)$.

So for fluid mechanics, why do we not just stick to the Eulerian approach? The reason is that the existing laws of physics apply most naturally to fluid parcels. For example,

- if you apply heat to a fluid parcel, its temperature will increase: dT/dt = heating rate;
- if you apply a force to a fluid parcel, it will accelerate: $d\vec{u}/dt = \vec{F}/m$.

So in fluid mechanics we must be bilingual. When we set out to develop the equations of motion (next chapter), we'll do it in two steps:

1. apply the known laws of physics to fluid parcels
2. translate the results into Eulerian form for mathematical analysis.

But how do we accomplish this translation between Lagrangian and Eulerian perspectives? The key is an operation called the *material derivative*, which we discuss next.

5.1.1 The material derivative

Consider a function of time and space $\phi(t,\vec{x})$. For specificity, this could be some ocean property such as temperature or salinity. Now suppose that we evaluate $\phi(t,\vec{x})$ along some arbitrary trajectory $\vec{x}(t)$. That could be the course of an oceanographic vessel from which ϕ is measured. Let the velocity of the measurement point (i.e., the ship) be $\vec{v} = d\vec{x}/dt$. At what rate will our measured value of ϕ change in time? The answer is given by the chain rule:

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial x_j}\frac{dx_j}{dt} = \frac{\partial\phi}{\partial t} + v_j\frac{\partial\phi}{\partial x_j}.$$

(Note that derivatives of ϕ are partial derivatives because ϕ is a function of several variables, whereas derivatives of $\vec{x}$ are total derivatives because $\vec{x}$ is a function of time only.)

Now consider the special case in which $\vec{x}(t)$ is the trajectory of a fluid parcel, and the observer is following the same trajectory (e.g., a boat allowed to drift with the current). The velocity $\vec{v}$ is now the velocity of the flow *at the parcel's location* at any given time: $\vec{u}(t)$. In this special case, the rate of change we measure will be

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial t} + u_j\frac{\partial\phi}{\partial x_j}.$$

The expression on the right-hand side is called the *material derivative*, i.e., the time derivative following a material parcel. It is a total time derivative, but is distinguished by the use of an uppercase "D", e.g., $D\phi/Dt$. It can be written equivalently in index form or in vector form:

$$\boxed{\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u_j\frac{\partial}{\partial x_j} \equiv \frac{\partial}{\partial t} + \vec{u}\cdot\vec{\nabla}.} \tag{5.1.1}$$

The material derivative has a dual character: it expresses Lagrangian information (the rate of change following a fluid parcel), but does so in an Eulerian way, i.e., in terms of partial derivatives with respect to space and time.

It is instructive to solve (5.1.1) for the partial time derivative. Suppose, for example, that the field in question is air temperature. Then

$$\underbrace{\frac{\partial T}{\partial t}}_{\text{thermometer reading}} = \overbrace{\frac{DT}{Dt}}^{\text{heating/cooling}} - \underbrace{\vec{u}\cdot\vec{\nabla}T}_{\text{advection}}. \tag{5.1.2}$$

This tells us that the temperature at a given location can change for two reasons corresponding to the two terms on the right-hand side. The first term, DT/Dt, is nonzero only if the air parcels are actually being heated or cooled (heated by the sun, perhaps). The second term is due to the wind: if the wind is blowing from a warm place, the local temperature will rise[1]. This process, whereby local changes result from transport by the flow, is called *advection*[2].

Test your understanding by doing exercise 21.

5.2 The streamfunction

Many flows are approximately two-dimensional. For example, the thickness of the Earth's atmosphere relative to the planet is comparable to the skin on an apple. Large-scale atmospheric flows are therefore nearly two-dimensional. When a flow is two-dimensional and also incompressible ($\vec{\nabla}\cdot\vec{u}=0$), it can represented by means of a streamfunction Ψ. Suppose the two dimensions are x and y, with corresponding velocity components $u(x,y)$ and $v(x,y)$. Then Ψ is defined such that

$$u = -\frac{\partial\Psi}{\partial y}; \quad v = \frac{\partial\Psi}{\partial x}. \tag{5.2.1}$$

Curves of constant Ψ are called streamlines. The simple example of a pair of nearby minima is shown in figure 5.2(a).

Several properties of the streamfunction are noteworthy.

1. The definition (5.2.1) guarantees that $\vec{\nabla}\cdot\vec{u}$ will be zero. (Check this for yourself.)
2. The sign convention is arbitrary. The present convention leads to $\omega^{(z)} = +\nabla^2\Psi$.
3. You can add any fixed number to Ψ and it makes no difference to the resulting flow. In other words, *the streamfunction is defined only up to an additive constant.*
4. The direction of the flow vector $\vec{u}$ is perpendicular to $\vec{\nabla}\Psi$ or, equivalently, parallel to the streamlines:
$$\vec{u}\cdot\vec{\nabla}\Psi = u\frac{\partial\Psi}{\partial x} + v\frac{\partial\Psi}{\partial y} = uv - vu = 0,$$
using (5.2.1).
5. The flow direction is defined more fully by noting the signs of the derivatives in (5.2.1). These require that flow be clockwise around a maximum in Ψ and counterclockwise around a minimum (such as the pair of minima in figure 5.2). In figure 5.2a, the streamfunction increases (becomes less negative) from point "A" to point "B", hence $\partial\Psi/\partial x > 0$, hence $v > 0$.
6. The speed (velocity magnitude) is $\sqrt{u^2+v^2} = |\vec{\nabla}\Psi|$. Therefore, the flow is fastest where streamlines are clustered together, such as just outside the two peaks on figure 5.2a. Flow is slow where streamlines are widely spaced.

Given the velocity components u,v, one can easily invert (5.2.1) to obtain the streamfunction. We can start with either equation; here we'll pick the first one. Integrating:

$$\Psi = -\int u dy + f(x),$$

where f is an unknown function independent of y. Substituting this into the second of (5.2.1) gives

$$\frac{\partial\Psi}{\partial x} = -\int \frac{\partial u}{\partial x} dy + f'(x) = v.$$

[1]The minus sign in the advection term of (5.1.2) indicates that temperature change is determined by the direction the wind is *from*. This is why meteorologists (and sailors, and folksingers) traditionally name a wind by its origin, e.g., a "west wind" or "westerly wind" blows *from* the west. Oceanographers (such as the present author) eschew this perverse tradition; they refer instead to the direction a current flows *toward*, e.g., an eastward current.

[2]This is why we call $\vec{u}\cdot\vec{\nabla}$ the *advective derivative*, cf. section 4.1.5.

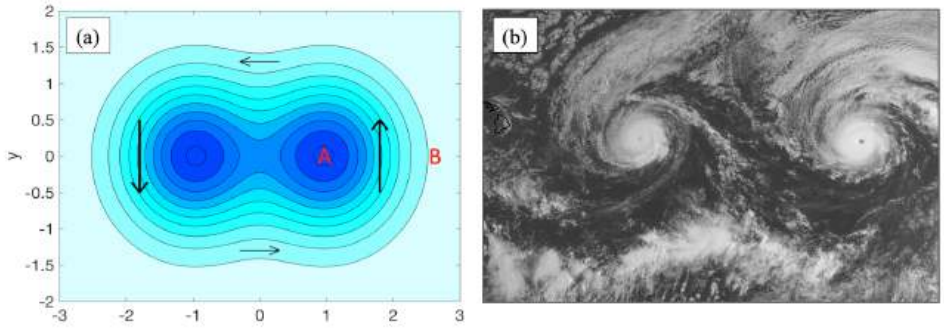


Figure 5.2: (a) Streamfunction for a pair of Gaussian vortices located at $x = \pm 1$: $\Psi = -e^{-(x-1)^2-y^2} - e^{-(x+1)^2-y^2}$. Blue indicates negative values. The sign of the streamfunction corresponds to positive vorticity (counterclockwise rotation as indicated by arrows). (b) An example in nature: satellite photo of hurricanes Madeline and Lester approaching Hawaii, August 29, 2016 (NASA Earth Observatory).

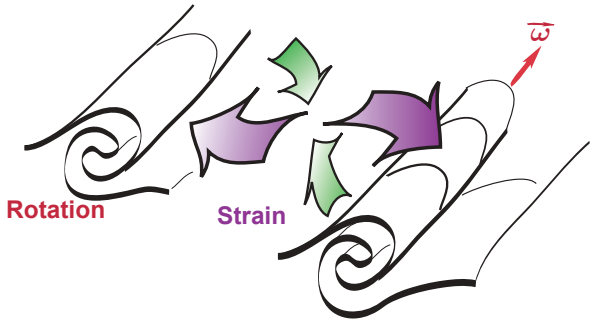


Figure 5.3: Schematic of co-rotating vortices, similar to figure 5.2, showing regions of rotation and strain.

This is readily solved for f', which we integrate to obtain f. Note that the constant of integration is arbitrary because of point 3 above.

Test your understanding by doing exercise 23, parts (a) and (b).

5.3 Rotation and strain: the relative motion of two nearby particles.

The complexity of flow can be halved, in a sense, by thinking of it as a combination of two simpler kinds of motion: rotation and strain (figure 5.3), with one or the other dominating at each point. We can then learn something useful by considering idealized flows consisting of rotation alone or strain alone.

Consider the instantaneous relative motion of two nearby fluid particles separated by the vector $\Delta\vec{x}$ (figure 5.4). Their velocities are

$$\frac{D}{Dt}\vec{x} = \vec{u}, \quad \text{and} \quad \frac{D}{Dt}(\vec{x}+\Delta\vec{x}) = \vec{u}+\Delta\vec{u},$$

and we can then subtract to see that

$$\frac{D}{Dt}\Delta\vec{x} = \Delta\vec{u}. \tag{5.3.1}$$

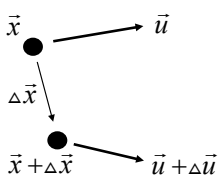


Figure 5.4: Relative motion of two nearby fluid particles.

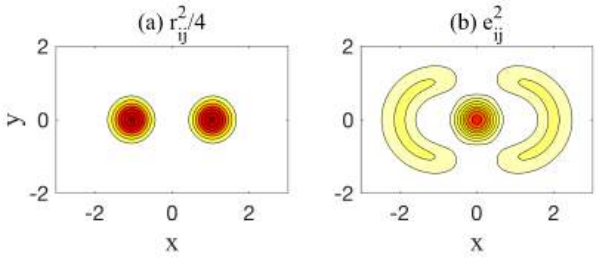


Figure 5.5: Strain and rotation magnitudes for the pair of two-dimensional, co-rotating Gaussian vortices shown in figure 5.2. Note that this flow geometry is also qualitatively similar to that sketched in figure 5.3. (a) the rotation magnitude $\frac{1}{4}r_{ij}^2$, (b) the strain magnitude e_{ij}^2.

The i^{th} component of the velocity difference $\Delta\vec{u}$ can be written as

$$\Delta u_i = \frac{\partial u_i}{\partial x_j}\Delta x_j. \tag{5.3.2}$$

The *velocity gradient tensor* $\partial u_i/\partial x_j$ can be decomposed into symmetric and antisymmetric parts:

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right),$$

or

$$\frac{\partial u_i}{\partial x_j} = e_{ij} + \frac{1}{2}r_{ij}, \tag{5.3.3}$$

where the symmetric part

$$\boxed{e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)} \tag{5.3.4}$$

is called the *strain rate tensor*[3] and the antisymmetric part (times two) is called the *rotation tensor*[4]:

$$\boxed{r_{ij} = \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}.} \tag{5.3.5}$$

Substituting in (5.3.2), we now write the velocity differential as

$$\Delta u_i = \frac{\partial u_i}{\partial x_j}\Delta x_j = e_{ij}\Delta x_j + \frac{1}{2}r_{ij}\Delta x_j \tag{5.3.6}$$

Any region of a flow can be characterized as strain-dominated or rotation-dominated depending on the relative magnitudes of the two terms on the right-hand side of (5.3.3). More specifically, we can multiply (5.3.3) by itself and get

$$\left(\frac{\partial u_i}{\partial x_j}\right)^2 = \left(e_{ij} + \frac{1}{2}r_{ij}\right)^2 = e_{ij}e_{ij} + \frac{1}{4}r_{ij}r_{ij}. \tag{5.3.7}$$

The cross term $e_{ij}r_{ij}$ is the product of a symmetric and an antisymmetric matrix and therefore vanishes identically (exercise 12). The remaining two terms are positive definite measures of the degree of strain and the degree of rotation. Vortices, not surprisingly, are rotation-dominated, e.g., the pair of corotating vortices shown in figure 5.5. (The stream function for this vortex pair is shown in figure 5.2.) The rotation term is greatest in the two vortex cores located at $x = \pm 1$ (figure 5.5a), while strain dominates in the region around the vortices, and especially between them (figure 5.5b, near $x = 0$).

The example in figure 5.5 is highly simplified; in a real flow the strained and rotating regions are intertwined in very complex ways, but are still recognizable. Figure 5.6 shows the evolution of turbulence in a shear layer. It begins (figure 5.6a) with the growth of co-rotating vortices (cf. figure 5.3). These become unstable (figure 5.6b) and break down into turbulence (figure 5.6c). The turbulence eventually decays, leaving a stable shear layer thickened by turbulent mixing. In the phase of vigorous turbulence, the strain magnitude e_{ij}^2 displays an intricate structure (figure 5.7). In the next two sections, we look more closely at the properties of rotation- and strain-dominated regions.

[3] *Strain* quantifies the *net deformation* of a material. The strain *rate* is its time derivative.

[4] It is only a matter of historical accident that e_{ij} is defined with the factor 1/2 and r_{ij} is not.

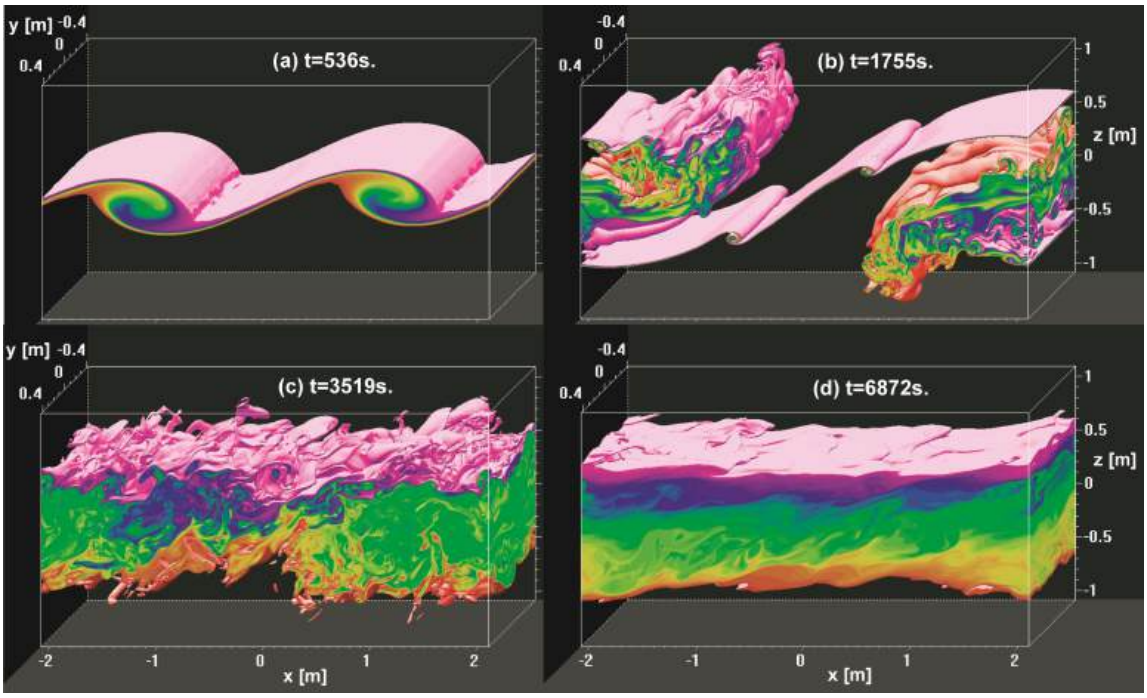


Figure 5.6: Evolution of turbulence between an upper layer of fresh water (blue-magenta) flowing to the right and a lower, salty layer (red) flowing to the left. Colors indicate intermediate salinity in the transition layer, with the outer layers rendered transparent for clarity. The flow was obtained as a numerical solution of the partial differential equations describing fluid motions in the Eulerian framework. These will be derived in chapter 6. (Image from Smyth and Thorpe 2012).

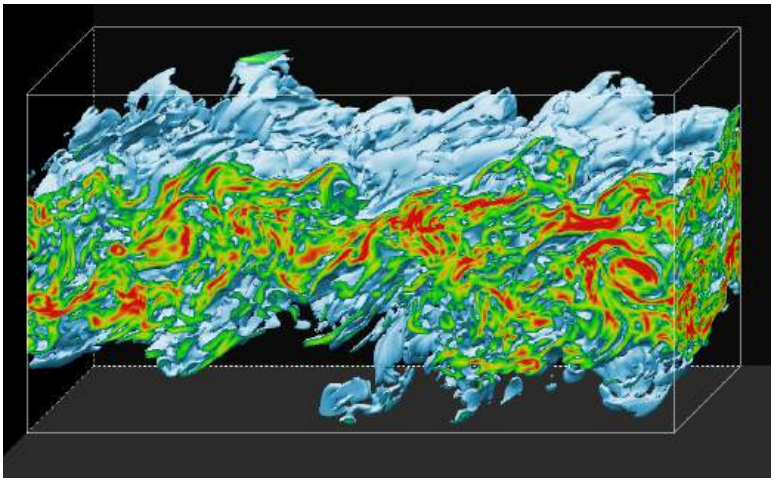


Figure 5.7: A snapshot of the strain magnitude e_{ij}^2 at t=3451s in the turbulent shear layer shown in figure 5.6. Red indicates regions of intense strain. The lowest values are rendered transparent.

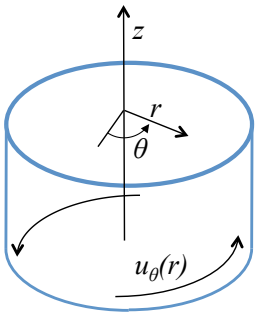


Figure 5.8: Axisymmetric (circular) vortex with cylindrical coordinates.

5.3.1 Rotation

The rotation tensor is closely related to a more familiar object: the vorticity vector $\vec{\omega}$:

$$\underset{\sim}{r} = \begin{bmatrix} 0 & \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} & \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} & 0 & \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} \\ \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} & \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix} \tag{5.3.8}$$

Reverting to index notation, we may write this relationship in a much more compact form:

$$r_{ij} = -\varepsilon_{ijk}\omega_k. \tag{5.3.9}$$

The contribution of rotation to the velocity differential in (5.3.6) is now

$$\frac{1}{2} r_{ij}\Delta x_j \;=\; -\frac{1}{2}\varepsilon_{ijk}\omega_k \Delta x_j \;=\; \frac{1}{2}(\vec{\omega}\times\Delta\vec{x})_i. \tag{5.3.10}$$

Thus, the change in velocity due to rotation is perpendicular to both the separation vector and the local vorticity. A consequence of this is that rotation does not change the distance $|\Delta\vec{x}|$ between the two particles; only strain can accomplish that. To show this explicitly, we write the equation for $|\Delta\vec{x}|$ (or, equivalently, $|\Delta\vec{x}|^2/2$) in vector form:

$$\frac{D}{Dt}\frac{1}{2}|\Delta\vec{x}|^2 \;=\; \Delta\vec{x}\cdot\frac{D}{Dt}\Delta\vec{x} \;=\; \Delta\vec{x}\cdot\Delta\vec{u} \;=\; \Delta\vec{x}\cdot(\underset{\sim}{e}\Delta\vec{x} + \frac{1}{2}\vec{\omega}\times\Delta\vec{x}) \;=\; \Delta\vec{x}\cdot\underset{\sim}{e}\Delta\vec{x}.$$

The second step above makes use of (5.3.1). Thus, changes in the distance between particles are cause only by strain.

5.3.2 Axisymmetric vortex models

Vortex motion is often approximately *axisymmetric*, i.e., invariant with respect to rotation about the vortex axis. Here we examine some very simple, axisymmetric vortex models. These are also called cylindrical, or circular, vortices.

Until now, we have measured space using Cartesian coordinates, but in some situations, curvilinear coordinates simplify the math. All of the mathematical constructs derived up to now can be expressed in curvilinear coordinates, and these expressions are listed in appendix I. The study of axisymmetric vortices is simplified using cylindrical polar coordinates (figure 5.8). In this case, every position in space has coordinates $\{r,\theta,z\}$ corresponding to the radial, azimuthal and axial directions, respectively. The corresponding velocity components are $\{u_r, u_\theta, u_z\}$. The vorticity is then given as the curl of the velocity vector:

$$\vec{\nabla}\times\vec{u} = \left\{\frac{1}{r}\frac{\partial u_z}{\partial\theta} - \frac{\partial u_\theta}{\partial z},\;\; \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r},\;\; \frac{1}{r}\frac{\partial (r u_\theta)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right\}.$$

In an axisymmetric vortex, the vorticity is purely axial and depends only on the radial coordinate:

$$\vec{\omega} = \omega(r)\hat{e}^{(z)}\;; \quad \omega(r) = \frac{1}{r}\frac{\partial (r u_\theta)}{\partial r}. \tag{5.3.11}$$

The circulation around such a vortex at any radius r is just $\Gamma(r) = 2\pi r u_\theta$ (show this). We'll look at three kinds of vortex motion in this geometrical context.

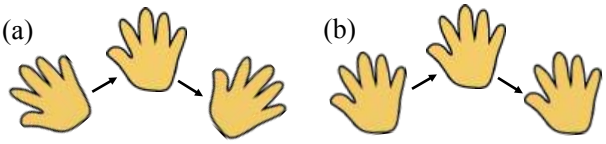


Figure 5.9: Waving (a) and wiping (b) motions.

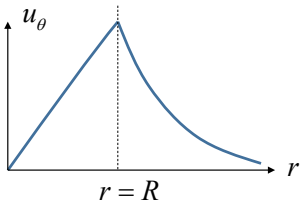


Figure 5.10: Azimuthal velocity in the Rankine vortex.

Rigid rotation

In this case the vorticity is uniform. Solving (5.3.11) gives

$$u_\theta = \frac{\omega}{2} r.$$

Note that the velocity is unbounded.

An irrotational vortex

In this case the motion is circular but the vorticity is zero. Solving (5.3.11) gives

$$u_\theta = \frac{C}{r},$$

where C is a constant of integration. The circulation is constant: $\Gamma = 2\pi C$. This gives us a meaningful way to identify the constant:

$$u_\theta = \frac{\Gamma}{2\pi r}.$$

This velocity distribution is unbounded at the origin.

To understand how motion can be circular but irrotational, consider the hand motions illustrated in figure 5.9. When you wave to someone (figure 5.9a), the orientation of your hand changes. When you wipe a window (figure 5.9b), your hand moves in a circle but its orientation doesn't change. Likewise, an object floating in an irrotational vortex would move in a circle without changing its orientation.

The Rankine vortex

The Rankine vortex[5] is a useful model for localized vortices such as tornadoes. The vorticity is uniform out to a radius $r = R$, and zero (irrotational) beyond that. The azimuthal velocity is sketched in figure 5.10. It is left as an exercise for the student to work out the mathematical expressions for u_θ and Γ.

Test your understanding by doing problems 24 and 25.

An isolated vortex

Consider the following vorticity distribution, sketched in figure 5.11:

$$\omega = \begin{cases} 2\dot{\theta}, & 0 \le r < R_1 \\ -2\dot{\theta}, & R_1 \le r \le R_2 \\ 0, & r > R_2 \end{cases} \tag{5.3.12}$$

[5] William John Macquorn Rankine (1820-1872) was a Scottish engineer whose primary interest was the thermodynamics of steam engines.

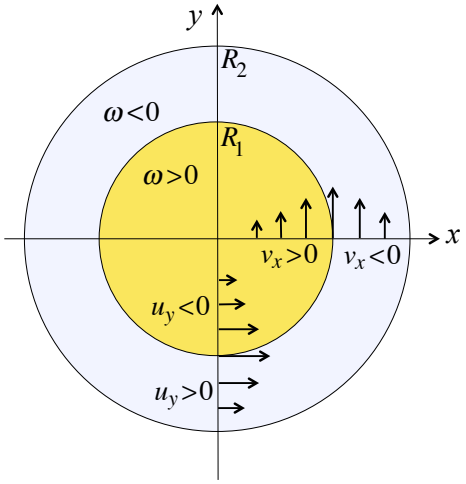


Figure 5.11: Vortex with a velocity maximum at $r = R_1$ and velocity dropping to zero by $r = R_2$. Vorticity $\omega = v_x - u_y$ (where subscripts denote partial derivatives) is positive in the inner region and negative in the outer ring.

where the constant $\dot{\theta}$ is the angular velocity (i.e. the time derivative of θ). On the x axis, azimuthal velocity is the same as the Cartesian velocity v, and is a maximum at $r = R_1$. Similarly, on the y axis, azimuthal velocity is u. From the signs of the derivatives of u and v, it is easy to see that the vorticity $\omega = v_x - u_y$ is positive for $r \le R_1$ and negative for $R_1 < r \le R_2$ as in (5.3.12).

A vortex is called *isolated* if its total circulation is zero. Here, the total circulation is

$$\Gamma_{total} = 2\pi \int_0^\infty \omega(r') r' \, dr'$$

can be evaluated for any $r \ge R_2$ since there is no vorticity (hence no change in circulation) in that region. The total circulation is zero if $R_2 = \sqrt{2} R_1$ (check for yourself).

5.3.3 Strain

The strain rate tensor is symmetric by definition:

$$\underset{\sim}{e} = \begin{bmatrix} u_x & \frac{1}{2}(u_y + v_x) & \frac{1}{2}(u_z + w_x) \\ \frac{1}{2}(u_y + v_x) & v_y & \frac{1}{2}(v_z + w_y) \\ \frac{1}{2}(u_z + w_x) & \frac{1}{2}(v_z + w_y) & w_z \end{bmatrix}. \tag{5.3.13}$$

Here, subscripts represent partial derivatives. The diagonal elements of $\underset{\sim}{e}$, namely u_x, v_y and w_z, represent normal strain. These can be either extensional or compressive depending on the sign (compare figures 5.12a,b). Off-diagonal components represent transverse strain, which may also be called tangential strain or shear (figure 5.12c).

Two points about normal strains are noteworthy.

- Imagine an irrotational straining motion in which the direction of the separation vector between two particles does not change, e.g., the normal strains shown in figures 5.12a and b. The separation vector must be an eigenvector of the strain rate tensor:

  $$\frac{d}{dt}\Delta\vec{x} = \Delta\vec{u} = \underset{\sim}{e}\Delta\vec{x} = \lambda\Delta\vec{x}.$$

  The solution of the above is

  $$\Delta\vec{x} = \Delta\vec{x}_0 \exp(\lambda t),$$

  i.e., the length of the separation vector grows or decays exponentially in time, and the corresponding eigenvalue gives the rate of growth/decay.

- The sum of the normal strains is the trace of the strain rate tensor, which is also equal to the divergence of the velocity field:

  $$\boxed{\mathrm{Tr}(\underset{\sim}{e}) = e_{ii} = \frac{\partial u_i}{\partial x_i} = \vec{\nabla}\cdot\vec{u}.} \tag{5.3.14}$$

  In an incompressible fluid, where $\vec{\nabla}\cdot\vec{u} = 0$, the normal strains must add to zero, i.e., extension and compression balance.

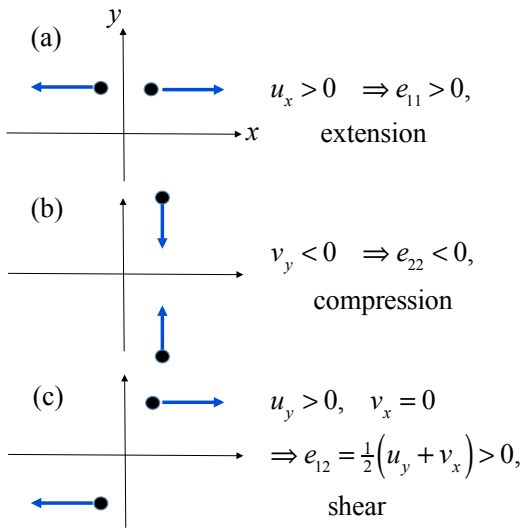


Figure 5.12: Pairs of particles in relative motion on the $x-y$ plane, illustrating the types of strain.

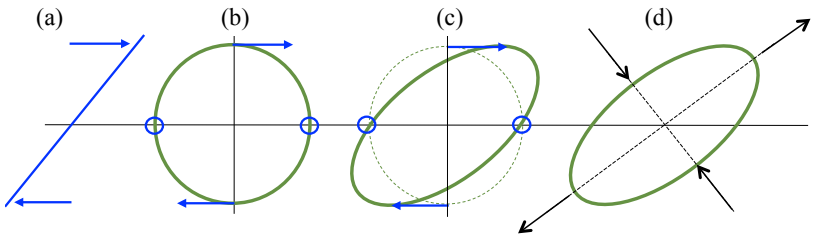


Figure 5.13: Evolution of an initially circular distribution of fluid particles in a uniform, transverse strain. (a) Velocity profile. (b) Initial circle. Arrows indicate the motion of the top and bottom of the circle; small circles indicate points that do not move. (c) After a short time the circle becomes a tilted ellipse. (d) When viewed in a tilted coordinate frame (dashed axes), the strains are purely extensional and compressive (black arrows).

5.3.4 The principal strains

Consider the evolution of a circular distribution of fluid particles advected by a uniformly sheared flow (figure 5.13), for which the strain is purely transverse (similar to figure 5.12c). Arrows show the velocity profile, with rightward motion in the upper half of the figure changing linearly to leftward motion in the lower half. As a result, the top of the circle moves to the right, the bottom moves to the left, and the sides don't move. After a short time, the circle becomes an ellipse with major axis tilted 45 degrees from the horizontal. When viewed in a coordinate frame tilted at the same angle (dashed lines), the circle is being expanded in one direction and compressed in the other. In other words, the transverse strain now appears as a purely normal strain.

This raises a crucial point: the distinction between normal and transverse strains depends on the choice of coordinates. In fact, we will now show that *any* strain is purely normal in an appropriately chosen coordinate system. Like any second-order tensor, the strain rate tensor can be expressed in a rotated reference frame using the transformation rule (3.3.5):

$$e'_{ij} = e_{kl} C_{ki} C_{lj}. \tag{5.3.15}$$

Recall that the columns of the rotation matrix $\underset{\sim}{C}$ are the basis vectors of the rotated coordinate system. Now, suppose that we transform $\underset{\sim}{e}$ into the special reference frame whose basis vectors are the *eigenvectors of* $\underset{\sim}{e}$. Then the j^{th} column of $\underset{\sim}{C}$ is the j^{th} eigenvector:

$$C_{lj} = v_l^{(j)}, \tag{5.3.16}$$

and $\lambda^{(j)}$ is the corresponding eigenvalue:

$$e_{kl} v_l^{(j)} = \lambda^{(j)} v_k^{(j)} \qquad \text{(no sum on } j\text{)}. \tag{5.3.17}$$

Now assume that the eigenvectors have been chosen to be orthogonal with length equal to 1 (and therefore $v_k^{(i)} v_k^{(j)} = \delta_{ij}$) and ordered so that $\det(\underset{\sim}{C}) = +1$. In other words $\underset{\sim}{C}$ represents a proper rotation. We now reorder (5.3.15) and substitute (5.3.16) and (5.3.17):

$$e'_{ij} = C_{ki}\, e_{kl} C_{lj} = v_k^{(i)}\, e_{kl} v_l^{(j)} = v_k^{(i)}\, \lambda^{(j)} v_k^{(j)} = \lambda^{(j)} \delta_{ij} \qquad \text{(no sum on } j\text{)}.$$

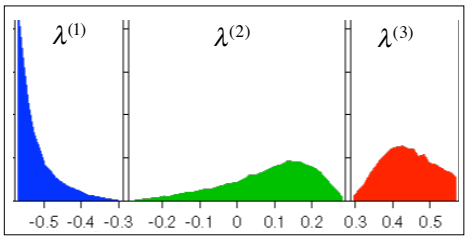


Figure 5.14: Probability distributions of the strain eigenvalues in a numerical simulation of sheared turbulence. The values are normalized by $\sqrt{2e_{ij}e_{ij}}$ (Smyth 1999).

The result is just a diagonal matrix with the eigenvalue $\lambda^{(j)}$ as the j^{th} diagonal element:

$$\underset{\sim}{e}' = \begin{bmatrix} \lambda^{(1)} & 0 & 0 \\ 0 & \lambda^{(2)} & 0 \\ 0 & 0 & \lambda^{(3)} \end{bmatrix}. \tag{5.3.18}$$

This special reference frame is called the *principal frame*. The basis vectors (the eigenvectors of $\underset{\sim}{e}$) are the *principal axes of strain* and the normal strains appearing on the main diagonal (the eigenvalues of $\underset{\sim}{e}$) are the *principal strains*.

Test your understanding by doing problem 23 .

The fine print: In this discussion, we have made two implicit assumptions about the strain rate tensor. First, we have assumed that its eigenvalues and eigenvectors are all real; otherwise, the geometrical interpretation of the principal axes and strains would make no sense. Second, we have assumed that the eigenvectors can be chosen to be orthogonal. Happily, these properties are guaranteed for *real, symmetric* matrices, of which the strain rate tensor is one. For further details see any linear algebra text (e.g., Bronson and Costa 2009).

5.4 Pancakes and noodles: the geometry of turbulence

In incompressible flow, the sum of the principal strains is zero:

$$\lambda^{(1)} + \lambda^{(2)} + \lambda^{(3)} = Tr(\underset{\sim}{e}) = \vec{\nabla} \cdot \vec{u} = 0.$$

This is easily seen in the principal frame, and can be shown to be true in any frame because both the eigenvalues and the trace are scalars. (Exercise: prove this.)

If we order the eigenvalues $\{\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}\}$ from smallest to largest, then $\lambda^{(1)} < 0$ and $\lambda^{(3)} > 0$, whereas $\lambda^{(2)}$ can have either sign. In any region of space where $\lambda^{(2)} > 0$ there are two extensional strains and one compressive strain. As a result, a spherical fluid parcel would be distended into an oblate spheroid or, less technically, a "pancake". In regions where $\lambda^{(2)} < 0$, there are two compressive strains and one extensional strain resulting in a prolate spheroid, or "noodle".

In the 1990s, computational hardware advances allowed the simulation of turbulent flows to determine which flow geometry is dominant. The procedure was to compute the eigenvalues of the strain rate tensor at every point in space, then see whether positive or negative values of $\lambda^{(2)} < 0$ were more common. Invariably, it was found that $\lambda^{(2)}$ has a tendency to be positive, i.e., a turbulent strain field is more likely to produce pancakes than noodles. An example from a flow similar to that shown in figures 5.6 and 5.7 is shown in figure 5.14 (middle frame).

The rotation field is dominated by one-dimensional regions of rapid rotation as sketched in figure 5.3, i.e., vortices. These tend to wrap the pancakes around themselves to form what we might think of as crepes.

Back to Table of Contents.

Chapter 6

Fluid dynamics

Fluid dynamics (as opposed to kinematics) inquires into the *causes* of fluid motion. Our discussion will be based on the two great theories of classical physics: Newton's laws of motion and the laws of thermodynamics. Newton's laws were designed to apply to rigid objects like apples and planets. How do we apply them to an object that not only accelerates but also changes its shape in response to an applied force? That challenge will occupy us for the first half of this chapter. Newton's laws lead naturally to a consideration of kinetic and potential energy. To these concepts we will add internal energy in accordance with the laws of thermodynamics. Finally, the addition of an equation of state (through which we describe the thermodynamic properties of the specific fluid we're interested in) results in a complete set of equations that will, we hope, describe the motion of real fluids.

Our approach to applying classical laws of physics to fluid motion involves the use of conservation laws. Specifically, we assume that

- The mass of a fluid parcel does not change.
- The momentum of a fluid parcel changes due to the applied forces in accordance with Newton's second law.
- The total energy of a parcel changes due work done and heat added in accordance with the first law of thermodynamics.

Mathematically, conservation laws for a fluid parcel take the Lagrangian form:

$$\frac{D}{Dt}\int_{V_m(t)} (\text{something})\, dV = \cdots, \tag{6.0.1}$$

where $V_m(t)$ is the fluid parcel: a volume of space that changes in time but always contains the same fluid. (The subscript m indicates a material volume; another term for a fluid parcel.) The equation says that the net amount of some property contained by the fluid parcel (e.g., mass, momentum, energy) changes in time according to whatever is on the right hand side. While the physical meaning of (6.0.1) is easy to understand, the result is an integro-differential equation whose solution would be quite daunting. We'll start by establishing Leibniz' rule, which allows us to convert Lagrangian conservation laws of the form (6.0.1) into Eulerian partial differential equations, which we at least have some hope of solving.

6.1 The Leibniz rule

Leibniz's rule[1] allows us to take the time derivative of an integral over a domain that is itself changing in time. Suppose that $f(\vec{x},t)$ is the volumetric concentration of some unspecified property we'll call "stuff". The Leibniz rule is mathematically valid for any function $f(\vec{x},t)$, but it's easiest to interpret physically if we imagine that f is something per unit volume. For a concrete example, imagine that the "stuff" is air, and f is then the mass of air molecules per unit volume, i.e., the density. Now consider a closed surface that can change arbitrarily in time (*not* a material volume, in general). Its area is $A(t)$ and the volume it encloses is $V(t)$ (figure 6.1a). The quantity of "stuff" contained in the volume at any given time is $\int_V f(\vec{x},t)dV$. That quantity can change in time in two ways.

First, the concentration f may change in time, e.g., the density of the air may change due to heating or cooling. If this were the only source of change, we could write:

$$\frac{d}{dt}\int_V f(\vec{x},t)dV = \int_V \frac{\partial f}{\partial t} dV. \tag{6.1.1}$$

[1] Gottfried Wilhelm Leibniz (1646-1716) was a German philosopher and mathematician who invented calculus independently of Isaac Newton. The notation "d/dx" that we use today comes from Leibniz's version. As a philosopher, Leibniz espoused the theory of "optimism", which holds that the universe we inhabit is the best one that God could have created. This cheery attitude is especially admirable given that Newton got all the credit for inventing calculus.

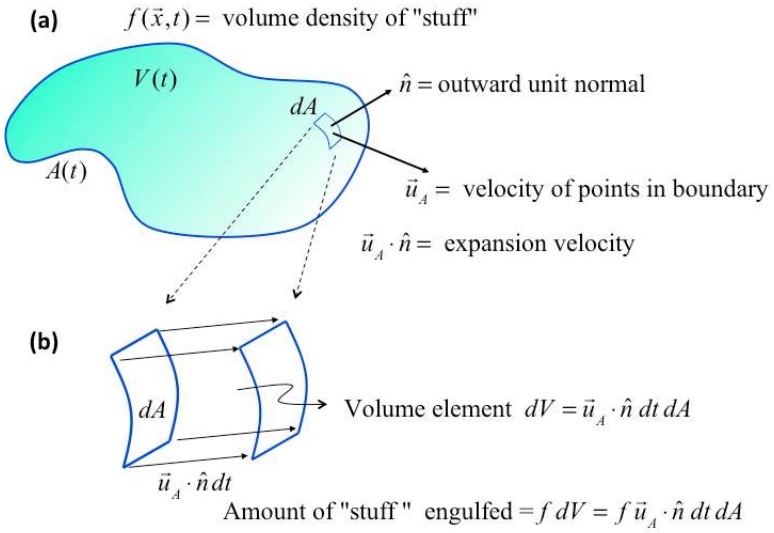


Figure 6.1: Definition sketch for Leibniz rule. (a) An arbitrary volume $V(t)$ enclosed by the closed surface $A(t)$. (b) A small volume engulfed as the surface element dA evolves over a brief time interval dt.

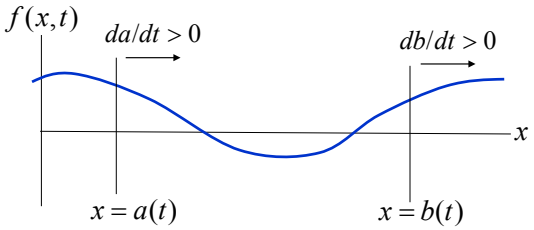


Figure 6.2: Definition sketch for the 1-dimensional form of Leibniz rule.

Second, the volume itself can change, for example, the volume could grow, thereby engulfing more "stuff". Quantifying this second contribution requires a bit more thought. At any point on the boundary we define $\hat{n}$ to be the outward-pointing normal vector (figure 6.1a). The points that make up the boundary have velocity $\vec{u}_A$, which varies over the boundary and also in time as the surface evolves. The expansion velocity, $\vec{u}_A\cdot\hat{n}$, is the component of $\vec{u}_A$ that is perpendicular to the boundary and directed outward. Now consider the motion of a small surface element over a brief time dt (figure 6.1b). The surface element moves by a distance $\vec{u}_A\cdot\hat{n}\,dt$, thereby enclosing a small volume $dV=\vec{u}_A\cdot\hat{n}\,dt\,dA$. The amount of "stuff" contained in this small volume is $f\,dV$, or $f\vec{u}_A\cdot\hat{n}\,dt\,dA$. If we now integrate this quantity over the whole surface, we get the amount of "stuff" engulfed (or ejected, if $\vec{u}_A\cdot\hat{n}<0$) in time dt: $\int_A f\vec{u}_A\cdot\hat{n}\,dt\,dA$. Dividing by dt and taking the limit $dt\to 0$, we have the second term that controls the change in the amount of "stuff" enclosed by our surface A:

$$\boxed{\frac{d}{dt}\int_{V(t)} f(\vec{x},t)dV=\int_{V(t)}\frac{\partial f}{\partial t}dV+\int_{A(t)} f\,\vec{u}_A\cdot\hat{n}\,dA.} \tag{6.1.2}$$

This is the most general form of Leibniz's rule.

Three special cases

1. If the surface is unchanging in time, then $\vec{u}_A=0$ and (6.1.2) is the same as (6.1.1).
2. Suppose that f is a function of only one spatial coordinate and time: $f=f(x,t)$. The integral is then an ordinary integral from, say, $x=a$ to $x=b$, but the boundaries a and b can vary in time (figure 6.2). In that case Leibniz' rule becomes
$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)dx=\int_{a(t)}^{b(t)}\frac{\partial f}{\partial t}dx+f(b,t)\frac{db}{dt}-f(a,t)\frac{da}{dt}. \tag{6.1.3}$$
The second and third terms on the right-hand side are the contributions due to the motion of the boundaries. This is the version of Leibniz' rule commonly found in calculus textbooks.
3. The most important case of (6.1.2) for fluid mechanics is that in which $A(t)$ is a material surface $A_m(t)$, always composed of the same fluid particles, and $V=V_m(t)$ is therefore a material volume (or fluid parcel). In this case $\vec{u}_A$ is

just $\vec{u}(\vec{x},t)$, the velocity of the motion, and the time derivative is D/Dt:

$$\frac{D}{Dt}\int_{V_m(t)} f(\vec{x},t)dV = \int_{V_m(t)} \frac{\partial f}{\partial t}dV + \int_{A_m(t)} f\vec{u}\cdot\hat{n}\,dA.$$

Note that the time derivative is defined as D/Dt because it is evaluated in a reference frame following the motion. It does not, however, have the form (5.1.1), as it does when applied to a continuous field. Now notice that, in the final term, the integrand is the dot product of the vector $f\vec{u}$ and the outward unit normal $\hat{n}$. According to the divergence theorem[2], we can convert this term to a volume integral:

$$\int_A (f\vec{u})\cdot\hat{n}\,dA = \int_V \vec{\nabla}\cdot(f\vec{u})\,dV.$$

We now have Leibniz' rule for a material volume: [3]

$$\boxed{\frac{D}{Dt}\int_{V_m(t)} f(\vec{x},t)\,dV = \int_{V_m(t)} \left(\frac{\partial f}{\partial t} + \vec{\nabla}\cdot f\vec{u}\right)\,dV.} \tag{6.1.4}$$

6.2 Mass conservation

In the introduction to this chapter we listed three assumptions that our theory of flow will be based on. We now have the tools to convert those assumptions into Eulerian form.

Our first assumption is that mass is neither created nor destroyed, which appears to be true absent nuclear reactions. In that case, the time derivative of the mass of a fluid parcel, taken on the parcel's trajectory, is

$$\frac{D}{Dt}\int_{V_m(t)} \rho(\vec{x},t)\,dV = 0, \tag{6.2.1}$$

where V_m is a material volume. This statement is in Lagrangian form. It is physically clear but difficult to handle mathematically. Invoking (6.1.4), we can write this equation as a volume integral:

$$\int_{V_m(t)} \left(\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\rho\vec{u}\right)\,dV = 0. \qquad \forall\, V_m \tag{6.2.2}$$

Now here is a critical point that we will encounter repeatedly: *the material volume V_m is chosen arbitrarily*; we could just as easily choose some different volume and the integral would still be zero. So, in what case may an integral be zero over *every* possible domain of integration? This can happen only if the integrand itself is zero everywhere. Therefore, mass conservation requires that

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\rho\vec{u} = 0} \tag{6.2.3}$$

at every point. This is commonly known as the continuity equation. Note that the product $\rho\vec{u}$ in 6.2.3 is the mass flux. The equation therefore tells us that the density at each point increases or decreases depending on whether the mass flux converges or diverges. We have now succeeded in converting the Lagrangian statement of mass conservation (6.2.1) into the Eulerian form (6.2.3). In other words, we have a partial differential equation which can, in principle at least, be solved.

A second form of (6.2.3) is equally useful. We use the product rule to split the second term into two,

$$\frac{\partial \rho}{\partial t} + \vec{u}\cdot\vec{\nabla}\rho + \rho\vec{\nabla}\cdot\vec{u} = 0,$$

and note that the first two terms now form the material derivative of ρ. Thus,

$$\boxed{\frac{D\rho}{Dt} + \rho\vec{\nabla}\cdot\vec{u} = 0.} \tag{6.2.4}$$

An important special case is that of an incompressible fluid, for which $\vec{\nabla}\cdot\vec{u} = 0$. In that case, the mass of any material volume remains constant:

$$\frac{D\rho}{Dt} = 0. \tag{6.2.5}$$

Density can still vary in space and time, but if you follow a fluid particle along its trajectory, its density will not change.

[2] The divergence theorem reads $\oint_A \vec{v}\cdot\hat{n}dA = \int_V \vec{\nabla}\cdot\vec{v}dV$, where $\vec{v}$ is a general vector and $\vec{n}$ is the unit normal to the closed surface A. See section 4.2.3 for details.

[3] Note that the integral on the left-hand side of (6.1.4) depends only on time. We write its time derivative as $D/Dt = \partial/\partial t + \vec{u}\cdot\vec{\nabla}$, even though only the term $\partial/\partial t$ is nonzero, and the derivative could be written just as accurately as d/dt. We choose the symbol D/Dt to remind ourselves that this time derivative is measured by an observer moving with the flow.

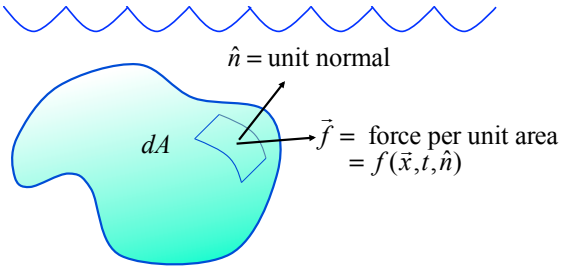


Figure 6.3: Definition sketch for the stress vector acting across an arbitrary surface A with unit normal $\hat{n}$.

6.3 Momentum conservation

The second assumption is that momentum changes only in accordance with Newton's second law, i.e., the rate of change of momentum of a fluid parcel equals the sum of forces applied. This assumption must be modified for relativistic flows, which occur in solar flares and supernovae, for example, but is otherwise very nearly exact.

6.3.1 Forces acting on a fluid parcel

We can write Newton's second law for a material volume $V_m(t)$ as an integro-differential equation analogous to (6.2.1):

$$\frac{D}{Dt}\int_{V_m} \rho\,\vec{u}\,dV = \Sigma_n\,\vec{\mathscr{F}}^{[n]}. \tag{6.3.1}$$

The superscript "$[n]$" is a label to enumerate the various forces that may be active. While the physical meaning of (6.3.1) is clear, its mathematical solution is less obvious. Our strategy is to convert it to a partial differential equation, as we did with mass conservation. First, we need to specify the individual forces $\vec{\mathscr{F}}^{[n]}$ acting on the parcel V_m. These fall into two categories: body forces and contact forces, which we now describe.

- **Body forces** act throughout the fluid parcel. The only body force we'll be concerned with here is gravity.[4] We'll assume that the gravitational field is uniform and constant, and hence can be represented by a vector $\vec{g}$ with units of acceleration (force per unit mass). The force per unit volume is then $\vec{F} = \rho\,\vec{g}$. To get the net gravitational force on a fluid parcel, we simply integrate the force per unit volume over the volume. The j-component is

  $$\int_{V_m} \rho\, g_j dV$$

  We often work in *gravity-aligned coordinates*, where the direction of gravity is taken to be $-\hat{e}^{(z)}$ (i.e., "down"), so that $\rho\,\vec{g}$ becomes $-\rho\, g\hat{e}^{(z)}$, g being the magnitude of $\vec{g}$. In this case the j component of the gravitational force per unit volume is $F_j = -\rho\, g\delta_{j3}$. On the Earth's surface, a typical value for g is $9.81 ms^{-2}$.

- **Contact forces** act only at the boundary of the fluid parcel. They are the macroscopic expression of intermolecular forces acting between molecules on opposite sides of the boundary. The mathematical description of these forces is one of the primary challenges of fluid dynamics, and we must overcome it before we can apply (6.3.1). The next subsection is devoted to the Cauchy stress tensor, through which we incorporate contact forces into (6.3.1).

6.3.2 The Cauchy stress tensor

Imagine pushing on a solid object with your fingertip, causing it to move. You are only actually pushing on a small part of the object, namely the part in direct contact with your finger, yet the whole object moves. How? Obviously the molecules pushed directly by your finger subsequently push their neighbors, which in turn push their neighbors, and so on. If the object is a rigid solid, the force is transmitted so that all parts of the object accelerate together, greatly simplifying the application of Newton's second law. But if the object is able to change its shape in response to your push, application of the second law presents a highly interesting challenge!

Stress is the means by which forces are transmitted through the interior of a continuous medium. A fluid is one example of a continuous medium, but the theory applies to elastic solids as well. Stress is the macroscopic expression of forces that act directly between neighboring molecules. It is the force per unit area $\vec{f}$ acting between molecules on either side of an arbitrary surface, which may be at a physical boundary of the medium (e.g., the ocean surface) or may be an imaginary interior surface (figure 6.3).

[4] Electromagnetic forces are another example of a body force. Their inclusion leads one into the fascinating realm of magnetohydrodynamics, which is unfortunately beyond our scope here (but see Choudhuri 1996).

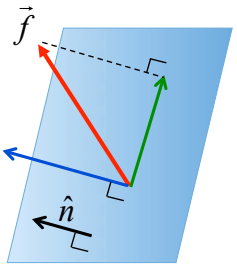


Figure 6.4: Stress vector at a surface, with unit normal $\vec{n}$, resolved into normal (blue) and tangential (green) components.

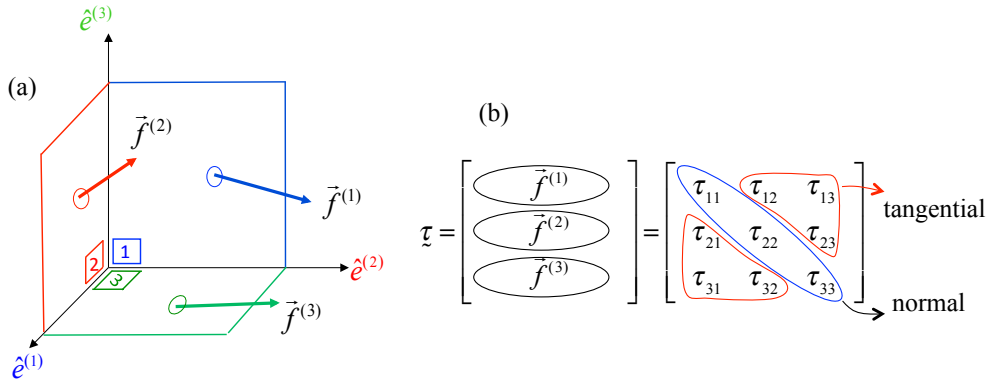


Figure 6.5: Stress on coordinate planes. (a) Vectors illustrate stress (force per unit area) acting on three coordinate planes. Colors indicate the force vector and the unit normal associated with each plane. (b) Array formed by the stress components, $\tau_{ij} = f_j^{(i)}$. Normal and tangential components are colored as in figure 6.4.

The stress vector $\vec{f}(\vec{x},t,\hat{n})$ represents the force per unit area acting at a point $\vec{x}$ on a surface where the unit normal is $\hat{n}$ (figure 6.3). It is useful to think of the force as the resultant of two separate forces, one normal to the plane (figure 6.4, shown in blue) and one tangential (green). The normal force is gotten by projecting $\vec{f}$ onto the normal vector: $(\vec{f}\cdot\hat{n})\hat{n}$. The tangential is what's left when we subtract the normal component: $\vec{f} - (\vec{f}\cdot\hat{n})\hat{n}$.

Some nomenclature: A normal stress may be referred to as *compression* (if it is directed toward the plane) or *tension*[5] (if it is directed away from the plane, as in figure 6.4). A tangential stress may also be called a *transverse*, or a *shear* stress. At the ocean surface, for example, the atmosphere exerts a compressive normal stress (pressure) and a tangential stress (wind).

Consider the simple case of stress on a coordinate plane, a plane perpendicular to one of the basis vectors. For example, the plane labelled 1 in figure 6.5a (shown in blue) is perpendicular to the coordinate basis vector $\hat{e}^{(1)}$. The stress (force per unit area) acting on this surface is defined as the vector $\vec{f}^{(1)}$. The normal stress is $f_1^{(1)}\hat{e}^{(1)}$; the tangential stress is $f_2^{(1)}\hat{e}^{(2)} + f_3^{(1)}\hat{e}^{(3)}$. Analogous vectors can be defined for the other two coordinate planes.

The three vectors $\vec{f}^{(i)}$ have a total of 9 components which may be collected to form an array we'll call the Cauchy array[6]:

$$\tau_{ij} = f_j^{(i)}. \tag{6.3.2}$$

Each row of the array is one of the vectors $\vec{f}^{(i)}$ (figure 6.5b). The first index of τ_{ij} denotes the coordinate plane that the force acts on. The second index denotes the direction of the force component. Each element on the main diagonal represents a normal stress, while off-diagonal elements represent the tangential stresses.

Reversing (6.3.2), we can write the j'th component of the force on the coordinate plane whose unit normal is $\vec{n}$ as

$$\boxed{f_j \;=\; \tau_{ij}\, n_i.} \tag{6.3.3}$$

For example, if the unit normal is $\hat{e}^{(1)}$, then $f_j = \tau_{ij}\delta_{i1} = \tau_{1j}$.

Now suppose we could show that the Cauchy array actually transforms as a tensor. If that were true, then (6.3.3) could describe the force acting on *any* plane, because we could rotate the coordinates so that the plane in question becomes a coordinate plane and (6.3.3)

[5] This is the origin of the word "tensor".

[6] Augustin-Louis Cauchy (1789-1857) was a French mathematician who pioneered group theory and complex analysis, as well as many areas of mathematical physics including the one we are interested in here: the mechanics of deformable materials. He is reputed to have more concepts and theorems named after him than any other mathematician, and you will see evidence of this.

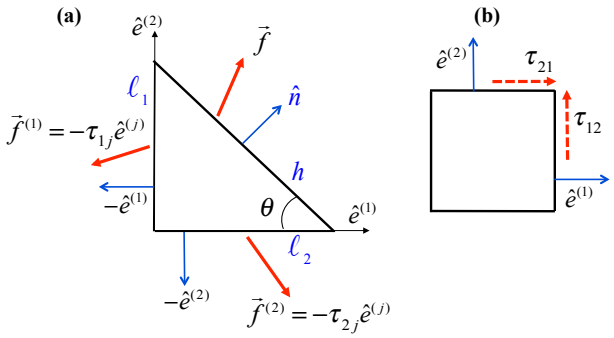


Figure 6.6: Two-dimensional representations of stresses acting on (a) a right triangle, (b) one corner of a rectangle. As the size of each figure goes to zero, net forces and torques must vanish.

is valid by definition. The proof that the Cauchy array is a tensor is somewhat involved. It is described in detail in appendix F. Here we give a simplified version.

Consider the net force acting on a fluid particle (a fluid parcel of infinitesimal size). In the limit as the mass becomes small, any nonzero net force results in infinite acceleration, contrary to what we observe. Therefore, the net force on a fluid particle must be zero. This is called the requirement of *local equilibrium*.

Consider a fluid parcel with the shape of a right triangle of uniform thickness, as shown on figure 6.6a. Look first at the left-hand face. The length of the face is ℓ_1, and its outward unit normal (blue arrow) is $-\hat{e}^{(1)}$. Therefore, (6.3.3) tells us that the force per unit area acting on the left-hand face is $-\tau_{1j}\hat{e}^{(j)}$. Similarly, the bottom face has length ℓ_2, outward normal $-\hat{e}^{(2)}$, and force/area $-\tau_{2j}\hat{e}^{(j)}$. Finally, let the hypotenuse have length h and force/area f.

In local equilibrium, those forces balance. For the jth component,

$$f_j h = \tau_{1j}\ell_1 + \tau_{2j}\ell_2.$$

(We have assumed that the thickness of the shape is uniform and therefore cancels out.) Now, note that $\ell_1 = h\sin\theta$ and $\ell_2 = h\cos\theta$; hence

$$f_j h = \tau_{1j} h\sin\theta + \tau_{2j} h\cos\theta.$$

Dividing by h and noting that $\hat{n}$ has components $\{\sin\theta, \cos\theta\}$, we can write this as

$$f_j = \tau_{1j} n_1 + \tau_{2j} n_2,$$

which is equivalent to (6.3.3). We conclude that the stress acting *at a point* obeys (6.3.3).

Equivalently, we can say that the array $\underset{\sim}{\tau}$ transforms as a 2nd-order tensor: $\tau'_{ij} = \tau_{kl} C_{ki} C_{lj}$. We call it the Cauchy stress tensor, or just the stress tensor. It describes a frame-independent relationship between two vectors: the force on a surface per unit area and the unit normal to that surface.

We can also show that the stress tensor must be symmetric. Consider the tangential stresses acting on two adjacent sides of a square (figure 6.6b). The torques exerted by these stresses act oppositely. Like the net force, the net torque must vanish as the size of the square goes to zero; otherwise there would be infinite angular acceleration. This requires that τ_{12} and τ_{21} be equal. By extending the argument to all three dimensions, we can show that the stress tensor at a point is symmetric (appendix F).

The symmetry of the stress tensor has three important consequences:

- The stress tensor has just six independent elements: three normal, and three tangential, to the coordinate planes.
- The formula (6.3.3) for the stress vector can be written in the equivalent form

$$\boxed{f_i \;=\; \tau_{ij}\, n_j,} \tag{6.3.4}$$

- Like the strain rate tensor (sections 5.3.3, 5.3.4), the stress tensor is a 2nd-order, real, symmetric tensor. It therefore has real eigenvalues, called the *principal stresses*, and real, orthogonal eigenvectors, the *principal axes of stress*. In the reference frame of the principal axes, the stress is entirely normal.

6.3.3 Cauchy's equation

The net contact force on a fluid parcel is the area integral of the force per unit area (defined in 6.3.3) over the boundary:

$$\int_{A_m} f_j dA = \int_{A_m} \tau_{ij} n_i dA.$$

The generalized divergence theorem (4.2.7) shows that the right-hand side is equal to the volume integral of the divergence of the stress tensor:

$$\int_{V_m} \frac{\partial \tau_{ij}}{\partial x_i} dV.$$

The sum of the forces forces appearing on the right-hand side of equation (6.3.1) can therefore be written entirely in terms of volume integrals:

$$\frac{D}{Dt}\int_{V_m} \rho u_j dV = \int_{V_m} \left(\rho g_j + \frac{\partial \tau_{ij}}{\partial x_i}\right) dV. \tag{6.3.5}$$

Now how about the left-hand side? We can convert it to a volume integral using (6.1.4) as we did in the case of mass conservation (section 6.2), but there is a very useful lemma that will make the result simpler.

Lemma:
Let χ be any quantity. Using Leibniz's rule applied to a material volume

$$\frac{D}{Dt}\int_{V_m} \rho\chi dV = \int_{V_m} \left[\frac{\partial}{\partial t}\rho\chi + \vec{\nabla}\cdot(\rho\chi\vec{u})\right] dV.$$

Expanding via the product rule, this becomes

$$\int_{V_m} \left[\rho\frac{\partial \chi}{\partial t} + \chi\underline{\frac{\partial \rho}{\partial t}} + \underline{\chi\vec{\nabla}\cdot(\rho\vec{u})} + \rho\vec{u}\cdot\vec{\nabla}\chi\right] dV.$$

The second and third terms (underlined) add up to zero, as we can see by removing the common factor χ and recognizing that what's left is the left-hand side of the mass equation (6.2.3). This leaves us with

$$\int_{V_m} \left[\rho\frac{\partial \chi}{\partial t} + \rho(\vec{u}\cdot\vec{\nabla})\chi\right] dV,$$

which you will recognize as

$$\int_{V_m} \rho\frac{D\chi}{Dt} dV.$$

Combining, we have

$$\boxed{\frac{D}{Dt}\int_{V_m} \rho\chi dV = \int_{V_m} \rho\frac{D\chi}{Dt} dV.} \tag{6.3.6}$$

Note that this equality follows only from conservation of mass and Leibniz's rule. Therefore, χ can be either a scalar or a component of a vector or tensor.

Note also that (6.3.6) is most useful when χ is a *specific concentration*, i.e., the amount of "something" per unit mass of fluid. In that case, $\rho\chi$ is the amount per unit volume (or volumetric concentration) and the volume integral is the total amount in the fluid parcel.

Using (6.3.6) with u_j as χ, we can rewrite (6.3.5) as

$$\frac{D}{Dt}\int_{V_m} \rho u_j dV = \int_{V_m} \rho\frac{Du_j}{Dt} dV = \int_{V_m} \rho g_j dV + \int_{V_m} \frac{\partial \tau_{ij}}{\partial x_i} dV,$$

or

$$\int_{V_m} \left\{\rho\frac{Du_j}{Dt} - \rho g_j - \frac{\partial \tau_{ij}}{\partial x_i}\right\} dV = 0 \qquad \forall\, V_m.$$

Now remember that the boundary of our fluid parcel V_m is chosen *arbitrarily*, so this volume integral must be zero for *any* fluid parcel we might choose. That can only be true if the integrand is identically zero. From this, we get the partial differential equation expressing Newton's second law for a deformable medium, called *Cauchy's equation*:

$$\boxed{\rho\frac{Du_j}{Dt} = \rho g_j + \frac{\partial \tau_{ij}}{\partial x_i}.} \tag{6.3.7}$$

6.3.4 Stress and strain in a Newtonian fluid

Cauchy's equation applies not only to fluids but also to other elastic materials such as rock, metal and ice. The same theory is therefore used by seismologists, engineers, glaciologists and many others. But now we must part company with these colleagues and press on into the realm of fluids. Note that the equations we have are not *closed*. Cauchy's equation represents 3 PDE's, and we already had one from mass conservation for a total of four. But there are more unknowns than this: density, three components of velocity, plus the six independent elements of the stress tensor. To close this system, we must specify in more detail the physical nature of the material we're talking about, and that will lead us to the idea of a "Newtonian" fluid.

We define a fluid as a material that flows in response to a shear (or transverse) stress (also see 1.2). If you apply a shear stress to a solid object it may bend or even break, but it will not flow. If it flows in response to a shear stress, it's a fluid. This includes air, water, paint, blood, ketchup, sand, ... all of which flow in response to a shear stress, though not all in the same way. We'll do this in two stages, talking first about a stationary (motionless) fluid, then introducing motion.

Stress in a fluid at rest

The condition for a fluid to be motionless is that there be no shear stress. In other words, the stress tensor must be *diagonal*, since shear stresses appear off the main diagonal. Notice, though, that the existence of shear stress depends on the choice of coordinates. Every stress tensor is diagonal in its principal coordinate frame (like the strain rate tensor as discussed in section 5.3.4). But if we rotate away from those axes, shear terms may appear off the main diagonal.

So, *for a fluid to be motionless, there can be no shear stresses in* **any** *coordinate frame*. This requires that the stress tensor be diagonal *in every coordinate frame*. This condition is satisfied only by tensors proportional to the identity tensor $\underset{\sim}{\delta}$ (see homework problem 28), meaning that the principal stresses must all be equal. So, we'll write the stress tensor for a motionless fluid as $\underset{\sim}{\delta}$ times some scalar, which is the common value of the principal stresses. It will turn out to be most convenient if we build a minus sign into the definition of that scalar, so $\underset{\sim}{\tau} = -p\underset{\sim}{\delta}$. We'll soon recognize that the scalar p is a familiar quantity, namely the pressure. Note that the corresponding stress vector is

$$f_j = \tau_{ij} n_i = -p\delta_{ij} n_i = -p n_j,$$

or

$$\vec{f} = -p\vec{n}.$$

The action of pressure on any surface is normal to the surface itself.

Substituting this stress tensor into Cauchy's equation, we have:

$$\rho \frac{Du_j}{Dt} = \rho g_j - \frac{\partial p}{\partial x_j},$$

and specifying $\vec{u} = 0$,

$$\frac{\partial p}{\partial x_j} = \rho g_j, \quad \text{or} \qquad \vec{\nabla} p = \rho \vec{g}.$$

This is the state of *hydrostatic equilibrium*: a balance between pressure and gravity that allows a fluid to remain motionless. In gravity-aligned coordinates, $g_j = -g\delta_{j3}$, so:

$$\frac{\partial p}{\partial x_j} = -\rho g \delta_{j3}.$$

The pressure cannot vary in the two horizontal directions ($j = 1, 2$, perpendicular to gravity), but varies in the vertical ($j = 3$) as prescribed by:

$$\frac{\partial p}{\partial z} = -\rho g. \tag{6.3.8}$$

For many applications it is useful to integrate the hydrostatic equation and thereby solve for pressure:

$$\boxed{p = \int_z^\infty \rho g dz.} \tag{6.3.9}$$

This tells us that the pressure at height z equals the *weight* of all fluid above z. In the atmosphere, $\rho \sim 1.2 kg/m^3$ at sea level and decreases to unmeasureably small values a few tens of kilometers up. The SI unit for pressure is Newtons per square meter, i.e., force per unit area

(which makes sense because it's a stress.) An abbreviation for this unit is the pascal (Pa). At sea level, the mean atmospheric pressure is about 10^5 Pa, which is sometimes referred to as one bar. Meteorologists express pressure in millibars, so mean surface pressure is 1000mbar. The old unit for pressure, which is still in common use, is pounds per square inch, or psi. Mean atmospheric pressure at the surface is 14 psi.

A well-inflated car tire holds about 28-30 psi in excess of the external (atmospheric) pressure, while pressures in bike tires can be as much as 130psi. Pressurized tires maintain their shape thanks to Pascal's Law: pressure acts equally in all directions. This is just another way to say that, in hydrostatic equilibrium, the stress tensor is isotropic. The force (per unit area) is $f_j = n_i \tau_{ij} = n_i(-p\delta_{ij}) = -pn_j$, so it acts opposite to the outward normal from any surface (real or fictitious) regardless of its orientation.

In oceanography, we usually define a Cartesian coordinate system such that $z = 0$ is at the ocean surface, and z-values in the ocean are therefore negative. The pressure is given by (8.2.3), but we can now break the range of integration into two parts; the integral from z to 0 (the ocean surface) and the integral from 0 to infinity, which is just atmospheric pressure at the surface. So pressure beneath the ocean surface is the sum of atmospheric pressure and the weight of the water above a given depth:

$$p = \int_z^0 \rho_{water} g dz + \int_0^\infty \rho_{air} g dz. \tag{6.3.10}$$

Stress in a moving fluid

> *"Air, I should explain, becomes wind when it is agitated."* - Titus Lucretius Carus, On the Nature of Things

We now allow our fluid to flow. To begin, we'll define a new quantity, a tensor expressing the added stress due to the fact that the fluid is in motion, i.e.

$$\tau_{ij} = -p\delta_{ij} + \sigma_{ij}. \tag{6.3.11}$$

This new tensor σ is called the "deviatoric" stress tensor. Despite the complicated name, this stress is familiar; it's basically just friction.[7]

And how do we write the deviatoric stress tensor? To do that, we need to specify the *kind* of fluid we're talking about, because deviatoric stress is different in different fluids. Remembering that deviatoric stress is just friction, you can imagine that it would be different in gooey maple syrup than in thin air, and the differences actually get more complicated than that.

We'll now make a sequence of three assumptions that define a hypothetical substance called *Newtonian fluid*. These will be labelled N1, N2 and N3.

> **Assumption N1**: **The deviatoric stress depends only on the velocity gradient tensor.** Is this assumption plausible? Yes, it makes intuitive sense that friction should depend on differences of velocity between different fluid parcels.

> **Assumption N2**: **Deviatoric stress is a *linear* function of the velocity gradients.** This is best understood in terms of a counterexample: paint (figure 6.7). Paint is manufactured so that, when you apply stress to it using your paintbrush, it flows easily. But after it's on the wall, the stress applied by gravity tends to make it run (i.e., strain), and you want to minimize that. So good quality paint has a nonlinear relationship between stress and strain, maximizing strain under the strong stress of the brush, but minimizing it under the weak stress of gravity. The same is true of ketchup. In an old-fashioned glass bottle, mild shaking causes the ketchup to stay right where it is. But if you shake harder, you reach a threshold where suddenly the ketchup flows, typically onto your shirt. So the assumption we're making here is that we're NOT talking about paint or ketchup, we're talking about a fluid in which strain increases with stress at a uniform rate. In this case we can write the stress-strain relationship using a 4-dimensional tensor:
>
> $$\sigma_{ij} = K_{ijkl} \frac{\partial u_k}{\partial x_l}. \tag{6.3.12}$$

You can also think of this as the first term in a Taylor series approximation of a more general function consistent with N1. In that case, we expect (6.3.12) to be valid for velocity gradients that are not too intense.

But how do we write K? Note that we seem to be going in the wrong direction: we started with 10 unknowns, and we've just added another 81! Bear with me.

> **Assumption N3**: **The fluid is "frictionally isotropic"**, i.e., friction acts the same in every direction. A counterexample to this would be blood, which has platelets in it that cause the fluid to flow more easily in some directions than in others. So, if

[7] Note also that, in this definition, the pressure p need not be in hydrostatic balance.

we're not dealing with paint, ketchup, blood or the like, we can assume that the deviatoric stress and the strain are connected by a 4-dimensional tensor K that is isotropic. In that case, the most general form for K is

$$K_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}, \tag{6.3.13}$$

where λ, μ and γ are scalars (appendix C).

So a Newtonian fluid is defined by the assumption that deviatoric stress tensor is proportional to strain tensor, and the 4-dimensional tensor of proportionality constants is isotropic. How does this make life better? Well, we've just reduced 81 unknowns to 3: λ, μ and γ. But we can do even better.

We already know is that the stress tensor is symmetric, and therefore so is the deviatoric stress tensor. From (6.3.12), it is obvious that $\underset{\sim}{K}$ is symmetric in its first and second indices. Knowing this, we can now show that γ and μ are the same. Working from (6.3.13), we write

$$\begin{aligned} K_{ijkl} - K_{jikl} &= 0 \\ &= \lambda(\delta_{ij} - \delta_{ji})\delta_{kl} + \mu(\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il}) + \gamma(\delta_{il}\delta_{jk} - \delta_{jl}\delta_{ik}). \end{aligned}$$

The first term on the right-hand side is zero because δ is symmetric. The products of δ's in the second term are the same as those in the third, just reversed; hence:

$$K_{ijkl} - K_{jikl} = (\mu - \gamma)(\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il}) = 0. \tag{6.3.14}$$

Note that this equation holds for *every* combination $ijkl$. The expression $\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il}$ on the right hand side will be zero for some such combinations, but not for all. The only way that (6.3.14) can hold for all $ijkl$ is if $\mu - \gamma = 0$.

As a result, (6.3.13) becomes

$$K_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{6.3.15}$$

eliminating one more unknown.

We can now substitute (6.3.15) in (6.3.12):

$$\begin{aligned} \sigma_{ij} &= K_{ijkl}\frac{\partial u_k}{\partial x_l} \\ &= \lambda\delta_{ij}\delta_{kl}\frac{\partial u_k}{\partial x_l} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\frac{\partial u_k}{\partial x_l} \\ &= \lambda\delta_{ij}\frac{\partial u_k}{\partial x_k} + \mu(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) \\ &= \lambda\delta_{ij}e_{kk} + 2\mu e_{ij}. \end{aligned}$$

We now substitute this result into (6.3.11), arriving at

$$\boxed{\tau_{ij} = -p\delta_{ij} + \lambda\delta_{ij}e_{kk} + 2\mu e_{ij}.} \tag{6.3.16}$$

This is an example of a *constitutive* relation. A fluid that obeys this particular constitutive relation is called a Newtonian fluid. Air and water are Newtonian to a very close approximation; ketchup is not. Before we substitute our stress tensor back into the Cauchy equation, let's look at the two new parameters that have appeared: μ and λ . Each represents a different kind of friction. The term involving μ contains e_{ij}, which you'll recall is the symmetric part of the velocity gradient tensor. The quantity μ is called the *dynamic viscosity*. Dynamic viscosity depends on temperature, but in many real-life situations it is nearly constant. Typical values are

$$\mu = \begin{cases} 10^{-3} kg\ m^{-1}s^{-1}, & \text{in water} \\ 1.7\times10^{-5} kg\ m^{-1}s^{-1}, & \text{in air.} \end{cases} \tag{6.3.17}$$

The quantity λ is called the "second viscosity''. As with pressure, the stress involving λ is proportional to the identity tensor, and therefore involves purely normal stresses. Moreover, it is proportional to $e_{kk} = \vec{\nabla}\cdot\vec{u}$. This friction responds to either expansion or contraction of the fluid, and is zero in an incompressible fluid. The second viscosity term is negligible in most applications to the atmosphere and oceans. Before long we're going to assume that it's zero and forget about it, but we can easily go back and retrieve it should we ever want to.

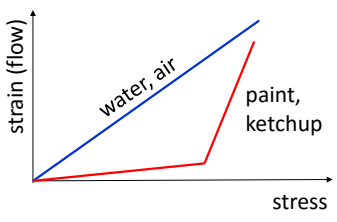


Figure 6.7: Schematic examples of constitutive relationships between stress and strain.

6.3.5 The Navier-Stokes equation

Now we can substitute the Newtonian stress tensor (6.3.16) into Cauchy's equation (6.3.7). For the special case of uniform λ and μ, the result is

$$\begin{aligned}\rho\frac{Du_j}{Dt} &= \rho g_j + \frac{\partial}{\partial x_i}\tau_{ij} \\ &= \rho g_j + \frac{\partial}{\partial x_i}\left[-p\delta_{ij} + \lambda\delta_{ij}e_{kk} + \mu(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})\right] \\ &= \rho g_j - \frac{\partial p}{\partial x_j} + \lambda\frac{\partial}{\partial x_j}e_{kk} + \mu\frac{\partial^2 u_j}{\partial x_i^2} + \mu\frac{\partial^2 u_i}{\partial x_j \partial x_i}. \end{aligned} \tag{6.3.18}$$

This is called the Navier-Stokes equation, and it's the central equation of fluid dynamics. In vector form it looks like this:

$$\boxed{\rho\frac{D\vec{u}}{Dt} = \rho\vec{g} - \vec{\nabla}p + \mu\nabla^2\vec{u} + (\lambda+\mu)\vec{\nabla}(\vec{\nabla}\cdot\vec{u})} \tag{6.3.19}$$

The final two terms on the right-hand side have been interchanged for clarity. For incompressible flow, the final term vanishes.

To summarize, the Navier-Stokes equation (6.3.19) describes how the velocity at a point evolves in response to four forces, corresponding to the four terms on its right-hand side:

- Gravity acts just as it does on solids.
- The pressure gradient force acts oppositely to the gradient of the pressure, pushing fluid from high- to low-pressure regions.
- The viscosity term represents the effect of friction as neighboring molecules collide. This tends to smooth out gradients in the velocity.
- The fourth term, nonzero only in compressible flow, describes viscous resistance to either expansion or contraction.

Test your understanding: It's important that you understand the assumptions that had to be made to arrive at (6.3.19). Go back through section 6.3 and identify the parts that were essential for this argument. Then list as many circumstances as you can think of in which (6.3.19) would *not* be valid.

6.3.6 Eddy viscosity

We think of viscosity as being due to molecular forces but it is also useful to think of it as being due to turbulence. Turbulence can be considered a phenomenon separate from the "main" flow. For example, flow over an airplane wing is smooth up to a certain speed, after which it develops little swirls and eddies. The "main" flow still goes over the wing, but it has this added, small-scale component. Turbulent eddies exert a drag on the plane, very similar to friction. To the pilot, it feels like the air has become more viscous. Models of this effect are essential in the design of airplane wings. Turbulence modeling is also necessary in every atmospheric and ocean model.

To use such a model, we define an "effective" viscosity (or eddy viscosity) due to turbulence. This "viscosity" is usually much greater than the molecular viscosity. In the simplest models, eddy viscosity is assumed to be a constant. But it is much more accurate to let eddy viscosity vary in space and time in response to changes in the large-scale flow. This requires re-deriving (6.3.18) and (6.3.19) with derivatives of μ and λ retained. A further generalization recognizes that turbulence may not be isotropic. In that case, assumption N3 must be revisited. We will normally assume that eddy viscosity is a scalar constant, like molecular viscosity.

6.4 Energy conservation in a Newtonian fluid

By assuming that mass and momentum are conserved, we have developed equations for density and flow velocity. To arrive at a closed set of equations, we must also invoke conservation of energy. We'll do this in a rather roundabout fashion. The discussion of energy conservation leads us to an intuitively appealing summary of the factors affecting the motion and evolution of a fluid parcel which we'll take some time to explore. It also frustrates our attempt at closure by introducing new variables, necessitating some additional assumptions about the nature of the fluid and the changes that it undergoes. Finally, we arrive at a closed system of equations that we can, in principle, solve to predict fluid behavior in a wide variety of situations.

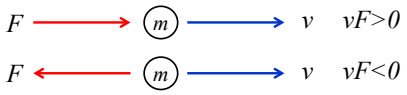


Figure 6.8: Schematic of the rate of working by a force on a solid object.

We can often gain greater understanding of a physical system by identifying its evolution as an exchange of energy among two or more" reservoirs'', or kinds of energy. In a Newtonian fluid, energy is exchanged between kinetic, potential and internal forms through various identifiable processes. Recall that a fluid is in fact made of molecules (section 1.2). What we call the flow velocity is really the average velocity of many molecules occupying a small space. Besides this average velocity, each molecule is doing its own complicated dance, whizzing around, spinning, oscillating, and colliding randomly with its neighbors. The kinetic energy of these microscopic motions is manifested macroscopically as the temperature of the fluid. In energetic terms, it is regarded as part of the *internal* energy. In a compressible flow, squeezing molecules together requires that work be done against intermolecular forces. The result is analogous to the storage of potential energy in a compressed spring, and is treated as part of the internal energy.

6.4.1 Kinetic energy

We begin by recalling some basic concepts from solid body mechanics. A solid object of mass m (see figure 6.8), moving at speed v, has kinetic energy

$$KE = \frac{1}{2}mv^2.$$

The object also has momentum mv, which changes in time according to Newton's second law when a force F is applied:

$$\frac{d}{dt}mv = F. \tag{6.4.1}$$

The connection between momentum and kinetic energy is made by multiplying both sides of (6.4.1) by v:

$$v\frac{d}{dt}mv = \frac{d}{dt}\frac{1}{2}mv^2 = vF.$$

The product vF is called the *rate of working* of the force F upon the object. If $vF > 0$, i.e., if the force acts in the direction that the object is already moving, it tends to increase the object's kinetic energy. The opposite is true if the force is opposite to the motion.

The kinetic energy of a fluid parcel is given by

$$KE = \int_{Vm} \frac{1}{2}\rho u_j^2 dV.$$

The analogue of Newton's second law is Cauchy's equation (6.3.7). Multiplying both sides of (6.3.7) by u_j, we have

$$\rho u_j \frac{Du_j}{Dt} = \rho u_j g_j + u_j \frac{\partial \tau_{ij}}{\partial x_i}. \tag{6.4.2}$$

The left-hand side of (6.4.2) is easily transformed using the product rule of differentiation (omitting the factor ρ for simplicity):

$$u_j \frac{Du_j}{Dt} = u_j \frac{\partial}{\partial t}u_j + u_j u_i \frac{\partial}{\partial x_i}u_j \tag{6.4.3}$$

$$= \frac{\partial}{\partial t}\frac{1}{2}u_j^2 + u_i \frac{\partial}{\partial x_i}\frac{1}{2}u_j^2 = \frac{D}{Dt}\left(\frac{1}{2}u_j^2\right). \tag{6.4.4}$$

Restoring ρ and integrating over the fluid parcel then gives

$$\int_{V_m} \rho u_j \frac{Du_j}{Dt} dV = \int_{V_m} \rho \frac{D}{Dt}\left(\frac{1}{2}u_j^2\right) dV = \frac{D}{Dt}\int_{V_m} \rho \frac{1}{2}u_j^2 dV = \frac{D}{Dt}KE,$$

where Cauchy's lemma (6.3.6) has been used for the second step. We therefore have an evolution equation for the kinetic energy of the fluid parcel:

$$\frac{D}{Dt}KE = \int_{V_m} \rho u_j g_j dV + \int_{V_m} u_j \frac{\partial \tau_{ij}}{\partial x_i} dV.$$

The terms on the right hand side represent the rates of working by gravity and by contact forces, respectively. The contact term is worth a closer look. Using the product rule, we can rewrite its integrand in two parts,

$$u_j \frac{\partial \tau_{ij}}{\partial x_i} = \frac{\partial}{\partial x_i}(u_j \tau_{ij}) - \tau_{ij}\frac{\partial}{\partial x_i}u_j, \tag{6.4.5}$$

which we will investigate seperately. The volume integral of the first term can be converted to a surface integral using the generalized divergence theorem (section 4.2.3)

$$\int_{V_m} \frac{\partial}{\partial x_i}(u_j \tau_{ij}) dV = \oint_{A_m} u_j \tau_{ij} n_i dA,$$

where $\hat{n}$ is the outward normal to the parcel boundary A_m. Noting that $\tau_{ij} n_i = f_j$, we can write this area integral as

$$\oint_{A_m} \vec{u} \cdot \vec{f} dA.$$

Analogous in form to (6.4.1), this is the rate of working by contact forces at the parcel boundary.

The second term in (6.4.5) is the rate of working by contact forces in the interior of the parcel. The integrand is split into two parts by recalling (5.3.3), the symmetric-antisymmetric decomposition of the deformation tensor:

$$\frac{\partial u_j}{\partial x_i} = e_{ji} + \frac{1}{2} r_{ji} = e_{ij} - \frac{1}{2} r_{ij}.$$

Therefore

$$-\tau_{ij} \frac{\partial u_j}{\partial x_i} = -\tau_{ij}(e_{ij} - \frac{1}{2} r_{ij}) = -e_{ij} \tau_{ij},$$

where the final equality results from the fact that $\underset{\sim}{r}$ is antisymmetric while $\underset{\sim}{\tau}$ is symmetric. This term can be further subdivided by substituting (6.3.16):

$$-e_{ij}\tau_{ij} = -e_{ij}(-p\delta_{ij} + \lambda \delta_{ij} e_{kk} + 2\mu e_{ij}) = p e_{jj} - 2\mu e_{ij} e_{ij} - \lambda e_{kk}^2$$

The three terms on the right-hand side represent distinct physical processes.

- The first term, $p e_{jj}$ represents the rate of working by pressure, or the *expansion work*. If the fluid is expanding, $e_{jj} > 0$, the outward force of pressure accelerates the expansion, increasing kinetic energy, whereas contraction is opposed by pressure.

- The second term is negative definite and is important enough to have its own symbol:

  $$-2\mu e_{ij} e_{ij} = -\rho \varepsilon.$$

  This term represents the action of ordinary viscosity, which decreases kinetic energy whenever strain is nonzero. This process is called *dissipation*, and ε is called the *kinetic energy dissipation rate*. [8] It is most commonly written as

  $$\varepsilon = 2\nu e_{ij}^2,$$

  where $\nu = \mu/\rho$ is the *kinematic viscosity*. Typical values are

  $$\nu = \begin{cases} 10^{-6} m^2 s^{-1}, & \text{in water} \\ 1.4 \times 10^{-5} m^2 s^{-1}, & \text{in air.} \end{cases} \tag{6.4.6}$$

- The final term represents the action of the second viscosity. The term is negative semidefinite: zero if the divergence is zero, negative if the divergence is nonzero. Therefore, the second viscosity opposes *any divergent motion*, either expansion or contraction. The second viscosity term is small in most naturally-occurring flows and will be neglected from here on, but it is easily retrieved if needed.

We can now assemble these various terms to make the evolution equation for the kinetic energy of the fluid parcel:

$$\boxed{\frac{D}{Dt} KE = \underbrace{\int_{V_m} \rho \vec{u} \cdot \vec{g} dV}_{\text{gravity}} + \underbrace{\oint_{A_m} \vec{u} \cdot \vec{f} dA}_{\text{surface contact}} + \underbrace{\int_{V_m} p \vec{\nabla} \cdot \vec{u}\, dV}_{\text{expansion work}} - \underbrace{\int_{V_m} \rho \varepsilon dV}_{\text{viscous dissipation}}}. \tag{6.4.7}$$

[8] This should not be confused with the Levi-Civita tensor $\underset{\sim}{\varepsilon}$ defined in section 3.3.7.

6.4.2 Mechanical energy

Further insight into the gravity term can be gained by working in gravity-aligned coordinates. The gravity vector is $\vec{g} = -g\hat{e}^{(z)}$, where g is taken to be a constant and $\hat{e}^{(z)}$ defines the vertical direction. We can now write

$$\vec{u}\cdot\vec{g} = -g\vec{u}\cdot\hat{e}^{(z)} = -gw,$$

where w is the vertical component of velocity. Now note that, as a parcel moves, w is the time derivative of its vertical coordinate:

$$\frac{Dz}{Dt} = \frac{\partial z}{\partial t} + u\frac{\partial z}{\partial x} + v\frac{\partial z}{\partial y} + w\frac{\partial z}{\partial z} = 0+0+0+w.$$

Since g is a constant, we have

$$\vec{u}\cdot\vec{g} = -\frac{D}{Dt}gz.$$

The gravity term in (6.4.7) now becomes

$$-\int_{V_m}\rho\frac{D}{Dt}(gz)dV = -\frac{D}{Dt}\int_{V_m}\rho\, gz dV,$$

where (6.3.6) has been used. The volume integral on the right hand side represents the potential energy of the fluid parcel; hence, the gravity term represents an exchange between kinetic and potential energies. We now have an equation for the sum of kinetic and potential energy, called the *mechanical energy*:

$$\boxed{\frac{D}{Dt}\int_{V_m}\left(\frac{1}{2}\rho|\vec{u}|^2+\rho gz\right) = \oint_{A_m}\vec{u}\cdot\vec{f}dA + \int_{V_m}p\vec{\nabla}\cdot\vec{u}\,dV - \int_{V_m}\rho\varepsilon dV.} \tag{6.4.8}$$

The concept of potential energy is equally valid in other coordinate frames. For the general case, we define Φ as the specific[9] potential energy such that the net potential energy of a fluid parcel is $PE = \int_{V_m}\rho\Phi dV$ and

$$\vec{u}\cdot\vec{g} = -\frac{D}{Dt}\Phi,$$

so that $\Phi = gz$ in the special case of gravity-aligned coordinates. We can write this as a Lagrangian evolution equation for the potential energy of a fluid parcel:

$$\frac{D}{Dt}\int_{V_m}\rho\Phi dV = -\int_{V_m}\rho\vec{u}\cdot\vec{g}dV. \tag{6.4.9}$$

6.4.3 Internal energy and heat

The *first law of thermodynamics* can be postulated as follows:

The change of energy stored in a physical object equals the work done on the object by its environment plus the heat added.

Before we explore the consequences of this for a fluid parcel, we represent the *heat flux* through a material as the vector field

$$\vec{q}(\vec{x},t) = -k\vec{\nabla}T + \vec{q}_{rad}, \tag{6.4.10}$$

where the two terms on the right hand side represent conduction and radiation, respectively. The scalar k is called the *thermal conductivity*. We define $\mathscr{I}$ as the internal energy per unit mass, so that $\rho\mathscr{I}$ is the internal energy per unit volume. We can now write the first law of thermodynamics as:

$$\frac{D}{Dt}\int_{V_m}\left(\frac{1}{2}\rho|\vec{u}|^2+\rho gz+\rho\mathscr{I}\right) = \oint_{A_m}\vec{u}\cdot\vec{f}dA - \oint_{A_m}\vec{q}\cdot\hat{n}\,dA. \tag{6.4.11}$$

Subtracting (6.4.8), we obtain an equation for the internal energy of the fluid parcel:

$$\frac{D}{Dt}\int_{V_m}\rho\mathscr{I}\,dV = \underbrace{-\oint_{A_m}\vec{q}\cdot\hat{n}\,dA}_{\text{heat input}} \underbrace{-\int_{V_m}p\vec{\nabla}\cdot\vec{u}\,dV}_{\text{loss to expansion}} + \underbrace{\int_{V_m}\rho\varepsilon\,dV}_{\text{viscous heating}}. \tag{6.4.12}$$

[9] per unit mass

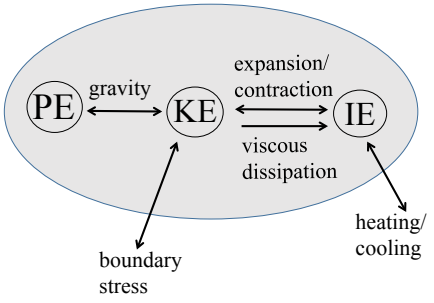


Figure 6.9: The energy associated with a fluid parcel is partitioned into three reservoirs: kinetic (KE), potential (PE) and internal (IE). Arrows show the various processes that mediate exchanges of energy between the reservoirs and between the parcel and its environment. These are described mathematically by (6.4.13), (6.4.14), and (6.4.15).

- The first term represents a gain of internal energy if heat is being absorbed by the parcel and a loss if heat is lost.
- If the parcel is expanding, the second term describes a conversion of the potential energy stored in the intermolecular forces to kinetic energy of expansion, and vice versa if the parcel is contracting.
- The third term is the conversion of kinetic energy to heat via visous dissipation, and is always positive (an expression of the second law of thermodynamics).

6.4.4 Summary

The Lagrangian equations for kinetic, potential and internal energy, collected below, can be summarized in the form of an energy budget diagram (figure 6.9). The boundary stress represents an interaction with the external environment, as does the heat flux term. These occur only once in the three equations. The remaining terms each occur twice with opposite signs; they therefore represent conversions between energy types within the parcel.

$$\frac{D}{Dt}\int_{V_m}\frac{1}{2}\rho u_i^2 dV = \underbrace{\int_{V_m}\rho\vec{u}\cdot\vec{g}dV}_{\text{gravity}} + \underbrace{\oint_{A_m}\vec{u}\cdot\vec{f}dA}_{\text{boundary stress}} + \underbrace{\int_{V_m}p\vec{\nabla}\cdot\vec{u}\,dV}_{\text{expansion}} - \underbrace{\int_{V_m}\rho\varepsilon dV}_{\text{dissipation}}. \tag{6.4.13}$$

$$\frac{D}{Dt}\int_{V_m}\rho\Phi dV = \underbrace{-\int_{V_m}\rho\vec{u}\cdot\vec{g}dV}_{\text{gravity}}. \tag{6.4.14}$$

$$\frac{D}{Dt}\int_{V_m}\rho\mathscr{I}\,dV = \underbrace{-\oint_{A_m}\vec{q}\cdot\hat{n}\,dA}_{\text{heat output}} - \underbrace{\int_{V_m}p\vec{\nabla}\cdot\vec{u}\,dV}_{\text{expansion}} + \underbrace{\int_{V_m}\rho\varepsilon\,dV}_{\text{dissipation}}. \tag{6.4.15}$$

6.5 The temperature (heat) equation

We now resume our quest for a closed set of equations to describe the flow of a Newtonian fluid. We previously assumed mass and momentum conservation, resulting in the density equation and the Navier-Stokes momentum equation (6.3.19). This collection of equations totals 4, but involves 5 unknowns: ρ, p, and the three components of $\vec{u}$.

We have now invoked a new assumption, namely energy conservation in the form of the 1st law of thermodynamics. This will allow us to add a new equation to the set. We first convert (6.4.12) to Eulerian form. Two terms require conversion to volume integrals.

- We apply Cauchy's lemma to the left -hand side, resulting in:

$$\frac{D}{Dt}\int_{V_m}\rho\mathscr{I}\,dV = \int_{V_m}\rho\frac{D\mathscr{I}}{Dt}\,dV.$$

- The heat loss can be converted using the divergence theorem (section 4.2.3):

$$\oint_{A_m}\vec{q}\cdot\hat{n}\,dA = \int_{V_m}\vec{\nabla}\cdot\vec{q}\,dV$$

We now have

$$\int_{V_m} \left(\rho \frac{D\mathscr{I}}{Dt} + \vec{\nabla} \cdot \vec{q} + p\vec{\nabla} \cdot \vec{u} - \rho\varepsilon \right) dV = 0. \qquad \forall\, V_m.$$

We conclude as usual that the integrand must be zero everywhere, resulting in:

$$\boxed{\rho \frac{D\mathscr{I}}{Dt} = k\nabla^2 T - \vec{\nabla} \cdot \vec{q}_{rad} - p\vec{\nabla} \cdot \vec{u} + \rho\varepsilon,} \tag{6.5.1}$$

where (6.4.10) has been used for the heat flux. We have gained a new equation, but have also introduced two new unknowns, the internal energy $\mathscr{I}$ and the temperature T. (Note that neither ε nor $\vec{q}_{rad}$ counts as an unknown. The former is determined by the velocity field via the strain, $\varepsilon = 2\nu e_{ij}^2$, while we assume that the latter is specified independently.) This leaves us short two pieces of information.

6.5.1 Specific heat capacity

Our next goal is to eliminate internal energy from the problem by establishing a relationship between it and temperature. We'll consider two possible approaches, each based on an assumption about the nature of the temperature changes that can be illustrated with a simple lab experiment.

Incompressible heating

Suppose that a fluid sample is contained in a closed, rigid vessel wherein it is slowly heated. We keep track of the heat input and the resulting temperature rise, and the two turn out to be approximately proportional:

$$\left(\frac{\partial Q}{\partial T} \right)_{\upsilon} = C_{\upsilon}. \tag{6.5.2}$$

Here Q is the specific heat content, i.e., heat content per unit mass, typically measured in joules per kilogram. The subscript υ on the partial derivative specifies that the specific volume $\upsilon = \rho^{-1}$ is held fixed while Q and T are changing. C_{υ} is called the *specific heat capacity at constant volume*, and can be regarded as constant if the range of temperatures is not too wide. Typical values are

$$C_{\upsilon} = \begin{cases} 4200J\ kg^{-1}K^{-1}, & \text{in water} \\ 700J\ kg^{-1}K^{-1}, & \text{in air.} \end{cases} \tag{6.5.3}$$

Now because this incompressible fluid does not expand or contract, changes in internal energy are due entirely to changes in heat content, so

$$\frac{D\mathscr{I}}{Dt} = \frac{DQ}{Dt} = C_{\upsilon} \frac{DT}{Dt}.$$

We can now rewrite (6.5.1) as

$$\rho C_{\upsilon} \frac{DT}{Dt} = k\nabla^2 T - \vec{\nabla} \cdot \vec{q}_{rad} + \rho\varepsilon, \tag{6.5.4}$$

keeping in mind that $\vec{\nabla} \cdot \vec{u} = 0$. We have now succeeded in eliminating E, but the solution only works if $\vec{\nabla} \cdot \vec{u} = 0$. Compressibility effects are not always negligible, especially in gases. To allow for that possibility, we imagine a slightly different experiment.

Isobaric heating

Suppose that we once again apply heat to a sample of fluid, but that the fluid is enclosed not in a rigid container but rather in a flexible membrane, like a balloon. As a result, the fluid can expand or contract freely, but the pressure does not change. (We assume that our balloon does not change altitude significantly, as would a weather balloon.) The process is therefore designated as *isobaric*.

For this process we define a new thermodynamic variable called the specific *enthalpy*, H. When a system changes slowly, the change in enthalpy is given by $\Delta H = \Delta\mathscr{I} + \Delta(p\upsilon)$. In an isobaric process, this becomes $\Delta H = \Delta\mathscr{I} + p\Delta\upsilon$. For a given change in temperature, the change in enthalpy is given by

$$\left(\frac{\partial H}{\partial T} \right)_{p} = C_p. \tag{6.5.5}$$

C_p is called the specific heat capacity at constant pressure, and is approximately constant for small temperature changes. Typical values are

$$C_p = \begin{cases} 4200J\ kg^{-1}K^{-1}, & \text{in water} \\ 1000J\ kg^{-1}K^{-1}, & \text{in air.} \end{cases} \tag{6.5.6}$$

For a fluid parcel undergoing this isobaric change, the material derivative of the enthalpy is

$$\frac{DH}{Dt} = \frac{D\mathscr{I}}{Dt} + p\frac{D\upsilon}{Dt} = C_p\frac{DT}{Dt}. \tag{6.5.7}$$

The material derivative of υ can be written as

$$\frac{D\upsilon}{Dt} = \frac{D\rho^{-1}}{Dt} = -\rho^{-2}\frac{D\rho}{Dt} = -\rho^{-2}(-\rho\vec{\nabla}\cdot\vec{u}) = \rho^{-1}\vec{\nabla}\cdot\vec{u}, \tag{6.5.8}$$

where mass conservation (6.2.4) has been invoked. Multiplying (6.5.7) by density and substituting (6.5.8), we have

$$\rho\frac{DH}{Dt} = \rho\frac{D\mathscr{I}}{Dt} + p\vec{\nabla}\cdot\vec{u} = \rho C_p\frac{DT}{Dt}$$

which, together with (6.5.1), gives

$$\rho C_p\frac{DT}{Dt} = k\nabla^2 T - \vec{\nabla}\cdot\vec{q}_{rad} + \rho\varepsilon. \tag{6.5.9}$$

6.5.2 The heat equation

The two temperature equations that hold in the incompressible and isobaric approximations, (6.5.4) and (6.5.9), differ only in the choice of the specific heat capacity:

$$\rho\,(C_\upsilon \text{ or } C_p)\frac{DT}{Dt} = k\nabla^2 T - \vec{\nabla}\cdot\vec{q}_{rad} + \rho\varepsilon.$$

Which approximation is better? In water, there is virtually no difference, because C_υ is nearly equal to C_p. In air, compressibility can be important, so the isobaric approximation is preferable. For geophysical applications, then, we choose the isobaric version:

$$\boxed{\rho C_p\frac{DT}{Dt} = k\nabla^2 T - \vec{\nabla}\cdot\vec{q}_{rad} + \rho\varepsilon.} \tag{6.5.10}$$

This is a generalization of the "heat equation'' often discussed in physics texts: neglecting the radiation and dissipation terms, it becomes

$$\frac{DT}{Dt} = \kappa_T\nabla^2 T. \tag{6.5.11}$$

Here, κ_T is the thermal diffusivity, given by

$$\kappa_T = \frac{k}{\rho C_p} = \begin{cases} 1.4\times10^{-7}m^2s^{-1}, & \text{in water} \\ 1.9\times10^{-5}m^2s^{-1}, & \text{in air} \end{cases} \tag{6.5.12}$$

By using (6.5.10) instead of (6.5.1) we are able to impose energy conservation while adding only one new unknown (instead of two), so at least we are no worse off in terms of closure. We now have 5 equations for 6 unknowns.

In the special case of incompressible fluid, the condition $\vec{\nabla}\cdot\vec{u} = 0$ represents an additional equation and the set is closed (i.e., no more equations are needed). For the more general case, we need to make a further assumption about the nature of the fluid. This will take the form of an equation of state. Again, there is more than one possibility.

6.6 Equations of state

Equations of state are laws that relate changes in density to changes in other thermodynamic variables. Each equation of state is an approximation. In a given situation, we want to choose the simplest approximation consistent with the level of accuracy we need. Here we will list several possibilities grouped into four categories of increasing complexity and (one hopes) accuracy.

6.6.1 Homogeneous fluid

The simplest approximation we can make is to assume that density is uniform. We refer to such a fluid as *homogeneous*. In this case, the equation of state is simply

$$\rho = \rho_0,$$

and our set of equations is closed.

6.6.2 Barotropic fluid

A slightly more general choice is to assume that density varies only due to changes in pressure:

$$\rho = \rho(p).$$

A fluid for which this is true is called *barotropic*. It can be shown that the variation of density with pressure supports the propagation of sound waves, and

$$\boxed{\left(\frac{\partial \rho}{\partial p}\right)_T = \frac{1}{c^2}} \tag{6.6.1}$$

where c is the speed of sound. The subscript T reminds us that the partial derivative is to be evaluated at fixed temperature. Typical values are

$$c = \begin{cases} 1500 ms^{-1}, & \text{in water} \\ 300 ms^{-1}, & \text{in air.} \end{cases}$$

6.6.3 Temperature-dependent: $\rho = \rho(p,T)$

To increase realism, we can allow for density to be governed by both pressure and temperature. We'll look at two examples:

1. Perhaps the best-known equation of state is the *ideal gas law*:

$$\rho(p,T) = \frac{p}{RT}, \tag{6.6.2}$$

   where R is the gas constant, equal to $287 J\ kg^{-1}K^{-1}$ for dry air. The ideal gas law can be derived from the assumption that molecular collisions conserve kinetic energy (Curry and Webster 1998).

2. For liquids, the equation of state can only be determined empirically, i.e., density is measured over a range of pressures and temperatures and the results are fitted to some mathematical function by adjusting values of constants. For example, a useful approximation for liquid water has the form

$$\rho(p,T) = \frac{\sum_{n=0}^{5} A_i T^n}{1 - p/p_0}$$

   with $A_0 = 999.842594$, $A_1 = 6.793952 \times 10^{-2}$, $A_2 = 9.09529 \times 10^{-3}$, $A_3 = 1.001685 \times 10^{-4}$, $A_4 = -1.120083 \times 10^{-6}$, $A_5 = 6.536332 \times 10^{-9}$, $p_0 = 19652$, T in Celsius and p in bars (Gill 1982). (If you're serious about computing the density of water, I suggest downloading a software package such as seawater, described in subsection 6.6.4).

The thermal expansion coefficient quantifies the tendency for a material to expand when heated.

$$\boxed{\alpha = -\frac{1}{\rho_0}\left(\frac{\partial \rho}{\partial T}\right)_{p,S}},$$

with typical values

$$\alpha = \begin{cases} 1 \times 10^{-4} K^{-1}, & \text{in water} \\ 3 \times 10^{-3} K^{-1}, & \text{in air.} \end{cases}$$

The thermal expansion coefficient can vary significantly and should only be treated as constant in nearly-uniform conditions.

6.6.4 Composition effects

The next step toward increased realism is to allow density to depend not only on pressure and temperature but also on the chemical composition of the fluid. We consider three examples.

1. The air in Earth's atmosphere has almost uniform composition except for a variable fraction of water vapor. We define the specific humidity q to be the mass of water per unit mass of air, measured in parts per thousand or g/kg. Accounting for humidity, the ideal gas law becomes

$$\rho(p,T) = \frac{p}{RT}\ \frac{1}{1+0.61q}. \tag{6.6.3}$$

   The constant 0.61 is determined by the molecular masses of air and water(Gill 1982).

2. In salt water, density is affected by salinity, defined as the mass of salt per unit mass of water. Salinity can be measured in parts per thousand, g/kg, commonly called *practical salinity units* (psu). Values range from zero in fresh water to 41psu in the Red Sea; 35psu is typical. The equation of state for seawater is entirely empirical and is too complicated to reproduce here. You can look it up in (for example) Gill (1982), or you can evaluate it using standard software such as the "seawater" package, currently available at

   `http://www.cmar.csiro.au/datacentre/ext_docs/seawater.htm`

6.6.5 Linearized equations of state

The empirical equations of state for liquids are far too cumbersome for use in analytical calculations. Because many problems involve only small variations in density, it is useful to work with the equation of state in a linearized form. In the case of seawater, for example, we can assume that p, T and S vary only slightly from some fixed values p_0, T_0 and S_0 at which the density is ρ_0. We can then represent the density dependence, $\rho = \rho(p,T,S)$, as a first-order Taylor series expansion

$$\rho = \rho_0 + \left(\frac{\partial \rho}{\partial p}\right)_{T,S}(p-p_0) + \left(\frac{\partial \rho}{\partial T}\right)_{p,S}(T-T_0) + \left(\frac{\partial \rho}{\partial S}\right)_{p,T}(S-S_0).$$

The partial derivatives are taken to be constants. One of these is the inverse square of the sound speed as discussed above:

$$\left(\frac{\partial \rho}{\partial p}\right)_{T,S} = \frac{1}{c^2},$$

where $c = c(p_0, T_0, S_0)$. A second is the thermal expansion coefficient α. We also use the saline density coefficient

$$\beta = \frac{1}{\rho_0}\left(\frac{\partial \rho}{\partial S}\right)_{p,T},$$

whose value in seawater remains fairly close to $7 \times 10^{-4} psu^{-1}$. With these definitions, the linearized equation of state can be written as

$$\frac{\rho - \rho_0}{\rho_0} = -\alpha(T-T_0) + \beta(S-S_0) + \frac{1}{c^2}(p-p_0). \tag{6.6.4}$$

6.7 The advection-diffusion equation for a scalar concentration

The composition-dependent equations of state all require adding a new variable to the equations of motion: q in the case of moist air, S in the case of seawater. Closure then requires that one more equation be added to account for the new variable. In the case of moist air, the new equation must account for the complex thermodynamics and chemistry of water vapor, and we will not go into it here (for a detailed discussion see Curry and Webster 1998). In the case of salinity, the solution is much simpler.

Salinity S is the mass of salt, in grams, per unit mass of salt water, in kilograms. The mass of salt in a fluid parcel is therefore given by $\int_{V_m} \rho S dV$. That mass of salt can change only by means of exchanges with the environment, represented by a salt flux $\vec{J}_S$:

$$\frac{D}{Dt}\int_{V_m} \rho S \, dV = -\oint_{A_m} \vec{J}_S \cdot \hat{n} dA = -\int_{V_m} \vec{\nabla} \cdot \vec{J}_S \, dV,$$

where the divergence theorem has been used for the final step. This Lagrangian statement can be converted to Eulerian form in the usual way, resulting in

$$\rho \frac{DS}{Dt} = -\vec{\nabla} \cdot \vec{J}_S.$$

The salt flux is well approximated by Fick's law:

$$\vec{J}_S = -\rho \kappa_S \vec{\nabla} S,$$

where κ_S is the molecular diffusivity of salt, with typical value $10^{-9} m^2 s^{-1}$. Neglecting small variations in $\rho \kappa_S$, we have a diffusion equation for salinity:

$$\boxed{\frac{DS}{Dt} = \kappa_S \nabla^2 S.} \tag{6.7.1}$$

6.8 Summary: the equations of motion

We now have the tools to specify a closed set of equations for most geophysical flows we are likely to encounter. Here we'll list the seven equations applicable to seawater:

$$\frac{D\rho}{Dt} = -\rho\vec{\nabla}\cdot\vec{u}, \tag{6.8.1}$$

$$\rho\frac{D\vec{u}}{Dt} = \rho\vec{g} - \vec{\nabla}p + \mu\nabla^2\vec{u} + \mu\vec{\nabla}(\vec{\nabla}\cdot\vec{u}), \tag{6.8.2}$$

$$\rho C_p\frac{DT}{Dt} = k\nabla^2 T - \vec{\nabla}\cdot\vec{q}_{rad} + \rho\varepsilon; \qquad \varepsilon = 2\nu e_{ij}^2, \tag{6.8.3}$$

$$\rho\frac{DS}{Dt} = -\vec{\nabla}\cdot\vec{J}_S; \qquad \vec{J}_S = -\rho\kappa_S\vec{\nabla}S, \tag{6.8.4}$$

$$\rho = \rho(p,T,S) \qquad \text{(the equation of state)}. \tag{6.8.5}$$

This set involves seven unknowns:

$$\rho, \vec{u}, p, T, \text{ and } S$$

and the following input parameters:

$$\vec{g}, \mu, C_p, k, q_{rad}, \text{ and } \kappa_S.$$

The second viscosity λ is neglected here as it is usually negligible in geophysical applications.

In atmospheric flows, the salinity equation is not needed, and the equation of state is the ideal gas law (6.6.2). If moisture effects are important, an advection-diffusion equation for the humidity q is added (section 6.7), the ideal gas law is modified as in (6.6.3) and the temperature equation (6.8.3) is extended to include latent heating effects (Curry and Webster 1998).

We have also neglected the effects of the Earth's rotation. The flows we deal with here happen on scales of distance and time that we can witness directly, say a few km or less and a few hours or less, and planetary rotation is unimportant for these motions. To deal with larger and/or slower flow phenomena (e.g., synoptic weather systems or ocean currents), we would have to account for the *Coriolis* and *centrifugal* accelerations. In the most common approximation, this requires adding a new term to the right-hand side of (6.8.2): $f\hat{e}^{(z)}\times\vec{u}$. Here $\hat{e}^{((z))}$ is the local vertical unit vector and $f = 1.46\times10^{-4}s^{-1}$ is proportional to the Earth's rotation rate (e.g., Vallis 2006). We will not go further into these effects in this book, but if you want to experience them directly, try playing catch on a merry-go-round.

6.8.1 Unpacking the equations of motion

The equations summarized above contain several vector differential operators of the sort described in section 4.1. Each of these is an abbreviation for one or more partial derivatives acting on one or more components of a vector. It is worthwhile to spend some time looking at more explicit versions of these equations to make sure we understand the operations involved.

The mass equation (6.2.4) can be written in index notation as

$$\left(\frac{\partial}{\partial t} + u_i\frac{\partial}{\partial x_i}\right)\rho = -\rho\frac{\partial u_i}{\partial x_i}.$$

Note that i is a dummy variable to be summed over. Using the familiar notation $\vec{u} = (u,v,w)$; $\vec{x} = (x,y,z)$, this can be expanded as

$$\frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial x} + v\frac{\partial\rho}{\partial y} + w\frac{\partial\rho}{\partial z} = -\rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right).$$

Similarly, we can write the momentum equation (6.8.2) as

$$\rho\left(\frac{\partial}{\partial t} + u_i\frac{\partial}{\partial x_i}\right)u_j = \rho g_j - \frac{\partial p}{\partial x_j} + \mu\frac{\partial^2}{\partial x_i^2}u_j + \mu\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_i}\right).$$

Here i is once again a dummy index, while j identifies the direction of the velocity component. For the case $j = 1$ we can write:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g_1 - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \mu\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right).$$

Exercise: Write out the corresponding equations for the cases $j = 2$ and $j = 3$, then repeat the exercise for (6.8.3) and (6.8.4). Also, familiarize yourself with the cylindrical and spherical versions summarized in appendix I.

6.9 Boundary conditions

> *"Land and water are not really separate things, but they are separate words, and we perceive through words."* - David Rains Wallace

Boundaries, in the real universe, are figments of the human imagination. We simplify the task of understanding reality by dividing it into parts (e.g., the atmosphere, the ocean) and trying to understand them individually. But our "parts" are not really separate - one blends smoothly into the other. For example, at the ocean surface the properties of the fluid change dramatically over a small distance, but a close look reveals a complex mixture of air, spray, foam, bubbles and water.

The equations of fluid motion remain valid at each point, and therefore any variable that appears differentiated with respect to space must be continuous to avoid infinities. Because the stress tensor is differentiated (as seen explicitly in 6.3.7), it must be continuous everywhere, and in particular at the surface of a body of water. One result of this is that the pressure immediately below the ocean surface must equal atmospheric pressure[10]. Another is that the transverse stress immediately below the surface must match that exerted by the wind.

We usually model the surface of a lake or ocean as a material surface (i.e., always composed of the same molecules) located where fluid properties change most rapidly. This surface is represented by its height at each horizontal location: $z = \eta(x,y,t)$, where the average of η is zero. Because the surface is material, we can write

$$\frac{D}{Dt}(z-\eta) = 0, \text{ or } w|_{z=\eta} = \frac{D\eta}{Dt} = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y}. \tag{6.9.1}$$

A special case of this is a fixed boundary, an example being the lower boundary of the atmosphere at the ground. This boundary can be modeled by (6.9.1), but with the vertical location $z = h(x,y)$ representing the Earth's surface:

$$\frac{D}{Dt}(z-h) = 0, \text{ or } w = \frac{Dh}{Dt} = u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} \text{ at } z = h.$$

At a solid boundary in a viscous fluid, we imagine fluid molecules becoming intermingled with boundary molecules (or crystals) and therefore require that the velocity itself be continuous. In the Earth's reference frame, the fluid velocity must approach zero at the boundary. This is called a "no-slip" boundary condition.

A useful idealization is the frictionless boundary, at which the velocity is not necessarily zero but the tangential (shear) stress must vanish. This minimizes the effect of friction on the flow and is realistic in cases where viscosity is unimportant. For example, frictionless flow over a horizontal boundary would obey

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0.$$

This is often called a "free-slip" boundary condition.

6.10 Solution methods

There is no general solution for the equations summarized in section 6.8, because they are inherently nonlinear. The main source of nonlinearity is in the advective part of the material derivative, e.g., $[\vec{u}\cdot\vec{\nabla}]\vec{u}$, which stymies standard solution methods as well as accounting for many of the most fascinating aspects of fluid motion. To make analytical progress, we must restrict our attention to very simple flow geometries. In recent decades numerical methods of solution have become increasingly important. While allowing progress on complex flows, numerical solutions have an important limitation. Each numerical solution pertains only to a single set of assumed parameter values. If we want to know how a flow varies with some parameter, we must create many such solutions, and we can never be sure that we've captured all of the variability.

For example, suppose we want to know how the wind speed over a mountain depends on the mountain's height. We could construct numerical solutions for mountains of height 1000m, 2000m, 3000m, etc., plot the results on a graph and draw a smooth curve connecting them. But what if something completely different happens for a mountain of height 1500m? No matter how closely we space our heights, we can never be certain that we are seeing the real picture. At what height is the speed a maximum? We can simulate forever and never be sure. The task is further complicated because wind speed over a mountain depends on many other parameters such as the width of the mountain and the upstream velocity. We can easily find ourselves doing thousands of simulations to describe one fairly simple flow geometry. Laboratory experiments, incidentally, suffer exactly the same limitation.

[10] neglecting the effect of surface tension; see exercise 33.

An analytical solution, even if it requires an extreme simplification of the physics, provides us with a mathematical description that we can examine in as much detail as we wish. For example, we can find the mountain height that maximizes wind speed simply by differentiating the solution. In the mountain example, the most useful solution follows from assumptions of this sort: "The flow varies mainly in the streamwise (x) direction and in height z, so derivatives with respect to t and y can be discarded."

In practice, progress in understanding fluids results from a combination of numerical solutions, analytical solutions and laboratory experiments, all of which must be compared with real-world observations to assess the validity of the underlying assumptions.

In what follows we will construct analytical solutions for a few very simple flow geometries that model phenomena we witness in everyday life. We do this to gain insight into the workings of these phenomena, but more importantly to test the validity of our model of Newtonian fluid mechanics by comparing its predictions with the behavior we observe.

Back to Table of Contents.

Chapter 7

Vortices

7.1 Introduction

Even in the most chaotic, turbulent flow, long-lived coherent vortices can be identified (e.g., figure 7.1). In this section we'll establish three theorems that tell us why vortices have such a remarkable tendency to stay together. To begin with, we'll study the basic aspects of vorticity in the simplest possible form, by neglecting complications due to viscosity and inhomogeneity, i.e., we'll assume that $\mu = \nu = 0$ and $\rho = \rho_0$. Later, we'll add the effects of viscosity and allow ρ to vary.

7.2 Vortex dynamics in a homogeneous, inviscid fluid

7.2.1 The vorticity equation

Assume $\rho = \rho_0$, $\nu = \mu = 0$. The mass and momentum equations (6.2.4, 6.8.2) are then

$$\vec{\nabla} \cdot \vec{u} = 0 \tag{7.2.1}$$

$$\frac{D\vec{u}}{Dt} = \vec{g} - \vec{\nabla}\frac{p}{\rho_0}. \tag{7.2.2}$$

Vorticity is defined as the curl of velocity:

$$\vec{\omega} = \vec{\nabla} \times \vec{u}. \tag{7.2.3}$$

To get an equation for $\vec{\omega}$, we take the curl of (7.2.2). Here are four vector identities that will be useful in that task:

1. $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{u}) \equiv 0$

Figure 7.1: A toroidal vortex, created in an eruption of Mt. Etna and made visible by volcanic steam. The bright spot is the sun (photo by Dr. J. Alean; details on page iii).

2. $\vec{\nabla}\times(\vec{\nabla}\phi)\equiv 0$
3. $[\vec{u}\cdot\vec{\nabla}]\vec{u}\equiv(\vec{\nabla}\times\vec{u})\times\vec{u}+\frac{1}{2}\vec{\nabla}(\vec{u}\cdot\vec{u})$
4. $\vec{\nabla}\times(\vec{u}\times\vec{v})\equiv[\vec{v}\cdot\vec{\nabla}]\vec{u}-\vec{v}(\vec{\nabla}\cdot\vec{u})-[\vec{u}\cdot\vec{\nabla}]\vec{v}+\vec{u}(\vec{\nabla}\cdot\vec{v})$

An immediate consequence of [1] is that the vorticity field is solenoidal:

$$\boxed{\vec{\nabla}\cdot\vec{\omega}=0.} \tag{7.2.4}$$

This is true for *every* flow, not just inviscid and homogeneous. Next, we apply the curl operator to (7.2.2), starting with the left hand side.

$$\begin{aligned}
\vec{\nabla}\times\frac{D\vec{u}}{Dt} &= \vec{\nabla}\times\left(\frac{\partial\vec{u}}{\partial t}+[\vec{u}\cdot\vec{\nabla}]\vec{u}\right)\\
&= \vec{\nabla}\times\left(\frac{\partial\vec{u}}{\partial t}+\vec{\omega}\times\vec{u}+\frac{1}{2}\vec{\nabla}(\vec{u}\cdot\vec{u})\right) \qquad [3]\\
&= \frac{\partial\vec{\omega}}{\partial t}+\vec{\nabla}\times(\vec{\omega}\times\vec{u})+0 \qquad [2]\\
&= \frac{\partial\vec{\omega}}{\partial t}+[\vec{u}\cdot\vec{\nabla}]\vec{\omega}-0-[\vec{\omega}\cdot\vec{\nabla}]\vec{u}+0 \qquad [4]\\
&= \frac{D\vec{\omega}}{Dt}-[\vec{\omega}\cdot\vec{\nabla}]\vec{u},
\end{aligned}$$

where the numbers indicate the vector identities employed. The right hand side is simpler:

$$\vec{\nabla}\times\left(\vec{g}-\vec{\nabla}\frac{p}{\rho_0}\right)=0,$$

because the first term is a constant and the second is a gradient (see [2]). The curl of (7.2.2) is therefore

$$\boxed{\frac{D\vec{\omega}}{Dt}=[\vec{\omega}\cdot\vec{\nabla}]\vec{u}.} \tag{7.2.5}$$

Aside: An interesting consequence of (7.2.5) is that, in homogeneous, inviscid flow, if the vorticity *of a fluid parcel* is zero at some initial time, it remains zero for all time. Therefore if the vorticity is zero *everywhere* at some initial time, it remains zero everywhere, even if the flow is accelerated by gravity or pressure. This leads to the idealization of *irrotational flow*, in which the math is simplified by assuming that $\vec{\omega}$ is identically zero. That idealization is useful in technological applications but less so in geophysical flows, so we will not pursue it here (for a discussion see Kundu et al. 2016).

7.2.2 Vortex filaments

Consider, as we did previously, the relative motion of two nearby fluid particles separated by a vector $\Delta\vec{x}$ [figure 7.2, also see (5.3.1)]. The material derivative of $\Delta\vec{x}$ is the velocity differential:

$$\frac{D}{Dt}\Delta x_i=\Delta u_i=\frac{\partial u_i}{\partial x_j}\Delta x_j=[\Delta x_j\frac{\partial}{\partial x_j}]\,u_i$$

or

$$\frac{D}{Dt}\Delta\vec{x}=[\Delta\vec{x}\cdot\vec{\nabla}]\vec{u}. \tag{7.2.6}$$

Note the correspondence in form between (7.2.5) and (7.2.6): the vorticity vector $\vec{\omega}$ and the particle separation vector $\Delta\vec{x}$ obey the same equation! Therefore, if $\vec{\omega}$ and $\Delta\vec{x}$ are parallel at some initial time, they will remain parallel for all future times. Consider the two points labelled $\vec{x}$ and $\vec{x}+\Delta\vec{x}$ at the upper left of figure 7.3. These are chosen so that the separation between them, $\Delta\vec{x}$, is parallel to the local vorticity vector $\vec{\omega}$. After a time interval Δt, the two points have moved to new positions $\vec{x}'$ and $\vec{x}'+\Delta\vec{x}'$, and the local vorticity vector has become $\vec{\omega}'$, but the separation vector is still parallel to the vorticity.

A vortex filament is a curve within the fluid that is everywhere parallel to the local vorticity, e.g., the upper yellow curve in figure 7.3. After the time interval Δt, each point on the vortex filament has moved to its new position, resulting in the lower yellow curve. Because

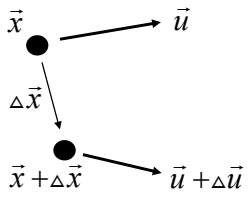


Figure 7.2: Relative motion of two nearby fluid particles, reproduced from figure 5.4).

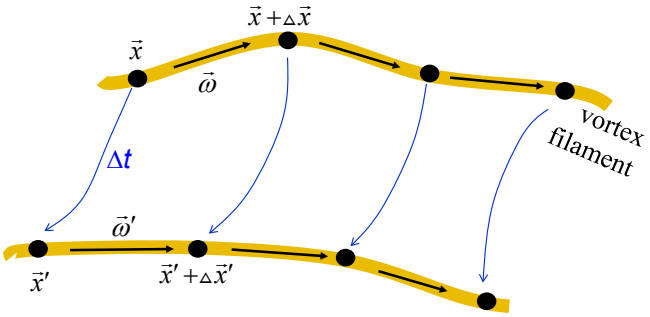


Figure 7.3: Motion over a time increment Δt of points lying on a vortex filament.

the separation vector between each pair of points on the curve remains parallel to the local vorticity vector, the yellow curve remains a vortex filament. We therefore have the following theorem:

Helmholtz's theorem #1 [1]: *In an inviscid, homogeneous fluid, vortex filaments move with the flow.* One may say that vortex filaments are "frozen" into the flow.

Remember that the time derivative of the separation vector can also be written in terms of the the strain rate and rotation tensors (cf. 5.3.6):

$$\frac{D}{Dt}\Delta\vec{x} = \Delta u_i = \frac{\partial u_i}{\partial x_j}\Delta x_j = (e_{ij} + \frac{1}{2}r_{ij})\Delta x_j. \tag{7.2.7}$$

It now follows from (7.2.5) that the same is true of the vorticity vector:

$$\frac{D}{Dt}\omega_i = (e_{ij} + \frac{1}{2}r_{ij})\omega_j. \tag{7.2.8}$$

Remembering that $r_{ij} = -\varepsilon_{ijk}\omega_k$, we note that the second term on the right-hand side must be zero:

$$r_{ij}\omega_j = -\varepsilon_{ijk}\omega_j\omega_k = 0,$$

as is easily seen by interchanging j and k. As a result, (7.2.8) becomes

$$\boxed{\frac{D}{Dt}\vec{\omega} = \underset{\sim}{e}\vec{\omega}.} \tag{7.2.9}$$

Equation (7.2.9) is equivalent to (7.2.5) , but it clearly shows the effect of strain on the vorticity.

- If the strain is *extensional*, it lengthens the vorticity vector just as it does the separation vector. In the ideal case where the vorticity is aligned with a principal strain, the vorticity grows exponentially, with growth rate equal to the corresponding strain rate eigenvalue. This exponential amplification of vorticity by extensional strain is called vortex stretching. Examples include tornadoes, where local rotation is amplified by stretching due to rising air, and a bathtub drain, where stretching is due to plunging water. If the local strain is *compressive*, the vorticity vector is compressed and its amplitude is correspondingly reduced.
- The off-diagonal elements of $\underset{\sim}{e}$ induce a tilting motion, converting one component of vorticity into another.

Vortex stretching is most efficient for vortices lying parallel to the principal axis of extensional strain. However, for a slowly-varying strain field, almost all vortices are rotated by the transverse strains so that they ultimately point in that optimal direction, as we now illustrate. Figure 7.4a shows a vorticity vector (red), with material particles delineating its ends, marked "1''. In the background is a strain field, drawn in two dimensions for simplicity. The vortex is oriented nearly parallel to the principal axes of compression. As a

[1] Hermann von Helmholtz (1821-1894) was a German physicist and physician. He is widely known among physiologists for his work on the physics of vision. He also wrote on philosophy, particularly the relationship of perception to physical laws. A well-known partial differential equation is named for him.

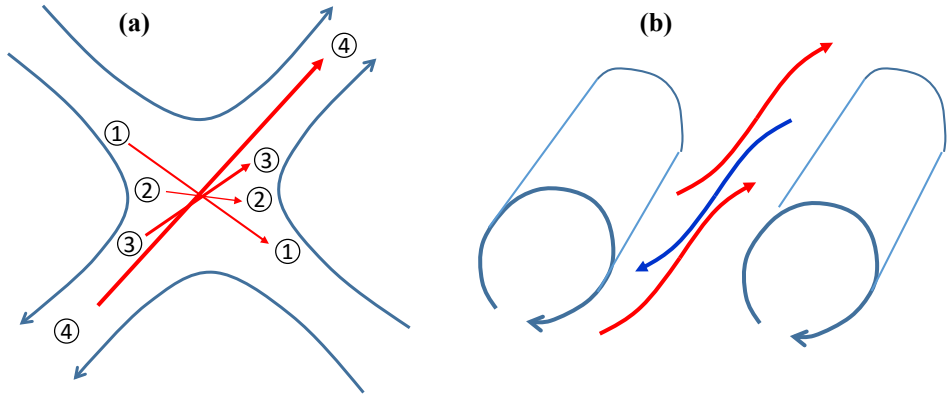


Figure 7.4: (a) A vorticity element (red) being tilted and stretched in a steady strain flield (blue). Numerals indicate the direction of time. (b) Counterrotating secondary vortices amplified by stretching in the extensional strain between corotating primary vortices.

result, the vortex is compressed and its magnitude (speed of rotation) decreases, indicated by thinner arrow at position 2. In the next stage of evolution (position 3), the particles are drawn toward the principal axis of extension. The vorticity vector is now stretched, and its magnitude increases accordingly. Ultimately (position 4), the vortex approaches perfect alignment with the principal axis of extension. In this limit the magnitude of the vorticity grows exponentially, with growth rate equal to the largest eigenvalue of the strain rate tensor.[2]

While the initial prientation of the vorticity vector shown in figure 7.4a is arbitrary, the end state is virtually always the same: the vector winds up growing exponentially in the direction of maximum extensional strain. (The only exception is if the initial vector is *exactly* perpendicular to the extensional strain axis, in which case it will compress to zero.)

Vortex stretching leads to a phenomenon often seen in nature - the creation of vorticity by a strain field. The flow may be almost entirely strain, but because stretched vorticity grows exponentially (in this simple geometry, at least), even a very weak vortical disturbance can rapidly become dominant. Figure 7.4b shows a strain field between two corotating vortices. In the region between the vortices, small-amplitude vortices are stretched by the process shown in figure 7.4a, soon becoming aligned with the extensional strain. If you observe waves breaking on a beach, you often see that the crest of the wave just before it breaks is riven with small ripples, only an inch or so wide, aligned perpendicular to the wave crests. These are a manifestation of vortex stretching by the strain created around the wave crest as the wave grows.

7.2.3 Vorticity conservation in planar flow

Consider a flow that lies entirely in the $x-y$ plane, so that the vorticity is $\omega(x,y)\hat{e}^{(z)}$. In this case the "3" component of (7.2.9) is

$$\frac{D}{Dt}\omega = e_{3j}\omega_j = \omega e_{33}.$$

But the only nonzero components of the strain rate tensor will correspond to $i = 1,2$ and $j = 1,2$ (see exercise 23, for example), so that $e_{33} = 0$. As a result $D\vec{\omega}/Dt = 0$. This result is true for all 2D flows: i.e., vorticity is conserved following the motion. There is no vortex stretching, tilting or compression in a 2D flow.

7.2.4 Vortex tubes

A vortex filament is infinitely thin, while vortices we observe have thickness. To model these, we define a vortex tube:

Definition: A vortex tube is a bundle of vortex filaments (figure 7.5).

The measure of the strength of a vortex tube is its circulation, taken around any cross section. This would not be a useful measure if it depended on the cross section chosen. Helmholtz's theorem #2 assures us that it does not:

Helmholtz theorem #2: *Circulation is the same for every cross-section through a vortex tube.* This theorem has two important corollaries:

[2] You can show this more generally by approximating $\underset{\sim}{e}$ as a constant matrix and writing the general solution of (7.2.9) as a linear combination of its eigenvectors. Each coefficient grows or decays exponentially at a rate given by the corresponding strain rate eigenvalue. Eventually, the term with the largest positive eigenvalue will dominate, i.e., $\vec{\omega}$ will approach the direction of greatest extensional strain.

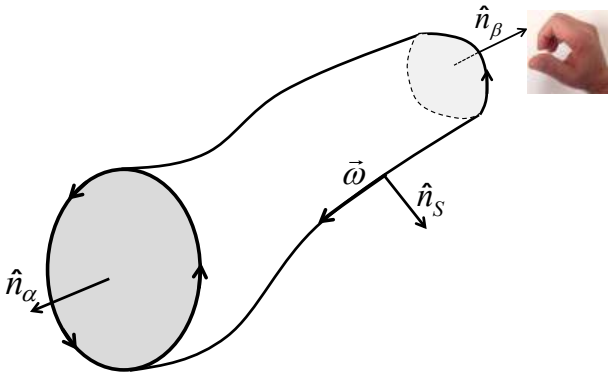


Figure 7.5: A vortex tube cut by two arbitrary cross sections α and β. Outward unit normals are drawn from each cross section and from the sidewall.

1. Thinner parts of a vortex tube have stronger vorticity. This is because $\Gamma = \oint \vec{\omega} \cdot \hat{n} dA$, so for Γ to be uniform a reduction of area must be compensated by an increase in vorticity.

2. A vortex tube cannot end within the fluid. It can either:

   - end at a boundary (e.g., tornado)
   - form a loop (e.g., smoke ring, as in figure 7.1).

Proof We will integrate the vorticity over the surface of the vortex tube segment (figure 7.5) in two ways. First, we will perform the whole integral at once using the divergence theorem, then we will integrate over the two cross-sections and the sidewall separately. Finally we'll use Stokes' theorem to express the result in terms of circulation.

The complete integral is

$$\oint_A \vec{\omega} \cdot \hat{n} \, dA = \int_V \vec{\nabla} \cdot \vec{\omega} \, dV = 0. \tag{7.2.10}$$

The first equality is due to the divergence theorem; the second follows from (7.2.4).

Alternatively, we can perform the integral over the three surfaces seperately:

$$\oint_A \vec{\omega} \cdot \hat{n} \, dA = \underbrace{\int_\alpha \vec{\omega} \cdot \hat{n}_\alpha \, dA}_{=\Gamma_\alpha} + \underbrace{\int_S \vec{\omega} \cdot \hat{n}_S \, dA}_{=0} + \underbrace{\int_\beta \vec{\omega} \cdot \hat{n}_\beta \, dA}_{=-\Gamma_\beta} = 0. \tag{7.2.11}$$

The first term is equal to the circulation around the cross-section α,

$$\Gamma_\alpha = \int_\alpha \vec{\omega} \cdot \hat{n}_\alpha dA,$$

by Stokes theorem (4.3.6). The integral over the sidewall (the second term) must vanish because that surface is composed of vortex filaments, and therefore its unit normal vector is everywhere perpendicular to $\vec{\omega}$.

Stokes' theorem requires that the unit normal be directed according to the right-hand rule, which in the third term would be into the vortex tube segment (figure 7.5), opposite to the outward normal $\hat{n}_\beta$. Therefore the third term in (7.2.11) equals $-\Gamma_\beta$. Because the three terms must add to zero, we have $\Gamma_\beta - \Gamma_\alpha = 0$ and the theorem is proven.

Kelvin's circulation theorem [3] : *If the fluid is inviscid and homogeneous, then circulation around a vortex tube does not change in time.*

This explains the extraordinary longevity of vortex tubes even as they twist and turn in a turbulent flow.

Proof As preparation, consider a scalar function of space: $\varphi = \varphi(\vec{x})$. If we move through a small displacement $d\vec{x}$, the change in φ is

$$d\varphi = \frac{\partial \varphi}{\partial x_i} dx_i.$$

[3] William Thompson, 1st Baron Kelvin, (1824-1907) was born in Belfast but spent most of his career at the University of Glasgow. He established the lower limit of temperature, absolute zero, and as a result the Kelvin temperature scale is named in his honor. He led the design of the first trans-Atlantic telegraph cable, for which he was knighted by Queen Victoria. He was the first British scientist elevated to the House of Lords.

We call this a *perfect differential*. Now consider the following integral

$$\int_{\vec{x}^{(1)}}^{\vec{x}^{(2)}} \frac{\partial \varphi}{\partial x_i} dx_i = \int_{\vec{x}^{(1)}}^{\vec{x}^{(2)}} d\varphi = \varphi(\vec{x}^{(2)}) - \varphi(\vec{x}^{(1)}).$$

For a closed circuit, $\vec{x}^{(1)} = \vec{x}^{(2)}$, and therefore

$$\oint \frac{\partial \varphi}{\partial x_i} dx_i = 0.$$

So, the integral of a perfect differential around a closed circuit is zero. Our strategy here is to get things into the form of perfect differentials.

OK, here goes. Consider the circulation around an arbitrary cross-section. By the definition of the integral, we can write this as the sum over many increments, as illustrated in figure 7.6, in the limit as the size of each increment goes to zero:

$$\Gamma = \oint \vec{u} \cdot d\vec{x} = \lim_{|\Delta\vec{x}| \to 0} \sum_n \vec{u}^{(n)} \cdot \Delta\vec{x}^{(n)}$$

We now take the material derivative:

$$\begin{aligned}
\frac{D\Gamma}{Dt} &= \lim_{|\Delta\vec{x}| \to 0} \frac{D}{Dt} \sum_n \vec{u}^{(n)} \cdot \Delta\vec{x}^{(n)} \\
&= \lim_{|\Delta\vec{x}| \to 0} \sum_n \frac{D}{Dt} \left(\vec{u}^{(n)} \cdot \Delta\vec{x}^{(n)} \right) \\
&= \lim_{|\Delta\vec{x}| \to 0} \sum_n \left(\frac{D\vec{u}^{(n)}}{Dt} \cdot \Delta\vec{x}^{(n)} + \vec{u}^{(n)} \cdot \frac{D\Delta\vec{x}^{(n)}}{Dt} \right)
\end{aligned}$$

Now we substitute from (7.2.2) and return the terms to integral form

$$\begin{aligned}
\frac{D\Gamma}{Dt} &= \lim_{|\Delta\vec{x}| \to 0} \sum_n \left[\left(\vec{g} - \frac{1}{\rho_0} \vec{\nabla} p^{(n)} \right) \cdot \Delta\vec{x}^{(n)} + \vec{u}^{(n)} \cdot \Delta\vec{u}^{(n)} \right] \\
&= \lim_{|\Delta\vec{x}| \to 0} \sum_n \left[\vec{g} \cdot \Delta\vec{x}^{(n)} - \frac{1}{\rho_0} \vec{\nabla} p^{(n)} \cdot \Delta\vec{x}^{(n)} + \vec{u}^{(n)} \cdot \Delta\vec{u}^{(n)} \right] \\
&= \oint \vec{g} \cdot d\vec{x} - \frac{1}{\rho_0} \oint dp + \oint \vec{u} \cdot d\vec{u}
\end{aligned}$$

Each of these three integrands is a perfect differential and therefore integrates to zero around a closed circuit. For clarity, we'll look at the terms individually.

- The first term is the line integral of the constant vector $\vec{g}$ around the closed curve. We could rewrite the integrand as $g_i\, dx_i$, or $d(g_i x_i)$ since g_i is a constant. So the integrand is a perfect differential.
- In the second term, $\vec{\nabla} p^{(n)} \cdot \Delta\vec{x}^{(n)}$ has been recognized as the small change in p over the spatial increment $\Delta\vec{x}^{(n)}$, i.e., another perfect differential.
- The third term is the perfect differential of $\vec{u} \cdot \vec{u}/2$:

$$\vec{u} \cdot d\vec{u} = u_i\, du_i = u_i \frac{\partial u_i}{\partial x_j} dx_j = \frac{\partial}{\partial x_j} \left(\frac{u_i^2}{2} \right) dx_j = d\left(\frac{u_i^2}{2} \right) = d\left(\frac{\vec{u} \cdot \vec{u}}{2} \right)$$

  and therefore integrates to zero.

So we have $D\Gamma/Dt = 0$, and the proof is complete.[4]

Summary: In an inviscid, homogeneous fluid we have the following:

- Helmholtz #1: Vortex tubes move with the fluid (because vortex filaments do).
- Helmholtz #2: A vortex tube cannot end within the fluid.
- Kelvin: Circulation around a vortex tube does not vary in time.

Taken together, these three theorems explain the extraordinary coherence and persistence of vortex tubes.

[4] Note that this proof does not actually require that the path of integration enclose a vortex tube; it applies to any closed circuit in an inviscid, homogeneous fluid. The theorem also holds if ρ is not uniform *provided* that ρ is a function of p only, i.e., if the fluid is barotropic (see section 6.6.2).

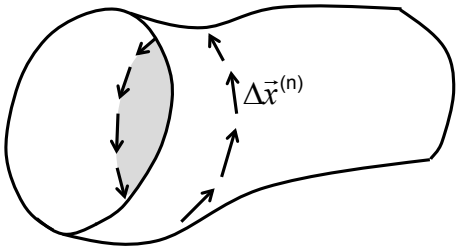


Figure 7.6: Line integral around a vortex tube represented as a sum of small increments.

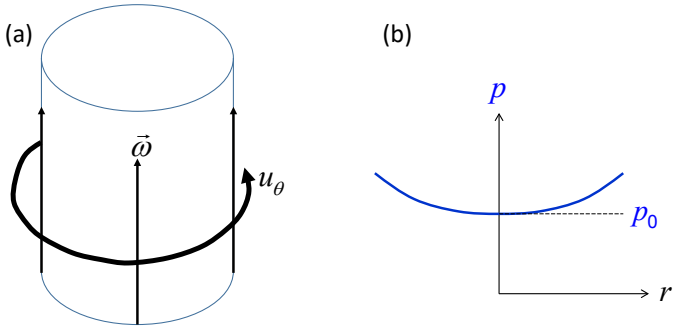


Figure 7.7: (a) A columnar vortex tube. (b) Radial pressure variation.

7.2.5 Pressure drop within a vortex tube

Inside a vortex tube, the outward centrifugal force is balanced by a reduction of pressure. Here, we will quantify this effect.

Consider a Rankine vortex with radius R in a homogeneous fluid (figure 7.7a; also section 5.3.2). In a cylindrical coordinate system, the velocity has radial component $u_r = 0$ and azimuthal component $u_\theta = \dot{\theta} r$ for $r < R$. The equation for radial velocity in this coordinate system (appendix I):

$$\frac{Du_r}{Dt} = \frac{u_\theta^2}{r} - \frac{\partial}{\partial r}\frac{p}{\rho_0}. \tag{7.2.12}$$

The first term on the right-hand side represents the centrifugal acceleration. To maintain $u_r = 0$, the right-hand side must vanish, i.e., the radial pressure gradient must balance the centrifugal acceleration:

$$\frac{\partial}{\partial r}\frac{p}{\rho_0} = \frac{u_\theta^2}{r} = \frac{(\dot{\theta} r)^2}{r} = \dot{\theta}^2 r.$$

Integrating, we have

$$\frac{p}{\rho_0} = \frac{1}{2}\dot{\theta}^2 r^2 + \frac{p_0}{\rho_0}, \tag{7.2.13}$$

where p_0 is the pressure at $r = 0$. The pressure profile therefore has the form of a parabola (figure 7.7b). The total pressure change from the center to the radius of maximum velocity ($r = R$) is

$$p - p_0 = \frac{1}{2}\rho_0 u_\theta^2,$$

where p and u_θ are values at $r = R$. It is left as an exercise for the reader to determine the pressure distribution for $r > R$. Note that $p - p_0$ is positive definite, i.e., the pressure inside a vortex tube is always reduced.

Example: pressure drop in a tornado A strong tornado (as defined for the purpose of nuclear reactor design) has maximum azimuthal velocity 130m/s. Using $\rho_0 = 1.2\text{kg/m}^3$, the mean density of air at sea level, the pressure drop is 1.4psi, or about 1/10 of normal atmospheric pressure. This is why a tornado can cause windows to burst: on a 1m^2 window, 1/10 of atmospheric pressure is about the weight of a small car.

Exercise: Think about all the ways in which a real tornado differs from the simple model considered here.

Surface elevation in rotating flow Consider a vertical, axisymmetric vortex in homogeneous fluid with a free surface, e.g., a stirred mug of coffee. Let the height of the free surface be $\eta(r)$. With no vertical motion, the fluid is in hydrostatic equilibrium, i.e., the pressure at each point is the weight of the fluid above it:

$$p(r,z) = \int_z^{\eta(r)} \rho_0 g\, dz = \rho_0 g(\eta - z).$$

Substitution into (7.2.13) gives

$$\eta = \eta_0 + \frac{\dot{\theta}^2}{2g} r^2, \tag{7.2.14}$$

where η_0 is the surface elevation at $r = 0$. The surface is therefore a paraboloid of revolution.[5]

Suppose we stir a cup of coffee at a rate of two revolutions per second, so that $\dot{\theta} = 4\pi s^{-1}$. If the radius of the cup is $4cm$ and $g = 9.81 ms^{-2}$, then the elevation difference should be about $1.3cm$.

You'll get to play with these concepts in exercises 30 and 31.

7.3 Viscous effects

Here we will extend the vorticity equation to cover viscous effects, then use the result to develop a simple model of a vortex in a viscous fluid.

7.3.1 The vorticity equation for a viscous fluid

Assume $\rho = \rho_0$ and that ν is uniform but nonzero. The momentum equation (6.8.2) is then

$$\frac{D\vec{u}}{Dt} = \vec{g} - \vec{\nabla}\frac{p}{\rho_0} + \nu\nabla^2\vec{u}. \tag{7.3.1}$$

As before, we obtain the vorticity equation by taking the curl of the momentum equation, in this case (7.3.1). Happily we have already done most of the work; we need only add the viscous term. This is simply

$$\vec{\nabla} \times (\nu\nabla^2\vec{u}) = \nu\nabla^2(\vec{\nabla} \times \vec{u}) = \nu\nabla^2\vec{\omega}.$$

The curl of (7.3.1) is therefore

$$\boxed{\frac{D\vec{\omega}}{Dt} = [\vec{\omega} \cdot \vec{\nabla}]\vec{u} + \nu\nabla^2\vec{\omega}.} \tag{7.3.2}$$

The final term tells us that vorticity is diffused by viscosity in the same manner as is velocity (cf. 7.3.1).

7.3.2 The Burgers vortex

In the presence of viscosity, vortex tubes are no longer immortal due to Kelvin's theorem. They can, however, be maintained against viscous diffusion by vortex stretching. The Burgers [6] vortex is a simple model of an axisymmetric vortex that is simultaneously amplified by extensional strain and diffused by viscosity (figure 7.8a) such that equilibrium is maintained, i.e., $\partial\vec{\omega}/\partial t = 0$. The flow has two components: a strain field with expansion in the axial (z) direction balanced by compression in the radial (r) direction and a vortex whose motion is entirely azimuthal. We will define these fields in turn, and then test the resulting solution as a model for naturally occurring vortices.

We begin by defining the extensional strain component in the simplest possible way:

$$\boxed{w = \lambda z.} \tag{7.3.3}$$

To deduce the corresponding radial velocity, we invoke incompressibility using the divergence in cylindrical coordinates (I.1.2):

$$\vec{\nabla} \cdot \vec{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial w}{\partial z} = 0.$$

Axisymmetry requires that $\partial/\partial\theta$ be zero, so the second term on the right-hand side vanishes, while the third term is just λ. This leads to a simple equation for u_r

$$\frac{\partial}{\partial r}(ru_r) = -\lambda r,$$

which integrates to give

$$\boxed{u_r = -\frac{1}{2}\lambda r.} \tag{7.3.4}$$

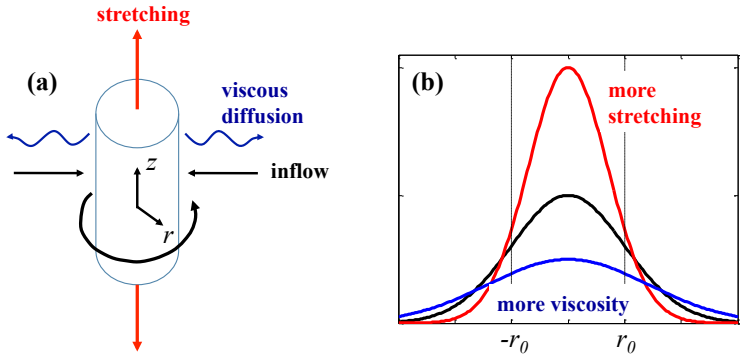


Figure 7.8: (a) An axisymmetric vortex tube maintained by a balance of stretching and diffusion. (b) The black curve is a vorticity profile for Burgers vortex (7.3.8). For the red (blue) curve, the circulation is the same but λ (ν) is doubled.

So we have the assumed vertical extension (7.3.3) balanced by a radial inflow (7.3.4), as illustrated in figure 7.8a.

We next solve for the vorticity. Recall that, for an axisymmetric vortex,

$$\vec{\omega} = \omega(r)\hat{e}^{(z)},$$

so we only have to determine the scalar function $\omega(r)$. We do this using the z-component of (7.3.2):

$$\frac{D\omega}{Dt} = [\omega\frac{\partial}{\partial z}]w + \nu\nabla^2\omega. \tag{7.3.5}$$

The first term on the right-hand side is just $\omega\lambda$, but there is complexity hidden in the cylindrical forms of the material derivative and the Laplacian. The material derivative of ω is I.1.6

$$\begin{aligned}\frac{D\omega}{Dt} &= \left(\frac{\partial}{\partial t} + u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial\theta} + u_z\frac{\partial}{\partial z}\right)\omega, \\ &= -\frac{1}{2}\lambda r\frac{d\omega}{dr}.\end{aligned} \tag{7.3.6}$$

The total derivative is written because ω depends only on r. The Laplacian of ω (I.1.4) is

$$\begin{aligned}\nabla^2\omega &= \left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}\right)\omega, \\ &= \frac{1}{r}\frac{d}{dr}\left(r\frac{d\omega}{dr}\right).\end{aligned} \tag{7.3.7}$$

Substituting (7.3.6) and (7.3.7) into (7.3.5) we obtain an ordinary differential equation for $\omega(r)$:

$$-\frac{1}{2}\lambda r\frac{d\omega}{dr} = \omega\lambda + \frac{\nu}{r}\frac{d}{dr}\left(r\frac{d\omega}{dr}\right).$$

Multiplying through by r and integrating yields

$$\nu r\frac{d\omega}{dr} + \lambda\frac{1}{2}r^2\omega = 0,$$

or

$$\frac{d\omega}{dr} = -\frac{\lambda}{2\nu}r\omega.$$

The solution is a Gaussian function:

$$\omega = \omega_0 e^{-\frac{\lambda}{4\nu}r^2},$$

where ω_0 is an arbitrary constant representing the maximum vorticity. This is more commonly written as

$$\boxed{\omega(r) = \frac{\Gamma}{2\pi r_0^2}\exp\left(-\frac{r^2}{2r_0^2}\right), \quad \text{where } r_0 = \sqrt{\frac{2\nu}{\lambda}}.} \tag{7.3.8}$$

The balance between stretching and viscosity is expressed in the radial scale r_0 (figure 7.8b). Stronger stretching gives a thinner, more intense vortex; stronger viscosity gives a thicker, weaker vortex.

[5] As a fluid dynamicist, always stir your coffee first and add cream second, so you can see the circulation patterns.

[6] Johannes Martinus (Jan) Burgers was a Dutch physicist best known for the Burgers equation, which describes nonlinear-diffusive systems.

Vortices at the dissipation scale In a turbulent flow, the vorticity field has the form of a spaghetti-like tangle of vortex tubes (e.g., Moffatt et al. 1994). Is the Burgers model applicable to these structures? The smallest turbulent motions are of the order of the Kolmogorov scale, $L_K = (\nu^3/\varepsilon)^{1/4}$, where ε is the kinetic energy dissipation rate $\varepsilon = 2\nu e_{ij}e_{ij}$. In geophysical turbulence, ε varies greatly, but L_K varies less because of the power 1/4; a typical value is $\sim 1cm$. In the Burgers model, the only nonzero strain rate component is $e_{33} = \lambda$, and therefore $\varepsilon = 2\nu\lambda^2$. The Burgers model predicts $r_0 = 1.7L_K$. The agreement is reasonable in an order-of-magnitude sense.

Test your understanding: Of the simplifying assumptions that underlie the Burgers model, which are most likely to be wrong in this case?

A tornado as a Burgers vortex Suppose that a tornado is driven by a vertical expansion $dw/dz = \lambda = 0.1s^{-1}$. This would correspond, for example, to an updraft of 50m/s at a height of 500m. For air, $\nu = 10^{-5}\mathrm{m}^2/\mathrm{s}$. These values lead to $r_0 \sim 10^{-2}\mathrm{m}$, far too small to be realistic.

Clearly at least one simplifying assumption is wrong. For example, tornados are not usually cylindrical but are more funnel-shaped. A more extreme discrepancy, though, is in the very simple, symmetric form of the velocity field. In reality, tornadoes are intensely turbulent. The effect of turbulence on the overall flow is similar to that of a greatly increased viscosity. To get $r_0 = 30\mathrm{m}$, a reasonable value, we must assume that this "turbulent" viscosity (or "eddy" viscosity, see section 6.3.6) is $10^2\mathrm{m}^2/\mathrm{s}$.

7.4 Buoyancy effects: the baroclinic torque

We now assume that the fluid is inviscid ($\nu = 0$) but not homogeneous (ρ is allowed to vary). The momentum equation (7.2.2) becomes

$$\frac{D\vec{u}}{Dt} = \vec{g} - \frac{1}{\rho}\vec{\nabla}p, \tag{7.4.1}$$

and once again we take its curl to form the vorticity equation. The first two terms are already computed:

$$\vec{\nabla}\times\frac{D\vec{u}}{Dt} = \frac{D\vec{\omega}}{Dt} - [\vec{\omega}\cdot\vec{\nabla}]\vec{u} \quad \text{and } \vec{\nabla}\times\vec{g} = 0.$$

The curl of the new term (obtained using identities 11 and 15 listed in appendix E) is:

$$\begin{aligned}\vec{\nabla}\times\left(-\rho^{-1}\vec{\nabla}p\right) &= -\left(\vec{\nabla}\rho^{-1}\times\vec{\nabla}p + \rho^{-1}\vec{\nabla}\times\vec{\nabla}p\right)\\ &= -\vec{\nabla}\rho^{-1}\times\vec{\nabla}p\\ &= \frac{1}{\rho^2}\vec{\nabla}\rho\times\vec{\nabla}p.\end{aligned}$$

This is called the baroclinic torque:

$$\vec{B} = \frac{1}{\rho^2}\vec{\nabla}\rho\times\vec{\nabla}p, \tag{7.4.2}$$

and the vorticity equation is

$$\frac{D\vec{\omega}}{Dt} = [\vec{\omega}\cdot\vec{\nabla}]\vec{u} + \vec{B}.$$

The baroclinic torque $\vec{B}$ is the mechanism by which density variations influence vorticity. This torque is zero in the case of a barotropic fluid (section 6.6.2), where $\rho = \rho(p)$, because $\vec{\nabla}\rho$ and $\vec{\nabla}p$ are then parallel.

To understand the mechanism physically, we first separate the pressure into two parts: hydrostatic and nonhydrostatic:

$$p = p_H + p^*.$$

The hydrostatic part is defined by the condition of hydrostatic balance:

$$\vec{\nabla}p_H = \rho\vec{g},$$

so that

$$\vec{\nabla}p = \rho\vec{g} + \vec{\nabla}p^*. \tag{7.4.3}$$

Substituting this decomposition into the momentum equation (7.4.1) gives

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho}\vec{\nabla}p^*. \tag{7.4.4}$$

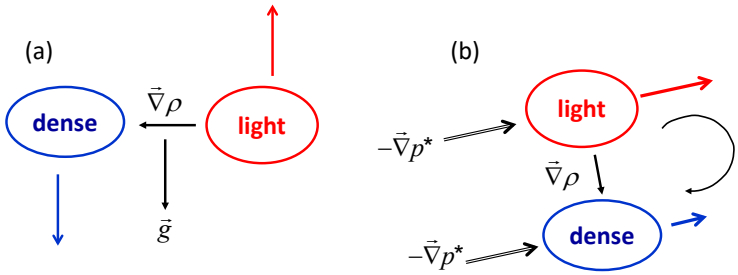


Figure 7.9: Schematics illustrating the two parts of the baroclinic torque via their action on two fluid parcels of different density: (a) buoyancy, (b) inertia.

So the hydrostatic pressure simply holds up the weight of the water at each point while the nonhydrostatic pressure actually causes motion.

Now we substitute the decomposition (7.4.3) into (7.4.2):

$$\begin{aligned} \vec{B} &= \frac{1}{\rho^2}\vec{\nabla}\rho\times\vec{\nabla}p \\ &= \frac{1}{\rho^2}\vec{\nabla}\rho\times(\rho\vec{g}+\vec{\nabla}p^*) \qquad (7.4.5) \\ &= \underbrace{\frac{1}{\rho}\vec{\nabla}\rho\times\vec{g}}_{\text{buoyancy}} + \underbrace{\frac{1}{\rho^2}\vec{\nabla}\rho\times\vec{\nabla}p^*}_{\text{inertia}}. \qquad (7.4.6) \end{aligned}$$

Evidently, the baroclinic torque has two parts. Illustrated in figure (7.9), these correspond to two distinct mechanisms by which density differences can alter the vorticity. The buoyancy term quantifies the tendency for dense fluid to sink and light fluid to rise. It is nonzero if the density gradient has a component perpendicular to gravity, as in figure 7.9a. Using the right-hand rule, verify for yourself that the cross product $\vec{\nabla}\rho\times\vec{g}$ is directed out of the page. The buoyancy term therefore imparts a counterclockwise rotation to the flow, consistent with the vertical motion of the fluid parcels.

The inertia term results from the fact that a force (e.g., $-\vec{\nabla}p^*$), acting on fluid parcels of different density, produces different rates of acceleration (figure 7.9b). The term is nonzero if the density gradient has a component perpendicular to the pressure gradient force. Again, use the right-hand rule to check that the inertial torque is directed into the page and therefore generates clockwise vorticity.

Which term is more important? Inspection of (7.4.5) shows that the inertial term is small compared to the buoyancy term if $|\vec{\nabla}p^*|/\rho \ll g$. The left-hand side of this inequality is just the net acceleration of the flow, as is shown by (7.4.4). Therefore, the buoyancy term dominates if the net acceleration is much less than the gravitational acceleration ("1 gee", in aeronautical lingo). This is true of the slow motions caused by weak density gradients in the interior of the ocean or the atmosphere. For that reason, we often neglect the inertial term when describing such motions. This leads to the *Boussinesq approximation* (see Appendix G).

In contrast, air-water interfaces have large density gradients. Accelerations are therefore comparable to gravity (imagine a breaking wave, for example), and buoyancy and inertia are both important. These motions are the topic of the upcoming chapter.

Back to Table of Contents.

Chapter 8

Waves

"I spin on the circle of wave upon wave of the sea." - Pablo Neruda

8.1 Introduction

Our second example of the application of the Navier-Stokes equation to natural flows is surface gravity waves, as would occur at the surface of an ocean or lake. We'll make several simplifying assumptions:

- The fluid is inviscid: $\nu = 0$.
- The fluid is homogeneous: $\rho = \rho_0$.
- The flow is confined to the $x-z$ plane, so there is no dependence on y and no motion in the y direction.
- The amplitude of the waves is small enough to allow neglect of nonlinearity.

The last assumption is new, and is tremendously important in the analysis of geophysical flows. By linearizing the equations, we filter out some very interesting phenomena such as large-amplitude breaking waves, but the resulting simplification allows us to understand the dynamics in detail. This provides a foundation for more sophisticated theories that include large-amplitude phenomena. In the next application, hydraulic flows, the restriction to small-amplitude motions will be removed.

Hyperbolic functions review[1]

[1] Also see problem 7.

Figure 8.1: *The Ninth Wave*, oil painting by Ivan Aivazovsky (1817-1900). Image courtesy of Wikimedia Commons.

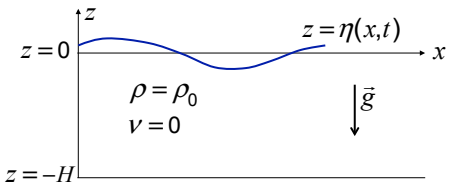


Figure 8.2: Two-dimensional flow in an inviscid, homogeneous fluid with a free surface. The vertical coordinate is directed opposite to gravity (upward). The mean surface is $z = 0$ and the mean depth is H.

In this section we'll make frequent use of the hyperbolic functions

$$\sinh x = \frac{e^x - e^{-x}}{2}; \quad \cosh x = \frac{e^x + e^{-x}}{2}; \quad \text{and } \tanh x = \frac{\sinh x}{\cosh x}.$$

These obey the relations

$$\frac{d}{dx}\sinh x = \cosh x; \quad \frac{d}{dx}\cosh x = \sinh x; \quad \frac{d^2}{dx^2}\sinh x = \sinh x; \quad \frac{d^2}{dx^2}\cosh x = \cosh x,$$

and have the following Taylor series approximations:

$$\sinh x \approx x; \quad \cosh x \approx 1; \quad \tanh x \approx x$$

These are valid for $|x| \ll 1$ and become exact in the limit $|x| \to 0$. As $x \to \pm\infty$,

$$\sinh x \to \pm\frac{e^{|x|}}{2}; \quad \cosh x \to \frac{e^{|x|}}{2}; \quad \tanh x \to \pm 1.$$

8.2 The dispersion relation

With $\rho = \rho_0$ and $\nu = \mu = 0$, the mass and momentum equations (6.2.4, 6.8.2) are

$$\rho_0 \frac{D\vec{u}}{Dt} = -\rho_0 g\hat{e}^{(z)} - \vec{\nabla}p. \tag{8.2.1}$$

$$\vec{\nabla}\cdot\vec{u} = 0 \tag{8.2.2}$$

The velocity field has components u and w, and the independent variables are x, z and t. As in our previous discussion of the baroclinic torque, we separate the pressure into two parts:

$$p = p_H + p^*, \quad \text{where } \vec{\nabla}p_H = -\rho_0 g\hat{e}^{(z)}, \quad \text{or } p_H = -\rho_0 gz. \tag{8.2.3}$$

Note that p_H is the hydrostatic pressure in the absence of surface deflections. Substituting this decomposition into the momentum equation (8.2.1) gives

$$\rho_0 \frac{D\vec{u}}{Dt} = -\vec{\nabla}p^*. \tag{8.2.4}$$

8.2.1 Linearizing the equations of motion

In the absence of motion, the fluid is in an equilibrium state defined by

$$\vec{u} = 0, p^* = 0, \eta = 0.$$

We'll assume that the system remains close to this equilibrium state. As a result of this assumption, any term that involves a product of two or more of $\vec{u}, p^*$ and η will be treated as negligible. For example:

$$\frac{D\vec{u}}{Dt} = \frac{\partial\vec{u}}{\partial t} + \underbrace{[\vec{u}\cdot\vec{\nabla}]\vec{u}}_{\approx 0} \approx \frac{\partial\vec{u}}{\partial t}. \tag{8.2.5}$$

As a result, (8.2.4) becomes

$$\boxed{\rho_0 \frac{\partial\vec{u}}{\partial t} = -\vec{\nabla}p^*.} \tag{8.2.6}$$

This equation is linear, and as a result is vastly easier to solve than the nonlinear version (8.2.4). We must bear in mind, though, that the solution becomes invalid if the amplitude is large enough that the advective term in (8.2.5) is not negligible. (How large is this? We'll take up this important question in section 8.2.4.)

The next step is to derive an equation that we can solve for p^*. Taking the divergence of (8.2.6) and using (8.2.2), we find

$$\rho_0 \vec{\nabla} \cdot \frac{\partial \vec{u}}{\partial t} = \rho_0 \frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{u} = 0 = -\vec{\nabla} \cdot \vec{\nabla} p^* = -\nabla^2 p^*.$$

The nonhydrostatic pressure is therefore a solution of Laplace's equation:

$$\frac{\partial^2 p^*}{\partial x^2} + \frac{\partial^2 p^*}{\partial z^2} = 0. \tag{8.2.7}$$

Combining this with the horizontal and vertical components of (8.2.6):

$$\rho_0 \frac{\partial u}{\partial t} = -\frac{\partial p^*}{\partial x}; \tag{8.2.8}$$

$$\rho_0 \frac{\partial w}{\partial t} = -\frac{\partial p^*}{\partial z}, \tag{8.2.9}$$

we have three equations for the three unknowns u, w, and p^*. Notice further than u appears only in (8.2.8), so (8.2.7) and (8.2.9) form a fully-determined system for the two unknowns p^* and w .

8.2.2 Boundary conditions

Boundary conditions are imposed at the surface and the bottom. We'll begin with the latter as is it is simpler. We assume that the bottom is flat and impermeable:

$$w = 0 \quad \text{at} \quad z = -H. \tag{8.2.10}$$

Since this is true for all time, it is also true that $\partial w/\partial t = 0$ at the lower boundary, in which case (8.2.9) provides a boundary condition on the nonhydrostatic pressure:

$$\frac{\partial p^*}{\partial z} = 0 \quad \text{at} \quad z = -H. \tag{8.2.11}$$

The surface is assumed to be material, i.e., it is always composed of the same fluid particles. As a result:

$$w = \frac{D\eta}{Dt} \quad \text{at} \quad z = \eta.$$

This boundary condition is both nonlinear and self-referential, i.e., you can't apply it without knowing $\eta(x,t)$, which requires knowing the solution. We know how to remove the nonlinearity:

$$\frac{D\eta}{Dt} = \frac{\partial \eta}{\partial t} + \underbrace{u \frac{\partial \eta}{\partial x}}_{\approx 0} \approx \frac{\partial \eta}{\partial t}.$$

The self-referential nature of the boundary condition is removed via another kind of linearization:

$$w|_{z=\eta} = w|_{z=0} + \underbrace{\left.\frac{\partial w}{\partial z}\right|_{z=0} \eta}_{\approx 0} \approx w|_{z=0}. \tag{8.2.12}$$

The result is a linearized boundary condition:

$$w = \frac{\partial \eta}{\partial t} \quad \text{at} \quad z = 0. \tag{8.2.13}$$

A second surface condition originates with the requirement that pressure be continuous. The total pressure $(p_H + p^*)$ immediately below the water surface must therefore be equal to atmospheric pressure p_A.[2] We assume that p_A is uniform and, since only gradients of pressure matter, we set it to zero. The total pressure p must therefore approach zero at the surface and, from (8.2.3),

$$p^* = -p_H = \rho_0 g \eta \quad \text{at} \quad z = \eta.$$

[2] If centimeter-scale waves are of interest, we must also take account of *surface tension*. The force that holds water molecules together is stronger on the water side than on the air side, creating an additional pressure at the surface given by $p = -\sigma \nabla^2 \eta$, where σ is a constant and small amplitude is assumed. You'll examine this effect in homework problem 33.

By the same logic employed in (8.2.12), we can apply this boundary condition at $z = 0$:

$$p^*|_{z=\eta} = \rho_0 g\eta = p^*|_{z=0} + \underbrace{\left.\frac{\partial p^*}{\partial z}\right|_{z=0}}_{\approx 0} \eta \approx p^*|_{z=0},$$

and therefore

$$p^* = \rho_0 g\eta \quad \text{at} \quad z = 0. \tag{8.2.14}$$

Taking inventory: Our problem now consists of the equations (8.2.7) and (8.2.9) and the boundary conditions (8.2.10), (8.2.11), (8.2.13) and (8.2.14). Once these are solved for p^* and w, the horizontal momentum equation (8.2.8) can be solved separately to obtain u.

8.2.3 The normal mode solution

We're looking for solutions for the nonhydrostatic pressure p^* and the vertical velocity w. To begin with, we'll seek a solution in which p^* has the following form:

$$\boxed{p^* = P(z)\cos(kx - \omega t).} \tag{8.2.15}$$

This is called a normal mode or a plane wave solution. Several features of this important functional form should be noted.

- $P(z)$ is a function yet to be determined.
- The x-dependence is determined by the x-wavenumber k, which is also equal to $2\pi/\lambda$, λ being the wavelength.
- The time-dependence is described by the radian frequency ω, which is related to the period T by $\omega = 2\pi/T$. It is often more convenient to use the cyclic frequency $f = \omega/2\pi$, with units of cycles per second (cps.) or Hertz (Hz.)
- The pattern moves with phase speed $c = \omega/k$.

The assumption (8.2.15) is not as restrictive as it may appear at first. By virtue of Fourier's theorem (see any applied math text), an arbitrary dependence on x and t can be expressed as a superposition of trigonometric functions of the form (8.2.15). Moreover, because the equations are linear, if (8.2.15) is a solution, any such superposition will also be a solution.

Substituting (8.2.15) into (8.2.7), we have

$$\frac{\partial^2 p^*}{\partial x^2} + \frac{\partial^2 p^*}{\partial z^2} = [-k^2 P + P'']\cos(kx - \omega t) = 0,$$

where primes denote derivatives with respect to z. Because the last equality must be valid for all x and t, the quantity in square brackets must be zero:

$$P'' = k^2 P.$$

This is a very common ordinary differential equation whose solutions can be expressed in terms of hyperbolic functions. To satisfy the bottom boundary condition (8.2.11), we choose

$$P = P_0 \cosh k(z + H), \tag{8.2.16}$$

where P_0 is an undetermined constant.

Next we need a corresponding solution for w. In the vertical momentum equation (8.2.9) we try a solution of the form

$$\boxed{w = W(z)\sin(kx - \omega t).} \tag{8.2.17}$$

This results in

$$-\rho_0 \omega W \cos(kx - \omega t) = -P' \cos(kx - \omega t).$$

Since this must be true for all x and t, the coefficients of the cosine functions must be equal. Therefore (8.2.17) is a solution provided that:

$$W = \frac{P'}{\rho_0 \omega},$$

or, substituting (8.2.16),

$$W = \frac{P_0 k}{\rho_0 \omega} \sinh k(z + H). \tag{8.2.18}$$

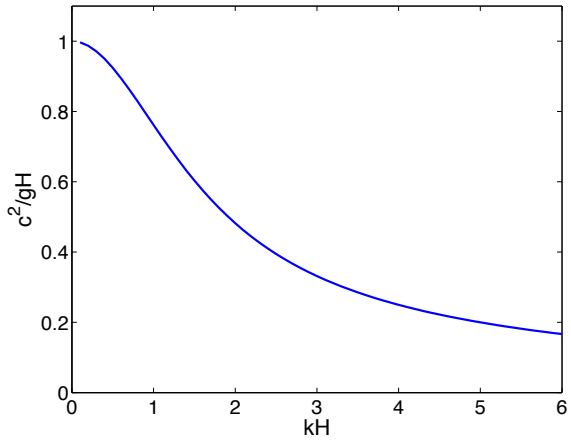


Figure 8.3: Dispersion relation (8.2.23) for small-amplitude surface gravity waves.

Note that (8.2.18) automatically satisfies (8.2.10), the bottom boundary condition on w. In summary, we now have solutions for p^* and w in the form of (8.2.15, 8.2.16, 8.2.17, 8.2.18), and these satisfy the bottom boundary conditions (8.2.11, 8.2.10).

To satisfy the *surface boundary conditions* (8.2.13, 8.2.14), we must consider the surface deflection $\eta(x,t)$. (8.2.14) requires $p^* = \rho_0 g\eta$ at $z = 0$. We solve for η and substitute (8.2.15) and (8.2.16) to get

$$\eta = \frac{1}{\rho_0 g} P_0 \cosh kH \cos(kx - \omega t).$$

Since the surface deflection is easier to observe than the pressure, we'll rewrite this solution as

$$\boxed{\eta = \eta_0 \cos(kx - \omega t),} \tag{8.2.19}$$

and note that $P_0 = \rho_0 g \eta_0 / \cosh kH$. Our solutions (8.2.16) and (8.2.18) for P and W become

$$\boxed{P = \rho_0 g\, \eta_0\, \frac{\cosh k(z+H)}{\cosh kH};} \tag{8.2.20}$$

and

$$\boxed{W = \frac{gk}{\omega}\, \eta_0\, \frac{\sinh k(z+H)}{\cosh kH}.} \tag{8.2.21}$$

We now have a complete solution for p^*, w and η, but we have not yet satisfied (8.2.13), the surface condition on w, i.e., we have an overdetermined system. The solution can only work for certain combinations of the wave parameters k and ω. Substituting our solutions (8.2.19, 8.2.21) into (8.2.13), we have

$$\frac{gk}{\omega}\, \eta_0\, \frac{\sinh kH}{\cosh kH} \sin(kx - \omega t) = \eta_0 \omega \sin(kx - \omega t) \quad \forall x,t,$$

and thus

$$\frac{gk}{\omega}\, \eta_0\, \frac{\sinh kH}{\cosh kH} = \eta_0 \omega,$$

or

$$\boxed{\omega^2 = gk \tanh kH.} \tag{8.2.22}$$

This is called the dispersion relation. The normal mode expressions (8.2.15,8.2.17,8.2.19) can satisfy the equations and *all* of the boundary conditions only if ω and k satisfy (8.2.22).

We can gain some insight into the propagation of surface gravity waves by writing the dispersion relation in terms of the phase speed $c = \omega/k$:

$$c^2 = \frac{g}{k} \tanh kH = gH\, \frac{\tanh kH}{kH}, \tag{8.2.23}$$

as shown in figure 8.3. The fastest propagation occurs when kH is small, i.e., in the limit of low wavenumber, or wavelength long compared to the water depth. This is in contrast to sound waves, in which the short waves travel fastest. (Imagine, for example, the sound of thunder. It begins with a high-frequency clap, then decays away to a low-frequency rumble.) In the ocean, surface waves are most often generated by storms. An observer some distance from the storm will see long, low-frequency waves first.

Test your understanding: In exercise 33 you will extend this derivation to include the effects of *surface tension*.

To complete the picture of small-amplitude surface waves, we now go back and add the horizontal velocity u. This is easily done by substituting our solution (8.2.15, 8.2.20) into the horizontal momentum equation (8.2.8). This results in

$$\boxed{u = U(z)\cos(kx-\omega t),} \tag{8.2.24}$$

where

$$U(z) = \frac{gk}{\omega}\,\eta_0\,\frac{\cosh k(z+H)}{\cosh kH}. \tag{8.2.25}$$

We can write this in a slightly more intuitive form if we note that (1) the dispersion relation (8.2.22) can be rearranged to give $kg/\omega = \omega/\tanh kH$ and (2) $\cosh x \tanh x = \sinh x$, and therefore

$$\boxed{U(z) = \omega\eta_0\,\frac{\cosh k(z+H)}{\sinh kH}.} \tag{8.2.26}$$

The velocity profile rises from a minimum value at the bottom to a maximum at the surface. Similarly, (8.2.21) can be rewritten as

$$\boxed{W(z) \;=\; \omega\eta_0\,\frac{\sinh k(z+H)}{\sinh kH},} \tag{8.2.27}$$

which rises from zero at the bottom to the maximum value $\omega\eta_0$ at the surface.

8.2.4 How small is small?

The solution derived above depends on the assumption that the amplitude of the waves is "small". How large can the amplitude be before this assumption is violated? It depends on how much inaccuracy we can tolerate. As a general introduction to the way theorists think about such questions, let's make order-of-magnitude estimates of the two terms in the material derivative of $\vec{u}$ (cf. 8.2.5). Suppose we are dealing with waves having period T, wavelength λ and velocity amplitude u_0. We estimate the time derivative as u_0/T and the space derivative as u_0/λ, leading to:

$$\frac{\partial\vec{u}}{\partial t} \approx \frac{u_0}{T}; \qquad [\vec{u}\cdot\vec{\nabla}]\vec{u} \approx \frac{u_0^2}{\lambda}. \tag{8.2.28}$$

We then require that the second term be much smaller than the first, which is equivalent to

$$u_0 \ll \lambda/T$$

In other words, the maximum fluid velocity should be much smaller than the phase speed of the waves.

8.2.5 Particle paths

Suppose that, in the absence of waves, a fluid particle is located at $\vec{x}_0$, and in the presence of waves its position is $\vec{x}_0+\vec{x}'(t)$. The particle's motion is described by

$$\frac{d\vec{x}'}{dt} = \vec{u}(\vec{x}_0,t)$$

recognising that the difference between $\vec{u}(\vec{x}_0,t)$ and $\vec{u}(\vec{x},t)$ is negligible.

Having solved for the components of $\vec{u}$ (8.2.24, 8.2.26, 8.2.17, 8.2.27), we can substitute the results and integrate in time to obtain the trajectory of the particle. The result is

$$x' = -\frac{U(z_0)}{\omega}\sin(kx_0-\omega t); \quad z' = \frac{W(z_0)}{\omega}\cos(kx_0-\omega t). \tag{8.2.29}$$

We can rewrite this as

$$\frac{x'^2}{L_x^2} + \frac{z'^2}{L_z^2} = 1, \tag{8.2.30}$$

where

$$L_x = \eta_0\frac{\cosh k(z_0+H)}{\sinh kH}; \quad L_z = \eta_0\frac{\sinh k(z_0+H)}{\sinh kH}. \tag{8.2.31}$$

We can now make three observations about the particle paths.

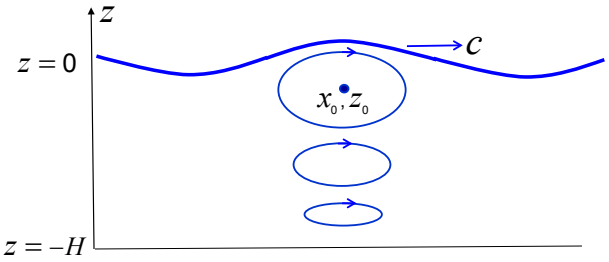


Figure 8.4: Elliptical particle trajectories under a small-amplitude surface wave propagating to the right with phase velocity c.

- Each path is an ellipse with radii L_x and L_z (figure 8.4). Both L_x and L_z decrease with depth, with $L_z = 0$ at the bottom (where $z_0 = -H$).
- The vertical excursion z' is proportional to the surface deflection (cf. 8.2.19):

$$z' = \eta(x,t)\frac{\sinh k(z_0+H)}{\sinh kH},$$

  and therefore reaches a maximum directly beneath the wave crests for all z_0.
- The horizontal particle velocity is also proportional to η:

$$u = \frac{g}{c}\eta(x,t)\,\frac{\cosh k(z_0+H)}{\cosh kH},$$

  where (8.2.25) and the definition of the phase velocity $c = \omega/k$ have been used. Its value at $z = 0$ is

$$u = \frac{g}{c}\eta(x,t).$$

  Directly beneath the wave crests, the horizontal motion has the same sign as the phase speed.

If we extend the theory to take nonlinear effects into account, we find that there is actually a slight drift in the direction of the wave propagation. Called the Stokes drift, it results from the fact that the particle speed at the top of each ellipse is slightly greater than the speed at the bottom of the ellipse. Appendix J gives a more detailed discussion.

8.3 Superposition phenomena

8.3.1 Beats

The "sneaker waves" that beach visitors are warned about, the spectacular spring tides (which have nothing to do with the season), and a musician's ability to hear when two strings are exactly in tune are all examples of the phenomenon of *beats* (figure 8.5). When two oscillations with slightly different frequencies ω_1 and ω_2 occur together, what we perceive is a single oscillation with frequency equal to the average $(\omega_1+\omega_2)/2$, modulated in amplitude by a slower frequency equal to the difference $|\omega_1-\omega_2|$.

Here, we'll see how these results follow from the addition rule for the sine function (e.g., exercise 10f). Consider the sum of two waves having equal amplitude but different wavenumbers and frequencies:

$$\begin{aligned}\frac{\eta(x,t)}{\eta_0} &= \frac{1}{2}\cos(k_1x-\omega_1t)+\frac{1}{2}\cos(k_2x-\omega_2t)\\ &= \cos\left(\frac{k_1+k_2}{2}\,x-\frac{\omega_1+\omega_2}{2}\,t\right)\,\cos\left(\frac{k_1-k_2}{2}x-\frac{\omega_1-\omega_2}{2}\,t\right)\\ &= \underbrace{\cos(\bar{k}x-\bar{\omega}t)}_{\text{average wave}}\;\underbrace{\cos\left(\frac{\Delta k}{2}x-\frac{\Delta\omega}{2}t\right)}_{\text{envelope}}.\end{aligned} \tag{8.3.1}$$

The first cosine function describes the "average wave", an oscillation with wavenumber and frequency equal to the means of the individual values. The amplitude of this oscillation is modulated by a slower oscillation called the *envelope*, shown by the dashed curve in figure 8.5.

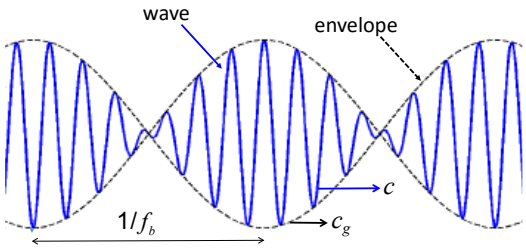


Figure 8.5: Two oscillations summed such that every seventh wave is especially large. The dashed curve shows the envelope function. Arrows indicate the phase velocity c, the group velocity c_g and the beat period $1/f_b$.

Let us now assume for simplicity that we are observing the waves at a fixed location, $x = 0$, so the surface deflection is

$$\frac{\eta(0,t)}{\eta_0} = \cos(\bar{\omega}t)\,\cos\left(\frac{\Delta\omega}{2}t\right) = \cos(2\pi\bar{f}t)\,\cos(\pi\Delta f t)$$

(The second form employs the cyclic frequency $2\pi f$.) A subtlety to notice is that the oscillation has maximum amplitude when the envelope $\cos\left(2\pi\frac{f_1-f_2}{2}t\right)$ is equal to *either* +1 or -1. Beats therefore occur with double the frequency of the envelope:

$$f_b = 2\times|(f_1-f_2)/2| = |f_1-f_2|;$$

just the absolute difference between the two frequencies.

Example 7: To tune a string, one begins by playing the string together with another oscillation having the correct pitch. If the string is slightly out of tune, the ear perceives beats. The tension in the string is then adjusted in whichever direction makes the beats slower. As the string approaches the correct frequency, the beat frequency approaches zero. You know the string is in tune when the beats are no longer heard.

Example 8: Suppose that the waves from two ocean storms reach the beach at the same time with frequencies 10s and 12s. The beat frequency is then

$$f_b = \frac{1}{10s} - \frac{1}{12s} = \frac{1}{60s}.$$

There'll be a big wave every 60s, or about every 5th wave.[3]

8.3.2 Group velocity

In (8.3.1) we added two waves and found the average wave modulated by the envelope function

$$\cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right),$$

which can also be written

$$\cos\left[\frac{\Delta k}{2}\left(x - \frac{\Delta\omega}{\Delta k}t\right)\right].$$

The amplitude of the envelope is therefore constant in a reference frame travelling at the speed $\Delta\omega/\Delta k$. More generally, a wave field consists of a continuous spectrum of components with different frequencies, wavenumbers and amplitudes. Wave pulses travel at the group velocity, given by $c_g = \partial\omega/\partial k$.

In the still more general case where motion is allowed in both x and y, i.e., $\eta = \eta_0\cos(kx+ly-\omega t)$,

$$\vec{c}_g = \{\frac{\partial\omega}{\partial k}, \frac{\partial\omega}{\partial l}\}. \tag{8.3.2}$$

[3] In Henri Charriere's classic adventure novel "*Papillon*", the hero escapes an island prison by hurling himself from a cliff into the ocean, having first determined that every 7th wave is large enough to carry him safely past the rocks and out to sea. The image was also used as a metaphor in the popular song "*Love is the Seventh Wave*", by Sting. To the contrary, Russian sailors traditionally believe that the 9th wave is the big one (figure 8, or the album "Hounds of Love" by Kate Bush). In reality, the ratio depends on the wind field and is as likely on any given day to be 5 or 10 as 7 or 9, or to not be discernible at all.

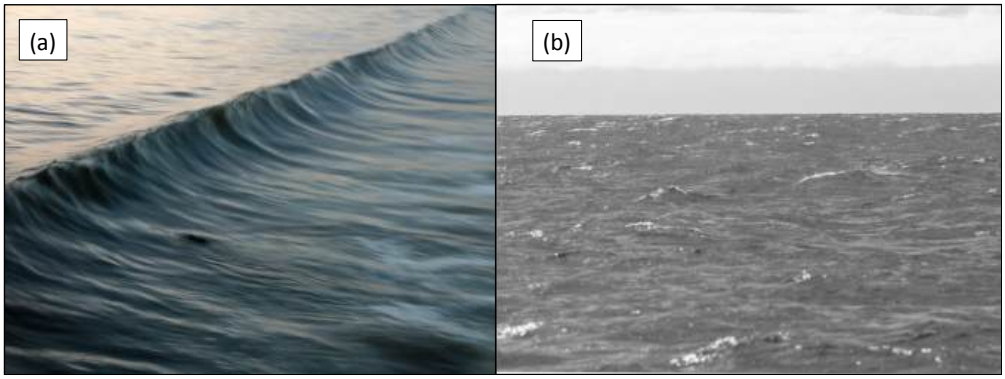


Figure 8.6: (a) The sport of surfing is possible because shallow-water waves can maintain a consistent shape, at least briefly. Photo by Barry Tuck. (b) Deep-water waves tend to be choppy and inconsistent, with random orientation and wavelength. Photo by Aurelie Moulin. Details on page iii.

8.4 Limiting cases

The nondimensional parameter kH is the ratio of water depth to wavelength (times 2π). Two important limiting cases are $kH \to \infty$ (short waves or deep water) and $kH \to 0$ (long waves or shallow water).

8.4.1 Deep water (short) waves

In the short wave limit $|kH| \to \infty$ we can simplify the dispersion relation using:

$$\lim_{kH\to\infty} \tanh kH = 1.$$

Substituting this into (8.2.22) or (8.2.23) gives

$$\omega = (gk)^{1/2}, \quad \text{or} \quad c = \left(\frac{g}{k}\right)^{1/2}.$$

The group velocity is obtained from the first of these:

$$c_g = \frac{\partial \omega}{\partial k} = \frac{1}{2}(gk)^{-1/2} g = \frac{1}{2}\left(\frac{g}{k}\right)^{1/2} = \frac{1}{2}c.$$

This relationship between group and phase velocities is often visible in the wake of a small boat or canoe. The wake propagates to either side of the boat's trajectory as a wave pulse, or envelope. If you look carefully, the envelope is constantly changing its shape as individual wave crests appear at the back, propagate to the front, and disappear. This is because they propagate with the phase velocity while the pulse as a whole propagates at the group velocity, only half as fast (see figure 8.5).

8.4.2 Shallow water (long) waves

We now consider the opposite limit, $kH \to 0$. When $kH \ll 1$, the tanh function is approximated as

$$\tanh kH = kH.$$

Using this approximation in (8.2.23) leads to

$$c = (gH)^{1/2}. \tag{8.4.1}$$

Note that this phase speed is the same for all frequencies. We therefore say that shallow water waves are nondispersive. An arbitrary wave shape made of many such waves will retain its shape rather than dispersing, because all of its Fourier components travel with the same speed (e.g., figure 8.6a). Bores and tsunamis are examples. Another indication of the nondispersive character of shallow water waves is the group velocity, computed from (8.4.1) as

$$c_g = \frac{\partial}{\partial k}(kc) = c,$$

i.e., the phase and group velocities are the same. In this regime a wave superposition such as that shown in figure 8.5 retains its shape.

In contrast, deep-water waves do not retain their shape but instead have an irregular, choppy appearance (figure 8.6b). This is because they are dispersive, i.e., their different Fourier components travel at different speeds.

Referring to figure 8.3 we see that $(gH)^{1/2}$ is the fastest speed possible for small-amplitude surface waves. In most of the ocean, H is about 4000m, so $(gH)^{1/2} = 200$m/s. This is the speed of a tsunami crossing the Pacific, for example. The shallow water limit applies, even though the depth is great, because the wavelength of a tsunami is greater. We'll refer to this speed as the linear long wave speed.

The Froude number

Suppose that something disturbs the surface of a channel so as to radiate waves both upstream and downstream. The fastest waves will travel at $c = \pm c_0$, where $c_0 = (gH)^{1/2}$ is the linear long wave speed. Now suppose that the water in the channel is flowing at speed u. If $u < c_0$, then long waves will still radiate both upstream and downstream, but the upstream waves will move more slowly (to a stationary observer). But if $u = c_0$ the motion of the upstream wave will be arrested, and if $u > c_0$ there will be no propagation upstream. This distinction is quantified by the Froude number[4]:

$$F = \frac{|u|}{(gH)^{1/2}}. \tag{8.4.2}$$

The flow is called critical if $F = 1$, supercritical (no upstream propagation) if $F > 1$ and subcritical (propagation in both directions) if $F < 1$.

If you turn back to figure 1.1 and look at the upper right frame, you will see a rather dramatic transition from smooth to turbulent flow in the Nile River. Upstream of the transition, the flow is smooth, almost glassy, apparently unaffected by the violent churning of the water only a short distance away. The upstream flow is *supercritical*, so that disturbances from the turbulent region downstream cannot reach it. At the transition, the flow changes from supercritical to *subcritical*. This happens because the flow slows down and gets deeper. (Verify by inspection of 8.4.2 that both of these changes act to reduce the Froude number.) This transition is called a hydraulic jump. In the downstream, subcritical region disturbances can propagate in all directions, and they clearly do. We will have much more to say about hydraulic jumps and the Froude number in chapter 9.

Vertical structure

We turn next to an examination of the vertical structure of the velocity and pressure fields in the shallow water regime. The vertical structure function for the horizontal velocity component is

$$U(z) = \omega\eta_0 \frac{\cosh k(z+H)}{\sinh kH},$$

reproduced from (8.2.26). The limit of $\cosh kH$ as $kH \to 0$ is 1. The same is true of $\cosh k(z+H)$ because $|z+H| \le H$. On the other hand, $\sinh kH \approx kH$ for $|kH| \ll 1$. Therefore, in the shallow water limit, we find that the horizontal velocity is independent of depth:

$$U = \frac{\omega\eta_0}{kH} = c\frac{\eta_0}{H}.$$

For the vertical velocity, we begin with

$$W(z) = \omega\eta_0 \frac{\sinh k(z+H)}{\sinh kH},$$

reproduced from (8.2.21). As $kH \to 0$, $\sinh kH \approx kH$, and also $\sinh k(z+H) \approx k(z+H)$ because $|z+H| \le H$. Meanwhile, $\cosh kH \approx 1$ as before. Therefore:

$$W(z) = \omega\eta_0 \frac{k(z+H)}{kH} = \omega\eta_0 \left(1 + \frac{z}{H}\right).$$

Vertical velocity is a linear function of depth with a nonzero surface value $W(0) = \omega\eta_0$.

The vertical distribution of the nonhydrostatic pressure is

$$P = \rho_0 g\, \eta_0 \frac{\cosh k(z+H)}{\cosh kH};$$

(reproduced from 8.2.20). Again, the limit of $\cosh kH$ as $kH \to 0$ is 1, and the same is true of $\cosh k(z+H)$. The nonhydrostatic pressure is independent of depth:

$$P = \rho_0 g \eta_0.$$

With the x and t dependence restored,

$$p^* = \rho_0 g \eta_0 \cos(kx - \omega t) = \rho_0 g \eta.$$

[4] William Froude (1810-1870) was a British naval architect concerned with designing an efficient hull shape for naval vessels. As a ship moves, it radiates waves, and those waves rob the ship of its momentum. The resulting drag increases when the ship speed exceeds the maximum wave speed (i.e., the motion becomes supercritical), and it is therefore dependent on what we now call the Froude number.

This is just the weight of the water between $z = 0$ and $z = \eta$ (negative if $\eta < 0$), i.e., what we have been calling the "nonhydrostatic pressure" actually becomes hydrostatic in the long-wave limit. This is because vertical accelerations are extremely small, so the vertical pressure gradient almost balances gravity. Adding this to the hydrostatic pressure for the motionless state $-\rho_0 gz$, we obtain the total pressure

$$p = \rho_0 g(\eta - z).$$

The total pressure is therefore hydrostatic even in the presence of long waves.

Back to Table of Contents.

Chapter 9

Nonlinear, hydrostatic flow over topography

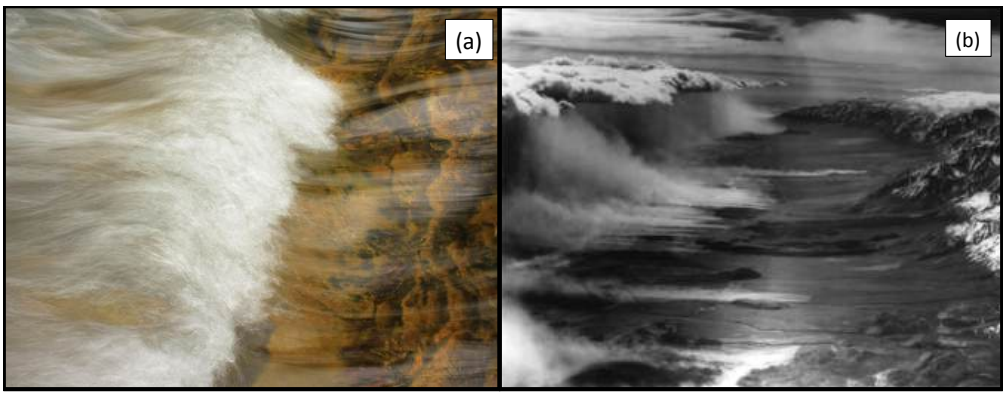


Figure 9.1: (a) River flow over an obstruction. Photo by Jonathan Ball.(b) "The Bishop Wave", a storm on the downwind slope of the Sierra Nevada mountain range. Wind is from right to left; clouds and dust at left indicate rapidly rising, turbulent air. Note the similarity to (a). (Photo by Robert Symons, 5 March 1950.) Further details are on page iii.

9.1 Introduction

Our third example of the application of the Navier-Stokes equations to natural flows is hydrostatic flows over topography. These occur in many natural settings such as downslope windstorms (figure 9.1b), tidal ocean currents and flow over dams. They are perhaps most recognizable in the flow of a small stream over a rocky bottom (figure 9.1a). For this discussion, we abandon the assumption that the amplitude of the disturbance is small. Instead, we assume that the flow is in hydrostatic balance, similar to the long-wave limit discussed above.

Assumptions:

1. The fluid is inviscid: $\nu = 0$.
2. The fluid is homogeneous: $\rho = \rho_0$.
3. The flow is confined to the $x-z$ plane, so there is no dependence on y and no motion in the y direction.
4. The channel width W is uniform and the walls are vertical.
5. The flow has the character of gravity waves in the long-wave limit:
   (a) The horizontal (streamwise) velocity is independent of depth: $\vec{u} = u(x,t)\hat{e}^{(x)}$.
   (b) The pressure is hydrostatic: $p = \rho_0 g(\eta - z)$.

Some of these assumptions will be relaxed as we go along.

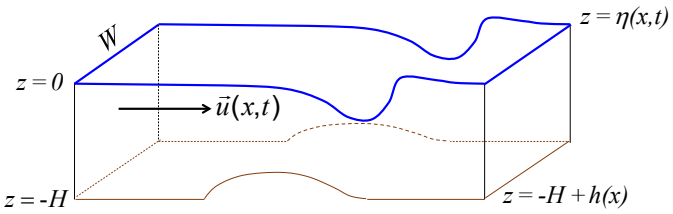


Figure 9.2: Two-dimensional flow in an inviscid, homogeneous fluid with a free surface. The vertical coordinate is directed opposite to gravity (upward). The mean surface is $z = 0$ and the mean depth is H.

9.2 Hydraulic control

Here we'll look at the phenomenon of hydraulic control, which constrains the state of flow over an obstacle. For this initial look we'll assume that the flow takes place in a channel of rectangular cross-section.

9.2.1 Equations of motion for flow in a rectangular channel

We now use these assumptions to simplify the equations of motion. The momentum equation for inviscid, homogeneous flow is

$$\rho_0 \frac{D\vec{u}}{Dt} = -\rho_0 g\hat{e}^{(z)} - \vec{\nabla} p.$$

Substituting assumption 5b, this becomes

$$\rho_0 \frac{D\vec{u}}{Dt} = -\rho_0 g\hat{e}^{(z)} - \rho_0 g(\vec{\nabla}\eta - \hat{e}^{(z)}) = -\rho_0 g\vec{\nabla}\eta,$$

and the streamwise (x) component is

$$\frac{Du}{Dt} = u_t + uu_x + \underbrace{vu_y}_{=0} + \underbrace{wu_z}_{=0} = -g\eta_x,$$

where subscripts indicate partial derivatives. Our streamwise momentum equation is now

$$\boxed{u_t + uu_x = -g\eta_x.} \tag{9.2.1}$$

Mass conservation is expressed by $\vec{\nabla}\cdot\vec{u} = 0$. Integrating this over z gives

$$\int_{-H+h(x)}^{\eta(x,t)} (u_x + w_z)dz = 0.$$

Because u is independent of z (assumption 5a), we can take the first term out of the integral, while the second term integrates trivially:

$$u_x(\eta + H - h) + w\big|_{-H+h}^{\eta} = 0. \tag{9.2.2}$$

Both the surface and the bottom boundary are material surfaces, so the boundary conditions on w are:

$$\begin{aligned} w\big|_{z=\eta} &= \frac{D\eta}{Dt} = \eta_t + u\eta_x. \\ w\big|_{z=-H+h(x)} &= \frac{D}{Dt}(-H+h) = \underbrace{h_t}_{=0} + uh_x = uh_x. \end{aligned}$$

Combining these, we have

$$w\big|_{-H+h}^{\eta} = \eta_t + u\eta_x - uh_x.$$

Substituting this into (9.2.2) gives

$$\boxed{\eta_t + u\eta_x - uh_x + u_x(\eta + H - h) = 0.} \tag{9.2.3}$$

Insight into (9.2.3) may be gained by interpreting it in terms of the volume flux:

$$Q = uA = uW\underbrace{(\eta + H - h)}_{\text{total depth}}.$$

In (9.2.3), note that the 2nd and 3rd terms are equivalent to $u(\eta + H - h)_x$, since H is a constant. Therefore:

$$\eta_t + u(\eta + H - h)_x + u_x(\eta + H - h) = \eta_t + [u(\eta + H - h)]_x = 0,$$

or,

$$\eta_t = -\frac{Q_x}{W}. \tag{9.2.4}$$

So (9.2.3) tells us that the surface moves up and down so as to balance convergences and divergences in the volume flux.

9.2.2 Steady flow and the Froude number

In a steady flow, time derivatives are zero and (9.2.1) and (9.2.3) can be written as

$$uu_x = -g\eta_x, \tag{9.2.5}$$
$$u\eta_x + u_x(\eta + H - h) = uh_x. \tag{9.2.6}$$

We now multiply (9.2.6) by u:

$$u^2\eta_x + (\eta + H - h)uu_x = u^2h_x$$

and substitute (9.2.5):

$$u^2\eta_x - (\eta + H - h)g\eta_x = u^2h_x.$$

Note that we now have η_x as a factor in both terms on the left-hand side. Dividing by $g(\eta + H - h)$, we have

$$\left[\underbrace{\frac{u^2}{g(\eta + H - h)}}_{F^2} - 1\right]\eta_x = \underbrace{\frac{u^2}{g(\eta + H - h)}}_{F^2} h_x.$$

The quantity $u^2/g(\eta + H - h)$ that appears on both sides is the squared velocity over g times the total water depth. We recognize this as the square of the Froude number (cf. section 8.4.2). In the case of small-amplitude waves over a flat bottom ($\eta \to 0; h = 0$), F^2 matches our previous definition u^2/gH. Recall that flow is supercritical (i.e., no information can propagate upstream) if $F > 1$ and subcritical if $F < 1$.

In summary, steady flow over an obstacle requires the following relation between the surface and bottom slopes:

$$\boxed{(F^2 - 1)\eta_x = F^2 h_x,} \tag{9.2.7}$$

where

$$\boxed{F = \frac{|u|}{\sqrt{g(\eta + H - h)}}.} \tag{9.2.8}$$

An obstacle of small amplitude

If the amplitudes of the surface deflection and the bottom topography are small compared to the water depth, the Froude number can be treated as a constant, allowing us to integrate (9.2.7). To see this, suppose that u is close to its upstream value u_0 and $|\eta| \ll H$ and $|h| \ll H$. In that case the Froude number is nearly constant:

$$F = F_0 + F',$$

where $F_0 = |u_0|/\sqrt{gH}$ and F' is the perturbation caused by nonzero (but small) perturbations in u, η and h. Then

$$F^2 = F_0^2 + 2F_0F' + F'^2.$$

As we did in section 8.2.1, we discard the term that is a product of small quantities, giving

$$F^2 = F_0^2 + 2F_0F'.$$

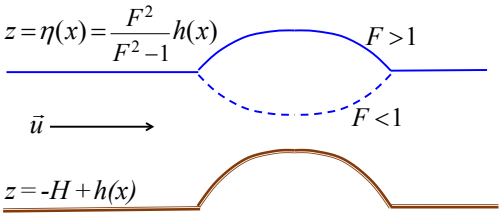


Figure 9.3: Steady, small-amplitude flow over an obstacle. The sign of the surface deflection depends on the Froude number in accordance with (9.2.11).

Substituting into (9.2.7), we now have

$$(F_0^2+2F_0F'-1)\eta_x'=(F_0^2+2F_0F')h_x',$$

where primes have been placed on η and h to indicate that they are small quantities. Once again we discard the products of primes, and arrive at

$$(F_0^2-1)\eta_x'=F_0^2h_x', \tag{9.2.9}$$

i.e., fluctuations in F can be neglected in the limit of small amplitude. We now dispense with the subscripts and primes as they have served their purpose.

Integrating (9.2.9), we obtain

$$(F^2-1)\eta=F^2h+C. \tag{9.2.10}$$

We assume that, far upstream, the bottom is flat and there is no surface deflection, so the constant of integration is zero. Hence:

$$\eta=\frac{F^2}{F^2-1}h. \tag{9.2.11}$$

So the surface deflection over a small bump depends on the flow speed, as shown in figure 9.3. If the flow is fast enough to make $F>1$, then the surface is elevated in proportion to the topography. If $F<1$, though, the surface deflection is opposite to the topography: low over a bump (dashed curve on figure 9.3); raised over a deep spot.

An obstacle of arbitrary amplitude: hydraulic control

From (9.2.7) we can infer a critical fact about flow over an obstacle:

If $h_x=0$, then either $\eta_x=0$ or $F=1$.
So at a high or low spot in the bottom topography, either the surface is flat or the flow is critical.

In the small-amplitude limit, we have seen that the first condition is satisfied; $\eta_x=0$ at the crest of the obstacle (figure 9.3). But in nature, we often observe plunging flow over an obstacle, in which case η_x is not zero (figure 9.4). In that case, the Froude number can only be 1. This restriction on the flow state is called hydraulic control. Since plunging flow does not happen in the small-amplitude limit, we identify it as a fundamentally nonlinear effect. To explore it we must consider disturbances of arbitary amplitude.

If the amplitudes are not small, we can no longer integrate (9.2.7) to obtain a quantitative relationship between η and h. Instead, we analyze (9.2.7) qualitatively. We will do this one step at a time for the case of flow over a bump, with careful reference to figure 9.4.

[1] At the crest of the bump, $F=1$.

[2] We therefore expect that $F<1$ upstream of the bump, where the flow is deeper and slower.

[3] When we first encounter the bump, $|\eta|$ and $|h|$ are small, and we therefore expect η to decrease as in the small-amplitude case with $F<1$ (figure 9.3).

[4] Approaching the crest of the bump, the disturbance is no longer small, but F is still less than 1 and h_x is still positive, so (9.2.7) tells is that η must continue to drop.

[5] On the lee side of the bump, $F>1$ and $h_x<0$, so we see from (9.2.7) that η_x must remain negative.

[6] Beyond the bump, the increase in F from subcritical to supercritical values produces relatively shallow, fast flow.

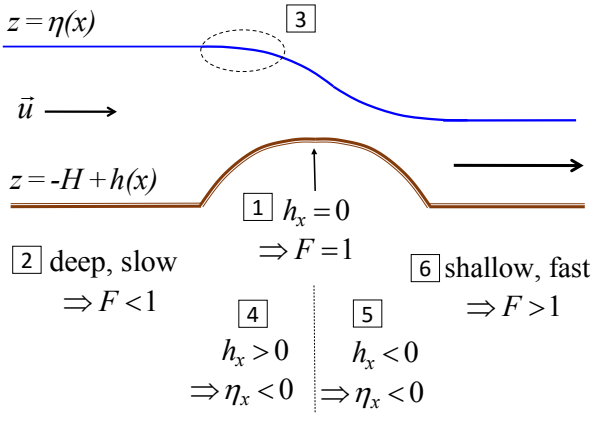


Figure 9.4: Steady, large-amplitude flow over an obstacle. Numerals indicate the points described in the text.

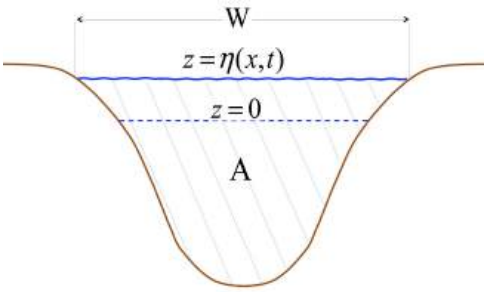


Figure 9.5: Cross section of a channel of arbitrary shape. $W(x,\eta)$ is the surface width; $A(x,\eta)$ is the wetted area.

9.2.3 Generalization: a channel of arbitrary cross-section

We now relax the assumption #4 of the previous section, namely that the the channel has rectangular cross-section. Instead, we let the channel have arbitrary shape. The width of the stream is now variable, both in x as the channel narrows and widens and in t as the surface elevation changes. We also define the "wetted" cross-sectional area A, which varies in both x and t for the same reasons. The only restriction is that we assume the existence of an "upstream" region where the channel shape is uniform in x, the velocity is even and $\eta = 0$. We would like to see how u and η vary in response to changes in the channel shape.

The streamwise momentum equation is now

$$\frac{Du}{Dt} = u_t + uu_x + vu_y + \underbrace{wu_z}_{=0} = -g\eta_x.$$

We add a new assumption, namely that the flow is "mostly" streamwise, in the sense that $|u| \gg |v|$, and/or $|u_x| \gg |u_y|$. As a result $|vu_y| \ll |uu_x|$, i.e., the spanwise advection term is negligible, leaving

$$u_t + uu_x = -g\eta_x$$

as before. We now assume that the flow is steady ($u_t = 0$), and substitute $u = Q/A$ to obtain

$$\frac{Q}{A}\frac{d}{dx}\left(\frac{Q}{A}\right) = -g\frac{d\eta}{dx}, \tag{9.2.12}$$

or

$$-\frac{Q^2}{A^3}\frac{dA}{dx} = -g\frac{d\eta}{dx}. \tag{9.2.13}$$

Can we solve this for η, and thereby predict the surface response to a given change in channel shape? No, because A is determined in part by η. Considering A as a function of x and $\eta(x)$, the total x-derivative of A has two parts:

$$\frac{dA}{dx} = \underbrace{\left(\frac{\partial A}{\partial x}\right)_\eta}_{\text{channel shape}} + \underbrace{\left(\frac{\partial A}{\partial \eta}\right)_x \frac{d\eta}{dx}}_{\text{water depth}}$$

The first term describes variations due only to the shape of the walls, the second only to the surface elevation. Note that the change in A due to a small change in surface elevation is $\delta A = W\delta\eta$, so

$$\left(\frac{\partial A}{\partial \eta}\right)_x = W,$$

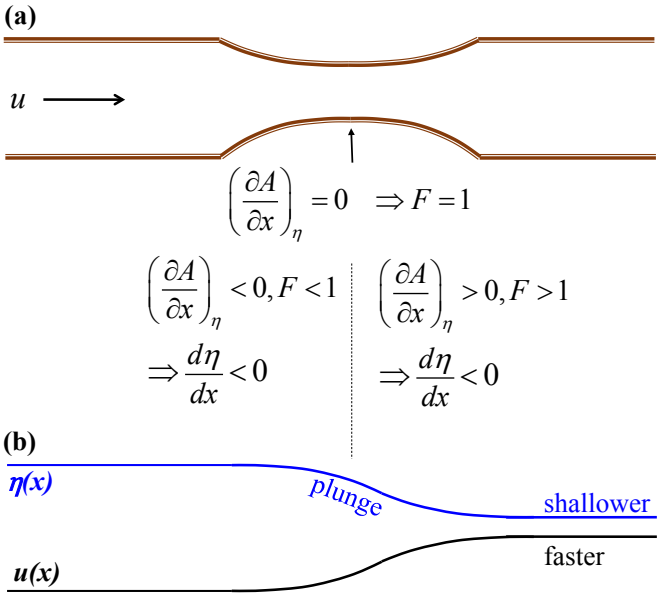


Figure 9.6: Flow through an arbitrary constriction. (a) Plan view. (b) Elevation and velocity profiles.

and we are left with

$$\frac{u^2}{A}\left[\left(\frac{\partial A}{\partial x}\right)_\eta + W\frac{d\eta}{dx}\right] = g\frac{d\eta}{dx}.$$

We now divide by g and collect terms proportional to $d\eta/dx$:

$$\left(\frac{Wu^2}{gA} - 1\right)\frac{d\eta}{dx} = -\frac{u^2}{gA}\left(\frac{\partial A}{\partial x}\right)_\eta.$$

Defining the Froude number as

$$\boxed{F = \frac{|u|}{\sqrt{gA/W}},} \tag{9.2.14}$$

we finally have

$$\boxed{(F^2-1)\frac{d\eta}{dx} = -\frac{F^2}{W}\left(\frac{\partial A}{\partial x}\right)_\eta.} \tag{9.2.15}$$

This is a generalization of our previous results (9.2.7, 9.2.8) for the rectilinear channel. In that previous case W is constant and $A = W(\eta + H - h)$, giving

$$\left(\frac{\partial A}{\partial x}\right)_\eta = -W\frac{dh}{dx},$$

from which we recover (9.2.7, 9.2.8).

We can understand plunging flow through an arbitrary constriction in the same way we did in the rectilinear case. Approaching the constriction (figure 9.6a), the wetted area decreases. If $F < 1$, the elevation must also decrease ($d\eta/dx < 0$; figure 9.6b). At the throat of the constriction, $F = 1$. Leaving the constriction, the wetted area increases but $F > 1$, so the surface continues to descend. Downstream, the flow is supercritical, i.e., shallower and faster.

Figure 9.7 shows an example from the Smith River in California. Here, rafters are crossing a hydraulic control caused by a constriction. They will now enjoy (one hopes) an exciting few seconds as they negotiate a hydraulic jump, the subject of the next section.

9.3 Hydraulic jumps and bores

9.3.1 A stationary hydraulic jump in a rectilinear channel

Downstream of a hydraulic control, the supercritical flow state is unstable. As a result it becomes turbulent and returns to a subcritical state (slower, deeper flow). This transition is called a hydraulic jump. As the name suggests, it can be quite sudden (figure 9.8).

Figure 9.7: Hydraulic control in flow through a constriction, followed by a hydraulic jump. River photo courtesy of Zachary Collier of the Northwest Rafting Company. (Further details are on page iii.)

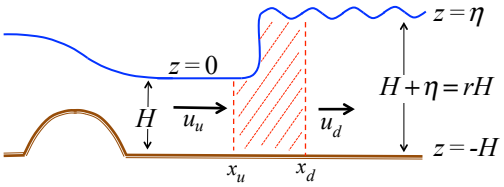


Figure 9.8: A hydraulic jump downstream of a constriction. The jump is enclosed by the region $x_u < x < x_d$. Upstream and downstream flows have velocity and depth equal to $\{u_u, H\}, \{u_d, rH\}$.

The hydraulic jump is a situation where we cannot ignore turbulence, but we will model its effects in the simplest way possible. The effect of turbulence on a unidirectional channel flow is to divert some of the mean downstream motion into chaotic swirls and eddies. These motions carry no net momentum (the swirling motion is as likely to go in one direction as another), but they do carry kinetic energy. Energy diverted into turbulence does not usually return to the mean flow. Instead, it is converted into progressively smaller eddies (e.g., figures 5.6, 9.9) and ultimately converted to internal energy via frictional dissipation (sections 6.4.2, 6.4.3).

Our goals here are to make testable predictions of the downstream flow state. Is it deeper or shallower than upstream? Slower or faster? And by how much? Our tools will be the familiar equations of momentum and mass conservation plus a new energy conservation law that accounts for turbulence.

To simplify the analysis, we return to the case of a rectilinear channel. We retain the assumptions that the fluid is inviscid and homogeneous, and we assume further that, in the vicinity of the hydraulic jump, the bottom is flat ($h = 0$). The momentum and mass equations describing the mean flow are

$$u_t = -uu_x - g\eta_x \tag{9.3.1}$$
$$\eta_t = -[u(H+\eta)]_x. \tag{9.3.2}$$

Now consider the vertically-integrated streamwise momentum

$$M(x,t) = \int_{-H}^{\eta} u\, dz = u(\eta + H).$$

The evolution of M is governed by:

$$\begin{aligned} M_t &= u_t(\eta+H) + u\eta_t \\ &= (-uu_x - g\eta_x)(\eta+H) - u[u(H+\eta)]_x \\ &= -u_x u(\eta+H) - g(H+\eta)_x(\eta+H) - u[u(H+\eta)]_x. \end{aligned}$$

We have used the fact that $\eta_x = (H+\eta)_x$, because H is constant. The first and third terms combine to form a complete derivative, as does the middle term:

$$M_t = -[u^2(\eta+H)]_x - g\frac{1}{2}[(H+\eta)^2]_x,$$

or

$$\boxed{M_t = -\mathscr{F}_x^{(m)}, \quad \text{where} \quad \mathscr{F}^{(m)} = u^2(\eta+H) + \frac{g}{2}(H+\eta)^2.} \tag{9.3.3}$$

This tells us that the vertically-integrated momentum is governed by the convergence of the momentum flux $\mathscr{F}^{(m)}$.

Now consider a steady hydraulic jump, as shown in figure 9.8. Equations (9.3.2) and (9.3.3) provide two constraints that determine the change in depth and velocity across the jump. First, the mass flux exiting the jump must be the same as that entering it:

$$[u(H+\eta)]_{x_u}^{x_d} = 0. \tag{9.3.4}$$

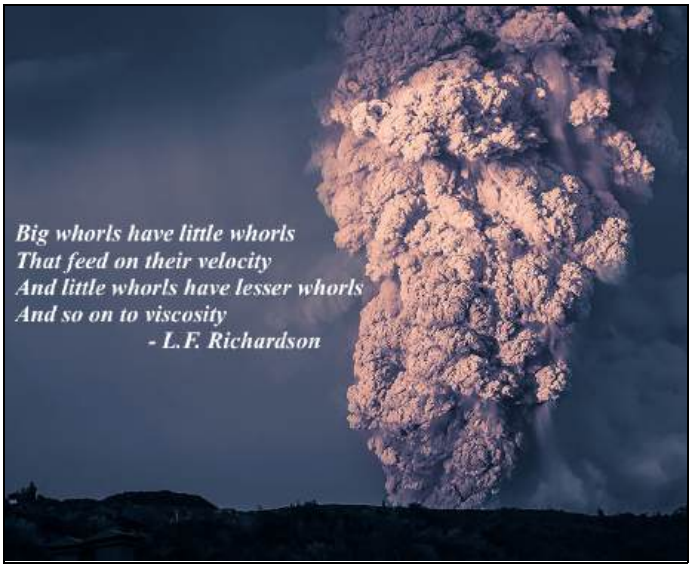


Figure 9.9: Based on the traditional nursery rhyme *Fleas*, Lewis Fry Richardson's tongue-in-cheek description of the turbulent energy cascade is both poetic and accurate. The "whorls" shown here, both big and little, were generated by the Colbuco volcano in Chile, photographed by Marcelo Reyes. Richardson, 1881-1953, was an English mathematician and psychologist. He invented the first algorithms for numerical weather forecasting decades before there were computers to execute them. He was also a dedicated pacifist and published important statistical studies on the causation and prevention of wars.

Second, the momentum flux must be the same exiting as entering:

$$\left[u^2(\eta+H)+\frac{g}{2}(H+\eta)^2\right]_{x_u}^{x_d}=0. \tag{9.3.5}$$

To apply these constraints, we first relabel the upstream and downstream velocities as u_u and u_d (see figure 9.8) . The upstream depth is H and the downstream depth is

$$H+\eta=rH,$$

where the depth ratio r is a constant. If the flow deepens across the jump as shown in figure 9.8, then $r>1$. Substituting into (9.3.4, 9.3.5),

$$u_uH = u_drH, \tag{9.3.6}$$

$$Hu_u^2+\frac{g}{2}H^2 = rHu_d^2+\frac{g}{2}r^2H^2. \tag{9.3.7}$$

From (9.3.6), we have

$$u_d=\frac{u_u}{r}. \tag{9.3.8}$$

Substituting this in (9.3.7) and cancelling a factor H, we obtain

$$u_u^2+\frac{g}{2}H=\frac{u_u^2}{r}+\frac{g}{2}r^2H.$$

This is easily solved for u_u:

$$\boxed{u_u=\sqrt{\frac{gH}{2}r(r+1)}.} \tag{9.3.9}$$

Using (9.3.8), we also have

$$\boxed{u_d=\sqrt{\frac{gH}{2}\frac{r+1}{r}}.} \tag{9.3.10}$$

To predict the downstream flow state, assume that we know the upstream velocity u_u and depth H, which gives the upstream Froude number:

$$F_u^2=\frac{u_u^2}{gH}=\frac{r(r+1)}{2}.$$

This is a quadratic equation for r with a single positive solution:

$$\boxed{r = \frac{\sqrt{1+8F_u^2}-1}{2}.} \tag{9.3.11}$$

We are now able to predict both the downstream velocity $u_d = u_u/r$ and the downstream depth rH. To complete the picture we add the downstream Froude number:

$$F_d^2 = \frac{u_d^2}{g(H+\eta)} = \frac{u_u^2/r^2}{grH} = \frac{F_u^2}{r^3} = \frac{r+1}{2r^2}.$$

Example: Consider a dam spillway with water depth $H = 1$m and velocity $u_u = 6$m/s. The Froude number is easily computed:

$$F_u = \frac{6\text{m/s}}{\sqrt{9.8\text{m/s}^2 \times 1\text{m}}} = 1.92.$$

Because the flow is supercritical, we expect to see a hydraulic jump. From (9.3.11) we compute $r = 2.26$. The downstream flow is therefore slower and deeper, with depth $rH = 2.26$m and velocity $u_u/r = 2.65$m/s. The downstream Froude number is subcritical: $F_d = F_u/r^{3/2} = 0.32$.

An important question remains to be addressed. So far, we have *assumed* that $r > 1$, in which case

$$F_u^2 = \frac{r(r+1)}{2} > 1, \quad \text{and} \quad F_d^2 = \frac{r+1}{2r^2} < 1.$$

Now, suppose that $r < 1$. The results are opposite: $F_u < 1$ and $F_d > 1$, i.e., the transition is from a subcritical to a supercritical state, with flow becoming shallower and faster. This is a perfectly valid solution of the momentum and mass equations as developed above. Yet, it is a phenomenon never seen in nature (figure 9.10). Why? This paradox exists because we have not yet considered the conservation of energy.

9.3.2 Energy flux and the effects of turbulence

At a given location, the specific mechanical energy of the mean flow (kinetic plus potential, per unit mass) is $\frac{1}{2}u^2 + gz$. Integrating this over the vertical extent of the flow, and remembering that u is independent of z, we have

$$\begin{aligned} E(x,t) &= \int_{-H}^{\eta} (\tfrac{1}{2}u^2 + gz)\, dz \\ &= \left[\tfrac{1}{2}u^2 z + \tfrac{1}{2}gz^2\right]_{-H}^{\eta} \\ &= \tfrac{1}{2}u^2(\eta + H) + \tfrac{1}{2}g(\eta^2 - H^2). \end{aligned}$$

Our task now is to find the equation that governs $E(x,t)$. Differentiating with respect to time, we obtain

$$E_t = uu_t(\eta + H) + \tfrac{1}{2}u^2\eta_t + g\eta\eta_t.$$

Now from (9.2.1),

$$u_t = -uu_x - g\eta_x = -(\tfrac{1}{2}u^2 + g\eta)_x$$

and from (9.2.4),

$$\eta_t = -\left(\frac{Q}{W}\right)_x,$$

where $Q/W = u(\eta + H)$. Substituting, we have

$$\begin{aligned} E_t &= -u\left(\frac{1}{2}u^2 + g\eta\right)_x (\eta + H) - \left(\frac{1}{2}u^2 + g\eta\right)\left(\frac{Q}{W}\right)_x \\ &= -\frac{Q}{W}\left(\frac{1}{2}u^2 + g\eta\right)_x - \left(\frac{Q}{W}\right)_x \left(\frac{1}{2}u^2 + g\eta\right) \\ &= -\left[\frac{Q}{W}\left(\frac{1}{2}u^2 + g\eta\right)\right]_x, \end{aligned}$$

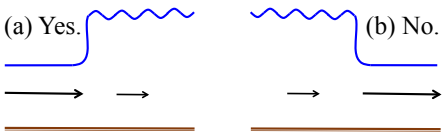


Figure 9.10: Why do hydraulic jumps always look like (a), and never (b)?

or

$$E_t = -\mathscr{F}_x^{(e)}, \tag{9.3.12}$$

in which

$$\mathscr{F}^{(e)} = \frac{Q}{W}\left(\frac{1}{2}u^2 + g\eta\right)$$

is the downstream flux of mechanical energy in the mean flow.

Now suppose we wanted to account for the fact that some mechanical energy is diverted into turbulent motions. The equations of motion would then include some very complicated additional terms, and (9.3.12) would take the form

$$E_t = -\mathscr{F}_x^{(e)} - \mathscr{E}, \tag{9.3.13}$$

where the new term $\mathscr{E}$ represents the rate of energy loss to turbulence. Explicit calculation of $\mathscr{E}$ is not practical. Happily, all we need to know here is that $\mathscr{E}$ is positive, i.e., turbulent dissipation only works one way, and that is to *reduce* the mechanical energy of the flow.

In steady state, then, $E_t = 0$ and (9.3.13) becomes

$$\mathscr{F}_x^{(e)} = -\mathscr{E} < 0.$$

This inequality simply states that *the flux of mechanical energy exiting a turbulent region is less than the flux entering it.* To apply this to the hydraulic jump shown in figure 9.8, we integrate from x_u to x_d and obtain

$$\int_{x_u}^{x_d} \mathscr{F}_x^{(e)}\, dx \;=\; \mathscr{F}^{(e)}\big|_{x_u}^{x_d} \;=\; \frac{Q}{W}\left(\frac{1}{2}u_d^2 + g\eta - \frac{1}{2}u_u^2\right) < 0.$$

We have oriented our coordinates such that $Q > 0$, and therefore

$$\frac{1}{2}u_d^2 + g\eta - \frac{1}{2}u_u^2 < 0.$$

Substitution from (9.3.8), (9.3.9) and (9.3.11) gives

$$\frac{1}{2}\,\frac{gH}{2}\,\frac{r+1}{r} + g(r-1)H - \frac{1}{2}\,\frac{gH}{2}\,r(r+1) < 0.$$

With a little algebra, this becomes

$$(1-r)^3 < 0, \quad \text{therefore} \quad r > 1.$$

If r were less than 1, this condition would be violated, meaning that the mean flow would gain energy from the turbulence, an impossibility. As a result, hydraulic jumps in the real world always carry the flow from a supercritical to a subcritical state, as in figure 9.10a.

9.3.3 A hydraulic bore

Until now we have assumed that our hydraulic jump is stationary, as it would be downstream of a fixed obstruction. A very similar phenomenon is a bore, which is basically a moving hydraulic jump. A good example is swash, i.e., a beach wave after it breaks.

Mathematical analysis requires only that we subtract the appropriate velocity from (9.3.8) and (9.3.9). In the stationary case considered previously, we have upstream velocity u_u and downstream velocity u_d, while the velocity of the jump, which we'll call u_J, is zero (figure 9.11a). Now suppose the jump is moving into a quiescent region (figure 9.11b). To use our previous results, we subtract u_u from all velocities, so that the new upstream velocity $u_u' = 0$. Similarly, the downstream velocity $u_d' = u_d - u_u$ and the jump velocity $u_J' = -u_u$, where u_u and u_d are still as given by (9.3.9) and (9.3.8).

Two features are worth noting:

- The bore propagates leftward relative to the upstream fluid at a velocity

$$u_J' = -u_u = -\sqrt{\frac{gH}{2}r(r+1)}. \tag{9.3.14}$$

  Its speed is then

$$|u_J'| \;=\; \sqrt{gHr}\,\sqrt{\frac{(r+1)}{2}} \;>\; \sqrt{gHr} \quad \text{for} \quad r > 1.$$

  This speed is greater than the linear long wave speed, the "speed limit" for small-amplitude waves. This is an example of *nonlinear speedup*, the tendency for nonlinear effects to increase the speed of a disturbance. The greater the height of the bore (i.e., r), the greater the speedup.

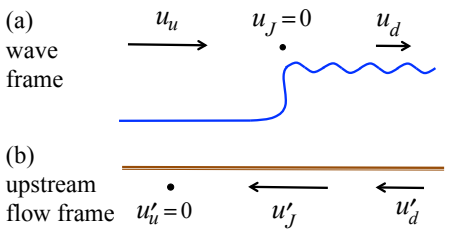


Figure 9.11: Definition sketch for velocities in (a) a stationary hydraulic jump (upper) and (b) a bore moving into still water (lower). The wave is the same; the only difference is the reference frame. The velocities u_u and u_d are given by (9.3.9) and (9.3.8).

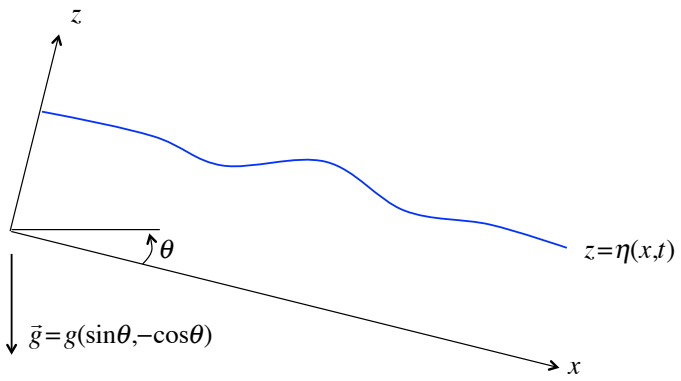


Figure 9.12: Definition sketch for two-dimensional, free-surface flow in a tilted channel.

- The flow behind the bore moves to the left with speed

$$|u'_d| = |u_d - u_u| = \sqrt{\frac{gH}{2}r(r+1)}\left(1-\frac{1}{r}\right) < \sqrt{\frac{gH}{2}r(r+1)}.$$

  This means that the current behind the bore moves more slowly than the bore itself. More specifically, the ratio of the speed of the water behind the bore to that of the bore itself is

$$\frac{u'_d}{u'_J} = 1 - \frac{1}{r}.$$

  Two limiting cases are of interest:

  - In the limit $r \to 1$, the water behind the bore is nearly stationary ($u'_d \to 0$). This is similar to the case of small-amplitude waves, where the surrounding water does not travel with the wave.
  - In the limit $r \to \infty$, $u'_d \to u'_J$, i.e. when a large bore travels into shallow water, it becomes a "wall of water", traveling almost as a solid object.

In some cases, both the bore and the upstream water are moving, e.g., a tidal bore advancing up a river. In that case, (9.3.14) still gives the speed of the wave *relative to the upstream flow that it is propagating into*. If you know the speed of the upstream (or the downstream) flow, you can calculate the speed of the wave relative to the shore, or relative to any other reference frame that may be relevant.

9.4 Flood waves in a turbulent river

In homework exercise 29, we predict river speed using the stress-strain relation for a Newtonian fluid, and find that the speed is ~ 8 km/s! The result is rendered reasonable only by imposing an eddy viscosity, much greater than the true viscosity, to represent the retarding effect of turbulent fluctuations. Here we will do that calculation again with turbulence represented in a more realistic fashion. We will also allow for a variable surface elevation $\eta(x,t)$ and hence the possibility of waves.

We begin with Cauchy's equation (6.3.7), with the stress tensor expanded into pressure and deviatoric parts as per (6.3.11):

$$\rho\frac{Du_j}{Dt} = \rho g_j - \frac{\partial p}{\partial x_j} + \frac{\partial \sigma_{ij}}{\partial x_i}. \tag{9.4.1}$$

We assume that the flow is homogeneous ($\rho = \rho_0$) and two-dimensional ($v = 0,\ \partial/\partial y = 0$). We also assume that the stress varies primarily in z, and therefore neglect its x-derivatives. Finally, we assume that the motion is nearly hydrostatic and therefore $\partial u/\partial z = 0$.

Writing out the components of (9.4.1), we have

$$\frac{Du}{Dt} = g\sin\theta - \frac{1}{\rho_0}\frac{\partial p}{\partial x} + \frac{1}{\rho_0}\frac{\partial \sigma_{31}}{\partial z} \tag{9.4.2}$$

$$\frac{Dw}{Dt} = -g\cos\theta - \frac{1}{\rho_0}\frac{\partial p}{\partial z} + \frac{1}{\rho_0}\frac{\partial \sigma_{33}}{\partial z}. \tag{9.4.3}$$

Assuming that the vertical acceleration Dw/Dt is negligible, the vertical momentum equation gives us an altered form of hydrostatic balance:

$$\frac{1}{\rho_0}\frac{\partial p}{\partial z} = -g\cos\theta + \frac{1}{\rho_0}\frac{\partial \sigma_{33}}{\partial z}.$$

Integrating over z and assuming $p = 0$ at $z = \eta$, we have

$$p = -\rho_0 g\cos\theta\,(\eta - z) + \sigma_{33}|_z^\eta.$$

We now differentiate with respect to x (remembering that stress is independent of x) to get

$$\frac{\partial p}{\partial x} = \rho_0 g\cos\theta\frac{\partial \eta}{\partial x}.$$

Substituting this into the x-momentum equation, we have

$$\frac{Du}{Dt} = g\sin\theta - g\cos\theta\frac{\partial \eta}{\partial x} + \frac{1}{\rho_0}\frac{\partial \sigma_{31}}{\partial z}.$$

In summary, the along-stream flow is driven by three forces: the downhill pull of gravity, pressure gradients due to surface deflection, and the stress divergence. The tangential stress σ_{31} is exerted on the fluid by the solid bottom boundary and by the air above the surface, then transmitted through the fluid interior by turbulent eddies.

Assuming again that u is independent of z (the long-wave approximation), we can integrate in the vertical to obtain

$$(H+\eta)\frac{Du}{Dt} = (H+\eta)\left(g\sin\theta - g\cos\theta\frac{\partial \eta}{\partial x}\right) + \frac{\sigma_{31}}{\rho_0}\Big|_{-H}^{\eta}. \tag{9.4.4}$$

Here we see a major advantage of working directly from Cauchy's equation: all we need to know about the stress tensor is the values of σ_{31} at the surface and the bottom. The bottom stress is given by a very well-tested empirical relationship:

$$\sigma_{31}|_{z=-H} = \rho_0 C_D u^2,$$

where C_D is the drag coefficient. Typically, C_D is in the range $10^{-3} - 10^{-2}$. Here we treat C_D as a constant. A similar relationship holds at the surface:

$$\sigma_{31}|_{z=\eta} = \rho_A C_D (u_A - u)^2,$$

where ρ_A is the density of air and u_A is the wind speed. For a typical river flow the effect of wind is negligible, so we will consider only the bottom stress. Substituting in (9.4.4) and dividing by $H+\eta$, we have

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \underbrace{g\sin\theta}_{gravity} - g\cos\theta\frac{\partial \eta}{\partial x} - \underbrace{\frac{C_D u^2}{H+\eta}}_{friction}. \tag{9.4.5}$$

We now assume that the terms involving $\partial/\partial t$ and $\partial/\partial x$ in (9.4.5) are negligible. In other words, the dominant balance is between the downhill pull of gravity and the retarding effect of bottom friction. This assumption makes it easy to solve for the flow velocity[1]:

$$\boxed{u = \sqrt{\frac{g(H+\eta)\sin\theta}{C_D}}.} \tag{9.4.6}$$

For example, consider the river described in homework exercise 29, with $H = 2m$, $\eta = 0$, $g = 9.8ms^{-2}$, $\sin\theta = 4.3\times10^{-4}$ and set $C_D = 3\times10^{-3}$ (Li et al. 2004). The result is $u = 1.7m/s$, a reasonable river speed.

[1] This is called the Chézy formula for river speed. It was developed by the French engineer Antoine de Chézy (1718-1798) and tested using measurements of the River Seine.

Exercise: Derive an expression for the Froude number based on (9.4.6). What is its value for the river parameters given above? Now derive a simple formula for the slope angle at which the flow becomes supercritical. If $C_D = 3 \times 10^{-3}$, what is this critical slope angle in meters per kilometer? Based on this model, could a river become supercritical as a result of increased flow rate (in flood conditions, for example)? [2]

We now allow for small but nonzero variations in x and t. Note that we have not yet had to invoke mass conservation: with such simple flow geometry mass is conserved automatically. To enforce mass conservation in the presence of variations in x and t, we write (9.2.2) with bottom topography omitted ($h = 0$):

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}[u(H+\eta)] = 0. \tag{9.4.7}$$

Substituting for u using (9.4.6), this becomes[3]:

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}\sqrt{\frac{g(H+\eta)^3 \sin\theta}{C_D}} = 0.$$

After carrying out the differentiation with respect to x, this becomes

$$\frac{\partial \eta}{\partial t} + \frac{3}{2}u\frac{\partial \eta}{\partial x} = 0.$$

What this tells us is that $\eta(x,t)$ is constant on trajectories $x(t)$ given by

$$\left.\frac{dx}{dt}\right|_{\eta=\text{const.}} = -\frac{\partial \eta/\partial t}{\partial \eta/\partial x} = \frac{3}{2}u, \tag{9.4.8}$$

i.e., signals propagate 50% faster than the current itself. In the river described in exercise 29, the current travels $1.7 m/s$. Suppose now that a rain event upstream causes a sudden increase in flow rate. Our model predicts that increase will propagate downstream at $2.6 m/s$.

Back to Table of Contents.

[2] In fact, measurements show that the drag coefficient decreases with increasing depth, approximately as depth$^{-1/3}$, so the Froude number is proportional to depth$^{1/6}$ (e.g., White 2003).

[3] Does it bother you that we discarded terms involving $\partial/\partial t$ and $\partial/\partial x$ in (9.4.5) but retain such terms in (9.4.7)? In (9.4.5), we neglected the terms involving $\partial/\partial t$ and $\partial/\partial x$ not on the grounds that these terms are zero (i.e., that nothing varies in x or t), but rather because *the remaining terms are much bigger*. In the mass equation (9.4.7), *both* terms involve partial derivatives, so there are no larger terms to dominate the balance. It is therefore permissible to retain the partial derivatives in (9.4.7).

Chapter 10

Postface

We have developed skills in mathematics and advanced scientific reasoning that allow us to take a set of assumptions (hypotheses) and develop from them a testable prediction in the form of a set of partial differential equations. The result of this is the Navier-Stokes equations and the accompanying mass equation, heat equation, etc.

To test these predictions is not easy, for one must not only solve the equations but measure a real flow with precision sufficient to tell whether the solution matches reality or not. We have played at this by extracting some extremely simple solutions for idealized model flows and comparing them with reality on scales that we can perceive easily without specialized equipment. In some cases, the match is good. When it is not, the culprit has most commonly been a failure to account for turbulence. This is not a failing of the Navier-Stokes equations but rather of the unrealistically simple flow geometries for which we are able to solve them.

Even in nature, where flows are invariably turbulent, we see things like waves, vortices, and hydraulic jumps, and they behave much as our simple idealizations predict. If predicted scales are seriously inaccurate, inserting a simple model of the turbulent energy cascade often gives realistic results.

So where do we go from here? The student of oceanography will go on to study the effects of density stratification and planetary rotation. The atmospheric physicist will need these and also the thermodynamics of water vapor. Plasmas found in the ionosphere and stellar atmospheres may be understood by adding Maxwell's equations for electromagnetism (Choudhuri 1996). In smaller systems like lakes, rivers and beaches, often of concern to civil and environmental engineers, density stratification is important but planetary rotation is less so. In all geophysical systems, turbulence must be accounted for to achieve a realistic level of understanding and a predictive capacity.

For further exploration, check out *Fluid Mechanics* (Kundu et al. 2016). It contains concise summaries of most of these advanced aspects of the discipline and many more (and I do mean many) not listed here.

As you walk through the world, you are surrounded by flow. You contain flow. You now carry with you a conceptual understanding developed over centuries by humans like you who also walked through, and were part of, this world of flow. Remember the words of Bruce Lee: *Be water, my friend.*

Bill Smyth

Back to Table of Contents.

Chapter 11

Bibliography

Aris, R., 1962: *Vectors, Tensors and the Basic Equations of Fluid Mechanics*. Cambridge Univ. Press.

Bronson, R. and G. Costa, 2009: *Matrix Methods: Applied Linear Algebra (3rd ed.)*. Academic Press, New York.

Choudhuri, A., 1996: *The Physics of Fluids and Plasmas, an Introduction for Astrophysicists*. Cambridge Univ. Press.

Curry, J. A. and P. J. Webster, 1998: *Thermodynamics of atmospheres and oceans*, volume 65. Academic Press.

Gill, A., 1982: *Atmosphere-Ocean Dynamics*. Academic Press, San Diego,, 662 pp.

Kundu, P., I. Cohen, and D. Dowling, 2016: *Fluid Mechanics (6th ed.)*. Academic Press, San Diego.

Li, C., A. Valle-Levinson, L. Atkinson, K. Wong, and K. Lwiza, 2004: Estimation of drag coefficient in James River Estuary using tidal velocity data from a vessel-towed ADCP. *J. Geophys. Res.*, **109**, doi:10.1029/2003JC001991, c03034.

Marion, J., 2013: *Classical dynamics of particles and systems*. Academic Press, New York.

Moffatt, H., S. Kida, and K. Ohkitani, 1994: Stretched vortices - the sinews of turbulence; large reynolds number asymptotics. *J. Fluid Mech.*, **259**, 241–264.

Smyth, W., 1999: Dissipation range geometry and scalar spectra in sheared, stratified turbulence. *J. Fluid Mech.*, **401**, 209–242.

Smyth, W. and S. Thorpe, 2012: Glider measurements of overturning in a kelvin-helmholtz billow train. *J. Mar. Res.*, **70**, 119–140.

Vallis, G., 2006: *Atmospheric and Oceanic Fluid Dynamics*. Cambridge Univ. Press.

White, F., 2003: *Fluid Mechanics*. McGraw-Hill, New York.

Appendices

Appendix A

Taylor series expansions

Taylor's theorem (which we won't prove here) gives us a way to take a complicated function $f(x)$ and approximate it by a simpler function $\tilde{f}(x)$. The price of this simplification is that $\tilde{f} \approx f$ only in a small region surrounding some point $x = x_0$ (figure A.1).

The formula is:

$$\tilde{f}(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{1}{2}f''(x_0)(x-x_0)^2 + \frac{1}{6}f'''(x_0)(x-x_0)^3 + \ldots \tag{A.0.1}$$

The sequence goes on forever, but we typically only use the first few terms. To apply (1), we choose a point x_0 where we need the approximation to be accurate. We then compute the derivatives at that point, $f'(x_0), f''(x_0), f'''(x_0)$, etc., for as far as we want to take it. For accuracy, use a lot of terms; for simplicity, use only a few.

For example, if

$$f(x) = (1-x)^{-1}$$

and we choose $x_0 = 0$, then the Taylor series is just the well-known expression

$$\tilde{f}(x) = 1 + x + x^2 + x^3 + \cdots .$$

Figure A.2 shows the expansion with successively larger numbers of terms retained. Near x_0, good accuracy can be achieved with only a few terms. The further you get from x_0, the more terms must be retained for a given level of accuracy.

The real benefit of Taylor series is evident when working with more complicated functions. For example, $f(x) = \sin(\tan(x))$ can be approximated by $\tilde{f} = x$ for x close to zero.

Note that the first - order terms in (A.0.1):

$$\tilde{f}(x) = f(x_0) + f'(x_0)(x-x_0),$$

give a valid approximation of $f(x)$ in the limit $x \to x_0$, and can be rearranged to form the familiar definition of the first derivative:

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

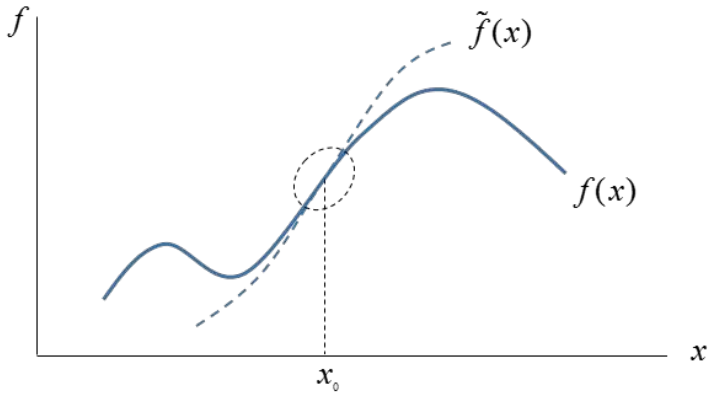


Figure A.1: Definition sketch for Taylor series expansions. The original function $f(x)$ is approximated by $\tilde{f}(x)$. The approximation is accurate near the expansion point x_0.

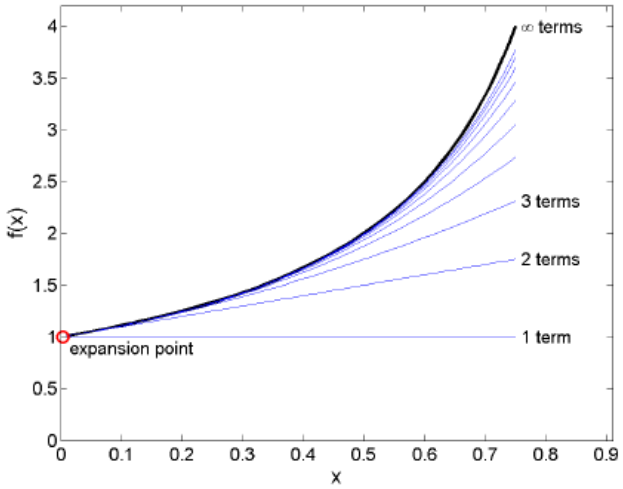


Figure A.2: Taylor series expansions of the function $f(x) = 1/(1-x)$ about $x = 0$. Accuracy increases as more terms are retained. The thick curve is the exact function.

The Taylor series expansion technique can be generalized for use with multivariate functions. Suppose that $f = f(\vec{x})$, where $\vec{x} = (x, y)$. Then

$$\tilde{f}(\vec{x}) = f(\vec{x}_0) + f_x(\vec{x}_0)\delta x + f_y(\vec{x}_0)\delta y + \frac{1}{2}f_{xx}(\vec{x}_0)\delta x^2 + f_{xy}(\vec{x}_0)\delta x\delta y + \frac{1}{2}f_{yy}(\vec{x}_0)\delta y^2 + \cdots, \tag{A.0.2}$$

where $\delta x = x - x_0, \delta y = y - y_0$ and subscripts denote partial derivatives. Note that the first - order terms in (A.0.2) can be written using the directional derivative:

$$f(\vec{x}) = f(\vec{x}_0) + \vec{\nabla} f(\vec{x}_0) \cdot \delta\vec{x}.$$

You'll notice that $\tilde{f}$ has been replaced by f; this is valid in the limit $\vec{x} \to \vec{x}_0$, or $\delta\vec{x} \to 0$.

Back to Table of Contents.

Appendix B

Torque and the moment of inertia

In this section we take a brief excursion into solid-body mechanics, specifically rotational motion. This will give us an example to use in the next section when we define a tensor, and also a simple result that we'll need later to understand forces acting within a fluid.

B.1 Torque

Newton's second law $\vec{F} = m\vec{a}$ has a rotational analogue. When a force $\vec{F}$ is exerted at a location $\vec{r}$ measured from some axis of rotation (e.g., the bolt in figure B.1), then the cross product $\vec{r} \times \vec{F}$ is called the *torque*, $\vec{T}$. The cross product is defined in (2.1.1), and is derived in detail in section D.3.1. For now, it is a vector perpendicular to both $\vec{F}$ and $\vec{r}$, with direction given by the right-hand rule. The magnitude is $|\vec{r} \times \vec{F}| = |\vec{r}||\vec{F}||\sin\phi|$, where ϕ is the angle between $\vec{r}$ and $\vec{F}$.

B.2 The moment of inertia tensor

The angle θ increases in time (if you push hard enough[1]) in accordance with

$$\vec{T} = \underset{\sim}{I}\vec{\alpha}, \tag{B.2.1}$$

in which $\vec{\alpha}$ is the angular acceleration and $\underset{\sim}{I}$ is a matrix called the *moment of inertia*. For the simple case shown in figure B.1, $\underset{\sim}{I}$ is proportional to the identity matrix $\underset{\sim}{\delta}$, $\vec{\alpha}$ is parallel to the axis of rotation (the bolt), and its magnitude $|\vec{\alpha}|$ is $d^2\theta/dt^2$.

The general definition of the moment of inertia matrix is

$$I_{ij} = \int_V dV \rho(\vec{x})(x_k x_k \delta_{ij} - x_i x_j), \tag{B.2.2}$$

where $\rho(\vec{x})$ is the density (mass per unit volume). Details can be found in most classical mechanics texts, e.g., Marion (2013).

[1] In the case shown here, $\vec{F}$ is really the sum of the force exerted by the person and the opposing force exerted by friction, and similarly for $\vec{T}$.

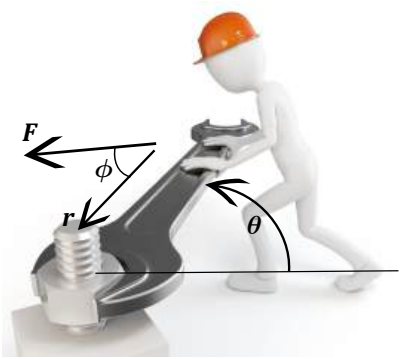


Figure B.1: Definition sketch for Newton's second law in rotational form. A force F is exerted at a distance r from the axis of rotation, changing the angle θ.

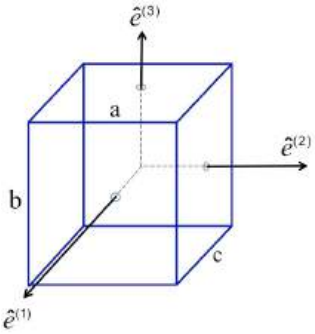


Figure B.2: Definition sketch for the moment of inertia matrix. The example shown is a rectangular prism with sides a, b, and c.

Example: The particular case illustrated in figure B.2 is the rotation of a rectangular prism, with uniform density and edge dimensions a, b and c, about the $\hat{e}^{(1)}$ axis. In this case both torque and angular acceleration are parallel to $\hat{e}^{(1)}$, and the only nonzero component of $\underset{\sim}{I}$ is I_{11}, computed as follows:

$$\begin{aligned} I_{11} &= \int_V dV \rho\,(x_2^2 + x_3^2) \\ &= \rho \int_{-a/2}^{a/2} dx_1 \int_{-b/2}^{b/2} dx_2 \int_{-c/2}^{c/2} dx_3\ (x_2^2 + x_3^2) \\ &= \rho\ \frac{abc(b^2+c^2)}{12}. \end{aligned}$$

For the simple case of a cube with $a = b = c = \Delta$,

$$I_{11} = \rho \frac{\Delta^5}{6}. \tag{B.2.3}$$

Is the moment of inertia matrix $\underset{\sim}{I}$ a tensor? We would expect so, since it connects two physically real vectors via (B.2.1). We can also establish this directly from (B.2.2), the general formula for $\underset{\sim}{I}$. Like any other integral, $\underset{\sim}{I}$ can be written as the limit of a sum:

$$I_{ij} = \sum \Delta V \rho\,(x_k x_k \delta_{ij} - x_i x_j),$$

where each term in the sum is evaluated at the center of a volume element ΔV. Now ΔV and ρ are scalars, and so is the dot product $x_k x_k$ (section 3.2). Moreover, we know that both δ_{ij} and the dyad $x_i x_j$ transform according to (3.3.5). Each term in the sum is therefore a tensor, and so then is the sum itself. Taking the limit as $\Delta V \to 0$, we conclude that $\underset{\sim}{I}$ transforms according to (3.3.5). We therefore refer to $\underset{\sim}{I}$ as the moment of inertia tensor.

Back to Table of Contents.

Appendix C

Isotropic tensors

Isotropic tensors play a fundamental role in tensor algebra as well as in fluid mechanics. We can easily test whether or not a given tensor is isotropic, but is there a way to identify *all* isotropic tensors?

Because an isotropic tensor is invariant under under *every* rotation, it must in particular be invariant under very slight rotations. In the limit of an infinitesimal rotation, the rotation matrix is just the identity plus a small added matrix. This allows us to simplify the algebra needed to derive requirements for isotropy.

C.1 Infinitesimal rotations

Any rotation matrix can be written as the identity plus something:

$$\underset{\sim}{C} = \underset{\sim}{\delta} + \underset{\sim}{r}$$

The orthogonality requirement $\underset{\sim}{C}^T\underset{\sim}{C} = \underset{\sim}{\delta}$ can then be expressed as:

$$\begin{aligned}
\underset{\sim}{C}^T\underset{\sim}{C} &= (\underset{\sim}{\delta}^T + \underset{\sim}{r}^T)(\underset{\sim}{\delta} + \underset{\sim}{r}) \\
&= \underset{\sim}{\delta}^T\underset{\sim}{\delta} + \underset{\sim}{\delta}^T\underset{\sim}{r} + \underset{\sim}{\delta}\underset{\sim}{r}^T + \underset{\sim}{r}^T\underset{\sim}{r} \\
&= \underset{\sim}{\delta} + \underset{\sim}{\delta}\underset{\sim}{r} + \underset{\sim}{\delta}\underset{\sim}{r}^T + \underset{\sim}{r}^T\underset{\sim}{r} = \underset{\sim}{\delta},
\end{aligned}$$

(using the identities $\underset{\sim}{\delta}^T = \underset{\sim}{\delta}$ and $\underset{\sim}{\delta}\underset{\sim}{\delta} = \underset{\sim}{\delta}$), and therefore

$$\underset{\sim}{r} + \underset{\sim}{r}^T + \underset{\sim}{r}^T\underset{\sim}{r} = 0. \tag{C.1.1}$$

Now suppose that the rotation is through a very small angle[1]. In this case, all components of $\underset{\sim}{r}$ are $\ll 1$, and the third term on the left-hand side of (C.1.1) is therefore negligible, leaving us with

$$\underset{\sim}{r} + \underset{\sim}{r}^T = 0. \tag{C.1.2}$$

So, for an infinitesimal rotation, the rotation matrix equals the identity plus an *antisymmetric* matrix whose elements are $\ll 1$.

C.2 1st-order isotropic tensors

A 1st-order tensor is a vector. If the vector $\vec{v}$ is isotropic, then under an infinitesimal rotation

$$v_i' = v_j C_{ji} = v_j(\delta_{ji} + r_{ji}) = v_i + v_j r_{ji} = v_i.$$

[1] More precisely, we take the limit as the angle goes to zero.

Therefore,

$$\boxed{v_j r_{ji} = 0.}$$

This represents three algebraic equations, one for each value of i:

$$\begin{aligned} v_1 r_{11} + v_2 r_{21} + v_3 r_{31} &= 0 \\ v_1 r_{12} + v_2 r_{22} + v_3 r_{32} &= 0 \\ v_1 r_{13} + v_2 r_{23} + v_3 r_{33} &= 0. \end{aligned} \tag{C.2.1}$$

Because $\underset{\sim}{r}$ is antisymmetric, $r_{11} = r_{22} = r_{33} = 0$, removing one term from each equation.

Now consider the first equation of (C.2.1):

$$v_2 r_{21} + v_3 r_{31} = 0. \tag{C.2.2}$$

Here is a crucial point: if $\vec{v}$ is isotropic, then (C.2.1) must be true *for all antisymmetric matrices* $\underset{\sim}{r}$, i.e., regardless of the values of r_{21} and r_{31}. The *only* way this can be true is if $v_2 = 0$ and $v_3 = 0$. The same considerations applied to the second equation of (C.2.1) tell us that v_1 must also be zero, hence the only isotropic 1st-order tensor is the trivial case

$$\vec{v} = 0.$$

C.3 2nd-order isotropic tensors

Let a 2nd-order isotropic tensor $\underset{\sim}{A}$ be subjected to an infinitesimal rotation $\underset{\sim}{\delta} + \underset{\sim}{r}$. Then

$$\begin{aligned} A'_{ij} = A_{kl} C_{ki} C_{lj} &= A_{kl}(\delta_{ki} + r_{ki})(\delta_{lj} + r_{lj}) = A_{kl}(\delta_{ki}\delta_{lj} + \delta_{ki} r_{lj} + r_{ki}\delta_{lj} + r_{ki} r_{lj}) \\ &= A_{ij} + A_{il} r_{lj} + A_{kj} r_{ki} = A_{ij} \end{aligned}$$

where we have neglected the product of the infinitesimal matrices $\underset{\sim}{r}$. This leaves us with

$$\boxed{A_{ik} r_{kj} + A_{kj} r_{ki} = 0,} \tag{C.3.1}$$

which must be true for all antisymmetric matrices $\underset{\sim}{r}$ whose elements are $\ll 1$. (Note that we have renamed the dummy index l as k for tidiness. There is no potential for confusion because the pairs of ks are in separate terms.)

(C.3.1) represents nine equations, one for each combination of the free indices i and j. It will be enough to consider three of these.

Case 1: $i = 1,\ j = 1$

$$A_{11} r_{11} + A_{12} r_{21} + A_{13} r_{31} + A_{11} r_{11} + A_{21} r_{21} + A_{31} r_{31} = 0.$$

Remembering that $r_{11} = 0$, we can write this as

$$(A_{12} + A_{21}) r_{21} + (A_{13} + A_{31}) r_{31} = 0.$$

Because this must be true for all values of r_{21} and r_{31}, the coefficients of those quantities must vanish separately:

$$A_{12} + A_{21} = 0\,; \quad A_{13} + A_{31} = 0. \tag{C.3.2}$$

Case 2: $i = 1,\ j = 2$

$$A_{11} r_{12} + A_{12} r_{22} + A_{13} r_{32} + A_{12} r_{11} + A_{22} r_{21} + A_{32} r_{31} = 0.$$

Because $r_{11} = r_{22} = 0$ and $r_{21} = -r_{12}$, we can write this as

$$(A_{11} - A_{22}) r_{12} + A_{13} r_{32} + A_{32} r_{31} = 0.$$

Because this must be true for all $\underset{\sim}{r}$,

$$A_{11} = A_{22}\,; \quad A_{13} = 0\,; \quad A_{32} = 0. \tag{C.3.3}$$

Case 3: $i=1,\ j=3$

The same reasoning leads to

$$A_{11}=A_{33}\ ;\quad A_{12}=0\ ;\quad A_{23}=0. \tag{C.3.4}$$

Combining (C.3.2), (C.3.3) and (C.3.4), we have

$$A_{11}=A_{22}=A_{33}\ ;\quad A_{12}=A_{21}=A_{32}=A_{23}=A_{13}=A_{31}=0. \tag{C.3.5}$$

If A_{11} has the value a, then $\underset{\sim}{A}$ must therefore be proportional to the identity matrix:

$$\underset{\sim}{A}=\begin{pmatrix} a & 0 & 0\\ 0 & a & 0\\ 0 & 0 & a\end{pmatrix}=a\underset{\sim}{\delta}.$$

We conclude that the only isotropic 2nd order tensors are those that are proportional to the identity.

C.4 3rd-order isotropic tensors

Let a 3rd-order isotropic tensor $\underset{\sim}{A}$ be subjected to an infinitesimal rotation $\underset{\sim}{\delta}+\underset{\sim}{r}$. Then

$$A'_{ijk}=A_{lmn}C_{li}C_{mj}C_{nk}=A_{ijk}.$$

Reasoning as in the last section (see derivation of (C.3.1), we obtain

$$\boxed{A_{ilk}r_{lj}+A_{ljk}r_{li}+A_{ijl}r_{lk}=0.} \tag{C.4.1}$$

Again, we have changed all dummy indices to l for tidiness. (And again, this is safe only because they appear in separate terms!)

(C.4.1) represents 27 equations, one for each combination of the index values 1, 2 and 3. It will simplify things if we classify those 27 combinations as follows:

- all values equal (111, 222 and 333)
- two values equal and one different (e.g., 223)
- all values different (123, 231, 312, 213, 321 and 132).

Case 1: $i=1,\ j=1,\ k=2$

With $\{i,j,k\}=\{1,1,2\}$, (C.4.1) becomes

$$\begin{aligned} A_{112}r_{11} &+ A_{122}r_{21} + A_{132}r_{31}\\ +A_{112}r_{11} &+ A_{212}r_{21} + A_{312}r_{31}\\ +A_{111}r_{12} &+ A_{112}r_{22} + A_{113}r_{32}=0.\end{aligned}$$

Using the antisymmetry of $\underset{\sim}{r}$, we can write this as

$$(A_{122}+A_{212}-A_{111})r_{21}+(A_{132}+A_{312})r_{31}+A_{113}r_{32}=0.$$

Since the coefficients must vanish separately, we have three equations:

$$A_{122}+A_{212} = A_{111} \tag{C.4.2}$$
$$A_{132} = -A_{312} \tag{C.4.3}$$
$$A_{113} = 0. \tag{C.4.4}$$

The third equation tells us that an element with two equal indices is zero. Let's *guess* that this is true for all such elements. If we've guessed right, then the first equation tells us that an element with all three indices equal is zero. Finally, in the second equation, interchanging two indices changes the sign.

Are these patterns generally true? Let's try another case to check.

Case 2: $i=2,\ j=3,\ k=2$

Repeating the previous case with $\{i,j,k\}$ changed to $\{2,3,2\}$, we have

$$\begin{aligned} A_{212}r_{13} &+ A_{222}r_{23} + A_{232}r_{33} \\ +A_{132}r_{12} &+ A_{232}r_{22} + A_{332}r_{32} \\ +A_{231}r_{12} &+ A_{232}r_{22} + A_{233}r_{32} = 0 \end{aligned}$$

or

$$A_{332}+A_{233} = A_{222} \tag{C.4.5}$$
$$A_{132} = -A_{231} \tag{C.4.6}$$
$$A_{212} = 0. \tag{C.4.7}$$

The pattern is the same as in case 1. The third equation tells us that an element with two equal indices is zero. If this is generally true, then the first equation tells us that an element with three equal indices is zero. Finally, interchanging two indices changes the sign.

You can check as many cases as you like; the results are always the same. Note that the only "rule" that matters here is the second one: *interchanging two indices changes the sign*. The other two rules follow from this one, because they both involve elements with two or more equal indices. Interchanging two equal indices makes no difference, but it also changes the sign. That only works if the value is zero.

We conclude that a 3rd-order tensor can be isotropic only if it is completely antisymmetric, i.e., interchanging any two indices changes the sign. In section D.1, we show that the only completely antisymmetric 3rd-order tensor is, to within a multiplicative constant, the Levi-Civita alternating tensor $\underset{\sim}{\varepsilon}$. The most general isotropic 3rd-order tensor is therefore

$$\underset{\sim}{A} = a\underset{\sim}{\varepsilon}$$

where a is any scalar.

C.5 4th-order isotropic tensors

To identify the isotropic 4th-order tensors, one uses the same logic as in the 3rd-order case (section C.4) but, as you might guess, there is considerably more of it. The details may be found, for example, in Aris (1962). Here we will just quote the result. The most general isotropic 4th-order tensor is a bilinear combination of 2nd-order isotropic tensors:

$$A_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}, \tag{C.5.1}$$

where λ, μ and γ are scalars.

Back to Table of Contents.

Appendix D

The Levi-Cevita alternating tensor

The importance of the alternating tensor follows from the fact that it is the only isotropic, 3rd-order tensor[1]. Appendix C shows that a 3rd-order tensor can be isotropic only if it is completely antisymmetric, meaning it changes sign when any two of its three indices are interchanged. In this appendix, we'll see how this property of complete antisymmetry defines the Levi-Civita tensor. We'll then see how some fundamental operations in algebra follow from it.

D.1 A completely antisymmetric, three-dimensional array.

We'll start by recalling a few important aspects of antisymmetry in ordinary matrices. The matrix shown in figure D.1 illustrates the idea of antisymmetry. If you flip the matrix about its main diagonal, you get the transpose. Now notice that every element in the transpose is the negative of the corresponding element in the original matrix. So $\underset{\sim}{A}^T = -\underset{\sim}{A}$, or in index notation, $A_{ji} = -A_{ij}$.

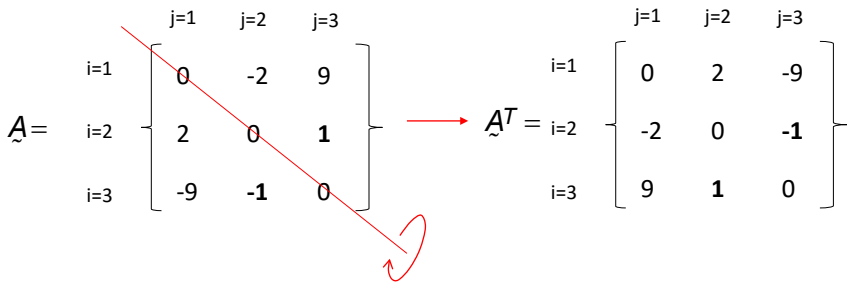


Figure D.1: (a) Example of an antisymmetric two-dimensional matrix. Transposing the matrix changes the sign of every element.

A property shared by *every* antisymmetric matrix is that the elements on the main diagonal are all zero. That's obvious if you think about it: transposing the matrix doesn't change those elements, so if transposing the matrix changes the sign, then those elements can only be zero (because only zero is its own negative). So there are an infinite number of antisymmetric matrices, but $A_{11} = A_{22} = A_{33} = 0$ in all of them.

Now imagine that there exists a three-dimensional array that has the property of *complete asymmetry*. We'll call it $\underset{\sim}{\varepsilon}$. Two additional properties of $\underset{\sim}{\varepsilon}$ follow from complete antisymmetry.

1. First, *any element with two equal indices must be zero*. The logic is the same as in the case of the two-dimensional matrix. Interchanging two indices changes the sign, but if the two indices are identical, the interchange makes no difference. So the value of the element equals its own negative and must therefore be zero.

2. A second property that results from complete antisymmetry is that *cyclic permutations*[2] *of the indices have no effect*. For example, starting with ε_{ijk} (where i, j and k are any combination of 1,2 and 3), move the third index back to the first position and shift the

[1] up to a multiplicative constant.

[2] Recall that cyclic permutations are accomplished by moving the final value to the beginning, or the first value to the end, as shown in figure 3.9. Interchanging any two values changes a cyclic to a non-cyclic permutation, or vice versa.

other two indices one place to the right. The result is ε_{kij}. Now, is the element ε_{kij} related to the original element, ε_{ijk}? Yes. We can tell this because we can recover the original ordering of the indices by making two successive interchanges, each of which changes the sign:

$$\varepsilon_{kij} \;=\; -\varepsilon_{ikj} \;=\; \varepsilon_{ijk}$$

The result is ε_{ijk}, just what we started with. This shows that ε_{ijk} is invariant under any cyclic permutation of its indices.

At this stage we know some things that must be true about this hypothetical array *if it exists*, but does it? If so what does it look like? We'll now deduce the specific form of $\underset{\sim}{\varepsilon}$ in three steps.

- The integers 1, 2 and 3 have 27 combinations, only six of which are non-repeating: 123, 312, 231, 213, 321, and 132. The corresponding six elements of $\underset{\sim}{\varepsilon}$ are the only ones that can be nonzero (by property 1 above).

- The first three combinations, 123, 312 and 231, are cyclic permutations, and so are the remaining three. Because cyclic permutations make no difference (property 2 above),

$$\varepsilon_{123} \;=\; \varepsilon_{312} \;=\; \varepsilon_{231}, \tag{D.1.1}$$

and

$$\varepsilon_{213} \;=\; \varepsilon_{321} \;=\; \varepsilon_{132}. \tag{D.1.2}$$

So there are only two different nonzero values that elements of $\underset{\sim}{\varepsilon}$ can have.

- Finally, note that these two values must be additive inverses. For example, consider the first member of each triplet: 123 and 213. These are related by an exchange of the first and second indices, and therefore

$$\varepsilon_{213} \;=\; -\varepsilon_{123}. \tag{D.1.3}$$

So, if we choose a value for ε_{123}, we can deduce the values of all the other elements.

To obtain the Levi-Civita tensor, we make the simplest choice $\varepsilon_{123} = 1$. We then have

$$\varepsilon_{123} \;=\; \varepsilon_{312} \;=\; \varepsilon_{231} \;=\; 1, \tag{D.1.4}$$
$$\varepsilon_{213} \;=\; \varepsilon_{321} \;=\; \varepsilon_{132} \;=\; -1, \tag{D.1.5}$$

and all other elements are zero. In summary:

$$\boxed{\varepsilon_{ijk} \;=\; \begin{cases} 1, & \text{if } ijk = 123, 312, 231, \\ -1, & \text{if } ijk = 213, 321, 132, \\ 0, & \text{otherwise.} \end{cases}} \tag{D.1.6}$$

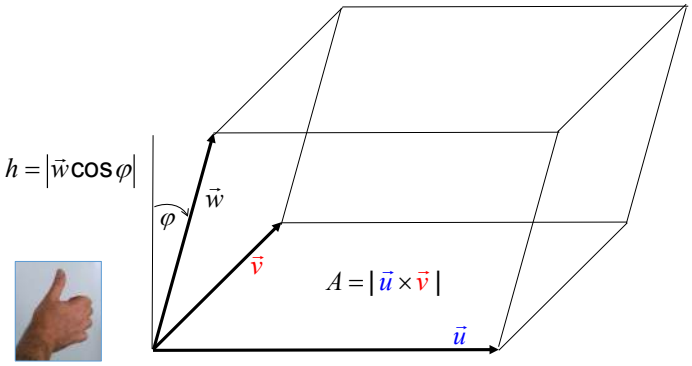
Note that *every* completely antisymmetric, three-dimensional array must be proportional to $\underset{\sim}{\varepsilon}$, i.e., equal to $\underset{\sim}{\varepsilon}$ times some scalar. To see this, recall that we chose $\varepsilon_{123} = 1$ arbitrarily. What if we had chosen $\varepsilon_{123} = 2$? The resulting array would be exactly the same except multiplied by 2.

D.2 The $\varepsilon - \delta$ relation.

As was stated without proof in section 3.3.7, the alternating tensor is related to the 2nd-order identity tensor by

$$\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}. \tag{D.2.1}$$

The easiest way to convince yourself of this is to try a few tests. First, set $i = j$ and verify that the right-hand side is zero, as it should be. Then try interchanging i and j and check that the right-hand side changes sign, as it should. The same tests work with l and m. To remember (D.2.1), note that the first δ on the right-hand side has subscripts i and l; these are the first free indices of the two ε's on the left-hand side. After this, the remaining pairs of indices fall into place naturally.



Right handed triple: $V = (\vec{u}\times\vec{v})\bullet\vec{w} > 0$

Figure D.2: Relation of the triple product to the volume of the parallelepiped bounded by three vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$.

D.3 Applications of ε

In this section we'll look at three fundamental algebraic operations that are based on $\underset{\sim}{\varepsilon}$: the cross product, the triple product and the determinant. We'll see how each can be interpreted geometrically in terms of areas and volumes, and derive their most useful properties.

D.3.1 The cross product

One function of a 3rd-order tensor is to define a mathematical relationship between three vectors, e.g., a way to multiply two vectors to get a third vector. To be generally useful, such an operation should work the same in every reference frame, i.e., the tensor should be isotropic. Because $\underset{\sim}{\varepsilon}$ is the only isotropic 3rd-order tensor, there is only one such operation:

$$z_k = \varepsilon_{ijk} u_i v_j. \tag{D.3.1}$$

This is the kth component of the familiar cross (or vector) product: $\vec{z} = \vec{u}\times\vec{v}$ whose properties were discussed in section 3.3.7.

D.3.2 The triple product

The triple product of three vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ is a *trilinear* scalar product given by

$$T = (\vec{u}\times\vec{v})\cdot\vec{w} = \varepsilon_{ijk} u_i v_j w_k. \tag{D.3.2}$$

Because $\underset{\sim}{\varepsilon}$ is the only isotropic 3rd-order tensor, the triple product is the only trilinear scalar product that is computed the same way in all reference frames. We now derive some algebraic and geometrical properties of the triple product.

1. The triple product is equal to the volume V of the parallelepiped enclosed by the three vectors (figure D.2), give or take a minus sign. The plane that encloses $\vec{u}$ and $\vec{v}$ divides space into two half-spaces, and the cross product $\vec{u}\times\vec{v}$ extends into one or the other of those half-spaces, determined by the right-hand rule. If $\vec{w}$ points into the same half-space as the cross product (i.e., the angle ϕ, defined in figure D.2, is acute), $\vec{u}$, $\vec{v}$ and $\vec{w}$ are called a right-handed triple, a distinction that will be of great use to us. In this case, the triple product T is positive and is equal to the volume V. Conversely, if $\vec{u}$, $\vec{v}$ and $\vec{w}$ form a left-handed triple (ϕ obtuse), $T = -V$.

2. The triple product is unchanged by a cyclic permutation of $\vec{u}$, $\vec{v}$ and $\vec{w}$, i.e.

$$(\vec{u}\times\vec{v})\cdot\vec{w} \;=\; (\vec{w}\times\vec{u})\cdot\vec{v} \;=\; (\vec{v}\times\vec{w})\cdot\vec{u}.$$

   This follows from (D.3.2) and the invariance of $\underset{\sim}{\varepsilon}$ to cyclic permutations. Geometrically, a cyclic permutation corresponds to a rotation of the parallelepiped, which obviously leaves its volume unchanged.

3. Interchanging any two of $\vec{u}$, $\vec{v}$ and $\vec{w}$ changes the sign of the triple product. This is seen easily as a consequence of (D.3.2) and the complete antisymmetry of $\underset{\sim}{\varepsilon}$. Geometrically, it means that the parity of the set $\{\vec{u},\vec{v},\vec{w}\}$ changes from right-handed to left-handed, and hence the triple product changes from $+V$ to $-V$.

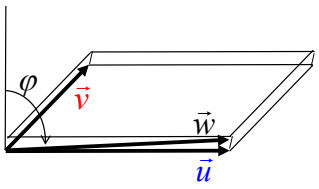


Figure D.3: As $\vec{w}$ turns into the plane of $\vec{u}$, $\vec{v}$, the volume of the parallelepiped shrinks to zero.

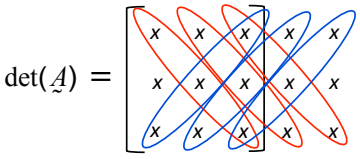


Figure D.4: Elementary recipe for computing the determinant of a 3x3 matrix for comparison with (D.3.4). Combinations marked in red (blue) are added (subtracted).

4. Now imagine that the parallelepiped is squashed so that $\vec{w}$ lies in the plane of $\vec{u}$ and $\vec{v}$ (figure D.3). Clearly the volume of the parallelepiped goes to zero; hence, the triple product must also go to zero. A special case is if $\vec{w}$ is equal to either $\vec{u}$ or $\vec{v}$. In fact, if any two of $\vec{u}$, $\vec{v}$ and $\vec{w}$ are equal, the triple product is zero.

Properties 2-4 above can be expressed succinctly using $\underset{\sim}{\varepsilon}$. Suppose we relabel our three vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ as $\vec{u}^{(i)}$, $\vec{u}^{(j)}$ and $\vec{u}^{(k)}$, where i, j and k are any combination of 1, 2 and 3 (including those with repeated values). Now let T stand for the triple product where the indices i, j and k actually are 1, 2 and 3, i.e., $T = (\vec{u}^{(1)} \times \vec{u}^{(2)}) \cdot \vec{u}^{(3)}$. Next, let the indices ijk be *any* combination of 1, 2 and 3. The aforementioned properties of the triple product (namely that cyclic permutations change nothing, interchanging changes the sign and setting two vectors equal makes the triple product zero) are equivalent to

$$(\vec{u}^{(i)} \times \vec{u}^{(j)}) \cdot \vec{u}^{(k)} = T\varepsilon_{ijk}. \tag{D.3.3}$$

As a special case, suppose that the $\vec{u}^{(i)}$ are the basis vectors of a right-handed Cartesian coordinate system $\hat{e}^{(i)}$, where $i = 1,2,3$. In that case the parallelepiped is just a unit cube and

$$\boxed{(\hat{e}^{(i)} \times \hat{e}^{(j)}) \cdot \hat{e}^{(k)} = \varepsilon_{ijk}.}$$

D.3.3 The determinant

Definition and elementary properties

The determinant of a 3x3 matrix $\underset{\sim}{A}$ can be written as

$$\det(\underset{\sim}{A}) = \varepsilon_{ijk} A_{i1} A_{j2} A_{k3}, \tag{D.3.4}$$

i.e., if we regard the columns as vectors, the determinant is their triple product.[3] As taught in elementary algebra classes, the determinant is calculated as a linear combination of multiples of three elements, one from each column (figure D.4). Three of those combinations are added, three are subtracted, and the rest have a coefficient of zero. Referring back to (D.1.6), you can check that the coefficients of that linear combination are just the corresponding elements of $\underset{\sim}{\varepsilon}$.

Referring again to figure D.4, turn the book sideways and verify that the determinant can just as easily be written as the triple product of rows:

$$\det(\underset{\sim}{A}) = \varepsilon_{ijk} A_{1i} A_{2j} A_{3k}. \tag{D.3.5}$$

The expressions (D.3.4) and (D.3.5) lead immediately to several elementary properties of the determinant:

1. Transposing the matrix does not change the determinant:

$$\det(\underset{\sim}{A}^T) = \det(\underset{\sim}{A}).$$

[3] This definition does not require that the columns actually transform as vectors; in general they do not.

2. Interchanging two columns of a matrix changes the sign of the determinant. This corresponds to the fact that $\underset{\sim}{\varepsilon}$ is antisymmetric with respect to every pair of indices.
3. Remembering that transposing doesn't change the determinant, we can make the same statement about rows: interchanging any two rows changes the sign of the determinant.
4. If any two columns (or two rows) are the same, the determinant is zero. This is because elements of $\underset{\sim}{\varepsilon}$ are zero if two of their indices are the same.
5. Finally, because cyclic permutations of the indices of $\underset{\sim}{\varepsilon}$ make no difference, cyclic permutations of either the rows or the columns of a matrix leave the determinant unchanged.
6. As an alternative to the pattern shown in figure D.4, (D.3.4) can also be arranged into a formula commonly called the method of cofactors. One chooses a row or column to expand along, then forms a linear combination of 2x2 subdeterminants. Here is the formula for expanding along the top row:

$$\det(\underset{\sim}{A}) = A_{11}\det\begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix} - A_{12}\det\begin{pmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{pmatrix} + A_{13}\det\begin{pmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{pmatrix}. \tag{D.3.6}$$

The other expansions work similarly. The minus sign applied to the middle term results from the antisymmetry of $\underset{\sim}{\varepsilon}$.

Easy mnemonics for the triple and cross products.

The identification of the triple product with $\underset{\sim}{\varepsilon}$ provides an easy way to calculate it. Just arrange the three vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ into a matrix:

$$(\vec{u}\times\vec{v})\cdot\vec{w} = \varepsilon_{ijk}u_iv_jw_k = \det\begin{pmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{pmatrix}, \tag{D.3.7}$$

and calculate the determinant using either the elementary formula shown in figure D.4 or the method of cofactors (e.g., D.3.6).

The cross product of two vectors can be computed by arranging the components of the vectors, together with the three basis vectors, to form a matrix.

$$\vec{u}\times\vec{v} = \varepsilon_{ijk}u_iv_j\vec{e}^{(k)} = \det\begin{pmatrix} u_1 & v_1 & \hat{e}^{(1)} \\ u_2 & v_2 & \hat{e}^{(2)} \\ u_3 & v_3 & \hat{e}^{(3)} \end{pmatrix}, \quad \text{or} \quad \det\begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ \hat{e}^{(1)} & \hat{e}^{(2)} & \hat{e}^{(3)} \end{pmatrix}. \tag{D.3.8}$$

This construct makes no sense as a matrix, because three of its elements are vectors. That's ok; it's just a device that allows us to compute the cross product using a pattern we've already memorized. We now calculate the determinant using the method of cofactors, expanding along the row or column that contains the unit vectors:

$$\vec{u}\times\vec{v} = \hat{e}^{(1)}(u_2v_3-u_3v_2) - \hat{e}^{(2)}(u_1v_3-u_3v_1) + \hat{e}^{(3)}(u_1v_2-u_2v_1)$$

(cf. 4.1.4).

Geometrical meaning of the determinant

Consider the volume V of the parallelepiped formed by three arbitrary vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ (figure D.5). Assume for now that $\vec{u}$, $\vec{v}$ and $\vec{w}$ form a right-handed triple, so that their triple product is positive. In that case, the triple product is equal to V. Now suppose that the three vectors are transformed by the same matrix $\underset{\sim}{A}$:

$$u'_i = A_{ij}u_j\ ; \quad v'_i = A_{ij}v_j\ ; \quad w'_i = A_{ij}w_j.$$

The volume of the new parallelepiped is

$$\begin{aligned} V' &= \varepsilon_{ijk}\,u'_iv'_jw'_k \\ &= \varepsilon_{ijk}\,A_{il}u_l\,A_{jm}v_m\,A_{kn}w_n \\ &= \varepsilon_{ijk}\,A_{il}A_{jm}A_{kn}\,u_lv_mw_n \qquad \text{(reordering)} \\ &= B_{lmn}\,u_lv_mw_n \end{aligned}$$

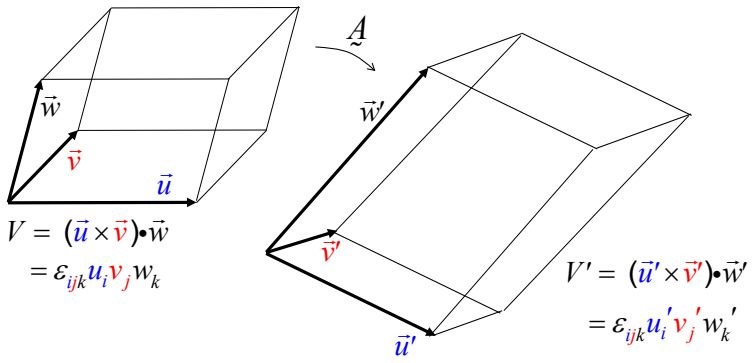


Figure D.5: Transformation of the parallelepiped enclosed by three arbitrary vectors when the vectors are transformed by the arbitrary matrix A.

where we have defined a new three-dimensional array $B_{lmn} = \varepsilon_{ijk}\, A_{il}A_{jm}A_{kn}$.

Now we explore the meaning of $\underset{\sim}{B}$. B_{lmn} is the triple product of columns l, m and n, where l, m and n can be any integers 1,2 or 3. In the special case $lmn = 123$, B_{123} is just the determinant of $\underset{\sim}{A}$ (cf. D.3.4). But what is $\underset{\sim}{B}$ for arbitrary lmn? This situation is familiar; the triple product of three arbitrary vectors $\vec{u}^{(i)}$, $\vec{u}^{(j)}$ and $\vec{u}^{(k)}$ is:

$$(\vec{u}^{(i)} \times \vec{u}^{(j)}) \cdot \vec{u}^{(k)} = T\varepsilon_{ijk},$$

where $T = (\vec{u}^{(1)} \times \vec{u}^{(2)}) \cdot \vec{u}^{(3)}$ (cf. D.3.3). By the same reasoning,

$$B_{lmn} = \det(\underset{\sim}{A})\varepsilon_{lmn}.$$

Assembling these results, we find that V' is $|\underset{\sim}{A}|$ times the triple product of the original vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$. Equivalently,

$$\boxed{V' = \det(\underset{\sim}{A})V.} \tag{D.3.9}$$

This is our geometrical interpretation of $\det(\underset{\sim}{A})$: it is the factor by which the volume of the parallelepiped changes after transformation by $\underset{\sim}{A}$. This interpretation also works for left-handed triples. *But*, if $\det(\underset{\sim}{A}) < 0$, a right-handed triple will be converted into a left-handed triple, and vice versa.

Incidentally, this tells us something interesting about matrix transformations in general. You can transform *any* three vectors by the same matrix, and the parallelepiped they form will always expand by the same factor! Therefore, you can think of a matrix transformation as expanding *all of space* by a uniform factor.

Further properties of the determinant

Having identified the determinant as an expansion factor, we can now understand more of its properties.

7. The determinant of a 2nd-order tensor is a scalar, clearly, because the expansion factor is invariant to rotations. If you double the volume of a parallelipped in one coordinate frame, you double it in all coordinate frames.

8. The product rule says that the determinant of the product of two matrices is the product of the determinants. This can be understood by considering two successive transformations. Let's say a vector $\vec{v}$ is transformed by the matrix $\underset{\sim}{B}$, and then the result is transformed by the matrix $\underset{\sim}{A}$. This can also be expressed as a single transformation by the product $\underset{\sim}{A}\underset{\sim}{B}$. The net expansion has to be the same, whether you impose the expansions sequentially or together. For example, suppose that in successive transformations by the two matrices, you expand by a factor 2 and then by a factor 3, so the net expansion is 2x3=6. When you apply the two transformations together, the expansion factor must also be 6.

9. The inverse rule states that the determinant of the inverse is the inverse of the determinant. The inverse matrix reverses the initial transformation, giving you back your original vectors. So the net expansion factor, which is the product of the expansion factors for $\underset{\sim}{A}$ and $\underset{\sim}{A}^{-1}$, must be one.

10. How about the fact that a matrix with zero determinant has no inverse? Well if $\underset{\sim}{A}$ flattens all vectors onto the same plane, then that transformation is not reversible. This corresponds to the fact that $\underset{\sim}{A}^{-1}$ doesn't exist.

11. In section 3.1.3 it was stated without proof that the determinant of an orthogonal matrix, i.e., a matrix whose inverse equals its transpose, is ± 1. We can now see why that's true. The determinant of the inverse equals the determinant of the transpose, which we know from property 1 is the same as the determinant of the original matrix. Therefore $\det(\underset{\sim}{A})$ is a number that equals its own reciprocal. The only two such numbers are +1 and -1. Therefore, every orthogonal matrix is a rotation matrix. Those with determinant -1 change the sign of the volume of the parallelepiped, effectively turning it inside-out: improper rotations. Orthogonal matrices with determinant +1 represent proper rotations.

Back to Table of Contents.

Appendix E

Vector identities

The following are true for all vectors $\vec{u}$, $\vec{v}$, $\vec{w}$, and $\vec{x}$ and scalars ϕ and ψ that vary continuously in space.

Algebraic identities:

1. $(\vec{u}\times\vec{v})\cdot\vec{w} = (\vec{w}\times\vec{u})\cdot\vec{v} = (\vec{v}\times\vec{w})\cdot\vec{u}$
2. $\vec{u}\times(\vec{v}\times\vec{w}) = (\vec{u}\cdot\vec{w})\vec{v} - (\vec{u}\cdot\vec{v})\vec{w}$
3. $(\vec{u}\times\vec{v})(\vec{w}\times\vec{x}) = (\vec{u}\cdot\vec{w})(\vec{v}\cdot\vec{x}) - (\vec{u}\cdot\vec{x})(\vec{v}\cdot\vec{w})$

Identities involving the gradient

4. $\vec{\nabla}(\phi+\psi) = \vec{\nabla}\phi + \vec{\nabla}\psi$
5. $\vec{\nabla}(\phi\psi) = \psi\vec{\nabla}\phi + \phi\vec{\nabla}\psi$
6. $\vec{\nabla}(\vec{u}\cdot\vec{v}) = [\vec{v}\cdot\vec{\nabla}]\vec{u} + \vec{v}\times(\vec{\nabla}\times\vec{u}) + [\vec{u}\cdot\vec{\nabla}]\vec{v} + \vec{u}\times(\vec{\nabla}\times\vec{v})$

Identities involving the divergence

7. $\vec{\nabla}\cdot(\vec{u}+\vec{v}) = \vec{\nabla}\cdot\vec{u} + \vec{\nabla}\cdot\vec{v}$
8. $\vec{\nabla}\cdot(\phi\vec{u}) = \vec{u}\cdot\vec{\nabla}\phi + \phi\vec{\nabla}\cdot\vec{u}$
9. $\vec{\nabla}\cdot(\vec{u}\times\vec{v}) = \vec{v}\cdot(\vec{\nabla}\times\vec{u}) - \vec{u}\cdot(\vec{\nabla}\times\vec{v})$

Identities involving the curl

10. $\vec{\nabla}\times(\vec{u}+\vec{v}) = \vec{\nabla}\times\vec{u} + \vec{\nabla}\times\vec{v}$
11. $\vec{\nabla}\times(\phi\vec{u}) = \vec{\nabla}\phi\times\vec{u} + \phi\vec{\nabla}\times\vec{u}$
12. $\vec{\nabla}\times(\vec{u}\times\vec{v}) = [\vec{v}\cdot\vec{\nabla}]\vec{u} - \vec{v}(\vec{\nabla}\cdot\vec{u}) - [\vec{u}\cdot\vec{\nabla}]\vec{v} + \vec{u}(\vec{\nabla}\cdot\vec{v})$
13. $\vec{\nabla}\times(\vec{\nabla}\times\vec{u}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{u}) - \nabla^2\vec{u}$
14. $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{u}) = 0$
15. $\vec{\nabla}\times(\vec{\nabla}\phi) = 0$

Identities involving the Laplacian

16. $\nabla^2(\phi\psi) = \psi\nabla^2\phi + \phi\nabla^2\psi + 2\vec{\nabla}\phi\cdot\vec{\nabla}\psi$
17. $\nabla^2(\phi\vec{u}) = \vec{u}\nabla^2\phi + \phi\nabla^2\vec{u} + 2(\vec{\nabla}\phi)\cdot\vec{\nabla}\vec{u}$

Identities involving the advective derivative

18. $[\vec{u}\cdot\vec{\nabla}](\phi\vec{v}) = (\vec{u}\cdot\vec{\nabla}\phi)\vec{v} + \phi([\vec{u}\cdot\vec{\nabla}]\vec{v})$
19. $[\vec{u}\cdot\vec{\nabla}](\vec{v}\cdot\vec{w}) = ([\vec{u}\cdot\vec{\nabla}]\vec{v})\cdot\vec{w} + \vec{v}\cdot([\vec{u}\cdot\vec{\nabla}]\vec{w})$
20. $[\vec{u}\cdot\vec{\nabla}](\vec{v}\times\vec{w}) = ([\vec{u}\cdot\vec{\nabla}]\vec{v})\times\vec{w} + \vec{v}\times([\vec{u}\cdot\vec{\nabla}]\vec{w})$

21. $[\vec{u}\cdot\vec{\nabla}]\vec{u} \equiv (\vec{\nabla}\times\vec{u})\times\vec{u} + \frac{1}{2}\vec{\nabla}(\vec{u}\cdot\vec{u})$

Back to Table of Contents.

Appendix F

The Cauchy stress tensor

In section 6.3.2 we defined the Cauchy array, whose elements are the components of the stress vector $\vec{f}$ acting on each of the three coordinate planes:

$$\tau_{ij} = f_j^{(i)}.$$

In this appendix we will demonstrate three additional properties of this array:

- The stress vector acting on *any* plane is given by $f_j = \tau_{ij} n_i$, where $\hat{n}$ is the unit normal to the plane in question.
- The array $\underset{\sim}{\tau}$ transforms as a 2nd-order tensor.
- $\underset{\sim}{\tau}$ is symmetric.

We will do this by applying Newton's second law to carefully chosen fluid parcels and imagining the result if we take the size of the parcel to zero.

F.1 Local equilibrium

Recall the famous story of Galileo dropping cannon balls from a tower to demonstrate that balls of different sizes fall at the same rate. This showed that gravity pulls more strongly on a big ball than on a small one. Suppose this was not so, and the force acting on both balls was in fact the same. In that case, by Newton's second law $a = F/m$, the smaller ball would accelerate faster. If it were light enough, it could zip to the ground like a bullet! (Conversely, a sufficiently massive ball would seem to hang in mid-air, a prisoner of its own inertia.)

In our universe that's not how it works: as the size of an object goes to zero, the force acting on it must go to zero also, so that the acceleration remains finite. This is true not only of gravity but of any force. It is also true of rotational motion: as an object's size shrinks to zero, the net torque acting on its surface must also vanish, or else it would spin infinitely fast.

These observations support the *principle of local equilibrium*, which goes a long way towards specifying how force is transmitted within a fluid. Picture an imaginary closed surface within the fluid, with intermolecular forces and torques acting on it, and now imagine that

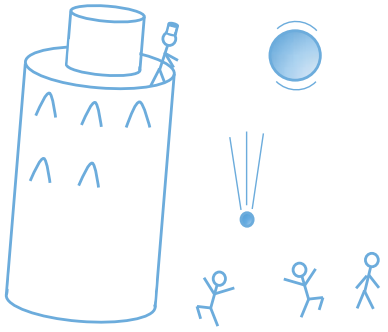


Figure F.1: If the gravitational force was independent of mass.

the closed surface shrinks to infinitesimal size. Both the net force and the net torque acting on the surface must go to zero. In the next two sections we'll see how these requirements lead to the three statements about the stress tensor listed at the beginning of this appendix.

F.2 Stress on a tilted plane

Consider a plane of arbitrary orientation as shown in figure F.2. The intersection of the tilted plane with the coordinate planes forms a tetrahedron. The tilted surface, outlined in red, is called the *hypotenuse* of the tetrahedron. We will first find an expression for the force acting on each face of the tetrahedron. We will then argue that this total force must vanish as the tetrahedron shrinks to infinitesimal size. Therefore, the force on the hypotenuse must balance the sum of the forces on the other surfaces.

Each of the coordinate faces has outward unit normal *opposite to the corresponding basis vector*. For example, the face labelled $A^{(1)}$ has outward unit normal $-\hat{e}^{(1)}$. Therefore, the stresses acting on those faces are minus the corresponding elements of the Cauchy array. The net force is the sum of the stress vectors times the areas of the faces act upon:

$$\vec{F} = \vec{f}A - \tau_{ij}\hat{e}^{(j)}A^{(i)}. \tag{F.2.1}$$

We now assume that the tetrahedon is filled with a continuous medium of density ρ. By Newton's second law, the sum of forces equals the product of the mass of the tetrahedron, m, and resultant acceleration $\vec{a}$. Taking the j component, we have

$$f_j A - \tau_{ij}A^{(i)} = ma_j. \tag{F.2.2}$$

To proceed, we need two facts from three-dimensional geometry that may be unfamiliar:

1. The volume of a tetrahedron is $Ah/3$, where A is the area of the hypotenuse and h is the perpendicular distance from the hypotenuse to the origin (figure F.3a). This is analogous to the formula for the area of a triangle: one half the base times the height. The mass is therefore
$$m = \rho Ah/3. \tag{F.2.3}$$

2. The area of each of the orthogonal coordinate planes is proportional to the area of the hypotenuse:
$$A^{(i)} = A\cos\theta^{(i)}.$$
Here, $\theta^{(i)}$ is the angle between $\hat{e}^{(i)}$ and $\hat{n}$, $\hat{n}$ being the outward unit normal to the tilted surface. (Figure F.3 shows the example $i = 3$.) This may be understood by recognizing that $\theta^{(i)}$ is the angle between the two planes, and the projection of the hypotenuse onto the coordinate plane is A times cosine of that angle. The cosine can also be expressed as the component of $\hat{n}$ perpendicular to the coordinate plane. To see this, note that the dot product of $\hat{n}$ and $\hat{e}^{(i)}$ is the product of their magnitudes (both equal to one) times $\cos\theta^{(i)}$. Moreover, that dot product is also equal to the i component of $\hat{n}$, so $\cos\theta^{(i)} = n_i$. As a result, we have
$$A^{(i)} = An_i. \tag{F.2.4}$$

Substituting (F.2.3) and (F.2.4) into (F.2.2), we obtain

$$f_j A - \tau_{ij}An_i = \rho A\frac{h}{3}a_j,$$

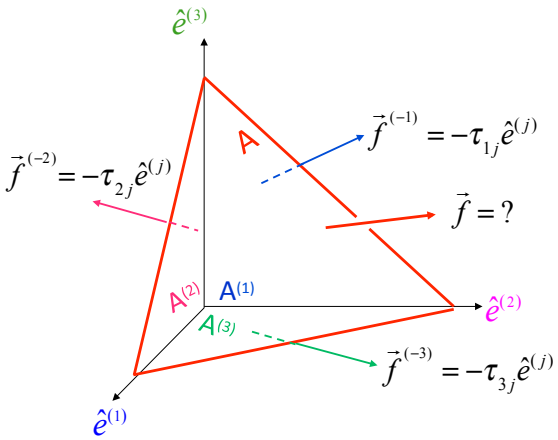


Figure F.2: Stresses on the faces of a tetrahedon. Colors indicate the area of each face and the stress vector acting upon it.

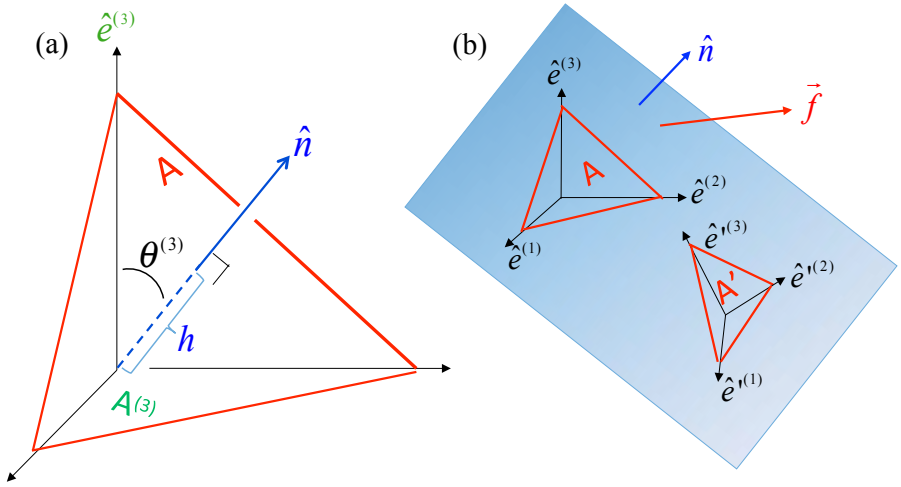


Figure F.3: (a) Geometry of a tetrahedon, showing the outward unit normal to the hypotenuse $\hat{n}$, the perpendicular distance h, and the angle θ from $\hat{e}^{(3)}$. The area $A^{(3)} = A\cos\theta^{(3)}$, and the volume is $Ah/3$. (b) Tetrahedra formed by the same plane (with unit normal $\hat{n}$ and stress vector $\vec{f}$) intersecting two different sets of Cartesian axes. The derivation of (F.2.5) works in either case, demonstrating that τ_{ij} is a tensor.

or

$$f_j - \tau_{ij} n_i = \rho \frac{h}{3} a_j.$$

We now shrink the size of the tetrahedron to zero by taking the limit $h \to 0$, resulting in

$$f_j = \tau_{ij} n_i. \tag{F.2.5}$$

From the foregoing, we can conclude that $\underset{\sim}{\tau}$ is a tensor. This can be seen in two ways.

1. In deriving (F.2.5), we did not assume a specific coordinate system; any Cartesian coordinate system would do. Referring to figure (figure F.3b), if we rotated the coordinate axes (keeping the tilted plane fixed), the areas A and $A^{(i)}$ would change, as would the values of the τ_{ij}, but we would still arrive at (F.2.5).
2. Both $\vec{f}$ and $\hat{n}$ are vectors, since they exist in physical space independently of the coordinate system. Therefore $\underset{\sim}{\tau}$, as a *relationship* between two vectors, must transform as a second order tensor (section 3.3.1).

We therefore refer to τ as the *Cauchy stress tensor*, or just the stress tensor. While the stress *vector* $\vec{f}(\vec{x},t,\hat{n})$ depends on the orientation of the plane it acts on, the stress *tensor* depends only on location and time: $\tau = \tau(\vec{x},t)$. It contains the information needed to evaluate the stress vector acting on any plane passing through $\vec{x}$.

F.3 Symmetry of the stress tensor

The symmetry of the stress tensor will be demonstrated in two ways. The first is fairly intuitive. We argue that stress components located above and below the main diagonal represent torques that are equal but opposite. If the tensor is symmetric, then, those torques add up to zero. This simple argument is deficient in that (1) it does not show why the torques should add to zero, and (2) it neglects the spatial variability of the stress tensor. We will address these issues in the more rigorous version that follows.

F.3.1 The hand-waving argument

Consider a cube with edge length Δ, as shown in figure F.4a, and the distribution of forces that act to rotate the cube counterclockwise about $\hat{e}^{(1)}$ (blue arrow). Assume that the stress tensor is uniform in space.

Consider the point labelled "A", which is located above the x_2-axis at $x_2 = \Delta/2$. The force (per unit area) $\vec{f}$ acting on the right hand face at this point can be resolved into two components vectors $\tau_{23}\hat{e}^{(3)}$ and $\tau_{22}\hat{e}^{(2)}$ as shown by the dashed arrows. As drawn here, $\tau_{23} > 0$ and $\tau_{22} < 0$, and each component exerts a counterclockwise torque about $\hat{e}^{(1)}$.

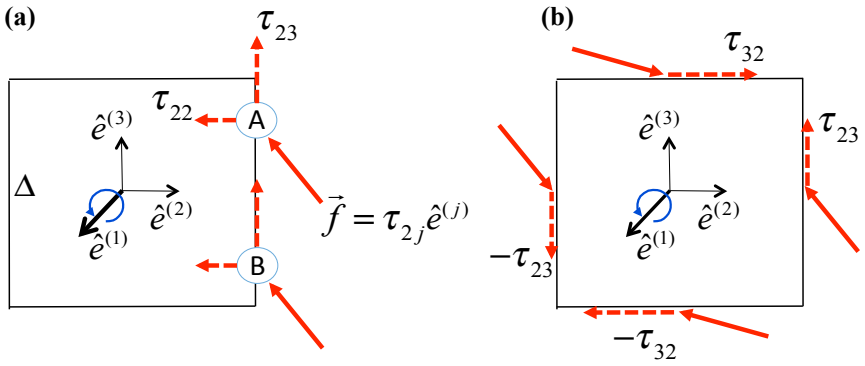


Figure F.4: Forces acting to rotate a cube about $\hat{e}^{(1)}$. (a) Effects of the same force acting at points above and below the x_2-axis. (b) Tangential forces acting on all four faces.

Now, consider the force acting at point B, which is located the same distance *below* the x_2-axis. Because the stress tensor is uniform, the force is the same, but now the normal component $\tau_{22}\hat{e}^{(2)}$ exerts a *clockwise* torque, equal but opposite to that exerted by the same component at point A. The torque exerted by the tangential component $\tau_{23}\hat{e}^{(3)}$ is unchanged. You can now imagine that, if we integrate over the right-hand face, the net torque exerted by the normal force component $\tau_{22}\hat{e}^{(2)}$ will vanish by symmetry, i.e., the net torque is due entirely to the tangential force component.

Now examine figure F.4b. On the left-hand face, the applied force is reversed because the unit normal is $-\hat{e}^{(2)}$. The tangential component is directed oppositely to that on the right-hand face, but the torque it exerts is the same.

Next we consider the upper face. The force acting there is $\tau_{3j}\hat{e}^{(j)}$, and its tangential component is $\tau_{32}\hat{e}^{(2)}$. As drawn here, $\tau_{32} > 0$, and the torque is *clockwise*. On the bottom face, the force is again opposite but the torque is the same. We now conclude that the net torque about $\hat{e}^{(1)}$ is proportional to the difference between τ_{23} and τ_{32}. If $\tau_{23} = \tau_{32}$, the net torque about $\hat{e}^{(1)}$ is zero.

Repeating this analysis for rotations about the other two axes, we find that the condition for equilibrium is $\tau_{ij} = \tau_{ji}$.

F.3.2 The quantitative argument

As before, consider a cube with edge length Δ, as shown in figure F.5a, and the distribution of stresses that act to rotate the cube about $\hat{e}^{(1)}$. Our plan is to compute the torque on each face and add the results.

On the right-hand face, the unit normal is $\hat{e}^{(2)}$. The force component τ_{23} acts in the "3" direction. The torque per unit area at any point is the stress vector crossed with the moment arm $\vec{r}$ (figure F.5b), which is the perpendicular distance from the "1" axis to the point where the force acts. The magnitude of that cross product is just $|\tau_{23}|$ times $|\vec{r}|\cos\theta$, where θ is the angle between $\vec{r}$ and the horizontal, and this in turn is equal to $|\tau_{23}|\Delta/2$. If τ_{23} is positive as shown, then the torque is positive (i.e., counterclockwise).

Now expand τ_{23} in a first-order Taylor series about the origin:

$$\tau_{23}(x, \frac{\Delta}{2}, z) = \tau^0_{23} + \frac{\partial \tau^0_{23}}{\partial x}x + \frac{\partial \tau^0_{23}}{\partial y}\frac{\Delta}{2} + \frac{\partial \tau^0_{23}}{\partial z}z + O(\Delta^2). \tag{F.3.1}$$

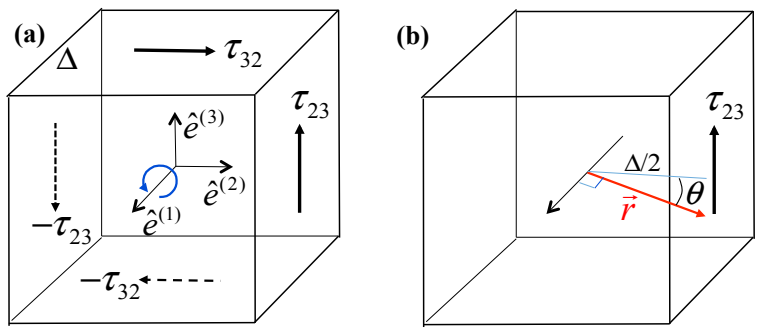


Figure F.5: Torques about the $\hat{e}^{(1)}$ axis on a cube with side Δ. (a) Tangential stresses. (b) The moment arm.

The superscript "0" denotes the value of τ_{23} or one of its derivatives evaluated at the origin. The y coordinate has been set to the uniform value $\Delta/2$ corresponding to the right-hand face. Integrating $\tau_{23}\Delta/2$ over the right-hand face, we find

$$\begin{aligned} T_1^{[right]} &= \int_{-\Delta/2}^{\Delta/2} dx \int_{-\Delta/2}^{\Delta/2} dz \, \frac{\Delta}{2} \, \tau_{23}(x, \frac{\Delta}{2}, z) \\ &= \Delta^2 \, \frac{\Delta}{2} \left(\tau_{23}^0 + \frac{\partial \tau_{23}^0}{\partial y} \frac{\Delta}{2} \right). \end{aligned}$$

Note that the terms in (F.3.1) proportional to x and z have integrated to zero.

The torque on the left-hand face is calculated similarly. The stress vector acts oppositely (downward, if $\tau_{23} > 0$), but the sign of $\cos\theta$ is reversed, and those two changes cancel. Ultimately, the only difference is that the stress is evaluated at $y = -\Delta/2$ rather than $y = \Delta/2$, so that

$$T_1^{[left]} = \Delta^2 \, \frac{\Delta}{2} \left(\tau_{23}^0 - \frac{\partial \tau_{23}^0}{\partial y} \frac{\Delta}{2} \right).$$

The net torque on the right and left faces is

$$T_1^{[right]} + T_1^{[left]} = \Delta^3 \, \tau_{23}^0.$$

The torques on the top and bottom faces are calculated in the same manner, and give $-\Delta^3 \, \tau_{32}^0$, so that the net torque about $\hat{e}^{(1)}$ is

$$T_1 = \Delta^3 \, (\tau_{23}^0 - \tau_{32}^0).$$

Now the rotational form of Newton's second law states that this torque equals $I_{11}\alpha_1$, where I_{11} is the moment of inertia for torque and rotation about $\hat{e}^{(1)}$ and α_1 is the corresponding angular rotation. For this cube, $I_{11} = \rho\Delta^5/6$ (appendix B), hence

$$\Delta^3 \, (\tau_{23}^0 - \tau_{32}^0) = \rho \, \frac{\Delta^5}{6} \alpha_1,$$

or

$$\tau_{23}^0 - \tau_{32}^0 = \rho \, \frac{\Delta^2}{6} \alpha_1.$$

We now take the limit as $\Delta \to 0$. The right-hand side goes to zero (provided α_1 is finite) while the superscripts on the left-hand side become superfluous, leaving us with

$$\tau_{23} - \tau_{32} = 0.$$

The same calculation can be repeated for rotation about $\hat{e}^{(2)}$ and $\hat{e}^{(3)}$ with analogous results, so that

$$\boxed{\tau_{ij} - \tau_{ji} = 0,} \tag{F.3.2}$$

i.e., the stress tensor is symmetric at every point in space.

Back to Table of Contents.

Appendix G

The Boussinesq approximation

The neglect of the inertial effect of inhomogeneity (section 7.4) permits a powerful simplification of the Navier-Stokes momentum equation.

G.1 The Navier-Stokes equation for nearly-uniform density

Let us look again at (6.3.19), this time in the form

$$\rho \frac{D\vec{u}}{Dt} = -\rho g \hat{e}^{(z)} - \vec{\nabla} p + \mu \nabla^2 \vec{u} + \mu \vec{\nabla}(\vec{\nabla} \cdot \vec{u}). \tag{G.1.1}$$

Here we are working in gravity-aligned coordinates, so that the gravity vector $\vec{g} = -g\hat{e}^{(z)}$.[1] The left-hand side is nonlinear in two respects. First, the material derivative contains a quadratic combination of unknown fields, $[\vec{u} \cdot \vec{\nabla}]\vec{u}$. In addition, the factor ρ multiplying the material derivative adds another layer of nonlinearity. In geophysical fluids, density variations are often small enough that this additional nonlinearity can be removed.

To begin, we write the density ρ as the sum of a uniform "background" value ρ_0 and a fluctuating part ρ':

$$\rho = \rho_0 + \rho'. \tag{G.1.2}$$

In addition, we decompose the pressure as

$$p = p_0 + p^*,$$

where the "background" part is in hydrostatic balance with the background density:

$$\vec{\nabla} p_0 = -\rho_0 g \hat{e}^{(z)},$$

so that

$$\vec{\nabla} p = -\rho_0 g \hat{e}^{(z)} + \vec{\nabla} p^*. \tag{G.1.3}$$

With the substitution of (G.1.2) and (G.1.3), (G.1.1) becomes

$$(\rho_0 + \rho') \frac{D\vec{u}}{Dt} = -\rho' g \hat{e}^{(z)} - \vec{\nabla} p^* + \mu \nabla^2 \vec{u} + \mu \vec{\nabla}(\vec{\nabla} \cdot \vec{u}), \tag{G.1.4}$$

Now assume that $|\rho'| \ll \rho_0$, and neglect ρ' in favor of ρ_0 on the left-hand side. Dividing through by the constant ρ_0, we then have

$$\boxed{\frac{D\vec{u}}{Dt} = b\hat{e}^{(z)} - \vec{\nabla}\frac{p^*}{\rho_0} + \nu \nabla^2 \vec{u} + \nu \vec{\nabla}(\vec{\nabla} \cdot \vec{u}).} \tag{G.1.5}$$

Here,

$$b = -\frac{\rho'}{\rho_0} g$$

[1] This restriction is not necessary; it just simplifies the discussion.

is the buoyancy and

$$\nu = \frac{\mu}{\rho_0}$$

is the kinematic viscosity. By replacing ρ with the constant ρ_0 everywhere except in the buoyancy, we simplify the solution of the momentum equation considerably. (G.1.5) is called the Boussinesq approximation of the momentum equation.

Note that we have yet to define the background density ρ_0. The only guidance we have for this choice is that the accuracy of the approximation depends on fluctuations about the background density being small. It therefore makes sense to define ρ_0 so as to minimize those fluctuations, e.g., by using the volume average.

G.2 Alternative derivation

Another way to derive (G.1.5) is by rearranging (G.1.4) as

$$\rho_0 \frac{D\vec{u}}{Dt} = \rho' \left(\vec{g} - \frac{D\vec{u}}{Dt} \right) - \vec{\nabla} p^* + \mu \nabla^2 \vec{u} + \mu \vec{\nabla} (\vec{\nabla} \cdot \vec{u}).$$

Now consider the first two terms in parentheses. If we assume that all accelerations are small compared with gravity, then the second term, $D\vec{u}/Dt$, can be discarded. We then divide through by ρ_0 to obtain the Boussinesq equation (G.1.5) as before. Note that the smallness of accelerations compared with gravity is the same assumption that justifies neglecting the inertial terms in the baroclinic torque (section 7.4).

Back to Table of Contents.

Appendix H

Bernoulli's equation

Recall the momentum equation for a homogeneous, inviscid fluid, written in gravity-aligned coordinates:

$$\frac{D\vec{u}}{Dt} = -g\hat{e}^{(z)} - \vec{\nabla}\frac{p}{\rho_0}.$$

Using the vector identity

$$[\vec{u}\cdot\vec{\nabla}]\vec{u} \equiv (\vec{\nabla}\times\vec{u})\times\vec{u} + \frac{1}{2}\vec{\nabla}(\vec{u}\cdot\vec{u}),$$

we can rewrite this as

$$\frac{\partial\vec{u}}{\partial t} + \vec{\omega}\times\vec{u} + \frac{1}{2}\vec{\nabla}(\vec{u}\cdot\vec{u}) = -g\hat{e}^{(z)} - \vec{\nabla}\frac{p}{\rho_0},$$

where $\vec{\omega} = \vec{\nabla}\times\vec{u}$ is the vorticity. Next, note that the vertical unit vector is the gradient of the vertical coordinate: $\hat{e}^{(z)} = \vec{\nabla}z$. We now substitute this and collect all of the terms that can be expressed as gradients:

$$\frac{\partial\vec{u}}{\partial t} + \vec{\omega}\times\vec{u} = -\vec{\nabla}\left(\frac{1}{2}(\vec{u}\cdot\vec{u}) + gz + \frac{p}{\rho_0}\right),$$

or

$$\frac{\partial\vec{u}}{\partial t} + \vec{\omega}\times\vec{u} = -\vec{\nabla}B,$$

where

$$\boxed{B = \frac{1}{2}(\vec{u}\cdot\vec{u}) + gz + \frac{p}{\rho_0}}$$

is called the Bernoulli function [1].

Now assume that the flow is in steady state, i.e., $\partial\vec{u}/\partial t = 0$:

$$\boxed{\vec{\nabla}B = \vec{u}\times\vec{\omega}.}$$

This tells us that the gradient of the Bernoulli function is perpendicular to both $\vec{u}$ and $\vec{\omega}$, and therefore that B does not vary in the direction of either of those vectors. In other words, in steady flow of a homogeneous, inviscid fluid,

- B is uniform along a vortex filament, and
- a fluid particle maintains a constant value of B (since $\vec{u}\cdot\vec{\nabla}B = DB/Dt = 0$) as it travels.

The second point, often called *Bernoulli's Law*, famously explains how an airplane flies. Because the upper and lower surfaces of the wing are convex, flow past them is forced to speed up, so that the first term in B increases. Variation in the second term is negligible, so the third term must decrease to maintain a constant value of B, i.e., the pressure must drop. Wings are designed with the upper surface more convex than the lower, so that the pressure drop is greater. The resulting pressure difference exerts a net upward force ("lift") on the wing.

Back to Table of Contents.

[1] This should not be confused with the baroclinic torque vector $\vec{B}$ that we discussed in section 7.4.

Appendix I

Vector operations in curvilinear coordinates

I.1 Cylindrical coordinates

Let ψ be a scalar and $\vec{u}$ be a vector expressed as a linear combination of the cylindrical basis vectors: $\vec{u} = u_r\hat{e}^{(r)} + u_\theta\hat{e}^{(\theta)} + u_z\hat{e}^{(z)}$.

Gradient of a scalar

$$\vec{\nabla}\psi = \hat{e}^{(r)}\frac{\partial\psi}{\partial r} + \hat{e}^{(\theta)}\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \hat{e}^{(z)}\frac{\partial\psi}{\partial z} \tag{I.1.1}$$

Divergence of a vector

$$\vec{\nabla}\cdot\vec{u} = \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{\partial u_z}{\partial z} \tag{I.1.2}$$

Curl of a vector

$$\vec{\nabla}\times\vec{u} = \hat{e}^{(r)}\left(\frac{1}{r}\frac{\partial u_z}{\partial\theta} - \frac{\partial u_\theta}{\partial z}\right) + \hat{e}^{(\theta)}\left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right) + \hat{e}^{(z)}\left(\frac{1}{r}\frac{\partial(ru_\theta)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right) \tag{I.1.3}$$

Laplacian of a scalar

$$\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial z^2} \tag{I.1.4}$$

Laplacian of a vector

$$\nabla^2\vec{u} = \hat{e}^{(r)}\left(\nabla^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2}\frac{\partial u_\theta}{\partial\theta}\right) + \hat{e}^{(\theta)}\left(\nabla^2 u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r^2}\right) + \hat{e}^{(z)}\nabla^2 u_z \tag{I.1.5}$$

Material derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial\theta} + u_z\frac{\partial}{\partial z} \tag{I.1.6}$$

The equations of motion are the same as those listed in section 6.8, except that the Navier-Stokes momentum equation (6.8.2) has added terms on the right-hand side, representing the centrifugal force. They arise mathematically because in curvilinear coordinates, the

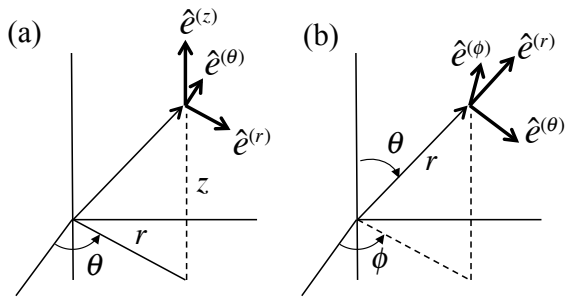


Figure I.1: Cylindrical (a) and spherical (b) coordinates.

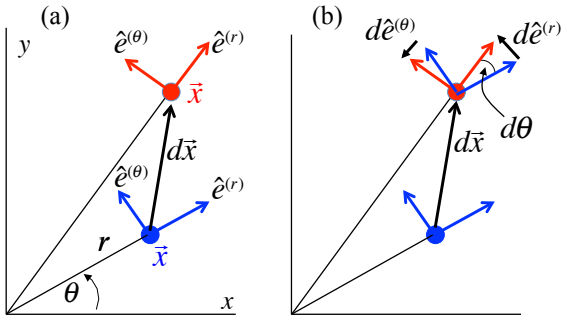


Figure I.2: Spatial variability of the basis vectors in polar coordinates. (a) The point at $\vec{x}$ (blue) moves to a new location (red). Basis vectors $\hat{e}^{(r)}$ and $\hat{e}^{(\theta)}$ are shown for both locations. (b) Basis vectors are copied from the initial point (blue) to the final point to show the changes $d\hat{e}^{(r)}$ and $d\hat{e}^{(\theta)}$.

directions of the basis vectors vary in space. Differentiating the velocity to get acceleration therefore involves differentiating the basis vectors as well.

In cylindrical coordinates, the basis vectors $\hat{e}^{(r)}$ and $\hat{e}^{(\theta)}$ vary in space but $\hat{e}^{(z)}$ does not. We can therefore consider the simpler case of polar coordinates $\{r,\theta\}$. Suppose a fluid particle at $\vec{x}$ has velocity

$$\vec{u} = u_r \hat{e}^{(r)} + u_\theta \hat{e}^{(\theta)}. \tag{I.1.7}$$

Over a short time interval dt, this velocity carries the particle to a new location $\vec{x}+d\vec{x}$. In figure I.2(a), the basis vectors at the initial and final locations are color coded red and blue, respectively. Now how does the velocity change? Differentiating (I.1.7) gives

$$\frac{d}{dt}\vec{u} = \frac{du_r}{dt}\hat{e}^{(r)} + u_r\frac{d\hat{e}^{(r)}}{dt} + \frac{du_\theta}{dt}\hat{e}^{(\theta)} + u_\theta\frac{d\hat{e}^{(\theta)}}{dt}. \tag{I.1.8}$$

Figure I.2(b) shows the increments of the basis vectors, $d\hat{e}^{(r)}$ and $d\hat{e}^{(\theta)}$. Because the basis vectors have length 1, the distance between their tips is approximated by the arc length $d\theta$ (accurate as $|d\theta| \to 0$). The direction of $\hat{e}^{(r)}$ is parallel to that of $\hat{e}^{(\theta)}$, while the direction of $\hat{e}^{(\theta)}$ is opposite to that of $\hat{e}^{(r)}$. Therefore:

$$d\hat{e}^{(r)} = d\theta\ \hat{e}^{(\theta)}\ ; \quad d\hat{e}^{(\theta)} = -d\theta\ \hat{e}^{(r)}$$

Dividing by dt and taking the limit $dt \to 0$, we obtain the time derivatives of the basis vectors. We now substitute these into (I.1.8) and find that

$$\frac{d}{dt}\vec{u} = \frac{du_r}{dt}\hat{e}^{(r)} + u_r\frac{d\theta}{dt}\hat{e}^{(\theta)} + \frac{du_\theta}{dt}\hat{e}^{(\theta)} - u_\theta\frac{d\theta}{dt}\hat{e}^{(r)}. \tag{I.1.9}$$

Note finally that $d\theta/dt = u_\theta/r$, giving

$$\frac{d}{dt}\vec{u} = \frac{du_r}{dt}\hat{e}^{(r)} + \frac{u_r u_\theta}{r}\hat{e}^{(\theta)} + \frac{du_\theta}{dt}\hat{e}^{(\theta)} - \frac{u_\theta^2}{r}\hat{e}^{(r)}. \tag{I.1.10}$$

The second and fourth terms on the right-hand side are the centripetal acceleration. If we write the material derivative as (I.1.6), we must subtract these extra terms from the right hand side:

$$\rho\frac{D\vec{u}}{Dt} = \rho\vec{a}_C + \rho\vec{g} - \vec{\nabla}p + \mu\nabla^2\vec{u} + \mu\vec{\nabla}(\vec{\nabla}\cdot\vec{u}), \tag{I.1.11}$$

where

$$\vec{a}_C = \frac{u_\theta^2}{r}\hat{e}^{(r)} - \frac{u_r u_\theta}{r}\hat{e}^{(\theta)}. \tag{I.1.12}$$

I.2 Spherical coordinates

Let ψ be a scalar and $\vec{u}$ be a vector expressed as a linear combination of the cylindrical basis vectors: $\vec{u} = u_r\hat{e}^{(r)} + u_\theta\hat{e}^{(\theta)} + u_\phi\hat{e}^{(\phi)}$.

Gradient of a scalar

$$\vec{\nabla}\psi = \hat{e}^{(r)}\frac{\partial\psi}{\partial r} + \hat{e}^{(\theta)}\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \hat{e}^{(\phi)}\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi} \tag{I.2.1}$$

Divergence of a vector

$$\vec{\nabla}\cdot\vec{u} = \frac{1}{r^2}\frac{\partial(r^2u_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(u_\theta\sin\theta)}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial u_\phi}{\partial\phi} \tag{I.2.2}$$

Curl of a vector

$$\vec{\nabla}\times\vec{u}=\hat{e}^{(r)}\frac{1}{r\sin\theta}\left[\frac{\partial\left(u_\phi\sin\theta\right)}{\partial\theta}-\frac{\partial u_\theta}{\partial\phi}\right]+\hat{e}^{(\theta)}\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial u_r}{\partial\phi}-\frac{\partial\left(ru_\phi\right)}{\partial r}\right]+\hat{e}^{(\phi)}\frac{1}{r}\left[\frac{1}{r}\frac{\partial\left(ru_\theta\right)}{\partial r}-\frac{\partial u_r}{\partial\theta}\right] \tag{I.2.3}$$

Laplacian of a scalar

$$\nabla^2\psi=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} \tag{I.2.4}$$

Laplacian of a vector

$$\begin{aligned}\nabla^2\vec{u} &= \hat{e}^{(r)}\left[\nabla^2u_r-\frac{2u_r}{r^2}-\frac{2}{r^2\sin\theta}\frac{\partial\left(u_\theta\sin\theta\right)}{\partial\theta}-\frac{2}{r^2\sin\theta}\frac{\partial u_\phi}{\partial\phi}\right]\\ &+ \hat{e}^{(\theta)}\left[\nabla^2u_\theta+\frac{2}{r^2}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r^2\sin^2\theta}-\frac{2}{r^2}\frac{\cos\theta}{\sin^2\theta}\frac{\partial u_\phi}{\partial\phi}\right]\\ &+ \hat{e}^{(\phi)}\left[\nabla^2u_\phi+\frac{2}{r^2\sin^2\theta}\frac{\partial u_r}{\partial\phi}+\frac{2}{r^2}\frac{\cos\theta}{\sin^2\theta}\frac{\partial u_\theta}{\partial\phi}-\frac{u_\theta}{r^2\sin^2\theta}\right]\end{aligned} \tag{I.2.5}$$

Material derivative

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+u_r\frac{\partial}{\partial r}+\frac{u_\theta}{r}\frac{\partial}{\partial\theta}+\frac{u_\phi}{r\sin\theta}\frac{\partial}{\partial\phi} \tag{I.2.6}$$

Centrifugal acceleration

$$\vec{a}_C=\hat{e}^{(r)}\frac{u_\theta^2+u_\phi^2}{r}+\hat{e}^{(\theta)}\left(-\frac{u_ru_\theta}{r}+\frac{u_\phi^2}{r\tan\theta}\right)+\hat{e}^{(\phi)}\left(-\frac{u_ru_\phi}{r}-\frac{u_\theta u_\phi}{r\tan\theta}\right).$$

Back to Table of Contents.

Appendix J

The Stokes drift

In the limit of small amplitude waves, fluid parcels oscillate in place (section 8.2.5), i.e., there is no overall current associated with the wave. But at finite amplitude, waves drive a nonzero mean motion called the Stokes drift. Floating objects such as driftwood are carried ashore by the Stokes drift. The essential reason for the Stokes drift is that the amplitude of the particle ellipses (figure 8.4) increases toward the surface. As a result, a particle moves slightly faster at the top of its ellipse than at the bottom.

Here we will estimate the speed of the Stokes drift using the small-amplitude theory developed in chapter 8. To begin with, write the horizontal velocity as a small perturbation $\{x', z'\}$ from its value at $\{x_0, z_0\}$:

$$\begin{aligned} u(x,z,t) &= u^0 + u' \\ &= u^0 + u^0_x x' + u^0_z z', \end{aligned}$$

where the superscript 0 denotes evaluation at $\{x_0, z_0\}$. Now substitute from (8.2.24, 8.2.17, 8.2.29):

$$u' = k\frac{(U^0)^2}{\omega}\sin^2(kx_0 - \omega t) + U^0_z \frac{W^0}{\omega}\cos^2(kx_0 - \omega t).$$

We now average this over one wave period, $2\pi/\omega$. The averages of the squared sine and cosine functions over a period are both equal to 1/2. Noting also that $U^0_z = kW^0$, we have

$$\begin{aligned} u^S = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} u'\, dt &= \frac{1}{2}\frac{k}{\omega}\left[(U^0)^2 + (W^0)^2\right] \\ &= \omega k \eta_0^2 \, \frac{\cosh^2 k(z_0+H) + \sinh^2 k(z_0+H)}{2\sinh^2 kH} \end{aligned}$$

The reader may check that w', calculated in similar fashion, averages to zero.

- In the short wave (deep water) limit $kH \to \infty$ this becomes

  $$u^S = c(k\eta_0)^2\, e^{2kz_0}.$$

  The Stokes drift is therefore proportional to the phase speed c, the square of the wave steepness $k\eta_0$, and a rapidly-decreasing function of depth.

- In the long wave (shallow water) limit $kH \to 0$, the Stokes drift is independent of depth:

  $$u^S = \frac{c}{2}\frac{\eta_0^2}{H^2}.$$

Back to Table of Contents.

Chapter 12

Exercises

All matrices are 3x3 and all vectors are 3-vectors, unless otherwise specified.

1. **Combining vectors**

   Let

   $$\vec{u} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \quad ; \quad \vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

   Compute the following:

   (a) $\vec{u} \cdot \vec{v}$

   (b) $|\vec{u}|$

   (c) $|\vec{v}|$

   (d) θ

   (e) component of $\vec{u}$ in the direction of $\vec{v}$

   (f) component of $\vec{v}$ in the direction of $\vec{u}$

2. **Self-consistency of free indices 1**

   In which of the following equations are the free indices consistent?

   (a) $A_{ij}B_j = \Gamma_i$

   (b) $A_{ij}B_i = \Gamma_i$

   (c) $A_{ij}B_i = \Gamma_j$

3. **Matrix multiplication**

   Which of the following expressions are equivalent to $A_{ij}B_{jk}$?

   (a) $A_{im}B_{mk}$

   (b) $A^T_{mi}B_{mk}$

   (c) $B_{jk}A_{ij}$

4. **Self-consistency of free indices 2**

   Fill in the indices on Γ to make the equation self - consistent:

   (a) $A_{ij}B_j = \Gamma_?$

   (b) $A_{kj}B_j = \Gamma_?$

   (c) $A_{ik}B_k = \Gamma_?$

   (d) $A_{iq}B_q = \Gamma_?$

   (e) $A_{ij}B_{jk} = \Gamma_?$

   (f) $A_{ik}B_{kj} = \Gamma_?$

   (g) $A_{lm}B_{ma} = \Gamma_?$

   (h) $A_{ij}B_{kj}C_{kl} = \Gamma_?$

5. **Using the identity matrix** Simplify the following by summing over the dummy indices. Assume that vectors have 3 elements and matrices are 3×3.

   (a) $\delta_{3k}p_k$

   (b) $\delta_{3i}\delta_{ij}$

   (c) $\delta_{i2}\delta_{i2}$

   (d) $\delta_{ij}\delta_{ij}$

   (e) $\delta_{i2}\delta_{ik}\delta_{3k}$

   (f) $\delta_{ij}v_iv_j$

   (g) $\delta_{i2}\delta_{j3}A_{ij}$

   (h) $\delta_{2i}A_{ji}$

   (i) $\delta_{q\alpha}\delta_{\beta\gamma}A_{mn}B_{nq}A_{\alpha\beta}$

6. **Vector and matrix properties**

   For each statement on the left, choose the appropriate statement from the list on the right.

   (a) $u_iv_i = 0$

   (b) $A_{pq}B_{qr} = \delta_{pr}$

   (c) $A_{ij} + A_{ji} = 0$

   (d) $u_iv_i = \pm|\vec{u}||\vec{v}|$

   (e) $|\vec{v}| = 1$

   (f) $A^T_{ij}A_{jk} = \delta_{ik}$

   - $\underset{\sim}{A}$ is an orthogonal matrix.
   - $\vec{u}$ is an eigenvector of $\underset{\sim}{A}$.
   - $\underset{\sim}{B}$ is the inverse of $\underset{\sim}{A}$.
   - $\vec{u}$ and $\vec{v}$ are orthogonal vectors.
   - $\underset{\sim}{A}$ is a symmetric matrix.
   - $\vec{v}$ is a unit vector.
   - $\underset{\sim}{A}$ is an antisymmetric matrix.
   - $\vec{u}$ is parallel to $\vec{v}$.

7. **Hyperbolic functions review**

   The hyperbolic sine and cosine functions are defined as:

   $$\sinh x = \frac{e^x - e^{-x}}{2}\ ; \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

   (a) Show that

   $$\frac{d}{dx}\sinh x = \cosh x\ ; \quad \frac{d}{dx}\cosh x = \sinh x. \quad \frac{d^2}{dx^2}\sinh x = \sinh x.$$

   (b) Calculate the Taylor series expansions about $x = 0$ of $f(x) = \sinh x$ and $f(x) = \cosh x$ up to (and including) the term proportional to x^2. [See Appendix A if you need a refresher on Taylor series.]

   (c) Sketch the sinh and cosh functions.

(d) The hyperbolic tangent function is

$$\tanh x = \frac{\sinh x}{\cosh x}$$

Using your results from (a) and (b) above, show that $\tanh x \simeq x$ for x close to zero.

(e) Determine the limiting values of $\tanh x$ as $x \to \infty$ and as $x \to -\infty$. (Hint: start by writing $\tanh x$ in terms of $e^x$and e^{-x}.)

(f) Sketch the tanh function.

8. **Matrix transformations 1**

For each matrix, compute the determinant and describe in words the effect the matrix has on a general column vector $\vec{v}$. Is the effect reversible?

(a)

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(b)

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

9. **Matrix transformations 2** Let

$$\underset{\sim}{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(a) Compute the determinant. Is $\underset{\sim}{A}$ singular?

(b) Describe in words the effect the matrix has on a general column vector $\vec{v}$.

(c) Consider a set of unit vectors emanating from the origin in all possible directions. Their tips comprise a sphere of radius 1, the unit sphere. Now suppose that each unit vector is multiplied by $\underset{\sim}{A}$. What becomes of the sphere?

(d) Compute eigenvalues and eigenvectors of $\underset{\sim}{A}$. Guided by these results, describe in words the class (or classes) of vectors whose direction is unchanged after multiplication by $\underset{\sim}{A}$.

10. **Rotation exercise**

    (a) Compute $\underset{\sim}{C}(\theta)$ for rotation around $\hat{e}^{(3)}$ by a nonzero angle θ.

    (b) Verify that $\underset{\sim}{C}(-\theta) = \underset{\sim}{C}^T = \underset{\sim}{C}^{-1}$.

    (c) Compute the eigenvalues of $\underset{\sim}{C}$. You should find that one eigenvalue is 1 and the other two are complex.

    (d) Show that $\hat{e}^{(3)}$ is the eigenvector with eigenvalue 1. [Hint: You don't have to compute the eigenvector, just show that it works, i.e. that (2.4.1) is satisfied by $\hat{e}^{(3)}$ with $\lambda^{(m)} = 1$.

    (e) Express the tensor

$$\underset{\sim}{A} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

    in the rotated frame defined by $\underset{\sim}{C}$. What special property does $\underset{\sim}{A}$ have when $a = b$ (i.e. how does the transformed matrix depend on θ when $a = b$)?

    (f) Now suppose we rotate the coordinates around $\hat{e}^{(3)}$ by an *additional* angle φ. Write down the corresponding rotation matrix $\underset{\sim}{C}(\varphi)$, then multiply onto the matrix $\underset{\sim}{C}(\theta)$ that you derived in (a) to get a matrix that describes the net rotation. From this, deduce the standard trigonometric identities for $\cos(\theta+\varphi)$ and $\sin(\theta+\varphi)$.

11. **Index notation exercise**

    Assuming that $\underset{\sim}{S}$ is symmetric and $\underset{\sim}{A}$ is antisymmetric, prove that $S_{ij}A_{ij} = 0$.

12. **Symmetric-antisymmetric decomposition**

    Let an arbitrary matrix T_{ij} be decomposed into a sum of symmetric and antisymmetric parts:

$$T_{ij} = S_{ij} + A_{ij}$$

    where

$$S_{ij} = \tfrac{1}{2}(T_{ij} + T_{ji}) \; ; \quad A_{ij} = \tfrac{1}{2}(T_{ij} - T_{ji}).$$

    (a) Show that $T_{ij}T_{ij} = S_{ij}S_{ij} + A_{ij}A_{ij}$. [Hint: Your result from 11 will be useful.]

    (b) Show that $u_i T_{ij} u_j = u_i S_{ij} u_j$, for any vector $\vec{u}$.

13. **Tensor order 1** Each of the following expressions represents a tensor (or a component of a tensor). Give the order of the tensor.

    (a) $A_{ij}v_j$

    (b) $A_{ij}B_{jk}$

    (c) $v_i A_{ij}$

    (d) $A_{ij}v_k$

    (e) $A_{ij}B_{kl}$

    (f) $B_{kl}A_{lm}$

    (g) $\underset{\sim}{A}\vec{v}$

    (h) $\vec{u}\cdot\vec{v}$

    (i) $\vec{u}\cdot\underset{\sim}{A}\vec{v}$

    (j) $u_i A_{ij} v_j$

14. **Tensor order 2**

    Each of the following expressions represents a component of a tensor that varies in space. Give the order of the tensor.

    (a) $\partial\phi/\partial x_i$

    (b) $\partial v_i/\partial x_i$

    (c) $\partial v_i/\partial x_j$

(d) $\partial A_{ij}/\partial x_j$

15. **Is it a tensor?**

    Assume that $\vec{u}$ is a vector. Show that $\partial u_i/\partial x_j$ transforms as a second order tensor under coordinate rotations.

16. **Rotation rule for a 3rd-order tensor**

    Assume that $\vec{u}$ is a vector and $\underset{\sim}{A}$ is a 2nd-order tensor. Derive the forward transformation rule for a 3rd-order tensor Z_{ijk} such that the relation

    $$u_i = Z_{ijk}A_{jk},$$

    remains valid after a coordinate rotation.

17. **The $\varepsilon-\delta$ relation**

    Rewrite using the $\varepsilon-\delta$ relation:

    (a) $\varepsilon_{ikj}\varepsilon_{mjl}$

    (b) $\varepsilon_{tom}\varepsilon_{jim}$

18. **Properties of the cross product**

    (a) Using the properties of alternating tensor, prove that $\vec{u}\times\vec{v}$ is perpendicular to both $\vec{u}$ and $\vec{v}$. (Hint: One way to show that two vectors are perpendicular is to show that their dot product is zero.)

    (b) Consider 3 vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ related by $\vec{w}=\vec{u}\times\vec{v}$. Using the $\varepsilon-\delta$ relation, prove that $|\vec{w}| = |\vec{u}||\vec{v}||\sin\theta|$, where θ is the angle between $\vec{u}$ and $\vec{v}$. (Hint: Start by computing $\vec{w}\cdot\vec{w}$.)

19. **Transforming matrix properties**

    Suppose a 2nd-order tensor $\underset{\sim}{A}$ has elements A_{ij} in one coordinate frame and A'_{ij} in a second frame which is related to the first by the rotation matrix $\underset{\sim}{C}$. Prove the following.

    (a) If $\underset{\sim}{A}$ is symmetric, it remains so after the coordinate rotation.

    (b) If $\underset{\sim}{A}$ is antisymmetric, it remains so after the coordinate rotation.

    (c) The trace is unchanged, i.e., $Tr(\underset{\sim}{A})$ is a scalar.

    (d) The eigenvalues of $\underset{\sim}{A}$ are scalars.

20. **Proving vector identities**

    Assume that ϕ and ψ are scalars and $\vec{u}$, $\vec{v}$ and $\vec{w}$ are vectors, all of which vary smoothly in space. Prove the following:

    (a) $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{u}) = 0$

    (b) $\vec{\nabla}(\phi\psi) = \psi\vec{\nabla}\phi + \phi\vec{\nabla}\psi$

    (c) $\vec{\nabla}\cdot(\phi\vec{u}) = \vec{u}\cdot\vec{\nabla}\phi + \phi\vec{\nabla}\cdot\vec{u}$

    (d) $\vec{\nabla}\times(\phi\vec{u}) = \vec{\nabla}\phi\times\vec{u} + \phi\vec{\nabla}\times\vec{u}$

    (e) $\vec{u}\times(\vec{v}\times\vec{w}) = (\vec{u}\cdot\vec{w})\vec{v} - (\vec{u}\cdot\vec{v})\vec{w}$

    (f) $\vec{\nabla}\times(\vec{u}\times\vec{v}) = (\vec{v}\cdot\vec{\nabla})\vec{u} - (\vec{\nabla}\cdot\vec{u})\vec{v} - (\vec{u}\cdot\vec{\nabla})\vec{v} + (\vec{\nabla}\cdot\vec{v})\vec{u}.$

21. **Advection**

    A rainstorm, moving from the west, has passed over Philomath and is just reaching Corvallis[1]. The storm moves eastward at a constant speed $V>0$. The distribution of the rain rate, R, remains steady as the storm moves, i.e., $R(x,t) = R(s)$, where $s = x - Vt$. (R is the amount of rain per unit time, e.g., 10mm/hour, as would be measured using a rain guage. $R(s)$ is a maximum at $s=0$ and drops off to zero as $s\to\pm\infty$.)

    (a) Using the chain rule, show that the rain rate at a fixed point evolves as:

    $$\frac{\partial R}{\partial t} = -V\frac{\partial R}{\partial x}.$$

    (b) Describe in words the meaning of each of the expressions V, $\partial R/\partial x$, and $\partial R/\partial t$ in this equation. Referring to the diagram, give and interpret the signs of these quantities, both at Philomath and at Corvallis.

[1] Corvallis, Oregon is the home of Oregon State University, and Philomath is a town a few miles to the west. A few miles further to the west is the Pacific ocean, which makes for a lot of rainstorms.

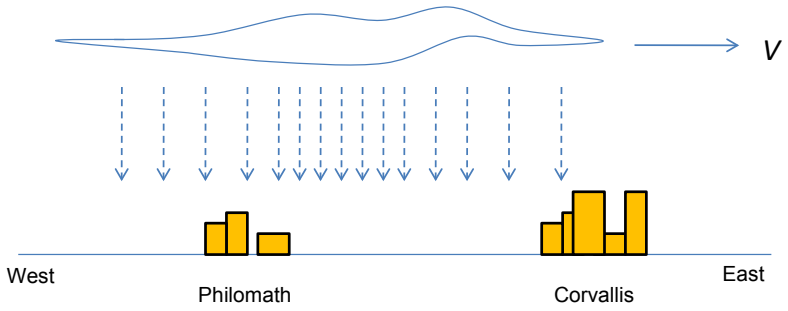


Figure 12.1: A rain storm moving eastward from Philomath to Corvallis.

22. **Proving more identities**

Prove the following:

(a) $(\vec{u}\times\vec{v})\cdot(\vec{w}\times\vec{x}) = (\vec{u}\cdot\vec{w})(\vec{v}\cdot\vec{x}) - (\vec{u}\cdot\vec{x})(\vec{v}\cdot\vec{w})$

(b) $\vec{\nabla}\cdot(\vec{u}\times\vec{v}) = \vec{v}\cdot(\vec{\nabla}\times\vec{u}) - \vec{u}\cdot(\vec{\nabla}\times\vec{v})$

23. **Streamlines and strains in a 2D flow**

A 2D steady flow has components $u = y,\ v = x$.

(a) Compute the streamfunction and show that the streamlines are hyperbolas: $x^2 - y^2 = const$.

(b) Sketch a few representative streamlines, showing the direction of the flow.

(c) Compute the vorticity vector $\vec{\omega}$ and the strain tensor $\underset{\sim}{e}$.

(d) Compute the eigenvalues and eigenvectors of $\underset{\sim}{e}$. Choose the eigenvectors to have unit length.

(e) Confirm that the eigenvectors are orthogonal.

(f) Form the rotation matrix that diagonalizes $\underset{\sim}{e}$, ensuring that its determinant is + 1 (reorder the eigenvalues and eigenvectors if necessary).

(g) Describe the rotation that this matrix represents. Give both angle and axis of rotation. (If this isn't clear, try reordering the eigenvalues and eigenvectors.)

(h) Use your rotation matrix to diagonalize the strain tensor.

(i) Identify the principal axes and principal strains. Indicate these with arrows on your sketch from part (b).

(j) Characterize the principal strains as extensional or compressive.

24. **An isolated vortex**

An isolated vortex is one such that the circulation in the limit $r \to \infty$ is zero. Consider an axisymmetric vortex (no dependence on the azimuthal angle θ) for which the azimuthal velocity u_θ is proportional to $r^{-\alpha}$. What is the range of possible values of the exponent α such that the circulation $\Gamma(r)$ is finite as $r \to \infty$?

25. **The Rankine vortex**

A Rankine vortex has angular velocity $\dot{\theta}$ for $0 \le r \le R$, and is irrotational for larger r.

(a) Compute and sketch the azimuthal velocity $u_\theta(r)$ for all r. Require that $u_\theta(r)$ be continuous.

(b) Compute and sketch the vorticity $\omega^{(z)}$ and the circulation $\Gamma(r)$.

(c) Interpret the result of (b) in terms of your result for the previous problem 24.

26. **Conservation of property** X

Suppose that X is the concentration of "something" per unit mass of fluid.

(a) Show that the concentration per unit *volume* is ρX, where ρ is the density of the fluid.

(b) Now suppose that the *flux* of X is given by

$$\vec{F}_X = -\rho\gamma\vec{\nabla}X,$$

where γ is a scalar. Write down a Lagrangian conservation equation stating that the net amount of X in a fluid parcel changes according to the flux normal to its surface. (More specifically, the net X increases if the total flux through the surface is inward, and vice versa.)

(c) Assuming that mass is conserved, show that X must obey this equation:

$$\frac{\partial X}{\partial t} + \vec{u}\cdot\vec{\nabla}X = \frac{1}{\rho}\vec{\nabla}\cdot(\rho\gamma\vec{\nabla}X).$$

27. **Force on a small cube**

(a) Compute the net contact force (per unit volume) on a cube with edge length Δ in the limit $\Delta \to 0$. Do this by integrating the stress vector $\vec{f} = \hat{n}\underset{\sim}{\tau}$ over all 6 surfaces. Follow the sequence of steps we used in section 4.2.3 to compute the net volume flux out of a cube.

(b) Obtain the same result using the generalized divergence theorem.

28. **Preserving diagonality**

Under what conditions does a diagonal tensor remain diagonal after an *arbitrary* coordinate rotation $\underset{\sim}{C}$?

[Hint: First assume that $A_{11} = A_{22} = A_{33}$, and show that the tensor remains diagonal after an arbitrary rotation. Now, are there any *other* diagonal matrices that remain diagonal after an arbitrary rotation? Pick an easy rotation, one for which you can calculate $\underset{\sim}{A}'$ explicitly. Identify the conditions under which it is diagonal. (You may want to review your answer to exercise 10e.) Now pick a different simple rotation and repeat the argument. If you can show that diagonality is preserved for these two particular rotations *only* if $A_{11} = A_{22} = A_{33}$, then you have shown that diagonality is preserved for *every* rotation only if $A_{11} = A_{22} = A_{33}$.]

29. **River flow**

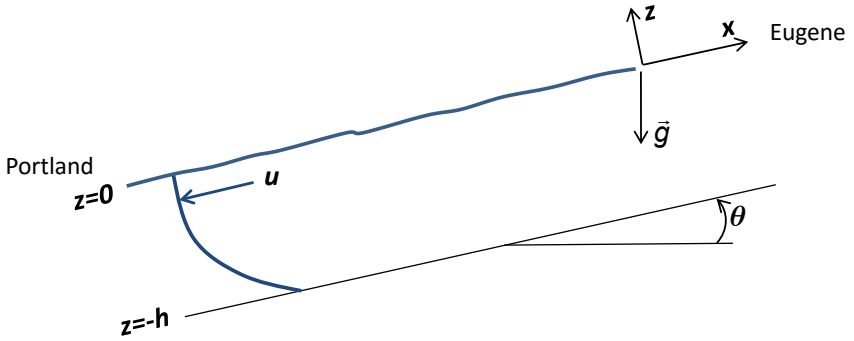


Figure 12.2: Definition sketch for the Willamette River, which flows northward between Eugene and Portland, Oregon.

In this problem you will use the Navier-Stokes equation, together with a list of simplifying assumptions, to predict the flow profile of the Willamette River as it flows past Corvallis (figure 12.2).

(a) Set up the equation for the flow velocity parallel to the bottom (u) as a function of perpendicular distance from the surface (z). Apply the appropriate boundary conditions and solve for $u(z)$.

Assume the following:

1. The flow obeys the Navier - Stokes equation for an incompressible flow ($\vec{\nabla}\cdot\vec{u} = 0$).
2. The current is steady (independent of time), so that $\partial/\partial t$ of anything is zero.
3. The density is uniform ($\rho = \rho_0$).
4. The current is a parallel shear flow $\vec{u} = u(z)\hat{e}^{(x)}$ in a coordinate frame parallel to the bottom.
5. No fluid property varies in the direction parallel to the bottom ($\partial/\partial x = 0$).
6. No fluid property varies in the cross - stream (y) direction.
7. At the river bottom, $u = 0$.
8. At the surface, $du/dz = 0$. (This is true when wind effects are not important.)

Hint: Two coordinate systems are useful here:

- *gravity - aligned*
- *tilted* so that $\hat{e}^{(x)}$ is parallel to the bottom.

Although the gravity vector is simplest in gravity - aligned coordinates, everything else is simpler in tilted coordinates. So, solve the equations in the tilted coordinate system after performing the appropriate rotation to express $\vec{g}$ in those coordinates.

(b) What is the velocity at the surface? Give a numerical value based on the following parameter values:

- The mean grade (slope) of the Willamette River is 43 cm per km.
- The kinematic viscosity $\nu = \mu/\rho = 10^{-6} m^2/s$.
- The water depth $h = 2$m.
- The vertical gravitational acceleration $g = 9.8 m/s^2$.

Does your result seem consistent with the observed velocity?

(c) Suppose that frictional effects are supplied not by molecular viscosity but by a much larger "effective" viscosity due to turbulence (see section 6.3.5). If this turbulent viscosity is $10^{-2} m^2 s^{-1}$, what is the maximum flow speed? Verify that this velocity is more consistent with observations than the value you obtained in (b).

30. **Pressure drop and surface deflection in a Rankine vortex**

Consider the Rankine vortex, a steady, axisymmetric vortex whose velocity is purely azimuthal and is given by

$$u_\theta(r) = \begin{cases} \dot{\theta} r & \text{for } r \le R \\ \dot{\theta} R^2 r^{-1} & \text{for } r \ge R \end{cases}$$

(see exercise 25).

(a) Beginning with the radial velocity equation for inviscid, homogeneous flow (Appendix I.1),

$$\frac{Du_r}{Dt} = \frac{u_\theta^2}{r} - \frac{1}{\rho_0}\frac{\partial p}{\partial r},$$

write down a differential equation whose solution is the r-dependence of the pressure $p(r)$ in a Rankine vortex.

(b) Solve the equation, requiring that $p(r)$ be a continuous function. Sketch a graph of your solution. (We did the inner part, $r < R$, in class. Your solution should include all r.)

(c) Show that the pressure drop between $r = \infty$ and $r = 0$ is $\rho_0 V^2$, where ρ_0 is the density and V is the maximum value of the azimuthal velocity u_θ.

(d) Suppose that the Rankine vortex in (a) exists in a body of water with surface $z = \eta(r)$, and that the vertical pressure gradient is hydrostatic. Compute the profile of surface elevation $\eta(r)$. You can assume that, as $r \to \infty$, both the surface pressure and the surface displacement approach zero. Show that $\eta(0) = -V^2/g$, and sketch $\eta(r)$.

(e) Consider the angular frequency $\dot{\theta}$. How does the surface deflection $\eta(0)$ vary with this frequency, all else being equal?

31. **Kitchen sink experiment**

Obtain a clear cylindrical container such as a big measuring cup. A glass coffee mug will do, but the bigger the better. You'll also need a watch (or other way to measure seconds), a ruler, and something to stir with. Fill the container about 3/4 full with water. Stir the water until it is rotating evenly, and notice that the water surface is depressed at the center and elevated at the outside. Notice also that the faster you stir, the greater the depression/elevation.

Your objective is to determine how the depression (or elevation) depends on the frequency of stirring. It will be a power - law dependence, i.e., depression/elevation will be proportional to frequency, or frequency^2, or frequency^3 or something, and you're going to determine the exponent.

(a) Devise some way to measure the depression/elevation to within, say, 1mm. This could be the depression, the elevation, or the difference between the two, whichever you find easiest to measure. Call this distance h. [Here's one way: with the fluid motionless, mark the level on the outside of the container. Then, with the fluid turning, mark the elevation again and measure the difference.]

(b) Devise a way to measure the frequency of stirring (e.g., 1 stir per second). You can probably use your watch or some app on your phone. Call that frequency f.

(c) Make the measurement at several frequencies and plot a graph of $\log h$ versus $\log f$. Try at least 4 frequencies (more is better) covering a wide range (a factor of 4 or more, say).

(d) If $h = af^n$, where a and n are constants, then this graph should give a straight line. Fit a straight line through your measurements, determine its slope and intercept, and from that information estimate values for a and n. (Don't forget to include units!)

(e) Compare your estimate of n with the theoretical prediction developed in exercise 30 for the Rankine vortex. List possible reasons why this comparison may be imperfect. (This should include both measurement inaccuracies and physical assumptions that may be wrong.) Say which you think is most likely to be important.

Submit a clear description of your method, results and interpretation.

32. **The Froude number**

The Froude[2] number is important in hydraulic channel flows. It is defined as

$$F = \frac{|u|}{\sqrt{gH}},$$

where $u(x)$ is the flow velocity in a unidirectional current (a river, say), g is gravity and $H(x)$ is the depth of the fluid. Consider a channel of uniform width where the flow velocity varies inversely with depth so as to conserve the volume flux: $uH = const.$ [3]

Compute the Froude number for each of the following situations. Approximate g as 10ms^{-2} for simplicity.

(a) $u = 1ms^{-1}$, $H = 0.1m$.

(b) Depth H increases to $0.4m$ (with the same volume flux).

(c) Depth H decreases to $0.025m$.

33. **Effects of surface tension on surface waves**

(a) Derive the dispersion relation $\omega = \omega(k)$ for surface waves in the presence of surface tension. Surface tension imposes an effective pressure at the surface given by $p = -\sigma \nabla^2 \eta$, where σ is a constant. This alters the surface boundary condition on pressure. Aside from this, make the same assumptions we made in class: the fluid is homogeneous and inviscid, and there is no variation in the y - direction.

(b) Show that the frequency ω is increased by a factor $\sqrt{1+\sigma k^2/\rho_0 g}$ over the case without surface tension.

(c) Compute the wavelength of waves for which the second term in the dispersion relation equals the first (i.e., the contribution from surface tension equals that from gravity). Give a numerical value for this wavelength for the water/air interface ($\sigma/\rho_0 = 7.4\times10^{-5}m^3s^{-2}$). Waves shorter than this are called *capillary waves*.

34. **Mass conservation in a channel flow**

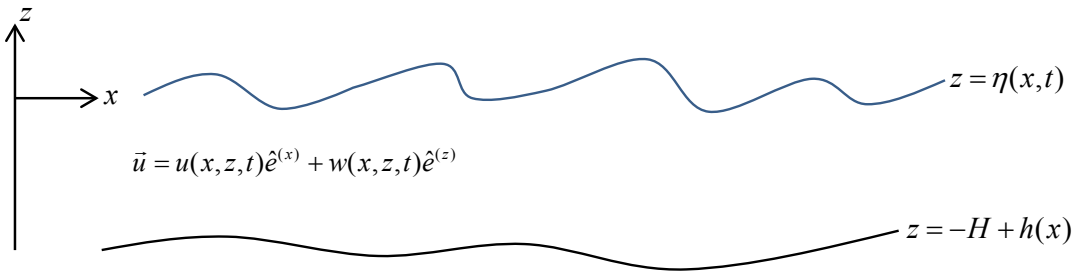


Figure 12.3: Definition sketch for two-dimensional channel flow.

(a) Consider inviscid, incompressible, 2 - dimensional flow (independent of y) in a channel with a corrugated, impermeable bottom $z = -H + h(x)$ and an undulating surface $z = \eta(x,t)$ (figure 12.3). Integrate the divergence vertically over the water column:

$$\int_{-H+h(x)}^{\eta(x,t)} \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) dz,$$

simplify where possible and set the result equal to zero. Use the result to show that

$$\frac{\partial \eta}{\partial t} = -\frac{\partial}{\partial x}\int_{-H+h(x)}^{\eta(x,t)} u\, dz.$$

[Hints: For the first term, use the 1 - dimensional form of Leibniz' rule (6.1.3) to bring the partial derivative outside the integral. Simplify the second term by using the appropriate boundary conditions. Do not assume that the disturbance is small-amplitude.]

(b) What difference does it make to **(a)** if the fluid is *viscous*, i.e., if the current goes to zero at the bottom boundary?

[2] Rhymes with "food".

[3] Hence the saying "Still waters run deep."

Index