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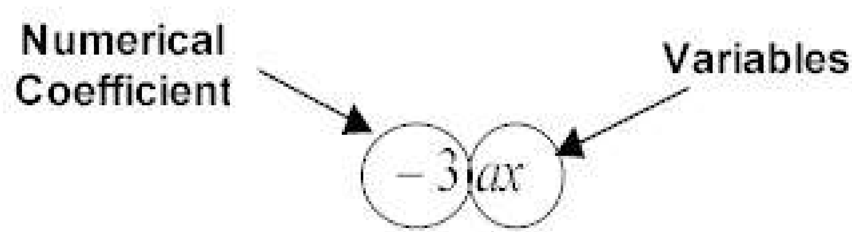
What is Algebra?

Algebra is a branch of mathematics that uses mathematical statements to describe relationships between things that vary. These variables include things like the relationship between the supply of an object and its price.

When we use a mathematical statement to describe a relationship, we often use letters to represent the quantity that varies, since it is not a fixed amount.

Algebraic Terms

The basic unit of an algebraic expression is a ***term***. In general, a ***term*** is either a number or a product of a number and one or more variables. Below is the term –3*ax*.



The numerical part of the term is what we refer to as the ***numerical coefficient.*** This numerical coefficient will take on the sign of the operation in front of it. In the term above, the numerical coefficient is –3; and the variables in the term are *a* and *x*.

Terms such as *xz* may not appear to have a numerical coefficient, but they do. The numerical coefficient in the term *xz* is 1, which is assumed.

Algebraic Expressions

An ***expression*** is a meaningful collection of numbers, variables and signs (positive or negative) of operations that must make mathematical and logical sense. Expressions:

- _contain any number of algebraic terms
- _use signs of operation: addition, subtraction, multiplication, and division

- _do not contain an equality sign (=)

An example of an expression is: $-3ax + 11wx^2y$

The signs of operation separate the expression into terms. The sign also becomes part of the term that follows. For example, the expression above contains two terms; the first term is $-3ax$ and the second term is $+11wx^2y$. The addition sign separates the two terms and is also part of the second term. Terms that do not have a sign listed in front of them are understood to be positive.

> **NOTE:** The operation of multiplication can be represented several ways. The statements $a \times b$, $a \bullet b$, $(a)(b)$, and ab are equivalent.

Algebraic Equations

An *equation* is a mathematical statement that two expressions are equal. The following three statements are equations:

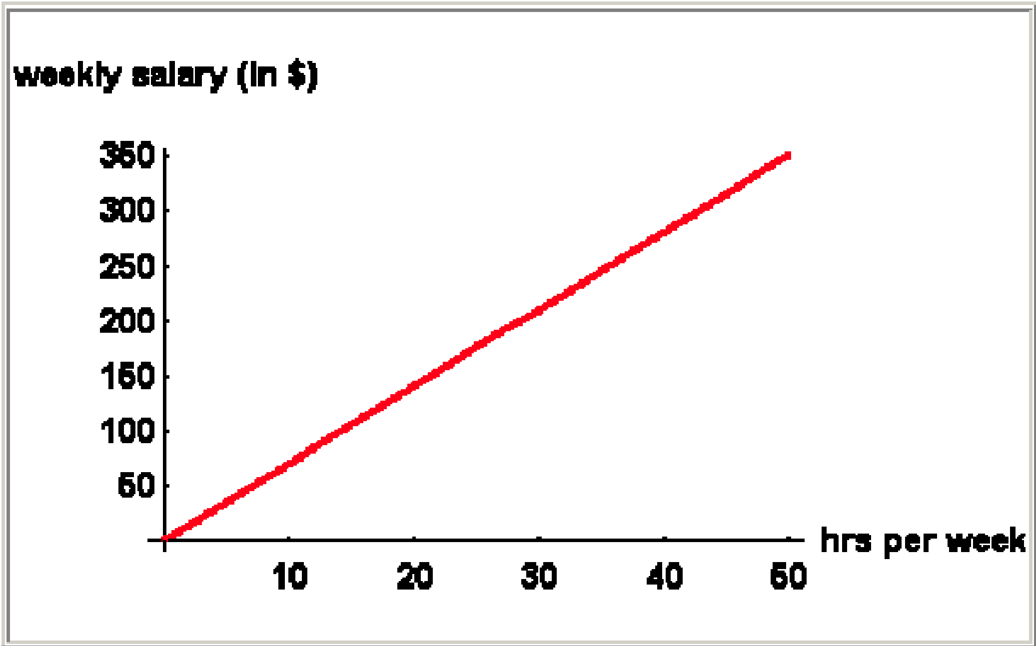
$4 + 5 = 9$ $\qquad$ $x - 35 = 56k^2 + 3$ $\qquad$ $x + 3 = 15$

The first equation, 4 + 5 = 9, contains only numbers; the other two, however, also contain variables. All three contain two *expressions* separated by an equal sign.

When an equation contains variables, you will often have to solve for one of those variables.

Functions

A function is just a rule that assigns a single value *y* or *f(x)* (read "f of *x*") to each value of *x*. The rule links one number to a second number in an orderly and specific manner. All the points on the graph of a function are made up of two parts: a number, and the function value at that number. For example, the number of hours worked in a week could be the first number, and the salary for the week could be the function value. If an hourly salary is $7.00, then the rule would be 7 times the number of hours worked.



You could identify a point on the graph of a function as (x,y) or (x, f(x)). You may have only one function value for each x number. Another way to think about it is that to any value of x, there cannot be more than one number, y, associated with x.

Suppose you have a function defined as: $f(x) = 2x$

If $x=1$, then $f(x) = 2$
If $x= -1$, then $f(x) = -2$

There could not be more than one value of f(x) associated with the same x.

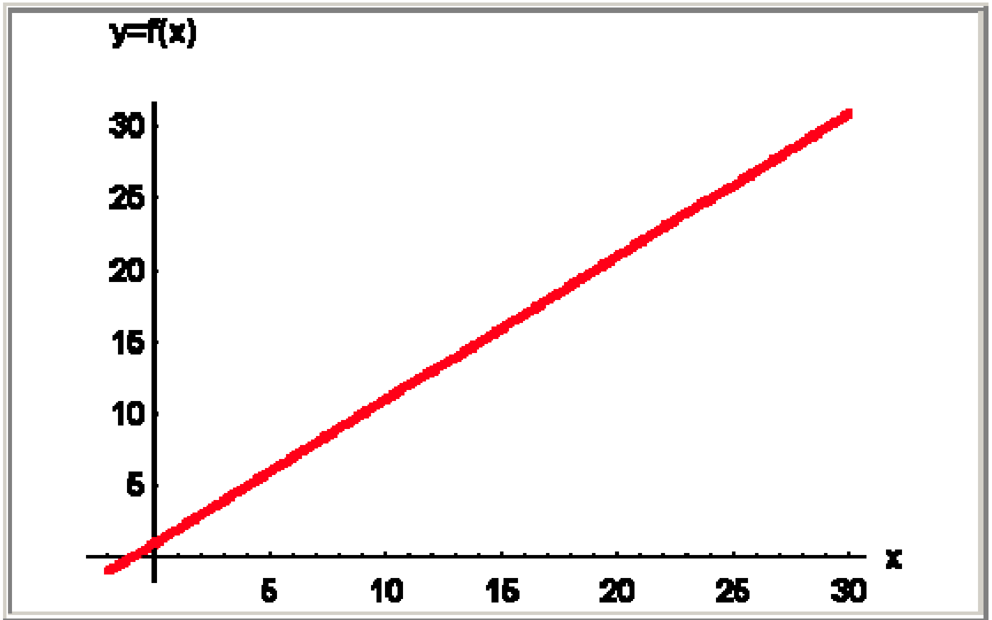
However, a function could take two or more distinct x's and associate them with the same f(x).

Consider the following function: $f(x) = x^2$

If $x = 3$, then $f(x) = 9$
If $x = -3$, then $f(x) = 9$

Example

If the points (2, 3), (4, 5), (10, 11), and (25, 26) are located on the graph of a function, you could easily figure out a corresponding rule. To get the function value, you just add 1 to the first number. The rule is $f(x) = x + 1$.



The points (3, 8) and (3, 18) could not be points on the graph of a function because there are two different function values for the same x value.

Operations with algebraic terms

Multiplying Algebraic Terms

(1) Multiply numerical coefficients together.

(2) List all the variables that occur in the terms being multiplied.

(3) Add the exponents of like variables.

$$x^2 \bullet 3x = 1 \bullet 3 \bullet x^{2+1} = 3x^3$$

In this example, the base of both terms is *x.* This means we only need to multiply the coefficient for each term, attach the variable, then add the exponents. Remember that when no exponent is explicitly stated, the exponent is assumed to be one.

$$5k^3 \bullet k^2 = 5 \bullet 1 \bullet k^{3+2} = 5k^5$$

Again, we multiply the numerical coefficients, keep the variable and add the exponents of like variables.

Both examples above contained terms with the same variables in their bases.

However, even when the bases are not the same, we follow the same procedure. We multiply the numerical coefficients and then simply list the variables from each term.

For example, if we multiply the terms $3xy$ and $5y^2$, we get the result $15xy^3$.

This works because we can view $5y^2$ as $5x^0y^2$. **NOTE**: Remember that $x^0 = 1$.

We can write this multiplication as:

$$3xy \bullet \_5y^2 = 3x^1y^1 \bullet \_5x^0y^2 = 3 \bullet \_5 \bullet \_x^{1+0 \bullet \_}y^{1+2} = 15xy^3$$

Example

Multiply the following terms.

1. $2x^2y \bullet 6x^2y$
2. $3ab \bullet 5xy^2$

Answers

First multiply the numerical coefficients, add the exponents of like variables.

1. $2x^2y \bullet 6x^2y = (2 \bullet 6)(x^2y \bullet x^2y) = 12x^{2+2}y^{1+1} = 12x^4y^2$
2. $3ab \bullet 5xy^2 = (3 \bullet 5)abxy^2 = 15abxy^2$

Dividing Algebraic Terms

The process for division of algebraic expressions is much like the multiplication process, with the exception that you divide the numerical coefficients and then subtract exponents instead of adding.

If there are numerical coefficients in the expressions to be divided, just divide the numerical coefficient and then use the exponent rule to divide the variables.

$$\frac{4x^a}{2x^b} = \frac{4}{2}x^{a-b} = 2x^{a-b}$$

To divide variables, keep the variables and subtract the exponents.

$$\frac{x^a y^c z}{x^b y^d} = x^{a-b} y^{c-d} z$$

If we divide $12xy^2$ by $3y$, we get $4xy$ as a result. This works because we can look at the term in the bottom of the fraction as being $3x^0y^1$.

$$\frac{12xy^2}{3y} = \frac{12}{3} x^{1-0}y^{2-1} = 4x^1y^1 = 4xy$$

Because we subtract exponents, it is possible to get a negative exponent. **NOTE**: When we get an answer in this form, with negative exponents, we may leave them in this form. We may also rewrite the expression according to the following rule:

$x^{-n} = 1/x^n$

Thus, if we divide $2kr$ by kr^4 we get:

$$\frac{2kr}{kr^4} = \frac{2}{1}k^{1-1}r^{1-4} = 2k^0r^{-3} = 2r^{-3} = \frac{2}{r^3}$$

Example

Divide the following terms.

1. b^6/b^3
2. k^4z^9/k^2z^5
3. $20a^4b^5 / 5a^7b^4$

Answers

1. $b^6/b^3 = b^{6-3} = b^3$
2. $k^4z^9/k^2z^5 = k^{4-2}z^{9-5} = k^2z^4$

3. $20a^4b^5 / 5a^7b^4 = (20 \div 5)a^{4-7}b^{5-4} = 4a^{-3}b$

This solution can be rewritten as $4a^{-3}b = 4b/a^3$

Adding and Subtracting Like Terms

There is a very simple property for adding and subtracting algebraic expressions. To be able to add or subtract expressions, we must have *like terms. Like terms* are terms that contain the same variable or group of variables raised to the same exponent, regardless of their numerical coefficient. For example:

- $3x$ and $6x$ are *like terms*. They both contain x.
- $6c^2$ and $19c^2$ are *like terms*. They both contain c^2.
- $2xy^3$ and $101xy^3$ are *like terms*. They both contain xy^3.
- $-km^2x^5$ and $17km^2x^5$ are *like terms*. They both contain km^2x^5.

Notice that to determine *like terms*, you must consider the variables in each term as a group. *Like terms* are those with exactly the same variables raised to the same exponent. If two terms have the same variables, but to different powers, they are not *like terms* and cannot be combined. For example:

- x^4 and $3x^2$ are not like terms since one contains x^4 and the other contains x^2 as variables.
- $5vk^3$ and vk^2 are not like terms since one contains vk^3 and the other contains vk^2 as variables.

To combine *like terms*, do the following:

1. Determine which terms contain the same variable or groups of variables raised to the same exponent.
2. Add or subtract the numerical coefficients.
3. Attach the common variables and exponents.

For example, $3x + 6x$ can be simplified to $(3 + 6)x = 9x$.

Example

If possible, simplify each of the following expressions.

1. $10a + 10b - 3a$
2. $5b^2 + 8b^3$
3. $3x^2y^2z - 5xyz + x^2y^2z$

Answers

1. $10a + 10b - 3a = 7a + 10b$
2. $5b^2 + 8b^3 = 5b^2 + 8b^3$
3. $3x^2y^2z - 5xyz + x^2y^2z = 4x^2y^2z - 5xyz$

Mathematical properties of real numbers

When you manipulate algebraic *expressions* and *equations,* you will need to understand the underlying properties of real numbers; these include the Commutative Properties, Associative Properties, Distributive Property, and Reflexive Property.

The Commutative Properties

These properties tell us that the order of numbers does not matter when performing addition or multiplication. The word commutative comes from the word commute, which means to move around, exchange, and change order.

1. *a + b = b + a* Commutative property of addition

For example: If you were to add 3 and 5, you would get 8. Likewise, if you add 5 and 3, you will also get 8. By the commutative property, we know that the order of terms can be reversed.

$5 + 3 = 3 + 5$
$8 = 8$

2. *a • b = b • a* Commutative property of multiplication

For example, multiplying 4 by −12 yields −48, as does multiplying −12 by 4.
$(-12)(4) = (4)(-12)$

$-48 = -48$
NOTE: the commutative property does *not* hold for subtraction or division. For example:

Compare: $5-3 = 2$ and $3-5 = -2$
$2 \neq -2$

Compare: $1 \div 2 = 1/2$ and $2 \div 1 = 2/1 = 2$
$1/2 \neq 2$

The Associative Properties

The associative properties also hold only for addition and multiplication. The associative properties demonstrate that the order in which terms are grouped does not matter. The word associative is derived from associate, which means to join together, connect, combine, or unite.

1. $a + (b + c) = (a + b) + c$ Associative property of addition

For example, if you add 7 to the sum of 2 and 10 you will obtain 19. You will get the same answer if you first add 7 and 2 together, and then add 10.

$7 + (2+10) = 7 + 12 = 19$
and
$(7 + 2) + 10 = 9 + 10 = 19$

The same applies when we add algebraic terms, as shown below.

$(2x + 3x) + 4x = 5x + 4x = 9x$
and
$2x + (3x + 4x) = 2x + 7x = 9x$

2. $a \bullet (b \bullet c) = (a \bullet b) \bullet c$ Associative property of multiplication

There is also an associative property of multiplication. It is similar in principle to the associative property of addition. For example, if we multiply 4 by 3, then multiply the result by 5, we get 60. We get the same result when we multiply 5 by 4, then multiply the result by 3.

$5 \bullet (4 \bullet 3) = 5 \bullet 12 = 60$
and
$(5 \bullet 4) \bullet 3 = 20 \bullet 3 = 60$

The same principles apply in multiplication of algebraic terms. For example, if we multiply 5 by the product of $4x$ by $3z$, we get the same result if we multiply 5 by $4x$, then multiply the result by 3*z*.

$5 \bullet (4x \bullet 3z) = 5 \bullet 12xz = 60xz$
and
$(5 \bullet 4x) \bullet 3z = 20x \bullet 3z = 60xz$

NOTE: The associative properties do *not* apply to subtraction or division. Observe what happens when we try to apply this property to both a division and subtraction problem.

Compare: $(4 \div 2) \div 3 = 2 \div 3 = 2/3$ and $4 \div (2 \div 3) = 4 \div 2/3 = 12/2 = 6$
$2/3 \neq 6$

Compare: $(5 - 1) - 3 = 4 - 3 = 1$ and $5 - (1 - 3) = 5 - (-2) = 7$
$1 \neq 7$

The Distributive Property This property tells us that we may distribute, or apply by multiplication, a term outside the parentheses to each term within the parentheses.

$\boldsymbol{a(b + c) = ab + ac}$

For example, if we multiply 3 by the sum of 5 and 2 we get the same result as if we multiply 3 by 5 then add this to the result of multiplying 3 by 2.

$3(5+2) = 3 \bullet 7 = 21$
or
$3 \bullet 5 + 3 \bullet 2 = 15+6 = 21$

In essence, the process of distributing a term from outside a set of parentheses to terms inside the parentheses is the opposite of factoring out a common term.

This is demonstrated below with equations that contain only numerical terms.

$3 \bullet 7+3 \bullet 13$ is equal to $3(7+13)$
$3(7+13) = 3(20) = 60$

If we are given an algebraic equation, this is applied as shown below. Note that in the second expression, the common factor of 4 is taken out.

$4x+4y$ is equal to $4(x+y)$

When performing the distributive property, we are putting a common term back in, rather than taking it out. You may wonder why we would ever want to do this. The answer is that often it is easier to perform two smaller multiplication operations and then take the sum, than it is to perform one large multiplication.

For example, suppose you are asked to multiply 13 by 14. Instead of reaching for the calculator, you can use mental math to multiply 13 (10 + 4). Note that this is still 13 by 14.

13•10 + 13• 4 = 130 + 52 = 182

When we have subtraction within the parentheses, the property still holds. For example, suppose you are buying six notebooks for $1.97 each. The total price is then 6(1.97). A simpler way to compute this may be to rewrite 1.97 as (2.00 - .03). Now find the total price by using the distributive property.

T = 6(1.97) = 6(2.00-.03) =
6(2.00) – 6(.03) =
12 - .18 = $11.82

When working with algebraic expressions, factoring may often be more useful in simplifying complex expressions than distributing the common factor to all terms. For example, if we are given the expression 3x + 3y)/3, this expression can be simplified using factoring.

In this case, the 3 on the top and bottom of the fraction cancel each other out.

$$\frac{3x+3y}{3} = \frac{\not{3}(x+y)}{\not{3}} = x+y$$

Reflexive Property

This property says that if two expressions are set equal to one another, it does not matter in which order they are presented.

***If* a = b, *then* b = a**

For example, if we are given that 5 is equal to x, then it is also true that x is equal to 5.

If *x* = 5, then 5 = *x*

This also holds for any algebraic equation.

If 2*x* +4*z* = *y*, then *y* = 2*x* +4*z*

This property is commonly used when working with algebraic equations. For example, if we are adding two equations $x + 2z = 23$ and $16 = x + 5z$, we write these as:

$x + 2z = 23$
$\underline{+ (16 = x + 5z)}$

But it would be more advantageous to write these as:
$x + 2z = 23$
$\underline{+ (x + 5z = 16)}$

The reflexive property shows us that these two representations are equivalent.

Order of Operations

The final tool you will need to evaluate complex expressions is the order of operations. The order in which you perform these steps makes a difference. There is a systematic way to evaluate expressions to ensure you do it correctly.

The *order of operations* tells us a step-by-step method for evaluating expressions:

1. Evaluate all expressions within **parentheses** and other grouping symbols.
2. Evaluate all expressions involving **exponents**.
3. Do the remaining **multiplication** and **division**, *as you come to them*, when working from left to right in the expression. (See below for a discussion of this step.)
4. Do the remaining **addition** and **subtraction**, *as you come to them*, when working from left to right in the expression. (See below for a discussion of this step.)

One trick to help remember the correct order of operations is to think of the acronym **PE[MD][AS]**. This stands for **Parentheses, Exponents, [Multiplication and Division], [Addition and Subtraction].**

For example, suppose you want to evaluate a simple expression. Look at the example shown below. If we follow the order of operations, we get a result of 16.

$$6 + 2 \text{ x } 5 = 6 + 10 = 16$$

If we simply move left to right, ignoring the order of operations, we get a result of 40.

$$6 + 2 \text{ x } 5 = 8 \text{ x } 5 = 40$$

Thus, it is important to follow the order of operations.

When dealing with algebraic expressions, or terms that contain variables, we follow the same order of operations. It is important to begin within the parentheses (or other grouping symbols) and follow the rest of the steps from there.

Example
Evaluate the following expressions.

1. $[(10 \bullet 2 + 8) \div 14]^2$
2. $12x^2y \div 4x + (6x - x)^2$
3. $(x + z) + (5x \bullet 2)$
4. $\{4\ [2xz^4\ (2c^3v \div 2cv^2)^3 \div 2]\} \div c^6x \bullet v^3$

Answers

1. $[(10 \bullet 2 + 8) \div 14]^2 = 4$
2. $12x^2y \div 4x + (6x - x)^2 = 3xy + 25x^2$
3. $(x + z) + (5x \bullet 2) = 11x + z$
4. $\{4\ [2xz^4\ (2c^3v \div 2cv^2)^3 \div 2]\} \div c^6x \bullet v^3 = 4z^4$