50 Chess and Mathematics Exercises for Schools

A (chess) game-based approach to problem solving

Erasmus+ Contract No. 2017-1-SK01-KA201-035424
with EU financial support.

INTRODUCTION

This book provides material for the much-sought-after link between chess and mathematics for the classroom. We have fully tried out all these exercises and found that most children are enthusiastic – often more so even than their teachers! Chess is a classic board game that children enjoy at all levels. We use the chessboard and the chess pieces to convey mathematical insights consistent with the syllabus for primary school (i.e. children from age 6-11) mathematics in most countries. Only a basic knowledge of chess is required – how the pieces move. It is not necessary to be a chess player to use this book. The main emphasis is on problem solving.

The 50 exercises are categorised by age and by the most natural grouping – individuals, pairs, quads (two pairs), groups or the whole class. The relevant topic in the mathematics syllabus is also displayed. Most exercises have preliminary questions or extensions. Solving the exercises should normally take no more than a lesson period. The materials required are minimal. Chess sets can be used but the exercises work just as well using 8x8 grids and coloured counters or markers. Printouts of board positions are useful for several exercises.

Gradually work your way through the book at the pace of the children. Some children will far exceed the stated age range. Problem-solving requires persistence. The children are asked to complete a task or conduct an investigation or play a game. Hints are often provided to overcome intellectual hurdles. A solution method is provided for each problem but there is always more than one solution method – and children should be encouraged to think for themselves. The teacher can develop their own preliminary and extension exercises to suit the ability of the children. The later exercises require a higher level of abstraction and solutions are prone to error hence requiring more teacher intervention.

The mathematical games are simple and easy to play. They illustrate some fundamental concepts such as parity and symmetry. The point of these games is not to win but to understand the underlying concepts. Children are fascinated to discover that many of these games can be won by recognising that there is an underlying pattern. This is part of the realisation that mathematics provides an underlying pattern to scientific laws. From a didactic perspective, children are delighted to learn “tricks” how to win a game or solve a puzzle.

These exercises are drawn from traditional sources and supplemented by some original exercises from the authors. Many of the exercises will be new to recreational mathematicians. What marks them as distinctive is that they are developed within a constrained domain of the chessboard to create a sense of familiarity as well as accessibility.

The primary school reference syllabus is from Singapore because this is regarded as being close to the problem-solving methodology promoted by this project. The Singapore “method” has gained widespread international attention following its success in the OECD Pisa rankings. The primary school syllabus is consistent with those in Europe as validated by the teaching experts on the project. In addition to mathematics, many of these exercises incorporate some principles of Game Theory which although not normally taught in primary school is within the capability of most children.

Perhaps the greatest value in working through these exercises comes from having to structure the problems. A well-structured problem is a delight to solve. Teachers are encouraged to develop and expand these exercises and to share their classroom experiences.

This project was funded through ErasmusPlus from the European Union. The partners on the CHAMPS project were the Slovak Chess Federation (co-ordinator), Chess in Schools and Communities (UK), Ludus (Portugal), University of Girona (Spain) and Veľká Ida Primary School, Slovakia. The following individuals also contributed to the project: Carlos Santos, Carme Saurina Canals, Josep Serra, Mark Szavin, Stefan Löffler, Alessandro Dominici, Malcolm Pein, Chris Fegan, Zdenek Gregor, Eva Repkova, Vladimir Szucs, Viera Kebluskova and Niki Vrbova. The project manager was Stefan Marsina. The chessboard diagrams were generated using the Logiq Board from LearningChess.net (Hungary).

John Foley *(@ChessScholar)*
Rita Atkins
Carlos Santos
Viera Haraštová

March 2019

©Copyright the authors.
This work is licensed under the Creative Commons Attribution-NonCommercial-ShareAlike 4.0 International License. See http://creativecommons.org/licenses/by-nc-sa/4.0/
Educators may use, reuse, adapt and share this material on a non-commercial basis provided that attribution is given to the authors.

age

The 50 Exercises

			Page	
1.	Each Square Has A Name	Individual	1	6
2.	Identify The Square Colour	Whole Class	2	
3.	Piece Categorisation	Groups	3	
4.	Cross The Line Game	Pairs	4	
5.	Diagonal Opposition Game	Pairs	5	
6.	Chess Arithmetic	Individual	6	
7.	Four Rooks Puzzle	Pairs	7	7
8.	The Fallen Pieces	Quads	8	
9.	Corner Attack Puzzle	Quads	9	
10.	Piece Power Contours	Quads	10	
11.	Avoid Three In A Line	Individual	11	8
12.	Counters On A Line [11]	Pairs	12	
13.	Wythoff's Game [Q]	Pairs	13	
14.	Rook Corner Puzzle	Pairs	14	
15.	Shortest Rook Tour	Individual	15	
16.	Race To The Corner [Rook]	Pairs	16	
17.	Chomp	Pairs	17	
18.	Army Power Puzzle	Pairs	18	
19.	Maximum Number Of Knights	Pairs	19	
20.	Mystery Combination Piece	Pairs	20	
21.	Positional Logic	Pairs	21	
22.	Counters On A Line [16]	Pairs	22	9
23.	Minimum Moves Map	Pairs	23	
24.	How Many Routes? [Pawn]	Pairs	24	
25.	The Light Pawn	Pairs	25	
26.	Wythoff's Game [QQ]	Pairs	26	
27.	The Domino Tiling Puzzle	Pairs	27	
28.	The Tromino Tiling Puzzle	Pairs	28	
29.	Two Queens Puzzle	Quads	29	
30.	Equidistant Checkmate	Pairs	30	
31.	Subtraction Game	Pairs	31	
32.	Nim	Pairs	32	10
33.	Golden Coin Game	Pairs	33	
34.	Northcott's Game	Pairs	34	
35.	Sliding Rooks Game	Pairs	35	
36.	How Many Routes? [Rook]	Pairs	36	
37.	Rook Tour On A Mutilated Board	Pairs	37	
38.	Bishop Zigzag Puzzle	Pairs	38	
39.	Five Queens Puzzle	Quads	39	
40.	Dividing The Board	Individual	40	
41.	Tiling The Board	Individual	41	
42.	Twelve Knights Problem	Pairs	42	
43.	Eight Queens Puzzle	Pairs	43	
44.	How Many Squares On A Board?	Pairs	44	
45.	Tournament Scoring Puzzle	Individual	45	
46.	Kings Random Jump	Pairs	46	
47.	King's Random Walk	Pairs	47	11
48.	The Invention Of Chess	Group	48	
49.	How Many Rectangles on a Board?	Group	49	
50.	Leaper Problem	Pairs	50	

MATHEMATICAL TOPICS

1. CO-ORDINATES, POSITIONS, MOVEMENTS, PENCIL
2. PARITY, VISUALISATION, PENCIL
3. PATTERNS, SORTING, ORDER, SEQUENCES, PENCIL
4. PARITY, SYMMETRY
5. PARITY, SYMMETRY
6. ARITHMETIC, SYMBOLS, EQUATIONS, PENCIL
7. GEOMETRY, SPATIAL NOTIONS, ENUMERATION, INTERSECTION
8. ENUMERATION, PIGEONHOLE PRINCIPLE, MAXIMUM/MINIMUM
9. ARITHMETIC, TRIAL AND ERROR, INPUT/OUTPUT
10. ENUMERATION, SYMMETRY
11. GEOMETRY, STRAIGHT LINES, SLOPE, RULER
12. GEOMETRY, STRAIGHT LINES, SLOPE, RULER, PENCIL
13. WORKING BACKWARDS FROM THE TARGET
14. PARITY, PENCIL
15. ENUMERATION
16. SYMMETRY
17. SYMMETRY, ELIMINATION
18. SHAPES, SYMMETRY
19. ENUMERATION, VENN DIAGRAMS, TRIAL AND ERROR
20. ENUMERATION, PARITY
21. DISJOINT PROPERTIES, UNION, SYMMETRY
22. LOGIC
23. STRAIGHT LINES, SLOPES, TRIAL AND ERROR
24. ENUMERATION, SPATIAL NOTIONS
25. ENUMERATION, SPATIAL NOTIONS
26. LOGIC, INFORMATION, TREE DIAGRAMS
27. SYMMETRY, WORKING BACKWARDS FROM THE TARGET
28. SHAPES, PARITY
29. TRIAL AND ERROR, SYMMETRY, MULTIPLES, DIVIDE TO CONQUER
30. ENUMERATION, TRIAL AND ERROR
31. GEOMETRY, DISTANCE MEASURES, PEER LEARNING
32. SYMMETRY, MULTIPLES
33. SYMMETRY, MULTIPLES, POWERS
34. SYMMETRY, MULTIPLES, TREE DIAGRAMS, WORKING BACKWARDS
35. SYMMETRY, REPRESENTATION
36. SYMMETRY, PARITY, WORKING BACKWARDS
37. ENUMERATION, PASCAL'S TRIANGLE
38. ENUMERATION, PASCAL'S TRIANGLE
39. ENUMERATION, PASCAL'S TRIANGLE
40. DECOMPOSITION OF SHAPES, TRIAL AND ERROR
41. DECOMPOSITION OF SHAPES, ENUMERATION, TRIAL AND ERROR, AREA MEASURES, SQUARE NUMBERS
42. ENUMERATION, TRIAL AND ERROR
43. ENUMERATION, TRIAL AND ERROR
44. ENUMERATION, SHAPES, ARITHMETIC, ORGANISING INFORMATION IN TABLES
45. LOGIC, ORGANISING INFORMATION IN TABLES
46. ENUMERATION, TRIAL AND ERROR, SYMMETRY
47. ENUMERATION, PROPORTION, ORGANISING INFORMATION IN TABLES
48. EXPONENTIAL GROWTH, GEOMETRIC SEQUENCE
49. ENUMERATION, SHAPES, ORGANISING INFORMATION IN TABLES
50. ENUMERATION, SYMMETRY, ANGLES

1. Each Square has a Name — Individual

Age 6+	*Co-ordinates, positions, movements, pencil*

This exercise is fundamental to understanding location on the board.

Hands-on exercise

Explain each square is named according to its co-ordinates:
column (file) letter and row (rank) number. Example b3 in diagram:
Walk up the street (b) to the house number 3.
For small children, draw a line from the square to its co-ordinates.

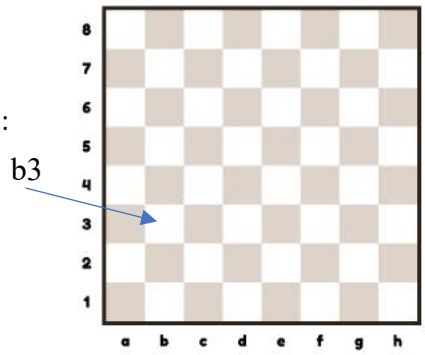


Figure 1: Empty Board

Hand out printouts of a chessboard. Use pencils.

Task: Write the name of each square on the chessboard.

Games

(a) Teacher calls out names of squares and ask class to put a piece on them (not easy at first).
(b) Trace the route (e.g. a1-a5)
(c) Make a movement (e.g. c8-h3) and ask class for the co-ordinates
(d) Place a rook on a1 and ask class how it can get to f5
(e) Print a maze on the chessboard and ask for the co-ordinates for the escape route
(f) Using Velcro strips, get children to place pieces on the squares blindfold using only touch.

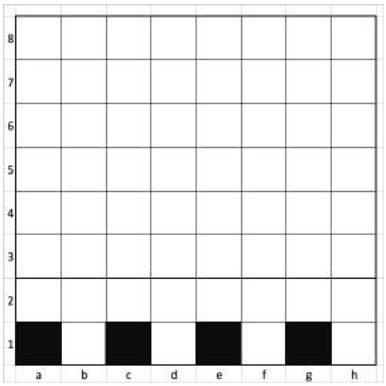
Naming the squares with a blindfold

2. Identify the Square Colour — Whole Class

Age 6+	*Parity, Visualisation, Pencil*

Hand out a sheet with only the first row shaded.
Explain alternating shading.

Task: complete the chessboard pattern.



Whole class exercise

Teacher faces away from an empty chess display board and asks the children to call out a square. Teacher identifies the colour of the square e.g. c4 is white. Repeat with the children

Class Discussion: What methods can you use to arrive at the answer?

Possible methods:

(a) **Snake Pattern**: a1 = black, b1 = white, c1 = black, …h1 = white, h2 = black, g2 = white etc.

(b) **Diagonals:** The long diagonals (Black a1-h8; White a8-h1) travel through the board. Find the nearest square to the one you need and adjust colour accordingly.

(c) **Parity addition** (age 11+) Relabel the rows from a, b, c … to 1, 2, 3 and add the co-ordinates of the required square. If the sum is even, then the square is black.

Extension

Visualisation (age 8+)

Pupils close their eyes and raise their right hand for white or left hand for black square called out.

Photo: Children answering c4

3. Piece Categorisation — Groups

Age 6+	*Patterns, sorting, order, sequences, pencil*

Hands-on exercise

Show the class that the pawns are all the same size and smaller than the pieces
Show that the physical pieces get shorter the closer they are to the corner

♚ = ♛ > ♝ > ♞ > ♜

Tasks: find

(a) 5 ways to classify chess pieces e.g. according to colour, whether they slide or jump etc.
(b) 5 ways to order chess pieces e.g. according to their exchange value, size of their base etc.

Using piece tokens makes sorting quicker.

Extension

Teacher shows children groups of pieces and asks class to explain the grouping principle

Solution Method

A combination of approaches is required to generate ideas

- Brainstorming involves calling out ideas in a group
- Compare and contrast two piece types
- Discuss how a third piece type differs from another two types
- Analyse the pieces alone, in their starting position, or during play.

Answer

Classification

- ☐ Colour of piece
- ☐ Can/cannot move from starting position
- ☐ Colour of starting square occupied
- ☐ Long-range v short-range
- ☐ Heavy v light
- ☐ Pawns v pieces
- ☐ Major v minor
- ☐ Sliders/Jumpers
- ☐ Whether they can stand upside down

Ordering

- ☐ Height
- ☐ Diameter of base
- ☐ Exchange value
- ☐ Number of pieces of each type
- ☐ Fastest to other side of board from unblocked starting square
- ☐ Alphabetical (language dependent)
- ☐ Number of piece-type that can fit onto one playing square

Solutions offered by children:

4. Cross the Line Game — Pairs

Age 6+	*Parity, symmetry*

Place the two kings on their starting squares. The kings move as in chess. White goes first. Who wins – the first player or the second player? – in each of these positions.

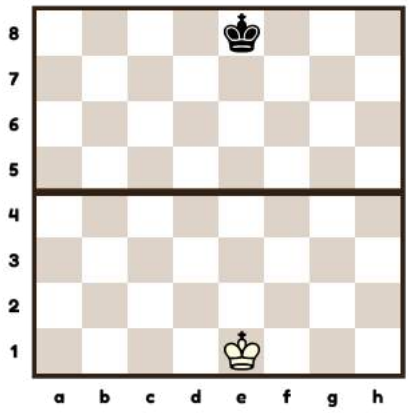


Figure 4(a) The first player to cross the middle line is the winner

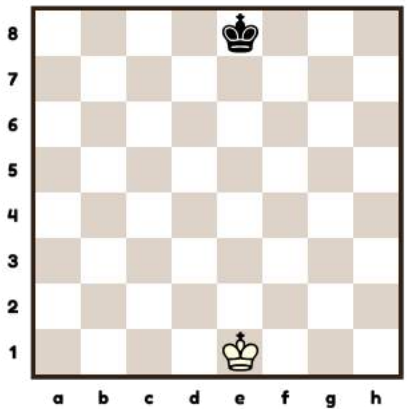


Figure 4(b) White wins by reaching the 8th rank.

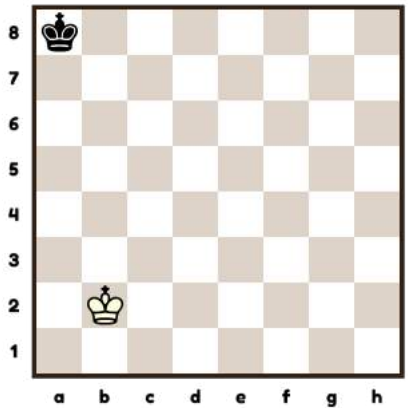


Figure 4(b) Black to play and prevent White from reaching the 8th rank

Game Strategy

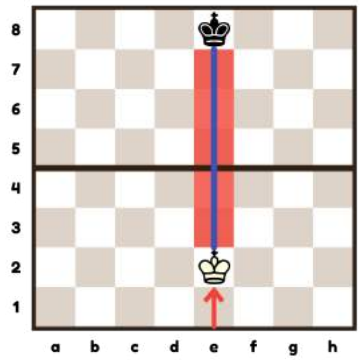
This game is related to the concept of Opposition in chess. A direct opposition occurs on a file when two kings face each other with only one square between them. The distant opposition occurs on a file when two kings face each other with an odd number of squares (>1) between them. Thus, this game is closely related to the concepts of parity and symmetry.

Answers

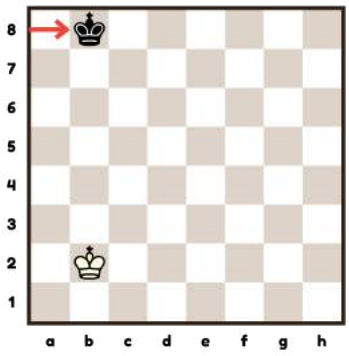
4(a) If White goes first, 1. ♔e2 is the winning move. If we observe the blue mirror line, Black has to be the first to choose one side, leaving the way open for White to choose the other. If Black tries to stay on the mirror line, the fact that there is an odd number of squares between the kings proves to be the determining factor for White victory.

4(b) The best move is the same as in 4(a)

4(c) Black wins by keeping the distant opposition with 1.. ♚b8



4(a) and 4(b) The best move for White

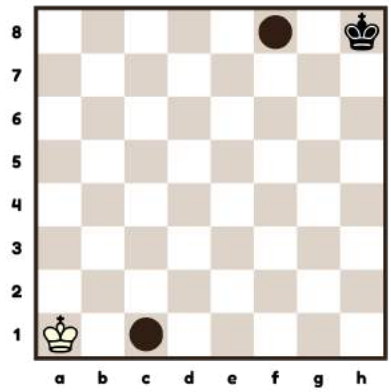


5. The Diagonal Opposition Game — Pairs

Age 6+	*Parity, symmetry*

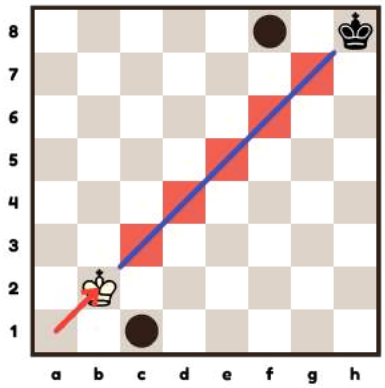
Place the two kings on diagonally opposite corners.
The kings move as in chess. White goes first.
The first player to occupy the starting square of the opposing king or the marked square on the other side wins.

Who wins - the first player or the second player?



Game Strategy

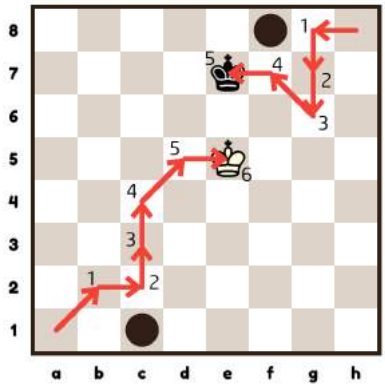
Opposition along a diagonal (instead of a rank or file) is called diagonal opposition.
This concept is the main idea behind this game.



Answer

1.Kb2 is the winning move. If we observe the blue mirror line, as in the previous game, Black has to be the first to choose one side, leaving the way open for White to choose the other.

Note that, at some point, the diagonal opposition may turn into direct opposition. Consequently, the pupil can use previously learned knowledge in a new situation - something important in mathematics.



6. Chess Arithmetic — Individuals

Age 7+	*Arithmetic, symbols, equations, pencil*

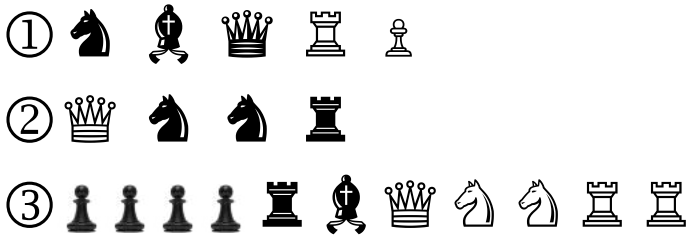
Explain the conventional points value for each chess piece.

♕ = 9
♖ = 5
♗ = 3
♘ = 3
♙ = 1

Explain: ♕ = ♖ + ♗ + ♙ because 9 = 5 + 3 + 1

Tasks:

(a) ♘ + ♜ + ♝ = ?
(b) ♕ = ♗ + ♗ + ?
(c) Can you find four pieces that add together to the same value as a queen?
What difference would it make if the queen was worth 10?
(d) These are the captured pieces during a game, which side is leading in material, black or white?



Answers

(a) 9
(b) 3
(c) Not possible because the pieces all have odd numbers, assuming the king is not worth 0.
It is possible to find four pieces adding to 10 e.g. ♗ + ♗ + ♗ + ♙
(d)
① White
② White
③ Black

7. Four rooks puzzle — Pairs

Age 7+	*Geometry, spatial notions, enumeration, intersection*

Tasks:

(a) Place four rooks on black squares so that all the white squares are attacked.

(b) Find another position to achieve the same task.

Advise the children to check their solutions.

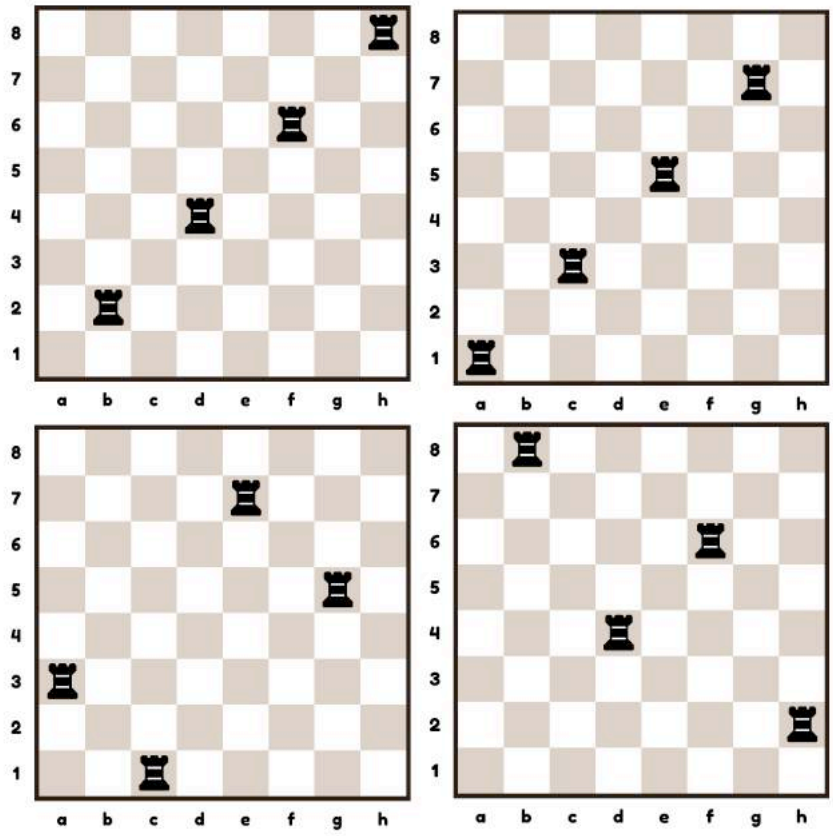
Solution Method

The chessboard has 32 white squares. Each rook covers 8 white squares (4x8 = 32). Note that if a rook stands on a white square it would only cover 6 other white squares. Standing on a black square a rook can cover 8 white squares. Therefore, we are seeking four black squares on which to place the rooks.

If the rooks stand on black squares in adjacent lines e.g. on a1 and b2, then the coverage of the rooks would intersect on the white squares b1 and a2 which means that instead of covering 16 white squares between them, they would only cover 14 white squares, which is insufficient. By inspection we find that the rooks should be an even number of squares away from each other given that their intersections need to be restricted to black squares.

Hint: Start by finding solutions on the long black diagonal. Then find the other solutions.

Answer



8. The Fallen Pieces — Quads

Age 7+	*Enumeration, pigeonhole principle, maximum/minimum*

After chess club there are many chess pieces on the floor. You pick them up and count them. How many chess sets did they come?

(a) There are 17 black pawns

(b) In addition to the 17 black pawns you also find 3 white knights.

(c) The classroom has only just enough chess sets for 20 children to play in pairs. Investigate the maximum solutions for each scenario above.

Solution Method

Hands-on investigation using counters.
Explain meaning of maximum and minimum and ask to find these for each question.
Note that we are not talking about probable an outcome is, only if it is possible.

Answer

The answer should be a range: one or more pieces could have come from each set. It is important to be aware that not all answers are single numbers.-

(a) Between 3 and 17
At least 3 sets are required because a set contains only 8 pawns. If one pawn belongs to each set, then we require the maximum number of sets.

(b) Between 3 and 20
We do not need any more than 3 sets because they already include 3 white knights. 20 pieces are missing which could have come from 20 sets at worst. The colour is irrelevant.

(c) Between 3 and 10
The minimum number does not change. The maximum number is restricted by the requirement that the number of boards is 10 (i.e. 20÷2).

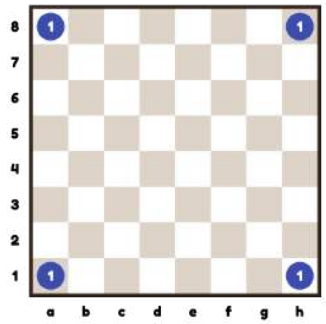
9. Corner attack puzzle — Quads

Age 7+	Arithmetic, trial and error, input/output

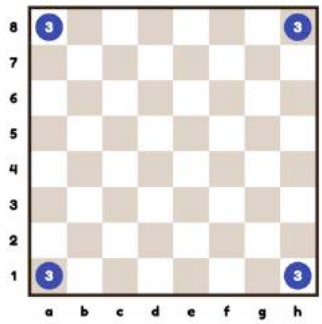
Each corner of an otherwise empty board is occupied by an unknown chess piece. These pieces may be under attack from one or more unknown pieces in the other corners. Ignore colours.

Each of the corner squares contains a number. This is the total number of attacks on that square from the pieces in the other corner squares. Each piece attacks at least one other piece.

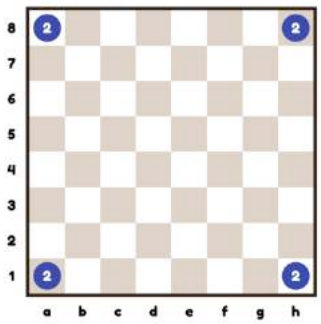
Hands-on tasks: Deduce what the four corner pieces are from the information given below:



Task 9(a)



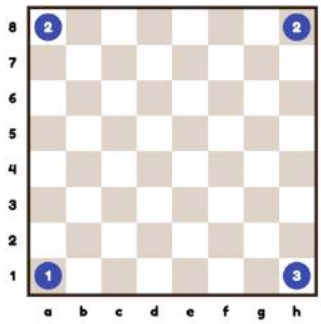
Task 9(b)



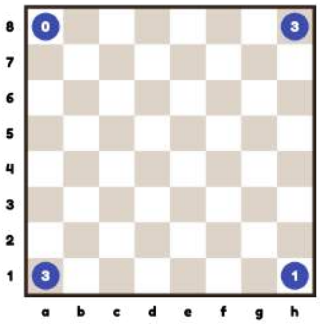
Task 9(c)

(d) Is there any other answer to 9(c)?

Extension

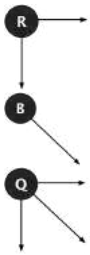


Task 9(e)



Task 9(f)

Solution Method



- Establish that the only relevant pieces are the long-distance pieces ♗ ♖ ♕
- Establish that the number of squares attacked by each piece is ♗=1 ♖=2 ♕=3
- Explain that no. of attacks output equals no. of attacks input
  i.e. the sum of the piece attack values = sum of the corner numbers
  Note that there is only one way to obtain totals of 4 and 12,
  being minimum and maximum respectively.

Answers

(a) ♗ ♗ ♗ ♗
(b) ♕ ♕ ♕ ♕
(c) ♖ ♖ ♖ ♖
(d) ♗ ♗ ♕ ♕
(e) ♗ ♖ ♕ ♖
(f) ♕ ♗ ♗ ♖

10. Piece power contours — Quads

Age 7+	*Enumeration, symmetry*

The power of a piece varies according to its position on the board. Its power is the number of squares the piece attacks.

Hands-on Task: (using coloured pencils on a printed board or coloured counters on a board)

Colour-code each square according to the number of squares attacked from that square.
The resulting pattern is the "power contour" for that piece. Find the contours for:

- Rook
- Bishop
- Queen

Which piece is depicted by these power contours?

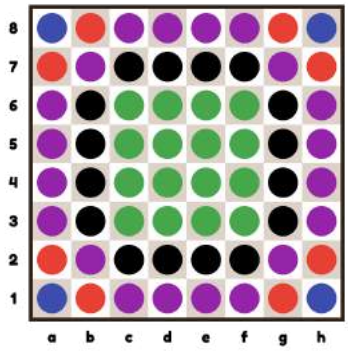


Figure 10(a)

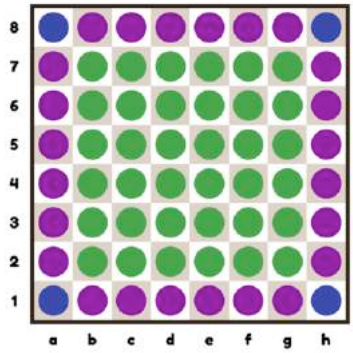
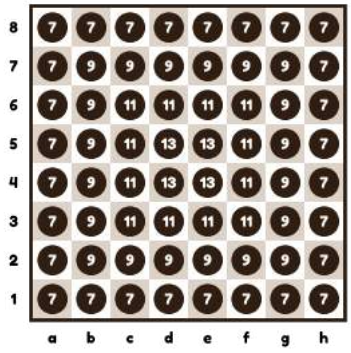


Figure 10(b)

Answers

Rook	14 uniform distribution
Bishop	See diagram
Queen	= Bishop + Rook
	Add 14 to each square of the bishop contour

10(a) Knight
10(b) King



Bishop Power Contour

11. Avoid Three in a Line Game — Pairs

Age 8+	*Geometry, straight lines, slope, ruler*

Starting with an empty chessboard, two players take turns to place same-colour counters on the board. A player loses the game by making a straight line of three counters. There is no obligation to notify your opponent that you have reached a losing condition. Use a ruler placed through the centre of the counters to check for straight lines. The game is easy enough but there may be confusion about how to find out if a third counter is on the same line as the first two.

Playing method

Check the board after each move. A counter may be on a straight line of three even if it is some distance away.

Explain that there are different directions for straight lines. The straight lines used in the sliding chess moves of the rook are the easiest to recognise:

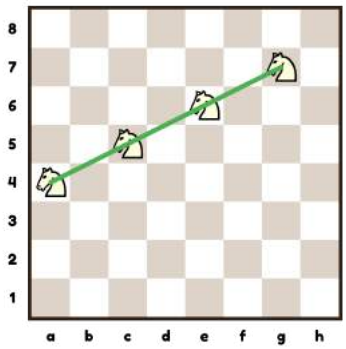
+ Rook (Orthogonal)
- vertical (up, down)
- horizontal (left, right)

The straight lines used in the sliding chess moves of the bishop are also easy to recognise:

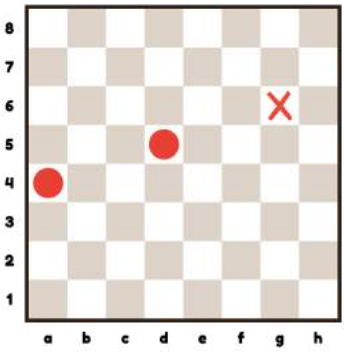
x Bishop (Diagonal)
- slope bottom left to top right (slope = 1)
- slope top left to bottom right (slope = -1)

The slope between any two counters is the ratio between the vertical distance and the horizontal distance. If the slope between two counters is the same as the slope from them to a third counter, then the counters are on the same straight line. Lines pointing up to the right have a positive slope and those pointing down to the right have a negative slope.

The knight moves in a 2:1 L-shape. Straight lines also radiate from the knight but these are harder to recognise. The slope is important when considering the knight. In the diagram on the right, the knights all appear on the same line and their slope is ½ because the vertical distance (1) is divided by the horizontal distance (2).



The slope of the knight move is 1/2



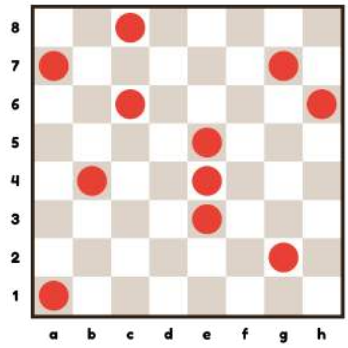
Placing a counter on g6 loses

In the diagram on the left, there are two counters (a4, d5) with a vertical (up) distance of 1 and a horizontal (across) distance of 3 (like a long reach knight). Hence their ratio is 1/3. By inspection, the square g6 is also on a 1/3 slope from d5. If a4-d5 is a 1/3 slope and d5-g6 is a 1/3 slope, then a line runs through the three squares a4-d5-g6. The player to moves should select another square than g6.

12. Counters on a Line [11] Pairs

Age 8+	*Geometry, straight lines, slope, ruler, pencil*

Hand out printouts of a chessboard with 11 counters in the pattern shown.



Tasks

(a) How many sets of three counters on a straight line can you find?

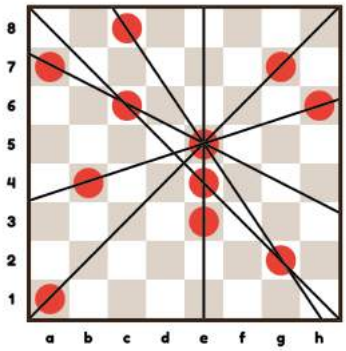
(b) Put a twelfth counter on the board and create four more sets of three in a line.

It is easier to complete this exercise by first playing the game in Exercise 11.

Solution Method

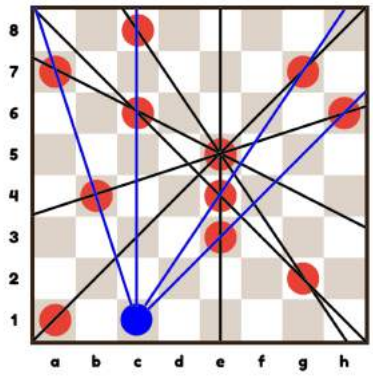
(a) The children can find the vertical and diagonal lines easily (a7-c6-e5; e5-e4-e3; g7-e5-a1). It is more difficult for them to find lines whose slope is other than diagonal.

Place a ruler between the centre of any two counters. Ensure the ruler goes through the centres. Find if there is a third counter along the straight edge. Test for all pairs of counters.



Alternatively, find the slope of the line between any two points. The slope is the ratio between the vertical and horizontal distance from one counter to the next e.g. two up and three across. Then from the second point find if there is another point which has the same slope. If so, then all three points are on the same line. Test using a ruler (more details on Exercise 11).

(b) Move systematically counter-by-counter and, using a ruler, check if you can find some lines. Children tend lack a systematic approach.



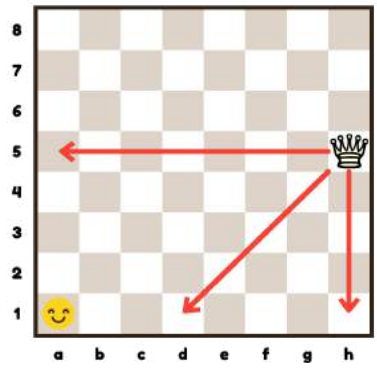
Hint: start at a1 and move to the nearest counter.
From c1, the lines c1-c6-c8; c1-e3-h6; c1-b4-a7; c1-e4-g7 radiate.

13. Wythoff's Game [Q] Pairs

Age 8+	*Working backwards from the target*

Place a queen on h5 on an empty chessboard. This queen can only move West, South West or South. The players take turns moving the queen. The first player to move the queen to a1 is the winner.

Who wins – the first player or the second player?

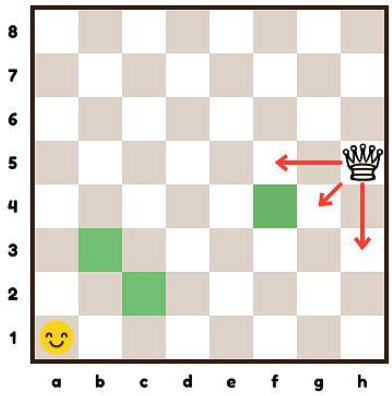


Solution Method

First play the game to get a feeling of possible best play.

Identify the 'safe squares' where you would like to move to in order to win. To do this, work backwards from a1. Use counters to mark the safe squares.

There are three safe squares: b3, c2 and f4 (marked green).



Answer

The second player wins with best play.

On their first move, the first player cannot avoid allowing the second player to either reach a safe square or go directly to a1.

14. Rook Corner Puzzle — Pairs

Age 8+	*Parity, pencil*

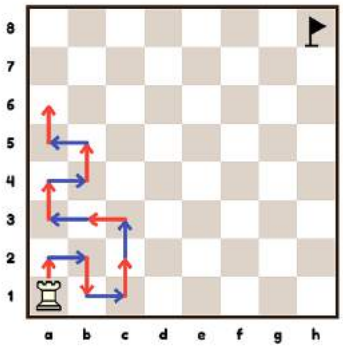
Use pencil and paper with many prints of a chessboard with a rook on a1.
Task: The rook must visit all the squares on the board and finish on h8.
Condition: The rook is not allowed to visit the same square twice.

Solution Method

Having tried a few different routes the children will conclude that there is no solution. They may need some convincing to grasp that failing to find a solution does not mean that there is none to be found. Proving that something is impossible is a giant leap in children's mathematical understanding.

Hints and leading questions to guide towards the solution:

- The rook should proceed in one-step moves on its journey. Use two colours: e.g. red for odd (1st, 3rd, 5th etc.) and blue for even moves as shown in the diagram.
- How many squares does the rook visit on its way to h8?
- What is the colour of the 1st, 2nd, 3rd, 4th etc. squares?
- What is the colour of the 63rd square?
- What is the colour of the h8 target square?



Answer

Every odd visited square is a light square, so the 63rd last square should also be light, but it is in fact a dark square so the rook can never complete the mission.

Extension

Question: On what size boards would the rook tour work?
Answer: The task is only possible for odd sized boards (7x7, 5x5 3x3 etc.)

15. Shortest Rook Tour — Individual

Age 8+	*Enumeration*

Use pencil and paper with many prints of a chessboard with a rook on a1.
Alternatively, use plastic sheets over a board template. Show a rook on a1 and a marker on h8.
Task: The rook must visit all the squares on the board and return to a1.
Condition: The rook is not allowed to visit the same square twice.

Suggested activities:

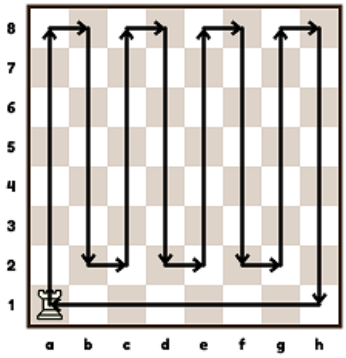
- Find as many ways as you can to complete the rook's tour.
- Draw each route on paper.
- Find the length of each tour that you have drawn.

Note that the squares passed through by the rook *en route* from one stopping place to another are 'visited' during that move. It is important to emphasise that the rook returns to its starting point, a1. Let children explore the various tours and state their lengths.

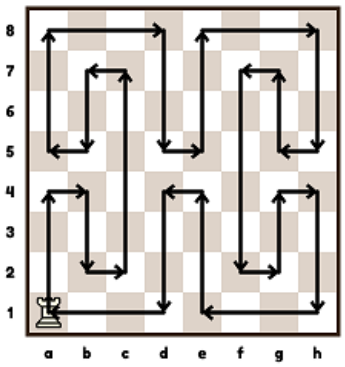
Exploration

- What is the smallest number of moves in which the tour can be completed?
- What is the longest tour (the largest number of moves) that you can find

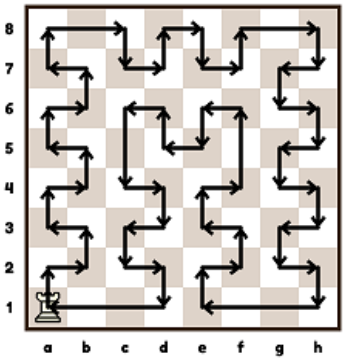
Answer



A tour with the minimum 16 moves



A tour with 28 moves



The longest tour with 56 moves

It can be proved that the shortest tour consists of 16 moves. Some interesting properties of the rook's tour, however, can be investigated. For discussion:

- If a tour starts with a vertical move, it must end with a horizontal move.
- Vertical and horizontal moves alternate.
- There are equal number of horizontal and vertical moves.
- The length of each tour is even.

56 moves - wrong answer!

16. Race to the corner [rook] Pairs

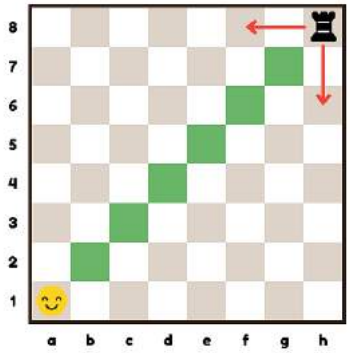
Age 8+	*Symmetry*

Use a chessboard, a rook, and counters to mark the moves. Place a rook on h8 and a marker on a1.
This rook can only move South or West.
Players take turns to move the rook.
The first player to reach a1 wins the game.

This is a game that children like to play. Who wins – the first player or the second player?

Playing method

The second player has the winning strategy. He /she must move the rook to the a1-h8 long diagonal. Here is how to guide children towards the solution:



- First children should explore the game freely. Ask them to look for a way to win. Ask if they prefer to be the first or second player in this game.
- Play as second player with correct strategy against a child or a group of children.
- If the winning strategy is still not found, then tell them to think about the importance of the a1-h8 long diagonal.

17. Chomp — Pairs

Age 6+	*Shapes, symmetry*

Using a printout of a board with a "poison" marker on h8. See Figure 18(a).

Each player takes turns to fold ("chomp") the paper chessboard horizontally or vertically along any marked line. See Figure 18(b).

There are no pieces in this game. The section further from the marker is discarded. The loser is the one left holding the poison. In the original game they played with a large bar of chocolate!
Who wins – the first or second player?

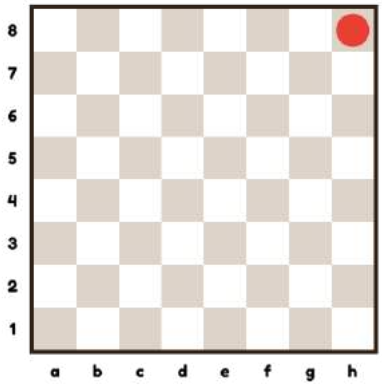


Figure 18(a) Avoid holding h8

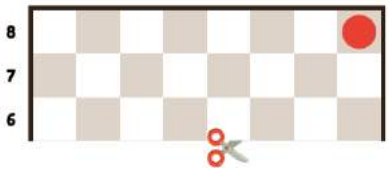


Figure 18(b) Chomping horizontally

Playing method

In order to win, a player should use a geometrical strategy. Whatever the first player does, the second player should fold the paper so as to leave a square-shaped board. See Figure 18(c).

In other words, the winning strategy is to chomp along the diagonal to h8.
The opponent will have no other choice but to fold into an oblong.
Successively, in each move, the strategy of leaving square-shaped shapes is repeated until the poisoned square is achieved.

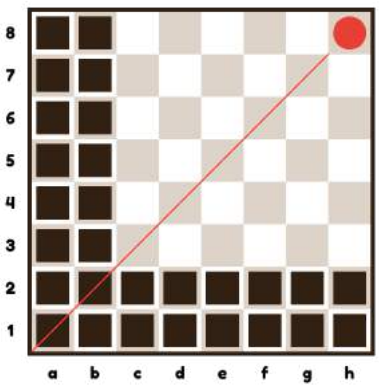


Figure 18(c) The next player can only make an oblong

18. Army Power Puzzle — Pairs

Age 8+	*Enumeration, Venn diagrams, Trial and Error*

Place one of each of the six piece types of the same colour on their starting squares. See Figure 19(a).

Q1: How many squares does White attack?
Squares attacked more than once count only once.

Q2: Identify the squares which are attacked once, twice, thrice.

Q3: Reposition the pieces to find the minimum number of squares that are being attacked.

Q4: (advanced) Reposition the pieces to find the maximum number of squares under attack.

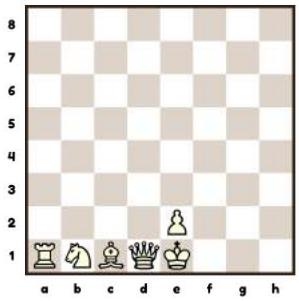


Figure 19(a) Army Power Puzzle

Answers

Q1: 25, see Figure 19(b)

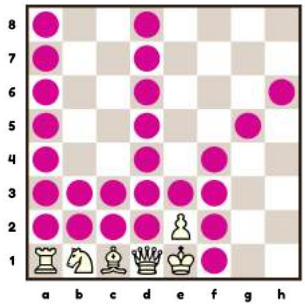


Figure 19(b) 25 squares covered

Q2: See Figure 19(c)

- 1 attack
- 2 attacks
- 3 attacks

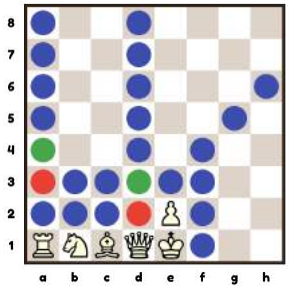


Figure 19(c) Multiple attacks

Q3: The minimum number of attacks is 5, achieved various ways. See Figure 19(d).

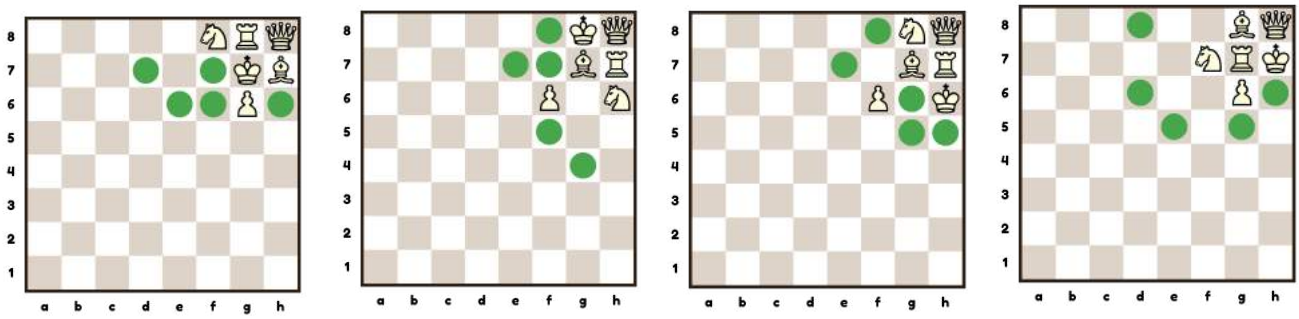


Figure 19(d) Minimum number of attacks

Q4: The maximum number of squares attacked is 51.

6 squares are occupied by pieces.
7 squares are unattacked.

Extension

Q5 Give the locations unoccupied squares and ask the class to figure out the position of the pieces.

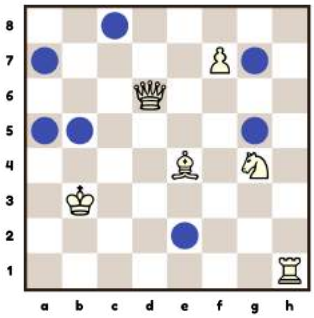


Figure 19(c) Maximum number of attacks

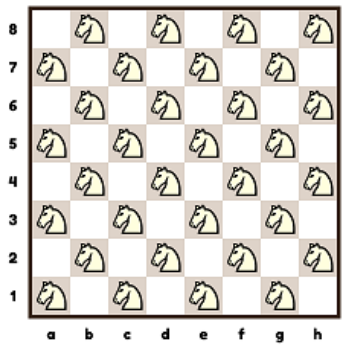
19. Maximum number of knights — Pairs

Age 8+	*Enumeration, parity*

What is the maximum number of knights that can be placed on a chessboard such that no knight attacks another?

Use counters instead of knights.

Solution Method



Knights that do not attack each other are called independent knights. Notice that a knight changes colour on each move (from a light square to a dark one and vice versa). Knights placed on the same colours will be independent.

Answer

The maximum number is 32.

20. Mystery combination piece — Pairs

Age 8+	*Disjoint properties, union, symmetry*

A combination piece combines powers from different chess pieces.

A mystery combination piece is attacking the crossed squares (*Fig 20*):

(a) Mark where the mystery piece is it located
(b) Identify the pieces it combines

Review Exercise 10 for piece patterns.

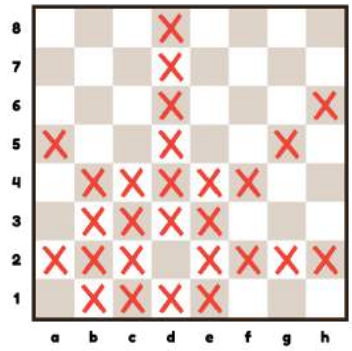


Figure 20: Mystery Combination Piece

Solution Method

Detect the symmetry around d2.

(a) Find the longest lines and join them together with a ruler and pencil.
These are the ranges of the long-distance pieces. See where they intersect.
d8-d1, a5-e1, h6-c1, a2-h2

(b) Notice that a queen on d2 is consistent with the pattern of crosses. Subtract the queen's pattern and see what is left. This is consistent with the moves of a knight.

Answers

(a) d2
(b) queen + knight

21. Positional logic — Pairs

Age 8+	*Logic*

Starting pieces ♚ ♛ ♜ ♝ ♞ ♟

This is a hands-on activity. The children must set up the correct position on the chessboard.
Task: Set up a position with the above pieces obeying the following rules:

1. The pieces stand next to each other on the same line
2. The king is next to the queen which is on c4
3. The rook is between the bishop and a pawn
4. The bishop is the furthest piece from the queen
5. The knight stands on a higher rank than the king

Solution Method

This can be solved using guesswork, trial and error. Alternatively, a more structured approach can be deployed. The key is to find a good starting point. Make use of the information such that no guessing is necessary. For example:

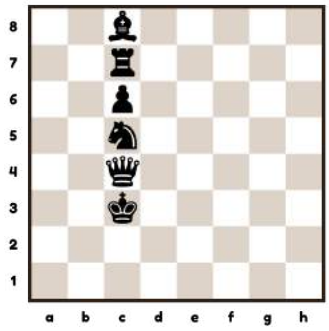
(a) From (5) we learn that the pieces are arranged vertically along the c file

(b) From (2) we know that the position of the queen is fixed at c4

(c) From (3), there is a 3-piece segment bishop+rook+pawn
There are only four squares above the c4 queen and three below.
To satisfy (4), the bishop must be on c8 the furthest point from the queen on the c file

(d) From (5) the knight is above the king which, since c4 is occupied, implies that it must be at c5 or above

(e) By elimination, the king sits on c3.



21. Solution

Extension

Creative activity: Ask children to devise a positional logic problem. The 'best problem' award goes to the one with a unique solution!

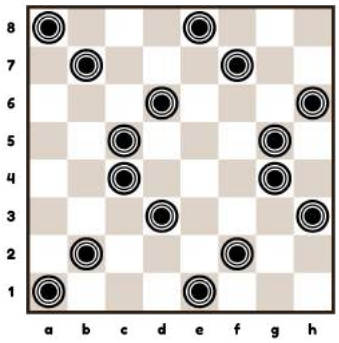
22. Counters on a Line — Pairs

Age 8+	*Straight lines, slopes, trial and error*

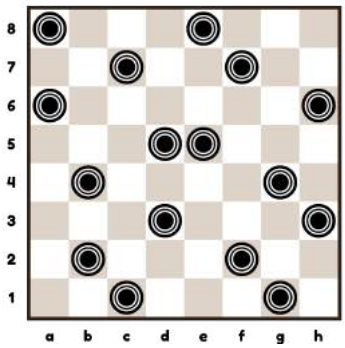
Children recognise horizontal, vertical or diagonal lines. They find it difficult to identify lines with other slopes especially when the counters are spaced apart.

Task 1: Hand out printouts of these two positions and ask the class to find the spoiling lines where three counters are in a row.

(a)



(b)



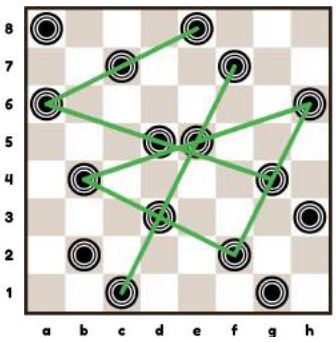
Task 2: Place 16 counters on the chessboard so that no three are in a line.

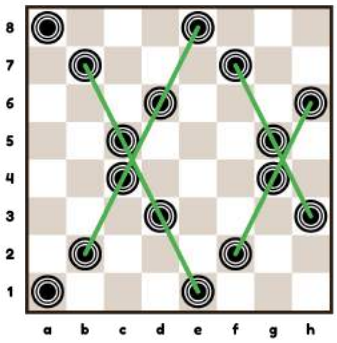
Solution Method

Solving this requires trial and error whilst checking for all possible lines. Here are some hints:

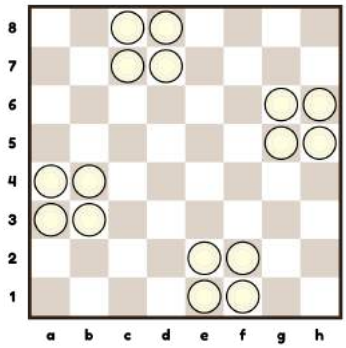
- Start by putting the counters in groups (e.g. of four) which simplifies the search task.
- Place two counters in each column and each row whilst checking they do not touch on the diagonals. Then check for the other slopes with a ruler through the centre of the counters.
- If you get near a solution, try adjusting the adjacent counters systematically.

Task 1





Task 2



One solution: Four groups of four

23. Minimum Moves Map — Pairs

Age 9+	*Enumeration, spatial notions*

Task

Use the coloured counters to build a colour-coded map of the board showing the minimum number of moves it takes a piece to reach every other square from d4 for (a) a king; (b) a knight.

Solution Method

Starting at d4, mark all the squares reachable in one move. Then from each of these repeat the process using different coloured counters. Repeat until there are no more squares.

Answer

24. How Many Routes? [pawn] Pairs

Age 9+	*Enumeration, spatial notions*

The pawn can make any move one square forwards whether straight or diagonal. Note the pawn can also make a move by capturing an imaginary piece – hence diagonal moves must be counted as well.

Task: (Using printouts of these positions) How many different routes are there for the pawn:

	From	To
(a)	a2	Other side
(b)	d5	Other side
(c)	d4	d8

Solution Method

For each possible position of the pawn count the number of ways the pawn can reach that square and write that number in the square. Start from the original position. Some leading questions:

- How do you proceed to the next rank?
- What do you do with the numbers when you move up a rank?
- Is the pawn equally likely to occupy every possible square?

Answer

21(a) 35 ways to reach back rank

21(b) 27 ways to reach the back rank

21(c) 19 ways to get to d8

25. The Light Pawn — Pairs

Age 9+	*Logic, information, tree diagrams*

There are eight white pawns that look the same but one of the pawns is slightly lighter than the others. The difference is too small to judge by hand. You have an imaginary weighing balance in which you can place one or more pawns either side. Use this to find the light pawn.
Question: How many weighings do you need to find the lighter pawn?

Solution Method

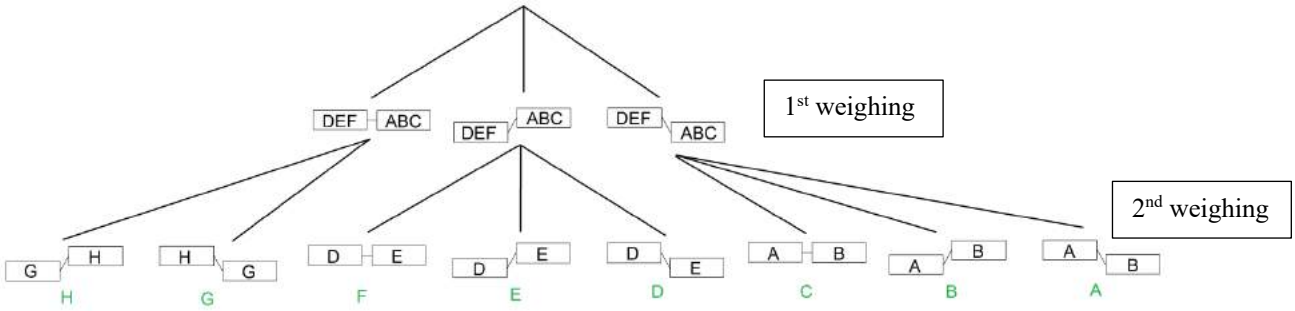
In this thought experiment, the key idea is that if you know that one item is a different weight, then if you weigh two items, you can derive conclusions about the third item.

Consider eight pawns with labels A, B, C, D, E, F, G, H.
Let your first weighing be ABC v DEF.

If it is balanced, then you know that the lighter pawn is either G or H. In that case, execute the final single weighing G v H.

If it is unbalanced, with ABC heavier than DEF, then you know that the lighter pawn is either D, E or F. In that case, execute the final single weighing D vs E (if it is balanced, the lighter pawn is F; if D is heavier than E, the lighter pawn is E; if E is heavier than D, the lighter pawn is D).
Vice versa if the first weighing is unbalanced, with DEF heavier than ABC.

A pedagogical way to proceed is to organize a tree on the whiteboard:



The pupils should understand that the eight possibilities (eight pawns) match the eight terminals of the tree that describes the strategy.

Answer Only two weighings are required.

Extension

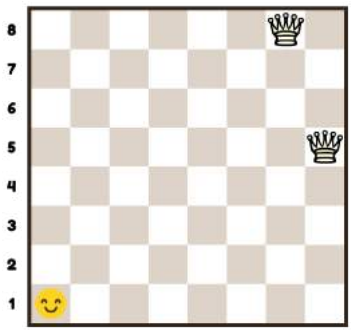
What happens if there are nine pawns, one of them slightly lighter than the others?

The strategy still works, with one more branch (for the balanced ABC vs DEF). There are three possible results (left<right; left=right; left>right). There are three ways to walk through a one-weighing tree and nine ways (3x3) to walk through a two-weighings tree, and so on. Note that 10 pawns cannot be solved with only two weighings – three weighings are required.

26. Wythoff's Game [QQ] Pairs

Age 9+	*Symmetry, working backwards from the target*

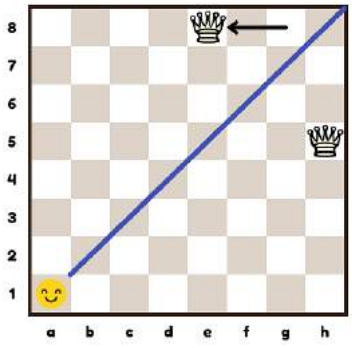
Place two queens of the same colour on h5 and g8 on an otherwise empty chessboard. Stipulate that these queens can only move West, South West or South. The players take turns moving either queen. Provide markers to place on the board. The first player to move the queen to a1 is the winner.
Who wins – the first player or the second player?



- See also Exercise 13 Wythoff's Game with one queen.
- First children should explore the game freely. Ask them to look for a way to win. Ask if they prefer to be the first or second player in this game.
- Play as first player with correct strategy against a child or a group of children.
- If the winning strategy is still not found, then tell them to think about the importance of the a1-h8 long diagonal

Solution Method

Set up a symmetric position around the a1-h8 main diagonal.
Thereafter the first player copies the moves of the second player – a strategy known as Tweedledum-Tweedledee.



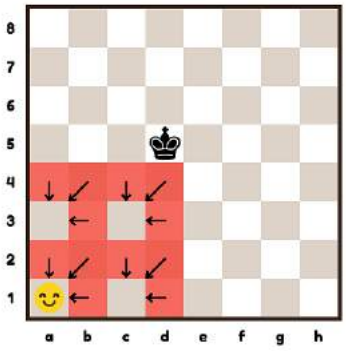
Answer

The first player can win by moving a queen to e8.

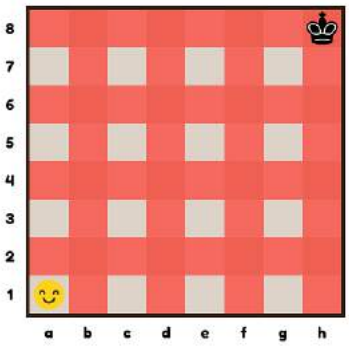
Extension

Let's play with a single king starting from h8. Like before, the first player to move the king to a1 wins.

Hint: Work backwards from the solution. Identify the squares where you do not want to land the king (marked in red). Winning strategy: avoid the red squares.
The first player can win with best play: King to g7, then copy the opponent's move.



Wythoff's king game: The final moves



Wythoff's king game: full pattern

27. The domino tiling puzzle — Pairs

Age 9+	*Shapes, parity*

This exercise benefits from access to 32 dominoes of size 2x1 that cover exactly two squares of the board. These can be card cut-outs.

Q1: Can the dominoes cover all 64 squares of the chess board?

Q2: Can the dominoes cover all 62 squares of a board from which two opposite corners are removed?

Q3: Can the dominoes cover all 62 squares of a board from which two different coloured squares are removed?

Answers

Q1 Yes, depicted in Figure 28(a). There are many ways to cover the board.

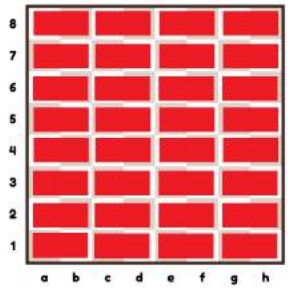


Figure 28(a) Covering a chessboard with dominoes

Q2 No. The second question has an ingenious chromatic solution. A domino piece always covers one white square and one black square. So, a group of dominoes covers an equal number of squares of each color. The mutilated board (Figure 28(b)) has 30 white squares and 32 black squares, so the answer to the given question is negative.

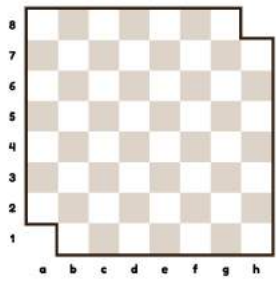


Figure 28(b) Multilated chessboard

Extension

Q3: Yes. Suppose that you remove two squares of different colors.

Yes. There is an ingenious geometric solution. The line in Figure 28(c) goes through all squares.

Remove one black and one white square. The removal divides the line into two segments (See Figure 28(d)). One connects the yellow sides and the other connects the blue sides).

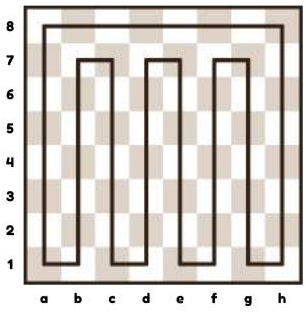


Figure 28(c) Continuous line

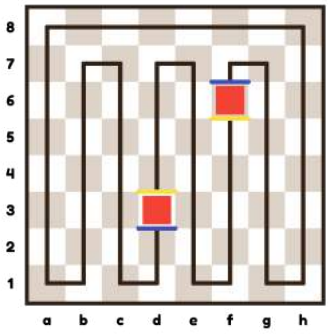


Figure 28(d) Continuous line in two segments

Now, all that must be done is to cover each segment with dominoes. This is possible because there is a balanced number of black and white squares in each segment. This method works for any pair of removed opposite coloured squares.

28. The tromino tiling puzzle — Individuals

Age 9+	*Trial and Error, symmetry, multiples, divide to conquer*

Using printouts of empty chessboard prints with pencil and 22 trominoes.
A tromino is a shape of size 3x1. Each tromino covers exactly three squares of the board.
Can you tile the chessboard with 3x1 trominoes if a corner square has been removed?
This exercise is easy to explain and can be done alone. Reserve for a "special day".

(a) Can the trominoes cover all 64 squares of the chess board?
(b) Remove one corner of the chessboard.
Can the trominoes cover all 63 squares of the mutilated board?
(c) Is there any other square, apart from the corner square, which you can remove to make the trominoes fit?

Introductory questions

- Can the trominoes cover all 64 squares of the chessboard?
No, since 64 is not a multiple of 3.
- How many trominoes do you need to cover 63 squares?
63 / 3 = 21: 21 trominoes will cover 63 squares.
- Does it matter which corner of the chessboard is removed?
No, it does not matter due to symmetry.

Getting into the problem

- Try to cover the chessboard that has a corner removed with trominoes. Make several attempts with different tile arrangements. [It is not possible to complete the task.]
- Remove a different square from the chessboard (cross it out on your sheet). Perhaps try to remove a square from the long diagonal. Removing b7 does not seem to work as shown in Figure 29(a).

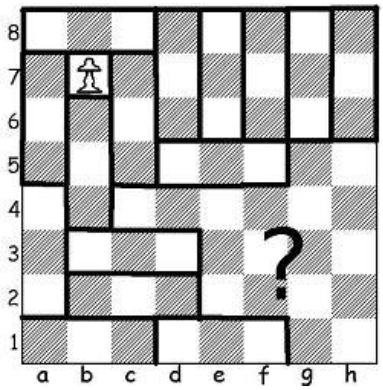


Figure 29(a)

Answer

If c3, c6, f3 or f6 is removed, the tiling can be done!

Solution Method

With Trial and Error children will find one of the four symmetrical solutions - Figure 29(b).
The proof is beyond primary school level.

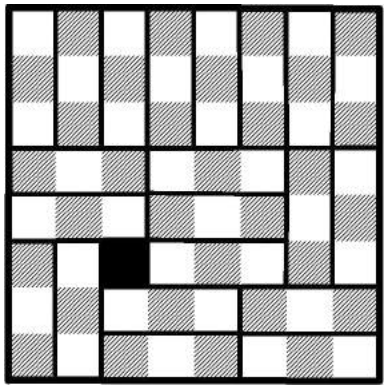


Figure 29(b)

29. Two queens puzzle — Quads

Age 9+	*Enumeration, trial and error*

Tasks: Place two queens on a chessboard such that they attack the:

(a) maximum number of squares
(b) minimum number of squares

Only count unoccupied squares. A second attack on a square is not counted. Ignore colours.

Solution Method

(a) We know from its power contour that the queen is strongest at the centre of the board. Start by placing a queen in one of the central squares (e.g. d4). Then systematically try placing the second queen as close as possible. The maximum number of unoccupied squares attacked 42 arises where the queens are (i) orthogonally adjacent in the centre, or (ii) are a knight's distance away on the roomy side of the first queen (e.g. d4 & e6).

(b) We know from the power contour that the queen is weakest on the edge of the board. We can also restrict movements by placing two queens on the same line. As a first attempt, placing the queens on the corners (opposite or lateral) gives 32. However, there is a lower number of 31 which arises with (i) a1 & a3/a5/a7 or (ii) b1 & d1/f1/h1 (plus symmetries). Figure 30(c) is a two queens joint contour map for the number of unoccupied squares under attack given one queen on a1 – which incorporates the solution in Figure 30(b)(i).

Answer

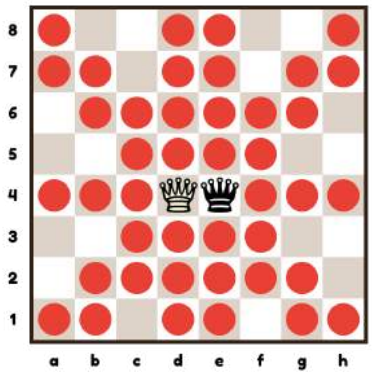


Figure 30(a) maximum (i)

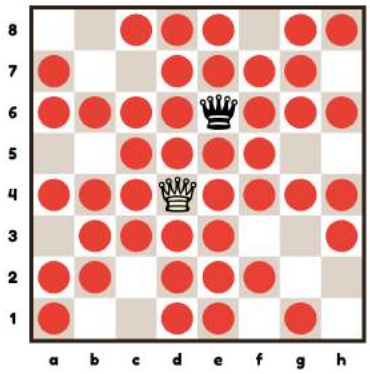


Figure *30(a) maximum (ii)*

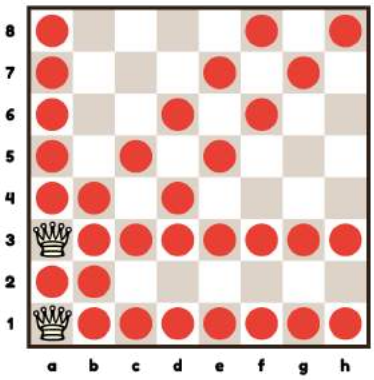


Figure 30(b) minimum (i)

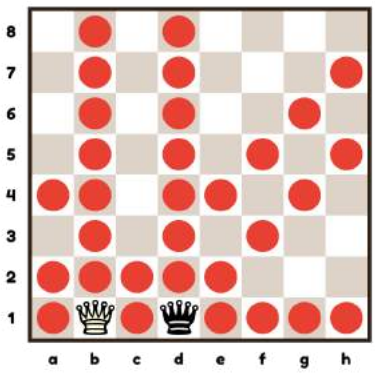


Figure 30(b) minimum (ii)

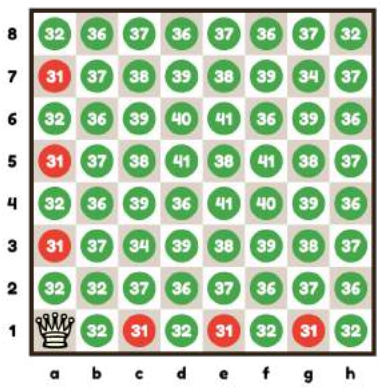


Figure 30(c) Joint contour with Qa1

30. Equidistant checkmate — Pairs

Age 9+	*Geometry, distance measures, peer learning*

This is a hands-on exercise with pieces and a chessboard best done in pairs or small groups. As it is necessary to know about checkmate to play chess, at least one member of the group or pair must know the rules of chess. After this investigation they will all know what checkmate is!

Introduction

Question: One piece gives check but what is the role of the second piece in checkmate?
Answer: The second piece guards the escape squares i.e. holds the king 'in prison'.

Solution Method

Set up positions with two white pieces that are the same distance away from the a1 square.
Now place the white king on c3 and the black king on a1

i. Find two white pieces to deliver checkmate
ii. Ensure that these two white pieces are the same distance away from the black king

Answers

There are a few different solutions with legal checkmate. In Figures 31(a) and 31(b) the queen can swap with the other piece and still deliver checkmate. Note Figure 31(c) is a false solution because the white queen can be captured. Nor is Figure 31(d) a solution because the position is illegal – the black king was already in check in the previous move.

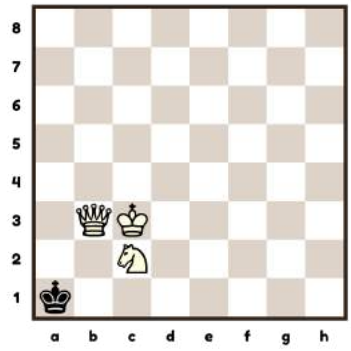


Figure 31(a)

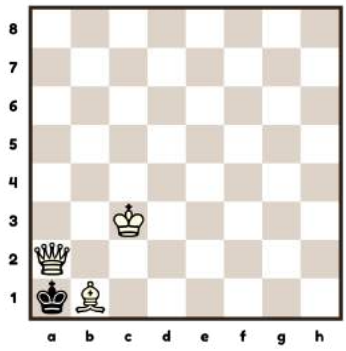


Figure 31(b)

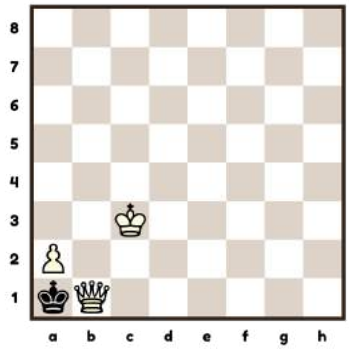


Figure 31(c) X

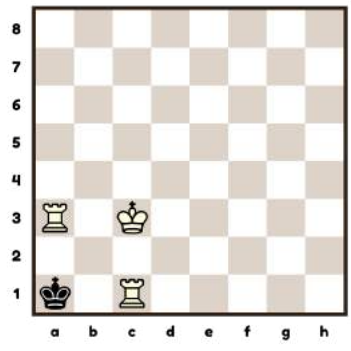


Figure 31(d) X

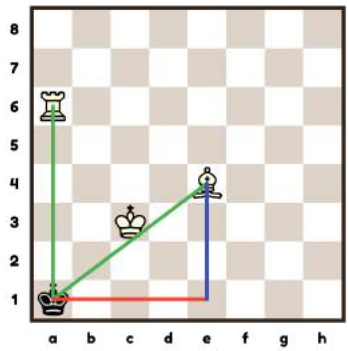


Figure 31(e)

In Figure 31(e) the pieces are 5 units (a unit being the length of the side of a square) away from the black king; more precisely the centre point of a1. The centre points of a1, e1 and e4 form a right-angled triangle. The two shorter sides are of 3 and 4 unit length and, using Pythagoras' theorem, the length of the hypotenuse must be 5 units. [Age 10+]

31. The Subtraction Game — Pairs

Age 9+	Symmetry, multiples

Place 10 counters on three rows. Players take turns to **remove 1, 2 or 3 counters** from the end of any row. The player to remove the last counter is the winner.

This game is related to the 3 times table (or the 4 times table if the maximum number of counters to be removed is 4, etc.)

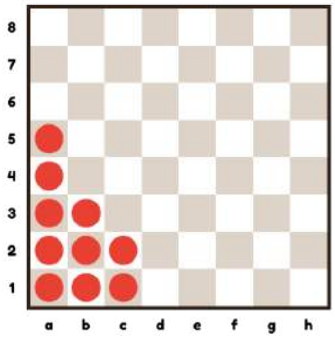
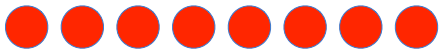


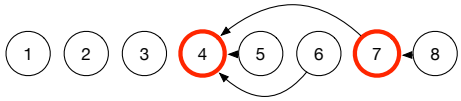
Figure 32(a) Subtraction Game

Preliminary question

Find a winning move in a row of size 8



The winning move is the removal of one counter.
In general, the winning play is to leave 4, 7, (10) … counters i.e. a multiple of 3 plus 1, which is one more than the maximum number of counters that can be removed. In the end, the opponent must leave 1, 2 or 3 counters after which the first player removes the last counter.



Game strategy

The best strategy in last-move-wins games is to create **independent pairs of identical components**. Thereafter, for each pair, mimic the move of your opponent - the "Tweedledum and Tweedledee" strategy (see also Exercise 26).

Winning Plan

The winning move is to remove the third row.
In Figure 32(b) the pairs of identical components are bracketed together.
From this position, copy anything the opponent does.

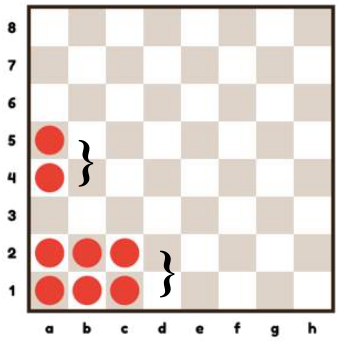


Figure 32(c) Pairs of identical components

Extension

Play the position in 32(c).
The winning move is to remove one counter from the first rank.
For a solution method, see next Exercise 33 Nim.

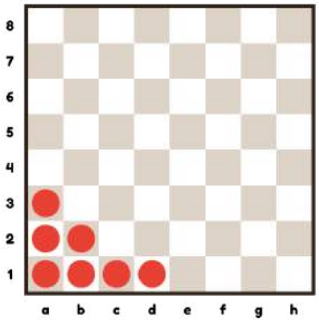


Figure 32(c) Extension exercise

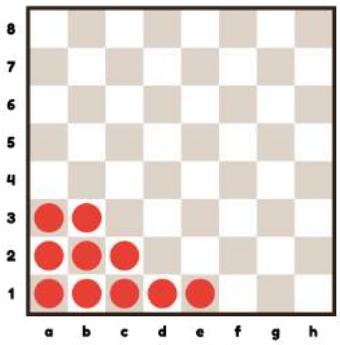
32. Nim — Pairs

Age 10+	*Symmetry, multiples, powers*

Place ten counters on three ranks: 5,3,2

Players take turns to **remove one or more counters** from the end of any rank. The person to remove the last counter is the winner.

Who wins, the first player or the second player?



Game strategy

A winning plan is to try to obtain symmetry and then copy your opponent's moves. Symmetry arises when there are pairs of rows with the same number of counters in each row. If your opponent removes one or more counters, you remove the same number of counters in the other row: Tweedledum and Tweedledee.

Winning Plan

The winning move for the first player is to remove four counters from the first row to leave {1,3,2}. Whatever your opponent does, you can create symmetry on your next turn. For example:

- Your opponent removes row 1 leaving {0, 3, 2}. On your move, remove one counter from the second row to leave {0, 2, 2}. This is a symmetrical and you can win as described above.
- Your opponent removes the third row leaving {0, 3, 5}. On your move, remove two counters from the first row to leave {0, 3, 3}. This is a symmetrical and you can win as described above.

[Age 11+] More generally, the winning plan is to achieve symmetry regarding the number of "powers of 2" (1,2,4,8). For example, a row of 7 comprises three powers of 2: 1, 2, and 4. In the starting position, the groups are {5,3,2} decompose as follows:

$$5 = 4 + 0 + 1$$
$$3 = 2 + 1$$
$$2 = 2 + 0$$

This gives us

Powers of 2	**1**	**2**	**4**
No. of these	2	2	1

There are two 1's, two 2's and one 4.

By removing the one 4 we are left with these paired powers of 2: two 1's and two 2's
Hence the winning play on the first move is to remove four counters from the first row.

33. Golden Coin Game — Pairs

Age 10+	*Symmetry, multiples, tree diagrams, working backwards*

Place ten red counters on three rows: 5,3,2. In addition to these counters, append one golden counter to the longest row. See Figure 34(a).
Players take turns removing **any number of** counters from anywhere within any row. The golden counter must be the last one to be removed and it cannot be removed along with a red counter.

The player to remove the golden counter is the winner. Who wins?

This game benefits from having played the Subtraction Game (Exercise 32) and Nim (Exercise 33).

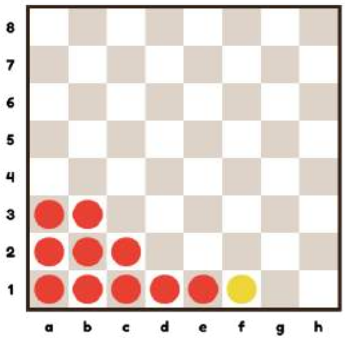


Figure 34(a) Golden Coin Game

Game strategy

The conventional strategy when playing subtraction games is to be the person to remove the last counter. The twist in this game is that the person who removes the last red counter is the loser not the winner - this is the misère version of the subtraction game. The best strategy is to follow the conventional winning strategy as if the game were played just with red counters. However, just before the end, give the other side no choice but to remove the last red counter.

Winning Plan

To simplify matters, we ignore the golden counter and play to lose the game with red counters. Notice that the arrangement of red counters is the same as in Exercise 33. Hence, to take control of the game, the first player should remove 4 red counters from the first row. The play then follows the game tree in Figure 34(b) which sets out the responses from Player 1 to moves by Player 2. Player 1 wins by ultimately leaving one unit (Box 10) which is reached by various routes. Player 2 must pick up the last red counter after which Player 1 picks up the Golden Counter. Player 1 reaches a winning position by leaving either **two pairs** (Box 7) or **three units** (Box 8).

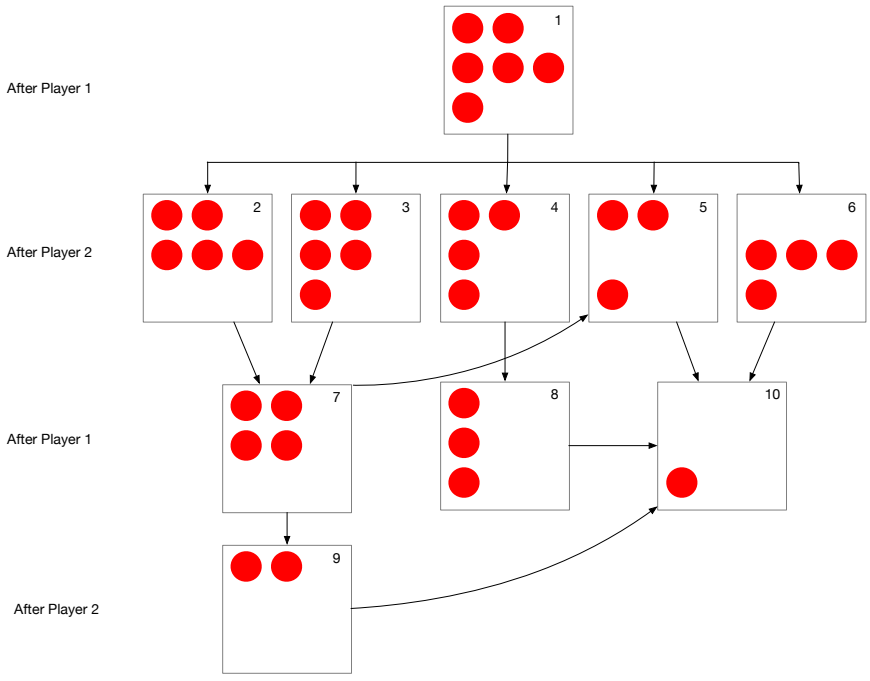


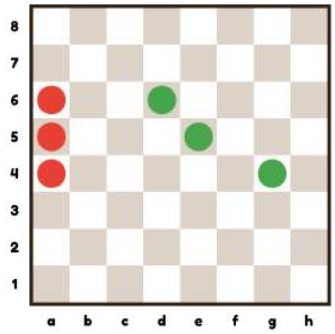
Figure 34(b) Game Tree for Golden Coin Game

34. Northcott's Game — Pairs

Age 10+	*Symmetry, representation*

Player 1 has counters on a4, a5 and a6 and Player 2 has counters on g4, f5 and e6.

Moving alternately, each player moves one of their coloured counters along the rank any number of moves towards their opponent. No captures or jumping are permitted. The last person to move is the winner. Who wins?



Northcott's Game

Solution Method

This exercise is equivalent to Nim (Exercise 32) but with spaces representing counters. The horizontal distance between each of the differently coloured counters is {2,3,5}. Each move reduces the distance, equivalent to removing a counter in Nim. The playing strategy is therefore the same as in Nim.

35. Sliding Rooks Game — Pairs

Age 10+	*Symmetry, parity, working backwards*

Place eight white rooks on the left side of the board and eight black rooks on the right-hand side of the board (Figure 36). These rooks can only slide sideways, not up and down. The white rooks can only move rightwards, and the black rooks can only move leftwards. Capturing or jumping over pieces is not permitted. The last person to move is the winner.
Who wins?

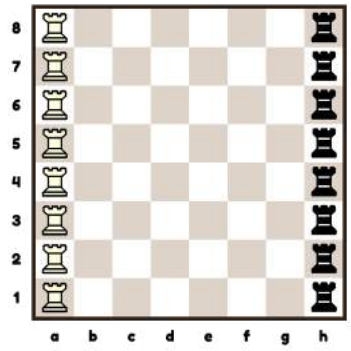


Figure 36(a) Sliding Rooks Game

Playing Method

The rows are independent of each other because the rooks can only move horizontally. Work backwards and consider a single row and then take the other rows into account. The first player can win the first row by moving their rook next to their opponent's rook. Similarly, the second player can win the second row. If there are only two rows, the second player is the last player to move and hence wins the game. The second player will win following this symmetrical strategy because there is an even number of rows.

The first player must try and "lose a move" to obtain symmetry. One trick is for the first player is tempt the second player to unwittingly break symmetry. For example, on the first move, the first player could leave one square between their rook and their opponent's (e.g. ♖a1-f1) tempting the second player to close the gap. However, the second player's correct response would be to mirror the move on another row e.g. ♜h8-c8. As long as the second player is alert, he or she should win.

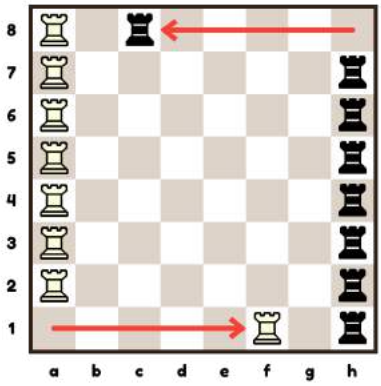


Figure 36(b) White tries 1. ♖f1 but Black responds 1.. ♜c8

36. How Many Routes? [rook] Pairs

Age 10+	*Enumeration, Pascal's triangle*

Resources: Printouts of these positions with a rook on a1 and the target square marked in colour.

Task

The rook can only move North or East, alternating direction on every move.
Calculate how many different routes are there for the rook to reach:

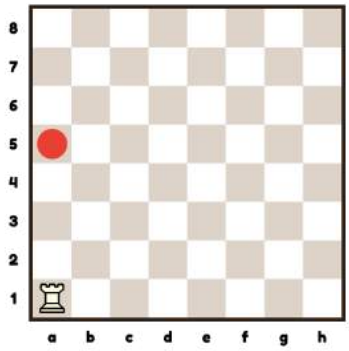


Figure 37(a)

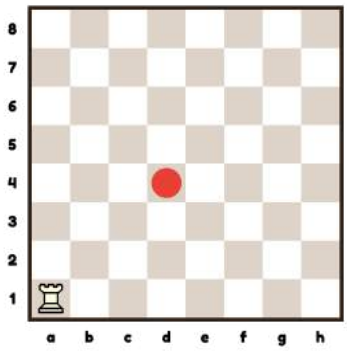


Figure 37(b)

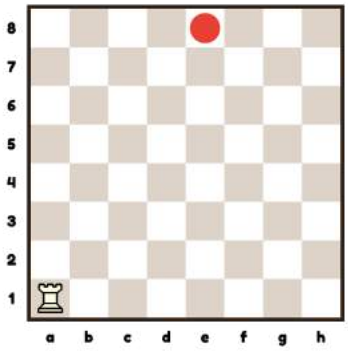


Figure 37(c)

Introductory questions

- Why do you think that the rook can only move North or East? *So that the rook gets closer to the target square on every move. Otherwise there could be countless ways to get there.*
- Why is it important to alternate direction on every move? *We are looking for different routes, but subsequent moves in the same direction would be part of the same route.*

How many ways can the rook get to a5 from a1? (Figure 37(a)
As it alternates direction on every move, there is only one route, the single move from a1 to a5.

Solution Method

- Start from the original position. For each possible position of the rook, count the number of ways it can reach that square and write that number in the square.
- Solving trick: look for the neighbouring squares in the direction where the rook could have come from. For example, for the rook to reach c2 it could have come through b2 or c1, so the sum of the numbers in b2 and c1 gives the total number of routes to c2; see Figure 37(b).
- Continue numbering each square this until you get to the target square.

Answers

There are 20 routes to d4 as shown on Figure 37(e). There are 330 ways to get to e8.

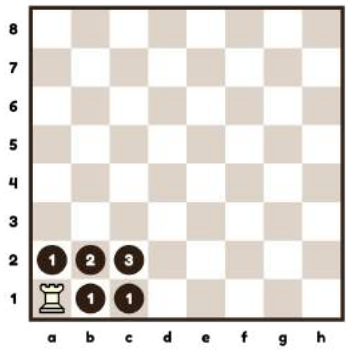


Figure 37(d)

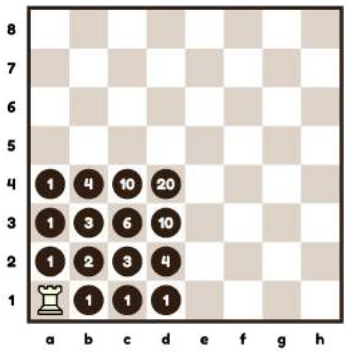


Figure 37(e)

37. Rook's Tour on a Mutilated Board — Pairs

Age 10+	*Symmetry, Elimination*

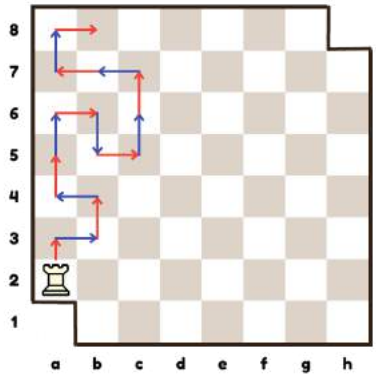
Remove the opposite corners of a chessboard. Place a rook on any square. Can the rook visit all the squares of the mutilated chessboard without landing twice on the same square?
This is an advanced version of Exercise 14, the Rook Corner Puzzle.

Solution method

Use pencil and paper and chessboard printouts. Children should have an unguided investigation first. Let them ask for clarification about the problem. Expect questions like these:

- Does it matter where the rook starts?
  No, the rook can start from any square.
- Does the rook have to go back to the starting square?
  No, the rook can finish on any square provided it has visited every square once and only once.

The children will not be able to find a solution. Now you can guide them towards the proof with the following hints and questions:



- How many dark and how many light squares does the rook have to visit?
  30 dark and 32 light squares.
- The rook should proceed in one-step moves on its journey. Use two colours, for example red for odd (1st, 3rd, 5th etc) and blue for even (2nd, 4th, 6th etc.) moves as shown in the diagram.
- How many one-step moves would the rook have to do to complete the tour?
  61 moves.
- How many odd and how many even one-step moves would the rook have to do to complete the tour?
  31 odd and 30 even moves.
- Suppose the rook starts on a light square, such as a2. What can you say about the colour of the target square (where the rook lands) on odd and on even one-step moves?
  The target square of odd moves is always a dark square, while the target square of even moves is always a light square.
- Suppose the rook starts on a light square, such as a2. Count the number of dark target squares it needs to complete the tour. Can you explain why the rook cannot complete the tour?
  The target square of odd moves is always a dark square and there are 31 odd moves altogether. Hence there are 31 dark target squares. The rook cannot complete its tour as it would need to visit only 30 dark squares.
- Suppose the rook starts on a dark square, such as b2. Count the number of light target squares it needs to complete the tour. Can you explain why the rook cannot complete the tour?
  The target square of odd moves is always a light square and there are 31 odd moves altogether. Hence there are 31 light target squares. The rook cannot complete its tour as it would need to visit 32 light squares.

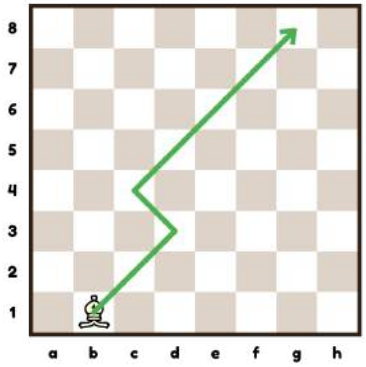
Answer

By a process of elimination the answer is no, the rook cannot complete a tour of the board.

38. Bishop Zigzag Puzzle — Pairs

Age 10+	*Enumeration, Pascal's triangle*

The diagram shows the bishop crossing the board diagonally.
On each move, it moves up a rank and alternates direction from North-East to North-West and vice versa. Use pencil and chessboard prints.



Questions:

(a) How many routes from b1 to g8?
(b) How many routes from b1 to the other side of the board?
(c) What is the total number of white-square paths starting on the first rank and ending on the eighth rank?

This problem has a similar solution method to that of Exercise 37. Children who understood the enumeration of the rook's routes will succeed with this task about the bishop.

Introductory questions

- Why do you think that the bishop must move up a rank on each move?
  So that it would get closer to the target square on each move.
- Why is it important to alternate direction on every move?
  We are looking for different routes, and just like in Exercise 37, subsequent moves in the same direction are part of the same route.

Solution Method

Use a chessboard to find ways for the bishop to get to g8 from b1. Write down your routes. Here are two examples:

1st route: b1→d3→c4→g8
2nd route: b1→e4→d5→ g8

How many different routes can you find? There are altogether seven.

For each possible position of the bishop count the number of ways it can reach that square and write that number in the square. Start from the original position.

Answers

(a) There are 7 routes from b1 to g8.
(b) There are 14+28+20+7=69 routes to the other side; see Figure 38(a).
(c) There are 296 different routes. Add all numbers in the 8th rank of Figures 38(a), 38(b), 38(c) and 38(d).

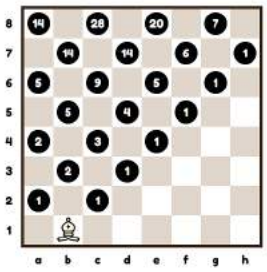


Figure 38(a)

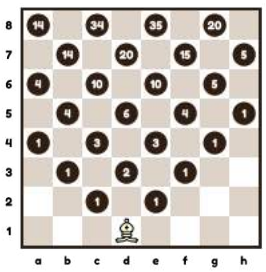


Figure 38(b)

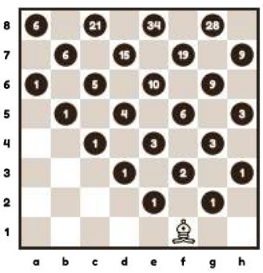


Figure 38(c)

Figure 38(d)

39. Five queens domination puzzle Quads

Age 10+	*Enumeration, Pascal's triangle*

Task

Rearrange five queens such that they attack every square on the board at least once.
The attacks must cover all the occupied squares. Pieces that attack every unoccupied square are said to dominate the chessboard. A minimum of five queens are required to dominate the chessboard.

Introductory question

How many squares can a queen attack?
Note that the queen does not attack the square on which she stands.
21, 23, 25 or 27 depending on her position. See the queen's contour in question 40.

Solution Method

Place five queens on the chessboard.

Mark every square that is under attack by one or more queens with a counter.
How many free (unoccupied and not attacked) squares can you count?

Try to reduce the number of free squares by moving a queen to a different square.
Can you reduce the number of free squares to zero? If so, then you have solved the task.

Answers

There are exactly 4860 different ways to accomplish the task.
Let's look at some interesting arrangements:

Figure 42(a) All queens are placed on a long diagonal
Figure 42(b) All queens are placed on a diagonal that is not the long diagonal
Figure 42(c) All queens are placed along a single file of the chessboard
Figure 42(d) None of the queens attack one another

Note that in in (a) - (c) the attacks cover all the occupied squares as well.

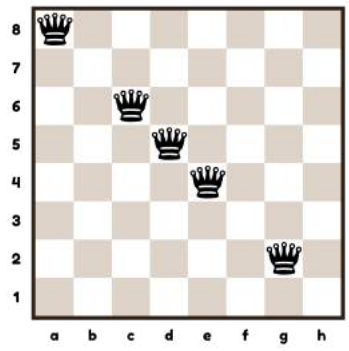


Figure 42(a)

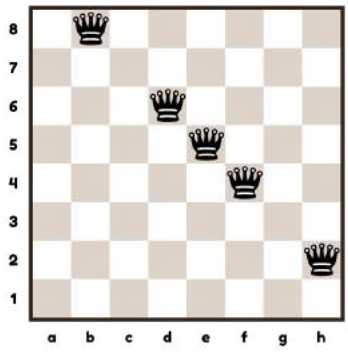


Figure 42(b)

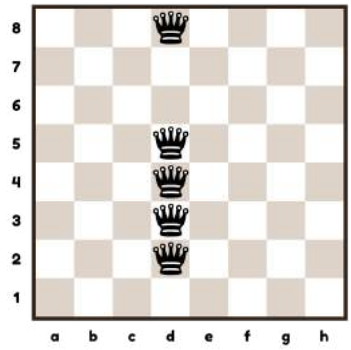


Figure 42(c)

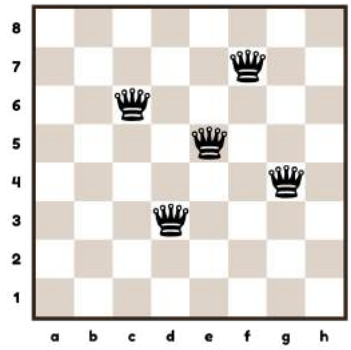


Figure 42(d)

40. Dividing the board — Individual

Age 10+	*Decomposition of shapes, trial and error*

Hand out printouts of a chessboard marked up with crosses as in Figure 40(a).

Task

Divide the chessboard into eight smaller squares so that each one contains at least one cross. Not all the drawn squares need to be the same size.

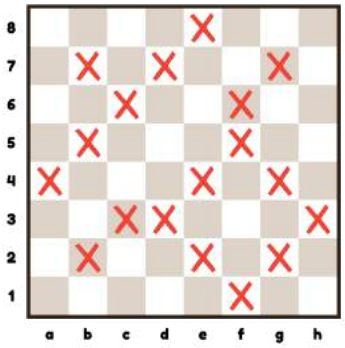


Figure 40(a) Divide the board

Solution Method

Try different size squares from 2x2. There are several potential squares which contain one cross. Start from a corner of the board and then systematically move down and across.

Answer

Figure 40(b) shows the solution.

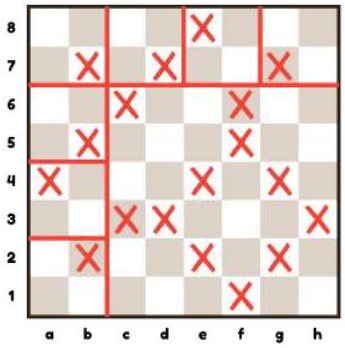
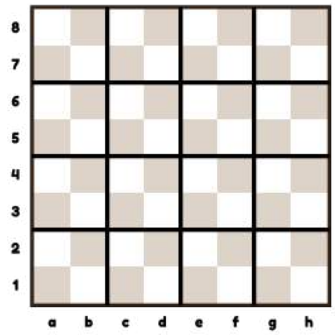


Figure 40(b) Board divided into 8 squares

41. Tiling the board — Quads

Age 10+	*Decomposition of shapes, enumeration, Trial and Error, area measures, square numbers*

Display a chessboard tiled into 16 squares.
Hand out printouts of an empty chessboard.



Task: Show two other ways to partition a chessboard into 16 squares.

Introductory questions

1. A single square of the chessboard has area of 1 unit.
   What is the area of the chessboard?
   64 units.
2. Squares are drawn on the chessboard. List the area of these squares.
   What name is given to these numbers?
   1, 4, 9, 16, 25, 36, 49 and 64. These are square numbers.

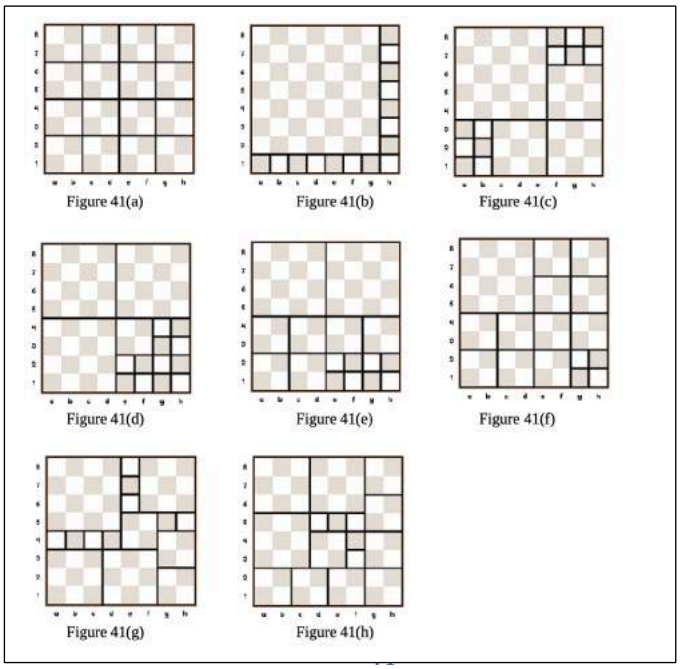
Getting into the problem

1. Tile the chessboard with squares. Try to use many different sizes of squares.
   Count the number of squares that you used.
2. Try to use 16 squares to tile the chessboard. Find as many different solutions as you can.

Solution Method

Rearranging the same set of squares does not constitute a new answer. The task can be completed in eight different ways. The teacher may give hints to guide the children towards each solution.

1. All squares are the same size: Figure 41(a)
2. There is a 7x7 square: Figure 41(b)
3. Use a 5x5 square and three 3x3 squares: Figure 41(c)
4. Use three 4x4 squares: Figure 41(d)
5. There are two 4x4 and six 2x2 squares: Figure 41(e)
6. There is a 4x4 square and four squares of unit area: Figure 41(f)
7. Use a 4x4 square and the same number of 3x3 and 2x2 squares: Figure 41(g)
8. There are eight 2x2 squares in this solution: Figure 41(h)



42. Twelve knights problem — Pairs

Age 10+	*Enumeration, Trial and Error*

Task

Place twelve Knights on a chessboard so that every square is either attacked or occupied.

If you do not have enough knights, use counters to represent them.

Introductory question

How many squares can a knight attack?

2, 3, 4, 6 or 8 depending on its position. See the knight's contour in Exercise 10.

Solution Method

1. Put twelve knights on the chessboard. Mark every square that is under attack (by one or more knights) with a counter. How many free (unoccupied and not attacked) squares can you count?
   a) Try to reduce the number of free squares by moving a knight to a different square.
   b) Can you further reduce the number of free squares?
2. Place three knights on b6, c6 and c5. Mark every square that is under attack by one or more knights with a counter. Now one quarter of the board is almost fully covered with knights and counters. Try to use this knight arrangement to cover the rest of the board.

Answers

If every square is either occupied or attacked by the knights, then we say that the knights *dominate* the chessboard. The two solutions are shown in Figure 39(a) and Figure 39(b).

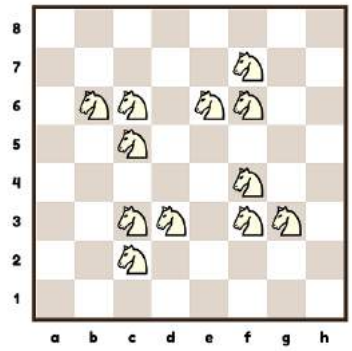


Figure 39(a)

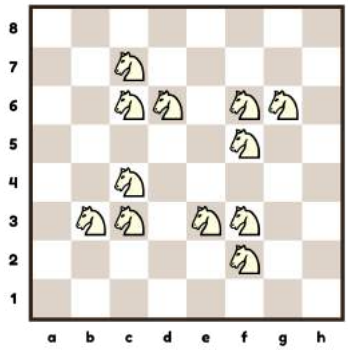


Figure 39(b)

Note if we reflect either solution around the horizontal or vertical central axis, or along the main diagonals, then we get the other solution. Also, the diagrams have a rotational symmetry of order four (i.e. they look the same when rotated a quarter turn).

43. Eight queens puzzle — Groups

Age 10+	*Enumeration, Trial and Error*

Task: Rearrange eight queens such that no queen attacks any other.
You may find it helpful to mark the attacked squares with counters.

This is the oldest and most famous chess & maths problem. Pieces that do not attack one another are called independent.

Solution Method

- Put two queens on the chessboard that do not attack each other
  e.g. a knight's move apart
- Add another queen such that no queen attacks any other
- Add another queen. Check that still no queens are under attack.
  Increase the number of queens one by one. Can you get to eight queens?

There is no formula for doing this. Computer scientists use a method called "backtracking": every time you find a conflict, backtrack to the last placement and instead try the next square.

In a simplified version of this investigation one, two or three queens are missing from the solution and the task is to place the remaining queens on the correct squares.
Best done on paper with prepared diagrams.

Answers

Altogether there are 92 positions which satisfy the requirements. If we exclude all rotations and reflections, then there are only 12. Some of these are shown below:

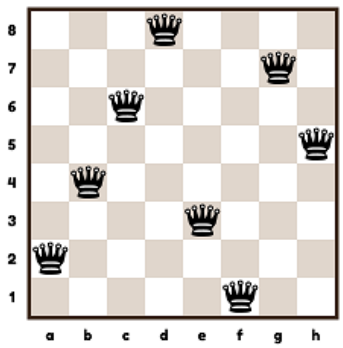


Figure 43(a) Four in a row solution

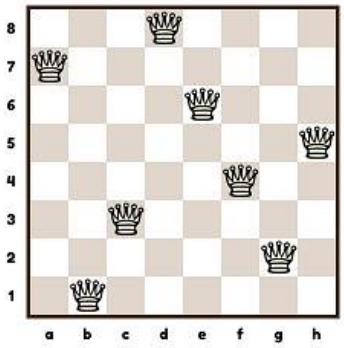


Figure 43(b) Another four in a row

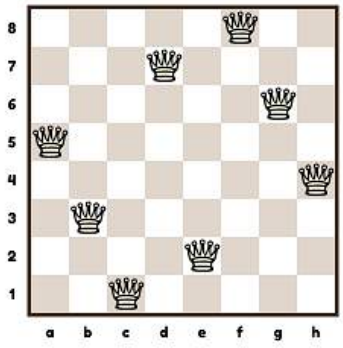


Figure 43(c) The only symmetrical solution

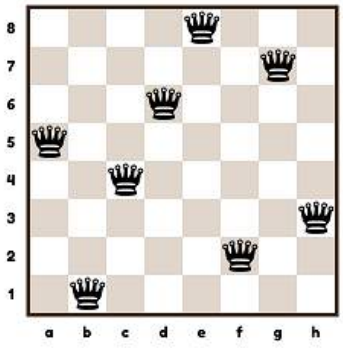


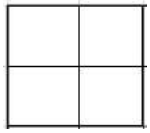
Figure 43(d)

44. How many Squares on a Chessboard? Individuals

Age 10+	*Enumeration, shapes, arithmetic, organising information in tables*

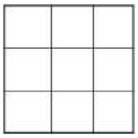
Task 1: How many squares of all sizes can you find in this grid?

Size of square	No. of squares
1 by 1	4
2 by 2	1
Total	5



Task 2: How many squares of all sizes can you find in this grid?

Size of square	No. of squares
1 by 1	9
2 by 2	□
3 by 3	1
Total	14



Solution Method

Start from a small grid and step by step expand the size of the square.

Task 3: How many squares of all sizes can you find in a 4x4 grid?

Task 4: How many squares of all sizes can you find on a chessboard? Complete the grid.

	Number of squares								
Grid size	1x1	2x2	3x3	4x4	5x5	6x6	7x7	8x8	Total
1x1	1								1
2x2	4	1							5
3x3	9	4	1						14
4x4	16	9	4	1					30
5x5	25	16	9	4	1				55
6x6	36	25	16	9	4	1			91
7x7	49	36	25	16	9	4	1		140
8x8	64	49	36	25	16	9	4	1	204

At some point, the pattern may be noticed.
The total number of geometrical squares on a chessboard is $1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2$

Answers

1) 5
2) Missing number is 4
3) 30
4) 204

45. Tournament Scoring Puzzle — Quads

Age 10+	*Logic, organising information in tables*

The scoring system for a school tournament is:

Win = 3 points Draw = 2 points Loss = 1 point

Four children (Albert, Bridget, Cecilia, and Dirk) play each other once in an all-play-all tournament. You are told that:

1) Bridget was the winner
2) Dirk came last
3) Cecilia scored a win, a draw and a loss
4) All the players scored a different number of points.

How many points did Albert score?

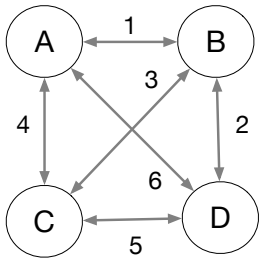
Introductory Question (the relevance of this will become apparent later)
Distribute 10 counters into three squares subject to minimum=2 and maximum=4 per square.
4,4,2; 4,3,3 excluding permutations

Solution Method

There are intricate solutions going through all the game outcomes. We can take a shortcut by examining the implications of condition (4) that each player scored a different number of points.

Establish that in a 4-player all-play-all there are 6 games (see diagram).
Show that each game results in 4 points giving a tournament total of 24 points. (24 is an invariant.)
We want to distribute 24 points across 4 players with a minimum of 3 and a maximum of 9.
The combinations with unique point counts are:

Winner	Second	Third	Last	Total
9	8	4	3	24
8	7	6	3	24
8	7	5	4	24



Only one of these combinations has 6 points.
This gives the final points table.

	Played	Points	Results	
Bridget	3	8	WWD	3+3+2
Albert	3	7	WWL	3+3+1
Cecilia	3	6	WDL	3+2+1
Dirk	3	3	LLL	1+1+1

The corresponding all-play-all table is

	Bridget	Albert	Cecilia
Albert	Albert = Bridget		
Cecilia	Bridget win	Cecilia = Albert	
Dirk	Bridget win	Albert win	Cecilia win

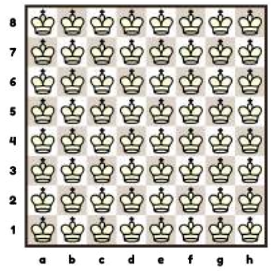
Answer

Albert scored 7 points.

46. Kings random jump — Pairs

Age 10+	Enumeration, Trial and Error, Symmetry

Each square of a chessboard is occupied by a king. Each king moves randomly to an adjacent square. Assume each square can accommodate more than one king. What is the greatest number of empty squares that could remain?



Hint: use counters that have different-coloured sides. This ensures that all jumps are 'recorded': if a king jumps, the counter is turned over and put on the arrival square. Older children can also work on paper with a set of empty chessboard prints.

Introductory questions

- What is the greatest number of kings that can end up on the same square?
  A square can be surrounded by a maximum of eight squares, but the resident king must jump away from this square, so the answer is 8.
- Is it possible that all squares are occupied after the jumps?
  Yes, it is, for example if neighbouring kings on the same file swap places.
- Is it possible to have an isolated king after the jumps? An isolated king has no neighbours on adjacent squares.
  No, it is not possible. If there were an isolated king, then the king that was originally on that square would have had nowhere to jump to.

Solution Method

1. Place the counters on every square of the chessboard with the same colour up. Make every king jump to an adjacent square by moving each counter to an adjacent square and turning it over at the same time. When all kings jumped count the number of free squares on the board.
2. Repeat 1. and try to increase the number of unoccupied squares.

Answer

Having explored the problem children may come to the correct conclusion that at most 52 squares can remain empty, so all the kings can gather on 12 squares. See 46(a) and 46(b).

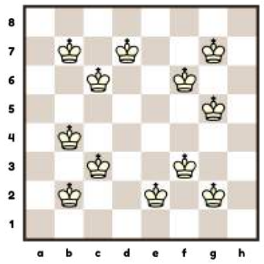


Figure 46(a)

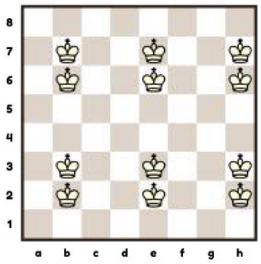


Figure 46(b)

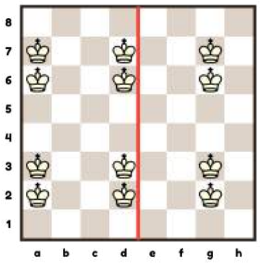


Figure 46(c)

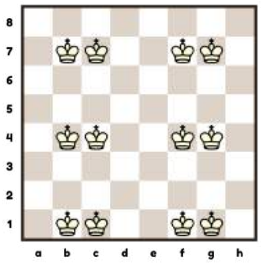


Figure 46(d)

Extension

Ask the class to collect as many possible answers with 52 unoccupied squares as they can. Discuss how some of these are related to each other by symmetry. For example, Figure 46(c) is a reflection of Figure 46(b) around the vertical bisector of the board, and Figure 46(d) is a rotation by 90 degrees clockwise of Figure 46(b).

Some solutions have their own symmetry: Figure 46(a) has a rotational symmetry of order 4 around the centre, while Figure 46(b) has a line symmetry with the horizontal bisector of the board as the mirror line. The colour of the squares is ignored.

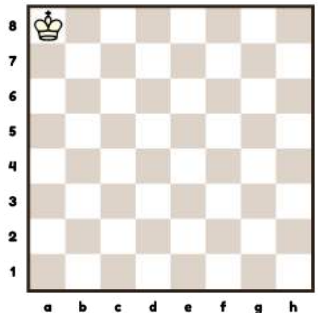
47. King's random walk — Pairs

Age 11+	*Enumeration, Proportion, Organising information in tables*

A king moves randomly starting at a8. What is the probability that the king returns to a8 after:

(a) Two moves?

(b) Three moves?



Introductory questions

- How many squares can the king move to from a8?
  3: a7, b7 and b8
- How many squares can the king move to from a7 (and b8 by symmetry)?
  5: a6, b6, b7, b8 and a8
- How many squares can the king move to from b7?
  8: a8, a7, a6, b6, c6, c7, c8 and b8

Solution Method

(a) Complete this table to account for all possibilities:

First move to	No. of possible moves from there	No. Of moves when king returns to a8
a7	5	1
b7	8	1
b8	5	1
TOTAL:	**18**	**3**

The king returns to a8 three times out of 18 i.e. 1 out of 6 times on average (16.7%).

(b) Collect information about his majesty's walk in a larger table:

First 2 moves	No. of possible moves from there	No. of moves when king returns to a8
a7, a8	3	0
a7, b8	5	1
a7, b7	8	1
a7, b6	8	0
a7, a6	5	0
b7, a8	3	0
b7, b8	5	1
b7, c8	5	0
b7, c7	8	0
b7, c6	8	0
b7, b6	8	0
b7, a6	5	0
b7, a7	5	1
b8, a8	3	0
b8, a7	5	1
b8, b7	8	1
b8, c7	8	0
b8, c8	5	0
TOTAL:	**105**	**6**

The king returns to a8 six times out of 105 i.e. 2 out of 35 times on average (5.7%).

48. The Invention of Chess — Groups

Age 11+	*Exponential Growth, Geometric Sequence*

There is a legend about the invention of Chess. The modern version is this. When the inventor showed the Persian king his new game, the Shah was very impressed and offered him a choice of two rewards. Either the inventor could have €1 million for every square on the chessboard i.e. €64 million for the whole board, or he could have 1 cent for the first square, 2 cents for the second square, 4 cents for the third square, doubling each time, all the way up to sixty-four squares.
Which option would you choose?

Small scale trial

To understand the inventor's dilemma, use a reduced chessboard (4x4). Fill all the squares of the board with rice grains. Children can «feel» the very rapid growth through this simple procedure. The growth rate gets faster the more rice grains we get – the illustration of exponential growth. With a standard 8x8 board, we get the following which is impossible to handle manually.

Intermediate question

Which is greater:

a) The total of the grains of rice in squares 1-8
b) The number of grains of rice in square 9

Answer: b)
Repeat question for comparisons {1-16 v 17} and {1-32 v 33} with the same answer.

Solution Method

This exercise involves doing a lot of long calculations. A calculator or spreadsheet should be used to save time and achieve accuracy. The formula for the number of grains of rice for n squares is $2^n - 1$.
Multiply 2 by itself according to the number of squares required and subtract 1.
To convert cents to euros, divide by 100.

Answer

The doubling option gives a much higher figure. Although starting slowly, once half the board is covered, the situation has changed dramatically. By square 33, the doubling total has reached nearly €43 million overtaking the first option of €33 million. Filling the complete board would cost more than all the money in the world.

49. How many rectangles on a chessboard? Pairs

Age 11+	*Enumeration, Shapes, Organising information in tables*

This is a challenging task for primary students, therefore best done as a teacher-led activity.
First present this table without the numbers in the second column:

Rectangle type	No. of this rectangle on the chessboard	
1x1	8x8	= 64
1x2	7x8x2	= 112
1x3	6x8x2	= 96
1x4	5x8x2	= 80
1x5	4x8x2	= 64
1x6	3x8x2	= 48
1x7	2x8x2	= 32
1x8	1x8x2	= 16
2x2	7x7	= 49
2x3	6x7x2	= 84
2x4	5x7x2	= 70
2x5	4x7x2	= 56
2x6	3x7x2	= 42
2x7	2x7x2	= 28
2x8	1x7x2	= 14
3x3	6x6	= 36
3x4	5x6x2	= 60
3x5	4x6x2	= 48
3x6	3x6x2	= 36
3x7	2x6x2	= 24
3x8	1x6x2	= 12
4x4	5x5	= 25
4x5	4x5x2	= 40
4x6	3x5x2	= 30
4x7	2x5x2	= 20
4x8	1x5x2	= 10
5x5	4x4	= 16
5x6	3x4x2	= 24
5x7	2x4x2	= 16
5x8	1x4x2	= 8
6x6	3x3	= 9
6x7	2x3x2	= 12
6x8	1x3x2	= 6
7x7	2x2	= 4
7x8	1x2x2	= 4
8x8	1x1	= 1
Total:		**1296**

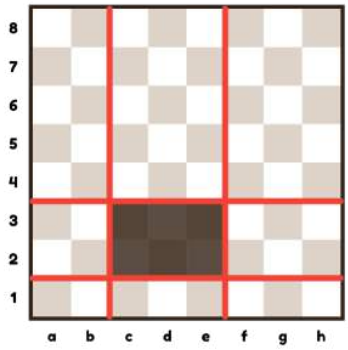


Figure 49(a)

Solution Method

Ask pairs of children to cut out one or more of the rectangle types from paper. Their task is to place this on the chessboard as many different ways as possible and record their result. Encourage them to find a systematic way to do the counting.

Beware of pitfalls such as counting **axb** but not counting **bxa** rectangles. Collect all answers and correct if necessary.

The total number of rectangles that can be found on a chessboard is 1296.

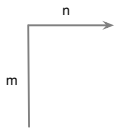
Advanced Solution Method: A rectangle is bounded by two vertical and two horizontal lines as shown on Figure 49(a). Every rectangle is uniquely determined by a vertical and a horizontal pair of lines. There are as many horizontal pairs as vertical ones, so it is enough to count the total number of vertical pairs that can be drawn.

There are 9 ways to choose the first vertical line and 8 ways to choose the second. We have double-counted each pair, so the number of vertical pairs is 9x8/2 = 36. There are 36 horizontal pairs as well. For each vertical pair we can choose any of the 36 horizontal pairs, hence there are 36x36 choices, giving a total of 1296 rectangles.

50. Leaper problem — Pairs

Age 11+	*Enumeration, symmetry, angles*

Leapers are pieces which move m squares in one direction and then n squares at a right angle. The only leaper used in chess is the knight which is a (2,1) leaper.



Tasks

(a) Produce a Power Contour for each of these leapers below : (*Fig. 50*)

(2,1) Knight
(2,2) Alfil
(3,1) Camel
(3,2) Zebra [optional]
(3,3) Tripper [optional]
(4,1) Giraffe [optional]

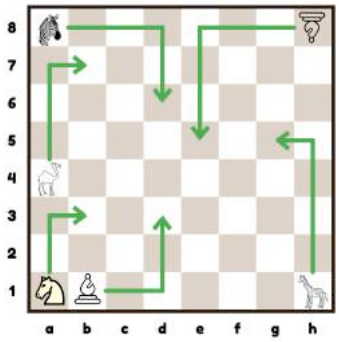


Fig. 50 Leapers

Hint: To build a piece contour, take a look at some examples of piece power contours (Exercise 10).

(b) Identify the leapers defined by the contour maps below

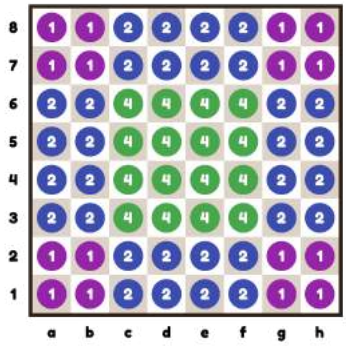


Fig. 50(a)

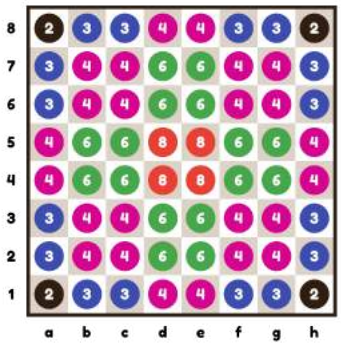


Fig. 50(b)

Solution Method

Visually compare the diagrams drawn in (a) with the contours given in (b).

Solution

(a) Alfil (2,2)
(b) Camel (3,1)