Dynamics

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Lecture and Problem Class Times: (all in D110)
Monday 15.00, Thursday 14.00; Lecture or Problem Class - Friday 17.00

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Books
Theoretical mechanics, Spiegel
Introduction to classical mechanics, French & Ebison

Contents

These lecture notes are based on the notes originally prepared by Prof. Sutcliffe. We thank him for making his notes available to us.

1 Introduction and Newton's Laws

Dynamics - how things move and interact.
Math model - classical mechanics - good approx.
Need to be more sophisticated for objects which are:
• very small - quantum mechanics
• very fast - special relativity
• very heavy - general relativity.

Math model

1. Physical quantities → math objects
2. Make simplifications
3. Physical laws → equations
4. Solve the equations
5. (Compare results with experiment to see if the model is good.)

Physical laws: Newton's laws of motion (1665, age 22).

N1: First Law:

Every body remains in a state of rest or in uniform motion in a straight line unless acted upon by some external force.

N2: Second Law:

The rate of change of momentum of a body is equal to the net force acting on it. ($F = ma$, if m constant).

N3: Third Law:

Action and reaction are equal and opposite.

2 Kinematics

An object (eg. ball, planet,...) is idealized as a point particle (zero size) with a quantity of matter called *mass*.
Good approx if size of object $\ll$ trajectory, and rotation not important.
A point particle has position vector $\mathbf{r}(t)$ at time t, given a chosen origin $\mathcal{O}$.
Write down the equation of motion for $\mathbf{r}(t)$ (ODE) and solve it to find the trajectory $\mathbf{r}(t)$ ie. a curve in space.

2.1 Definitions

Definitions of some quantities.

velocity

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}}$$

is tangent to the trajectory.
speed, $v = |\mathbf{v}| \geq 0$, magnitude of the velocity.
momentum, $\mathbf{p} = m\mathbf{v}$.
acceleration

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \ddot{\mathbf{r}}$$

kinetic energy, $T = \frac{1}{2}mv^2$.

2.2 Cartesian coordinates

$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$
where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are fixed orthogonal unit vectors ie $\mathbf{i} \cdot \mathbf{i} = 1, \mathbf{i} \cdot \mathbf{j} = 0$, etc, eg. $\mathbf{i} = (1, 0, 0)$.
In mechanics do not write just the components, $\mathbf{r} = (x, y, z)$, but include the basis vectors ie. $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

This is because sometimes the basis vectors are not constant, and then we would get the wrong answer for the velocity if we just differentiated the components.
If $\mathbf{r} = \alpha\mathbf{e}$, then $\dot{\mathbf{r}} = \dot{\alpha}\mathbf{e} + \alpha\dot{\mathbf{e}} \neq \dot{\alpha}\mathbf{e}$ if $\dot{\mathbf{e}} \neq 0$.

eg. $\mathbf{r} = t\mathbf{i} + \mathbf{j} + t^2\mathbf{k}$ with $m = 2$.
$\mathbf{v} = \dot{\mathbf{r}} = \mathbf{i} + 2t\mathbf{k}$, $\quad v = |\mathbf{v}| = \sqrt{1+4t^2}$, $\quad \mathbf{p} = m\mathbf{v} = 2\mathbf{v} = 2\mathbf{i} + 4t\mathbf{k}$, $\mathbf{a} = \dot{\mathbf{v}} = 2\mathbf{k}$, $\quad T = \frac{1}{2}mv^2 = (1 + 4t^2)$.

Note: Acceleration can be non-zero even if the speed is constant, since the direction of the velocity might not be constant.

Given the acceleration at all times and initial position and velocity, the position can be found by integration.
eg. $\mathbf{a} = 2\mathbf{k}$, $\quad \mathbf{r}(0) = \mathbf{j} + \mathbf{k}$, $\quad \mathbf{v}(0) = \mathbf{i}$.
$\mathbf{v} = 2t\mathbf{k} + \mathbf{c}$, but $\mathbf{v}(0) = \mathbf{c} = \mathbf{i}$ therefore $\mathbf{v} = 2t\mathbf{k} + \mathbf{i}$.
$\mathbf{r} = t^2\mathbf{k} + t\mathbf{i} + \mathbf{d}$, but $\mathbf{r}(0) = \mathbf{d} = \mathbf{j} + \mathbf{k}$, therefore $\mathbf{r} = (t^2 + 1)\mathbf{k} + t\mathbf{i} + \mathbf{j}$.

2.3 Polar coordinates and vectors

Consider motion in a plane, using polar coordinates r, θ, where $x = r\cos\theta$ and $y = r\sin\theta$.
$\mathbf{r} = x\mathbf{i} + y\mathbf{j} = r(\cos\theta\mathbf{i} + \sin\theta\mathbf{j})$.
The **radial unit vector** $\mathbf{e}_r$ is a vector in the direction of $\mathbf{r}$,
$\mathbf{e}_r = \frac{\mathbf{r}}{r} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$.
The **tangential unit vector** $\mathbf{e}_\theta$ is a vector perpendicular to $\mathbf{e}_r$, and is
$\mathbf{e}_\theta = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$, $\quad$ (increasing θ is anti-clockwise).

If the particle is moving then r and θ can depend on time.
$\dot{\mathbf{e}}_r = \frac{d\theta}{dt}\frac{d}{d\theta}\mathbf{e}_r = \dot{\theta}(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) = \dot{\theta}\mathbf{e}_\theta$.
$\dot{\mathbf{e}}_\theta = \frac{d\theta}{dt}\frac{d}{d\theta}\mathbf{e}_\theta = \dot{\theta}(-\cos\theta\mathbf{i} - \sin\theta\mathbf{j}) = -\dot{\theta}\mathbf{e}_r$.

Note: $\mathbf{e}_r \cdot \mathbf{e}_r = \mathbf{e}_\theta \cdot \mathbf{e}_\theta = 1$ and $\mathbf{e}_r \cdot \mathbf{e}_\theta = 0$ for all time.

$\mathbf{r} = r\mathbf{e}_r$ therefore $\dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\mathbf{e}}_r = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta$.

$$\ddot{\mathbf{r}} = \ddot{r}\mathbf{e}_r + \dot{r}\dot{\mathbf{e}}_r + \dot{r}\dot{\theta}\mathbf{e}_\theta + r\ddot{\theta}\mathbf{e}_\theta + r\dot{\theta}\dot{\mathbf{e}}_\theta = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_\theta.$$

$$\boxed{\mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta, \qquad \mathbf{a} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_\theta.}$$

Eg. Motion in a circle with constant speed
$r = \rho$ with ρ constant.
$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta = \rho\dot{\theta}\mathbf{e}_\theta$.
Note that for circular motion $\mathbf{v} \cdot \mathbf{r} = 0$ since $\mathbf{e}_\theta \cdot \mathbf{e}_r = 0$.
$v = |\mathbf{v}| = |\rho\dot{\theta}||\mathbf{e}_\theta| = |\rho\dot{\theta}|$.
Hence for constant speed $\dot{\theta} = \omega$ with ω constant (choose $\omega > 0$.)
$v = \omega\rho$ hence $\omega = v/\rho$.
Since $\dot{\theta}$ is constant then $\ddot{\theta} = 0$, so
$\mathbf{a} = -\rho\omega^2\mathbf{e}_r$. Hence $a = |\mathbf{a}| = \rho\omega^2 = v^2/\rho$.
The acceleration is directed radially **inwards**.
This is called centripetal (centre-seeking) acceleration.
Warning: do not confuse with centrifugal (centre-fleeing) – see later.

2.4 Units and dimensions

Generally use SI units (often drop units altogether).
Mass kg, length m, time s. Remember to convert eg. mins to seconds.

Dimensions are similar to units but more significant.

Quantity	Dimension
Mass	M
Length	L
Time	T

Write [mass] = M etc
[velocity] = $[\frac{\text{length}}{\text{time}}] = \text{LT}^{-1}$, [acceleration] = $[\frac{\text{velocity}}{\text{time}}] = \text{LT}^{-2}$.

Correct equations must have the same dimensions on each side.
Can check consistency using dimensional analysis.

Eg. Period of a pendulum
Pendulum of length l and mass m swings under gravity (acceleration due to gravity g). Its period is $2\pi\sqrt{l/g}$. Check this has the correct dimensions

$$[2\pi\sqrt{\frac{l}{g}}] = \sqrt{\frac{[l]}{[g]}} = \sqrt{\frac{L}{LT^{-2}}} = T$$

.
An expression like mg/l is obviously wrong, since

$$[mg/l] = MLT^{-2}L^{-1} = MT^{-2} \neq T.$$

Can calculate the dimensions of constants in expressions.
Eg. suppose a force is given by κA, where A is the surface area of an object.

$$[\text{force}] = [\text{mass} \times \text{acceleration}] = MLT^{-2} = [\kappa A] = [\kappa]L^2$$

hence $[\kappa] = ML^{-1}T^{-2}$, so could be given in units of $kg/m/s^2$.

2.5 Relative motion

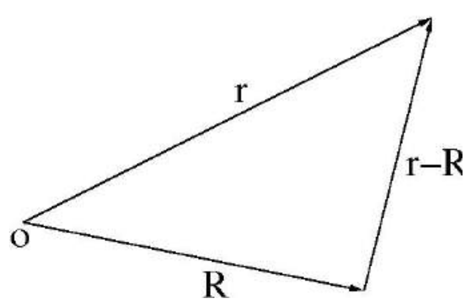

Figure 1: Relative position

$\mathbf{r}$ is the position of an object with respect to a fixed origin $\mathcal{O}$. Let an observer (possibly moving) have position $\mathbf{R}$. Then the **relative position** of the object to the observer is

$$\tilde{\mathbf{r}} = \mathbf{r} - \mathbf{R}.$$

Relative velocity $\dot{\tilde{\mathbf{r}}} = \dot{\mathbf{r}} - \dot{\mathbf{R}}$, relative acceleration $\ddot{\tilde{\mathbf{r}}} = \ddot{\mathbf{r}} - \ddot{\mathbf{R}}$, *Eg. Bart is going north (direction* $\mathbf{j}$*) on his skateboard at 10mph and feels a headwind of 25mph. What is the windspeed (velocity) relative to the ground?*

$\mathbf{R}$ is Bart's position, and $\mathbf{r}$ is the position of an air particle.
(Units are miles and hours).
$\dot{\mathbf{R}} = 10\mathbf{j}, \quad \dot{\tilde{r}} = -25\mathbf{j} = \dot{\mathbf{r}} - \dot{\mathbf{R}}$
$\dot{\mathbf{r}} = -25\mathbf{j} + 10\mathbf{j} = -15\mathbf{j}$
Windspeed relative to the ground is 15mph southward.

If Bart now goes east at 15mph what wind does he feel?
$\dot{\mathbf{R}} = 15\mathbf{i}, \quad \dot{\mathbf{r}} = -15\mathbf{j}$, so $\dot{\tilde{\mathbf{r}}} = \dot{\mathbf{r}} - \dot{\mathbf{R}} = -15\mathbf{j} - 15\mathbf{i}$
$|\dot{\tilde{\mathbf{r}}}| = 15\sqrt{2}$, so feels a wind of $15\sqrt{2}$mph in the direction $-(\mathbf{i}+\mathbf{j})/\sqrt{2}$ ie. southwest.

Centrifugal acceleration
This is a result of viewing centripetal acceleration in rotating coordinates.
Let $\mathbf{R}$ be the position of an observer moving in circular motion with radius ρ and constant speed v eg. child on a roundabout.
$\ddot{\mathbf{R}} = -\frac{v^2}{\rho}\mathbf{e}_R$ hence $\ddot{\tilde{\mathbf{r}}} = \ddot{\mathbf{r}} - \ddot{\mathbf{R}} = \ddot{\mathbf{r}} + \frac{v^2}{\rho}\mathbf{e}_R$,
so even for an object with no forces acting in this plane $\ddot{\mathbf{r}} = \mathbf{0}$,
eg. ball released by the child, then $\ddot{\tilde{\mathbf{r}}} = \frac{v^2}{\rho}\mathbf{e}_R$, so observer sees a relative acceleration directed radially **outwards**.
This is **centrifugal** (centre-fleeing) acceleration.
eg. Child sees the ball flying outwards.

2.6 Inertial frames

The above example appears to contradict **N1** – no forces, no acceleration.
In fact **N1** defines the type of observer (or better reference frame) for which **N2** holds.

An **inertial frame** is one which is not accelerating ie. $\ddot{\mathbf{R}} = \mathbf{0}$,
then $\ddot{\tilde{\mathbf{r}}} = \ddot{\mathbf{r}} - \ddot{\mathbf{R}} = \ddot{\mathbf{r}}$ so see the 'true' acceleration.

N2 can be applied only in an inertial frame.

Eg. coordinates on the earth gives an approximately inertial frame
Radius of the earth $\rho \approx 6,400km = 6.4 \times 10^6 m$.
Rotation frequency $\omega = \frac{2\pi}{T} = \frac{2\pi}{24\times60\times60} \approx 7.3 \times 10^{-5} radians/s$
At the equator, the centripetal acceleration has magnitude
$\rho\omega^2 = 6.4 \times 10^6 \times (7.3 \times 10^{-5})^2 = 0.034 ms^{-2}$
over a 100 times weaker than the acceleration due to gravity $g = 9.8ms^{-2}$.
Regarding these coordinates as exactly inertial is a typical simplification.

3 Forces

A force is an influence (push, pull, gravity, friction,...) that makes a body change its momentum (velocity when m constant). It is modelled by a vector.

$$\textbf{N2}: \qquad \frac{d}{dt}(m\mathbf{v}) = \mathbf{F}$$

where $\mathbf{F}$ is the sum of all forces that act.
If the mass m is constant then

$$\mathbf{F} = m\mathbf{a}.$$

The SI unit of force is the Newton; $1\,N = 1\,kg\,m\,s^{-2}$.

Generally, $m\ddot{\mathbf{r}} = \mathbf{F}(t, \mathbf{r}, \dot{\mathbf{r}})$ is a set of 3 coupled 2nd order ODE's - tricky.
Simplify by (i) choice of coordinates, (ii) use of conservation laws.

3.1 One-dimensional problems

If the relevant forces all act along a line that coincides with the initial motion, then a suitable choice of coordinates will reduce the problem to a 1D problem.

eg. push a coin along a table top with constant velocity.
$\mathbf{i}$ horizontal direction of push, $\mathbf{k}$ vertical direction.
Forces: $\mathbf{i}$ push, friction; $\mathbf{j}$ none; $\mathbf{k}$ weight, reaction of table.
$\mathbf{v}$ = const so $\mathbf{F} = 0 = \mathbf{i}$(push+friction) + $\mathbf{k}$(weight+reaction).
push+friction = 0 and weight+reaction = 0, **N3**.
If now push along $\mathbf{i}$ direction so that speed is not constant, only need to worry about $\mathbf{i}$ component of $\mathbf{F} = m\mathbf{a}$ ie. 1D problem.

eg. A beer glass of mass m slides along a horizontal bar, slowed by a frictional force of magnitude be^{av}, where v is the speed and $a, b > 0$ constants. If its initial speed is u, how long does it take to come to rest?
Choose $\mathbf{i}$ as the horizontal direction along the bar.
$\mathbf{r} = x\mathbf{i}$ and $\mathbf{F} = -be^{av}\mathbf{i}$, where $v = \dot{x} \geq 0$

$$m\dot{v} = m\frac{dv}{dt} = F = -be^{av}, \quad \int me^{-av}\,dv = \int -b\,dt$$

$$-\frac{m}{a}e^{-av} + c = -bt, \quad \text{at } \ t = 0, v = u, \ \text{hence } \frac{m}{a}(e^{-av} - e^{-au}) = bt$$

Comes to rest at $t = T$ when $v = 0$ hence

$$T = \frac{m}{ab}(1 - e^{-au}).$$

Vertical motion near the Earth's surface
Acceleration due to gravity $g \approx 9.8ms^{-2}$ is independent of mass (Galileo) and is approximately constant (make this simplification) near the Earth's surface. The gravitational force of magnitude mg on a mass m is called its *weight*.

Choose $\mathbf{i}$ to be vertically up and x to be height above the ground.
$\mathbf{F} = -mg\mathbf{i}$, so $m\ddot{x}\mathbf{i} = -mg\mathbf{i}$ and hence $\ddot{x} = -g$.
eg. If an object is thrown upwards from the ground with speed u find the maximum height it reaches.
$\ddot{x} = -g$ so by integrating $\dot{x} = -gt + c$, but at $t = 0$ we have $\dot{x} = u = c$, therefore $\dot{x} = -gt + u$ and integrating once more $x = -\frac{1}{2}gt^2 + ut + d$ where $d = 0$ since at $t = 0$ we have $x = 0$.

Maximum height is when $\dot{x} = 0$ hence at $t = u/g$ at which time

$$x = -\frac{1}{2}g(\frac{u}{g})^2 + u\frac{u}{g} = \frac{u^2}{2g}.$$

$x_{\text{max}} = \frac{1}{2}u^2/g$, eg. if $u = 15ms^{-1}$ then $x_{\text{max}} \approx 11.5m$.
Note 1: the answer is independent of mass.
Note 2: we ignored air resistance.

Terminal velocity
Friction, eg. air resistance, opposes the motion and slows it.
$\mathbf{F} = -b\mathbf{v}$, where $b > 0$, is a simple but reasonable model at low speeds.
Note: force depends on the velocity and not on the position.

eg. Parachutist dropping vertically from rest ie. $\mathbf{v}(0) = \mathbf{0}$.
Choose $\mathbf{i}$ direction as vertically down, so $v = \dot{x} \geq 0$.
$m\dot{v} = mg - bv$.
A *terminal velocity* is reached at which there is no further acceleration, ie. $\dot{v} = 0$ if $v = mg/b \equiv v_T$, when gravity and air resistance balance.
Note that v_T depends on m (and b) so different objects fall at different speeds.

$$\dot{v} + \frac{b}{m}v = g$$

Solve using an integrating factor,

$$I = \exp \int \frac{b}{m}\, dt = e^{bt/m}$$

$$v = e^{-bt/m} \int g e^{bt/m}\, dt = e^{-bt/m} g(\frac{m}{b}e^{bt/m} + \alpha)$$

$v(0) = 0$ gives $\alpha = -m/b$ and hence

$$v = \frac{mg}{b}(1 - e^{-bt/m}).$$

Note 1: As $t \to \infty$ then $v \to mg/b = v_T$.
Note 2: Can also solve with $v(0) = u$ and see that v_T does not depend on u.

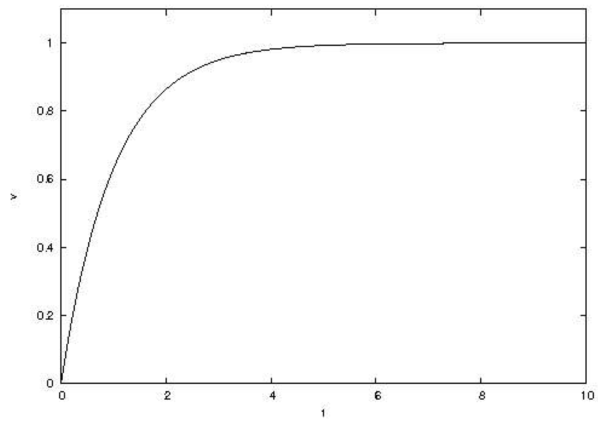

Figure 2: Sketch of $v(t)$ showing approach to terminal velocity

Note 3: Terminal velocity is not reached in finite time, it is an asymptotic limit as $t \to \infty$.

A better model of air resistance is $\mathbf{F} = -C\rho A v \mathbf{v}$ where C is the drag coefficient, ρ is the air density, A is the cross-sectional area of the object. This model is quadratic in speed and is slightly harder to solve – try it as an exercise.

Sliding with friction on an inclined plane

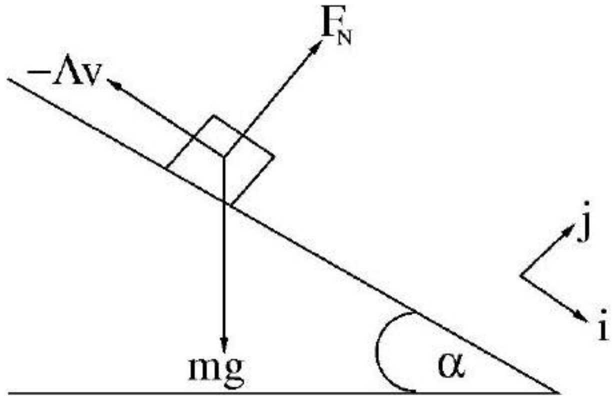

Figure 3: An object on a slope at angle α with $\mathbf{i}$ down slope and $\mathbf{j}$ normal to slope. Forces are gravity acting down, normal part of planes reaction force and friction up slope.

Friction $-\Lambda\mathbf{v}$ and normal part of reaction force is $\mathbf{F}_N$.

$$m\dot{\mathbf{v}} = m\mathbf{g} - \Lambda\mathbf{v} + \mathbf{F}_N.$$

$\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, with $\mathbf{r}\cdot\mathbf{k} = 0$.
$\mathbf{g} = g(\sin\alpha\,\mathbf{i} - \cos\alpha\,\mathbf{j})$, and $\mathbf{F}_N = F_N\mathbf{j}$.

$$\ddot{\mathbf{r}} = \ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} = g\sin\alpha\,\mathbf{i} - g\cos\alpha\,\mathbf{j} - \mu(\dot{x}\mathbf{i} + \dot{y}\mathbf{j}) + \frac{F_N}{m}\mathbf{j}$$

where $\mu = \Lambda/m$.
$\ddot{y} = -g\cos\alpha - \mu\dot{y} + \frac{F_N}{m}$
The motion remains on the plane so $y = \dot{y} = \ddot{y} = 0$ giving $F_N = mg\cos\alpha$.

$\ddot{x} = g\sin\alpha - \mu\dot{x}$, which can be written as $\dot{v} + \mu v = g\sin\alpha$ where $v = \dot{x}$.
Given $v(0) = 0$ then solving using an integrating factor gives

$$v = \frac{g\sin\alpha}{\mu}(1 - e^{-\mu t}).$$

Given an initial condition $x(0) = 0$ then

$$\dot{x} = \frac{g\sin\alpha}{\mu}(1 - e^{-\mu t})$$

can be directly integrated to give

$$x = \frac{g\sin\alpha}{\mu}\Big(t + \frac{1}{\mu}(e^{-\mu t} - 1)\Big).$$

3.2 Restoring forces and simple harmonic motion

Elastic/springy forces increase with deformation eg. springs, beams in bridges, inter-molecular forces,. . . .
Usually, such forces depend only on position and not on velocity.

eg. spring, $\mathbf{F} = -Kx\mathbf{i}$, where $K > 0$ is called the *spring constant*.
Force is along a fixed line $\mathbf{i}$ and is always directed towards $x = 0$, hence a restoring force.
$m\ddot{x} = -Kx$, so define $K/m = \omega^2$ to get $\ddot{x} = -\omega^2 x$.

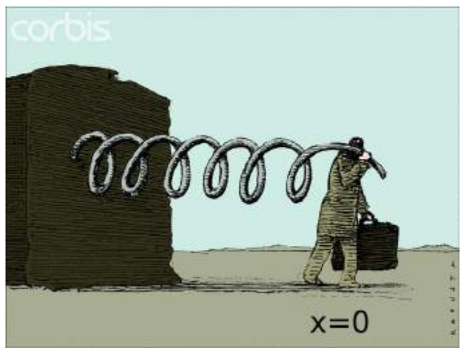

Figure 4: A spring attached to a wall, with $x = 0$ as equilibrium.

This is the ODE of *simple harmonic motion* (SHM), with frequency ω.
General solution is $x = A\cos(\omega t) + B\sin(\omega t) \quad (*)$.
The motion is periodic $x(t + T) = x(t)$ with period (shortest repeat time)

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{K}}.$$

Another way to write $(*)$ is as $x = C\sin(\omega t - \alpha)$ which agrees with $(*)$ if $C = \sqrt{A^2 + B^2}$ and $\tan\alpha = -A/B$.
In this form it is clear that the amplitude is $|x|_{\max} = C$.
SHM is important because it occurs universally.
Note: oscillation can be about $X = a$ not $x = 0$. If we shift coordinates and write $X = x + a$ then $\ddot{X} = \ddot{x}$ hence $\ddot{X} = -\omega^2(X - a)$.

3.3 Ballistics

Ballistics - motion under gravity near the Earth's surface.

Assume gravity is constant, ignore air resistance, ignore Earth's rotation.

$$\ddot{\mathbf{r}} = \mathbf{g}, \quad \text{solve to give} \quad \mathbf{r} = \frac{1}{2}\mathbf{g}t^2 + \mathbf{u}t + \mathbf{r}_0$$

where $\mathbf{u} = \dot{\mathbf{r}}(0)$ and $\mathbf{r}_0 = \mathbf{r}(0)$.

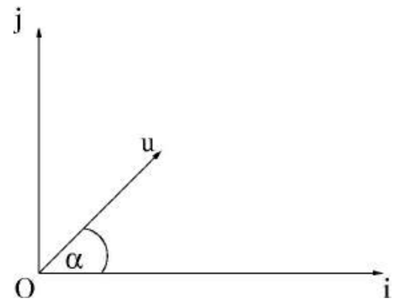

Figure 5: Coordinate system for 2D ballistics

Choose $\mathbf{j}$ vertically upwards, with $\mathbf{k}$ perpendicular to the plane of the trajectory. Choose the origin to be the starting position.
$\mathbf{r}_0 = \mathbf{0}$, and $\mathbf{u} = u(\cos\alpha\mathbf{i} + \sin\alpha\mathbf{j})$.
Then $\mathbf{r} = x\mathbf{i} + y\mathbf{j} = -\frac{1}{2}gt^2\mathbf{j} + ut(\cos\alpha\mathbf{i} + \sin\alpha\mathbf{j})$.
The horizontal component is free motion $x = tu\cos\alpha$.
The vertical motion is under gravity $y = tu\sin\alpha - \frac{1}{2}gt^2$.
Note: rate of fall is independent of horizontal motion.

Time of flight and range
Let T be the time taken to return to the ground ($y = 0$).
$Tu\sin\alpha - \frac{1}{2}gT^2 = 0$, gives ($T = 0$ is not the required solution)

$$T = \frac{2u\sin\alpha}{g}.$$

Let X be the range ie. $x(T) = X$ then

$$X = Tu\cos\alpha = \frac{2u^2\cos\alpha\sin\alpha}{g} = \frac{u^2\sin(2\alpha)}{g}.$$

Note 1: for a given u the range is maximal if $\alpha = \pi/4 = 45°$.
Note 2: doubling the speed increases the range by a factor of 4.

The trajectory is a parabola and can be found by eliminating t to write $y(x)$.
From $x = tu\cos\alpha$ then $t = x/(u\cos\alpha)$.
Put this into $y = tu\sin\alpha - \frac{1}{2}gt^2$ to get

$$y = \frac{xu\sin\alpha}{u\cos\alpha} - \frac{gx^2}{2u^2\cos^2\alpha} = x\tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha}.$$

Maximum height is attained at the turning point

$$\frac{dy}{dx} = 0 = \tan\alpha - \frac{gx}{u^2\cos^2\alpha},$$

which gives the solution

$$x = \frac{u^2\sin\alpha\cos\alpha}{g} = \frac{u^2\sin(2\alpha)}{2g} = \frac{X}{2}.$$

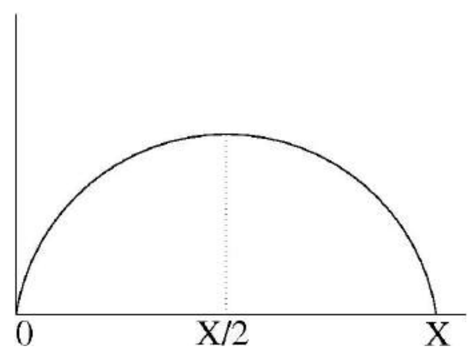

Figure 6: Parabolic trajectory with turning point at half range

Hitting a target

For a given u we want to find the angle α to hit a target at (x_0, y_0),
ie. we want a solution with $y(x_0) = y_0$.
This gives the condition

$$y_0 = x_0 \tan\alpha - \frac{g x_0^2}{2u^2 \cos^2\alpha}$$

to be solved for α.
This can be written as a quadratic in $\tan\alpha$ by using

$$\frac{1}{\cos^2\alpha} = \sec^2\alpha = 1 + \tan^2\alpha$$

to give

$$\left(\frac{g x_0^2}{2u^2}\right)\tan^2\alpha - x_0 \tan\alpha + \frac{g x_0^2}{2u^2} + y_0 = 0.$$

This has two real roots if the target is in range, which merge to a repeated root if the target is *just* in range. There are no real roots if the target is not in range (too high or far away for given u).
Real solution(s) if

$$x_0^2 - 4\left(\frac{g x_0^2}{2u^2}\right)\left(\frac{g x_0^2}{2u^2} + y_0\right) \geq 0$$

$$\frac{u^2}{2g} \geq \frac{g x_0^2}{2u^2} + y_0$$

$$y_0 \leq \frac{u^2}{2g} - \frac{g x_0^2}{2u^2}.$$

Ballistics with air resistance

Include air resistance as a force $-\Lambda\mathbf{v}$ where $\Lambda > 0$.

$$m\dot{\mathbf{v}} = m\mathbf{g} - \Lambda\mathbf{v}.$$

Let $\mu = \Lambda/m$ to write as

$$\dot{\mathbf{v}} + \mu\mathbf{v} = \mathbf{g}.$$

Solve with the integrating factor

$$I = \exp \int \mu \, dt = e^{\mu t}$$

to give

$$\mathbf{v} = e^{-\mu t}\int e^{\mu t}\mathbf{g}\,dt = e^{-\mu t}(\frac{1}{\mu}e^{\mu t}\mathbf{g} + \mathbf{c}) = \frac{1}{\mu}\mathbf{g} + e^{-\mu t}\mathbf{c}.$$

With initial conditions $\mathbf{v}(0) = \mathbf{u}$ then $\mathbf{c} = \mathbf{u} - \frac{1}{\mu}\mathbf{g}$ so

$$\mathbf{v} = \frac{1}{\mu}\mathbf{g} + e^{-\mu t}(\mathbf{u} - \frac{1}{\mu}\mathbf{g}).$$

Note: as $t \to \infty$ then $\mathbf{v} \to \frac{1}{\mu}\mathbf{g} = \frac{m}{\Lambda}\mathbf{g}$
so asymptotically the particle has a terminal velocity along $\mathbf{g}$ where gravity and friction balance.
Integrate the equation once more to get

$$\mathbf{r} = \frac{1}{\mu}\mathbf{g}t - \frac{1}{\mu}e^{-\mu t}(\mathbf{u} - \frac{1}{\mu}\mathbf{g}) + \mathbf{d}$$

Given the initial condition $\mathbf{r}(0) = \mathbf{r}_0$ then

$$\mathbf{d} = \mathbf{r}_0 + \frac{1}{\mu}(\mathbf{u} - \frac{1}{\mu}\mathbf{g})$$

hence finally

$$\mathbf{r} = \frac{1}{\mu}\mathbf{g}t + \mathbf{r}_0 + \frac{1}{\mu}(1 - e^{-\mu t})(\mathbf{u} - \frac{1}{\mu}\mathbf{g}).$$

Exercise: Check that $\mu \to 0$ yields the previous result $\mathbf{r} = \frac{1}{2}\mathbf{g}t^2 + \mathbf{u}t + \mathbf{r}_0$.

3.4 Motion of a charged particle in an electromagnetic field

A particle with electric charge q placed in an electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ feels the *Lorentz force*

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

The magnetic part of the force is perpendicular to the motion and this curves the trajectory.

$$m\dot{\mathbf{v}} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}).$$

This is generally difficult to solve since $\mathbf{E}$ and $\mathbf{B}$ could depend on $\mathbf{r}$. Even if they are constant then it is still not easy as all 3 components of the ODE can be coupled.

Simplified case: **E** and **B** are constant and parallel.
Choose $\mathbf{E} = E\mathbf{k}$ and $\mathbf{B} = B\mathbf{k}$, then

$$\mathbf{v} \times \mathbf{B} = \mathbf{i}\,\dot{y}B - \mathbf{j}\,\dot{x}B.$$

The equation of motion then becomes

$$m(\ddot{x}\mathbf{i} + \ddot{y}\mathbf{j} + \ddot{z}\mathbf{k}) = qE\mathbf{k} + \mathbf{i}q\dot{y}B - \mathbf{j}q\dot{x}B.$$

The **k** component is simple to solve
$m\ddot{z} = qE$, which is just a constant force

$$\dot{z} = \frac{qE}{m}t + \dot{z}(0), \text{ we choose } \dot{z}(0) = 0.$$

$$z = \frac{qE}{2m}t^2 + z(0), \text{ we choose } z(0) = 0.$$

$$z = \frac{qE}{2m}t^2$$

The x and y equations are more difficult because they are coupled

$$m\ddot{x} = qB\dot{y}, \quad m\ddot{y} = -qB\dot{x}.$$

Solve these equations with $\mathbf{r}(0) = \mathbf{0}$ and $\dot{\mathbf{r}} = u\mathbf{i}$.

$$m\dot{x} = qBy + c_1, \quad m\dot{y} = -qBx + c_2$$

but at $t = 0$ then $x = y = 0$ and $\dot{x} = u$ with $\dot{y} = 0$, hence
$c_1 = mu$ and $c_2 = 0$ to give

$$m\dot{x} = qBy + mu, \quad m\dot{y} = -qBx$$

Two first order ODE's can be written as one second order ODE

$$m\ddot{x} = qB\dot{y} = -\frac{q^2B^2}{m}x$$

which we can write in the form

$$\ddot{x} = -\omega^2 x, \text{ where } \omega^2 = \frac{q^2B^2}{m^2}.$$

This is simple harmonic motion with solution
$x = a\cos(\omega t) + b\sin(\omega t)$. Using $x(0) = 0$ gives $a = 0$.
Then $\dot{x} = b\omega\cos(\omega t)$, so $\dot{x}(0) = b\omega = u$ hence $b = u/\omega$.

$$x = \frac{u}{\omega}\sin(\omega t).$$

Next
$$y = \frac{m}{qB}(\dot{x} - u) = \frac{m}{qB}(u\cos(\omega t) - u) = \frac{u}{\omega}(\cos(\omega t) - 1).$$

Note that the above solution satisfies the relation

$$x^2 + \left(y + \frac{u}{\omega}\right)^2 = \frac{u^2}{\omega^2}$$

hence the motion in the (x, y)-plane is in a circle with radius u/ω centre $(x, y) = (0, -u/\omega)$, and angular frequency $\omega = qB/m$.

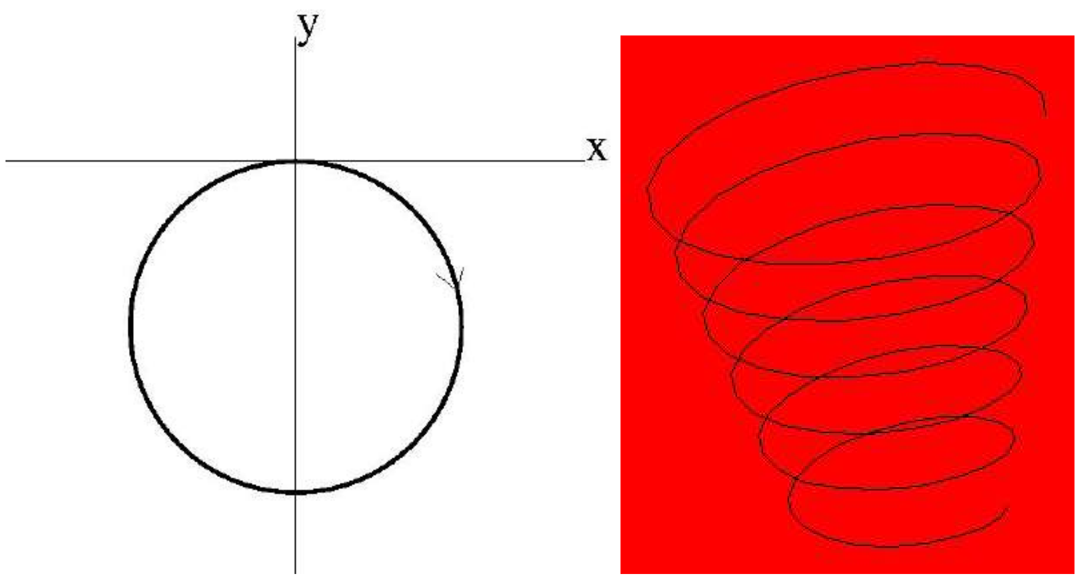

Figure 7: Cyclotron and helix trajectories

This is called *cyclotron motion* and ω is the *cyclotron frequency.*
Magnetic fields are used in this way to constrain charged particles to move in circular paths. Accelerated charged particles radiate electromagnetic waves – this is how radios/TVs/phones etc work.

Recall that for our example z is also increasing quadratically with t, because of the electric field. Hence the path of the particle is a helix with increasing *pitch* (the distance in the z direction between points after one complete turn).

4 Energy

4.1 Energy conservation for one-dimensional problems

Suppose the force $F(x)$ depends only on the position x and not on time t or velocity $\dot{x}$.
Such a force is called a *conservative force.*
Friction for example is NOT a conservative force.
For a conservative force

$$m\ddot{x} = F(x)$$

and we can integrate by first multiplying by $\dot{x}$

$$m\dot{x}\ddot{x} = \dot{x}F(x)$$

$$\frac{m}{2}\frac{d}{dt}(\dot{x}^2) = -\frac{d}{dt}V(x) = -\dot{x}\frac{dV}{dx} = \dot{x}F$$

where we have defined $V(x)$ by

$$V = -\int F\,dx \text{ so that } -\frac{dV}{dx} = F.$$

Trivially integrating the equation gives

$$\frac{m}{2}\dot{x}^2 = -V + E$$

where the constant of integration E is called *energy.*
This is the familiar relation

$$E = \frac{m}{2}v^2 + V$$

total energy = kinetic energy + potential energy.

Note 1: $V(x)$ is only defined up to a constant, so we are free to choose a convenient value as the *zero of energy.*

Note 2: Motion is constrained by the inequality $E \geq V$ since $\frac{m}{2}v^2 \geq 0$.

Note 3: The equation of motion $m\ddot{x} = F$ is recovered from the *conservation of energy* $\frac{dE}{dt} = 0$.

Eg. Gravity near the Earth's surface.
Choose $\mathbf{i}$ to be up and $x = 0$ to be the Earth's surface.
$\mathbf{F} = -mg\mathbf{i} = F(x)\mathbf{i}$, hence $F = -mg$.

$$V = -\int F\,dx = \int mg\,dx = mgx, \quad \{\text{chosen } V(0) = 0\}.$$

$$E = \frac{m}{2}v^2 + mgx.$$

Suppose at $x = x_1$ then $v = v_1$ and at $x = x_2$ then $v = v_2$.

$$E = \frac{m}{2}v_1^2 + mgx_1 = \frac{m}{2}v_2^2 + mgx_2.$$

$$v_2^2 - v_1^2 = 2g(x_1 - x_2)$$

so the speed gained can be obtained from the distance fallen.

Eg. if a stone is thrown upwards from the ground with a speed u then it reaches its maximum height X when $v = 0$.

$$E = \frac{m}{2}u^2 = mgX, \text{ hence } X = \frac{u^2}{2g}$$

as obtained before by solving the ODE.

The one-dimensional energy equation is a separable first order ODE for $x(t)$.

$$\frac{m}{2}\left(\frac{dx}{dt}\right)^2 + V = E$$

can be re-written as

$$\int \frac{dx}{\sqrt{E - V}} = \pm\sqrt{\frac{2}{m}}\int dt = \pm\sqrt{\frac{2}{m}}(t - t_0).$$

To write an explicit solution depends on being able to do the integral, which depends on how complicated $V(x)$ is.

Eg. Restoring force $F = -Kx$, with $K > 0$.

$$V = -\int F\,dx = \int Kx\,dx = \frac{1}{2}Kx^2.$$

$$\pm\sqrt{\frac{2}{m}}(t - t_0) = \int \frac{dx}{\sqrt{E - \frac{1}{2}Kx^2}}.$$

The integral can be done in this case (exercise) to give

$$x = \pm\sqrt{\frac{2E}{K}}\sin(\omega(t - t_0))$$

where $\omega = \sqrt{K/m}$ as before when we solved this problem of SHM.

4.2 Energy conservation for three-dimensional problems

In 3D the energy of a conservative system is

$$E = \frac{m}{2}\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} + V(\mathbf{r})$$

where

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{i} + \frac{\partial V}{\partial y}\mathbf{j} + \frac{\partial V}{\partial z}\mathbf{k} = -\mathbf{F} = -F_x\mathbf{i} - F_y\mathbf{j} - F_z\mathbf{k}.$$

Then

$$\frac{dE}{dt} = m\dot{\mathbf{r}}\cdot\ddot{\mathbf{r}} + \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} + \frac{\partial V}{\partial z}\dot{z} = \dot{\mathbf{r}}\cdot\mathbf{F} - \mathbf{F}\cdot\dot{\mathbf{r}} = 0$$

so energy is again conserved.

5 Motion near equilibrium

5.1 One-dimensional systems with $V(x)$

An equilibrium position (or point) $x = x_0$ is where $F(x_0) = 0$.

ie. the particle remains at rest if placed at $x = x_0$.
eg. restoring force,
$m\ddot{x} = -Kx$, where $K > 0$, only equilibrium point is $x = 0$.
For conservative systems $F = -V'(x)$ so equilibrium positions are stationary points (max and min) of the potential energy V.
For the above eg. $V' = Kx$ so $V = \frac{1}{2}Kx^2$ and equilibrium position is $x = 0$, where V is a minimum.

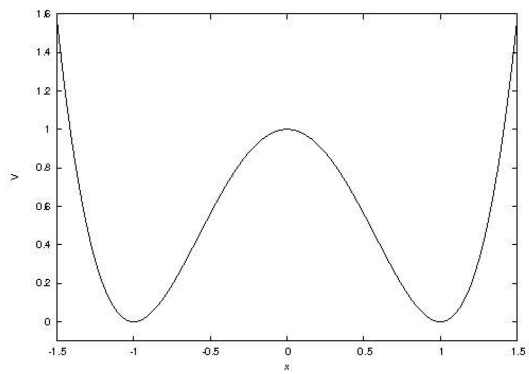

Figure 8: A typical potential with some maximum and minimum points

An equilibrium position x_0 is a local minimum and is **stable** if $V''(x_0) > 0$.
An equilibrium position x_0 is a local maximum and is **unstable** if $V''(x_0) < 0$.
Eg. $V = x^3 - x$, so $V' = 3x^2 - 1$, giving $V' = 0$ if $x = \pm\frac{1}{\sqrt{3}}$.
Note that $V(\pm\infty) = \pm\infty$, $V(0) = 0$, $V(\pm\frac{1}{\sqrt{3}}) = \mp\frac{2}{3\sqrt{3}}$ and $V(\pm 1) = 0$.
For this example $V'' = 6x$, hence $V''(\frac{1}{\sqrt{3}}) = 2\sqrt{3} > 0$, confirming that $x = \frac{1}{\sqrt{3}}$ is a stable equilibrium, whereas $V''(-\frac{1}{\sqrt{3}}) = -2\sqrt{3} < 0$, confirming that $x = -\frac{1}{\sqrt{3}}$ is an unstable equilibrium.

Small motion about a stable equilibrium is well described by SHM.
Consider motion close to a stable equilibrium x_0.
Put $x = x_0 + \epsilon$ then
$m\ddot{x} = m\ddot{\epsilon} = F = -V'(x) = -V'(x_0 + \epsilon)$
Taylor expand to give
$m\ddot{\epsilon} = -V'(x_0) - \epsilon V''(x_0) + \ldots$
But x_0 is an equilibrium so $V'(x_0) = 0$, thus neglecting $\ldots$ gives

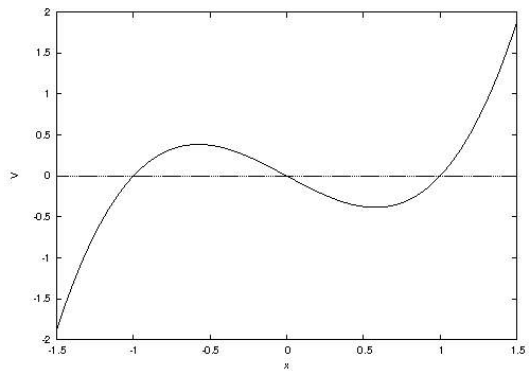

Figure 9: Sketch of the cubic $V = x^3 - x$ to illustrate that $x = \frac{1}{\sqrt{3}}$ is a local minimum and $x = -\frac{1}{\sqrt{3}}$ is a local maximum.

$\ddot{\epsilon} = -\omega^2\epsilon$, where $\omega^2 = V''(x_0)/m > 0$.

RESULT: Motion is approximately SHM with frequency $\omega = \sqrt{V''(x_0)/m}$
In the above example $x_0 = \frac{1}{\sqrt{3}}$ and $V''(\frac{1}{\sqrt{3}}) = 2\sqrt{3}$ hence $\omega = \sqrt{2\sqrt{3}/m}$

Eg. particle of mass m moves in the potential $V(x) = k(x^2 - 2)e^{-2x}$, where $k > 0$.

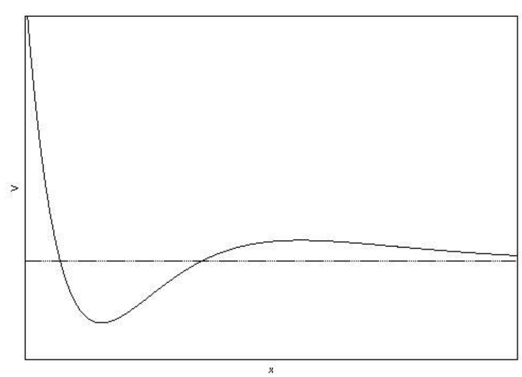

Figure 10: Sketch of $V(x) = k(x^2 - 2)e^{-2x}$.

Important values for plot are
$V(\infty) = 0, V(-\infty) = +\infty, V(\pm\sqrt{2}) = 0, V'(-1) = 0, V'(2) = 0.$
$V' = -2ke^{-2x}(x + 1)(x - 2)$, hence stationary points are $x = -1$ and $x = 2$.

$V'' = 2ke^{-2x}(2x^2 - 4x - 3)$,
hence $V''(-1) = 6ke^2 > 0$ and $V''(2) = -6ke^{-4} < 0$.
Therefore $x = -1$ is a stable equilibrium and $x = 2$ is unstable (which agrees with the sketch).
For the stable position $\omega = \sqrt{\frac{6ke^2}{m}} = \sqrt{\frac{6k}{m}}e$.
The period is

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{e}\sqrt{\frac{m}{6k}}.$$

Alternative method to calculate frequency of approx SHM.
Rather than calculating $V''(x_0)$ the frequency can be found by reading off the linear term in ϵ in $V'(x_0 + \epsilon)$.
For the above example $V'(x) = -2ke^{-2x}(x + 1)(x - 2)$ so $m\ddot{x} = F = -V'$ gives
$m\ddot{x} = 2ke^{-2x}(x + 1)(x - 2)$.
Now put $x = -1 + \epsilon$ and keep only terms linear in ϵ to get
$m\ddot{\epsilon} = 2ke^2\epsilon(-3) = -6ke^2\epsilon$.
Therefore $\ddot{\epsilon} = -\omega^2\epsilon$, where $\omega = \sqrt{\frac{6k}{m}}e$ as before.

Energy conservation and possible motion.
Energy conservation can be used to determine possible motion.
Eg. in the above problem, starting at the equilibrium position $x = -1$ with speed u, then what value of u ensures that $x \to +\infty$ as $t \to \infty$.

$$E = \frac{m}{2}\dot{x}^2 + V = \frac{m}{2}u^2 + V(-1) = \frac{m}{2}u^2 - ke^2.$$

The particle will escape to $x = +\infty$ if and only if it has enough energy to get to the top of the hill at $x = 2$ with some kinetic energy left. This requires $E > V(2) = 2ke^{-4}$ and hence

$$\frac{m}{2}u^2 - ke^2 > 2ke^{-4} \Leftrightarrow u > \sqrt{\frac{2k}{m}\Big(\frac{2}{e^4} + e^2\Big)}.$$

Note 1: Minimum kinetic energy = potential energy difference.

Note 2: In this example the particle can start with $\dot{x} < 0$ or $\dot{x} > 0$ because if motion is first to the left it will then return and pass through the starting point with the same speed but moving to the right.

5.2 Simple pendulum

Model as a point mass m on a light rod of length l, making an angle θ to the vertical, and ignore air resistance, flexing, friction,...

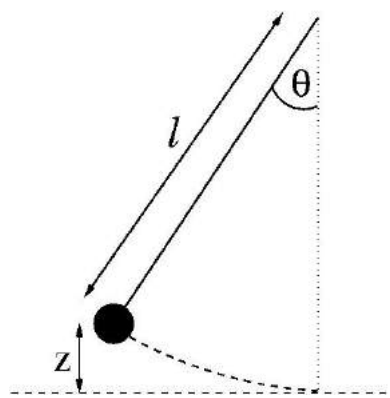

Figure 11: A pendulum of length l, making an angle θ to the vertical.

The height above the hanging down position is $z = l - l\cos\theta = l(1 - \cos\theta)$.
$V = mgz = mgl(1 - \cos\theta)$, having chosen $V(0) = 0$.
Equilibrium points are where $V'(\theta) = 0 = mgl\sin\theta$ giving
$\theta = 0$ (hanging down) and $\theta = \pi$ (standing up).
From earlier work on motion in polar coordinates we have $v = l|\dot{\theta}|$, hence the kinetic energy is

$$\text{kinetic energy } = \frac{m}{2}l^2\dot{\theta}^2.$$

$$E = \frac{m}{2}l^2\dot{\theta}^2 + mgl(1 - \cos\theta)$$

This separable ODE can be solved but requires elliptic integrals (too hard).
The equation of motion follows from energy conservation

$$\frac{dE}{dt} = 0 = ml^2\dot{\theta}\ddot{\theta} + mgl\dot{\theta}\sin\theta$$

so the equation of motion is

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0.$$

For small oscillations around $\theta = 0$ then $\sin\theta \approx \theta$ giving the approximate equation

$$\ddot{\theta} = -\omega^2\theta, \text{ where } \omega = \sqrt{\frac{g}{l}}.$$

Thus SHM with period $T = 2\pi/\omega = 2\pi\sqrt{l/g}$.

Use energy conservation to answer questions about the motion.
Eg. What angular speed $\Omega = \dot{\theta}(0)$ needed for propeller rotation from $\theta(0) = 0$?
From the initial condition at $t = 0$ we have that $E = \frac{m}{2}l^2\Omega^2$.
For propeller motion need to get to $\theta = \pi$ with some kinetic energy left ie.

$$E = \frac{m}{2}l^2\Omega^2 > V(\pi) = 2mgl, \ \Leftrightarrow |\Omega| > 2\sqrt{\frac{g}{l}}.$$

6 Damped vibrations

So far we have not included damping (friction) terms when studying restoring forces. We shall now include these.

Consider a mass m hanging vertically on a spring (with spring constant K) and moving under the influence of gravity and a frictional (damping) force of magnitude Λv.
$m\ddot{x} = mg - Kx - \Lambda\dot{x}$, with $\Lambda > 0$ the damping constant.
Equilibrium is where $x = x_0$ so that $\dot{x} = \ddot{x} = 0$ ie
$0 = mg - Kx_0$ hence $x_0 = mg/K$.
Let $u = x - x_0 = x - mg/K$ be the deformation from equilibrium

$$m\ddot{u} = mg - K(u + \frac{mg}{K}) - \Lambda\dot{u}$$

$$\ddot{u} = -\frac{K}{m}u - \frac{\Lambda}{m}\dot{u}$$

Note that if $\Lambda = 0$ (no damping) then SHM with frequency $\omega_0 = \sqrt{K/m}$, as we have seen before.

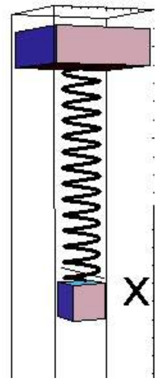

Figure 12: A spring hanging down with x measured downwards

Let $p = \Lambda/m$ and $q = K/m$ then
$\ddot{u} + p\dot{u} + qu = 0$, where $p, q > 0$.
Solve by finding roots of the characteristic equation
$\lambda^2 + p\lambda + q = 0$, where p relates to friction and q to spring constant.
There are 3 cases to consider:

Case (i). Complex roots ie. $p^2 - 4q < 0$, which is small damping.
Define $p^2 - 4q = -4\omega^2$ then the roots are

$$\lambda = \frac{-p \pm \sqrt{-4\omega^2}}{2} = -\frac{p}{2} \pm i\omega$$

$$u = e^{-pt/2}(A\cos\omega t + B\sin\omega t).$$

Note that $\omega_0^2 = q$ hence $\omega^2 = \omega_0^2 - \frac{p^2}{4} < \omega_0^2$.
Friction slows the oscillation.

Case (ii). Distinct real roots ie. $p^2 - 4q > 0$, which is large damping.
Define $p^2 - 4q = 4k^2$ then the roots are

$$\lambda = \frac{-p \pm \sqrt{4k^2}}{2} = -\frac{p}{2} \pm k$$

$$\lambda_+ = -\frac{p}{2} + k, \quad \lambda_- = -\frac{p}{2} - k,$$

are both negative.

$$u = Ae^{\lambda_+ t} + Be^{\lambda_- t}$$

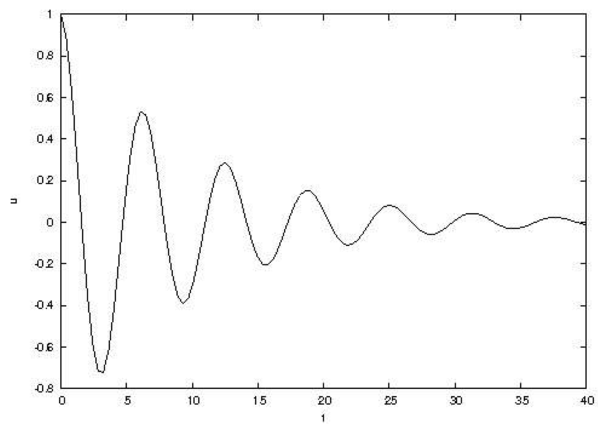

Figure 13: Case (i): a damped oscillation

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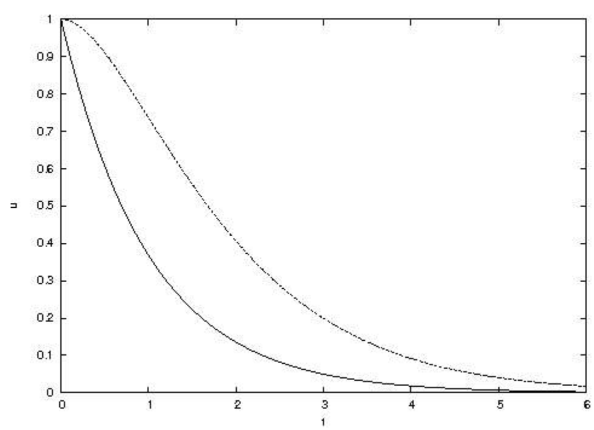

Figure 14: Cases (ii) and (iii): $u \to 0$ without oscillation

.

Note: u may change sign once if $AB < 0$.

Case (iii). Repeated root ie. $p^2 - 4q = 0$, which is **critical damping**.
$\lambda = -p/2$ is a repeated root

$$u = e^{-pt/2}(A + Bt).$$

Note: u may change sign once if $AB < 0$.

In all 3 cases energy is lost through friction.
Damped oscillations occur in everyday life eg. shock absorbers, bed springs, ...
If friction can be balanced with spring stiffness then it is close to critical damping. This gives a reasonable response and a smooth return to equilibrium.

eg. The deformation from equilibrium $u(t)$ of a damped spring is described by the equation

$$\ddot{u} + 2\dot{u} + 2u = 0.$$

Given that the spring is initially undeformed and given a velocity $\dot{u}(0) = -3$, find the subsequent deformation and sketch it.
Characteristic equation is $\lambda^2 + 2\lambda + 2 = 0$ with roots $\lambda = -1 \pm i$
hence the general solution is $u = e^{-t}(A\cos t + B\sin t)$.
$u(0) = 0 = A$ giving $u = Be^{-t}\sin t$.
Then $\dot{u} = Be^{-t}(-\sin t + \cos t)$ so $\dot{u}(0) = -3 = B$.
The subsequent deformation is therefore $u = -3e^{-t}\sin t$.

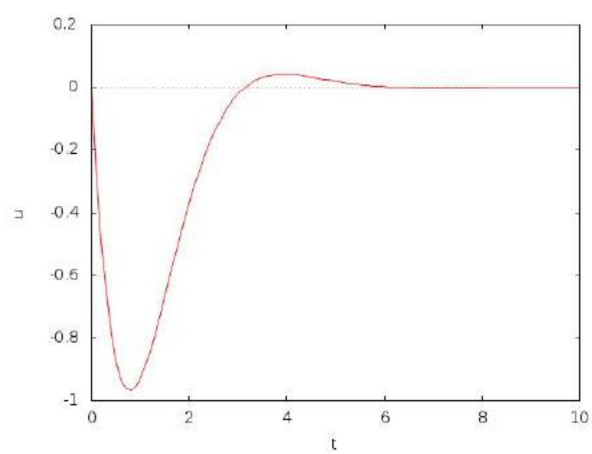

Figure 15: A plot of $u = -3e^{-t}\sin t$

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6.1 Forcing and resonance

Forced oscillations occur in many physical systems eg. a car shock absorber on a bumpy road, wind blowing a bridge,...

To illustrate the main idea let us first consider a simple undamped spring with unit mass and unit spring constant. Now introduce a forcing term with unit amplitude and frequency $\alpha > 0$. The equation of motion is

$$\ddot{u} + u = \sin(\alpha t).$$

General solution = complementary function + particular integral. The complementary function is the solution describing SHM with frequency $\omega = 1$

$$u_{CF} = A\cos t + B\sin t$$

with A and B determined by the initial conditions.
If $\alpha \neq 1$ then the particular integral has the form

$$u = a\cos(\alpha t) + b\sin(\alpha t)$$

and putting this in the equation of motion gives

$$a(1-\alpha^2)\cos(\alpha t) + b(1-\alpha^2)\sin(\alpha t) = \sin(\alpha t)$$

with solution $a = 0$ and $b = 1/(1-\alpha^2)$.
The general solution is therefore

$$u = A\cos t + B\sin t + \frac{\sin(\alpha t)}{1-\alpha^2}$$

Note that the final term does not depend upon the initial conditions and its amplitude increases as the forcing frequency α gets closer to 1, which is the frequency of the unforced spring.

This solution is not applicable if $\alpha = 1$ ie. if the forcing frequency is equal to the natural frequency of the spring. In this case the particular integral has the form

$$u = t(a\cos t + b\sin t)$$

putting this into the equation of motion gives

$$-2a\sin t + 2b\cos t = \sin t$$

with solution $a = -\frac{1}{2}$ and $b = 0$. The general solution is therefore

$$u = A\cos t + B\sin t - \frac{1}{2}t\cos t$$

Again the final term does not depend upon the initial conditions and in this case its amplitude increases without limit as t increases. The applied force is synchronized with the oscillation of the spring and adds constructively to continually increase the amplitude – this is an example of **resonance**.

More generally, consider a damped spring and forcing by a periodic force of the form

$$m\ddot{x} = mg - Kx - \Lambda\dot{x} + \Gamma\sin(\alpha t).$$

As before put $u = x - \frac{mg}{K}$, and define $p = \Lambda/m$, $q = K/m$ and $r = \Gamma/m$. Then the equation becomes

$$\ddot{u} + p\dot{u} + qu = r\sin(\alpha t), \quad (*).$$

General solution = complementary function + particular integral
$u = u_{CF} + u_{PI}$,
where u_{CF} is the general solution of the problem $(*)$ with $r = 0$, which we have solved already, and u_{PI} is **any** solution of the problem $(*)$.

From earlier we know $u_{CF} \to 0$ as $t \to \infty$, hence this **transient response** vanishes rapidly - as does the memory of the initial condition.
Thus for large times $u \approx u_{PI}$ is the **steady state response**.

Look for a solution of the form (method of undetermined coefficients)

$$u_{PI} = A\cos(\alpha t) + B\sin(\alpha t)$$

comparing the $\cos(\alpha t)$ and $\sin(\alpha t)$ terms gives

$$-\alpha^2 A + \alpha p B + qA = 0, \quad \text{and} \quad -\alpha^2 B - \alpha p A + qB = r.$$

The solution of these linear equations for A and B is

$$A = \frac{-p\alpha r}{(q-\alpha^2)^2 + p^2\alpha^2}, \quad B = \frac{(q-\alpha^2)r}{(q-\alpha^2)^2 + p^2\alpha^2},$$

Hence we have the steady state response

$$u_{PI} = \frac{r}{(q-\alpha^2)^2 + p^2\alpha^2}\{(q-\alpha^2)\sin(\alpha t) - p\alpha\cos(\alpha t)\} = \frac{r\sin(\alpha t - \phi)}{\sqrt{(q-\alpha^2)^2 + p^2\alpha^2}}$$

where $\tan\phi = p\alpha/(q-\alpha^2)$.
ϕ is the **phase difference** between forcing and response.
If p^2 is much smaller than $2q$ (small damping) then the response amplitude is maximal if $\alpha^2 \approx q$, ie. $\alpha \approx \sqrt{q} = \sqrt{K/m}$, which is the natural frequency of the SHM without damping and forcing.
This is called **resonance**.
At resonance, with forcing amplitude r, the amplitude of response is $\approx r/(p\alpha)$ and $\phi \approx \pi/2$.
This could be very large (even if r is small) if $p\alpha$ is small (note that $p\alpha = 0$ is not allowed for the solution we have found.)

eg. $r = 1, \;\; p = 1/2, \;\; q = 1$.
If $\alpha = 2$, this is not at resonance, $u_{max} = 1/\sqrt{(1-4)^2+1} = 1/\sqrt{10} \ll 1 = r$.
However, if $\alpha = 1 = \sqrt{q}$ then this is resonance and $u_{max} = 2 > 1 = r$.

Musical instruments use resonance, and sometimes bridges are destroyed by it.

7 Conservation of momentum

Consider two particles of masses m_1 and m_2 with velocities $\mathbf{v}_1$ and $\mathbf{v}_2$, which interact and there are no other forces (they are *isolated*). Then

$$\frac{d}{dt}(m_1\mathbf{v}_1) = \mathbf{F}_1$$

where $\mathbf{F}_1$ is the force felt by particle 1 due to particle 2. Similarly

$$\frac{d}{dt}(m_2\mathbf{v}_2) = \mathbf{F}_2.$$

Now by **N3** we have that $\mathbf{F}_1 = -\mathbf{F}_2$ hence

$$\frac{d}{dt}(m_1\mathbf{v}_1 + m_2\mathbf{v}_2) = \mathbf{F}_1 + \mathbf{F}_2 = 0.$$

Total momentum is conserved.
For constant masses, consider the relative position $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ where $\mathbf{F}_1(\mathbf{r}_1, \mathbf{r}_2) = \mathbf{F}(\mathbf{r}) = -\mathbf{F}_2(\mathbf{r}_1, \mathbf{r}_2)$. Then

$$\ddot{\mathbf{r}} = \ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2 = \Big(\frac{1}{m_1} + \frac{1}{m_2}\Big)\mathbf{F} = \frac{1}{\mu}\mathbf{F},$$

where the **reduced mass** μ is

$$\mu = \frac{m_1 m_2}{m_1 + m_2}.$$

This shows that the relative motion is the same as the motion of a particle of mass μ.
Note that if $m_2 \gg m_1$ (eg. earth and a ball, or the sun and the earth) then $\mu \approx m_1$. The larger mass is effectively fixed and the smaller mass moves under its force.

7.1 Particle collisions

A collision is called **elastic** if the total energy is conserved. In this case kinetic energy may be converted into potential energy during the interaction but all the kinetic energy re-emerges after the interaction.

A collision is called **inelastic** if energy is lost as a result of the interaction eg. lost into heat, deformation, sound,...

Eg. drop a golf ball onto a hard floor, then it regains most of the kinetic energy it had when it hit the floor as it bounces, and therefore bounces almost to *the height it was dropped from – approx elastic. But a bean bag just deforms and doesn't bounce at all – inelastic.*

Use conservation of momentum to determine the outcome of a collision.

Eg. A particle of mass $2m$ and speed v collides head-on with a particle at rest of mass m and the two continue to move in the same direction, with the heaviest particle moving at speed $v/2$. What is the speed u of the lightest particle? What % of the initial kinetic energy is lost in the collision?

By conservation of momentum
$2mv + m(0) = 2mv/2 + mu$, giving $u = v$.
The initial energy is $E_i = \frac{2m}{2}v^2 + \frac{m}{2}(0^2) = mv^2$.
The final energy is $E_f = \frac{2m}{2}(v/2)^2 + \frac{m}{2}v^2 = \frac{3m}{4}v^2$.
Thus $E_i - E_f = mv^2(1 - \frac{3}{4}) = \frac{m}{4}v^2 > 0$, hence the collision is inelastic.
The % of kinetic energy lost is

$$\frac{(E_i - E_f)}{E_i} \times 100\% = \frac{mv^2/4}{mv^2} \times 100\% = 25\%.$$

If we know that a given collision is elastic, then the speed of both particles after the collision can be found.

Eg. A particle of mass $2m$ and speed v collides **elastically** *with a particle at rest of mass m and the two continue to move in the same direction. What is the speed of each particle after the collision?*

Let w be the speed of the heavy particle and u the speed of the light particle after the collision.
By conservation of momentum
$2mv = 2mw + mu$, giving $w = v - \frac{1}{2}u$.
The initial energy is $E_i = \frac{2m}{2}v^2 = mv^2$.
The final energy is $E_f = \frac{2m}{2}w^2 + \frac{m}{2}u^2$.
The collision is elastic hence $E_i = E_f$ giving $v^2 = w^2 + \frac{1}{2}u^2$.

Use the above result for w to get $v^2 = (v - u/2)^2 + \frac{1}{2}u^2$, which gives $vu = \frac{3}{4}u^2$. Cancel the factor of u, since $u = 0$ is not the required solution (this is when the particle passes through the other one) hence $v = \frac{3}{4}u$.
Thus finally $u = \frac{4}{3}v$ and $w = \frac{1}{3}v$.

8 Angular momentum and central forces

8.1 Angular momentum

So far we have seen examples of useful conserved quantities, such as energy and momentum. We now introduce *angular momentum* which is a 3D concept.

Consider a particle of mass m, with position $\mathbf{r}$ and velocity $\mathbf{v} = \dot{\mathbf{r}}$.
Its **angular momentum** (about $\mathbf{r} = \mathbf{0}$) is defined as

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\mathbf{r} \times \mathbf{v}.$$

The angular momentum about $\mathbf{r}_0$ is defined similarly as $\mathbf{L} = m(\mathbf{r} - \mathbf{r}_0) \times \mathbf{v}$, but we wont need this.
$\mathbf{L}$ is associated with rotation since $\mathbf{r}$ parallel to $\mathbf{v}$ gives $\mathbf{L} = \mathbf{0}$.

$$\frac{d\mathbf{L}}{dt} = m\mathbf{v} \times \mathbf{v} + m\mathbf{r} \times \dot{\mathbf{v}} = \mathbf{r} \times \mathbf{F}.$$

This is called the *torque* or *moment* and is an angular force.

Note that by its definition $\mathbf{L} \cdot \mathbf{r} = \mathbf{L} \cdot \mathbf{v} = 0$.

Eg. $\mathbf{r} = t\mathbf{i} + \mathbf{j} + t^2\mathbf{k}$. *Calculate* $\mathbf{L}$.

$\mathbf{v} = \dot{\mathbf{r}} = \mathbf{i} + 2t\mathbf{k}$

$$\mathbf{L} = m\mathbf{r} \times \mathbf{v} = m(t\mathbf{i} + \mathbf{j} + t^2\mathbf{k}) \times (\mathbf{i} + 2t\mathbf{k}) = m\{-2t^2\mathbf{j} - \mathbf{k} + 2t\mathbf{i} + t^2\mathbf{j}\}$$

$$\mathbf{L} = m(2t\mathbf{i} - t^2\mathbf{j} - \mathbf{k}).$$

Exercise: Check that $\mathbf{L} \cdot \mathbf{r} = \mathbf{L} \cdot \mathbf{v} = 0$.

8.2 Central forces

A **central force** involves attraction or repulsion from a fixed point (*centre of force*).
Choose this point to be the origin then $\mathbf{F}$ is either parallel or anti-parallel to $\mathbf{r}$ ie. $\mathbf{F} = f(r)\mathbf{e}_r$.
In this case

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \mathbf{F} = r\mathbf{e}_r \times f\mathbf{e}_r = rf\mathbf{e}_r \times \mathbf{e}_r = \mathbf{0}.$$

Thus under a central force $\mathbf{L}$ is constant ie. angular momentum is conserved.

Since $\mathbf{L} \cdot \mathbf{r} = 0$, we have that the motion lies in a plane perpendicular to the constant $\mathbf{L}$.

Note: Writing $\mathbf{F} = f(r)\mathbf{e}_r$ then $f < 0$ is attraction, and $f > 0$ is repulsion.

Eg. Gravity is a central force.
Consider two particles of masses m and M, then there is an inverse square attraction ie. the force on m due to M is

$$\mathbf{F} = -G\frac{mM}{r^2}\mathbf{e}_r$$

where G is Newton's gravitational constant $G \approx 6.67 \times 10^{-11}\ m^3(kg)^{-1}s^{-2}$ and $\mathbf{e}_r$ is the unit radial vector pointing from M to m.

Close to the surface of the earth (radius R and mass M)

$$\mathbf{F} \approx -G\frac{mM}{R^2}\mathbf{e}_r = -mg\mathbf{e}_r,$$

hence $g = GM/R^2$.
Using $R \approx 6,400\,km$ and $M \approx 6 \times 10^{24}\ kg$ gives

$$g \approx \frac{6.67 \times 10^{-11} \times 6 \times 10^{24}}{(6.4 \times 10^6)^2}\ ms^{-2} = \frac{6.67 \times 60}{(6.4)^2}\ ms^{-2} \approx 9.8\ ms^{-2}.$$

8.3 Motion under a central force

Recall that $\mathbf{L}$ is constant, so choose coordinates so that $\mathbf{L}$ is in the $\mathbf{k}$ direction ie. $\mathbf{L} = L\mathbf{k}$.
Also recall that the motion takes place in a plane perpendicular to $\mathbf{L}$.
Let r and θ be polar coordinates in this plane.
From earlier we have that

$$\mathbf{r} = r\mathbf{e}_r, \qquad \dot{\mathbf{r}} = \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta, \qquad \ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_\theta.$$

We can work out the kinetic energy in these coordinates

$$\text{kinetic energy} = \frac{m}{2}|\dot{\mathbf{r}}|^2 = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2).$$

Similarly, the angular momentum is

$$\mathbf{L} = m\mathbf{r} \times \dot{\mathbf{r}} = mr\mathbf{e}_r \times (\dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta) = mr^2\dot{\theta}\mathbf{k}.$$

Therefore, $L = mr^2\dot{\theta}$.

The equation of motion is

$$m\ddot{\mathbf{r}} = \mathbf{F} = f(r)\mathbf{e}_r = m(\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + m(2\dot{r}\dot{\theta} + r\ddot{\theta})\mathbf{e}_\theta$$

Hence there is a radial and an angular equation

$$m(\ddot{r} - r\dot{\theta}^2) = f, \qquad 2\dot{r}\dot{\theta} + r\ddot{\theta} = 0.$$

Multiplying the angular equation by r we see that

$$2r\dot{r}\dot{\theta} + r^2\ddot{\theta} = 0 = \frac{d}{dt}(r^2\dot{\theta})$$

hence $r^2\dot{\theta} = \text{constant} = L/m$, using the result $L = mr^2\dot{\theta}$.
The angular equation is therefore equivalent to conservation of angular momentum.
The radial equation can be written as an *effective 1D problem* by eliminating $\dot{\theta}$ using $\dot{\theta} = L/(mr^2)$. This gives

$$m\ddot{r} - \frac{L^2}{mr^3} = f, \qquad (*).$$

Introduce the potential V as $f = -\frac{dV}{dr}$, with $V(\infty) = 0$ if $f(\infty) = 0$.
The energy is

$$E = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + V$$

Again use $\dot{\theta} = L/(mr^2)$ to write the energy as

$$E = \frac{m}{2}(\dot{r}^2 + \frac{L^2}{m^2r^2}) + V$$

and note that the equation (*) is eqivalent to $\frac{dE}{dt} = 0$.

At $r = \infty$, then $V = 0$ hence if the particle is at infinity then $E \geq 0$.
This shows that an orbit will be **bound** (ie. particle can not escape to infinity) if $E < 0$, since E is conserved.

Eg. A unit mass particle moves under the influence of the attractive central force $f = -1/r^2$. It starts at the radius $r = a$, with speed u perpendicular to $\mathbf{r}$. *Show that the particle cannot escape to infinity if $au^2 < 2$. In this case find the smallest and largest values of the radius r during the motion.*

$m = 1$ and $f = -1/r^2$ hence $V = -\int f\, dr = -1/r$.

$$E = \frac{1}{2}(\dot{r}^2 + \frac{L^2}{r^2}) - \frac{1}{r}.$$

$$L = r^2\dot{\theta} = r(r\dot{\theta}) = au$$

where we have used the initial condition $r = a$ and speed is u.

$$E = \frac{1}{2}(\dot{r}^2 + \frac{a^2u^2}{r^2}) - \frac{1}{r} = \frac{a^2u^2}{2a^2} - \frac{1}{a} = \frac{u^2}{2} - \frac{1}{a},$$

where again we have used the initial condition $r = a$ and $\dot{r} = 0$.
The particle cannot escape to infinity if $E < 0$, which gives $au^2 < 2$, ie. the particle cannot escape if it starts too close or too slow.

For a bound orbit, $au^2 < 2$, the trajectory is an ellipse (see later).

$$E = \frac{1}{2}(\dot{r}^2 + \frac{a^2u^2}{r^2}) - \frac{1}{r} = \frac{u^2}{2} - \frac{1}{a} < 0.$$

The maximum and minimum values of r are turning points where $\dot{r} = 0$, ie.

$$\frac{a^2u^2}{2r^2} - \frac{1}{r} = \frac{u^2}{2} - \frac{1}{a}.$$

Rearranging this gives the quadratic

$$(2 - au^2)r^2 - 2ar + a^3u^2 = 0 = (r - a)\{(2 - au^2)r - a^2u^2\}.$$

The roots are $r = r_+ = a$ and $r = r_- = a^2u^2/(2 - au^2) > 0$ for bound orbit.

8.4 Kepler's laws of planetary motion

K1 *The planets move in elliptical orbits with the sun at one focus.*

K2 *The radius from the sun to a planet sweeps out equal areas in equal times.*

K3 *The square of the period of the orbit of a planet is proportional to the cube of the semi-major axis.*

An ellipse can be written as $(X/a)^2 + (Y/b)^2 = 1$, where $a \geq b$.
The distance a is called the semi-major axis, and b is called the semi-minor axis.
The *eccentricity* ε of the ellipse is defined as

$$\varepsilon = \sqrt{1 - \frac{b^2}{a^2}}$$

and satisfies $0 \leq \varepsilon < 1$. If $\varepsilon = 0$ the ellipse becomes a circle.
Using ε the ellipse equation can be written as

$$X^2 + \frac{1}{1 - \varepsilon^2}Y^2 = a^2.$$

The points $(X, Y) = (\pm a\varepsilon, 0)$ are called the *foci*.
A geometrical definition of an ellipse is that the sum of the distances to the two foci is constant for all points on the ellipse.

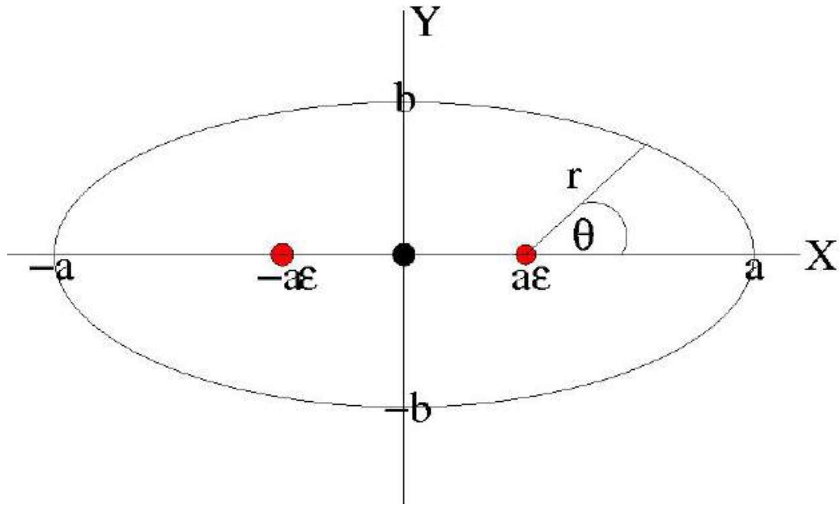

Figure 16: An ellipse

Let r and θ be polar coordinates, with one of the foci as the origin.
$X = a\varepsilon + r\cos\theta$
$r^2 = (X - a\varepsilon)^2 + Y^2 = (X - a\varepsilon)^2 + (a^2 - X^2)(1 - \varepsilon^2) = (a - X\varepsilon)^2$
Hence $r = a - X\varepsilon$. Using this and the earlier expression
$X\varepsilon = a - r = a\varepsilon^2 + \varepsilon r\cos\theta$ giving

$$r = \frac{a(1-\varepsilon^2)}{1+\varepsilon\cos\theta}.$$

Proof of K1.
Let M be the mass of the sun and m the mass of a planet.

$$f = -\frac{GMm}{r^2}, \qquad \text{therefore} \qquad V = -\int f\,dr = -\frac{GMm}{r}.$$

$$E = \frac{m}{2}(\dot{r}^2 + \frac{L^2}{m^2r^2}) - \frac{GMm}{r}$$

We want to determine $r(\theta)$ to show that it is an ellipse.

$$\dot{r} = \frac{dr}{dt} = \dot{\theta}\frac{dr}{d\theta} = \frac{L}{mr^2}\frac{dr}{d\theta}.$$

Put this into E to get

$$E = \frac{L^2}{2m}\Big(\frac{1}{r^4}\Big(\frac{dr}{d\theta}\Big)^2 + \frac{1}{r^2}\Big) - \frac{GMm}{r}.$$

We can either solve this differential equation (see exercise A below) or (see exercise B below) simply check that the ellipse form is a solution

$$r = \frac{C}{1+\varepsilon\cos\theta}, \quad \text{where } C = \frac{L^2}{GMm^2}, \text{ and } \varepsilon = \sqrt{1+\frac{2EL^2}{G^2M^2m^3}}.$$

Note $\varepsilon \in [0,1)$ since $E<0$.

<u>*Exercise A*</u>

$$E = \frac{L^2}{2m}\Big(\frac{1}{r^4}\Big(\frac{dr}{d\theta}\Big)^2 + \frac{1}{r^2}\Big) - \frac{GMm}{r}.$$

Make the change of variable $r = 1/u$ where $u(\theta)$. Then

$$\frac{dr}{d\theta} = \frac{dr}{du}\frac{du}{d\theta} = -\frac{1}{u^2}\frac{du}{d\theta} = -\frac{u'}{u^2}$$

Putting this into the above and rearranging gives

$$u'^2 + u^2 - \frac{2GMm^2}{L^2}u = \frac{2Em}{L^2} \qquad (\star)$$

Differentiating $(\star)$ and dividing by $2u'$ yields the second order constant coefficient equation

$$u'' + u = \frac{GMm^2}{L^2}$$

with solution

$$u = A\cos\theta + B\sin\theta + \frac{GMm^2}{L^2}.$$

If we choose the coordinate θ so that $\theta = 0$ is a turning point then $u'(0) = 0$ hence $B = 0$ and

$$u = A\cos\theta + \frac{GMm^2}{L^2}.$$

Putting this expression back into $(\star)$ we find

$$A^2\sin^2\theta + \Big(A\cos\theta + \frac{GMm^2}{L^2}\Big)^2 - \frac{2GMm^2}{L^2}\Big(A\cos\theta + \frac{GMm^2}{L^2}\Big) = \frac{2Em}{L^2}$$

which simplifies to

$$A = \frac{GMm^2}{L^2}\sqrt{1+\frac{2EL^2}{G^2M^2m^3}}$$

hence the solution

$$u = \frac{GMm^2}{L^2}\Big(\cos\theta\sqrt{1+\frac{2EL^2}{G^2M^2m^3}}+1\Big)$$

and finally

$$r = \frac{1}{u} = \frac{C}{1+\varepsilon\cos\theta}$$

where

$$C = \frac{L^2}{GMm^2} \qquad \text{and} \qquad \varepsilon = \sqrt{1+\frac{2EL^2}{G^2M^2m^3}}.$$

<u>*Exercise B*</u>

Check that the following form gives a solution

$$r = \frac{C}{1+\varepsilon\cos\theta}.$$

Then

$$\frac{dr}{d\theta} = \frac{C\varepsilon\sin\theta}{(1+\varepsilon\cos\theta)^2}$$

Putting these expressions in

$$E = \frac{L^2}{2m}\Big(\frac{1}{r^4}\Big(\frac{dr}{d\theta}\Big)^2+\frac{1}{r^2}\Big)-\frac{GMm}{r}$$

and simplifying gives

$$E = \frac{L^2}{2mC^2}(\varepsilon^2+1+2\varepsilon\cos\theta)-\frac{GMm}{C}(1+\varepsilon\cos\theta).$$

E is a constant, which means that it cannot depend on θ, hence the coefficient of the $\cos\theta$ term in the above expression for E must vanish. This gives that

$$C = \frac{L^2}{GMm^2}$$

and then E simplifies to

$$E = \frac{G^2M^2m^3}{2L^2}(\varepsilon^2-1)$$

which determines that ε is given by

$$\varepsilon = \sqrt{1 + \frac{2EL^2}{G^2M^2m^3}}.$$

Proof of K2.

For a small change in position $\mathbf{r} \mapsto \mathbf{r} + \delta\mathbf{r}$, the change in area (see figure) is

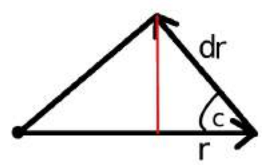

Figure 17: A small change in position

$$\delta A = \frac{1}{2}|\mathbf{r}||\delta\mathbf{r}|\sin c = \frac{1}{2}|\mathbf{r} \times \delta\mathbf{r}|.$$

Therefore

$$\frac{dA}{dt} = \frac{1}{2}|\mathbf{r} \times \dot{\mathbf{r}}| = \frac{L}{2m} = \text{constant}$$

proving that there is a constant rate of change of area swept out.

Proof of K3.

This law states that $T^2 = Ka^3$ where K is a constant, T is the period and a is the semi-major axis of the ellipse.

We will prove this only in the special case that the ellipse is a circle, in which case a is the radius of the circle.

For a circle the radius is constant $r = a$, ie. $\dot{r} = \ddot{r} = 0$.
As $L = mr^2\dot{\theta} = ma^2\dot{\theta}$ is constant then $\dot{\theta} = \omega$ is constant.
The radial equation

$$m\ddot{r} - \frac{L^2}{mr^3} = f$$

becomes

$$-ma\omega^2 = -\frac{GMm}{a^2}, \text{ giving } \omega^2 = \frac{GM}{a^3}.$$

The period T satisfies

$$T^2 = \left(\frac{2\pi}{\omega}\right)^2 = \frac{4\pi^2}{GM}a^3$$

giving the required result that $T^2 = Ka^3$, with $K = (4\pi^2)/(GM)$.

To prove the result for an ellipse we need to use $r = C/(1 + \varepsilon\cos\theta)$ and do an integral.

9 Waves on a string

9.1 Model

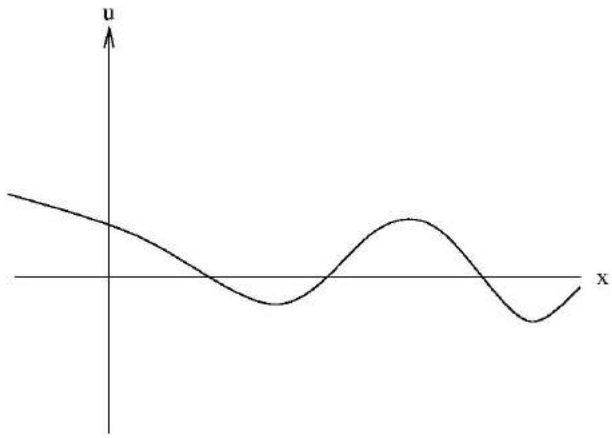

Figure 18: Transverse displacement of a string

$u(x,t)$ is the transverse displacement of a string at position x and time t.
We will assume a thin and perfectly flexible string, neglect friction, gravity, air resistance, …
Let the string have constant density ρ (mass per unit length) and constant tension T (force).
The transverse component of the equation of motion of the segment of the string between x and $x+\delta x$ is

$$\rho \delta x \frac{\partial^2 u}{\partial t^2} = (T\sin\theta)_{x+\delta x} - (T\sin\theta)_x.$$

For small angles

$$\sin\theta \approx \tan\theta = \frac{\delta u}{\delta x},$$

$$\rho\frac{\partial^2 u}{\partial t^2} = \frac{T}{\delta x}\Big\{\frac{\delta u}{\delta x}(x+\delta x) - \frac{\delta u}{\delta x}(x)\Big\}.$$

Taking the limit $\delta x \to 0$ gives

$$\rho\frac{\partial^2 u}{\partial t^2} = T\frac{\partial^2 u}{\partial x^2}, \quad \text{ie.} \quad \frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \quad (*)$$

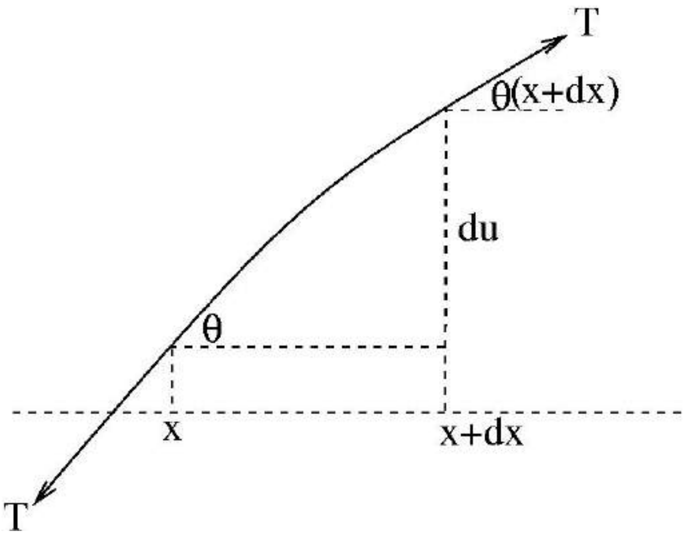

Figure 19: Force on a segment of the string

where $c = \sqrt{T/\rho}$ is the wave speed (dimensions are that of a speed).
The partial differential equation (*) is the *wave equation* in 1D.

9.2 d'Alembert's formula for an infinite string

It is easy to prove that

$$u(x,t) = f(x-ct) + g(x+ct)$$

solves the wave equation for *any* functions f and g.
Proof:

$$\frac{\partial u}{\partial x} = f'(x-ct) + g'(x+ct), \quad \frac{\partial^2 u}{\partial x^2} = f''(x-ct) + g''(x+ct).$$

$$\frac{\partial u}{\partial t} = -cf'(x-ct) + cg'(x+ct), \quad \frac{\partial^2 u}{\partial t^2} = c^2 f''(x-ct) + c^2 g''(x+ct) = c^2 \frac{\partial^2 u}{\partial x^2}.$$

Solution is a linear combination of right- and left-moving waves.

To see that this gives all solutions we must be able to choose functions f and g to give any initial conditions

$$u(x,0) = R(x), \quad \frac{\partial u}{\partial t}(x,0) = S(x),$$

where $R(x)$ and $S(x)$ can be any functions giving the initial position and initial speed.
The choice that satisfies these initial conditions gives *d'Alembert's formula*

$$u(x,t) = \frac{1}{2}\Big\{R(x-ct) + R(x+ct) + \frac{1}{c}\int_{x-ct}^{x+ct} S(z)\,dz\Big\}.$$

Proof that the initial conditions are satisfied is as follows

$$u(x,0) = \frac{1}{2}\{R(x) + R(x)\} = R(x).$$

$$\frac{\partial u}{\partial t}(x,0) = \frac{1}{2}\Big\{-cR'(x) + cR'(x) + \frac{1}{c}(cS(x) + cS(x))\Big\} = S(x).$$

Eg1. Consider a plucked string ($S = 0$) with $R = e^{-x^2}$.

$$u(x,t) = \frac{1}{2}\Big\{R(x-ct) + R(x+ct)\Big\} = \frac{1}{2}\Big\{e^{-(x-ct)^2} + e^{-(x+ct)^2}\Big\}.$$

Initial wave splits into two identical half-size waves travelling in opposite directions.

Eg2. Consider a struck string ($R = 0$) with $S = 2cxe^{-x^2}$.

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} S(z)\,dz = \frac{1}{2c}\int_{x-ct}^{x+ct} 2cze^{-z^2}\,dz = \frac{1}{2}[-e^{-z^2}]_{x-ct}^{x+ct}$$

$$= -\frac{1}{2}(e^{-(x+ct)^2} - e^{-(x-ct)^2}).$$

Two waves of *opposite sign* moving apart.

9.3 Finite string and separation of variables

Consider a finite piece of string, of length L, fixed at the two ends.

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} \qquad \text{for } x \in (0,L) \text{ with boundary conditions } u(0,t) = u(L,t) = 0.$$

Initial conditions are still

$$u(x,0) = R(x), \quad \frac{\partial u}{\partial t}(x,0) = S(x).$$

Could solve using d'Alembert's formula but it is difficult to include the boundary conditions using this approach. Better to try a separation of variables

$$u(x,t) = X(x)T(t),$$

Important: T should not be confused with the earlier tension.

$$\frac{\partial^2 u}{\partial x^2} = X''T, \quad \frac{\partial^2 u}{\partial t^2} = XT''$$

hence the wave equation becomes

$$XT'' = c^2 X''T.$$

Dividing by XTc^2 gives

$$\frac{T''}{Tc^2} = \frac{X''}{X} = -\alpha^2,$$

where α is a constant. The solution of $X'' = -\alpha^2 X$ is

$$X = a\cos(\alpha x) + b\sin(\alpha x)$$

and the boundary conditions $X(0) = X(L) = 0$ give $a = 0$ and $\alpha L = n\pi$ where n is a positive integer. Similarly, the solution of $T'' = -\alpha^2 c^2 T$ is

$$T = \tilde{a}\cos(\alpha ct) + \tilde{b}\sin(\alpha ct)$$

giving the solution for u as

$$u = \sin(\frac{n\pi x}{L})\Big(A\cos(\frac{n\pi ct}{L}) + B\sin(\frac{n\pi ct}{L})\Big)$$

where $A = b\tilde{a}$ and $B = b\tilde{b}$.

This is a solution for any positive integer n, and as the equation is linear then any linear combination of such solutions is also a solution. This gives the final form of the solution

$$u(x,t) = \sum_{n=1}^{\infty} \sin(\frac{n\pi x}{L})\Big(A_n\cos(\frac{n\pi ct}{L}) + B_n\sin(\frac{n\pi ct}{L})\Big).$$

The set of constants A_n and B_n are determined by the initial conditions.

$$u(x,0) = R(x) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{L})$$

$$\frac{\partial u}{\partial t}(x,0) = S(x) = \sum_{n=1}^{\infty} B_n \frac{n\pi c}{L} \sin(\frac{n\pi x}{L}).$$

An expansion of the form

$$R(x) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{L}) \quad (\dagger)$$

is called a half-sine fourier series, and there is an integral formula for the coefficients

$$A_n = \frac{2}{L} \int_0^L R(x) \sin(\frac{n\pi x}{L})\, dx.$$

Proof:
It is easy to prove (see tutorial exercise) that for positive integers n and m

$$\frac{2}{L} \int_0^L \sin(\frac{n\pi x}{L}) \sin(\frac{m\pi x}{L})\, dx = \delta_{nm}$$

where δ_{nm} equals 1 if $n = m$ and zero otherwise.
Multiplying both sides of $(\dagger)$ by $\frac{2}{L}\sin(\frac{m\pi x}{L})$ and integrating gives

$$\frac{2}{L} \int_0^L R(x) \sin(\frac{m\pi x}{L})\, dx = \sum_{n=1}^{\infty} A_n \frac{2}{L} \int_0^L \sin(\frac{n\pi x}{L}) \sin(\frac{m\pi x}{L})\, dx = \sum_{n=1}^{\infty} A_n \delta_{nm} = A_m.$$

Similarly, $B_n \frac{n\pi c}{L}$ are the coefficients of $S(x)$ hence

$$B_n \frac{n\pi c}{L} = \frac{2}{L} \int_0^L S(x) \sin(\frac{n\pi x}{L})\, dx.$$

Eg. A string is pulled aside and released from rest so that $u(x,0) = 1$ if $0 < x < L/2$ and $u(x,0) = -1$ if $L/2 < x < L$. Find the series solution.

As the string is released from rest then $S(x) = 0$ hence $B_n = 0\ \forall n$.

$$A_n = \frac{2}{L} \int_0^L R(x) \sin(\frac{n\pi x}{L})\, dx = \frac{2}{L} \Big\{ \int_0^{L/2} \sin(\frac{n\pi x}{L})\, dx - \int_{L/2}^{L} \sin(\frac{n\pi x}{L})\, dx \Big\}$$

$$= \frac{2}{L} \frac{L}{n\pi} \Big\{ \Big[-\cos(\frac{n\pi x}{L}) \Big]_0^{L/2} + \Big[\cos(\frac{n\pi x}{L}) \Big]_{L/2}^{L} \Big\}$$

$$= \frac{2}{n\pi} \Big\{ -2\cos(\frac{n\pi}{2}) + 1 + \cos(n\pi) \Big\} = \frac{2}{n\pi} \Big\{ 1 + (-1)^n - 2\cos(\frac{n\pi}{2}) \Big\}$$

Now $\cos(n\pi/2)$ equals 0 if n is odd and equals $(-1)^{n/2}$ if n is even, hence

$$A_{2r-1} = 0$$

$$A_{4r} = 0$$

$$A_{4r-2} = \frac{8}{(4r-2)\pi} = \frac{4}{(2r-1)\pi}.$$

Finally,

$$u(x,t) = \sum_{r=1}^{\infty} \sin(\frac{(4r-2)\pi x}{L})\frac{4}{(2r-1)\pi}\cos(\frac{(4r-2)\pi ct}{L})$$

$$= \frac{4}{\pi}\Big\{\sin(\frac{2\pi x}{L})\cos(\frac{2\pi ct}{L})+\frac{1}{3}\sin(\frac{6\pi x}{L})\cos(\frac{6\pi ct}{L})+\frac{1}{5}\sin(\frac{10\pi x}{L})\cos(\frac{10\pi ct}{L})+\ldots\Big\}$$

Or Eg. A string is pulled aside and released from rest so that $u(x,0) = \sin(2\pi x/L)$, $u_t(x,0) = 0$. *Find the series solution.*

Now, as before $B_n = 0$ and

$$R(x) = \sum_{n=1}^{\infty} A_n \sin(\frac{n\pi x}{L}).$$

However, as

$$R(x) \; = \; \sin(\frac{2\pi x}{L})$$

we see that only $A_2 \neq 0$, and is, in fact, 1 so the series consists of only one term and we have

$$u(x,t) = \sin(\frac{2\pi x}{L})\cos(\frac{2\pi ct}{L}).$$

we can also consider a string that is pulled aside and released from rest so that $u(x,0) = kx(L-x)$ *and* $u_t(x,0) = 0$.

In such a case we have to work harder.

We note the symmetry of $u(x,0) = kx(L-x)$ about $x = L/2$. Hence $A_{\text{even}} = 0$.

Also for $n =$ odd:

$$A_n = \frac{2k}{L}\int_0^L x(L-x)\sin(\frac{n\pi x}{L})\,dx = \frac{4k}{L}\int_0^{L/2} x(L-x)\sin(\frac{n\pi x}{L})\,dx = \frac{8kL}{(n\pi)^3}.$$

Thus

$$u(x,t) = \frac{8kL}{\pi^3} \sum_{r=0}^{\infty} \frac{1}{(2r+1)^3} \sin\left(\frac{2r+1}{L}\pi x\right) \cos\left(\frac{2r+1}{L}\pi c t\right).$$