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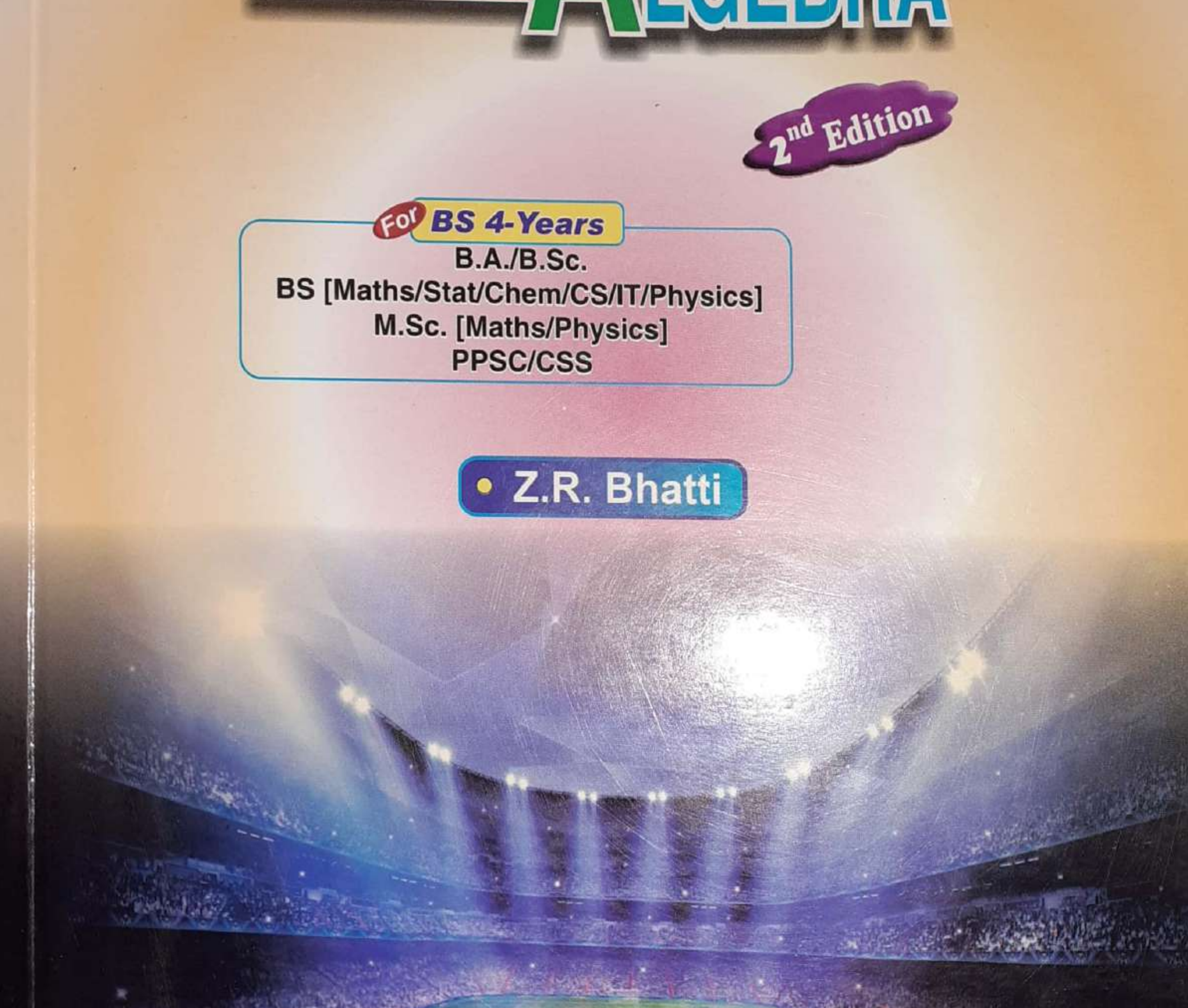
# Introduction to LINEAR ALGEBRA

2nd Edition

For BS 4-Years
B.A./B.Sc.
BS [Maths/Stat/Chem/CS/IT/Physics]
M.Sc. [Maths/Physics]
PPSC/CSS

• Z.R. Bhatti



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According to New Syllabus Approved by the
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# Preface
**(2nd Edition)**

The book is thoroughly revised to fulfil the needs of the students of BS Mathematics, BS Statistics, BS Chemistry, BS IT, BS Physics, BS Computer Science, BS Engineering and BS Software Engineering. Keeping in mind the fact that our students face troubles in understanding foreign books, I have tried my best to use easy and understandable langue. I have also restricted to the topics which are essential and cover the course outlines of our universities.

The book is divided into 5 chapters and covers all the syllabus of University of the Punjab for affiliated colleges. At the same time, it is equally useful for all the Universities of Pakistan for the students of Mathematics, Statistics, Chemistry, Physics, IT, CS, SE.

I hope that the readers of this book will continue to give their valuable suggestions.

**Lahore**
**July, 2019**

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# Contents

## *Chapter-4*
## VECTOR SPACES



## *Chapter-5*
## INNER PRODUCT SPACES

Chapter

# MATRICES

# 1



## 1.1 Definitions and Examples

### Matrix

A ***matrix*** is a rectangular array of numbers. The numbers in the array are called the ***entries*** in the matrix.

Example 1: Some examples of matrices are

$$\begin{bmatrix} 3 & 5 \\ 2 & 1 \\ 0 & -2 \end{bmatrix}, \quad [1 \ \ 2 \ \ 3 \ \ 0], \begin{bmatrix} e & 0 & \sqrt{3} \\ 5 & -1 & 7 \\ 2 & \sqrt{2} & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad [4]$$

The ***size*** of a matrix is described in terms of the number of rows (horizontal lines) and columns (vertical lines) it contains. For example, the first matrix in Example 1 has three rows and two columns, so its size is 3 by 2 (written $3 \times 2$).

In a size description, the first number always denotes the number of rows, and the second denotes the number of columns. The remaining matrices in Example 1 have sizes $1 \times 4$, $3 \times 3$, $3 \times 1$, and $1 \times 1$, respectively.

### Row Vector or Row Matrix

A matrix with only one row is called a ***row vector*** (or a ***row matrix***). For example

$$[1 \ \ 2 \ \ 3 \ \ 0], \quad [2 \ \ 2 \ \ 3 \ \ -1 \ \ 5]$$

are row matrices.

## Column Vector or Column Matrix

A matrix with only one column is called a ***column vector*** (or a ***column matrix***). For example

$$\begin{bmatrix}1\\0\\2\end{bmatrix}, \quad \begin{bmatrix}3\\4\end{bmatrix}, \quad \begin{bmatrix}7\\9\\-1\\-3\end{bmatrix}$$

are column matrices.

Note that [4] is both a row matrix and a column matrix.

We will use capital letters to denote matrices and lowercase letters to denote numerical quantities; thus we might write

$$A=\begin{bmatrix}1&5&0\\2&3&7\end{bmatrix} \quad \text{or} \quad B=\begin{bmatrix}a&b&c\\d&e&f\end{bmatrix}$$

When discussing matrices, it is common to refer to numerical quantities as ***scalars***. Unless stated otherwise, scalars may be real numbers or complex numbers.

The entry that occurs in row $i$ and column $j$ of a matrix $A$ will be denoted by $a_{ij}$. Thus a general $3 \times 4$ matrix might be written as

$$A=\begin{bmatrix}a_{11}&a_{12}&a_{13}&a_{14}\\a_{21}&a_{22}&a_{23}&a_{24}\\a_{31}&a_{32}&a_{33}&a_{34}\end{bmatrix}$$

and a general $m \times n$ matrix as

$$A=\begin{bmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&a_{2n}\\\vdots&\vdots&&\vdots\\a_{m1}&a_{m2}&\cdots&a_{mn}\end{bmatrix} \qquad \text{...(1)}$$

When a compact notation is desired, the preceding matrix can be written as

$$[a_{ij}]_{m\times n} \quad \text{or} \quad [a_{ij}]$$

the first notation being used when it is important in the discussion to know the size, and the second when the size need not be emphasized. Usually, we will match the letter denoting a matrix with the letter denoting its entries; thus, for a matrix $B$ we would generally use $b_{ij}$ for the entry in row $i$ and column $j$, and for a matrix $C$ we would use the notation $c_{ij}$.

The entry in row $i$ and column $j$ of a matrix $A$ is also commonly denoted by the symbol $(A)_{ij}$. Thus, for matrix (1) above, we have