

ILMI®

Introduction to **LINEAR ALGEBRA**

2nd Edition

For **BS 4-Years**

B.A./B.Sc.

BS [Maths/Stat/Chem/CS/IT/Physics]

M.Sc. [Maths/Physics]

PPSC/CSS

• **Z.R. Bhatti**



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According to New Syllabus Approved by the
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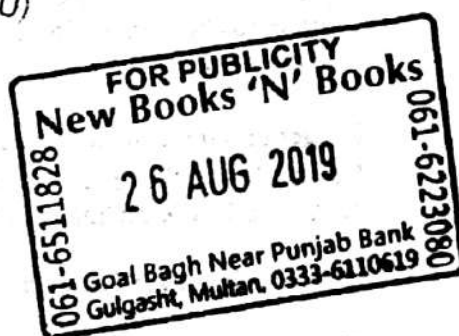
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Preface

(2nd Edition)

The book is thoroughly revised to fulfil the needs of the students of BS Mathematics, BS Statistics, BS Chemistry, BS IT, BS Physics, BS Computer Science, BS Engineering and BS Software Engineering. Keeping in mind the fact that our students face troubles in understanding foreign books, I have tried my best to use easy and understandable language. I have also restricted to the topics which are essential and cover the course outlines of our universities.

The book is divided into 5 chapters and covers all the syllabus of University of the Punjab for affiliated colleges. At the same time, it is equally useful for all the Universities of Pakistan for the students of Mathematics, Statistics, Chemistry, Physics, IT, CS, SE.

I hope that the readers of this book will continue to give their valuable suggestions.

Lahore
July, 2019

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Contents

Syllabi
Preface

Chapter-1 **MATRICES**

1.1	Definitions and Examples	1
1.2	Addition and Multiplication of Matrices	8
1.3	Algebraic Properties of Matrix Operations	15
	Exercise 1.1	27
1.4	Partitioning of Matrices	33
1.5	Elementary Row Operations	35
1.6	Elementary Column Operations	36
1.7	Echelon Form of a Matrix	37
1.8	Rank of a Matrix	39
1.9	Inverse of a Matrix	40
1.10	Canonical Form of a Matrix	48
	Exercise 1.2	52
1.11	Zero-One Matrices	60
	Miscellaneous Exercise 1	68
	Summary	76

Chapter-2 **DETERMINANTS**

2.1	Determinant of a Square Matrix	79
2.2	Minors and Cofactors	80
2.3	Properties of Determinants	82
2.4	Evaluating Determinants by Row Reduction	84
2.5	Adjoint, Inverse and Rank of a Matrix	89
	Exercise 2	95
	Summary	104

Chapter-3 **SYSTEMS OF LINEAR EQUATIONS**

3.1	Introduction	105
3.2	Homogeneous Systems of Linear Equations	109
3.3	Non-Homogeneous Systems of Linear Equations	118
3.4	Applications of Systems of Linear Equations	134
	Exercise 3	160
	Summary	169

Chapter-4 **VECTOR SPACES**

4.1	Fields	171
4.2	Definition and Examples of Vector Spaces	173
4.3	Subspaces	181
	Exercise 4.1	188
4.4	Linear Combinations and Spanning Sets	189
	Exercise 4.2	202
4.5	Linear Independence	208
	Exercise 4.3	216
4.6	Basis and Dimension	219
	Exercise 4.4	233
4.7	Row Space, Column Space and Null Space	235
	Exercise 4.5	246
4.8	Linear Transformation	248
	Exercise 4.6	256
4.9	Matrix of Linear Transformation	258
	Exercise 4.6	264
	Miscellaneous Exercise 4	267
	Summary	269

Chapter-5 **INNER PRODUCT SPACES**

5.1	Inner Product and Norm	273
5.2	Orthogonal and Orthonormal Sets	277
	Exercise 5.1	286
5.3	Eigenvalues and Eigenvectors	290
	Exercise 5.2	302
5.4	Similar Matrices	304
5.5	Diagonalization of Matrices	307
	Miscellaneous Exercise 5	314
	Summary	315
	Answers	317
	University Question Papers	335

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MATRICES

- 1.1 Definitions and Examples
- 1.2 Addition and Multiplication of Matrices
- 1.3 Algebraic Properties of Matrix Operations
- 1.4 Partitioning of Matrices
- 1.5 Elementary Row Operations
- 1.6 Elementary Column Operations
- 1.7 Echelon Form of a Matrix
- 1.8 Rank of a Matrix
- 1.9 Inverse of a Matrix
- 1.10 Canonical Form of a Matrix
- 1.11 Zero-One Matrices

1.1 Definitions and Examples

Matrix

A **matrix** is a rectangular array of numbers. The numbers in the array are called the **entries** in the matrix.

Example 1: Some examples of matrices are

$$\begin{bmatrix} 3 & 5 \\ 2 & 1 \\ 0 & -2 \end{bmatrix}, \quad [1 \ 2 \ 3 \ 0], \quad \begin{bmatrix} e & 0 & \sqrt{3} \\ 5 & -1 & 7 \\ 2 & \sqrt{2} & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad [4]$$

The **size** of a matrix is described in terms of the number of rows (horizontal lines) and columns (vertical lines) it contains. For example, the first matrix in Example 1 has three rows and two columns, so its size is 3 by 2 (written 3×2).

In a size description, the first number always denotes the number of rows, and the second denotes the number of columns. The remaining matrices in Example 1 have sizes 1×4 , 3×3 , 3×1 , and 1×1 , respectively.

Row Vector or Row Matrix

A matrix with only one row is called a **row vector** (or a **row matrix**). For example

$$[1 \ 2 \ 3 \ 0], \quad [2 \ 2 \ 3 \ -1 \ 5]$$

are row matrices.

Column Vector or Column Matrix

A matrix with only one column is called a **column vector** (or a **column matrix**).
For example

$$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 7 \\ 9 \\ -1 \\ -3 \end{bmatrix}$$

are column matrices.

Note that $[4]$ is both a row matrix and a column matrix.

We will use capital letters to denote matrices and lowercase letters to denote numerical quantities; thus we might write

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 2 & 3 & 7 \end{bmatrix} \quad \text{or} \quad B = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

When discussing matrices, it is common to refer to numerical quantities as **scalars**. Unless stated otherwise, scalars may be real numbers or complex numbers.

The entry that occurs in row i and column j of a matrix A will be denoted by a_{ij} . Thus a general 3×4 matrix might be written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix}$$

and a general $m \times n$ matrix as

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \quad \dots(1)$$

When a compact notation is desired, the preceding matrix can be written as

$$[a_{ij}]_{m \times n} \quad \text{or} \quad [a_{ij}]$$

the first notation being used when it is important in the discussion to know the size, and the second when the size need not be emphasized. Usually, we will match the letter denoting a matrix with the letter denoting its entries; thus, for a matrix B we would generally use b_{ij} for the entry in row i and column j , and for a matrix C we would use the notation c_{ij} .

The entry in row i and column j of a matrix A is also commonly denoted by the symbol $(A)_{ij}$. Thus, for matrix (1) above, we have