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Critical Percolation and A+B→2A Dynamics

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Abstract
We study an interacting particle system in which moving particles activate dormant particles linked by the components of critical bond percolation. Addressing a conjecture from Beckman, Dinan, Durrett, Huo, and Junge for a continuous variant, we prove that the process can reach infinity in finite time i.e., explode. In particular, we prove that explosions occur almost surely on regular trees as well as oriented and unoriented two-dimensional integer lattices with sufficiently many particles per site. The oriented case requires an additional hypothesis about the value of a certain critical exponent. We further prove that the process with one particle per site expands at a superlinear rate on integer lattices of any dimension. Some arguments use connections to critical first passage percolation, including a new result about the existence of an infinite path with finite passage time on the oriented two-dimensional lattice.

Keywords Percolation · Frog model · Particle system

1 Introduction

The goal of this work is to better understand expansion in a growing system of random walks with $A + B \to 2A$ "frog model" dynamics. In such systems A-particles, traditionally referred to as "active frogs," are mobile. They perform simple random walk and activate stationary B-particles, referred to as "sleeping frogs," upon contact. We introduce a variant in which sleeping frogs are linked by the geometry of critical bond percolation; a visit to a site containing a sleeping frog simultaneously wakes every sleeping frog in the connected cluster containing that frog. This makes it so large regions are activated instantly, which dramatically increases the spread of activated particles compared to $A + B \to 2A$ dynamics without linkage. Although we keep with convention and continue to use the frog imagery, these dynamics better correspond to combustion; A-particles represent diffusing heat and B-particles entwined fuel cells (as in [28,29]).

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We first informally describe some related graph percolation models. *First passage percolation* is the random metric induced by assigning independent passage times to each edge according to a distribution function F. *Bond percolation* is the special case that edges are p-open (have passage time 0) with probability p and otherwise are p-closed (the passage time is infinite). The critical value is the parameter p_c above which an infinite component of p-open edges emerges. *Critical first passage percolation* is the case in which $F(0) = p_c$.

The *critical frog model on G with m particles per site*, which we will often abbreviate as CFM(G, m), begins by placing m sleeping frogs at each vertex. Call each component of vertices that are pathwise connected via p_c-open edges a *cluster*. The process begins by waking up all frogs in the cluster containing the root. When a frog wakes up, it diffuses according to a simple random walk on G with independent exponential(1)-distributed delays between jumps. Whenever a frog jumps on a cluster with sleeping frogs, every frog in the cluster wakes up and begins its own independent random walk. We will call every vertex in such a cluster *activated*.

Recently, Dinan, Durrett, Beckman, Huo, and Junge generalized the frog model to d-dimensional Euclidean space [4]. They replaced simple random walks with Brownian motions and the underlying graph structure by distributing particles according to a Poisson point process and including a disk of radius r around each point. Whenever an active particle comes within r of a cluster of overlapping disks, all dormant particles in that cluster activate and begin their own Brownian motions.

Since entire clusters of overlapping disks activate simultaneously, it is important to understand the cluster geometry. This is a special case of continuum percolation called the *Boolean model*. It is known that there is a critical value r_d such that for $r > r_d$ an infinite cluster of overlapping disks forms in $\mathbb{R}^d$ [25]. Once this cluster is discovered, the Brownian frog model has infinitely many moving particles. The main result of [4] was that for $r < r_d$ the set of visited sites scaled linearly by time has the Euclidean ball with a deterministic radius (that depends on r) as its limiting shape.

The main open question from [4] was to describe the rate the Brownian frog model expands at criticality ($r = r_d$). The expansion was conjectured to be superlinear, but the authors did not try to guess the exact rate. The inspiration for the critical frog model comes from this conjecture. By moving to graphs, where percolation is better understood, we are able to prove concrete results. Continuum percolation in Euclidean space and bond percolation on the integer lattice are believed to have similar behavior at criticality. So, a result in either setting is suggestive of what occurs in the other.

Similar continuous time $A + B \to 2A$ dynamics were studied by Ramírez and Sidoravicius in [28]. They viewed the active/asleep dynamics as a toy model for heat diffusing and igniting fuel. The motivation was to rigorously study a caricature of chemical reactions associated to the steady-state burning of a homogeneous solid [29, Chapter 9]. Note that they considered the process without clusters linking particles, so a visit to a vertex only activates the particles at that site. Using different methods, Alves et al. in [1] concurrently proved a shape theorem similar to [28], but in discrete time. Subsequently, the discrete-time process has been extensively studied on lattices and trees (see [10,20,27] for a start). Note that bond percolation plays an important role in [11].

Another closely related $A + B \to 2A$ model was studied in a sequence of papers by Sidoravicius and Kesten [22–24]. With the intention of modeling the spread of a rumor or infection, they considered the process in which both A- and B-particles perform continuous time simple random walks. Starting with one A-particle at the origin, they proved that the set of locations of A-particles expands linearly in time. One could define an analogous critical process with these dynamics; when an A-particle meets a B-particle, it and all other B-

particles that started in the same cluster convert to A-particles. It would be interesting, and perhaps more tractable, to prove that this process explodes on lattices and trees.

1.1 Results

Let $G = (\mathcal{V}, \mathcal{E})$ be a connected, infinite graph with root $\mathbf{0}$. The examples we focus on are:

- The d-ary tree $\mathbb{T}_d$ which is the rooted tree in which each vertex has d children.
- The integer lattice $\mathbb{Z}^d$ has the standard basis $e_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ for $1 \le i \le d$. The set of vertices are all integer linear combinations of (e_i), and edges exist between any vertices separated by distance 1.
- The oriented lattice $\vec{\mathbb{Z}}^d$ which is the usual integer lattice in which each edge $(v, v + e_i)$ is directed towards $v + e_i$.

Let ξ_t be the set of activated vertices at time t. We say that CFM(G, m) is *explosive* if there exists an almost surely finite time T such that $|\xi_T| = \infty$. Note that the possibility of an explosion requires that the process evolves in continuous time. We use a comparison to critical first passage percolation to show that the critical frog model is explosive with enough particles per site. The statement for the oriented case relies on a bound for the critical exponent associated with the correlation length in the temporal direction. Although the exponent is not known to exist, we work with a well-defined upper bound.t

Theorem 1 CFM(G, m) *is explosive for:*

(i) $\mathbb{Z}^2$ *with* $m = 4$,
(ii) $\vec{\mathbb{Z}}^2$ *with* $m = 2$ *assuming* $\nu_{||}$ *defined at* (5) *is strictly less than 2, and*
(iii) $\mathbb{T}_d$ *with* $m = 1$.

The result for $\mathbb{Z}^2$ follows from an extension of [9] to the case $F(+\infty) < 1$. We prove a new result, Theorem 3, for $\vec{\mathbb{Z}}^2$ to establish (ii). As for our hypothesis on $\nu_{||}$, no rigorous proof exists, but it is widely believed that $\nu_{||} \approx 1.74$ [15]. To prove (iii), we work with an embedded process restricted to the leaves of percolation clusters and apply a result about explosive branching processes with heavy tails from [2].

Note that by a straightforward coupling, the results of Theorem 1 are also true for larger values of m. Our results for two-dimensional lattices require more than one particle per site, so that the critical frog model can be coupled to critical first passage percolation with independent passage times. For $\mathbb{Z}^2$, this allows us to use [9, Corollary 1.3] (see Sect. 1.2 for more details). We conjecture that the critical frog model is explosive with $m = 1$ on $\mathbb{Z}^d$ and $\vec{\mathbb{Z}}^d$ for $d \ge 2$, but prove a weaker result. Let $M_t = \max\{\|v - \mathbf{0}\| : v \in \xi_t\}$ be the maximum distance in the graph metric between sites of ξ_t and the root.

Theorem 2 *Consider* CFM$(G, 1)$. *Let* $\epsilon > 0$ *be arbitrary and* $d \ge 2$. *The following hold almost surely:*

(i) *If* $G = \vec{\mathbb{Z}}^2$, *then* $M_t / t^{5-\epsilon} \to \infty$.
(ii) *If* $G = \mathbb{Z}^d$ *or* $\vec{\mathbb{Z}}^d$, *then* $M_t / t \to \infty$.

The proof for the oriented case goes by tracking the maximally displaced point in one ancestry line. A lower bound for box-crossing probabilities from [12] guarantees that the jump size in some direction is at least n with probability $ct^{-1/5}$ for some fixed $c > 0$. Since we only consider one particle, we wait an exponential(1)-distributed time between each jump. The law of large numbers then gives the claimed growth of this ancestry line, which serves as a lower bound for M_t. For higher dimensions we do not have as good of a tail estimate, but it is not so hard to prove that the expected jump size is infinite at criticality, which is enough to deduce $M_t/t \to \infty$.

The unoriented setting has two extra complications. First, we use a subprocess that jumps when a new half-space with a sufficiently large cluster is reached in the e_d-direction. A similar argument as for the oriented case shows that the cluster size has infinite expectation. This requires an additional observation from [17] that p_c is the same on $\mathbb{Z}^d$ and on the d-dimensional half-space. The second difficulty is that, unlike in the oriented case, frogs might move back towards the origin. Thus, we need to control the number of active frogs near the front of the subprocess in order to ensure that the time between jumps has finite expectation. This relies on a relatively simple estimate for the hitting time of a set for multiple one-dimensional simple random walks.

1.2 Critical First Passage Percolation

Assign independent uniform(0, 1) random variables $(\omega_e)_{e\in\mathcal{E}}$ to each $e \in \mathcal{E}$. In *p-bond percolation*, $e \in \mathcal{E}$ is *p-open* if $\omega_e \leq p$. Otherwise, it is *p-closed*. The *critical value* is the infimum over p such that there is almost surely an infinite path of adjacent p-open edges:

$$p_c = p_c(G) = \inf\{p\colon P(\text{there exists an infinite } p - \text{open path}) = 1\}.$$

It is easy to show that $p_c(\mathbb{T}_d) = 1/d$ by comparison to a critical branching process. In [21], Kesten proved that $p_c(\mathbb{Z}^2) = 1/2$. The exact value of $p_c(\mathbb{Z}^d)$ is not known in higher dimensions, nor is $p_c(\vec{\mathbb{Z}}^d)$ known for any d.

Let F be a distribution function with $F(x) = 0$ for $x < 0$ with $F(+\infty) = 1$. The inverse of F is defined as

$$F^{-1}(y) = \inf\{y\colon F(y) \geq x\}.$$

Define the *passage time along edge* e as $t_e = F^{-1}(\omega_e)$. Assign to each *vertex self-avoiding path*, $\gamma = (x, v_1, v_2, \ldots, v_n, y)$ with all vertices distinct, the passage time $T(\gamma) = \sum_{(u,v)\in\gamma} t_{(u,v)}$, and define the *passage time from x to y* as

$$T(x, y) = \inf\{T(\gamma)\colon \gamma \text{ is a path from } x \text{ to } y\}.$$

The random metric space induced by $T(\cdot,\cdot)$ is referred to as *first passage percolation*. Define the *passage time to infinity* as the limit of the minimal passage times to distance n:

$$\rho = \rho(G, F) = \lim_{n\to\infty} \inf\{T(\mathbf{0}, x)\colon \|x\| = n\}.$$

Note that the limit exists for trees and lattices by monotonicity. The Kolmogorov 0–1-law ensures that $P(\rho < \infty) \in \{0, 1\}$.

Considerable attention has been devoted to the geometry of p_c-bond percolation. On trees, a simple comparison to a critical branching process shows that there are no infinite p_c-open paths. The same is believed to hold for $\mathbb{Z}^d$. This is only known in dimension 2 [21] and in high-dimensions [16,18]. Critical first passage percolation is a natural way to relax p_c-bond percolation.

Zhang initially studied F with $F(0) > p_c$ in [30]. It follows immediately that $\rho < \infty$ for such F, since there is an infinite zero-passage time path. On the other hand, for $F(0) < p_c$ it is not hard to show that $\rho = \infty$ almost surely [3, Proposition 4.4]. For these reasons, first passage percolation with any F satisfying $F(0) = p_c$ is referred to as *critical first passage percolation*.

In a followup work [31], Zhang investigated the critical case and proved that a "double behavior" occurs on $\mathbb{Z}^2$. In particular, he showed that for

$$F_a(x) = \begin{cases} 0, & x < 0 \\ p_c + x^a, & p_c + x^a \leq 1 \\ 1, & p_c + x^a > 1 \end{cases}$$

it holds that $\rho(\mathbb{Z}^2, F_a) < \infty$ almost surely for small enough a. On the other hand, for distributions that approach p_c more rapidly,

$$G_b(x) = \begin{cases} 0, & x < 0 \\ p_c + \exp(-1/x^b), & p_c + \exp(-1/x^b) \leq 1 \\ 1, & p_c + \exp(-1/x^b) > 1 \end{cases},$$

it holds that $\rho(\mathbb{Z}^2, G_b) = \infty$ for large enough b. Rather recently, Damron, Lam, and Wang in [9] gave exact conditions on F with $F(+\infty) = 1$ that lead to finite ρ:

$$\rho(\mathbb{Z}^2, F) < \infty \text{ if and only if } \sum_{k=1}^{\infty} F^{-1}(p_c + 2^{-k}) < \infty. \tag{1}$$

Their approach relies on the relative wealth of knowledge about percolation in two dimensions. It appears to be difficult to prove analogues of (1) for $\mathbb{Z}^d$ with $d > 2$. In fact, proving that ρ is infinite in the case of bond-percolation on $\mathbb{Z}^3$ is one of the major open problems in the field. However, a result of Bramson for branching random walk that predates the introduction of critical first passage percolation tells us what happens on trees. A direct consequence of [5, Theorem 2] is that

$$\rho(\mathbb{T}_d, F) < \infty \text{ a.s. if and only if } \exists \lambda > 0 \text{ with } \sum_{k=1}^{\infty} F^{-1}\left(p_c + \exp(-\lambda^k)\right) < \infty. \tag{2}$$

When $\rho = \infty$, Bramson's theorem additionally describes the growth rate:

$$T_n(\mathbb{T}_d, F) \approx \sum_{k=1}^{\log\log n} F^{-1}(p_c + \exp(2^{-k})). \tag{3}$$

It seems to us that this reformulation of Bramson's work in terms of critical first passage has not been observed in the literature. This is especially interesting because (2) is markedly weaker than the recently proven criterium at (1). For instance, $\rho(\mathbb{T}_d, G_b) < \infty$, but $\rho(\mathbb{Z}^2, G_b) = \infty$ for all $b > 0$. Also, (3) implies that when F is such that p_c-closed edges have passage time 1 rather than infinity, we have $T_n(\mathbb{T}_d, F) \approx \log\log n$. This contrasts with the result of Chayes, Chayes, and Durrett that $T_n(\mathbb{Z}^2, F) \approx \log n$ in this setting [6].

Obtaining a similar necessary and sufficient condition as either (1) or (2) for $\vec{\mathbb{Z}}^2$ appears challenging. The main difficulty is that we do not understand p-bond percolation with p near p_c nearly as well as with the unoriented case. We instead prove a result tailored for the critical frog model in which we assume a critical exponent bound. We delay defining it until just before the proof. Below we write $F(x) = p_c + O(x^a)$ to mean $0 \leq F(x) - p_c \leq Cx^a$ for some $C > 0$ and all $0 < x < \epsilon$ for some small $\epsilon > 0$.

Theorem 3 *Let $\nu_{||}$ be as defined at (5). If $F(x) = p_c + O(x^a)$ with $\nu_{||} < 1 + a^{-1}$, then $\rho < \infty$ a.s.*

Our proof of Theorem 3 uses the block construction for oriented percolation devised by Durrett in [13]. The modified proof is inspired by what was recently done for an inhomogeneous percolation model studied by Cristali et al. [7]. The difference is that they worked with p-bond percolation for $p = p_c + \epsilon + o(1)$ for fixed $\epsilon > 0$. In this work, we must consider $p = p_c + o(1)$. This makes critical exponents (i.e., the behavior of p-bond percolation very near p_c) a relevant quantity.

The idea is to map a lattice of interlocking parallelograms to a copy of $\vec{\mathbb{Z}}^2$. If one such parallelogram has a "centered crossing," then we declare the corresponding bond open in the renormalized lattice. The sizes and slopes of the parallelograms depend on their distance to $\mathbf{0}$ and on $\nu_{||}$. Making them suitably larger than the correlation length causes one-dependent bond percolation on the renormalized process to be supercritical and have an infinite connected path. We show that this path has a passage time that is summable so long as a is sufficiently small.

1.3 Organization

Section 2 uses results from critical first passage percolation to prove Theorem 1. Section 3 contains the the proof of Theorem 2. Section 4 is dedicated to Theorem 3.

2 Explosions

The following arguments rely on a connection between critical first passage percolation and the critical frog model. Let

$$B_t(G, F) = \{x \in \mathcal{V} \colon T(\mathbf{0}, x) \leq t\}$$

be the set of sites that can be reached from $\mathbf{0}$ by time t in first passage percolation. To prove the first two parts of Theorem 1, we will show that $\mathrm{CFM}(G, m)$ can be coupled to critical first passage percolation so that $B_t(G, F) \subseteq \xi_t$ for an appropriate choice of F. A technical difficulty is that $F(+\infty) < \infty$, so there are infinite passage time edges. This case is not directly covered by (1) or Theorem 3. We explain in the proofs how the results are extended to this setting.

Proof of Theorem 1 (i) We claim that the set of activated sites in $\mathrm{CFM}(\mathbb{Z}^2, 4)$ dominates $B_t(\mathbb{Z}^2, F)$ with passage times specified by

$$F(x) = p_c + \frac{(1 - p_c)(1 - e^{-x})^2}{16}. \tag{4}$$

Equivalently,

$$t_e = \begin{cases} 0, & \text{with probability } p_c \\ \max(X_e^{(1)}, X_e^{(2)}), & \text{with probability } (1 - p_c)/16 \\ \infty, & \text{with probability } 1 - p_c - (1 - p_c)/16 \end{cases}$$

with $X_e^{(1)}$ and $X_e^{(2)}$ independent unit exponential random variables. For the remainder of this proof we let F and t_e refer to this distribution and random variable.

The coupling between the $\mathrm{CFM}(\mathbb{Z}^2, 4)$ and F-critical first passage percolation is fairly straightforward to describe. At each vertex x assign the edges attached to x in a one-to-one manner to the four sleeping frogs at x. The frogs, when woken up, still move to a random neighboring vertex, but we will only use that frog if it moves across its pre-assigned edge and that edge was not already p_c-open.

More precisely, we give an edge $e = (x, y)$ passage time 0 if it is open in the underlying p_c-bond percolation. If e is not open, but both frogs at x and y assigned to e move across it in their first step, then we assign the maximum of their two passage times to e. Otherwise e has passage time ∞. The set of activated sites in this subprocess is dominated by the critical frog model in which frogs are only allowed to move one step along their preassigned edge. This is because t_e is 0 with the same probability in each process. Alternatively, if $t_e > 0$, it is stochastically larger than the time it takes for a frog to cross e. This is because we take the maximum of the two crossing times of the frogs at each end of e.

The criterium at (1) does not immediately apply when infinite passage times are possible. However, it does apply to $F^{(1)}$ induced by the truncated passage times $t_e^{(1)} = t_e \wedge 1$. It is straightforward to check that $F^{(1)}(x) - p_c = O(x^{-2})$ as $x \to 0$, and so the criterium at (1) to have $\rho(\mathbb{Z}^2, F^{(1)}) < \infty$ a.s. is satisfied. It is proven in [3, Lemma 4.3] that $\rho < \infty$ implies there exists an infinite vertex self-avoiding path γ from $\mathbf{0}$ to infinity. Since the passage time is finite, such a path uses only finitely many weight-1 edges. So, all edges beyond some almost surely finite distance R have passage time less than 1. Using the fact that t_e and $t_e^{(1)}$ are coupled via ω_e, it follows that there is an infinite vertex self-avoiding path $\gamma = (v_0, v_1, \ldots)$ satisfying: $\|v_0\| \leq R + 1$ and the passage time along γ is finite in F-critical first passage percolation. Since simple random walk is recurrent on $\mathbb{Z}^2$, it follows that the frog started at the origin will visit a site on γ in an almost surely finite amount of time. After which, the process will explode. □

The proof for the oriented lattice is similar.

Proof of Theorem 1 (ii) The analogous distribution to (4) to compare to CFM$(\vec{\mathbb{Z}}^2, 2)$ is

$$F(x) = p_c + (1 - p_c)(1 - e^{-x})/2$$

so that

$$t_e = \begin{cases} 0, & p_c \\ X_e, & (1 - p_c)/2 \\ \infty, & \text{otherwise} \end{cases}$$

with X_e a unit exponential random variable. The function and coupling are a little cleaner here since frogs must move away from $\mathbf{0}$ and there are only two outward edges at each site. Once again, we only allow frogs to move one step and along a predetermined outward edge assigned in a bijective manner. If the frog moves along its assigned edge, then the passage time is X-distributed. If the frog does not move along that edge, then the passage time is infinite.

It is easy to check that the truncated passage times $t_e^{(1)} = t_e \wedge 1$ satisfy the hypotheses of Theorem 3 with $a = 1$. Thus, assuming the other hypothesis $\nu_{||} < 2$ (defined in Sect. 4) holds, we have $\rho(\vec{\mathbb{Z}}^2, F^{(1)}) < \infty$ a.s. with $F^{(1)}$ the distribution induced by $t_e^{(1)}$. As this path can contain only finitely many edges with weight-1, it follows that $\rho(\vec{\mathbb{Z}}^2, F) < \infty$ with positive probability. To translate this into an almost sure statement about the frog model, notice that each time M_t increases for the frog model, a new embedded subprocess can be started from that point because the edges beyond have yet to be discovered. With positive probability the embedded first passage percolation model reaches infinity in finite time, and thus the frog model is explosive. Since this happens infinitely many times, the probability of an explosion is 1. □

The idea with the tree is similar, but now we can use independence among subtrees to construct a subprocess that follows a growing number of lineages.

Proof of Theorem 1 (iii) We can use the edge weights ω_e and information about the movement of frogs to induce a Galton Watson tree $\mathcal{T}$ whose vertices are a subset of $\mathbb{T}_d$. A *leaf* in a bond percolation cluster is a vertex whose parent edge is open, but all child edges are closed. Let $L(v)$ be the set of all leaves in the p_c-bond percolation cluster restricted to just v and its descendents. For each $u \in L(v)$ let A_u be the event that the first step of the frog at u is to a child vertex of u. Set $C_v = \{u \in L(v) \colon \mathbf{1}\{A_u\} = 1\}$. Notice that

$$|C_v| \sim \text{Binomial}(|L(v)|, d/(d+1)),$$

where the notation $X \sim \text{Bin}(Y, p)$ means that $X \sim \text{Bin}(k, p)$ conditional on $\{Y = k\}$.

The tree $\mathcal{T}$ starts with root $\mathbf{0}$. The first level of $\mathcal{T}$ consists of all vertices from $C(L(\mathbf{0}))$. The vertices attached to each v at level $i+1$ are the vertices from $C(L(v))$. To each edge (u, v) in $\mathcal{T}$, we attach an independent unit exponential passage time. Using independence, and self-similarity of $\mathbb{T}_d$, it follows that $\mathcal{T}$ is a Galton-Watson tree with offspring distribution $Z = |C_{\mathbf{0}}|$. Another advantage of $\mathcal{T}$ is that it tracks the edges with non-zero passage time in the frog model in which we only consider frogs at the leaves of critical bond percolation clusters that jump away from the root on their first step. After this jump these frogs are removed.

Let Z' be the size of the cluster containing the root in critical percolation on $\mathbb{T}_d$. It is a classical result that

$$P(Z' \geq n^2) = \frac{c'}{n}(1 + o(1))$$

for some $c' > 0$ (see [19, Theorem 2.1]). It is proven in [26, Theorem 2] that the number of leaves in a Galton–Watson tree, call it Z'', corresponds to a Galton–Watson process with the same asymptotic right-tail behavior as Z'. Thus,

$$P(Z'' \geq n^2) \geq \frac{c''}{n}$$

for all n and some $c'' > 0$. Recall that our offspring distribution of interest is $Z = |C_{\mathbf{0}}| \sim \text{Bin}(Z'', d/(d+1))$. Since a binomial random variable is greater than or equal to its mean with probability at least 1/2, we have

$$P(Z \geq n^2) \geq 2^{-1} P(Z'' > ((d+1)/d)n^2) \geq \frac{c}{n}$$

for all n and some $c = c''/2$. The above line ensures that the offspring distribution Z satisfies the "plumpness" criterium of [2, Theorem 1.3]. The condition to check for an explosion to occur is that $F^{-1}(e^{-\lambda^k})$ is summable for some $\lambda > 0$. As we are assigning independent unit exponential edge weights, it is straightforward to verify that this is summable for any $\lambda > 0$. Thus, conditional on $\mathcal{T}$ being infinite, it almost surely contains an infinite path with finite passage time.

As noted during the construction of $\mathcal{T}$, the set of sites visited by time t in on $\mathcal{T}$ is equivalent to the set of activated sites in a modified frog model in which we remove all frogs at non-leaf vertices of critical percolation clusters and only allow the remaining frogs to take one step. It follows that this process reaches infinity in finite time with positive probability. To turn this into an almost sure statement consider a slightly modified process. The frog started at the root will almost surely escape to ∞ along a unique, uniformly sampled vertex self-avoiding path. Each vertex v along this path contains a frog and also a copy of $\mathbb{T}_d$ rooted at the child vertex of v. With positive probability each of these subtrees will have infinitely many frogs activated in finite time. It follows that after an almost surely finite time one of these subtrees will be discovered by the frog at the root, and thus in finite time infinitely frogs will be activated. □

3 Results with One Particle per Site

3.1 The Oriented Lattice

Proof of Theorem 2 (i) Analogous to Theorem 1 (iii), we will again use an embedded subprocess. Let $\mathcal{C}(v)$ denote the cluster containing the vertex v. Let x_1 be the point in $\mathcal{C}(\mathbf{0})$ with the largest distance from $\mathbf{0}$. If there are infinitely many candidates, then the claimed result holds. If there are a finite number of candidates, then choose one uniformly at random. Let τ_1 be the time for the frog started at x_1 to jump to a neighboring vertex y_1. We then iteratively define x_{i+1} to be the vertex of $\mathcal{C}(y_i)$ with maximal distance from $\mathbf{0}$, breaking ties as before. We set y_{i+1} to be the vertex that the frog started at x_{i+1} first jumps to after waiting τ_{i+1} time units.

Since we are on the oriented lattice and we are choosing points at maximal distance, we have the jump sizes $J_i = \|y_i - x_{i-1}\|$ are independent and identically distributed. So, we can decompose $\|y_n\| = \sum_{i=1}^n J_i$. The lower bound from [12, Theorem 1.1] gives $P(J \geq n) \geq cn^{-1/5}$ for some $c > 0$ and all $n \geq 1$. As the expected time between each jump has mean 1, it follows by applying [14, Theorem 3.8.2] to the embedded jump chain in (M_t) that almost surely

$$P(t^{-5+\epsilon} M_t \to \infty) = P(n^{-5+\epsilon} M_n \to \infty) = 1.$$

The proof in higher dimension is similar, however we only know that $EJ = \infty$, in which case we invoke the strong law of large numbers to deduce that $M_t/t \to \infty$.

While we could not find a reference, one way to show that $EJ = \infty$ goes by contradiction. Suppose that $EJ < \infty$. Then, if J_h is the expected number of sites in the cluster at height h, monotonicity ensures there is a value h_0 such that $EJ_{h_0} < 1$. The total size of the cluster containing the origin is dominated by starting independent oriented percolation clusters of height h_0 at each site counted by J_{h_0}. The dominance holds because it can be thought of as adding additional edges to the graph. It follows that the cluster containing the origin is dominated by a subcritical branching process and is almost surely finite.

Since there are only finitely many accessible edges from the origin up to height h_0, the quantity EJ_{h_0} is a polynomial and thus continuous. It follows that there is a value $p_c' > p_c$ such that $E_{p_c'} J_{h_0} < 1$. The same reasoning as in the previous paragraph implies that the cluster containing the origin in p_c'-bond percolation is almost surely finite. This contradicts the definition of p_c. Thus $EJ = \infty$. □

3.2 The Unoriented Lattice

We begin by giving an overview of the proof before turning to the details in the next paragraph. The idea with the unoriented lattice is to use that the cluster size when the process crosses a new half-space in the e_d-direction is independent if restricted to sites in the half-space. An issue is that the expected time for a single frog to cross into a new region is infinite. However, an elementary result for one-dimensional simple random walk tells us that the expected time is finite for three or more particles. Accordingly, we wait some amount of time to have three frogs. Next, we let them diffuse until they discover a cluster of size at least 3.

We then use the three nearest particles to the boundary to discover the next cluster of size ≥ 3. The jumps at each new discovery have infinite expectation, and the distance between large enough discoveries, though random, is controllable. This yields an embedded process with superlinear growth.

We start by stating and proving the lemma for simple random walk. In essence, it says that at least one of three independent random walks started at -1, -2 and -3 will reach a site a geometrically distributed distance away in finite expected time.

Lemma 4 *Let $S_t^{(1)}, S_t^{(2)}, S_t^{(3)}$ be three simple random walks on $\mathbb{Z}$ with $S_0^{(i)} = -i$ for $i = 1, 2, 3$. Let Y be an independent geometric random variable supported on the nonnegative integers with parameter $q \in (0, 1)$. Let $\sigma = \inf\{t : S_t^{(i)} = 2Y$ for some $i\}$ be the number of steps to first reach $2Y$. It holds that $E\sigma < \infty$.*

Proof If T is the first time that a simple random walk started at 0 reaches 1, then it is a well known fact (see [14, Chapter 4]) that $P(T > t) \leq Ct^{-1/2}$ for some $C > 0$ and all $t > 0$. If T_k is the number of steps to reach k, then we can write T_k as a sum of k i.i.d. copies of T. The event $\{T_k > t\}$ is contained in the event that one of these copies is larger than t/k. Applying a union bound gives

$$P(T_k > t) \leq kP(T > t/k) \leq Ck(t/k)^{-1/2} = Ck^{3/2}t^{-1/2}.$$

Conditional on the event $\{Y = k\}$, it is easy to see that,

$$P(\sigma \geq t \mid Y = k) \leq C^3(2k)^{9/2}t^{-3/2}.$$

Letting $C_0 = 2^{9/2}C^3$, it follows that

$$E\sigma = \sum_{t=1}^{\infty} P(\sigma \geq t) \leq C_0 \sum_{k=1}^{\infty} k^{9/2} P(Y = k) \sum_{t=1}^{\infty} t^{-3/2} < \infty.$$

The above line is finite because $P(Y = k)$ decays exponentially fast, and $t^{-3/2}$ is summable. □

Proof of Theorem 2 (ii) Let $H = \max\{x_d : (x_1, \ldots, x_d) \in \mathcal{C}(\mathbf{0})\}$ be the displacement in the x_d-direction of the cluster containing $\mathbf{0}$ in critical bond percolation on the upper-half space $\mathbb{H}^d = \{(x_1, \ldots, x_d) : x_d \geq 0\}$. The main theorem of [17] is that the critical value for bond percolation on the half-space is equal to $p_c(\mathbb{Z}^d)$. A similar argument as at the end of the proof of Theorem 2 (i) then shows that $EH = \infty$ for all $d \geq 2$. We consider a restricted process which we now define.

Initially, only the frog started at $\mathbf{0}$ is allowed to diffuse. It does not wake any frogs until it has visited three distinct sites: v_1, v_2, v_3. This happens after a random time with right tail that decays exponentially, and thus the time elapsed has finite expectation. Let $\Gamma_1 = \cup_{i=1}^3 \mathcal{C}(v_i)$. Now we define an iterative exploration process.

Let z_1 be the point with largest x_d coordinate in Γ_1. As $|\Gamma_1| \geq 3$, choose three frogs $f_1(1), f_2(1), f_3(1)$ sleeping at vertices in Γ_1 that are nearest to the half-space $\mathcal{P}_1 = \{x : x_d > z_1\}$. If there are several candidates, then break ties arbitrarily. Wake these three frogs up. For all future steps we will disregard the other frogs (active or asleep) in $\mathcal{P}_1^c$.

Note that $\|f_i(1) - \mathbf{P}_1\| \leq 3$ for $i = 1, 2, 3$, since we are choosing the three nearest sleeping frogs in Γ_1 to $\mathbf{P}_1$. The displacement of each frog in the e_d direction is a lazy simple random walk that stays in place with probability $(d-1)/d$. Each time one of the $f_i(1)$ reaches a new maximal x_d coordinate in the union of their ranges, a new cluster in the half-space at that site is discovered. Since we are restricting to newly revealed sites in the half-space, the geometry of the just-discovered cluster is independent of the past and distributed like $\mathcal{C}(\mathbf{0}) \cap \mathbb{H}^d$. After a geometric number of trials Y with a positive probability of success, a cluster of size at least 3 will be discovered. Each failure requires the frogs to traverse at most an additional 2 units in the e_d direction to find another independent cluster. Thus, $f_1(1)$, $f_2(1)$, and $f_3(1)$ must traverse distance $2Y$ in the x_d direction. By Lemma 4, the time to discover a large enough cluster is dominated by σ, which has finite expectation, say, $\mu < \infty$.

Let Γ_2 be the newly discovered cluster with size at least 3 and let z_2 be the maximal x_d value of points in Γ_2. Define $\mathbf{P}_2 = \{x : x_d > z_2\}$ be the boundary half-space. We repeat the procedure from before. That is, we select three frogs—$f_1(2)$, $f_2(2)$, $f_3(2)$—nearest to $\mathbf{P}_2$, then disregard all frogs, except for the three newly activated ones. As in the previous step, these three frogs will discover a new cluster of size three after at most μ steps in expectation. Iterate this procedure indefinitely.

At each iteration of the discovery procedure, an independent jump with distributed like H conditioned on the event $\{|\mathcal{C}(0)| \geq 3\}$ is made. As mentioned in the first paragraph of this argument, the EH is infinite (conditioning the $\mathcal{C}(0)$ cluster to have size at least 3 only increases this quantity). Moreover, the time between each jump has expectation uniformly bounded by μ. Letting R_t be the maximal x_d coordinate among the explored sites in this process, it follows from the strong law of large numbers that $R_t/t \to \infty$ almost surely. Since $M_t \geq R_t$, the claimed result follows. □

4 Critical First Passage Percolation on the Oriented Lattice

It is often convenient to represent $\vec{\mathbb{Z}}^2$ with a rotated and rescaled lattice $\mathcal{L} = \{(m, n) \in \mathbb{Z}^2 : m + n \text{ is even}\}$ with oriented edges from $(m, n) \to (m + 1, n + 1)$ and $(m, n) \to (m - 1, n + 1)$. We start by defining the critical exponent from the statement of Theorem 3.

As mentioned earlier, $p_c(\vec{\mathbb{Z}}^2) \in (0, 1)$. Let r_n be the rightmost point at height n that can be reached by a p-open path started from a vertex on the negative half-line $(-\infty, 0] \times \{0\}$. It is known that $\alpha(p) = \lim_{n\to\infty} r_n/n$ exists, is positive for $p > p_c$ [13], and that $\alpha(p_c) = 0$. Supercritical percolation clusters grow like a cone with slope $1/\alpha(p)$. Looking towards defining the correlation length, let $\sigma_p(L, \delta)$ be the set of vertices contained in the parallelogram with boundary vertices

$$\begin{aligned} u_0 &= (-1.5\delta L, 0) \qquad u_1 = ((1 + 1.5\delta)L, (1 + 3\delta)L/\alpha(p)) \\ v_0 &= (-0.5\delta L, 0) \qquad v_1 = ((1 + 2.5\delta)L, (1 + 3\delta L)/\alpha(p)) \end{aligned}$$

We say that $\sigma_p(L, \delta)$ contains a *centered crossing* if there is a p-open path contained in $\sigma_p(L, \delta)$ that starts and ends near the middle of the boundary. More precisely, the crossing must start at a vertex in $[\lceil -1.25\delta L, -.75\delta L\rceil] \times \{0\}$ and end at a vertex in $[\lceil(1+1.75\delta)L, (1+2.25\delta L)\rceil] \times \{\lceil(1 + 3\delta)L/\alpha(p)\rceil\}$. Given $0 < \epsilon < 3^{-36}$, let $L(p, \epsilon)$ be the minimum value such that the probability of a centered crossing is at least $1 - \epsilon$:

$$L(p, \epsilon) = \min\{L : P(\text{ there exists a centered crossing of } \sigma_p(L, 1/10)) > 1 - \epsilon\}.$$

We assume that $\epsilon < 3^{-36}$ to ensure that $L(p, \epsilon) \to \infty$ as $p \downarrow p_c$. The 3^{-36} comes from a contour argument in [13, Sect. 10] which implies, if $L(p, \epsilon)$ increased to a finite limit, then we would have a block construction that guaranteed survival at the critical value, and by continuity of finite block probabilities also at a $p < p_c$. Define the *critical exponent in the*

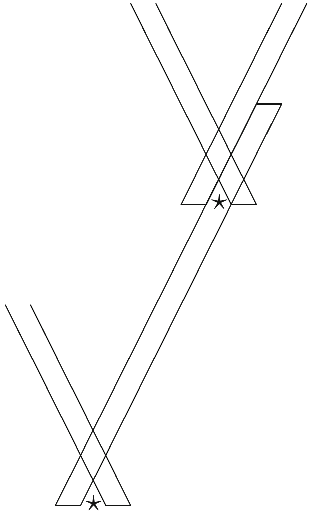

Fig. 1 Picture of the block construction. Stars mark points of the renormalized lattice. When a parallelogram has an open path we declare the corresponding edge open

time direction as

$$\nu_{||} := -\liminf_{p \downarrow p_c} \frac{\log L(p, \epsilon)}{\log(p - p_c)}. \tag{5}$$

Rearranging, this ensures that

$$L(p, \epsilon) = O((p - p_c)^{-\nu_{||}}). \tag{6}$$

Thus, $\nu_{||}$ is an upper bound on the exponent of the rate at which a parallelogram's length must diverge to have a centered crossing as p approaches p_c . We stress that the critical exponent obtained from (5) with the lim inf replaced with a limit is widely believed to exist, but its existence has not been rigorously established for oriented or unoriented percolation.

Proof of Theorem 3 To ensure that our parallelograms are not too long and that our block construction will percolate, we fix $\lambda < 1/\nu_{||}$ and $0 < \epsilon < 3^{-36}$. Set $p_n = p_c + 2^{-\lambda n}$ and let $L_n = L(p_n, \epsilon)$. The bound at (6) and our choice of λ ensure that

$$L_n = O((p - p_c)^{-\nu_{||}}) = O(2^{\lambda \nu_{||} n}) = o(2^n).$$

Choose n_1 large enough so that $L_n < 2^{n-1}$ for all $n \geq n_1$.

As in [8], we divide the space into strips $\mathbb{Z} \times [2^n, 2^{n+1})$. In each strip we lower bound the growth by considering p_{n+1}-bond percolation and make a block construction as described in [13, Sect. 9]. When we get to the top of the strip and the parallelograms have height above 2^{n+1}, we regard them as part of the next strip. As n is increased, the slope and length of the parallelograms increase.

By identifying each parallelogram to an edge in a renormalized lattice, we obtain a 1-dependent version of oriented bond percolation on a new copy of $\mathcal{L}$. Declare edges in this renormalized lattice open if the corresponding parallelogram contains a centered crossing (Fig. 1). It follows from our construction that each edge is open with probability at least $1 - \epsilon$. Our choice of ϵ ensures that this 1-dependent process on the renormalized lattice almost surely contains an infinite component of open edges because of the uniform bound $p_c < 1 - \epsilon$ for all 1-dependent percolation models proven in [13, Sect. 10].

Returning to the original lattice $\mathcal{L}$, we now show that the expected passage time along an open path in the infinite component is finite. The only edges that contribute to the passage time of a $(p_c + 2^{-\lambda n})$-open crossing from height 2^n to 2^{n+1} are those with $\omega_e \in (p_c, p_n]$. Since there are 2^n edges and the ω_e, conditional on being part of an open crossing, are uniform random variables on $(p_c, p_n]$, this has expected value that is

$$O((p_n - p_c)2^n) = O(2^{-\lambda n}2^n).$$

Worst case, the time to cross each of these edges is $F^{-1}(p_n) = O(2^{-(\lambda/a)n})$. This bound on $F^{-1}(p_n)$ comes from our hypothesis that $F(x) = p_c + O(x^a)$. Thus, conditional on there being a crossing, the expected passage time from height 2^n to height 2^{n+1} is

$$O(2^{-\lambda n}2^n 2^{-(\lambda/a)n}) = O(2^{[1-\lambda(1+1/a)]n}).$$

If $1 - \lambda(1 + 1/a) < 0$, then the exponent is negative, and thus the total expected passage time along a connected chain of parallelograms is summable. Choosing λ near $1/\nu_{||}$, the condition on a becomes the extra hypothesis in Theorem 3. Recall that the origin in the renormalized lattice corresponds to the height 2^{n_1}, and the infinite connected chain begins at some almost surely finite height above this. Since we assume that $F(+\infty) = 1$, the time to reach this infinite connected chain is almost surely finite. It follows that $\rho(\vec{\mathbb{Z}}^2, F) < \infty$ almost surely. □

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