11 MAGNETOSTATICS AND ELECTRODYNAMICS

11.1 INTRODUCTION

We know that static, electric and magnetic fields are independent of time. The fields are not coupled with each other *i.e.*, they can exist without each other. Now, we consider the electric and magnetic fields that change with time *i.e.*, time varying fields. These fields cannot exist without the other. These fields are called as ***electromagnetic fields.***

In time varying fields, a changing magnetic field gives rise to an electric field and vice versa. Therefore, there is a mutual dependence between electric and magnetic field vectors. In time varying fields, the electric field produced by changing magnetic field is a result of experimental researches of Michael Faraday. On the other hand, a magnetic field produced by changing electric field is a result of theoretical researches of James Clark Maxwell.

We will begin the study of time varying fields from ***Faraday's law of electromagnetic induction***. Then we will study the conduction current and displacement current. Based on these fundamentals, we will develop the ***Maxwell's equations***. We will also consider the Maxwell's equations in ***free space*** and ***conducting media***.

The varying electric and magnetic fields are mutually perpendicular to each other. The frequency of these fields is same as the frequency of oscillations of charged particles. The wave associated with these oscillations is called as electromagnetic wave. Therefore, "***an electromagnetic wave is a wave of oscillations of electric and magnetic fields in mutually perpendicular planes and these oscillations are perpendicular to the direction of propagation of wave***" After this, we will apply Maxwell's equation to introduce the fundamental theory of wave motion and ***basic principles of wave propagation***.

11.2 FARADAY'S LAW

When a current is passed through a conductor, magnetic field of constant magnitude is produced at nearby points of the conductor. But if an alternating current is passed through the conductor (*i.e.*, the current passed through the conductor is changing from time to time), the magnetic field produced at nearby points of the conductor will also change from time to time. The magnetic field so produced is called as time varying magnetic field. When a conducting circuit is placed in a time varying magnetic field, an induced emf is produced in it. Due to this induced emf, a current flows through the circuit.

Faraday observed the following points:

(i) ***Whenever the magnetic flux linked with a circuit is changed, an emf is induced in the circuit.***

(ii) ***The magnitude of induced emf is directly proportional to the negative rate of magnetic flux linked with the circuit.***

If Φ be the magnetic flux linked with the circuit at any instant and V_e be the induced emf, then

$$V_e = -\frac{d\Phi}{dt} \quad \text{...(1)}$$

If there are N turns in a coil, then induced emf becomes

$$V_e = -N\frac{d\Phi}{dt} \quad \text{...(2)}$$

The negative sign indicates that the direction of induced emf (or current) in a closed circuit is such that it opposes the original cause that produces it. Thus, is known as Lenz's law.

Consider a time varying magnetic field is produced and closed circuit C of any shape is placed in this field. The circuit encloses a surface S in the field as shown in Fig. (1). Let $\mathbf{B}$ be the magnetic flux density in the neighborhood of the circuit.

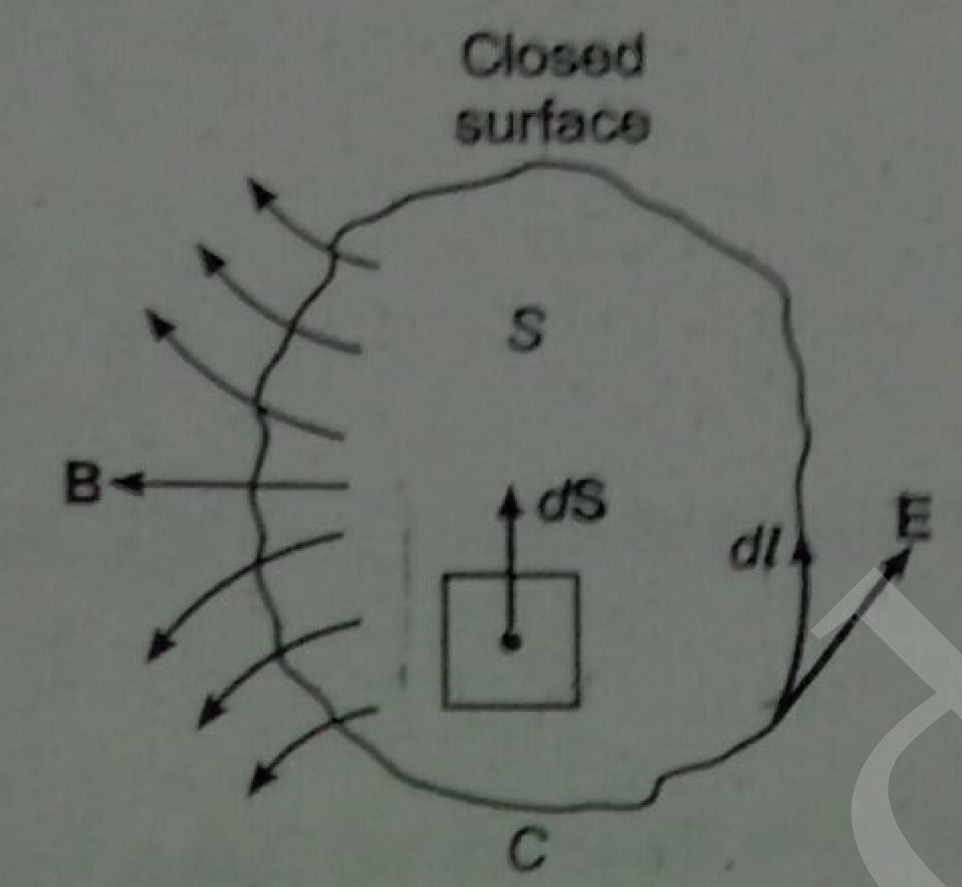

Fig. (1) Closed surface in time varying magnetic field

The magnetic flux through a small area dS will be $\mathbf{B} \cdot d\mathbf{S}$. Now, the flux, through the entire circuit is

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S} \quad \text{...(3)}$$

When magnetic flux is changed, an electric field is induced around the circuit. The line integral of the electric field gives the induced emf in the closed circuit. Thus,

$$V_e = \oint \mathbf{E} \cdot dl \quad \text{...(4)}$$

From eqs. (2) and (3), $$V_e = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S} \quad \text{...(5)}$$

From eqs. (4) and (5), we get

$$\boxed{V_e = \oint \mathbf{E} \cdot dl = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}} \quad \text{...(6)}$$

When the loop or closed circuit is stationary, then eq. (6) reduces to

$$V_e = -\int_S \frac{\partial \mathbf{B}}{\partial t} d\mathbf{S} \quad \text{...(7)}$$

The emf produced by changing field within a stationary circuit is called as transformer emf. Eq. (7) is known as *transformer induction equation*.

Faraday's law in ***integral form*** can be expressed as

$$V_e = \oint \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t}\, d\mathbf{S} \qquad \text{...(8)}$$

By Stoke's theorem $\oint \mathbf{E} \cdot dl = \int_S (\vec{\nabla} \times \mathbf{E}) \cdot d\mathbf{S}$ ...(9)

From eqs. (8) and (9), we get

$$\int_S (\vec{\nabla} \times \mathbf{E}) \cdot d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t}\, d\mathbf{S}$$

or

$$\vec{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{...(10)}$$

This is the ***differential form*** of Faraday's law. The eq. (10) gives a relation at a point.

11.3 LENZ'S LAW

According to Lenz's law, the direction of induced e.m.f. (or current) in a closed circuit is such that it opposes the original cause that produces it.

The law is based on the principle of conservation of energy. Thus, when the applied flux density B in a closed circuit is increasing, the e.m.f. or current induced in the closed circuit is in such a direction as to produce a field which tends to decrease B. On the other hand, when the applied flux density is decreasing in magnitude, the current in the closed circuit is in such a direction as to produce a field which tends to increase B. Thus, the induced current is in a direction such that it produces a magnetic flux tending to oppose the original change of flux *i.e.*, tending to keep the total flux constant in the circuit.

SOLVED EXAMPLES

EXAMPLE 1 *A sliding conductor of length l is situated in X-Y plane as shown in Fig. (2). The conductor is moving with velocity $v = v_0\, \hat{\mathbf{a}}_x$ where v_0 is constant. Find the induced emf.*

Solution The situation is shown in Fig. (2).

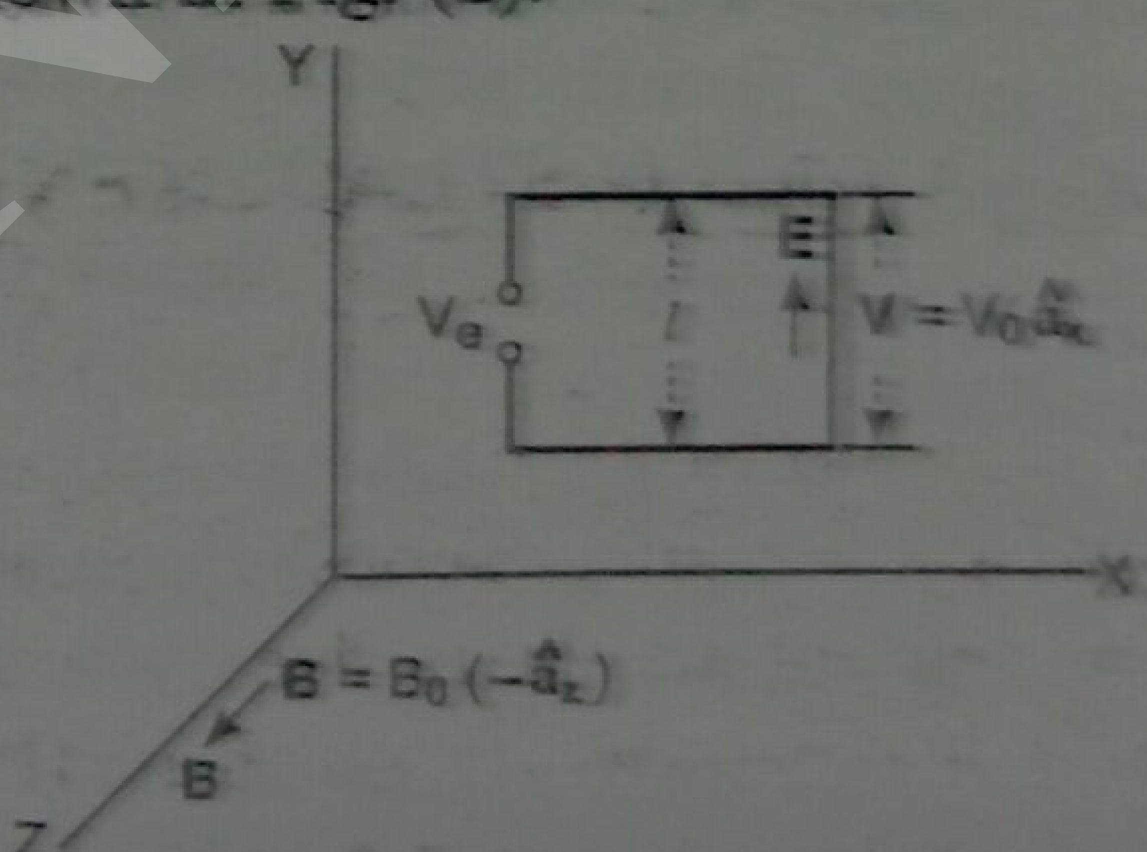

Fig. (2)

$$V_e = \int (\mathbf{v} \times \mathbf{B}) \cdot dl \qquad \text{...(1)}$$

Given $\quad \mathbf{v} = v_0\, \hat{\mathbf{a}}_x \quad \text{and} \quad \mathbf{B} = B_0 (-\hat{\mathbf{a}}_z)$

$\therefore$

$$\mathbf{v} \times \mathbf{B} = (v_0\, \hat{\mathbf{a}}_x) \times (-B_0\, \hat{\mathbf{a}}_z)$$
$$= v_0\, B_0\, \hat{\mathbf{a}}_y$$

and

$$dl = dy\,\hat{\mathbf{a}}_y$$

$$V_e = \int (v_0\, B_0\, \hat{\mathbf{a}}_y) \cdot dy\,\hat{\mathbf{a}}_y = v_0\, B_0 \int_0^l dy$$

$$= v_0\, B_0\, l$$

❑ **EXAMPLE 2** *A straight conductor of* 0.2 m *lies on the X-axis with one end at origin as shown in Fig.* (3). *The conductor is subjected to a magentic flux density* $\mathbf{B} = 0.04\,\hat{\mathbf{a}}_y$ tesla *and velocity* $\mathbf{v} = 2.5 \sin 10^3 t\, \hat{\mathbf{a}}_z$ m/s. *Calculate the motional electric field intensity and emf induced in the conductor.*

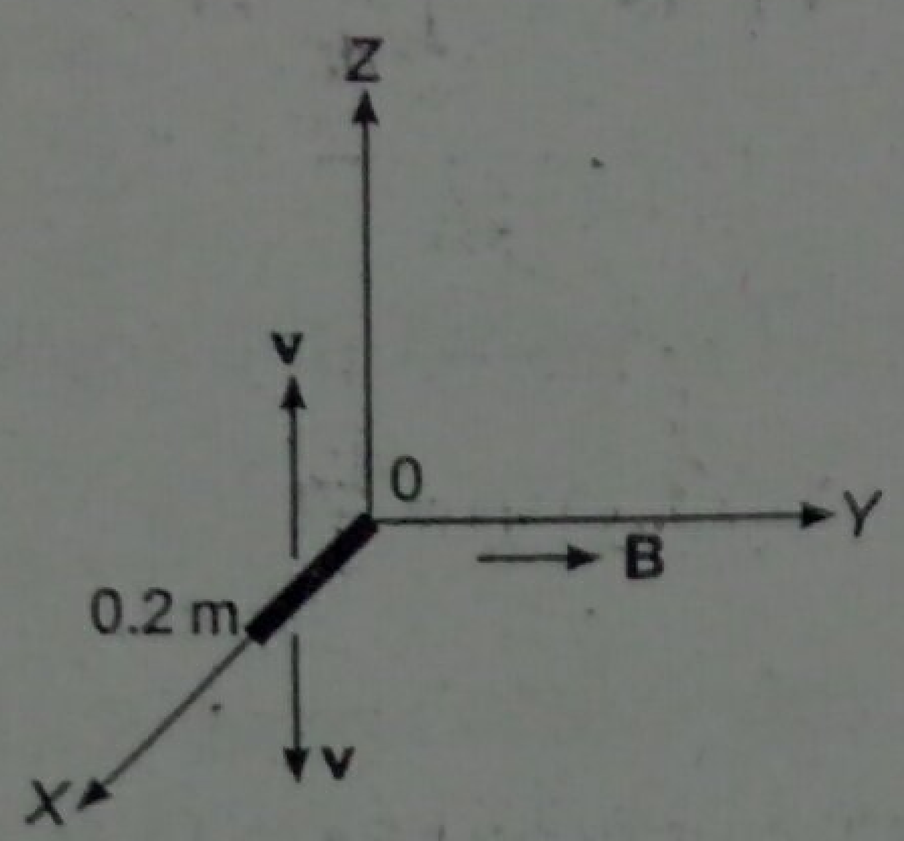

Fig. (3)

Solution The motional emf, *i.e.*, induced voltage producing field intensity

$$\mathbf{E} = \mathbf{v} \times \mathbf{B} = (2.5 \sin 10^3 t\, \hat{\mathbf{a}}_z) \times (0.04\, \hat{\mathbf{a}}_y)$$

$$= 0.10 \sin 10^3 t\, (-\hat{\mathbf{a}}_x) \text{ V/m}$$

Therefore, emf induced in the conductor

$$V_e = \int (\mathbf{v} \times \mathbf{B}) \cdot dl$$

$$= \int_0^{0.20} 0.10 \sin 10^3\, t\, (-\hat{\mathbf{a}}_x) \cdot dx\, \hat{\mathbf{a}}_x$$

$$= -0.10 \sin 10^3\, t\, [x]_0^{0.20} = -0.10 \sin 10^3\, t\, (0.20)$$

$$= -0.020 \sin 10^3 t \text{ volt}$$

The conductor first moves in $\hat{\mathbf{a}}_z$ direction and afterward it moves in $-\hat{\mathbf{a}}_z$ direction.

11.4 CONDUCTION CURRENT AND DISPLACEMENT CURRENT

Physical Interpretation

Consider a parallel plate capacitor is connected to an alternating generator through a resistance in series circuit as shown in Fig. (4). We know that in a series circuit, the current has the same value at all cross-sections. The conduction current which is due to electrical charges is same at all cross-sections of the circuit except at all cross-sections (such as *AB*) of the dielectric inside capacitor. The dielectric is air with permittivity ε_0. It is important to mention here that conduction electrons (free electrons) are flowing into one plate of the capacitor and forcing other free electrons out of the second plate. Therefore, *there is no flow of free electrons through the dielectric.* In other words, we can say that there is a discontinuity of the current in the space between the plates of a capacitor. Now, the question is that how this discontinuity of the current

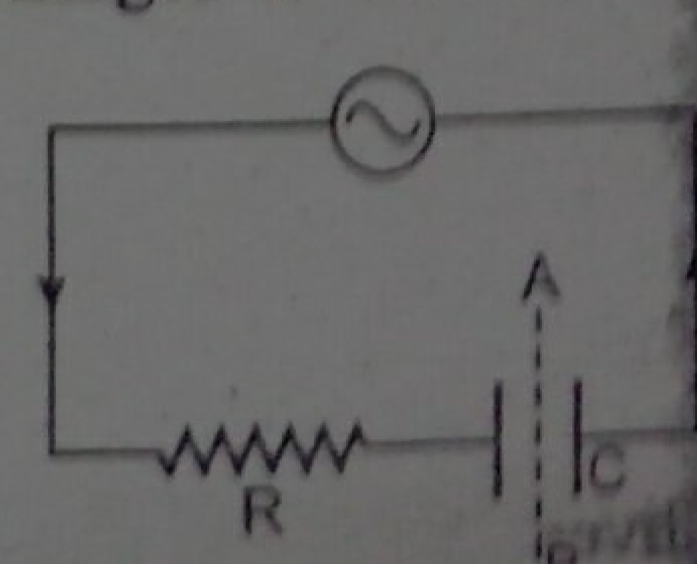

Fig. (4)

can be explained. In order to remove this discontinuity J.C. Maxwell revised and extended the definition of current by introducing the idea of *displacement current*.

Let at any particular instant, q be the charge on capacitor plate. The conduction current (i_c) is defined as the time rate of flow of charge, *i.e.*,

$$i_c = \frac{dq}{dt} \quad ...(1)$$

We have studied the electrical displacement D in dielectrics. The electrical displacement D is given by

$$D = \sigma = \frac{q}{A} \quad ...(2)$$

where σ is the surface charge density and A is the area of each plate.

From eq. (2),

$$q = D\,A \quad ...(3)$$

Substituting the value of q from eq. (3) in eq. (1), we get

$$i = \frac{d}{dt}(D\,A) = A\frac{dD}{dt} \quad ...(4)$$

Maxwell suggested that the term $A\left(\frac{dD}{dt}\right)$ should be considered as the current inside the dielectric. This current is called as *displacement current* and is denoted by i_d. Therefore,

$$\boxed{i_d = A\frac{dD}{dt}} \quad ...(5)$$

Consequently, the displacement current density $\mathbf{J}_d$ is given by

$$\boxed{\mathbf{J}_d = \frac{d\mathbf{D}}{dt}} \quad ...(6)$$

The vector **D** may vary with space, hence,

$$\boxed{\mathbf{J}_d = \frac{\partial \mathbf{D}}{\partial t}} \quad ...(7)$$

Thus, inside the dielectric, there will be a displacement current i_d which is equal to conduction current i in the line.

Important Points

1. The concepts of conduction current and displacement current are shown in Fig. (5).

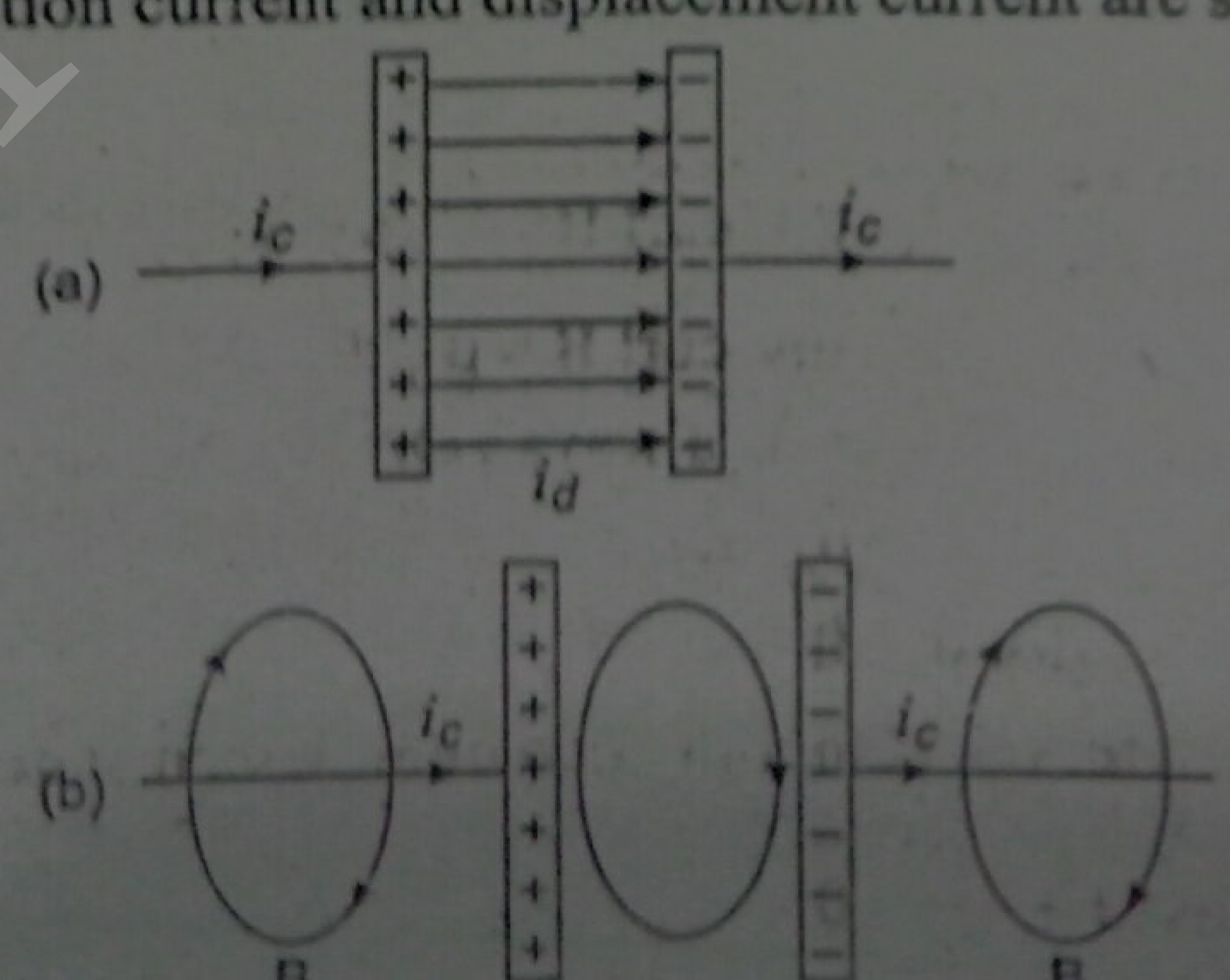

Fig. (5) Concepts of conduction and displacement current

2. The conduction current i_C flows from one plate to another plate of the capacitor through conducting wires. Maxwell suggested that during time varying electric field between the plates of the capacitor, an electric current, called the displacement current i_d, also flows across the space between the plates of the capacitor.
3. The displacement current density $\mathbf{J}_d$ does not represent a current which directly passes through the capacitor. It is only the apparent current representing the rate at which flow of charge takes place from one plate to another plate. Hence, the term displacement is justified.
4. The conduction current and displacement current are equal, *i.e.*,

$$i_C = i_d$$

5. Like conduction current, the displacement current is also a source of magnetic field.
6. In good conductors, the displacement current is negligible small as compared to conduction current at frequencies less than optical frequencies. At ultra high frequencies, the displacement current in conductors become quite important.
7. The displacement current exists only so long as electric field **E** is changing with time, *i.e.*, displacement **D** is changing with time. When the capacitor is fully charged upto the value of applied emf, the current in the line drops to zero. Now, the electric field **E** between the plates of the capacitor attains at steady value. As a result, $\left(\frac{d\mathbf{E}}{dt}\right)$ and $\left(\frac{d\mathbf{D}}{dt}\right)$ becomes zero.
8. The displacement current is called a current in the sense that it has the same dimensions as that of current.

11.5 MODIFIED AMPERE'S LAW AND DISPLACEMENT CURRENT

We have studied that a current in conductor produces a magnetic field. Maxwell proved that a changing electric field in vacuum or in a dielectric also produces a magnetic field. So, a changing electric field is equivalent to a current which flows as long as the electric field is changing. This produces the same magnetic effect as the ordinary conduction current. This is known as ***displacement current***. Now, we shall show that how the Ampere's law is modified by using the idea of displacement current.

Ampere's law in vector form can be expressed as

$$\vec{\nabla} \times \mathbf{B} = \mu_0 \, \mathbf{J} \qquad \text{...(1)}$$

where **J** is current density.

Taking divergence of this equation

$$\vec{\nabla} \cdot (\vec{\nabla} \times \mathbf{B}) = \text{div. curl } \mathbf{B} = \vec{\nabla} \, (\mu_0 \, \mathbf{J})$$

or

$$\text{div. curl } \mathbf{B} = \mu_0 \text{ div. } \mathbf{J} \qquad \text{...(2)}$$

We know that div. of curl of a vector is always zero and hence,

$$0 = \mu_0 \text{ div. } \mathbf{J}$$

$\therefore$

$$\text{div. } \mathbf{J} = 0 \qquad (\because \mu_0 \neq 0) \qquad \text{...(3)}$$

This is in contradiction with the continuity equation, which states that

$$\text{div. } \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \qquad \text{...(4)}$$

where ρ represents the charge density.

A close inspection of eqs. (3) and (4) shows that the Ampere's law in the form of div. $\mathbf{J} = 0$ is valid only when the charge density is static and not varying with time.

Maxwell concluded that eq. (1) is incomplete. He suggested that something must be added in eq. (1) such that the divergence of both side is same. Thus,

$$\vec{\nabla} \times \mathbf{B} = \mu_0 \, \mathbf{J} + \text{Something} \qquad \ldots(5)$$

In order to know this something, Maxwell postulated that similar to electric field due to hanging magnetic field (Faraday's law of induction), there must be a magnetic field due to hanging electric field. Thus, a changing electric field is equivalent to a current which flows as long s the electric field is changing. This produces the same magnetic effect as ordinary conduction urrent. Let us add an unknown term $\mathbf{J}_d$ for something in eq. (5). Therefore,

$$\vec{\nabla} \times \mathbf{B} = \mu_0 \, (\mathbf{J} + \mathbf{J}_d)$$

Taking div. of both sides,

$$\vec{\nabla} \cdot (\vec{\nabla} \times \mathbf{B}) = \mu_0 \, [\vec{\nabla} \cdot \mathbf{J} + \vec{\nabla} \cdot \mathbf{J}_d]$$

$$0 = \mu_0 \, [\vec{\nabla} \cdot \mathbf{J} + \vec{\nabla} \cdot \mathbf{J}_d]$$

$$\therefore \quad \vec{\nabla} \cdot \mathbf{J} + \vec{\nabla} \cdot \mathbf{J}_d = 0 \qquad (\because \mu_0 \neq 0)$$

or

$$\vec{\nabla} \cdot \mathbf{J}_d = -\vec{\nabla} \cdot \mathbf{J} \qquad \ldots(6)$$

Using eq. (4), we get

$$\vec{\nabla} \cdot \mathbf{J}_d = \frac{\partial \rho}{\partial t} \qquad \ldots(7)$$

According to Gauss's law in electrostatic

$$\vec{\nabla} \cdot \mathbf{D} = \rho \qquad \ldots(8)$$

Substituting the value of ρ from eq. (8) in eq. (7), we get

$$\vec{\nabla} \cdot \mathbf{J}_d = \frac{\partial}{\partial t} (\vec{\nabla} \cdot \mathbf{D}) = \vec{\nabla} \cdot \frac{\partial D}{\partial t}$$

$$\therefore \quad \boxed{\mathbf{J}_d = \frac{\partial \mathbf{D}}{\partial t}} \qquad \ldots(9)$$

where $\mathbf{J}_d$ is displacement current density.

Now, the modified form of Ampere's law is

$$\boxed{\vec{\nabla} \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right)} \qquad \ldots(10)$$

or

$$\boxed{\vec{\nabla} \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}} \qquad \ldots(11)$$

Displacement Current

The displacement current i_d is given by

$$i_d = \int_S \mathbf{J}_d \cdot d\mathbf{S} = \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{S}$$

$$= \frac{\partial}{\partial t} \int_S \mathbf{D} \cdot d\mathbf{S} = \varepsilon_0 \frac{\partial}{\partial t} \int_S \mathbf{E} \cdot d\mathbf{S}$$

$$= \varepsilon_0 \frac{d\Phi_E}{dt}$$

$$\therefore \quad \boxed{i_d = \varepsilon_0 \frac{d\Phi_E}{dt}} \quad \ldots(12)$$

where Φ_E is the electric flux.

Modified Ampere's Law in Integral Form

Integrating eq. (11) over the surface, we have

$$\int_S (\vec{\nabla} \times \mathbf{H}) \cdot d\mathbf{S} = \int_S \mathbf{J} \cdot d\mathbf{S} + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{S} \quad \ldots(13)$$

Applying Stoke's theorem, we get

$$\boxed{\oint \mathbf{H} \cdot d\mathbf{l} = i + i_d = \int_S \mathbf{J} \cdot d\mathbf{S} + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{S}} \quad \ldots(14)$$

In this way, Maxwell succeeded in bringing the complete symmetry between electric and magnetic fields by introducing the concept of displacement current.

SOLVED EXAMPLES

EXAMPLE 1 *A dielectric medium is kept in variable field. Show that the value of displacement current is equal to the conduction current.*

Solution The displacement current density $\mathbf{J}_d$ is given by

$$\mathbf{J}_d = \varepsilon_0 \frac{\partial E}{\partial t}$$

The displacement current i_d is given by

$$i_d = \int_S \mathbf{J}_d \cdot d\mathbf{S} = \varepsilon_0 \frac{\partial}{\partial t} \int_S \mathbf{E} \cdot d\mathbf{S}$$

According to Gauss's theorem

$$\int_S \mathbf{E} \cdot d\mathbf{S} = \frac{q}{\varepsilon_0}$$

$$\therefore \quad i_d = \varepsilon_0 \frac{\partial}{\partial t}\left(\frac{q}{\varepsilon_0}\right) = \frac{\partial q}{\partial t} = i$$

Thus, displacement current is equal to conduction current.

EXAMPLE 2 *A parallel plate capacitor with plate area of* $5\,\text{cm}^2$ *and plate separation of* 3 mm *has a voltage of* $50 \sin 10^3 t$ *applied to its plates. Calculate the displacement current assuming* $\varepsilon = 2\varepsilon_0$.

Solution We know that, $\mathbf{D} = \varepsilon \mathbf{E}$

Further, $\mathbf{E} = \frac{V}{d}$

$$D = \frac{\varepsilon V}{d}$$

Now $$j_d = \frac{\partial D}{\partial t} = \frac{\varepsilon}{d}\frac{dV}{dt}$$

$$i_d = j_d \times A = \frac{\varepsilon A}{d}\frac{dV}{dt} = C\frac{dV}{dt}$$

$$= \frac{2\varepsilon_0 A}{d}\frac{d}{dt}[50 \sin 10^3 t]$$

$$= \frac{2\varepsilon_0 A}{d}[(10)^3\, 50 \cos 10^3 t]$$

$$= \frac{2\times(8.83\times10^{-12})(5\times10^{-4})}{3\times10^{-3}}[5\times10^4 \cos 10^3 t]$$

$$= \mathbf{147 \cos 10^3 t} \text{ nA}$$

11.6 A REVIEW OF BASIC LAWS

The basic equations which we have studied are summarised in the following four equations:

1. $$\oint \mathbf{E}\cdot d\mathbf{S} = \left(\frac{q}{\varepsilon_0}\right) \quad \text{or} \quad \oint \mathbf{D}\cdot d\mathbf{S} = q$$

This is ***Gauss's law of electrostatic.*** This law states that the electric flux through a closed surface is equal to the net charge enclosed by the surface divided by the permittivity constant ε_0.

2. $$\oint \mathbf{B}\cdot d\mathbf{S} = 0$$

This is ***Gauss's law for magnetism.*** This law states that the magnetic flux through a closed surface is zero.

3. $$\oint \mathbf{E}\cdot d\mathbf{l} = -\frac{d\Phi_B}{dt}$$

This is ***Faraday's law*** of electromagnetic induction. This law states that an eletric field is produced by changing magnetic field.

4. $$\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 I \quad \text{or} \quad \oint \mathbf{H}\cdot d\mathbf{l} = \mu_0$$

This is ***Ampere's law*** for magnetic field due to steady current. This law states that the amount of workdone in carrying a unit magnetic pole one around a closed arbitrary path linked with the current is μ_0 times the current I.

The different symbols used are:

E = electric field intensity vector
D = electric flux density or electric displacement vector
B = magnetic flux density vector
H = magnetic field intensity vector

For a linear and isotropic media, E and D are related as

$$\mathbf{D} = \varepsilon\, \mathbf{E}$$

For linear isotropic media $\quad \mathbf{B} = \mu\, \mathbf{H}$

In non time varying fields (static case) E and D and B and H are independent pairs while in time varying fields vectors E and D are properly related to B and H.

11.7 MAXWELL'S EQUATIONS

We have studied that when the electric and magnetic fields are changing very rapidly in space with time, then the varying electric fields give magnetic field and vice-versa. Maxwell in 1862 formulated the basic laws of electromagnetic in the form of four fundamental equations. These equations are known as Maxwell's electromagnetic equations. These equations are based upon the well known laws such as Gauss's law of electrostatic, Gauss's law of magnetostatic, Faraday's law of electromagnetic induction and Ampere's circuital law.

The integral forms of these equations are given below:

$$\oint \mathbf{E} \cdot d\mathbf{S} = \left(\frac{q}{\varepsilon_0}\right) \quad \text{or} \quad \oint_S \mathbf{D} \cdot d\mathbf{S} = q \quad \text{...(1)}$$

$$\oint \mathbf{B} \cdot d\mathbf{S} = 0 \quad \text{...(2)}$$

$$\int_C \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi_B}{dt} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \quad \text{...(3)}$$

$$\left.\begin{aligned} \oint \mathbf{B} \cdot dl &= \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right) \\ \oint \mathbf{H} \cdot dl &= I + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{S} \end{aligned}\right] \quad \text{...(4)}$$

The differential forms of these equation are given below:

$$\text{div.}\,\mathbf{E} = \frac{\rho}{\varepsilon_0} \quad \text{or} \quad \vec{\nabla} \cdot \mathbf{D} = \rho \quad \text{...(1)}$$

$$\text{div.}\,\mathbf{B} = 0 \quad \text{or} \quad \vec{\nabla} \cdot \mathbf{B} = 0 \quad \text{...(2)}$$

$$\text{curl}\,\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{or} \quad \vec{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad \text{...(3)}$$

$$\text{curl}\,\mathbf{B} = \mu_0 \left(\mathbf{J} + \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right) \quad \text{or} \quad \vec{\nabla} \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad \text{...(4)}$$

11.8 DEVIATION OF MAXWELL'S EQUATIONS IN DIFFERENTIAL FORM

1. Maxwell's first equation

$$\vec{\nabla} \cdot \mathbf{D} = \rho$$

According to Gauss's law $\oint \mathbf{E} \cdot d\mathbf{S} = \frac{q}{\varepsilon_0}$

If ρ be the charge density and dV, the small volume considered, then

$$q = \int_V \rho \, dV$$

$$\therefore \quad \oint \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0} \int_V \rho \, dV$$

or

$$\varepsilon_0 \oint \mathbf{E} \cdot d\mathbf{S} = \int_V \rho \, dV$$

or

$$\oint \mathbf{D} \cdot d\mathbf{S} = \int_V \rho \, dV \quad (\because \varepsilon_0 \mathbf{E} = \mathbf{D}) \quad \text{...(1)}$$

According to divergence theorem,

$$\oint \mathbf{D}\cdot d\mathbf{S} = \int_V (\vec{\nabla}\cdot\mathbf{D})\, dV \qquad ...(2)$$

From eqs. (1) and (2), we get

$$\int_V (\vec{\nabla}\cdot\mathbf{D})\, dV = \int_V \rho\, dV$$

$$\boxed{\vec{\nabla}\cdot\mathbf{D} = \rho} \qquad ...(a)$$

From eq. (a), $\vec{\nabla}\cdot\mathbf{E} = \dfrac{\rho}{\varepsilon_0}$ or $\text{div.}\,\mathbf{E} = \dfrac{\rho}{\varepsilon_0}$ (3)

2. Maxwell's second equation

$$\vec{\nabla}\cdot\mathbf{B} = 0$$

According to Gauss's law for magnetism

$$\oint \mathbf{B}\cdot d\mathbf{S} = 0$$

Transforming the surface integral into volume integral by Gauss's divergence theorem, we have

$$\oint_S \mathbf{B}\cdot d\mathbf{S} = \int_V (\vec{\nabla}\cdot\mathbf{B})\, dV$$

$$\int_V (\vec{\nabla}\cdot\mathbf{B})\, dV = 0$$

As the volume is arbitrary, the integral must be zero. Hence,

$$\boxed{\vec{\nabla}\cdot\mathbf{B} = 0} \qquad ...(b)$$

or

$$\text{div.}\,\mathbf{B} = 0$$

3. Maxwell's third equation

$$\vec{\nabla}\times\mathbf{E} = -\left(\frac{\partial \mathbf{B}}{\partial t}\right)$$

We know that

$$\oint \mathbf{E}\cdot dl = -\frac{d\Phi_B}{dt} = -\frac{\partial}{\partial t}\int_S \mathbf{B}\cdot d\mathbf{S}$$

or

$$\oint \mathbf{E}\cdot dl = -\int_S \frac{\partial \mathbf{B}}{\partial t}\, d\mathbf{S} \qquad ...(4)$$

Applying Stoke's theorem

$$\oint \mathbf{E}\cdot dl = \int_S (\vec{\nabla}\times\mathbf{E})\cdot d\mathbf{S} \qquad ...(5)$$

From eqs. (4) and (5), we get

$$\int_S (\vec{\nabla}\times\mathbf{E})\cdot d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t}\, d\mathbf{S} \qquad ...(6)$$

Eq. (6) is true for all surfaces, therefore,

$$\boxed{\vec{\nabla}\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}} \qquad ...(c)$$

$$\text{curl}\,\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

This equation signifies that electric field is produced by a changing magnetic field.

4. Maxwell' fourth equation:

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

According to Ampere's circuital law, the line integral of magnetic field intensity over a closed path is equal to the total current enclosed by the path

$$\oint \mathbf{H} \cdot d\mathbf{l} = I = i_c + i_d \qquad \ldots(7)$$

where I is the total current.

The total current is the sum of conduction current i_c and displacement current i_d, i.e. $I = i_c + i_d$. In terms of conduction current density $\mathbf{J}_c$ and displacement current density $\mathbf{J}_d$, i_c and i_d can be expressed as

$$i_c = \int_S \mathbf{J}_c \cdot d\mathbf{S} \text{ and } i_d = \int_S \mathbf{J}_d \cdot d\mathbf{S} \qquad \ldots(8)$$

Substituting these values in eq. (7), we get

$$\oint \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J}_c \cdot d\mathbf{S} + \int_S \mathbf{J}_d \cdot d\mathbf{S}$$

or

$$\oint \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J}_c \cdot d\mathbf{S} + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{S} \qquad \ldots(9)$$

($\because$ Normally, conduction current $\mathbf{J}_c$ is represented by $\mathbf{J}$ and $\mathbf{J}_d = \partial \mathbf{D}/\partial t$ where $\mathbf{D}$ is electric flux density)

Eq. (9) can be expressed as

$$\oint \mathbf{H} \cdot d\mathbf{l} = \int_s \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{S} \qquad \ldots(10)$$

According to Stoke's law

$$\oint \mathbf{H} \cdot d\mathbf{l} = (\vec{\nabla} \times \mathbf{H}) \cdot d\mathbf{S} \qquad \ldots(11)$$

From eqs. (10) and (11), we get

$$\int_s (\vec{\nabla} \times \mathbf{H}) \cdot d\mathbf{S} = \int_s \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{S}$$

Taking derivative of both sides, we get

$$(\vec{\nabla} \times \mathbf{H}) = \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \qquad \ldots(12)$$

This equation gives a new concept of generation of magnetic field by displacement current associated with it.

11.9 MAXWELL'S EQUATIONS IN INTEGRAL FORM

The Maxwell's equations in integral from can be obtained with the help of Maxwell's equations in differential form. The integral forms can be obtained form differential forms simply by taking surface integrals and volume integrals.

1. Maxwell's first equation in integral form:

The first Maxwell's equation in differential form is

$$\vec{\nabla} \cdot \mathbf{D} = \rho \qquad \ldots(1)$$

Taking the volume integral of above equation over an arbitrary volume V, we get

$$\int_V \vec{\nabla} \cdot \mathbf{D} \, dV = \int_V \rho \, dV \qquad \ldots(2)$$

Changing the volume integral of L.H.S. of eq. (2) into surface integral by Gauss divergence theorem, we get

$$\int_V \vec{\nabla} \cdot \mathbf{D} \, dV = \int_S \mathbf{D} \cdot d\mathbf{S} \qquad \ldots(3)$$

where S is the surface which bounds volume V.

From eqs. (2) and (3), we get

$$\boxed{\int_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho \, dV} \qquad \ldots(a)$$

This is the integral form of Maxwell's first equation.

The R.H.S. of eq. (a) represents the net charge contained in volume V

$$\int_V \rho \, dV = q$$

Equation (a) signifies that ***the new outward flux of electric displacement vector* D *through the surface enclosing a volume V is equal to the net charge (q) contained within that volume.***

2. Maxwell's second equation in integral form:

The second Maxwell's equation in differential form is

$$\boxed{\vec{\nabla} \cdot \mathbf{B} = 0} \qquad \ldots(4)$$

Integrating this over an arbitrary volume V, we get

$$\int_V \vec{\nabla} \cdot \mathbf{B} \, dV = \int_V 0 \, dV = 0 \qquad \ldots(5)$$

Using Gauss divergence theorem, we get

$$\int_V \vec{\nabla} \cdot \mathbf{B} \, dV = \int_S \mathbf{B} \cdot d\mathbf{S} \qquad \ldots(6)$$

From eqs. (5) and (6), we have

$$\int_S \mathbf{B} \cdot d\mathbf{S} = 0 \qquad \ldots(b)$$

This is integral form of Maxwell's second equation.

This equation signifies that ***the net outward of magnetic induction* B *through any closed surface is equal to zero.***

3. Maxwell's third equation in integral form:

The third Maxwell's equation in differential form is

$$\int_S \vec{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \ldots(7)$$

Taking surface integral of both sides of eq. (7) over a surface S, we get

$$\int_S (\vec{\nabla} \times \mathbf{E}) \cdot d\mathbf{S} = -\int_S \left(\frac{\partial \mathbf{B}}{\partial t}\right) \cdot d\mathbf{S} \qquad \ldots(8)$$

Using Stoke's theorem, the surface integral of L.H.S. of eq. (8) can be converted into line integral along the boundary of C, *i.e.*,

$$\int_S (\vec{\nabla} \times \mathbf{E}) \cdot d\mathbf{S} = \int_C \mathbf{E} \cdot d\mathbf{l} \qquad \ldots(9)$$

From eqs. (8) and (9), we get

$$\boxed{\int_C \mathbf{E} \cdot d\mathbf{l} = -\int_s \left(\frac{\partial \mathbf{B}}{\partial t}\right) \cdot d\mathbf{S}} \quad \text{...(c)}$$

This is integral form of Maxwell's fourth equation.

Equation (c) signifies that the electromotive force (emf, $e = \int_C \mathbf{E} \cdot d\mathbf{l}$) ***around a closed path is equal to negative rate of change of magnetic flux linked with the path.***

4. Maxwell's fourth equation in integral form:

The fourth Maxwell's equation in differential form is

$$\vec{\nabla} \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \quad \text{...(10)}$$

Taking surface integral of both sides of eq. (1) over the surface S, we get

$$\int_s \vec{\nabla} \times \mathbf{H}\, d\mathbf{S} = \int_s \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) d\mathbf{S} \quad \text{...(11)}$$

Using Stoke's theorem

$$\int_s \vec{\nabla} \times \mathbf{H}\, d\mathbf{S} = \int_C \mathbf{H} \cdot dl \quad \text{...(12)}$$

From eqs. (11) and (12), we get

$$\boxed{\int_C \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) d\mathbf{S}} \quad \text{...(d)}$$

This is integral form of Maxwell's fourth equation.

This equation signifies that ***magnetomotive force*** (m.m.f. $\int_C \mathbf{H} \cdot d\mathbf{l}$) ***around a closed path is equal to the conduction current plus displacement current through any surface bounded by the path.***

11.10 ELECTROMAGNETIC POTENTIALS : VECTOR AND SCALAR POTENTIALS

Consider the Maxwell's equation $\vec{\nabla} \cdot \mathbf{B} = 0$. It follows that the magnetic field $\mathbf{B}$ is solenoidal field. When divergence of a vector is zero, it is called as solenoidal vector. When the vector has a certain value, then it is called as non-solenoidal. Therefore, magnetic field $\mathbf{B}$ can be expressed as the curl of a vector point function $\mathbf{A}$ *i.e.*, $\mathbf{B} = \vec{\nabla} \times \mathbf{A}$. Here $\mathbf{A}$ is called the vector potential of electromagnetic fields.

Substituting $\mathbf{B} = \vec{\nabla} \times \mathbf{A}$ in the Maxwell's equation

$$\vec{\nabla} \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

we get

$$\vec{\nabla} \times \mathbf{E} = -\frac{\partial}{\partial t}(\vec{\nabla} \times \mathbf{A}) = -\vec{\nabla} \times \frac{\partial \mathbf{A}}{\partial t} \quad \left(\because \frac{\partial}{\partial t} \text{ and } \vec{\nabla} \text{ commute each other}\right)$$

or

$$\vec{\nabla} \times \mathbf{E} + \vec{\nabla} \times \frac{\partial \mathbf{A}}{\partial t} = 0$$

or

$$\vec{\nabla} \times \left[\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right] = 0$$

This shows that the curl of vector $\left[\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t}\right]$ is zero.

Hence, the vector $\left[\mathbf{E}+\left(\frac{\partial \mathbf{A}}{\partial t}\right)\right]$ shows an irrotational field.

So, this can be expressed as the negative gradient of a scalar point function *i.e.* $\left[\mathbf{E}+\frac{\partial \mathbf{A}}{\partial t}\right]=-\text{grad}\,\phi$.

or
$$\mathbf{E}+\frac{\partial \mathbf{A}}{\partial t}=-\vec{\nabla}\phi \quad \text{or} \quad \mathbf{E}=-\vec{\nabla}\phi-\frac{\partial \mathbf{A}}{\partial t}$$

Here ϕ is called scalar potential of electromagnetic fields.

■ SOLVED EXAMPLES

❑ **EXAMPLE 1** *Use Maxwell's equation to establish the charge-current equation of continuity.*

Solution The Maxwell's fourth equation is

$$\text{curl}\,\mathbf{B}=\mu_0\,\mathbf{J}+\mu_0\,\varepsilon_0\,\frac{\partial \mathbf{E}}{\partial t}$$

Taking divergence of both sides, we get

$$\text{div.curl}\,\mathbf{B}=\text{div.}\mu_0\,\mathbf{J}+\mu_0\,\varepsilon_0\,\frac{\partial}{\partial t}(\vec{\nabla}\cdot\mathbf{E})$$

Now,
$$\text{div.curl}\,\mathbf{B}=\vec{\nabla}\cdot(\vec{\nabla}\cdot\mathbf{B})=0$$

∴
$$0=\text{div.}\mu_0\,\mathbf{J}+\mu_0\,\varepsilon_0\,\frac{\partial}{\partial t}(\vec{\nabla}\cdot\mathbf{E})$$

From Maxwell's first equation

$$\vec{\nabla}\cdot\mathbf{E}=\frac{\rho}{\varepsilon_0}$$

∴
$$0=\mu_0\,\vec{\nabla}\cdot\mathbf{J}+\mu_0\,\varepsilon_0\,\frac{\partial}{\partial t}\left(\frac{\rho}{\varepsilon_0}\right)$$

or
$$\vec{\nabla}\cdot\mathbf{J}=-\frac{\partial\rho}{\partial t}$$

❑ **EXAMPLE 2** *Using the Maxwell's relation*

$$curl\,\mathbf{B}=\mu_0\left(\mathbf{J}+\frac{\partial \mathbf{D}}{\partial t}\right)$$

prove that
$$div.\,\mathbf{D}=\rho$$

Solution Taking the divergence of the given Maxwell's equation, we have

$$\frac{1}{\mu_0}[\text{div.curl}\,\mathbf{B}]=\text{div.}\,\mathbf{J}+\frac{\partial}{\partial t}(\text{div.}\,\mathbf{D})$$

But
$$\text{div.curl}\,\mathbf{B}=\vec{\nabla}\cdot\vec{\nabla}\times\mathbf{B}=0$$

∴
$$\text{div.}\,\mathbf{J}=-\frac{\partial}{\partial t}(\text{div.}\,\mathbf{D})$$

or $$-\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}(\text{div}.\,\mathbf{D})$$ (Using continuity equation)

$$\text{div}\cdot\mathbf{D} = \rho$$

❑ **EXAMPLE 3** *Use Maxwell's equation*

$$curl\,\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

prove that

$$\text{div}\cdot\mathbf{B} = 0$$

Solution Taking the divergence of the given equation, we get

$$\text{div}\cdot\text{curl}\,\mathbf{E} = -\frac{\partial}{\partial t}(\text{div}\cdot\mathbf{B})$$

or $$0 = -\frac{\partial}{\partial t}(\text{div}\cdot\mathbf{B})$$

or $$\text{div}\cdot\mathbf{B} = 0$$

11.11 WAVE EQUATIONS IN FREE-SPACE

Here, we shall derive the wave equations using Maxwell's equations for homogeneous medium (Free-space). In homogeneous medium, we have no charges and no conduction current, *i.e.*, $\rho_v = 0, \sigma = 0$ and $\mathbf{J} = 0$.

The Maxwell's equations in free-space are:

$$\vec{\nabla}\times\mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} = \varepsilon_0\frac{\partial \mathbf{E}}{\partial t} \quad ...(1)$$

$$\vec{\nabla}\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = -\mu_0\frac{\partial \mathbf{H}}{\partial t} \quad ...(2)$$

$$\vec{\nabla}\cdot\mathbf{D} = 0, \qquad \vec{\nabla}\cdot\mathbf{E} = 0 \quad ...(3)$$

$$\vec{\nabla}\cdot\mathbf{B} = 0, \qquad \vec{\nabla}\cdot\mathbf{H} = 0 \quad ...(4)$$

(1). Taking the curl of both sides of eq. (2), we get

$$\vec{\nabla}\times\vec{\nabla}\times\mathbf{E} = -\mu_0\,\vec{\nabla}\times\frac{\partial \mathbf{H}}{\partial t}$$

or $$\vec{\nabla}\times\vec{\nabla}\times\mathbf{E} = -\mu_0\frac{\partial}{\partial t}(\vec{\nabla}\times\mathbf{H}) \quad ...(5)$$

We know the vector identity

$$\vec{\nabla}\times\vec{\nabla}\times\mathbf{E} = \vec{\nabla}\times(\vec{\nabla}\cdot\mathbf{E}) - \nabla^2\mathbf{E}$$

$$= \vec{\nabla}\times(0) - \nabla^2\mathbf{E} \quad \text{[Using eq. (3)]}$$

or $$\vec{\nabla}\times\vec{\nabla}\times\mathbf{E} = -\nabla^2\mathbf{E} \quad ...(6)$$

From eqs. (5) and (6), we get

$$-\nabla^2\mathbf{E} = -\mu_0\frac{\partial}{\partial t}(\vec{\nabla}\times\mathbf{H})$$

$$= -\mu_0 \frac{\partial}{\partial t}\left(\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right)$$ [(Using eq. (1)]

$$= -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}$$

$\therefore$

$$\boxed{\nabla^2 \mathbf{E} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}} \quad ...(a)$$

This expression is the wave equation for propagation of electric field **E** in free space.

In cartesian coordinates, eq. (a) can be expressed as

$$\left.\begin{aligned}
\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} &= \mu_0 \varepsilon_0 \frac{\partial^2 E_x}{\partial t^2} \\
\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial z^2} &= \mu_0 \varepsilon_0 \frac{\partial^2 E_y}{\partial t^2} \\
\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2} &= \mu_0 \varepsilon_0 \frac{\partial^2 E_z}{\partial t^2}
\end{aligned}\right\} \quad ...(a)$$

(2) Similarly, taking curl of both sides of eq. (1), we get

$$\vec{\nabla} \times \vec{\nabla} \times \mathbf{H} = \varepsilon_0 \frac{\partial}{\partial t}(\vec{\nabla} \times \mathbf{E})$$

or

$$-\nabla^2 \mathbf{H} = \varepsilon_0 \frac{\partial}{\partial t}\left(-\mu_0 \frac{\partial \mathbf{H}}{\partial t}\right)$$ [Using eq. (2)]

or

$$-\nabla^2 \mathbf{H} = -\mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{H}}{\partial t^2}$$

$\therefore$

$$\boxed{\nabla^2 \mathbf{H} = \mu_0 \varepsilon_0 \frac{\partial^2 \mathbf{H}}{\partial t^2}} \quad ...(b)$$

This expression is the wave equation for the propagation of **H** in free-space. Replacing μ_0 by μ and ε_0 by ε in eqs. (a) and (b), we get wave equations in perfect dielectric (or non-conducting) medium.

It is obvious from eqs. (a) and (b) that electromagnetic wave propagation involves electric and magnetic fields with more than one component. Each component depends on the three coordinates, in addition to time. However, if the electromagnetic waves consist of electric and magnetic fields which are perpendicular to each other and to the direction of propagation and are uniform in planes perpendicular to the direction of propagation, the waves are called as ***uniform plane waves***.

11.12 WAVE EQUATION FOR CONDUCTING MEDIA

In previous section, we have solved Maxwell's equations for perfect dielectric such as free space in which there are no charges, *i.e.*, no conduction currents. The aim of this article is to consider the wave equation for a conducting medium, *i.e.*, conduction current exists. Let us consider a uniform linear medium having dielectric constant ε, permeability μ and conductivity σ. Now,

$$\mathbf{D} = \varepsilon\,\mathbf{E} \quad \text{and} \quad \mathbf{J} = \sigma\,\mathbf{E}$$

Consider the following Maxwell equations

$$\vec{\nabla} \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \qquad \ldots(1)$$

and

$$\vec{\nabla} \times \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J} = \varepsilon \frac{\partial \mathbf{E}}{\partial t} + \sigma\,\mathbf{E} \qquad \ldots(2)$$

Taking curl of both sides of eq. (1), we get

$$\vec{\nabla} \times \vec{\nabla} \times \mathbf{E} = -\mu \vec{\nabla} \times \frac{\partial \mathbf{H}}{\partial t}$$

or

$$\vec{\nabla} \times \vec{\nabla} \times \mathbf{E} = -\mu \frac{\partial}{\partial t} [\vec{\nabla} \times \mathbf{H}] \qquad \ldots(3)$$

Substituting the value of $(\vec{\nabla} \times \mathbf{H})$ in eq. (3) from eq. (2), we get

$$\vec{\nabla} \times \vec{\nabla} \times \mathbf{E} = -\mu \frac{\partial}{\partial t} \left[\varepsilon \frac{\partial \mathbf{E}}{\partial t} + \sigma\,\mathbf{E} \right]$$

or

$$\vec{\nabla} (\vec{\nabla} \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\mu\,\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} - \mu\,\sigma \frac{\partial \mathbf{E}}{\partial t} \qquad \ldots(4)$$

As

$$\mathbf{D} = \varepsilon\,\mathbf{E}, \quad \mathbf{E} = \frac{1}{\varepsilon} \mathbf{D}$$

or

$$\vec{\nabla} \cdot \mathbf{E} = \frac{1}{\varepsilon} \vec{\nabla} \cdot \mathbf{D} = \frac{1}{\varepsilon} (\rho) \qquad (\because \vec{\nabla} \cdot \mathbf{D} = \rho)$$

We know that there is no net charge within a conductor because the charge resides on the surface of the conductor, $\rho = 0$

$$\therefore \quad \vec{\nabla} \cdot \mathbf{E} = \frac{1}{\varepsilon} (0) = 0 \qquad \ldots(5)$$

Substituting the value of $(\vec{\nabla} \cdot \mathbf{E})$ from eq. (5) in eq. (4), we get

$$\vec{\nabla}(0) - \nabla^2 \mathbf{E} = -\mu\,\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} - \mu\,\sigma \frac{\partial \mathbf{E}}{\partial t}$$

or

$$\nabla^2 \mathbf{E} = \mu\,\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu\,\sigma \frac{\partial \mathbf{E}}{\partial t}$$

$$\therefore \quad \boxed{\nabla^2 \mathbf{E} - \mu\,\varepsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} - \mu\,\sigma \frac{\partial \mathbf{E}}{\partial t} = 0} \qquad \ldots(a)$$

This is the wave equation for **E**.

Similarly, taking curl of both sides of eq. (2), we get

$$\vec{\nabla} \times \vec{\nabla} \times \mathbf{H} = \varepsilon \frac{\partial}{\partial t} (\vec{\nabla} \times \mathbf{E}) + \sigma (\vec{\nabla} \times \mathbf{E}) \qquad \ldots(6)$$

$$= \varepsilon \frac{\partial}{\partial t} \left(-\mu \frac{\partial \mathbf{H}}{\partial t} \right) + \sigma \left(-\mu \frac{\partial \mathbf{H}}{\partial t} \right) \qquad \text{[Using eq. (1)]}$$

or
$$\vec{\nabla}(\vec{\nabla}\cdot\mathbf{H})-\nabla^2\mathbf{H}=-\mu\varepsilon\frac{\partial^2\mathbf{H}}{\partial t^2}-\sigma\mu\frac{\partial\mathbf{H}}{\partial t}$$

As $\nabla\cdot\mathbf{B}=0$, therefore, $\vec{\nabla}\cdot\mathbf{H}=0$

$$-\nabla^2\mathbf{H}=-\mu\varepsilon\frac{\partial^2\mathbf{H}}{\partial t^2}-\sigma\mu\frac{\partial\mathbf{H}}{\partial t}$$

$\therefore$
$$\boxed{\nabla^2\mathbf{H}-\mu\varepsilon\frac{\partial^2\mathbf{H}}{\partial t^2}-\sigma\mu\frac{\partial\mathbf{H}}{\partial t}=0} \quad \text{...(b)}$$

This is wave equation for **H**.

SOLVED EXAMPLES

EXAMPLE 1 *Using eqs. (a) and (b) obtain wave equations of* **E** *and* **H** *for free space.*

Solution For free space, $\rho=0$, $\sigma=0$, $\varepsilon=\varepsilon_0$ and $\mu=\mu_0$.

Substituting these values in equations (a) and (b), we get wave equations of **E** and **H** respectively

$$\nabla^2\mathbf{E}=\mu_0\varepsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2}$$

and
$$\nabla^2\mathbf{H}=\mu_0\varepsilon_0\frac{\partial^2\mathbf{H}}{\partial t^2}$$

EXAMPLE 2 *Using eqs. (a) and (b) obtain wave equations of* **E** *and* **H** *for isotropic dielectric or non-conducting medium.*

Solution For a non-conducting medium

$$\rho=0 \quad\text{and}\quad \sigma=0$$

Therefore, for non-conducting medium, the wave equations are

$$\nabla^2\mathbf{E}=\mu\varepsilon\frac{\partial^2\mathbf{E}}{\partial t^2}$$

and
$$\nabla^2\mathbf{H}=\mu\varepsilon\frac{\partial^2\mathbf{H}}{\partial t^2}$$

11.13 PROPAGATION OF EM WAVE

(A) Solution of Maxwell's equation for EM wave

Consider that an electromagnetic wave is propagating in Z-direction. In this case, there will be no E_z component, *i.e.*, $E_z=0$. There will be only E_y and E_x components. Further, there will be no variations of **E** and **H** in X and Y directions. Therefore, for a uniform plane wave travelling in Z-direction, we have

$$E_z=0,\quad \frac{\partial E_x}{\partial x}=0 \quad\text{and}\quad \frac{\partial E_y}{\partial y}=0$$