125 FINAL REVIEW OUTLINE ©

****Note this is just me thinking about what might be on the test. I could be missing stuff and I have no actual idea what will be on your exam. This just gives you an idea of the general type of question, but you should definitely dig into as many old final questions as possible to prepare.*

1. Pro Tips

Take ≥ 3 Practice Finals: And I mean actually sit there with it for 1.5-2.5 hours without looking at solutions. This will help you identify your weaknesses. To put it in perspective – each point on the final is worth at least 100 webassign points. Think how much time you spent on homework. Now is not the time to drop off!

Write Neatly: Having graded a lot of exams, I can promise that having neat solutions will positively effect your score. For instance, it makes getting partial credit much more likely.

Find the FTC problem: So many people miss this question on the final. My advice is that the first thing to do when taking the test is locate this question and mark it. Look for either

- A variable as a bound in the integral.
- Derivative language like *maximum, minimum, rate of change*.

A nice formula is if $F(x) = \int_{g(x)}^{f(x)} h(t)dt$ then

$$F'(x) = h(f(x))f'(x) - f(g(x))g'(x).$$

A caveat though. Sometimes you have to be careful about the bounds and break it up to where $f(x) \leq g(x)$ and $g(x) \leq f(x)$. For example if $F(x) = \int_x^{x^2} e^{t^2} dt$ then it is best to write $F(x) = \int_x^0 e^{t^2} dt + \int_0^{x^2} e^{t^2} dt$ and handle each part separately (nothing special about 0, any number works).

Try a u-sub first: A lot of people try to use integration by parts when it should be a u-sub. In particular on integrals with a square root (or rational exponent). It is very common to have an integral like

$$\int \frac{1}{t(1+\sqrt{t})^2} dt$$

where you have to take $u = \sqrt{t}$ then use the fact that $u^2 = t$ and $du = \frac{1}{2}udt$.

Keep in mind that square roots also often mean *Trig Sub* is necessary. Look for there to be a quadratic term as well though.

Every problem has an answer: You should always be trying something. If you get stuck, ask yourself if either you made a mistake or if there is another technique you could try. Don't just stare at an integral!

Write the bounds: On work, area between curves, centroid, arc length and volume problems if you just write and integrand with the correct bounds you will often get at least 1 or 2 points. And in work problems if you write the correct expression for the distance the thing moves you often get a point. For example when lifting a 30 foot rope that weighs 3 lb/ft up a 30 foot well. You could get partial credit for writing the *wrong* answer:

$$\int_0^{30} (30 - y)$$

You would get full credit for writing $\int_0^{30} 3(30-y)dy$.

Outline

(1) Riemann Sums
   (a) Left

   (b) Right
   (c) Midpoint
   (d) Abstract Sums
(2) Antiderivatives
   (a) General Antiderivatives
   (b) Keeping track of units
   (c) Initial Conditions
(3) Fundamental Theorem of Calculus (FTC)
   (a) Differentiating under the integral sign
   (b) Taking definite integrals
   (c) Total vs. Net vs. Absolute Change
(4) Techniques of Integration
   (a) u-substitution
(5) Applications
   (a) Integrating with respect to dx or with respect to dy
   (b) Area enclosed by curves
   (c) Volumes of Solids
      (i) Washer
      (ii) Shells
(6) Work
   (a) Pumping water
   (b) Lifting Ropes
   (c) Spring Stuff
(7) Average Value
(8) Integration by Parts
(9) Trigonometric Identity Integrals
(10) Trig Sub
(11) Making Stuff Look Like an Arctrig Function
(12) Partial Fractions
(13) Approximation Techniques
   (a) Trapezoid Rule
   (b) Simpsons' Rule
(14) Improper Integrals
(15) Arc Length
(16) Centers of Mass
(17) Differential Equations
   (a) Separable Equations
   (b) Exponential Growth

Example Problems

(1) Riemann Sums
   (a) Left
   (b) Right
   (c) Midpoint
   (d) Abstract Sums
(2) Fundamental Theorem of Calculus (FTC)
   (a) Differentiating under the integral sign

Find the derivative of $G(x) = \int_x^{x^2} e(t^2)dt$. $G'(x) = e^{x^4}(2x) - e^{x^2}$

Tough one (to test mastery)... Find the arc length of the following curve on $0 \leq x \leq 4$

$$F(x) = \int_{e^{\sqrt{x}}}^{e^{\sqrt{2x}}} \ln^2 t\, dt$$

$$= \left(\frac{4}{9}\right)\left(\frac{2}{3}\right)\left(1 - 10^{\frac{2}{3}}\right)$$

   (b) Total vs. Net vs. Absolute Change

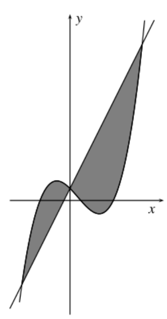
A ghost is moving along the x-axis with acceleration at time $t \geq 0$ given by $a(t) = -\frac{2000}{(t+2)^2}\ ft/sec^2$. At time $t = 0$ its velocity is 240 ft/sec. Find the total distance traveled by the ghost from $0 \leq t \leq 1$. $= \int_0^t |\frac{2000}{t+2} - 760| dt \approx 86.81$

(3) Techniques of Integration
   (a) u-substitution

$$\int_0^4 \frac{\arctan \sqrt{t}}{\sqrt{t}}\, dt = 4 \arctan 2 - \ln \frac{5}{2} - \frac{\pi}{2}$$

(4) Applications
   (a) Integrating with respect to dx or with respect to dy
   (b) Area enclosed by curves

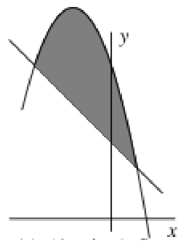
Find the area of the region between the curves $y = x^3 - x^2 - 2x + 1$ and $y = 4x + 1$.



$= 253/12$

   (c) Volumes of Solids

Let R be the region bounded by $y = -x^2 - 3x + 6$ and $x + y - 3 = 0$. Find R.



(i) Shells

SET UP BUT DO NOT EVALUATE THE INTEGRAL expressing the volume obtained by rotating R about the vertical line $x = 3$.

$$= 2\pi \int_{-3}^{1} \left((-x^2 - 3x + 6) - (3 - x)\right)(3 - x)\, dx$$

(ii) Washer

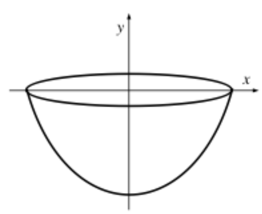
SET UP BUT DO NOT EVALUATE THE INTEGRAL expressing the volume obtained by rotating R about the horizontal line $y = 0$.

$$= \pi \int_{-3}^{1} (-x^2 - 3x + 6)^2 - (3 - x)^2\, dx$$

(5) Work

(a) Pumping water

A tank full of moon goo is on the surface of the moon, where the gravitational acceleration is 3.7 m/sec^2. The tank is in the shape of the curve $y = \frac{1}{3}x^2 - 3$ (the part below the x-axis) is rotated around the y-axis ; the units are meters. Find the work required to pump the moon goo out of an outlet at the top of the tank, which is at the level of the x-axis. (Note: Density of moon goo = 1000 kg.m^3)



$$3700\pi \cdot (13.5) \approx 157,000\ J$$

(b) Lifting Ropes

An 8 foot chain weighs 120 pounds. Matt is holding on end of the chain above the ground so that 3 feet of the chain is being supported and the other 5 feet is on the ground. How much work does Matt do to lift his end of the chain from 3 feet above the ground to a height of 10 feet above the ground?

$$652.5 ft - lb$$

(c) Spring Stuff

If 7 J of work are needed to stretch a spring from 10 cm to 13 cm and 13 J are needed to stretch it from 13 cm to 16 cm, what is the natural length L of the spring?

$$L = 8\ cm$$

(6) Average Value

Find $c > 0$ such that the average value of $f(x) = cx$ on $c \leq x \leq 2c$ is equal to c. $c = 2/3$

(7) Integration by Parts

$$\int \sqrt{x} e^{\sqrt{x}}\, dx = 2xe^{\sqrt{x}} - 4\sqrt{x}e^{\sqrt{x}} + 4e^{\sqrt{x}} + C$$

(the next one also requires a trig sub)

$$\int x \arctan(x) dx = \frac{1}{2}x^2 \arctan x - \frac{1}{2}x + \arctan x + C.$$

(8) Trigonometric Identity Integrals

$$\int_0^\pi \sin^4\theta\, d\theta = \frac{3}{8}\pi$$

$$\int (3\sin\theta - 1)^2 d\theta = \frac{11}{2}\theta + \frac{9}{2}\sin(\theta)\cos(\theta) + 6\cos\theta$$

(9) Trig Sub

$$\int \frac{1}{t^2 + 2t + 2)^{\frac{3}{2}}}\, dx = \frac{t+1}{\sqrt{t^2 + 2t + 2}} + C$$

$$\int_2^{\frac{7}{2}} \frac{x+1}{\sqrt{5 + 4x - x^2}\, dx} = -\frac{3\sqrt{3}}{2} + \frac{\pi}{2} + 3$$

(10) Making Stuff Look Like an Arctrig Function

$$\int \frac{e^x}{\sqrt{1 - e^{2x}}\, dx} = \frac{\arcsin(e^x)}{\sqrt{3}} + C$$

(11) Partial Fractions

$$\int \frac{1}{x^2 + x^3}\, dx = -\ln x - \frac{1}{x} + \ln|x+1| + C$$

(12) Approximation Techniques Just make your own problem up and do it

(a) Trapezoid Rule
(b) Simpsons' Rule

(13) Improper Integrals

$$\int_e^\infty \frac{1}{t \ln t}\, dt = \infty$$

$$\int_e^\infty \frac{1}{t \ln^2 t}\, dt = 1$$

$$\int_0^4 \frac{1}{(x-3)^3}\, dx = diverges$$

(14) Arc Length

Set up but do not evaluate the integral expressing the arc length of the curve $y = \sin(2x)$ for $0 \leq x \leq \frac{2\pi}{3}$. Plus, use the Trapezoid Rule with $n = 4$ subintervals to approximate the integral.

$$\int_0^{\frac{2\pi}{3}} \sqrt{1 + 4\cos^2(2x)}\, dx$$

$$\frac{\pi}{12}\left(5\sqrt{2} + 3\sqrt{5}\right)$$

(15) Centers of Mass

Let R be the region in the first quadrant enclosed by the graph of the function $y = 2\sin x$, the x-axis and the vertical line $x = \pi/2$. Find the centroid of $\mathbb{R}$. Also, find the volume of the solid obtained by rotating R around the line $y = -1$.

$$(\bar{x}, \bar{y}) = \left(1, \frac{\pi}{4}\right), \qquad V = 12\pi^2$$

(16) Differential Equations

(a) Separable Equations

A tank contains 100 liters of fresh water. Water containing x grams of salt per liter enters the tank at the rate of 5 liters/minute, and the well-mixed solution leaves at the same rate. Let $m(t)$ denote the amount of salt in grams in the tank at time t. If the concentration of salt in the tank is 3 grams/liter after 10 minutes, find x.

$$x = \frac{3}{1 - e^{-\frac{1}{2}}} \approx 7.62$$

Find the function $y(x)$ which satisfies $\frac{dy}{dx} = \frac{x(y^2+1)}{\sqrt{x^2-1}}$ such that $y = 1$ when $x = \sqrt{2}$.

$$\tan\left(\sqrt{x^2 - 1} + \frac{\pi}{4} - 1\right)$$

(b) Newton's Law of Cooling

When a cake is removed from an oven, the temperature of the cake is 210 F. The cake is left to cool at room temperature (70 F) and after 30 minutes the temperature of the cake is 140 F. According to Newton's law of cooling, the rate of change of temperature of a body is proportional to the temperature difference between the body and the environment. Set up and solve a differential equation to find when the temperature of the cake will be 100 F. $y(t) = 210e^{\frac{1}{30}\ln(\frac{1}{2})t} + 70$

(c) Exponential Growth

The population P of the moon is increasing steadily for two reasons (1) the birth rate exceeds the death rate resulting in an increase of 2% a year and (2) More people move to the moon than move away, resulting in an additional increase of 1000 people per year.

(i) Write down a formula for dP/dt. $\frac{dP}{dt} = (.02)P + 1000$

(ii) The moon was first colonized in 1995 with 25,000 moon people (i.e. $P(1995) = 25000$)). Solve the equation from part (i) $P(t) = 75000e^{.02t} - 50000$

(iii) In what year will the population of the moon be 60,000? 2014