San Francisco State University
ECON 715

Michael Bar
Fall 2019

Midterm Exam 1

September 24, 2019

Time: 1 hour, 30 minutes

Name: ______________________________

Instructions:

1. One double-sided sheet with any content is allowed.
2. Calculators are NOT allowed.
3. Show all the calculations, and explain your steps.
4. If you need more space, use the back of the page.
5. Fully label all graphs.

1. (15 points). Let X be $n \times k$ matrix.

   (a) Prove that the matrix products $X'X$ and XX' are always well defined.

   $X'_{k\times n} X_{n\times k}$ is well defined because #(columns of X') = #(rows of X) $= n$. Similarly, $X_{n\times k} X'_{k\times n}$ is well defined because #(columns of X) = #(rows of X') $= k$.

   (b) Let Y be $n \times 1$ vector and b be a $k \times 1$ vector. What is the dimension of $Y'Xb$?

$$\underbrace{Y'}_{1\times n}\underbrace{X}_{n\times k}\underbrace{b}_{k\times 1} \text{ is } 1 \times 1, \text{ scalar}$$

(c) Prove that $Y'Xb$ is symmetric.

A symmetric matrix is equal to its transpose. Since in part b we proved that $Y'Xb$ is a scalar, and transpose of a 1×1 scalar is the scalar itself, it follows that $Y'Xb$ is symmetric.

2. (5 points). Let B be idempotent matrix. Is $B - I$ also idempotent matrix? Prove your answer.

A square matrix B is idempotent if $BB = B$. We need to check if $(B - I)(B - I) = B - I$.

$$\begin{aligned} (B - I)(B - I) &= BB - BI - IB + II \\ &= B - B - B + I \\ &= I - B \neq B - I \end{aligned}$$

3. (5 points). Let A, B, C be square matrices, and suppose that the inverse $(ABC)^{-1}$ exists. Prove that $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$. You are allowed to use the result proved in the notes that $(AB)^{-1} = B^{-1}A^{-1}$ (inverse of a product of two matrices is equal to the product of inverses, in reverse order).

$$\begin{aligned} (ABC)^{-1} &= ((AB)C)^{-1} \\ &= C^{-1}(AB)^{-1} \\ &= C^{-1}B^{-1}A^{-1} \end{aligned}$$

We applied the property $(AB)^{-1} = B^{-1}A^{-1}$ twice.

4. (10 points). Consider the following system of equations:

$$\begin{aligned} x_1 + x_2 + 3x_3 &= d_1 \\ x_2 + x_3 &= d_2 \\ x_3 &= d_3 \end{aligned}$$

(a) Using Cramer's rule, solve for x_2. Denote the solution by x_2^*.

$$x_2^* = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 1 & d_1 & 3 \\ 0 & d_2 & 1 \\ 0 & d_3 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}} = \frac{1 \cdot \begin{vmatrix} d_2 & 1 \\ d_3 & 1 \end{vmatrix}}{1} = d_2 - d_3$$

(b) Based on your result in the last section, find the change in x_2^* due to an increase in d_3 by 1 unit (i.e. $\Delta d_3 = 1$).

$$\Delta x_2^* = -\Delta d_3 = -1$$

Thus, x_2^* will decrease by 1.

5. (10 points). Let $A_{3\times 3}$ and $B_{3\times 3}$ be two matrices, with determinants $|A| = 3$ and $|B| = 5$. Calculate the following (show steps):

(a) $|A'|$

$$|A'| = |A| = 3$$

(b) $|AB|$

$$|AB| = |A| \cdot |B| = 3 \cdot 5 = 15$$

(c) $|A^{-1}|$

$$\left|A^{-1}\right| = \frac{1}{|A|} = \frac{1}{3}$$

(d) $|2 \cdot B|$

$$|2 \cdot B| = 2^3 \, |B| = 8 \cdot 5 = 40$$

(e) $|A \cdot B^{-1}|$

$$\left|A \cdot B^{-1}\right| = |A| \cdot \left|B^{-1}\right| = 3 \cdot \frac{1}{5} = \frac{3}{5}$$

6. (20 points). Suppose that the input-output matrix for some economy, consisting of 3 sectors, is given by:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

(a) Find the technology (Leontief) matrix T.

$$T = I - A = \begin{bmatrix} 1 - a_{11} & -a_{12} & -a_{13} \\ -a_{21} & 1 - a_{22} & -a_{23} \\ -a_{31} & -a_{32} & 1 - a_{33} \end{bmatrix}$$

(b) Suppose the consumers' demand vector is $d = \begin{bmatrix} d_1 & d_2 & d_3 \end{bmatrix}'$. The social planner wants to find the output levels in each industry, $x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}'$, which will satisfy simultaneously the inter-industry demand and the consumers' demand. Write the social planner's problem in matrix form.

$$\underbrace{\begin{bmatrix} 1 - a_{11} & -a_{12} & -a_{13} \\ -a_{21} & 1 - a_{22} & -a_{23} \\ -a_{31} & -a_{32} & 1 - a_{33} \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}}_{d}$$

$$Tx = d$$

(c) Write the solution to $x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}'$, obtained via matrix inversion. No need to calculate the output; just write the formula.

$$x^* = T^{-1}d$$

(d) Consumers' demand in some country changed from $d = \begin{bmatrix} 5 & 10 & 10 \end{bmatrix}'$ to $\tilde{d} = \begin{bmatrix} 10 & 10 & 10 \end{bmatrix}'$. This can reflect an increase in demand for healthcare as the nation's population ages. As a result, the social planner's production plan changed from $x = \begin{bmatrix} 60 & 70 & 80 \end{bmatrix}'$ to $\tilde{x} = \begin{bmatrix} 70 & 75 & 85 \end{bmatrix}'$. Explain intuitively, why the production of all goods can increase even though the demand increased only for good 1.

The production of all goods requires inputs from all industries, potentially. Thus, higher demand for good 1 by the consumers causes higher demand by industry 1 for inputs produced by *all* industries.

7. (15 points). Consider the multiple regression model

$$Y_i = \beta_1 + \beta_2 X_{2,i} + \beta_3 X_{3,i} + ... + \beta_k X_{k,i} + u_i, \ i = 1, ..., n$$

(a) Write the above model in matrix notation.

$$Y = X\beta + u$$

(b) Let $b = [b_1, ..., b_k]'$ denote the OLS estimator of $\beta = [b_1, ..., b_k]'$. Write the OLS problem (in matrix notation) which b solves. **No need to solve the problem.**

$$\min_b RSS = (Y - Xb)' (Y - Xb)$$

Here the residuals are given by $e = Y - Xb$.

(c) We showed that $b = (X'X)^{-1} X'Y$. Prove that $X'e = 0_{k\times 1}$.

$$\begin{aligned} X'e &= X'(Y - Xb) \quad \text{(substituting } e\text{)} \\ &= X'Y - \underbrace{X'X (X'X)^{-1}}_{I} X'Y \quad \text{(substituting } b\text{)} \\ &= X'Y - X'Y = 0_{k\times 1} \end{aligned}$$

8. (5 points). Suppose you want to solve for b from the system $X'Xb = X'Y$, using *Gaussian Elimination* method, where $X_{n\times k}$ and $Y_{n\times 1}$ are matrices. Which Matlab command will perform this calculation? (circle the correct answer)

(a) `b = adjoint(X'*X)*X'*Y;`

(b) `b = inv(X'*X)*X'*Y;`

(c) `b = (X'*X)\*X'*Y;`

(d) `b = det(X'*X)*X'*Y;`

(e) `b = rank(X'*X)*X'*Y;`

9. (15 points). Consider the following Matlab code.

```
3   A = sym('a', [3,3]);
4   d = sym('d', [3,1]);
5   x = sym('x', size(d));
6   d_A = det(A);
7   for j = 1:size(A,2)
8       Aj = A; Aj(:,j) = d;
9       x(j,1) = det(Aj)/d_A;
10  end
11  simplify(x - inv(A)*d)
```

(a) What is the purpose of the command in line 3?

Creating symbolic matrix of size 3×3, and generic element a_{ij}, where i is the row index and j is the column index. The result is:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{22} \end{bmatrix}$$

(b) What is the purpose of the command `Aj(:,j) = d;`?

Replacing the content of column `j` in matrix `Aj` with `d`.

(c) What is the output of the last line in this code, `simplify(x - inv(A)*d)`? Explain.

Vector of zeros `[0 0 0]'`, since Cramer's rule and matrix inversion solution methods should produce the same result.