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INTRODUCERE

In the period May 2013 – December 2014, the team members of the grant PNII-RU-TE-2012-3-0161, with title "Algebraic study of some combinatorial problems and associated computational experiments", have written eight scientific papers [19], [24], [29], [20], [15], [16], [17], [9], from which four have been published or are accepted for publishing and four are submitted for publishing. At the same time, they have launched seven software packages (in several versions) which are available online [10], [4], [21]. Both for the scientific papers, as well as for the internet pages of the software packages one may find links from the grant web page, at the address https://dl.dropboxusercontent.com/s/lb9znyg9ov2gtkn/ASSCPACE.htm. In the the following we present in detail the results obtained.

1. MAIN SCIENTIFIC RESULTS

1. In the paper [19], published in the journal Experimental Mathematics (IF=1), we introduce an algorithm for computing the Hilbert depth of a finitely generated multigraded module M over the standard multigraded polynomial ring $R = K[X_1, \ldots, X_n]$. The algorithm is based on the method presented in [18] and some extra improvements. It may also be adapted for computing the Stanley depth of M if $\dim_K M_a \leq 1$ for all $a \in \mathbb{Z}^n$. Further, we provide an experimental implementation of the algorithm [19] in CoCoA [12] and we use it to find interesting examples. As a consequence, we give complete answers to the following open problems proposed by Herzog in [14]:

Problema 1. [14, Problem 1.66] *Find an algorithm to compute the Stanley depth for finitely generated multigraded R-modules M with* $\dim_K M_a \leq 1$ *for all* $a \in \mathbb{Z}^n$.

Problema 2. [14, Problem 1.67] *Let M and N be finitely generated multigraded R-modules. Then*

$$\operatorname{sdepth}(M \oplus N) \geq \operatorname{Min}\{\operatorname{sdepth}(M), \operatorname{sdepth}(N)\}.$$

Do we have equality?

Problema 3. [14, Text following Problem 1.67] *In the particular case that* $I \subset R$ *is a monomial ideal, does* $\operatorname{sdepth}(R \oplus I) = \operatorname{sdepth} I$ *hold?*

2. In the paper [24], published in Springer Proceed. in Math. and Statistics, we study the Stanley's Conjecture [34] in the monomial square free case.

Let $S = K[x_1, \ldots, x_n]$ be the polynomial algebra over the field K and $J \subset I \subset S$ two squarefree monomial ideals. Suppose that I is generated by some square free monomials $f_1, \ldots, f_r$ of degrees d and a set of square free monomials E of degrees $\geq d+1$. Let B be the set of square free monomials from $I \setminus J$ of degrees $d+1$. An easy remark says that if $d=1$ and all monomials from $B \setminus E$ are some least common multiples of two monomials f_i then $\text{depth}_S I/J = 3$. It gives a weak form of this result for $d > 1$.

The main result of this paper says that if $r \leq 3$ then a weak form of Stanley Conjecture holds, namely if the Stanley depth of I/J is $\leq d+1$ then $\text{depth}_S I/J \leq d+1$. The used method is taken from [27] and uses some important results from [25], [26] and [32]. Finally, it states a weaker form as above in a special case when $r = 4$.

3. In the paper [29], accepted for publication in Bull. Math. Soc. Sci. Math. Roumanie (IF=0.452), we extend the results on a weak form of the Stanley's Conjecture [34] from the monomial square free case (obtained in [25], [28]) to the monomial general case by essentially using a recent theorem obtained in [15].

Let $S = K[x_1, \ldots, x_n]$ be the polynomial algebra over the field K and $J \subset I \subset S$ two monomial ideals. It is known that if I, J are square free and I is generated by monomials of degrees d then $\text{depth}_S I/J \geq d$. The present paper extends this result for the case when the ideals are not square free.

The lower bound of depth given in this case is obtained from the polarization of a certain canonical form associated to I/J in [5]. Certainly after polarization we get a factor of square free monomial ideals where we may apply the usual lower bound given by degree.

Using this lower bound we get weak forms of Stanley Conjecture which extend those existing in the square free frame [25], [28]. Here it is essential the result of [15] concerning the behavior of the Stanley depth under polarization.

Finally a weak form of Stanley Conjecture is stated for the case when I is generated by 6 variables and J is square free monomial ideal.

4. In the paper [20], published in Proceedings of the Third Conference of Mathematical Society of Moldova IMCS-50, we make a brief presentation of the program [21].

5. The paper ”The behavior of Stanley depth under polarization” (see [15]) is written in collaboration by Bogdan Ichim, Lukas Katthän and Julio Moyano-Fernández, and it is submitted for publication.

As noted by Stanley himself in [34, p. 191], the Stanley Conjecture was formulated such that ”the question raised in (Garsia, Stanley) would follow affirmatively”. This question was reformulated by Stanley [35, Conjecture 2.7], and asks whether every Cohen-Macaulay simplicial complex is partitionable. While it is clear that the Stanley Conjecture implies the Garsia-Stanley Conjecture on Cohen-Macaulay simplicial complexes, in this paper we show that the converse is also true, that is, the Garsia-Stanley Conjecture on Cohen-Macaulay simplicial complexes implies what is (arguably) the most important case of the Stanley Conjecture. More precisely, we show that Stanley's Conjecture on Cohen-Macaulay simplicial complexes is equivalent to [2, Conjecture 1].

6. The paper ”Stanley depth and the lcm-lattice” (see [16]) is written in collaboration by Bogdan Ichim, Lukas Katthän and Julio Moyano-Fernández, and it is submitted for publication.

A particularly striking aspect of Stanley's conjecture is that it relates two numerical invariants coming from different fields. On the one hand, we have the depth, which is an algebraic invariant (homological in nature), very important in algebraic geometry since its introduction at the beginning of the last century; and on the other hand we have the Stanley depth, which is a purely combinatorial invariant. In this paper we propose the *lcm-lattice* as a common base for the study of both invariants.

In Section 5 the reader will find many applications which will illustrate the power of this approach.

7. The paper "Lcm-lattices and Stanley depth: a first computational approach" (see [16]) is written in collaboration by Bogdan Ichim, Lukas Katthän and Julio Moyano-Fernández, and it is submitted for publication.

Let $S = K[x_1, \ldots, x_n]$ be the polynomial algebra over the field K. Let I be a monomial ideal of S with up to 5 generators. In this paper, we present a computational experiment which allows us to prove that

$$\operatorname{depth}_S S/I = \operatorname{sdepth}_S S/I < \operatorname{sdepth}_S I.$$

This shows that the Stanley conjecture is true for S/I and I, if I can be generated by at most 5 monomials. The result also brings additional computational evidence for a conjecture made by Herzog.

8. The paper "The power of pyramid decomposition in Normaliz" (see [9]) is written in collaboration by Winfried Bruns, Bogdan Ichim and Christof Söger, and it is submitted for publication.

In this paper we describe the use of pyramid decomposition in Normaliz, a software tool for the computation of Hilbert bases and enumerative data of rational cones and affine monoids. Pyramid decomposition in connection with efficient parallelization and streamlined evaluation of simplicial cones has enabled Normaliz to process triangulations of size $\approx 5 \cdot 10^{11}$ that arise in the computation of Hilbert series related to combinatorial voting theory. In this article we document the mathematical ideas and the most recent development [1] resulting from them. It has extended the scope of Normaliz by several orders of magnitude.

Very recently, Normaliz was benchmarked against other top software available. We illustrate the data obtained with well-known classical examples proposed by independent authors in distinct scientific fields, as listed in the following.

(1) `CondPar`, `CEffPl` and `PlVsCut` come from combinatorial voting theory. `CondPar` represents the Condorcet paradox, `CEffPl` computes the Condorcet efficiency of plurality voting, and `PlVsCut` compares plurality voting to cutoff, all for 4 candidates. See Schürmann [31] for more details.

(2) `4x4`, `5x5` and `6x6` represent monoids of "magic squares": squares of size 4×4, 5×5 and 6×6 to be filled with nonnegative integers in such a way that all rows, columns and the two diagonals sum to the same "magic constant". They belong to the standard LattE distribution [22].

(3) `bo5` and `lo6` belong to the area of statistical ranking; see Sturmfels and Welker [36]. `bo5` represents the boolean model for the symmetric group S_5 and `lo6` represents the linear order model for S_6.

(4) `small` and `big` are test examples used in the development of Normaliz without further importance. `small` has already been discussed in [8].

(5) `cyclo36`, `cyclo38`, `cyclo42` and `cyclo60` represent the cyclotomic monoids of orders 36, 38, 42 and 60. They have been discussed by Beck and Hoşten [6].

(6) `A443` and `A553` represent monoids defined by dimension 2 marginal distributions of dimension 3 contingency tables of sizes $4 \times 4 \times 3$ and $5 \times 5 \times 3$. They had been open cases in the classification of Ohsugi and Hibi [30] and were finished in [7].

(7) `cross10`, `cross15` and `cross20` are (the monoids defined by) the cross polytopes of dimensions 10, 15 and 20 contained in the LattE distribution [22].

[1] Version 2.12.1 has been uploaded to `http://www.math.uos.de/normaliz` on October 23, 2014.

For Hilbert bases computations we have used the software package 4ti2 [1] developed by a team from the Technische Universität München. The following table contains data from these tests.

Input	4ti2	Nmz -d -x=1	Nmz -d -x=20	Nmz -N -x=1	Nmz -N -x=20
CondPar	0.025 s	0.018 s	0.031 s	5.628 s	1.775 s
PlVsCut	6.697 s	1.967 s	0.567 s	–	–
CEffPl	6:09 m	2:15 m	15.75 s	–	–
4x4	0.008 s	0.003 s	0.011 s	0.004 s	0.010 s
5x5	3.835 s	1.684 s	0.382 s	4:27 m	1:40 m
6x6	123:18:24 h	20:07:14 h	1:21:11 h	–	–
bo5	T	–	–	0.588 s	0.377 s
lo6	31:32 m	16:18 m	1:19 m	60:16 m	11:54 m
small	48:48 m	38:02 m	3:33 m	3.098 s	3.861 s
big	T	–	–	3:05 m	43.758 s
cyclo36	T	–	–	1.477 s	1.092 s
cyclo38	R	–	–	36:55:53 h	2:02:32 h
cyclo60	R	–	–	7:06 m	2:17 m
A443	T	–	–	1.080 s	0.438 s
A553	R	–	–	3:15:15 h	15:59 m

TABELA 1. Computation times for Hilbert bases

We stopped the computations when the time had exceeded 150 h (T) or the memory usage had exceeded 100 GB (R).

For Hilbert series computations we have used the software package [22] developed by a team from the University of California at Davis. The following table contains data from these tests.

Input	4ti2	Nmz -d -x=1	Nmz -d -x=20	Nmz -N -x=1	Nmz -N -x=20
CondPar	0.025 s	0.018 s	0.031 s	5.628 s	1.775 s
PlVsCut	6.697 s	1.967 s	0.567 s	–	–
CEffPl	6:09 m	2:15 m	15.75 s	–	–
4x4	0.008 s	0.003 s	0.011 s	0.004 s	0.010 s
5x5	3.835 s	1.684 s	0.382 s	4:27 m	1:40 m
6x6	123:18:24 h	20:07:14 h	1:21:11 h	–	–
bo5	T	–	–	0.588 s	0.377 s
lo6	31:32 m	16:18 m	1:19 m	60:16 m	11:54 m
small	48:48 m	38:02 m	3:33 m	3.098 s	3.861 s
big	T	–	–	3:05 m	43.758 s
cyclo36	T	–	–	1.477 s	1.092 s
cyclo38	R	–	–	36:55:53 h	2:02:32 h
cyclo60	R	–	–	7:06 m	2:17 m
A443	T	–	–	1.080 s	0.438 s
A553	R	–	–	3:15:15 h	15:59 m

TABELA 2. Computation times for Hilbert bases

We have stopped the computations when the time exceeded 150 hours (T), the memory usage was more than 100 GB RAM (R) or it has produced more than 400 GB of output (O). These limitation were imposed by the system available for testing. In two cases it has exceeded the system stack limit; this is marked by S.

2. Parallel Scientific Results

In June 2013 we have released the version 2.10 of the program Normaliz (see [10]) together with the graphical interface jNormaliz version 1.4 (see [4]). In November 2013 we have released the first version of the program Hdepth [21], this being the first program available for computing the multigraded Hilbert depth. In April 2013 we have released the version 2.11 of the program Normaliz (see [10]) together with the graphical interface jNormaliz version 1.5 (see [4]). In October 2013 we have released the version 2.12 of the program Normaliz (see [10]) together with the graphical interface jNormaliz version 1.6 (see [4]).

3. Results Presentation

In order to disseminate the scientific results twenty presentations were held by this grant team within the algebra seminars of the Simion Stoilow Institute of Mathematics of the Romanian Academy. Four presentations were held at the National School on Algebra ”Algebraic Methods in Combinatorics”, Bucuresti, 01–07 Septembrie 2013 and two presentations were held at the National School on Algebra ”Algebraic and Combinatorial Applications of Toric Ideals”, Bucuresti, 31 August – 06 Septembrie 2014 by Marius Vladoiu and Andrei Zarojanu (they were also members of the organizing committee).

Bogdan Ichim has participated at the conference ”Joint International Meeting of the American Mathematical Society and the Romanian Mathematical Society” where he gave a talk with the title ”How to compute the multigraded Hilbert depth of a module” at the University of Alba Iulia on 28-06-2013.

Bogdan Ichim has participated at three research stages abroad, two in Spain and one in Germany, which ended up with the writing of four articles in collaboration with professors from the host universities [9], [15], [16] and [17], all of them being submitted for publication. He gave a talk with the title ”An algorithm for computing the multigraded Hilbert depth of a module” at the University of Osnabrück on 05-11-2013 and a talk with the title ”Recent results in Computational Voting Theory” at the conference ”Trends in Commutative Algebra” at the Jaume I University on 18-09-2014.

Dorin Popescu presented his results at the conference in the honor of Constantin Nastasescu organized by the University of Bucharest in Mai 2013, at the conference ”Experimental and Theoretical Methods in Algebra, Geometry, and Topology”, Eforie Nord, 20–24 June, 2013 and at the 150 years anniversary conference organized by the University of Bucharest, 30 August – 1 September, 2013.

Marius Vladoiu gave a talk with the title ”Markov bases of lattice ideals” at the conference ”Meeting On Combinatorial Commutative Algebra 2014” on 12-09-2014.

Andrei Zarojanu gave a talk with the title ”An introduction to Hilbert depth” at the conference ”The Third Conference of Mathematical Society of the Republic of Moldova ICMS-50” on 20-08-2014.

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