Contents

1 Classification of Problem Types

1.1 Topology

- (2003 #2) A topological space is separable if it contains a countable dense subset.
-

1.2 Lebesgue Measure

1.2.1 Cantor-Set

- (2002 #1) A fat Cantor set, $\mathbb{C}_\alpha$, with $m(C_\alpha) = 1 - \alpha$ is constructed by deleting intervals of length $\alpha 3^{-k}$ with $\alpha \in (0,1)$.
- (2005 $1) Cantor-Lebesgue Function.

1.3 Actually Computing Integrals

1.3.1 Useful Theorems

Theorem 1 (Tonelli-Fubini). *Let $(X, \mathcal{M}, \mu), (Y, \mathbb{N}, \nu)$ be σ-finite and $f(x,y)$ is $\mathcal{M} \otimes \mathbb{N}$ measurable.*

(a) $f \in L^+(X \times Y)$ then $x \mapsto \int f_x(y) d\nu(y)$ is μ-measurable. and $y \mapsto \int f^y(x) d\mu(x)$ is ν-measurable. Also, we have

$$\int f(x,y) d(\mu \times \nu)(x,y) = \int \int f_x(y) d\nu(y) d\mu(x) = \int \int f^y(x) d\mu(x) d\nu(y)$$

(b) $f \in L^1(X \times Y)$ then $f_x \in L^1(Y)$ and $x \mapsto \int f_x(y) d\nu(y)$ is in L^1.

Theorem 2 (Lebesgue Differentiation Theorem). *If we have $f \in L^1(\mathbb{R}^d)$ then for all $B \ni x$ we have for almost every $x \in \mathbb{R}^d$ s*

$$\lim_{m(B) \to 0} \frac{1}{m(B)} \int_B f(y) dy = f(x)$$

Theorem 3 (Differentiating Under the Integral). *Let $f : \mathbb{R} \to \mathbb{R} \to \mathbb{R} : (t,x) \mapsto f(t,x)$ be a measurable functions satisfying*

(i) $f(t_0, x)$ is measurable and integrable for each $t_0 \in \mathbb{R}$.

(ii) $f_t(t,x) := \frac{\partial}{\partial t} f(t,x)$ is continuous in t.

(iii) There exists an integrable function $\Phi(x)$ such that $f_t(t,x) \le \Phi(x)$ for all x and t.

Then $F(t) = \int f(t,x) dx$ is differentiable with $F'(t) = \int f_t(t,x) dx$.

Proof. Let $h_n \to 0$ and fix $t_0 \in \mathbb{R}$. We have

$$\frac{F(t_0 + h_n) - F(t_0)}{h_n} = \int \frac{f(t_0 + h_n, x) - f(t_0, x)}{h_n} dx.$$

Since f_t exists we apply the mean value theorem and conclude for each n there exists $|r_n| < h_n$ such that

$$g_n(t_0, x) := f_t(t_0 + r_n, x) = \frac{F(t_0 + h_n) - F(t_0)}{h_n}.$$

Notice that by (ii) we have $g_n(t_0, x) \to f_t(t_0, x)$ and that $|g_n(t_0, x)| < \Phi(x)$ and therefore is dominated. Using (i) and dominated convergence we obtain

$$F'(t_0) = \lim_{n\to\infty} \frac{F(t_0 + h_n) - F(t_0)}{h_n} = \lim_{n\to\infty} \int g(t_0, x)dx = \int f_t(t_0, x)dx.$$

□

Theorem 4 (Minkowski's Integral Inequality)**.** *Let* $(X, \mathcal{M}, \mu)$ *and* $(Y, \mathbb{N}, \nu)$ *be* σ*-finite measure spaces with* f *measurable with respect to the product* σ*-algebra.*

(a) Suppose $f \geq 0$, $f \in L^+(X \times Y)$ *and* $1 \leq p < \infty$. *We have*

$$\left[\int \left(\int f(x, y)d\nu(y)\right)^p d\mu(x)\right]^{1/p} \leq \int \left[\int f(x, y^p)d\mu(x)\right]^{1/p} d\nu(y)$$

(Note these functions are measurable by Tonelli).

(b) Suppose $1 \leq p \leq \infty$. *We know that* $x \mapsto f(x, y)$ *is in* $L^p(\mu)$ *and* $y \mapsto \|f(\cdot, y)\|_p$ *is in* $L^1(\nu)$. *Then* $f(x, \cdot) \in L^1(\nu)$ *for almost every* x. *So* $x \mapsto \int f(x, y)d\nu(y)$ *is in* $L^p(\mu)$ *and*

$$\left\|\int f(\cdot, y)d\nu(y)\right\|_p \leq \int \|f(\cdot, y)\|_p \, d\nu(y)$$

Theorem 5 (Borel implies Radon on LCH with σ-compact)**.** *Let* X *be LCH with every open set in* X *is* σ*-compact. Then every Borel measure that is finite on compact sets is Radon.*

1.3.2 Tricks/Techinques

- (2002 #3) Differentiate under the integral.
- (2004 #2) $\sum_{m,n\in\mathbb{Z}} \frac{1}{m^2+n^2+c} > \infty$ by changing to polar coordinates.
- (2005 #6) Split up integral over $\mathbb{D}$ and $\mathbb{D}^c$.

1.4 Banach Spaces

1.5 Convergence Problems

1.5.1 Useful Theorems

Theorem 6 (Arezelá-Ascoli Theorem)**.** *(Compact Version) Let* X *be compact Hausdorff. Suppose that* $\mathcal{F} \subseteq C(X)$ *with uniform topology. If* $\mathcal{F}$ *is pointwise bounded (*$\forall x \in X, f \in \mathcal{F}$ *it holds that* $|f(x)| < M$*) and* $\mathcal{F}$ *is equicontinuous then* $\mathcal{F}$ *is totally bounded.*

Proof. Fix $\epsilon > 0$. For all $x \in X$ there exists $U_x \ni x$ such that $|f(y) - f(x)| < \epsilon/4$. For all $y \in U_x$ and for all $f \in \mathcal{F}$ we know that $\{U_x\}_{x\in X}$ covers X. By compactness there exists $x_1, \ldots, x_n : X = \bigcup_{j=1}^n U_{x_j}$. Because

$$Q = \bigcup_{j=1}^{n} \{f(x_j) : f \in \mathcal{F}\} \qquad \text{is bounded in } \mathbb{C}$$

So there exists an $\epsilon/4$-net in $\mathbb{C}$ for Q. i.e. there exist $\{z_1, \ldots, z_m\} \subseteq \mathbb{C}$ such that

$$\bigcup_1^m B(z_i, \epsilon/4) \supset \bigcup_1^n \{f(x_j) : f \in \mathcal{F}\}$$

Define the sets

$$A = \{x_1, \ldots, x_n\}, \qquad B = \{z_1, \ldots, z_m\}$$

For all $\varphi : A \to B$ we define the set

$$\mathcal{F}_\varphi := \{f \in \mathcal{F} : |f(x_j) - \varphi(x_j)| < \epsilon/4\}$$

We conclude that from our construction we have

$$\bigcup_{\varphi \in B^A} F_\varphi = \mathcal{F}.$$

It remains to show that $\text{diam}(\mathcal{F}_\varphi) < \epsilon$. Let $f, g \in F_\varphi$, we wish to prove that $\|f - g\|_n < \epsilon$. We can make the following bounds

$$\begin{aligned} |f(x) - g(x)| &\leq |f(x) - f(x_j)| + |f(x_j) - g(x_j)| + |g(x_j) - g(x)| \\ &< \epsilon/2 + |f(x_j) - g(x_j)| \\ &\leq \epsilon/2 + |f(x) - \varphi(x_j)| + |\varphi(x_j) - g(x_j)| \\ &< \epsilon/2 + \epsilon/2 \end{aligned}$$

□

Theorem 7 (General Arzelá-Ascóli)**.** *Let X be a compact Hausdorff space and Y a metric spcae. A subset $F \subseteq C(X, Y)$ is compact in the compact-open topology if and only if it is equicontinuous, pointwise relatively compact and closed.*

1.5.2 Tricks/Techniques

- Hölder Tricks:
  - (i) $\frac{p}{q} = p1-$
  - (ii)
- (2002 #2) $f_n \rightharpoonup f \implies \sup_n \|f_n\| < \infty$.
- (2002 #7) $|f| \leq \|f\|_\infty$ a.e.
- (2003 #4) Every $f \in L^p$ defines a bounded linear operator, T, on L^q by $Tg = \int fg$ for $g \in L^q$. Moreover, $\|T\| = \|f\|_p$.

2 Bass

(7.16) Let $(X, \mathcal{M}, \mu)$ be a finite measure space. We say that $\{f_n\}$ is uniformly integrable if given $\epsilon > 0$ there exists M such that for $A_{n,M} = \{|f_n| > M\}$ it holds that $\int_{A_{n,M}} |f_n| < \epsilon$ for all n. We say that $\{f_n\}$ is uniformly absolutely continuous if for all $\epsilon > 0$ there exists a δ such that for all $A \in \mathcal{M}$ with $\mu(A) < \delta$ it holds for all n that $\int_A |f_n| < \epsilon$.

Prove that $\{f_n\}$ is uniformly integrable if and only if $\sup \|f_n\|_1 < \infty$ and $\{f_n\}$ is uniformly absolutely continuous.

Proof. ($\Rightarrow$) Suppose $\{f_n\}$ is uniformly integrable. Fix $\epsilon > 0$ and take M as in the definition. For any n we can make a bound

$$\begin{aligned}\int |f_n| &= \int_{A_{n,M}} |f_n| + \int_{A^c_{n,M}} |f_n| \\ &\leq \epsilon + M\mu(X).\end{aligned}$$

Therefore $\sup \|f_n\|_1 < \infty$. Fix a set $B \in \mathcal{M}$ with $\mu(B) < \epsilon/M$. We have

$$\begin{aligned}\int_B |f_n| &= \int_{B\cap A_{n,M}} |f_n| + \int_{B \cap A^c_{n,M}} |f_n| \\ &\leq \epsilon + M\mu(B) \\ &\leq 2\epsilon.\end{aligned}$$

($\Leftarrow$) Fix $\epsilon > 0$ and let δ be as guaranteed by hypothesis. If the f_n are bounded the result follows immediately. So, WLOG suppose that all of the f_n are unbounded. For each f_n let M_n be such that $\int_{A_{n,M_n}} |f_n| < \epsilon$ and $\delta/2 < \mu(A_{n,M_n}) < \delta$. We can bound $\mu(A_{n,M_n})$ from below since the f_n are unbounded and we can bound from above since the f_n are integrable. We claim that $\limsup M_n < \infty$. If not then we would have a subsequence $M_{n_k} \to \infty$ and so

$$\int |f_{n_k}| \geq \int_{A_{n_k, M_{n_k}}} |f_{n_k}| \geq M_{n_k}\mu(A_{n_k,M_{n_k}}) \geq (\delta/2)M_{n_k} \to \infty.$$

Which contradicts that $\sup \||f_n|\| < \infty$.

□

(7.17) Suppose that $\{f_n\}$ is uniformly integrable, $f_n \to f$ a.e. and μ is a finite measure. Prove $f_n \to f$ in L^1.

Proof. Fatou's lemma implies that $\int |f| \leq \sup \int |f_n|$ and therefore the family $\{f\} \cup \{f_n\}$ is uniformly integrable. Let $g_n = f - f_n$. We claim that $\{g_n\}$ is uniformly integrable. This follows from 7.15 since the triangle inequality immediately implies that $\sup \|g_n\|_1 < \infty$ and $\{g_n\}$ is uniformly absolutely continuous. So, fix $\epsilon > 0$ and let M be the guaranteed uniform bound on the g_n. We then have

$$\begin{aligned}\int |f - f_n| &= \int |g_n| \\ &= \int_{A_{n,M}} |g_n| + \int_{A^c_{n,M}} |g_n| \\ &\leq \epsilon + \int \mathbb{1}_{A^c_{n,M}} |f - f_n|.\end{aligned}$$

Since $\mathbb{1}_{A^c_{n,M}}|f - f_n| \leq M\mu(X)$ and converges point wise to 0, we can apply DCT and conclude that $\int \mathbb{1}_{A^c_{n,M}}|f - f_n| \to 0$. Hence $f_n \to f$ in L^1.

By Egorov there exists a set B such that $f_n \to f$ uniformly on B and $\mu(B^c) < \epsilon$. Let N be such that for $n \geq N$ and $x \in B$ it holds that $|f_n - f| < \epsilon$. For any $n \geq N$ we now have

$$\begin{aligned} \int |f_n - f| &= \int_B |f_n - f| + \int_{B^c} |f_n - f| \\ &\leq \epsilon\mu(B) + \int_{B^c} |f_n| + |f| \\ &\leq \epsilon\mu(B) + \end{aligned}$$

□

(14.6) Let $\{F_n\}$ be a family of increasing right continuous functions on $[0, 1]$ which have the additional property that $F(x) = \sum_1^\infty F_n(x)$ satisfies $F(1) < \infty$. Prove that for a.e. x it holds that $F'(x) = \sum_1^\infty F'_n(x)$.

Proof. Define the functions $\psi_N(x) = \sum_N^\infty F_n(x)$. First we claim that for a.e x it holds that $\psi'_N(x) \geq \psi'_{N+1}(x)$. This is easy to see since

$$\psi'_N(x) = \frac{d}{dx}\sum_N^\infty F_n(x) = F'_N(x) + \frac{d}{dx}\sum_{N+1}^\infty F_n(x) = F'_N(x) + \psi_{N+1}(x) \geq \psi_{N+1}(x).$$

Next, we claim that $\psi'_N \stackrel{L^1}{\to} 0$. This follows since for any $\epsilon > 0$ choose N such that $\psi_N(1) < \epsilon$ (which we can do since $F(1) < \infty$). Since $\psi_N(x)$ is an increasing function we know for all $a < b$ it holds that $\int_a^b \psi'_N(x)dx \leq \psi_N(b) - \psi_N(a)$. In particular,

$$\int_0^1 \psi'_N(x)dx \leq \psi_N(1) - \psi_N(0) \leq \psi_N(1) < \epsilon.$$

This gives L^1 convergence and therefore we have a subsequence $\psi'_{N_k} \stackrel{a.e.}{\to} 0$. However, the fact that the ψ'_N are monotonically decreasing in N guarantees that $\psi'_N \to 0$ a.e. We conclude the argument by observing that for a.e. x

$$F'(x) = \sum_1^{N-1} F'_n(x) + \psi'_N(x) \stackrel{a.e.}{\to} \sum_1^\infty F'_n(x).$$

□

3 Other Schools

3.1 Purdue

(2009) Fix a measure space $(X, \mathcal{M}, \mu)$ and let f be a measurable function such that for all continuous $\alpha : \mathbb{R} \to \mathbb{R}$ the function $\alpha \circ f$ is integrable. Prove that $f \in \|L\|_\infty$.

Proof. Let $I_n = \{f \in [n, n+)$. Let $a_n = \mu(I_n)$. For any continuous α we let $\min_{[n,n+1]} \alpha = a_n$ and can write

$$\int \alpha \circ f = \sum_{-\infty}^{\infty} \int_{I_n} \alpha \circ f \geq \sum_{-\infty}^{\infty} \mu(I_n) a_n = \sum_{-\infty}^{\infty} a_n \alpha_n.$$

Suppose $f \notin L^\infty$. This gives a subsequence $a_{n_k} > 0$. If we let α be an increasing continuous function such that $\alpha_{n_k} > \frac{1}{a_{n_k}}$ (a piecewise linear function □

Real Analysis Qualifying Exam
January 2011

Each problem is worth ten points. Work each problem on a separate piece of paper.

1. Working directly from the definition of almost everywhere convergence, prove that if $\{f_n\}_{n=0}^{\infty}$ is a sequence of measurable functions on a measure space $(X, \mathscr{M}, \mu)$ such that $\int_X |f_n - f_0|^{1/4}\, d\mu < n^{-2}$ for each n then $\{f_n\}_{n=1}^{\infty}$ converges to f_0 μ-almost everywhere.

2. Let K be a compact metric space. Show that $C(K)$ is separable.

3. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions on a finite measure space $(X, \mathscr{M}, \mu)$. Recall that $\{f_n\}_{n=1}^{\infty}$ is said to be *uniformly integrable* if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|\int_E f_n\, d\mu| < \varepsilon$ for all measurable sets $E \subseteq X$ satisfying $\mu(E) < \delta$ and all n. Prove that if $\{f_n\}_{n=1}^{\infty}$ is uniformly integrable, $\sup_n \|f_n\|_1 < \infty$, and $\{f_n\}_{n=1}^{\infty}$ converges in measure to 0, then $\|f_n\|_1 \to 0$ as $n \to \infty$.

4. Let $1 \le p < \infty$ and let f be a positive element of $L^p[0,1]$. Prove that the set $\{f^{1/n} : n \in \mathbb{N}\}$ has compact closure in $L^p[0,1]$. Give an example to show that this is false when $p = \infty$.

5. Let X be a reflexive Banach space and K a nonempty closed convex subset of X. Prove that there exists an $x \in K$ such that $\|x\| = \inf_{y \in K} \|y\|$. Show that this x is unique in the case that X is a Hilbert space.

6. Let X be a Banach space such that X^* is separable. Prove that X is separable.

7. (a) State what it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be *absolutely continuous*.

   (b) Let $F : \mathbb{R} \to \mathbb{R}$ be a function and let $0 \le M < \infty$. Show that $|f(x) - f(y)| \le M|x - y|$ for all $x, y \in \mathbb{R}$ if and only if f is absolutely continuous and $|f'(x)| \le M$ almost everywhere with respect to Lebesgue measure.

8. For a function $f : [0,1] \to \mathbb{R}$ define

$$\|f\|_L = |f(0)| + \sup\left\{\frac{|f(x) - f(y)|}{|x - y|} : 0 \le x < y \le 1\right\}.$$

   Prove that the set of all functions $f : [0,1] \to \mathbb{R}$ satisfying $\|f\|_L < \infty$ is dense in $L^1[0,1]$.

9. Let $g : [0,1] \to [0,1]$ be a continuous function. Determine, with proof, conditions on g which are equivalent to the property that $\lim_{n\to\infty} \|g^n f\|_2 = 0$ for all $f \in L^2[0,1]$.

10. (a) State Fubini's theorem.

    (b) Let $\{f_n\}_{n=1}^{\infty}$ be a bounded sequence in $C([0,1]^2)$. Suppose that $f_{x,n} \to 0$ weakly in $L^2(\mu)$ for every $x \in [0,1]$, where $f_{x,n}(y) = f_n(x,y)$ for all $y \in [0,1]$ and μ is Lebesgue measure on $[0,1]$. Prove that $f_n \to 0$ weakly in $L^2(\mu \times \mu)$.

3.2 Texas A&M 2011

1. Fix $\epsilon > 0$ and let N be such that $\frac{1}{N} < \epsilon$. For any $n \geq N$ it follows from our hypothesis and Chebyshev's inequality that
$$\mu\left(\left\{|f_n - f|^{1/4} > \epsilon\right\}\right) \leq \mu\left(\left\{|f_n - f|^{1/4} > \frac{1}{n}\right\}\right) \leq \frac{1}{1/n}\int |f_n - f|^{1/4} \leq \frac{n}{n^2} = \frac{1}{n} < \frac{1}{N} < \epsilon.$$
This immediately implies $f_n \to f$ a.e.

2. Let $\{q_1, q_2, \ldots\} \subseteq K$ be a countable dense set. Define the functions $\varphi_i : K \to \mathbb{R} : x \mapsto \|q_i - x\|$. It is easy to verify using the reverse triangle inequality that each φ_i is Lipschitz and therefore continuous. Let S be the set consisting of all finite sums and products of the φ_i. Notice that S is countable and an algebra since it is closes under products and sums. Moreover S separates points since for $x \neq y$ we let q_i be such that $\|x - q_i\| < \|y - q_i\|$ and so $\varphi_i(x) \neq \varphi_i(y)$. Since the functions in S do not all simultaneously vanish at any points of K we apply Stone-Weirstrauss to conclude that S is dense in $C(K)$.

3. Fix $\epsilon > 0$ and let $\delta > 0$ be as guaranteed by uniform integrability. Since $f_n \to 0$ in measure we know there exists N such that for all $n \geq N$ we have $\mu(\{|f_n| > \epsilon) < \delta$. Let $A_{\epsilon,n} = \{|f_n| > \epsilon\}$. We can write
$$\begin{aligned}\int |f_n| &= \int_{A_{\epsilon,n}} |f_n| + \int_{A^c_{\epsilon,n}} |f_n| \\ &\leq \epsilon + \mu(A^c_{\epsilon,n})\epsilon \\ &\leq \epsilon(1 + \mu(X)).\end{aligned}$$
Since this holds for each $n \geq N$ we have $f_n \to 0$ in L^1. Note we did not use the fact that $\sup \|f_n\| < \infty$.

4. Let $f_n = f^{1/n}$. We claim that the closure of $\mathcal{F} = \{f_n\}$ in L^p is $\overline{\mathcal{F}} = \mathcal{F} \cup \{1\}$. To see this we let $A = \{f \leq 1\}$ and $A^c = \{f > 1\}$. Notice that $f_n(x)\mathbb{1}_A \to \mathbb{1}_A$ pointwise and since $f_n \mathbb{1}_A \leq 1$ for all n we can apply DCT to conclude
$$\lim \int_A |f_n - 1|^p = 0.$$
Similarly on A^c we know that each $f_n \mathbb{1}_{A^c} \leq f$ and again by DCT
$$\lim \int_{A^c} |f_n - 1|^p = 0.$$
It follows that $\int f_n = 1$. Therefore the closure $\overline{\mathcal{F}}$ is as claimed and is compact since any sequence will converge to 1 in L^p.

   I am not sure how to deal with $p = \infty$.

5.

6. Let $\{f_n\} \subseteq X^*$ be a dense countable set. Since subsets of a separable space are also separable we can assume that all of the f_n has $\|f_n\| = 1$. By definition of the norm, for each n choose x_n with $\|x_n\| = 1$ such that $|f(x_n)| > \frac{1}{2}$. Let $M = \overline{\{x_n\}}$. If $M \neq X$ take $x \in X \setminus M$ and by Hahn-Banach let f be such that f vanishes on M but is not identically zero. We then have for each n
$$\frac{1}{2} < |f_n(x_n)| = |f_n(x_n) - f(x_n)| \leq \|f_n - f\| \, \|x_n\| = \|f_n - f\|.$$
This implies the contradiction that the f_n are not dense.

4 Solomyak Practice Problems

(Bold Prediction) Suppose $E \subseteq \mathbb{R}^d$ has $m(E) > 1$. Show that $(E - E) \cap \mathbb{Z} \neq \{0\}$.

First do the case where $E \subseteq \mathbb{R}$ with $1 < m(E) = 1 + \epsilon$ for some $\epsilon > 0$. We can without loss of generality suppose that E is compact (since there exists a compact set $F \subseteq E$ with $1 < m(F) \leq f(E)$. To show a contradiction suppose that $(E - E) \cap \mathbb{Z} = \{0\}$. This implies that for all $N \in \mathbb{Z}^+$ we have the following is a disjoint union

$$E_N = \bigcup_{k=-N}^{N} E + k$$

Because we are assuming E is compact we have $E \subseteq [-n, n]$ for some fixed $n \in \mathbb{Z}^+$, thus $E_N \subseteq [-n - N, n + N]$. Notice that

$$m([-n - N, n + N) = 2N + 2n$$

and because E_N is a disjoint union of measurable sets we have

$$E_N = \sum_{k=-N}^{N} m(E) = 2Nm(E) = 2N(1 + \epsilon) = 2N + 2N\epsilon$$

So if we choose N large enough so that $2N\epsilon > 2n$ we have the contradiction

$$E_N \subseteq [-n - N, n + N], \qquad \text{but } m(E_N) > m([-n - N, n + N])$$

So the $(E - E) \cap \mathbb{Z}$ contains an integer.

It is easy to generalize to higher dimensions, we still use compactness and instead of the interval $[-n, n]$ we look at the cube $[-n, n]^d$.

4.1 Problems

1. Prove that a function $f : [0, 1] \to \mathbb{R}$ is absolutely continuous if and only if there exists a sequence of functions f_n with each f_n Lipshitz and $TV_0^1[f - f_n] \to 0$.
2. Let X be a normed vector space and let $S \subseteq X$ be a proper closed subspace.
   (a) Show that there exists $x \in X$ such that $\|x\| = 1$ and $\inf\{\|x - y\| : y \in S\} > 1/2$.
   (b) Prove that a normed vector space is finite dimensional if and only if the closed unit ball is compact.
3. Let $X_\alpha[0, 1]$ denote the space of all $f : [0, 1] \to \mathbb{R}$ which are Hölder with exponent α. Define the norm
$$\|f\|_\alpha = |f(0)| + \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\alpha}$$
   (a) Prove that $X_\alpha[0, 1]$ is a Banach Space.

(b) Let λ_α be the set of all $f \in X_\alpha$ such that

$$\frac{|f(x)-f(y)|}{|x-y|^\alpha} \to 0 \qquad \text{as } x \to y, \quad \text{for all } y \in [0,1] \ .$$

Prove that if $\alpha < 1$ then λ_α is a infinite dimensional closed subspace of X_α. If $\alpha = 1$ prove that λ_α consists of only the constants.

4. Let $f \in L^1[0,1]$. Suppose that

$$\int_0^1 e^{-nx} f(x)dx = 0, \qquad \forall \quad n = 0,1,2,\dots.$$

Prove that $f = 0$.

5. Suppose $\mathcal{H}$ is an infinite-dimensional Hilbert space. Show that for any sequence $\{f_n\} \subseteq \mathcal{H}$ with $\|f_n\| = 1$ for all n there exists $f \in \mathcal{H}$ and a subsequence $\{f_{n_k}\}$ such that for all $g \in \mathcal{H}$, one has

$$\lim_{k\to\infty} (f_{n_k}, g) = (f,g)$$

One says that $\{f_{n_k}\}$ *converges weakly* to f.

4.2 Solutions

1. *Proof.* ($\Rightarrow$) Suppose f is absolutely continuous. First assume $f(0) = 0$. Since f is AC we can write $f(x) = \int_0^x g(t)dt$. Since $[0,1]$ is bounded we know that F is of bounded variation and thus $\int_0^1 |g(t)|dt = TV(f) < \infty$. It follows that $g \in L^1([0,1])$. Because continuous functions are dense in $L^1([0,1])$ we can approximate g by continuous functions g_n satisfying $\|g - g_n\|_1 < 1/n$. Let $f_n = \int_0^x g_n(t)dt$. Clearly, each f_n is absolutely continuous and also satisfies $f_n'(x) = g_n(x) \leq \max_{t\in[0,1]} g(t) = M_n$. Absolutely continuous functions with bounded derivatives are Lipschitz. Lastly, we calculate TV:

$$TV\,{}_0^1|f - f_n| = \int_0^1 |f' - f_n'| = \int_0^1 |g - g_n| < 1/n \to 0$$

($\Leftarrow$) Suppose that there exists a sequence of Lipschitz functions $f_n \to f$ such that $TV_0^1[f - f_n] \to 0$. Fix $\epsilon > 0$. Let N be such that for all $n > N$ we have $TV\,{}_0^1(|f_n - f| < \epsilon/2$ and let M_n be the Lipschitz constant of f_N. Fix a partition $\mathcal{T}$ of $[0,1]$ satisfying $\sum_{\mathcal{T}} |b_j - a_j| < \epsilon/(2M_N)$ and consider

$$\begin{aligned}
\sum_{\mathcal{T}} |f(b_j) - f(a_j)| &\leq \sum_{\mathcal{T}} |f(b_j) - f(a_j) - (f_n(b_j) - f_n(a_j))| + |f_n(b_j) - f_n(a_j)| \\
&\leq TV\,{}_0^1[f - f_n] + M_N \sum_{\mathcal{T}} |b_j - a_j| \\
&< \epsilon/2 + \epsilon/2 \\
&= \epsilon
\end{aligned}$$

□

2. (a) The solution is an immediate corollary of the following lemma.

   Lemma 1 (Riesz Lemma)**.** *For a non-dense subspace Y of a Banach space X, given $0 < \alpha < 1$, there is a $\boldsymbol{b} \in X$ with $|\boldsymbol{b}| = 1$ but $\inf_{y \in Y} |y - \boldsymbol{b}| \geq \alpha$.*

   Proof. Choose $x_1 \in X \setminus \overline{Y}$ and let $R = \inf_{y \in Y} |y - x_1| > 0$. For $\alpha \in (0,1)$ let $y_1 \in Y$ be such that $|y_1 - x_1| < R/\alpha$. Let $\mathbf{b} = \frac{x_1 - y_1}{|y_1 - x_1|}$ so $|\mathbf{b}| = 1$. Also $\mathbf{b} \in X \setminus Y$ since if $\mathbf{b} \in Y$ then $x_1 - y_1 \in Y$ and thus $(x_1 - y_1) + y_1 = x_1 \in Y$ a contradiction. Now, for any $y \in Y$ we have

$$\begin{aligned}
\|\mathbf{b} - y\| &= \left\| \frac{x_1 - y_1}{\|x_1 - y_1\|} - y \right\| \\
&= \left\| \frac{x_1}{\|x_1 - y_1\|} - \frac{y_1}{\|x_1 - y_1\|} - y \right\| \\
&= \frac{1}{\|x_1 - y_1\|} \|x_1 - (y_1 + \|x_1 - y_1\| \cdot y)\| \\
&> (R/\alpha)^{-1} \cdot \inf_{y \in Y} |x_1 - y| \\
&= \frac{\alpha}{R} R \\
&= \alpha
\end{aligned}$$

   □

   (b) *Proof.* ($\Rightarrow$) Suppose X is finite dimensional. This implies X is isomorphic to $\mathbb{R}^n$ with the standard topology and since $\overline{\mathbb{D}}$ is closed and bounded the Heine-Borel theorem implies that $\overline{\mathbb{D}}$ is compact.

   ($\Leftarrow$) Suppose that X is not finite dimensional. Pick $x_1 \in X$ such that $|x_1| = 1$ and let S_1 be the subspace generated by x_1. The Riesz Lemma guarantees that there exists an element $x_2 \in X$ such that $|x_2| = 1$ and $d(x_2, S_1) \geq 1/2$. Let S_2 be the subspace generated by x_1 and x_2. Since X is infinite dimensional we can continue to apply the Riesz Lemma to proceed inductively and create a sequence $\{x_n\}$ such that each x_n satisfies $|x_n| = 1$ and $d(x_n, S_{n-1}) \geq 1/2$. Clearly $\{x_n\} \subseteq \overline{\mathbb{D}}$, however it has no convergent subsequence since $|x_n - x_m| \geq 1/2$ for all $m \neq n$. □

3. *Proof.* It suffices to prove that $(X_\alpha, \|\cdot\|_\alpha)$ is complete. Let $\{f_m\}$ be a Cauchy sequence with respect to $\|\cdot\|_\alpha$. We prove several facts about this sequence.

   (i) $\sup_m \|f_m\|_\alpha < A < \infty$ for some A.

   Proof. Let $\epsilon = 1$. Choose M such that for $m, n \geq M$ it holds that $\|f_m - f_n\|_\alpha < 1$. Let $A = \sup\{\|f_1\|_\alpha, \ldots, \|f_{M-1}\|_\alpha, \|f_M\|_\alpha + 1\}$. Since all of the elements f_n with $n \geq M$ satisfy $\|f_M - f_n\| < 1$ we know that $A > \sup \|f_n\|_\alpha$.

   □

   (ii) The $\{f_n\}$ is a Cauchy sequence in $C([0,1])$ where the convergence is in the uniform norm $\|\cdot\|_u$. It follows by completeness that there is some $f \in C([0,1])$ such that $\|f - f_n\|_u \to 0$.

Proof. Fix $\epsilon > 0$. Let M be such that for all $m, n \geq M$ it holds that $\|f_m - f_n\|_\alpha < \epsilon$. This says first off that $\{f_n(0)\}$ is a Cauchy sequence in $\mathbb{R}$ and so by completeness converges to some value $f(0)$. Moreover, for any $x \neq y$ we know that

$$|f_m(x) - f_n(x) - f_m(y) + f_m(y)| \leq A \cdot \epsilon |x - y|^\alpha.$$

Since $|x - y| \leq 1$ we conclude that for $n, m \geq M$

$$\|f_m - f\|_u < A\epsilon.$$

As $A < \infty$, this says that $\{f_n\}$ is Cauchy, and by completeness has a limit f in the uniform norm.

□

(iii) $\|f_m - f\|_\alpha \to 0$.

Proof. We know from the previous part that $f_m(0) \to f(0)$, so it suffices to prove that $\lim_{m\to\infty} \|f_m - f\|_\alpha = \lim_{m\to\infty} \sup_{x\neq y} \frac{|(f_m-f)(x)-(f_m-f)(y)|}{|x-y|^\alpha} = 0$. Again since $\{f_n\}$ is Cauchy in the α-norm fix ϵ and let M be such that for $n, m \geq M$ and any $x \neq y$

$$\frac{|f_m(x) - f_n(x) - [(f_m(y) - f_n](y)|}{|x - y|^\alpha} < \epsilon.$$

If we fix n and let $m \to \infty$ we obtain the inequality

$$\sup_{x\neq y} \frac{|f_m(x) - f(x) - [f_m(y) - f(y)]|}{|x - y|^\alpha} < \epsilon.$$

As x and y were arbitrary, it follows that for all $n \geq M$

$$\|f_n - f\|_\alpha < \epsilon.$$

Therefore $f_n \to f$ in the α-norm. □

(iv) $\|f\|_\alpha < \infty$.

Proof. It remains to prove that f is α-continuous. Since $f(0)$ is bounded we need to prove that there exists a constant $C > 0$ such that for $x \neq y$ it holds that $|f(x) - f(y)| < C|x - y|^\alpha$.
Fix $\epsilon > 0$ and let M be such that for $n \geq M$ it holds that $\|f_n - f\|_\alpha < \epsilon$. This means that

$$|f(x) - f_n(x) - f(y) + f_n(y)| < \epsilon |x - y|^\alpha.$$

For $x \neq y$ we can write

$$\begin{aligned}|f(x) - f(y)| &\leq |f(x) - f_n(x) - f(y) + f_n(y)| + |f_n(x) - f_n(y)| \\ &\leq \epsilon|x - y|^\alpha + A|x - y|^\alpha \\ &= (A + \epsilon)|x - y|^\alpha.\end{aligned}$$

Therefore $\|f\|_\alpha \leq |f(0)| + A$.

□

This means for each x the sequence $f_m(x)$ is Cauchy and converges to some value $f(x)$ uniformly on $[0,1]$ (by completeness of $\mathbb{R}$). We would like to prove that $f_m \to f$. By definition we know that, $f_m(0) \to f(0)$, so it remains to prove that

$$\lim_{m\to\infty} \|f_m - f\|_\alpha = \lim_{m\to\infty} \sup_{x\neq y} \frac{|(f_m - f)(x) - (f_m - f)(y)|}{|x-y|^\alpha} = 0 \tag{1}$$

By assumption we know that

$$\lim_{m,n\to\infty} \sup_{x\neq y} \frac{|f_m(x) - f_n(x) - [(f_m(y) - f_n](y)|}{|x-y|^\alpha} = 0$$

Hence for any $\epsilon > 0$ there exists N large enough such that

$$\sup_{x\neq y} \frac{|f_m(x) - f_n(x) - [(f_m(y) - f_n](y)||}{|x-y|^\alpha} < \epsilon, \qquad \forall n, m \geq N$$

Fixing $m > N$ and letting $n \to \infty$ gives

$$\sup_{x\neq y} \frac{|f_m(x) - f(x) - [f_m(y) - f(y)]|}{|x-y|^\alpha} < \epsilon$$

As ϵ was arbitrary this proves (1), and thus $f_m \to f$.

□

4. *Proof.* Let $\mathcal{A}$ be the subalgebra of $C([0,1])$ generated by $\{e^{-nx}\}_0^\infty$. Observe that $\mathcal{A}$ separates points and contains $1 = e^0$. Since $[0,1]$ is compact we can apply Stone-Weirstrass and conclude that $\mathcal{A}$ is dense in $C([0,1])$.

   Let (a,b) be an interval in $[0,1]$ and since $f \in L^1([0,1])$ let φ_ϵ be a continuous bump function on (a,b) which approximates f. More precisely, $\|f - \varphi_\epsilon\|_1 < \epsilon$. Since φ_ϵ is continuous and $\mathcal{A}$ is dense we let $h \in \mathcal{A}$ approximate φ_ϵ by ϵ. $|\varphi_\epsilon - g| < \epsilon$. We now have

$$\int_a^b |f| \to \int_a^b \varphi_\epsilon f \to \int_a^b hf \int_a^b f\varphi_\delta$$

□

5. *Proof.* Step 1: First we construct the subsequence. Fix an orthonormal basis $\{e_n\}_1^\infty$ in $\mathcal{H}$. Since $|(f_n, e_1)| \leq \|f_n\| \|e_1\| = 1$ we can extract a convergent subsequence $(f_{n_k^{(1)}}, e_j)$. The sequence $(f_{n_k}^{(1)}, e_2)$ is also bounded so we can select a convergent subsequence $(f_{n_k}^{(2)}, e_2)$. Proceeding inductively we obtain a family of subsequences given by indices $n_k^{(j)}$ with $\left(n_k^{(j+1)}\right) \subseteq \left(n_k^{(j)}\right)$. Let $n_k = n_k^{(k)}$. As $k \to \infty$ each sequence (f_{n_k}, e_j) converges to some $c_j \in \mathbb{C}$.

   Step 2: We prove that $\sum |c_j|^2 < \infty$. This follow from Fatou's lemma (under the counting measure) since

$$\sum_1^\infty |c_j|^2 = \sum_1^\infty \lim_{k\to\infty} |(f_{n_k}, e_j)|^2 \leq \lim_{k\to\infty} \sum_1^\infty |(f_{n_k}, e_j)|^2 \leq \lim_{k\to\infty} \|f_{n_k}\|^2 = 1.$$

   Therefore the function $f = \sum_1^\infty c_j e_j$ is well-defined.

Step 3: Consider finite linear combinations. We know that $(f_{n_k}, e_j) \to (f, e_j)$. Hence for any finite linear combination of basis elements (call this set S), we must have for $g = \sum_1^N a_j e_j$ that

$$(f, g) = \sum_1^N \overline{a}_j (f_{n_k}, e_j) \to \sum_1^N \overline{a}_j (f, e_j) = (f, g)$$

Since S is dense in $\mathcal{H}$ we know that for any $h \in \mathcal{H}$ and $\epsilon > 0$ there exists $g \in S$ such that $\|g - h\| < \epsilon$.

Step 4: We prove weak convergence. Fix $\epsilon > 0$. There exists $g \in S$ such that $\|g - h\| < \epsilon$ and there is N such that for $k > N$ we have $|(f_{n_k}, g) - (f, g)| < \epsilon$. We now have

$$|(f_{n_k}, h) - (f, h)| \leq |(f_{n_k}, g) - (f, g)| + |(f_{n_k} - f, g - h)| < \epsilon + \|f_{n_k} - f\| \cdot \|g - h\| \leq \epsilon + 2\epsilon = 3\epsilon$$

because $\|f_{n_k} - f\| \leq \|f_{n_k}\| + \|f\| = 2$.

□

The Final Exam will be on Wednesday, December 14, 2:30–5:30. There will be 4 to 6 problems on the exam. The instructions will be as follows:

If you refer to one of the texts or to a homework problem you must give precise citation. If you quote a theorem, state it explicitly and verify that the hypotheses are satisfied. If you use a problem or a theorem that was not discussed in class, you must give a complete solution/proof as part of your answer.

I. Short problems.

(1) Prove that for any set E in $\mathbb{R}^d$ with $m_*(E) < \infty$ (not necessarily measurable) there exists a set $F \supset E$ such that F is Borel and $m_*(F) = m_*(E)$.

(2) Suppose that $(X, \mathcal{M})$ is a measurable space and μ is a non-negative set function that is finitely additive and such that $\mu(\emptyset) = 0$. Suppose that whenever A_i is an increasing sequence of sets in $\mathcal{M}$, we have $\mu(\cup_i A_i) = \lim \mu(A_i)$. Show that μ is a measure.

(3) Suppose that $(X, \mathcal{M}, \mu)$ is a measure space and $A, B \in \mathcal{M}$. Prove that

$$\mu(A) + \mu(B) = \mu(A \cap B) + \mu(A \cup B).$$

(4) Let $\varepsilon > 0$. Find a measurable set $E \subset [0, 1]$ such that the closure of E is $[0, 1]$ and (Lebesgue measure) $m(E) = \varepsilon$.

(5) Suppose $f_n(x)$ are measurable functions. Prove that

$$A = \{x : \lim_{n\to\infty} f_n(x) \text{ does not exist } \}$$

is a measurable set.

(6) Give an example of a sequence of non-negative measurable functions f_n such that $f_n \to 0$ pointwise and $\int f_n \to 0$, but there is no integrable function g such that $f_n \le g$ for all n.

(7) Suppose f is a non-negative integrable function on a measure space $(X, \mathcal{M}, \mu)$. Prove that

$$\lim_{t\to\infty} t \cdot \mu(\{x : f(x) \ge t\}) = 0.$$

(8) Suppose that for each $\varepsilon > 0$ there exists a measurable set F such that $\mu(F^c) < \varepsilon$ and f_n converges to f uniformly on F. Prove that f_n converges to f a.e.

(9) Suppose f is real-valued and integrable with respect to the Lebesgue measure on $[0, 1]^2$, and

$$\int_0^a \int_0^b f(x, y)\, dy\, dx = 0$$

for all a and b in $[0, 1]$. Prove that $f = 0$ a.e.

(10) Find a real-valued f that is integrable with respect to the Lebesgue measure on $[0, 1]^2$, and

$$\int_0^a \int_0^1 f(x, y)\, dy\, dx = 0, \qquad \int_0^1 \int_0^b f(x, y)\, dy\, dx = 0$$

for all $a, b \in [0, 1]$, but f is not zero a.e.

II. Problems.

(1) Suppose μ_1, μ_2,..., are measures on a measurable space $(X, \mathcal{M})$ and $\mu_n(A) \uparrow$ for each $A \in \mathcal{M}$. Define

$$\mu(A) = \lim_{n\to\infty} \mu_n(A).$$

Is μ necessarily a measure? If not, give a counterexample. What if $\mu_n(A) \downarrow$ for each $A \in \mathcal{M}$ and $\mu_1(X) < \infty$?

(2) Suppose $(X, \mathcal{M})$ is a measurable space and $\mathcal{C}$ is a subset of $\mathcal{M}$. Suppose m and n are two σ-finite measures on $(X, \mathcal{M})$ such that $m(A) = n(A)$ for all $A \in \mathcal{C}$. Is it true that $m(A) = n(A)$ for all $A \in \sigma(\mathcal{C})$? (Recall that $\sigma(\mathcal{C})$ is the smallest σ-algebra containing $\mathcal{C}$.) What if m and n are finite measures?

(3) Suppose A is a Lebesgue measurable subset of $\mathbb{R}$ and

$$B = \bigcup_{x\in A} [x-1, x+1].$$

Prove that B is Lebesgue measurable.

(4) If $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable, prove that there exists a Borel measurable function g such that $g = f$ a.e.

(5) Suppose f_n and f are integrable, $f_n \to f$ a.e., and $\int |f_n| \to \int |f|$. Prove that $f_n \to f$ in L^1.

(6) Find the limit

$$\lim_{n\to\infty} \int_0^n \left(1 - \frac{x}{n}\right)^n \log(2 + \cos(x/n))\, dx$$

and justify your reasoning.

(7) Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\{f_n\}_{n\geq 1}$ be a sequence of non-negative measurable functions such that

$$\int_X f_n\, d\mu \leq 1$$

for all n. Prove that

$$\limsup_{n\to\infty} (f_n(x))^{1/n} \leq 1 \quad \mu\text{-a.e. } x \in X.$$

(8) Let $\{a_n\}$ and $\{r_n\}$ be two sequences of real numbers such that $\sum_{n=1}^{\infty} |a_n| < \infty$. Prove that

$$\sum_{n=1}^{\infty} \frac{a_n}{\sqrt{|x - r_n|}}$$

converges absolutely for a.e. $x \in \mathbb{R}$.

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((S8)) Let $\{a_n\}, \{r_n\} \subseteq \mathbb{R}$ with $\sum_1^\infty |a_n| < \infty$. Prove that $\displaystyle\sum_1^\infty \frac{a_n}{\sqrt{|x-r_n|}}$ converges absolutely for a.e. $x \in \mathbb{R}$.

Proof. Fix $M > 0$. We will show that

$$\sum_1^\infty \int_{-M}^M \frac{|a_n|}{\sqrt{|x-r_n|}} dx < \infty. \qquad (*)$$

If we prove this, it follows from Tonelli that the sum in question converges for a.e. $x \in [-M, M]$. If we let $M \to \infty$ we have the result. Since a_n and r_n are fixed, we can rewrite the expression at $*$ as

$$\sum_1^\infty |a_n| \int_{-M}^M \frac{1}{\sqrt{|x-r_n|}} dx.$$

We will show that $\int_{-M}^M \frac{1}{\sqrt{|x-r_n|}} dx < C$ independent of r_n. Suppose $r_n > M$, we then have $|x - r_n| = r_n - x$ for all $x \in [-M, M]$. Therefore

$$\int_{-M}^M \frac{1}{\sqrt{r_n - x}} = 2(\sqrt{r_n + M} - \sqrt{r_n - M}) < C_1.$$

Where we obtain a bound since $\lim_{y\to\infty} \sqrt{y+M} - \sqrt{y-M} = 0$ Similarly for $r_n < -M$ we obtain $\int_{-M}^M \frac{1}{\sqrt{|x-r_n|}} dx < C_2$. When $r_n \in [-M, M]$ we split up the integral and make a similar calculation

$$\int_{-M}^M \frac{1}{\sqrt{|x-r_n|}} dx = \int_{-M}^{r_n} \frac{1}{\sqrt{|x-r_n|}} dx + \int_{r_n}^M \frac{1}{\sqrt{|x-r_n|}} dx < C_3.$$

It follows that for any $r_n \in \mathbb{R}$ we have $\int_{-M}^M \frac{1}{\sqrt{|x-r_n|}} dx < \max\{C_1, C_2, C_3\} = C$. We can then conclude that

$$\sum_1^\infty \int_{-M}^M \frac{|a_n|}{\sqrt{|x-r_n|}} dx < \sum_1^\infty C|a_n| < \infty.$$

□

5 1992

(1992 #6) Let $A_n = \bigcup_{j=1}^{2^{n-1}} \left[\frac{2j-1}{2^n}, \frac{2j}{2^n}\right]$. Prove that if $f \in L^1$ then $\lim \int_{A_n} f = \frac{1}{2} \int f$. Hint: start with a nice type of function.

Proof. Fix $\epsilon > 0$ and let $0 < a < b < 1$. We let $f = \mathbb{1}_{[a,b]}$. Fix $\epsilon > 0$ and choose N such that there exists $p, q \in [1, 2^N]$ satisfying $|a - \frac{p}{2^N}| < \epsilon$ and $|b - \frac{q}{2^N}| < \epsilon$. Because we keep taking refinements, we can for all $n \geq N$ find integers $p_n, q_n \in [1, 2^n]$ which satisfy the same inequality. We claim that this implies for all $n \geq N$ it holds that

$$\int_{A_n} f = m([a,b] \cap A_n) \geq \frac{1}{2} m([a,b]) - 2\epsilon - 2^{-n+1}.$$

This is true, since for $n \geq N$ we know that at worst the intervals $[\frac{p_n - 1}{2^n}, \frac{p_n}{2^n}]$ and $[\frac{q_n}{2^n}, \frac{q_n+1}{2^n}]$ have empty intersection with $[a,b]$ and each interval has distance at most ϵ away from $[a,b]$. The remaining intervals of $A_n \cap [a,b]$ then account for half of the measure of the interval $\left[\frac{p_n+1}{2^n}, \frac{q_n-2}{2^n}\right]$. We can extend the inequality to arbitrary step functions $\varphi = \sum_1^M c_j \mathbb{1}_{[a_j,b_j]}$ by fixing $\epsilon > 0$ and choosing N_j so that for $n \geq N_j$ we have

$$m([a_j, b_j] \cap A_n) \geq \frac{1}{2} m([a_j, b_j]) - 2\frac{\epsilon}{M} - 2^{-n+1}.$$

Then take $N = \max\{N_j\}$ and sum to conclude that for any step function $\lim \int_{A_n} \varphi = \frac{1}{2} \int \varphi$.

Next, let $f \in L^1$ be such that $f \geq 0$. Fix $\epsilon > 0$ and take $\varphi \geq 0$ a step function such that $\int |f - \varphi| < \epsilon$. It follows that

$$\int_{A_n} f \leq \int_{A_n} |f - \varphi| + \int_{A_n} \varphi = \epsilon + \frac{1}{2} \int \varphi$$

□

6 1994

(1994 #2) Prove or disprove each of the following statements.

(i) If $f \colon \mathbb{R} \to \mathbb{R}$ and $g \colon \mathbb{R} \to \mathbb{R}$ are Borel measurable functions then the composition $f \circ g$ is also a Borel measurable funciton.

(ii) If neither of the functions $f \colon \mathbb{R} \to \mathbb{R}$ and $g \colon \mathbb{R} \to \mathbb{R}$ is Borel measurable then $f \circ g$ is not Borel.

Proof.
(i) By definition. (ii) let $A, B \subseteq \mathbb{R} \setminus \{0, 1\}$ be disjoint nonmeasurable sets. The functions $\mathbb{1}_A$ and $\mathbb{1}_B$ have the property that for any set $C \in \mathbb{R}$ it holds that $(\mathbb{1}_A \circ \mathbb{1}_B)^{-1}(C) = \{0\}$. □

(1994 #3) Let X and Y be Banach spaces over the real field and let $T \colon X \to Y$ be a linear operator.

(i) Define what it means for the graph of T to be closed and state the Closed Graph Theorem.

(ii) Suppose Z is also a real Banach space and that $\mathcal{F}$ is a family of continuous linear mappings $F \colon Y \to Z$ with the following property

$$F(y) = 0 \qquad \forall F \in \mathcal{F} \qquad \implies y = 0.$$

Show that if $F \circ T$ is continuous for each F then T is continuous.

Proof. (i) The graph $\Gamma_T = \{(x, T(x)) \colon x \in X\}$ is closed if it is closed in the product topology on $X \times Y$. The Closed Mapping Theorem states that if Γ_T is closed then T is continuous.

(ii) Let $U_F \colon X \times Y \to Z \colon (x, y) \mapsto F(T(x)) - F(y)$. Notice that U_F is continuous by hypothesis. Since $\{0\}$ is a closed set it follows that $Z_F = U_F^{-1}(\{0\})$ is closed. We can define an intersection then apply the hypotheses

$$\begin{aligned}\bigcap_{F\in\mathcal{F}} Z_f &= \bigcap_{F\in\mathcal{F}} \{(x,y) \colon F(T(x)) - F(y) = 0\}\\ &\overset{*}{=} \bigcap_{F\in\mathcal{F}} \{(x,y) \colon F(T(x) - y) = 0\}\\ &\overset{**}{=} \{(x,y) \colon y = T(x)\\ &= \Gamma_T.\end{aligned}$$

Where the equality at * follows from linearity of F and the equality at ** follows from the additional hypothesis on $\mathcal{F}$. Lastly, since the sets Z_f are closed we conclude that Γ_T is closed and therefore T is continous. □

(1994 #4) Consider the space $C[-1, 1]$ of continuous real-valued functions on the interval $[-1, 1]$ with the supremum norm $\|f\|_\infty$. Let $\{h_n\}$ be a bounded sequence of functions in $C[-1, 1]$. Prove that $L(h_n) \to 0$ for each L on $C[-1, 1]$ if and only if $h_n(x) \to 0$ for each $x \in [-1, 1]$.

Proof. Let $\sup_n \|h_n\|_\infty < M$.

($\Rightarrow$) Suppose for all L in the dual it holds that $L(h_n) \to 0$. Since $C[-1, 1]^*$ is the set of finite measures on $[-1, 1]$ we can write each $L(h_n) = \int_{-1}^1 h_n d\mu$ for some finite measure μ. In particular for each $x \in [-1, 1]$ we can consider $\mu_x = \mathbb{1}_{\{x\}}$ a point mass measure and we have

$$\int -1^1 h_n d\mu_x = h(x) \to 0.$$

($\Leftarrow$) Suppose $h_n \to 0$ pointwise. Let $L \in C[-1, 1]^*$ and let μ be the corresponding measure. We can apply DCT by dominating $|h_n|$ with M (which is integrable) and conclude

$$L(h_n) = \int |h_n| d\mu \to 0.$$

□

(1994 #5) Let $C[-1, 1]$ denote the Banach space defined in the previous problem and let S denote the subset $\{x^{3n}\}$. Describe the smallest subalgebra which contains S.

Proof. $S = \{f \in C[-1, 1] \colon f(0) = 0\}$. This follows from Stone-Weirstrauss since S separates points and is clearly not equal to all of $C[-1, 1]$ since, for instance, there does not exist a function which maps $0 \mapsto 1$. □

(1994 #6) Suppose that $f_n \colon \mathbb{R} \to [0, 1]$ is a sequence of nondecreasing functions which on $q \in \mathbb{Q}$ satisfy $\lim_{n\to\infty} f_n(q)$ exists. Prove that $\lim_{n\to\infty} f_n(y)$ exists for all $y \in \mathbb{R}$ except possible a countable number of y.

Proof. Let $\overline{f}(y) = \limsup f_n(y)$ and let $\underline{f}(y) = \liminf f_n(y)$. Notice that both $\overline{f}$ and $\underline{f}$ are monotonic since for $x < y$ we have $f_n(x) \leq f_n(y)$ and so the inequality holds in the limit. It is easy to prove that both $\overline{f}, \underline{f}$ have countably many discontinuities (since each jump must contain a rational number). Let $X \subseteq \mathbb{R}$ be the set of points at which both $\overline{f}$ and $\underline{f}$ are continuous. The function $\overline{f} - \underline{f}$ is continuous on X and identically 0 on $\mathbb{Q}$. As $\mathbb{Q}$ is dense it follows that $\overline{f} - \underline{f} \equiv 0$ on X.

□

(1994 #7) Let $\varphi(a, y) = \int_0^y e^{-ax^2} dx$ for $a, y > 0$. Prove that $\lim_{y\to\infty} \frac{\frac{\partial}{\partial \alpha}\varphi(a,y)}{\varphi(a,y)} = -\frac{1}{2a}$.

Proof. We would like to differentiate under the integral. Let $f(a, x) = e^{-ax^2}$. We need (i) $f(a, x)$ to be measurable and integrable for each a (ii) $\frac{\partial}{\partial a} f(a, x)$ is continuous in a and (iii) $|\frac{\partial}{\partial a} f(a, x)| \leq \Phi(x)$ for all a, x for some integrable function Φ. These three properties are easy to verify. Now we can differentiate and write

$$\frac{\partial}{\partial a}\varphi(a, y) = \int_0^y -x^2 e^{-ax^2} dx = \frac{-x}{2a} e^{-ax^2}\Big|_0^y - \frac{1}{2a}\int_0^y e^{-ax^2} dx.$$

Where we use integration by parts with $u = -x$ and $du = xe^{-ax^2} dx$. Taking $y \to \infty$ in the quotient by $\varphi(a, y)$ gives the desired result.

□

7 1995

(1995 #1) Derive Fatou's lemma from the MCT.

Proof. Let f_n be a collection of nonnegative integrable functions. We can write $\liminf f_n = \lim_{k\to\infty}(\inf_{n\geq k} f_n)$. The functions $\inf_{n\geq k} f_n$ are increasing and nonnegative and so by MCT

$$\lim_{k\to\infty} \int \inf_{n\geq k} f_n = \int \liminf f_n.$$

Since $\inf_{n\geq k} f_n \leq f_k$ we can write $\int \inf_{n\geq k} f_n \leq \int f_k$ and take $\liminf_{k\to\infty}$ of both sides to write

$$\liminf \int \inf_{n\geq k} f_n \leq \liminf \int f_k.$$

And we just proved that the righthand limit exists and is therefore equal to $\int \liminf_{n\geq k} f_n$. $\square$

8 1996

(1996 #1) Evaluate the following limits: (a) $\lim \int_0^\infty \frac{n\sin(x/n)}{1+x^4} dx$ and (b) $\lim \int_0^\infty \frac{n\cos^2(x/n)}{n+x^4} dx$.

Proof. (a) We can make the bound $\sin(x/n) \leq \frac{x}{n}$ and therefore $n\sin(x/n) \leq x$. This gives $\frac{n|\sin(x/n)|}{1+x^4} \leq \frac{|x|}{1+x^4}$. Which is integrable. It then follows from DCT that the limit is 0.

(b) We can write $\cos^2(x/n) = 1 - \sin^2(x/n)$. So we split up the integral and first consider function $\frac{n}{n+x^4}$.

□

9 1999

(1999 #4) Let (X, μ) be a measure space and let $1 \leq p < \infty$. Suppose that $\{f_n\}$ is a bounded sequence in L^p and that $f_n \to f$ a.e.

(a) Show that $f \in L^p$.

(b) Give an example where f_n does not converge to f in L^p.

(c) Show that if $|f_n| < |f|$ on X for every n then $f_n \to f$ in L^p.

Part (a)

Proof. As f_n is bounded in L^p we know that $\liminf \int |f_n|^p < \infty$. Applying Fatou's lemma we have

$$\int |f|^p = \int \liminf |f_n|^p \leq \liminf \int |f_n|^p < \infty.$$

□

Part (b)

Proof. Consider the functions $f_n = \mathbb{1}_{[n,n+1]}$. We have $\|f_n\|_1 = 1$ and $f_n \to 0$ pointwise. However, $\int |f_n - 0| = 1$ does not go to zero.

□

Part (c)

Proof. If $|f_n| < |f|$ we dominate $|f_n - f|^p$ and apply DCT since $|f_n - f|^p \leq (|f_n| + |f|)^p \leq 2^p |f|^p$ which is integrable by part (a).

□

(1999 #5) Let X be a real Banach space with dual X^*. A linear subspace $V \subseteq X^*$ is called total if it separates points of X (i.e. $x \neq y \in X$ implies there exists $\phi \in V$ such that $\phi(x) \neq \phi(y)$.

(a) Suppose that there exists a total subspace of X^* which is proper and closed. Show that X cannot be reflexive.

(b) Show that the space $X = L^1[0,1]$ is total in $C([0,1],\mathbb{R})^*$ where for $f \in L^1$ and $g \in C[0,1]$ we define $f(g) = \int_0^1 f(x)g(x)dx$.

Part (a)

Proof. Let $V \subseteq X^*$ be a proper closed total subspace. To show a contradiction suppose that X is reflective. Since V is proper let $\ell^* \in X^* \setminus V$. Since V is closed, the Hahn Banach theorem guarantees that there exists $\hat{x} \in X^{**}$ such that for $l \in V$ it holds that $\hat{x}(l) \equiv 0$ and for $l \in X^* \setminus V$ it holds that $\|\hat{x}\| > 0$ and also $\|\hat{x}(\ell^*)\| = 1$. Note that since x is reflexive we can write $\hat{x}(l) = l(x)$ for some $x \in X$. The above implies that for all $l \in V$ it holds that

$$0 = \hat{x}(l) = l(x).$$

Since $x \neq 0$ and $l(0) = 0$ for all $linV^*$ we obtain the contradiction that V is not total. □

Part (b)

Proof. Let $h \neq g$ in $C([0,1])$. Suppose WLOG that for some x_0 it holds that $g(x_0) < h(x_0)$. Since the function $h(x) - g(x)$ is continuous we conclude there exists $\epsilon > 0$ such that for $y \in I_\epsilon = (x_0 - \epsilon, x_0 + \epsilon)$ it holds that $g(y) < h(y)$. Let $f(x) = \mathbb{1}_{I_\epsilon}(x)$. It follows that

$$f(g) = \int_{I\epsilon} g(x)dx < \int_{I\epsilon} h(x)dx = f(h).$$

Therefore L^1 separates points.

□

10 2000

(2000 #1) Let X be a Banach space. Prove that if X^{**} is reflexive then X is reflexive.

Proof. Since $X \hookrightarrow X^{**}$ is a closed subspace, it suffices to show that given a reflexive Banach space Y and $M \subseteq Y$ a closed subspace then M is reflexive.

Since $M \hookrightarrow M^{**}$ via the evaluation map, it suffices to prove that the evaluation map is surjective. Fix $F \in M^{**}$. Define the functional $G \in Y^{**}$ on $\ell \in Y^*$ as

$$G(\ell) = F(\ell|_M).$$

Because Y is reflexive we know that G corresponds to some $y \in Y$ via

$$\ell(y) = G(\ell) = F(\ell|_M). \tag{1}$$

We claim that $y \in M$. To see this, we proceed by contradiction and suppose that $y \notin M$. The Hahn-Banach theorem implies that there exists $\ell \in Y^*$ such that $\ell|_M \equiv 0$ and $\ell(y) = 1$. However, (1) implies the contradiction $\ell(y) = F(\ell|_M) = 0$.

We now have $y \in M$. It is a consequence of Hahn-Banach that any $g \in M^*$ can be written as $g = \ell|_M$ for some $\ell \in X^*$. So we must have $F(g) = g(y)$ for all $g \in M^*$, which proves subjectivity. □

(2000 #5) Let $\{f_n\}$ be a sequence of non-negative functions defined on a measure space (X, μ) such that for every $n \geq 1$,

$$\int_X f_n d\mu \leq 1.$$

Prove that $\limsup_{n\to\infty}(f_n(x))^{1/n} \leq 1$ for μ-a.e. $x \in X$.

Proof. Let $E_n(k) = \{x : f_n(x) > (1 + \frac{1}{k})^n\}$. Chebyshev's inequality tells us that

$$\mu(E_n(k)) \leq \frac{1}{(1+\frac{1}{k})^n} \|f_n\|_1 \leq \frac{1}{(1+\frac{1}{k})^n}.$$

Let $E^k = \bigcap_{m=1}^\infty \bigcup_{n=m}^\infty E_n(k)$. Notice that

$$\mu\left(\bigcup_{n=1}^\infty E_n(k)\right) = \sum_{n=1}^\infty \mu(E_n(k)) \leq \sum_{n=1}^\infty \frac{1}{(1+\frac{1}{k})^n} < \infty.$$

Therefore $\mu(E^k) = 0$. It follows that for $E := \bigcup_{k=1}^\infty E^k$ has $\mu(E) = 0$. Unraveling the unions we see that if $x \in E^c$ then for all k there exists and N_k such that for $n > N_k$ it holds that $f(x) \leq (1 + \frac{1}{k})^n$. It follows that for $x \in E^c$ we have

$$\limsup_{n\to\infty} f(x)^{1/n} \leq 1.$$

□

(2000 #6) Show, with justification of each step, that

$$\int_0^1 \left(\sum_{k=1}^\infty x^k \frac{\cos(2^k \pi x)}{k}\right) dx = \sum_{k=1}^\infty \left(\int_0^1 x^k \frac{\cos(2^k\pi x)}{k}\right) dx.$$

Proof. Let $P_n(x) := \sum_{k=1}^n x^k \frac{|\cos(2^k\pi x)|}{k}$. Since the P_n are monotonically increasing we can apply the monotone convergence theorem at * and obtain

$$\begin{aligned}
\int_0^1 \sum_{k=1}^\infty x^k \frac{|\cos(2^k\pi x)|}{k} dx &= \int_0^1 \lim_{n\to\infty} P_n(x) dx \\
&\overset{*}{=} \lim_{n\to\infty} \int_0^1 P_n(x) dx \\
&= \lim_{n\to\infty} \sum_{k=1}^n \int_0^1 x^k \frac{|\cos(2^k\pi x)|}{k} \\
&= \sum_{k=1}^\infty \int_0^1 x^k \frac{|\cos(2^k\pi x)|}{k}.
\end{aligned}$$

Since $\int_0^1 x^k \frac{|\cos(2^k\pi x)|}{k} dx \le \int_0^1 \frac{x^k}{k} = \frac{1}{k(k+1)}$, we conclude that

$$\sum_{k=1}^{\infty} \int_0^1 x^k \frac{|\cos(2^k\pi x)|}{k} \le \sum_{k=1}^{\infty} \frac{1}{k(k+1)} < \infty.$$

Therefore

$$\int_0^1 \sum_{k=1}^{\infty} x^k \frac{|\cos(2^k\pi x)|}{k} < \infty,$$

and we must have $P = \sum_{k=1}^{\infty} x^k \frac{|\cos(2^k\pi x)|}{k}$ belongs to $L^1([0,1])$.

Next, we let $S_n := \sum_{k=1}^{n} x^k \frac{\cos(2^k\pi x))}{k}$. Notice that for each n we have for *a.e.* x it holds that $|S_n(x)| \le |P(x)|$. By the dominated convergence theorem we conclude that

$$\begin{aligned}\int_0^1 \sum_{k=1}^{\infty} x^k \frac{|\cos(2^k\pi x)|}{k} dx &= \int_0^1 \lim_{n\to\infty} S_n(x)dx \\ &\overset{**}{=} \lim_{n\to\infty} \int_0^1 S_n(x)dx \\ &= \sum_{n=1}^{\infty} \int_0^1 x^k \frac{|\cos(2^k\pi x)|}{k} dx.\end{aligned}$$

Where at $**$ we apply DCT. □

(2000 #8) Let $1 \le p < \infty$ and fix $f \in L^p(\mathbb{R})$ such that $f(x) = 0$ for $|x| \ge 1$. For $g \in L^q(\mathbb{R})$ where $\frac{1}{p} + \frac{1}{q} = 1$ let $f * g$ denote the convolution of f and g. Let $\mathcal{F}$ be the following family of functions

$$\mathcal{F} = \{f * g : g \in L^q(\mathbb{R}), \|g\|_q \le 1, \text{ and } g(x) = 0 \text{ for } |x| \ge 1\}.$$

(a) Show that $\mathcal{F} \subseteq C([-2,2])$.

(b) Show that the set $\mathcal{F}$ is precompact in $C([-2,2])$ with the uniform norm.

Part (a)

Proof. Let $f * g \in \mathcal{F}$. We know that $\operatorname{supp} f * g \subseteq [-2,2]$ since we proved in class that $\operatorname{supp}(f * g) \subseteq \operatorname{supp} f + \operatorname{supp} g = [-1,1] + [-1,2] = [-2,2]$. We claim that $f * g$ is actually uniformly continuous. Letting τ_y denote translation by y observe that

$$\|\tau_y(f) * g - f * g\|_u = \|(\tau_y f - f) * g\|_u \le \|(\tau_y f - f\|_p \|g\|_q \le \|\tau_y f - f\|_p \to 0.$$

□

Part (b)

Proof. We proved in Part (a) that $\mathcal{F}$ is equicontinuous (since our convergence was in the uniform norm and independent of our choice of g). We would like to apply Arzela-Ascoli to $\mathcal{F}$. So it remains to prove that $\mathcal{F}$ is pointwise bounded. Fix $x \in [-2,2]$ and let $f * g \in \mathcal{F}$. We have

$$|f * g(x)| = \left|\int_{-1}^1 f(x-y)g(y)dy\right| \le \int_{-1}^1 |f(x-y)||g(y)|dy \le \|f\|_q \|g\|_q \le \|f\|_q < \infty.$$

This says precisely that $\mathcal{F}$ is pointwise bounded. $\square$

11 2001

(2001 #1) Let X be a Banach space.

(a) Prove that if X is infinite dimensional then there are infinitely many elements e_k in X that are linearly independent in the following sense: if $\sum_{k=1}^{n} \lambda_k e_k$ converges in X to 0 then $\lambda_k = 0$ for every $k \geq 1$.

(b) Prove that X is finite dimensional if and only if every linear subspace of X is closed.

Part (a)

Proof. Suppose there are only finitely many elements $e_1, \ldots, e_m$ which are linearly independent. This implies that for any $x \notin \operatorname{Span} e_1, \ldots, e_m$ we can write $x = \sum_1^m \lambda_k e_k$ for some λ_k, which is the definition of being finite dimensional.

□

Part (b)

Proof. ($\Rightarrow$) Suppose that X is finite dimensional. Let $W \subseteq X$ be a linear subspace and let $w_n \to x$. We claim that $x \in W$. To see this notice that w_n is convergent and therefore Cauchy. Since finite dimensional normed spaces are complete we conclude that $w_n \to w$ with $w \in W$. By uniqueness of limits we then have $w = x$ and so $x \in W$.

($\Leftarrow$) Suppose that every linear subspace is closed and to show a contradiction that X is infinite dimensional. Let $\{e_k\}$ be a collection of linearly independent elements. Let $E_k = \operatorname{Span}\{e_1, \ldots, e_k\}$ be closed subspaces. The space $E = \operatorname{Span}\{e_1, \ldots\}$ is a linear subspace and therefore closed. This is a contradiction since we can write $E = \bigcup E_k$ as a countable union of closed sets, we conclude that one of the E_k must have nonempty interior. A contradiction since balls have infinite dimension. □

(2001 #2) In $L^2[a,b]$ let $S = \{e^{i2\pi nx} \colon n \in \mathbb{Z}\}$. Show that

(a) If $|b-a| < 1$ then the orthogonal complement space of S in $L^2[a,b]$ is $\{0\}$.

(b) If $|b-a| > 1$ then the orthogonal complement space of S in $l^2[a,b]$ is strictly larger than 0.

Part (a)

Proof. If $|b-a| \leq 1$ then we have S is a maximal orthogonal set and therefore a basis for the Hilbert space $L^2[a,b]$. It is clear that S is orthogonal. To see that it is maximal note that

□

Part (b)

Proof. If $|b-a| = 1 + \epsilon$ then the function which is

□

(2001 #3) Let (X, d) be a metric space and $K \subseteq X$ compact. Let $\{G_i\}$ be an open cover of K. Show that there exists $\epsilon > 0$ such that every subset $S \subseteq K$ of diameter less than ϵ is contained in one of the sets G_i (i.e. that there exists $i = i(S)$ such that $S \subseteq G_i$.

Proof. For each $x \in K$ let r_x be such that $B_x = \mathbb{B}(x, 2r_x) \subseteq G_i$ for some G_i. Reduce the cover $\bigcup_x B_x$ to a finite subcover $\{B_1, \ldots, B_m\}$. Let $\epsilon = \frac{1}{2}\min_{1 \leq j \leq m} r_j$. Now, for any $y \in K$ we claim the ball $\mathbb{B}(y, \epsilon)$ is contained in some G_i. This is true since $y \in B_j$ for some j and therefore □

(2001 #4) Suppose $b > a$ are two real numbers a $f \in L^1([a, b])$. Show that

$$\lim_{n\to\infty} \frac{\pi}{2} \int_a^b f(x)|\sin(nx)|dx = \int_a^b f(x)dx.$$

Proof. First we prove the special case for when $f(x) \equiv M$ on $[a, b]$. First notice that for any positive integers n, k we have

$$\int_{an}^{an+k2\pi} |\sin u|du = k\int_0^{2\pi} |\sin u|dy = 2k\int_0^{\pi} |\sin u|du = 4k. \qquad (1).$$

Next, we make a u-substitution and split up our integral to acquiesce to (1).

$$\begin{aligned}
\int_a^b |\sin(nx)|dx &= \frac{1}{n}\int_{an}^{bn} |\sin(u)|du \\
&= \frac{1}{n}\int_{an}^{an+2\pi k} |\sin(u)|du + \frac{1}{n}\int_{an+2\pi k}^{bn} |\sin(u)|du \\
&\overset{*}{=} \frac{4k}{n} + \frac{1}{n}\int_{an+2\pi k}^{bn} |\sin(u)|du.
\end{aligned}$$

If we choose k to be the largest integer so that $bn - (an + 2\pi k) = n(b-a) - 2\pi k < 1$ we have

$$k \to \frac{n(b-a)}{2\pi}, \qquad \text{as } n \to \infty. \qquad (2)$$

Also, notice that this choice of k along with the fact that $|\sin u| \leq 1$ guarantees that (for large n) the integral

$$\int_{an+2\pi k}^{b_n} |\sin(u)|du \leq n(b-a) - 2\pi k.$$

Recall that $f(x) \equiv M$ on $[a, b]$, letting $n \to \infty$ now gives:

$$\begin{aligned}
\lim_{n\to\infty} M\frac{\pi}{2}\int_a^b \sin(nx) &= M(\pi/2)\lim_{n\to\infty} \frac{4k}{n} + \frac{1}{n}\int_{an+2\pi k}^{bn} |\sin u|du \\
&= M(\pi/2)\frac{4(b-a)}{2\pi} \\
&= M(b-a) \\
&= \int_a^b f(x)dx.
\end{aligned}$$

Consider a step function on $[a,b]$ given by $\varphi(x) = \sum_1^N c_k \mathbb{1}_{[a_k,b_k]}(x)$ with $a \leq a_1 \leq b_1 \leq a_2 \leq \cdots \leq b$. It follows from our previous work that

$$\lim_{n\to\infty} (\pi/2) \int_a^b \varphi(x)|\sin(nx)|dx = \sum_{k=1}^N \lim_{n\to\infty} (\pi/2) \int_{a_k}^{b_k} c_k|\sin(nx)| = \sum_{k=1}^N \int_{a_k}^{b_k} c_k dx = \int_a^b \varphi(x)dx.$$

We finish by using the fact that the step functions are dense in L^1. Accordingly, let $\varphi_j \nearrow f$. Notice that the terms $\varphi_j|\sin(nx)|$ are dominated by $|f(x)|$, we can apply dominated convergence successively to write

$$\begin{aligned}\int_a^b f(x)dx &= \lim_{j\to\infty} \int_a^b \varphi_j(x)dx \\ &= \lim_{j\to\infty} \left(\lim_{n\to\infty} \frac{\pi}{2} \int_a^b \varphi_j(x)|\sin(nx)|dx \right) \\ &\overset{!}{=} \lim_{n\to\infty} \left(\lim_{j\to\infty} \frac{\pi}{2} \int_a^b \varphi_j(x)|\sin(nx)|dx \right) \\ &= \lim_{n\to\infty} \frac{\pi}{2} \int_a^b f(x)|\sin(nx)|dx.\end{aligned}$$

Note that the exchange of limits at ! is permissible since both limits exist.

□

(2001 #5) Let ℓ^2 be the Hilbert space of square summable real-valued sequences equipped with the Hilbert norm $\|a\|_2^2 = \sum a(i)^2$.

(a) Suppose $\{f_n, n > 1\}$ is a sequence of elements of ℓ^2 and $f_n \to 0$ weakly as $n \to \infty$. Show that $\sup \|f_n\|_2 < \infty$.

(b) Define elements $g_{nm}(1 \leq m < n < \infty)$ in ℓ^2 by $g_{nm}(k) = 0$ if $k \neq n$ and $k \neq m$ and $g_{nm}(m) = 1, g_{nm}(n) = m$. Show that 0 is in the weak closure $S = \{g_{nm}\}$ but that no sequence of elements converges to 0 in ℓ^2.

Part (a)

Proof. We prove the following, more general, statement: If X is a Banach space and $x_n \rightharpoonup x$ then $\sup \|x_n\| < \infty$. Let $\hat{x}_n$ be the image of $X \hookrightarrow X^{**}$, since $\|\hat{x}_n\| = \|x_n\|$ it suffices to prove the result for $\hat{x}_n$. For any $l \in X^*$ we have

$$\hat{x}_n(l) = l(x_n) \to l(x).$$

Therefore, $\sup_{\hat{x}_n} \hat{x}_n(l) < \infty$. It follows from the uniform boundedness principle that $\sup \|\hat{x}_n\| < \infty$. □

Part (b)

Proof. For $m \geq 2$, consider the sequence $h_m = g_{m-1,m}$. Notice that

$$h_m(k) = \begin{cases} 0, & k \neq m-1, k \neq m \\ 1, & k = m-1 \\ m, & k = m \end{cases}.$$

As $(\ell^2)^* = \ell^2$ we need to show that for any $a \in \ell^2$ it holds that $a(h_m) = \sum a(k)h_m(k) = a(m-1) + m \cdot a(m) \to 0$. This is straightforward since $a \in \ell^2$ implies by comparison to the harmonic series that $m \cdot a(m) \to 0$ as $m \to \infty$.

□

(2001 #6) Let $X = C([0,1])$ equipped with the uniform norm. Let K be a real-valued continuous function on $[0,1] \times [0,1]$. Define $T \colon X \to X$ by

$$(Tf)(x) = \int_0^1 K(x,y)f(y)dy.$$

(a) Prove that T has 1-dimensional range if and only if $K(x,y) = \phi(x)\psi(y)$ for some $\phi, \psi \in X$.

(b) Prove that T has finite dimensional range if and only if there exists an integer r and functions $u_i, v_i \in X$ such that

$$K(x,y) = \sum_1^r u_i(x)v_j(y).$$

(c) Prove that for any $\epsilon > 0$ there exists a continuous linear map $U \colon X \to X$ with finite range such that $\|T - U\|_X < \epsilon$. Here $\|\cdot\|_X$ denote the operator norm.

Part (a)

Part (b)

Proof. ($\Rightarrow$) Let $\{h_1, \ldots, h_d\}$ be an orthonormal (wrt to the Hilbert space $L^2([0,1])$) basis of continuous functions for Range(T). Complete this to an orthonormal basis $\{h_{d+1}, \ldots\}$ of $L^2([0,1])$ with each h_k continuous. Since the range is finite dimensional, we can write each $Th_n = \sum_1^d a_i^n h_i$. Since $K \in L^2([0,1] \times [0,1])$ and $\{h_m(x)h_n(y)\}$ is a basis we can compute the coefficients

$$\begin{aligned}(K, h_m h_n) &= \int_0^1 \int_0^1 K(x,y)h_m(x)h_n(y)dydx \\ &= \int_0^1 (Th_n)(x)h_m(x)dx \\ &= \int_0^1 \sum_1^d a_i^n h_i(x)h_m(x)dx \\ &= a_i^n \delta_i^m.\end{aligned}$$

Therefore $(K, h_m h_n) \neq 0$ if and only if $m, n \leq d$. Which implies

$$K(x,y) = \sum_{m,n\leq d}^{d} (K, h_m h_n) h_m(x) h_n(y).$$

($\Leftarrow$) Easy. □

Part (b)

Proof. □

Part (c)

Proof. □

(2001 #8) Let $f \in L^1([0,1])$ and suppose that for all $n \geq 1$ we have

$$\int_0^1 f^n = \int_0^1 f.$$

Show that f is the indicator function of some lebesgue measurable set $A \subseteq [0,1]$.

Proof. Let $A = \{f = 1\}$. Let E be a Lebesgue measurable set. First note that $f \leq 1$ since otherwise there would be a positive measure set on which $f \geq 1$ and therefore $f^n \to \infty$ on this set, which contradicts integrability. We can write

$$\begin{aligned}\int_E f &= \int_0^1 f^n - \int_{E^c} f \\ &= \int_{A\cap E} + \int_{A\cap E^c} f^n + \int_{A^c\cap E} f^n - \int_{A\cap E^c} f - \int_{A^c\cap E} f \\ &= m(A\cap E) + m(A\cap E^c) + 0 - m(A\cap E^c) - 0 \\ &= m(A\cap E).\end{aligned}$$

Where we use dominated convergence since on A^c we know that $f^n \to 0$ and is dominated by 1. This says that $\int_E f = m(A\cap E)$ for all E. Since this holds for $\mathbb{1}_A$ we use uniqueness of the Riesz representation theorem to conclude $f = \mathbb{1}_A$.

□

12 2002

(2002 #1) (a) Prove that there exists a closed nowhere dese set $A \subseteq [0,1]$ such that $m(A) > 0$
(b) Prove that there exists a Borel set $B \subseteq [0,1]$ such that for all $a, b \in [0,1]$ with $a < b$ we have $m(B \cap [a,b]) > 0$ and $m(B^c \cap [a,b]) > 0$.

Part (a)

Proof. Delete intervals of length $\alpha 3^{-k}$ with $\alpha \in (0,1)$ in a Cantor-like fashion. □

(2002 #2) Suppose $1 < p, q < \infty$ and $p^{-1} + q^{-1} = 1$. Let $f_n \in L^p(\mathbb{R})$ converge in $L^p(\mathbb{R})$ to f and let $g_n \rightharpoonup g$ in $L^q(\mathbb{R})$ with the $g_n \in L^q(\mathbb{R})$. Prove that $(f_n, g_n) \to (f, g)$.

Proof. Since $g_n \rightharpoonup g$ we know that $\sup_n \|g_n\|_q = M < \infty$. Fix $\epsilon > 0$ and let N large enough so that for $n > N$ we have $\|f_n - f\|_p < \frac{\epsilon}{2M}$ and $\left|\int f g_n - \int f g\right| < \epsilon/2$. We can now write

$$\begin{aligned}
\left|\int f_n g_n - fg\right| &= \left|\int f_n g_n - f g_n + f g_n - fg\right| \\
&\leq \left|\int f_n g_n - f g_n\right| + \left|\int f g_n - fg\right| \\
&= \int |f_n - f||g_n| + \left|\int f g_n - fg\right| \\
&\leq \|f_n - f\|_p \|g_n\|_q + \epsilon/2 \\
&\leq \epsilon.
\end{aligned}$$

□

(2002 #3)

(a) Prove that if $\{f_k\}$ is an equicontinuous sequence of real valued functions on $[0,1]$ with $f_k \to f$ in $L^1([0,1])$ then $f_k \to f$ uniformly.

(b) For a continuous functions $f : [0,1] \to \mathbb{R}$ let $f_1(x) = f(x)$ and

$$f_k(x) = |f_{k-1}(x)| - \|f_{k-1}\|_1, \qquad k \geq 2.$$

Prove that for every such f the sequence f_k converges uniformly and find the limit. (*Hint: One proof begins by comparing the L^2 norm of f_k with that of f_{k-1}.*)

Part (a)

Proof. First we claim that the $\{f_n\}$ are pointwise bounded. Suppose not, and so there exists $x_0 \in [0,1]$ with $\sup |f_n(x_0)| = \infty$. Choose a subsequence such that $|f_{n_k}(x_0)| > k$. Next, fix $\epsilon = 1$ and choose $\delta > 0$ such that $|f_{n_k}(x_0) - f_{n_k}(y)| < 1$ when $y \in \mathbb{B}(x_0, \delta)$. It follows that for all $y \in \mathbb{B}(x_0, \delta)$ it holds that $|f_{n_k}(y)| \geq k - 1$. We then have

$$\int_{\mathbb{B}(x_0,\delta)} |f_{n_k}(x)|dx \geq k\delta \to \infty.$$

Which contradicts that $f_{n_k} \to f$ in L^1. Therefore the family $\{f_n\}$ is pointwise bounded and equicontinuous and so by Arzela-Ascoli is normal.

To see that the f_{n_k} converge normally we fix a subsequence f_{n_k} and will show that there exists a further subsequence which converges uniformly to f. As $f_{n_k} \overset{L^1}{\to} f$ we know there is a subsequence of f_{n_k} which converges uniformly to some g. Again because $f_n \overset{L^1}{\to} f$ we know that if the f_{n_k} converge pointwise they must converge to the □

Part (b)

Proof. First we claim that the family $\{f_k\}$ is equicontinuous. Fix $\epsilon > 0$ and $x \in [0, 1]$. Let U be such that for $y \in U$ we have $|f(x) - f(y)| < \epsilon$. We prove, by induction, that for any $k \geq 1$ and $y \in U$ it holds that $|f_k(x) - f_k(y)| < \epsilon$. The base case is clearly satisfied. Suppose the $|f_k(x) - f_k(y)| < \epsilon$. Using the definition of f_{k+1} we can write

$$|f_{k+1}(x) - f_{k+1}(y)| = ||f_k(x)| - |f_k(y)|| \leq |f_k(x) - f_k(y)| < \epsilon.$$

Next, we claim that $f_n \overset{L^1}{\to} 0$. This is easy to see since for any $k \geq 2$ it holds that

$$\|f_k\| = \int |f_k| = \int |f_{k-1}(x)| - \|f_{k-1}\|_1 \, dx = \|f_{k-1}\| - \|f_{k-1}\| = 0.$$

It follows that $f_n \to 0$ uniformly. □

(2002 #4) Suppose $g : \mathbb{R} \to \mathbb{R}$ is a measurable function such that

$$\int |1 + |x|||g(x)|dx < \infty,$$

and let $F(t) = \int_{\mathbb{R}} g(x) \sin(tx)dx$. Show that F is differentiable and that

$$F'(t) = \int_{\mathbb{R}} xg(x) \cos(tx)dx.$$

Lemma 2. *Let $f : \mathbb{R} \to \mathbb{R} \to \mathbb{R} : (t, x) \mapsto f(t, x)$ be a measurable functions satisfying*

(i) $f(t_0, x)$ is measurable and integrable for each $t_0 \in \mathbb{R}$.

(ii) $f_t(t, x) := \frac{\partial}{\partial t} f(t, x)$ is continuous in t.

(iii) There exists an integrable function $\Phi(x)$ such that $f_t(t, x) \leq \Phi(x)$ for all x and t.

Then $F(t) = \int f(t, x)dx$ is differentiable with $F'(t) = \int f_t(t, x)dx$.

Proof. Let $h_n \to 0$ and fix $t_0 \in \mathbb{R}$. We have

$$\frac{F(t_0 + h_n) - F(t_0)}{h_n} = \int \frac{f(t_0 + h_n, x) - f(t_0, x)}{h_n} dx.$$

Since f_t exists we apply the mean value theorem and conclude for each n there exists $|r_n| < h_n$ such that

$$g_n(t_0, x) := f_t(t_0 + r_n, x) = \frac{F(t_0 + h_n) - F(t_0)}{h_n}.$$

Notice that by (ii) we have $g_n(t_0, x) \to f_t(t_0, x)$ and that $|g_n(t_0, x)| < \Phi(x)$ and therefore is dominated. Using (i) and dominated convergence we obtain

$$F'(t_0) = \lim_{n\to\infty} \frac{F(t_0 + h_n) - F(t_0)}{h_n} = \lim_{n\to\infty} \int g(t_0, x) dx = \int f_t(t_0, x) dx.$$

□

Proof of Problem Statement

Proof. Let $f(t, x) = g(x)\sin(tx)$. We check that $(i) - (iii)$ hold:

(i) As g and $\sin(tx)$ are both measurable functions we know that f is measurable. Moreover, we know that

$$\int |g(x)\sin(tx)| dx \leq \int |g(x)| < \int (1 + |x|)|g(x)| < \infty.$$

Hence $f(t, x)$ is integrable for all $t \in \mathbb{R}$.

(ii) We have $\frac{\partial}{\partial t} f(t, x) = xg(x)\cos(tx)$. Since cos is a continuous function, we conclude that f_t is continuous in t.

(iii) Let $\Phi(x) = (1 + |x|)|g(x)|$. Observe that by hypothesis Φ is integrable and that

$$|g(x)\sin(tx)| \leq |g(x)| \leq \Phi(x).$$

So, we can apply the lemma and conclude the desired result. □

(2002 #5) Let B be a Banach space and $V \subseteq B$ be a finite dimensional subspace. Show that there is a closed subspace $W \subseteq B$ such that $B = V \oplus W$ (i.e. every $x \in B$ can be written uniquely as $x = v + w$).

Proof. First we prove that V is closed. Suppose that $v_n \to u \in B$ with the $v_n \in V$. Let $\{e_1, e_2, \ldots e_n\}$ be a a basis for V. Let $V' = \text{Span}(e_1, \ldots, e_n, u)$. Since V' is a finite dimensional subspace it is clear that $V \subseteq V'$ is a closed subspace. Therefore $u \in V$ and V is closed.

As V is closed we can define the projection $P \colon B \to V : x \mapsto x\mathbb{1}_V(x)$. Note that P is linear and bounded since $\|P\| = \sup_{\|x\|=1} \|P(x)\| = 1$. Note that $\ker P$ is a closed subspace since P is bounded and therefore continuous so that if $x_n \to x$ with $x_n \in \ker P$ we have

$$P(x) = P(\lim x_n) = \lim P(x_n) = 0,$$

and so $x \in \ker P$. Lastly, by defnition we can write $B = V \oplus \ker P$..

□

(2002 #7) If $f : \mathbb{R} \to \mathbb{R}$ is any function and $\delta \in \mathbb{R}$, define $f_\delta : \mathbb{R} \to \mathbb{R}$ by $f_\delta(x) = f(x + \delta)$.

(a) If f is continuous and has compact support, show that $\lim_{\delta \to 0} \|f - f_\delta\|_\infty \to 0$.

(b) If $f \in L^p(\mathbb{R})$ for some p with $1 \le p < \infty$, show that $\lim_{\delta \to 0} \|f - f_\delta\|_p \to 0$.

(c) Prove that if $f \in L^\infty$ and $\lim_{\delta \to 0} \|f - f_\delta\|_\infty \to 0$ then there exists a continuous function $\tilde{f} : \mathbb{R} \to \mathbb{R}$ such that $f = \tilde{f}$ a.e.

Part (a)

Proof. Let $\operatorname{supp} f = K \subseteq \mathbb{R}$ and let $m = \sup\{|x| \in K\}$. Notice that $K \subsetneq [-2m, 2m] := I$. Since f is continuous on the compact set I we know that f is uniformly continuous. Hence for $\epsilon > 0$ let η be such that for all $|\delta| < \eta$ and $x \in I$ it holds that $|f(x) - f(x + \delta)| < \epsilon$. This says that $\|f - f_\delta\|_u \to 0$ as $\delta \to 0$.

□

Part (b)

Proof. First note that if $f \in L^p$ by translation invariance of Lebesgue measure we know that $f_\delta \in L^p$. Since $C_c(\mathbb{R})$ is dense in L^p we fix $\epsilon > 0$ and let $\|f - g\|_p < \epsilon$ and $\|f_\delta - g_\delta\| < \epsilon$ for functions $g, g_\delta \in C_c(\mathbb{R})$. Let $I_m = [-2m, 2m] \supset \operatorname{supp} g \cup \operatorname{supp} g_\delta$ as in part (a). Also, let δ be small enough so that $\|g - g_\delta\|_\infty < \frac{\epsilon}{4m}$. Using the fact that $|h(x)| \le \|h\|_\infty$ a.e., we can write

$$\begin{aligned}
\|f - f_\delta\|_p &\le \|f - g\|_p + \|g - g_\delta\|_p + \|g_\delta - f_\delta\|_p \\
&\le 2\epsilon + \int_I |g - d_\delta| \\
&\le 2\epsilon + 4m \|g - g_\delta\|_\infty \\
&< 3\epsilon.
\end{aligned}$$

□

Part (c)

Proof. Since $f \in L^\infty$ we know f is finite a.e. Let $A = \{x : |f(x)| < \infty\}$. We claim that f is uniformly continuous a.e. This is true since for fixed $\epsilon > 0$ the fact that $\lim_{\delta \to 0} \|f - f_\delta\|_\infty \to 0$ implies there exists η such that for all $|\delta| < \eta$ we have $\mu(\{x \in \mathbb{R} : |f(x) - f(x + \delta)| > \epsilon\}) = 0$. This says that f is uniformly continuous on a set of full measure. Let $B = \{x : |f(x) - f_\delta(x)| > \epsilon\}$ and let $Z = A \cup B$. We finish by appealing to the fact that a given a set $Z \subseteq \mathbb{R}$ with $m(Z) = 0$ it must be the case that Z^c is dense in Z. Therefore, if we let

$$\tilde{f}(x) = \begin{cases} f(x), & x \in \mathbb{R} \setminus Z \\ \lim_{k \to \infty} f(t_k), & x \in Z. \end{cases}$$

For any sequence $\{t_k\} \subseteq \mathbb{R} \setminus Z$ with $t_k \to x$. □

13 2003

(2003 #1) Suppose $\{f_n\}$ is a sequence of real-valued measurable functions on a finite measure space (X, μ). Show that there is a sequence of positive numbers $\{\lambda_n\}$ such that $\lim_{n\to\infty} \lambda_n f_n(x) = 0$ for μ-a.e. $x \in X$.

Hint: It may be useful to consider $g_k(x) = \max_{1\le n\le k} |f_n(x)|$.

Proof. Let g_k be as in the hint. Notice that $\{g_k\}$ is an increasing sequence. Let $A_{k,m} = \{x : g_k(x) \le m\}$. Note that since the f_n are finite valued it follows that the g_k are finite valued.

$$\lambda_n = \left(\sup_{x\in A_n} g_n(x)\right)^{-1}.$$

□

(2003 #2) Prove that in a separable metric space X, every uncountable set S contains a convergent sequence of distinct points (the limit need not lie in S).

Proof. Suppose that S contains no convergent subsequence. Hence there exist $\epsilon > 0$ such that for all $x, y \in S$ it holds that $\|x - y\| > \epsilon$. As X is separable, let $Q \subseteq X$ be a countable set such that $\overline{Q} = X$. For each $q_k \in Q$ let $D_k = \mathbb{D}(q_k, \frac{\epsilon}{2})$. As Q is dense we know that $\bigcup D_k$ is a countable cover. Moreover, at most one element of S belongs to each D_k, contradicting the fact that S is uncountable.

□

(2003 #3) Let V be the set of finite linear combinations of the functions $f_n(x) = x^{2n}$, $n = 1, 2, 3, \dots$. Describe with proof the closure of V in $C([a, b])$ with respect to the uniform norm where $-\infty < a < b < \infty$. (The answer depends on a and b and there are several cases to consider.)

(2003 #4) Let (X, μ) be a σ-finite measure space, f a measurable function on X, $p \in (1, \infty)$ and $p^{-1} + q^{-1} = 1$. Show that $f \in L^p(\mu)$ if and only if $\int |fg| d\mu < \infty$ for every $g \in L^q(\mu)$. (*Hint: for the if direction approximate f by a sequence of functions in $L^p(\mu)$.*)

Proof. $\boxed{\Rightarrow}$ Suppose that $f \in L^p$ and thus $|f| \in L^p(\mu)$. As $L^p(\mu) = (L^q(\mu))^*$ when (X, μ) is σ-finite, this implies that the bounded linear operator $T : L^q(\mu) \to \mathbb{R}$ given by $Tg = \int |f| g d\mu$ has norm $\|T\| = \|f\|_p$. Hence for any $g \in L^q(\mu)$ we have

$$\int |fg| d\mu = T|g| = \|g\|_q T\left(\frac{|g|}{\|g\|_q}\right) \le \|g\|_q \|T\| < \infty.$$

$\boxed{\Leftarrow}$ Suppose that $\int |fg| d\mu < \infty$ for every $g \in L^q$. Write $X = \bigcup X_n$ with each X_n finite measure. Approximate $|f|$ from below by the functions $f_n = \mathbb{1}_{\{|f|\le n\}\cap X_n}$. These converge pointwise to f and therefore for each $g \in L^q$ we have

$$\int |f_n g| \to \int |fg| < \infty.$$

By the uniform boundedness principle we know that the operator $T(g) = \int fg$ has bounded norm with $\|T\| = \|f\|_p$.

□

(2003 #5) Let X be a Banach space with norm $\|\cdot\|$ Suppose $V, W \subseteq X$ are vector subspaces of X such that $X = V \oplus W$, i.e. each $x \in X$ can be expressed uniquely as $x = v + w$. With this notation define $\|x\|' = \|v\| + \|w\|$. It is easy to check that $\|\cdot\|'$ is a norm on X (you don't have to do this). Prove that $\|\cdot\|$ and $\|\cdot\|'$ are equivalent if and only if both V and W are closed.

Proof. $\boxed{\Rightarrow}$ Suppose that the two norms are equivalent, since by the triangle inequality $\|x\| \leq \|x\|'$, this says that there exists a constant $A > 0$ such that

$$\|x\|' \leq A \|x\|, \qquad \forall x \in X. \qquad (1)$$

Suppose that $v_k \to x$ in $\|\cdot\|$ with the $v_k \in V$. By (1) we must also have $\|x - v_k\|' \to 0$ as $k \to \infty$. For each v_k we can write $x - v_k = v + w - v_k = (v - v_k) + w$. Therefore

$$\|x - v_k\|' = \|v - v_k\| + \|w\| \to 0.$$

It follows that $\|w\| = 0$ and thus $x = v \in V$. A similar argument proves that W is closed.

$\boxed{\Leftarrow}$ Suppose that V and W are closed. Since $\|x\| \leq \|x'\|$ we must find a constant A such that $A\|x'\| \leq \|x\|$ for all x. Let P_V and P_W be projection onto V and W respectively (i.e. $P_V(x) = v$). Since $\ker P_V = W$ and $\ker P_W = V$ we know that the kernels are closed and therefore P_V and P_W are continuous. This implies both maps are bounded. So we have for $x = v + w$

$$\|v\| = \|P_V x\| \leq A' \|x\|$$

,

$$\|w\| = \|P_W x\| \leq A'' \|x\|.$$

Putting these lines together yields $\|x\|' = \|v\| + \|w\| \leq (A' + A'') \|x\|$.

□

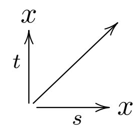
(2003 #6) Suppose that $f \in L^1(0, \infty)$. Define $f_1, f_2, \ldots$ on $(0, \infty)$ by $f_1(x) = \int_0^x f(t)$, $f_2(x) = \int_0^x f_1(t)dt$, ...

(a) Prove that $f_n(x) = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} f(t)dt$ for all $n \geq 1$.

(b) Prove that $\sum_1^\infty f_n(x)$ converges absolutely on $(0, \infty)$ and express it in terms of an integral with no summation.

Part (a)

Proof. We proceed by induction. The case $n = 1$ follows immediately. Suppose the formula holds for f_n. We use Fubini, but have to be careful since we are integrating over the top triangle of the square



$$\begin{aligned} f_{n+1}(x) &= \int_0^x f_n(t)dt \\ &= \int_0^x \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} f(s) ds dt \\ &= \frac{1}{(n-1)!} \int_0^x f(s) \int_s^x (t-s)^{n-1} dt ds \\ &= \frac{1}{(n-1)!} \int_0^x \frac{1}{n}(x-s)^n f(s) ds \\ &= \frac{1}{n!} \int_0^x (x-s)^n f(s) ds. \end{aligned}$$

□

Part (b)

Proof. By part (a) we can write

$$\begin{aligned} \sum_1^\infty |f_n(x)| &= \sum_{n=1}^\infty \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} |f(t)| dt. \\ &= \int_0^x |f(t)| \sum_{n=1}^\infty \frac{(x-t)^{n-1}}{(n-1)!} dt \\ &= \int_0^x |f(t)| e^{x-t} dt \\ &\leq \|f\|_1 \left\| e^{x-t} \right\|_\infty \\ &< \infty. \end{aligned}$$

The previous calculation shows that

$$\sum_1^\infty f_n(x) = \int_0^x f(t) e^{x-t} dt,$$

fairly surprising. □

(2003 # 8) Suppose f is absolutely continuous on $(0, 1)$, and that $f' \in L^p(0, 1)$, with $1 < p \leq \infty$. Let $p^{-1} + q^{-1} = 1$.

(a) Suppose $1 < p < \infty$. Show that

$$\lim_{x \to a} \frac{f(x) - f(a)}{|x - a|^{1/q}} = 0, \qquad \forall a \in (0,1).$$

(b) Suppose $p = \infty$. Show that the conclusion in part (a) is false in general and identify those f for which it is true. Derive a weaker result about the quotient in part (a) that is true for all f with $f' \in L^\infty$.

Part (a)

Proof. Using the fact that $(1/q) - 1 = -1/p$ we can write

$$\begin{aligned}
\lim_{x \to a} \frac{f(x) - f(a)}{|x - a|^{1/q}} &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h^{1/q}} \\
&= \lim_{h \to 0} \left(\frac{1}{h^{(1/q)-1}} \right) \left(\frac{1}{h} \int_a^{a+h} f'(t)dt \right) \\
&= \lim_{h \to 0} \left(h^{1/p} \right) \left(\frac{1}{h} \int_a^{a+h} f'(t)dt \right) \\
&\overset{a.e}{=} \lim_{h \to 0} h^{1/p} f(a) \\
&\to 0.
\end{aligned}$$

Where we apply the Lebesgue differentiation theorem at the line marked *a.e.* □

14 2004

(2004 #1) Let $X = C([0,1])$ be the Banach space using the unifrom norm. (a) State the Arzela-Ascoli crieterion for a closed subset of X to be compact, carefully defining all the terms you use. (b) Prove that the clousre of every nonempty open subset of X is not compact.

Part (a)

Theorem 8. *Arzéla Ascolí Criterion A subset $K \subseteq X$ is compact if and only if K is equicontinuous and K is pointwise bounded.*

Part (b)

Proof. Let $\mathcal{O} \subseteq X$ be an open set and let $\Delta = \mathbb{D}(f, r) \subseteq \mathcal{O}$ with $f \in \mathcal{O}$. Suppose that $|f(a)| = \|f\|_u =: M$ (that is suppose f attains it's supremum at $x = a$). Consider the family of functions $\mathcal{F} := \{f_n\} = \left\{ \frac{f(x)}{M} \cos(n(x-a)) \right\}$. We claim that $\mathcal{F} \subseteq \Delta$. This is true since for $f_n \in \mathcal{F}$ we have

$$\|f_n - f\|_u = \left\| \frac{f(x)}{M} \cos(nx) - f(x) \right\|_u \leq \|f\|_u \|\cos(n(x-a)) - 1\|_u$$

□

(2004 #2) Let $f \in L^1(\mathbb{R}^2)$ be a real-valued integrable function with respect to the usual 2-dimensional Lebesgue measure m. For $(x,y) \in \mathbb{R}^2$ let $D(x,y)$ denote the closed disk of radius 1 centered at (x,y). Define a function g by

$$g(x,y) = \int_{D(x,y)} f dm.$$

(a) Prove that g is continuous on $\mathbb{R}^2$.

(b) Show that $g(x,y) \to 0$ as $(x^2+y^2) \to \infty$.

(c) Prove that there exists at least one point (x,y) for which

$$|g(x,y)| \leq \frac{1}{x^2+y^2+100}.$$

Part (a)

Proof. Fix $\epsilon > 0$ and $(x,y) \in \mathbb{R}^2$. We know from Proposition 1.12 of SSII that the Lebesgue integral of f satisfies the property that for our $\epsilon > 0$ there exists $\delta > 0$ such that for all $E \subseteq \mathbb{R}^2$ with $m(E) < \delta$ it holds that $\int_E |f| < \epsilon/2$. Let δ be as stated and let $\eta > 0$ be such that $m(D(x,y)\Delta D(u,v)) < \delta$ for all $|(x,y)-(u,v)| < \eta$ (where Δ denotes symmetric difference). For convenience let us denote the set $D(x,y)\Delta D(u,v) =: \Omega_{x,y,u,v}$. We have

$$\begin{aligned}
|g(x,y) - g(u,v)| &= \left| \int_{D(x,y)} f dm - \int_{D(u,v))} f dm \right| \\
&= \left| \int_{D(x,y)\setminus D(u,v)} f dm - \int_{D(u,v)\setminus D(x,y)} f dm \right| \\
&\overset{*}{\leq} \left| \int_{D(x,y)\setminus D(u,v)} f dm \right| + \left| \int_{D(u,v)\setminus D(x,y)} f dm \right| \\
&\leq \int_{D(x,y)\setminus D(u,v)} |f| dm + \int_{D(u,v)\setminus D(x,y)} |f| dm \\
&\overset{**}{\leq} \int_{\Omega_{x,y,u,v}} |f| dm + \int_{\Omega_{x,y,u,v}} |f| dm \\
&= 2 \int_{\Omega_{x,y,u,v}} |f| dm \\
&\leq 2(\epsilon/2) \\
&= \epsilon
\end{aligned}$$

Where at * we use the fact that $|a-b| \leq |a| + |b|$ and at ** we use monotonicity of the integral. Thus proving continuity at (x,y).

□

Part (b)

Proof. To show a contradiction suppose that $g(x,y) \not\to 0$ as $|(x,y)| \to \infty$. Thus, there exists $\epsilon > 0$ such that for any choice of $n \in \mathbb{N}$ there exists $|(x_n, y_n)| > N$ such that $g(x_n, y_n) > \epsilon$. Create a sequence of such $\{(x_n, y_n)\}$ and let $\{(x_{n_k}, y_{n_k})\}$ be a subsequence satisfying $D(x_{n_k}, y_{n_k}) \cap D(x_{n_j}, y_{n_j}) = \emptyset$ for all $j \neq k$. Let $\Omega = \bigcup_{k=1}^{\infty} D(x_{n_k}, y_{n_k})$. We now have

$$\|f\|_1 = \int |f| dm \geq \int_\Omega |f| dm \geq \left| \int_\Omega f dm \right| > \sum_{k=1}^{\infty} g(x_{n_k}, y_{n_k}) > \sum_{k=1}^{\infty} \epsilon = \infty.$$

This contradicts the fact that $f \in L^1(\mathbb{R}^d)$.

□

Part (c)

Proof. To show a contradiction suppose that there did not exist any point (x,y) such that $|g(x,y)| \leq \frac{1}{x^2+y^2+100}$. Tile the plane with squares $S(2n, 2m)$ of side length two, centered at the point $(2n, 2m)$ with $n, m \in \mathbb{Z}$. Let $\Delta_{n,m} \subseteq S(2n, 2m)$ be the disk of radius one contained in each $S(2n, 2m)$ and take $\Delta = \bigcup_{n,m\in\mathbb{Z}} \Delta_{n,m}$. Our construction guarantees that Δ is a disjoint union. We can now contradict the fact the f is integrable via the following chain of inequalities

$$\|f\|_1 \geq \int_\Delta |f(x)| dx \geq \sum_{n,m\in\mathbb{Z}} \left| \int_{\Delta_n} f(x) dx \right| = \sum_{n,m\in\mathbb{Z}} |g(2n, 2m)| > \sum_{n,m\in\mathbb{Z}} \frac{1}{(2n)^2 + (2n)^2 + 100} \overset{*}{>} \infty.$$

Where we make the conclusion at * by noting that $\sum_{n,m\in\mathbb{Z}} \frac{1}{(2n)^2+(2n)^2+100} = \frac{1}{4} \sum_{n,m\in\mathbb{Z}} \frac{1}{n^2+n^2+100}$and comparring to the integral

$$\int\int \frac{1}{x^2+y^2+100} dxdy \overset{**}{=} \int\int \frac{1}{r^2+100} r dr d\theta \geq \int\int \frac{1}{r^2} r dr d\theta = \int\int \frac{1}{r} dr d\theta = \infty.$$

Where at ** we change to polar coordinates. □

(2004 #3) Let f be a fixed but arbitrary Lebesgue measurable function on $\mathbb{R}$. For $p \in \mathbb{R}$ define

$$F(p) = \int_{\mathbb{R}} |f(t)|^p dt,$$

where we use the convention $0^0 = 0$. Show that (a)

$$I = \{p : F(p) < \infty\}$$

is a connected subset of $\mathbb{R}$ and that (b) $\log F$ is a convex function on this set.

Part (a)

Proof. Since we are working in $\mathbb{R}$, to prove I is connected it suffices to show that for any $p, p' \in I$ with $p \leq p'$ it holds that $[p, p'] \subseteq I$. Let $E = \{x : |f(x)| \leq 1\}$ and fix $p, p' \in I$. Let $p \leq r \leq p'$.

Observe that

$$\begin{aligned}
F(r) &= \int |f(x)|^r dx \\
&= \int_E |f(x)|^r dx + \int_{E^c} |f(x)|^r dx \\
&\leq \int_E |f(x)|^p dx + \int_{E^c} |f(x)|^{p'} dx \\
&\leq \|f\|_p + \|f\|_{p'} \\
&< \infty.
\end{aligned}$$

Therefore $r \in I$ and I is connected.

□

Part (b)

Proof. Recall that a function G is convex on I if for all $s, t \in I$ and $\lambda \in (0,1)$ it holds that

$$G(\lambda s + (1-\lambda)t) \leq \lambda G(s) + (1-\lambda)G(t).$$

First we note that $F(p) = 0$ for some $p \in \mathbb{R}$ if and only if either (i) $f = 0$ a.e. or (ii) $f = \infty$ a.e. We consider both of these cases first.

(i) If $f = 0$ a.e. then we know that $I = [0, \infty)$ and that at each $p \in [0, \infty)$ we have $F(p) = 0$. Therefore $\log F \equiv \infty$, which is convex (vacuosly).

(ii) Similarly if $f = \infty$ a.e. then we know that $I = (-\infty, 0)$ and that $F(p) \equiv 0$ on I. Once again we have $\log F(p) \equiv \infty$ on I and is convex.

Fix s, t, λ as above. We have

$$\begin{aligned}
F(\lambda s + (1-\lambda)t) &= \int |f(x)|^{\lambda s + (1-\lambda)t} dx \\
&= \int |f(x)|^{\lambda s} |f(x)|^{(1-\lambda)t} dx \\
&= \left\| f^{\lambda s} f^{(1-\lambda)t} \right\|_1 \\
&\overset{*}{\leq} \left\| f^{\lambda s} \right\|_{1/\lambda} \left\| f^{\lambda t} \right\|_{1/(1-\lambda)} \\
&= \left(\int |f(x)|^s dx \right)^{\lambda} \left(\int |f(x)|^t dx \right)^{1-\lambda} \\
&= F(s)^{\lambda} F(t)^{1-\lambda}
\end{aligned}$$

Where at * we use Hölder's inequality with conjugates $\lambda + (1-\lambda) = 1$. Taking log of both sides yields

$$\log F = \lambda \log F(s) + (1-\lambda)F(t),$$

as desired. □

(2004 #6) Let $C^1(\mathbb{R})$ denote the space of continuously differentiable real-valued functions on $\mathbb{R}$. Suppose that $f_n \in C^1(\mathbb{R})$ for $n \geq 1$ and $f_n(x) = 0$ for all $|x| > 1$ and all n. Assume also that there is a $p > 1$ such that $\|f_n'\|_p \leq 1$ for all n. Prove that there is a subsequence of the f_n which converges uniformly on $\mathbb{R}$.

Proof. Let $\mathcal{F} = \{f_n\}$. Since each f_n satisfies $f_n(x) = 0$ for $|x| > 1$ we must have $\mathcal{F} \subseteq \mathbb{C}^1([-1,1])$. We seek to apply the Arzela-Ascoli theorem. Accordingly we must prove (i) that for each $x \in [0,1]$ we have $\sup_{f\in\mathcal{F}} |f(x)| < \infty$ and (ii) that $\mathcal{F}$ is equicontinuous.

(i) Since each $f \in \mathcal{F}$ is zero outside of $[-1,1]$ it suffices to consider fixed $x \in [-1,1]$. For any $f \in \mathcal{F}$ we have f' exists and is continuous on $[-1,1]$ and therefore bounded. So we can apply the fundamental theorem of calculus and write

$$|f'(x)| = \left|\int_{-1}^{x} f'(t)dt + f(1)\right| \stackrel{*}{=} \left|\int_{-1}^{x} f'(t)dt\right| \leq \int_{-1}^{x} |f'(t)|dt \stackrel{**}{\leq} \|f'\|_p \left\|\mathbb{1}_{[-1,x]}\right\|_q \stackrel{***}{\leq} 1 \cdot 2 = 2.$$

Where at * we use the fact that $f(1) = 0$ because f is continuous and at ** we use Hölder. Lastly, at *** we use the hypothesis that $\|f'\|_p \leq 1$ and that $m([-1,x]) \leq 2$, the fact that $p > 1$ implies that $q < \infty$ and thus $\left\|\mathbb{1}_{[-1,x]}\right\|_q = 2$. As f was arbitrary we conclude that (i) holds.

(ii) To see equicontinuity, first fix $\epsilon > 0$. For $x, y \in [-1,1]$ with $x < y$ and $f \in \mathcal{F}$ we perform a similar estimate as in (i)

$$\begin{aligned} |f(x) - f(y)| &= \left|\int_{-1}^{x} f'(t)dt - \int_{-1}^{y} f'(t)dt\right| \\ &\leq \int_{x}^{y} |f'(t)|dt \\ &= \int |f'(t)|\mathbb{1}_{[x,y]}(t)dt \\ &\leq \|f'\|_p \left\|\mathbb{1}_{[x,y]}\right\|_q \\ &\leq 1 \cdot \left\|\mathbb{1}_{[x,y]}\right\|_q . \end{aligned}$$

Since $q < \infty$ we know that $\left\|\mathbb{1}_{[x,y]}\right\|_q \to 0$ as $|x - y| \to 0$. And because this convergence is independent of f, we conclude that $\mathcal{F}$ is equicontinuous.

□

(2004 #7) Let f be a Borel measurable real-valued functio non $\mathbb{R}$ that is integrable with respect to Lebesgue measure. Let μ be a nonnegative Borel probability measure on $[0,1]$. Define Tf as

$$Tf(x) = \int_0^1 f(x+t)d\mu.$$

(a) Prove that $T(f)$ exists and is finite for m-almost every $x \in \mathbb{R}$.

(b) Show that $\|Tf\|_1 \leq \|f\|_1$.

Part (a)

Proof. We seek to apply Tonelli and then Fubini to Tf. First (supposing that $\int |Tf(x)|dx$ exists) consider

$$\begin{aligned}\int_{\mathbb{R}} |Tf(x)|dx &\leq \int_{\mathbb{R}} \int_0^1 |f(x+t)|d\mu(t)dx \\ &= \int_0^1 \int_{\mathbb{R}} |f(x+t)|dxd\mu(t) \\ &= \int_0^1 \|f\|_1 \, d\mu(t) \\ &= \|f\|_1 .\end{aligned}$$

By Tonelli we have $Tf \in L^1(\mathbb{R})$ and thus Tf exists and is finite m-almost everywhere.

□

Part (b)

Proof. Follows from the calculation in (a). □

(2004 #8) too long of problem statement

Part (a)

Proof. Follows from Stone-Weistrauss. □

Part (b)

Proof. The key observation is that for a.e. x it holds that $\{kx\}$ is dense in $[0,1]$. Therefore $\limsup_{k\to\infty} f(x,\{kx\}) = f(x,1)$ and $\liminf_{k\to\infty} f(x,\{kx\}) = f(x,0)$. By hypothesis $f(x,1) = f(x,0)$ for all x and therefore $\lim_{k\to\infty} f(x,\{kx\}) = f(x,1)$. We can use this to quickly prove that the equality holds for any $f_{m,n}(x)$. We then choose arbitrary f and for fixed $\epsilon > 0$ take $\|f_{m,n} - f\|_u < \epsilon$. The trick is to use the equality for $f_{m,n}$

$$\begin{aligned}\left|\int_0^1 f(s,1)ds - \int_0^1\int_0^1 f(s,t)dsdt\right| &= \left|\int_0^1 f(s,1)ds - \int_0^1 f_{m,n}(s,1)ds + \int_0^1\int_0^1 f_{m,n}(s,t)dsdt - \int_0^1\int_0^1 f(s,t)dsdt\right| \\ &= \left|\int_0^1 f(s,1) - f_{m,n}(s,1)ds + \int_0^1\int_0^1 f_{m,n}(s,t) - f(s,t)dsdt\right| \\ &\leq \int_0^1 |f(s,1) - f_{m,n}(s,1)|ds + \int_0^1\int_0^1 |f_{m,n}(s,t) - f(s,t)|dsdt \\ &\leq 2\epsilon.\end{aligned}$$

□

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(2005 #1) Prove that there exists a continuous nonnegative function $f : [0,1] \to [0,1]$ such that $f(0) = f(1) = 0$ and $f' > 0$ a.e.

Proof. Let F denote the Cantor-Lebesgue function. Consider the function

$$f(x) = x \sin(\pi F(x)).$$

Notice that f is continuous and nonnegative. Also, since $F(1) = 1$ we know that $f(0) = f(1) = 0$. Lastly, we know for a.e. x it holds that $F'(x) = 0$ and thus for a.e. $x \in [0,1]$

$$f'(x) = \sin(\pi F(x)) + \pi F'(x) \cos(\pi F(x)) = \sin(\pi F(x)) > 0.$$

□

(2005 #2) Let ℓ^p and ℓ^∞ be the spaces of p^{th} power summable functions and bounded functions respectively, on the positive integers.

(a) Suppose $1 < p < \infty$. Show that $f_n \rightharpoonup f$ in ℓ^p if and only if $f_n \to f$ pointwise and $\sup_n \|f_n\|_p < \infty$.

(b) Is the $\Rightarrow$ implication in part (a) true for $p = 1$? Is the $\Leftarrow$ implication? Justify.

Lemma 3. *If X is a normed vector space TFAE*

(i) $\sup_n \|x_n\| < \infty$ *and* $f(x_n) \to f(x)$ *for all* $f \in M \subseteq X^*$ *where* $\overline{span(M)} = X^*$.

(ii) $x_n \rightharpoonup x$.

Proof. Suppose (i). Then for any $g \in X^*$ we have $\{g_j\} \subseteq \text{span}(M)$ satisfying $g_j \to g$ in the operator norm. Therefore

$$\begin{aligned} \|g(x) - g(x_n)\| &\le \|g(x) - g_m(x)\| + \|g_m(x) - g_m(x_n)\| + \|g_m(x_n) - g(x_n)\| \\ &\le \|g - g_m\| \|x\| + \|g_m(x) - g_m(x_n)\| + \|g_m - g\| \sup_n \|x_n\|. \end{aligned}$$

Which goes to zero since the $g_m \in \text{span} M$. Hence (ii).

Next, suppose (ii). Thinking of the $\hat{x}_n$ as inhabiting X^{**} with $\hat{x}_n(f) = f(x_n)$ we know that for each f it holds that $\sup \hat{x}_n(f) \to x(f) < \infty$. By UBB we conclude that $\|\hat{x}_n\| < \infty$. By Hahn-Banach we know there exists $f \in X^*$ such that $\|f\| = 1$ and $f(x) = \|x\|$, so we can use that to prove that $\|x\| \ge \|\hat{x}\|$ for all $x \in X$ and the reverse is an easy application of the definition of $\|\cdot\|$. □

Part (a) follows immediately from the lemma.

Part (a)

Proof. ($\Rightarrow$) First we prove pointwise convergence. Let $t \in \mathbb{N}$ and let $g = \mathbb{1}_{\{t\}}$. Since $g \in \ell^q$ we have

$$\left|\sum f_n(x)g(x) - \sum f(x)g(x)\right| = |f_n(t) - f(t)| \to 0.$$

Next, we know that each f_n defines a linear functional on ℓ^q, denote this by $\hat{f}_n$. We can apply uniform boundedness to conclude that $\sup \left\|\hat{f}_n\right\| < \infty$ and thus $\sup \|f_n\| < \infty$. Both of these arguments do not depend on the fact that $p = 1$, so ($\Rightarrow$) holds when $p = 1$.

($\Leftarrow$) Pick N such that $\left\|g\mathbb{1}_{[N,\infty]}\right\|_q < \epsilon$.

$$\begin{aligned}\left|\sum f_n g - \sum fg\right| &\leq \sum |f_n - f||g| \\ &= \sum |f_n - f||g|\mathbb{1}_{[0,N]} + \sum |f_n - f||g|\mathbb{1}_{[N+1,\infty)}.\end{aligned}$$

The left term goes to zero since $f_n \to f$ pointwise and the sum is finite. The right hand term is bounded by

$$\sum |f_n - f||g|\mathbb{1}_{[N+1,\infty)} \leq \|f_n - f\|_p \left\|g\mathbb{1}_{[N+1,\infty]}\right\|_q \leq 2\sup \|f\|_n \epsilon.$$

□

Part (b)

Proof. We already discussed that ($\Rightarrow$) still holds when $p = 1$. However ($\Leftarrow$) fails via the example: $f_n = \mathbb{1}_{\{n\}}$ and $g = 1$. We have $f_n \to 0$ pointwise and $\|f_n\| = 1$ however, $|f_n g| \equiv 1$, a problem. □

(2005 #3) Let $B \subseteq \mathbb{R}$, let $\operatorname{int} B$ denote the interior of B and $\overline{B}$ denote the closure of B. Given $A \subseteq \mathbb{R}$ define the sequence of sets $\{A_k\}$ inductively by $A_1 = A$ and for $k \geq 1$, $A_{2k} = \overline{A_{2k-1}}$ and $A_{2k+1} = \operatorname{int} A_{2k}$.

(a) Find $A \subseteq \mathbb{R}$ such that the family $\{A_{2k}\}$ contains four distinct sets.

(b) Prove that for any $A \subseteq \mathbb{R}$ the family $\{A_k\}$ contains at most four distinct sets.

Part (a)

Proof. Let $\mathcal{C} \subseteq [0,1]$ denote the Cantor set. Consider $A_1 = \mathcal{C} \cup (2,3)$. Using the property that $\overline{\mathcal{C}} = \mathcal{C}$ and $\operatorname{int} \mathcal{C} = \emptyset$ and that when $\overline{X} \cap \overline{Y} = \emptyset$ it holds that $\operatorname{int}(X \cup Y) = \operatorname{int} X \cup \operatorname{int} Y$ we can write

$$\begin{aligned} A_1 &= \mathcal{C} \cup (2,3) \\ A_2 = \overline{A}_1 &= \mathcal{C} \cup [2,3] \\ A_3 = \operatorname{int}(\overline{A}_1) = \operatorname{int}(\mathcal{C}) \cup \operatorname{int}([2,3]) &= (2,3) \\ A_4 = \overline{(1,2)} &= [2,3]. \end{aligned}$$

Clearly $A_k \neq A_j$ for all $j \neq k$ with $j, k \leq 4$. □

Part (b)

Proof. We claim that for any set A it holds that $A_5 = A_3$, that is we claim that

$$\text{int}\left(\overline{\text{int}(\overline{A})}\right) = \text{int}(\overline{A}).$$

We will prove both containments.

($\supset$) Since $\overline{\text{int}\,\overline{A}} \supseteq \text{int}\,\overline{A}$ we make use of the fact that $\text{int}(\text{int}(\overline{A})) = \text{int}\,\overline{A}$ and take the interior of both sides to obtain $A_5 \supseteq A_3$.

($\subset$) Since $\text{int}\,\overline{A} \subseteq \overline{A}$ we can conclude from the fact that $\overline{\overline{A}} = \overline{A}$ that

$$\overline{\text{int}\,\overline{A}} \subseteq \overline{\overline{A}} = \overline{A}.$$

And since whenever $X \subseteq Y$ it holds that $\text{int}\,X \subseteq \text{int}\,Y$ we take the interior of the above and immediately obtain $A_5 \subseteq A_3$.

(2005 #4) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and there exists $k > 0$ such that for every $y \in \mathbb{R}$ there are at most k distinct $x \in \mathbb{R}$ with $f(x) = y$. Prove that the derivative $f'(x)$ exists a.e. $x \in \mathbb{R}$.

Proof. It suffices to prove that f is of bounded variation on all closed intervals $I_j = [-n, n]$. Since f is continuous we know that $f : I_n \to [a_n, b_n]$ where $a_n = \min_{x\in I_n} f(x), b_n = \max_{x\in I_n} f(x)$. We will prove that $\text{Var}_{I_n} f \le k(b_n - a_n)$. Fix a partition $-n = t_0 < t_1 < \cdots < t_j = n$ of I_n. We will show that

$$\sum_{i=1}^{j} |f(t_i) - f(t_{i-1})| \le k(b_n - a_n).$$

By the intermediate value theorem we know for any $y \in (f(t_{i-1}), f(t_j)) =: J_i$ there exists $t_i^* \in (t_{i-1}, t_i)$ such that $f(t_j^*) = y$. Therefore, for any $y \in \mathbb{R}$ there exist at most k indices $i = 1, \dots, j$ such that $y \in J_i$. Moreover, these t_i^* are distinct for different i since the intervals (t_{i-1}, t_i) are disjoint; and y can have no more than k pre images. It follows that

$$\sum_{i=1}^{j} \mathbb{1}_{J_i}(y) \le k.$$

Integrating with respect to y gives

$$\int_{a_n}^{b_n} \sum_{i=1}^{j} \mathbb{1}_{J_i}(y)dy = \sum_{1}^{j} \int_{a_n}^{b_n} \mathbb{1}_{J_i}(y)dy \le k(b_n - a_n).$$

Since $J_i \subseteq [a_n, b_n]$, the right-hand side is equal to

$$\sum_{i=1}^{j} m(J_i) = \sum_{i=1}^{j} |f(t_i) - f(t_{i-1})|.$$

□

□

(2005 #5) Suppose $f \in L^p(\mathbb{R})$ with $1 < p < \infty$.

(a) Show that f is integrable over every finite interval $[a, a+h]$ and that

$$\left| \int_a^{a+h} f(x)dx \right|^p \le h^{p-1} \int_a^{a+h} |f(x)|^p dx.$$

(b) Suppose there is a constant $C < 1$ such that for all $a \in \mathbb{R}$ and $h > 0$,

$$\left| \int_a^{a+h} f(x)dx \right|^p \le Ch^{p-1} \int_a^{a+h} |f(x)|^p dx.$$

Show that $f = 0$ a.e.

Part (a)

Proof. Fix h. We prove that f is integrable on $[a, a+h]$ via Hölder's inequality (letting q be such that $\frac{1}{p} + \frac{1}{q} = 1$).

$$\begin{aligned}
\int_a^{a+h} f(x)dx &= \int f(x)\mathbb{1}_{[a,a+h]}(x)dx \\
&\le \left\| f\mathbb{1}_{[a,a+h]} \right\|_1 \\
&\overset{*}{\le} \|f\|_p \left\| \mathbb{1}_{[a,a+h]} \right\|_q \\
&= \|f\|_p \, h^{1/q} \\
&< \infty.
\end{aligned}$$

Where at * we use Hölder and at the last step we use the fact that $f \in L^p$. We can make a similar calculation along with the fact that $\frac{p}{q} = p - 1$ to write

$$\begin{aligned}
\left| \int_a^{a+h} f(x)dx \right|^p &\le \left\| f\mathbb{1}_{[a,a+h]} \right\|_1^p \\
&\overset{*}{\le} \|f\|_p^p \left\| \mathbb{1}_{[a,a+h]} \right\|_q^p \\
&= \int |f(x)|^p dx \cdot h^{p/q} \\
&= h^{p-1} \int |f(x)|^p dx \\
&\le h^{p-1} \int_a^{a+h} |f(x)|^p dx.
\end{aligned}$$

□

Part (b)

Proof. Without loss of generality suppose that $h > 0$. Dividing both sides by h^p gives

$$\left| \frac{1}{h} \int_a^{a+h} f(x)dx \right|^p \leq C \frac{1}{h} \int_a^{a+h} |f(x)|^p dx.$$

The Lebesgue differentiation theorem tells us that the right hand side as $h \to 0$ is a.e. equal to $C|f(a)|^p$. By the same theorem, since $p > 1$ we know the left hand side a.e. goes to $|f(a)|^p$. Since the intersection of two full measure sets has full measure we conclude that $|f(a)|^p \leq C|f(a)|^p$ a.e. a which can only be true if $f = 0$ a.e.

□

> **(2005 #6)** Suppose that $f \in L^1(\mathbb{R}^3)$. Show that (a) the integral
>
> $$\varphi(x) = \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|} dy$$
>
> converges for a.e. $x \in \mathbb{R}^3$ and (b) that the resulting function φ is Lebesgue integrable over any bounded measurable set in $\mathbb{R}^3$.

Proof. Without loss of generality suppose that $f \geq 0$. Let $\psi(x) = \frac{1}{|x|} \mathbb{1}_{\mathbb{D}}$ we can write

$$\begin{aligned}
\varphi(x) &= \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|} dy \\
&= \int_{\mathbb{R}^3} \frac{f(x-y)}{|y|} dy \\
&= \int_{\mathbb{D}} \frac{f(x-y)}{|y|} dy + \int_{\mathbb{D}^c} \frac{f(x-y)}{|y|} dy \\
&\leq (\psi * f)(x) + \|f\|_1 .
\end{aligned}$$

Since $\psi, f \in L^1$, it follows from Young's theorem that $\psi * f \in L^1$. Therefore $\varphi(x)$ is finite for each x. Moreover, φ is integrable over any bounded set since it belongs □

Part (a)

Proof. Without loss of generality we can assume $f \geq 0$. It is equivalent to prove that the function $y \mapsto \frac{f(y)}{|x-y|}$ is in $L^1(\mathbb{R}^3)$.

$$\begin{aligned}
\varphi(x) &= \int_{\mathbb{R}^3} \frac{f(y)}{|x-y|} dy \\
&= \int_{\mathbb{R}^3} \frac{f(x-y)}{|y|} dy \\
&= \int_{\mathbb{D}} \frac{f(x-y)}{|y|} dy + \int_{\mathbb{D}^c} \frac{f(x-y)}{|y|} dy \\
&= I + II
\end{aligned}$$

Clearly, $II < \|f\|_1 < \infty$. To handle I we will use Fubini.

$$\int_{\mathbb{R}^3}\int_{\mathbb{D}} \frac{f(x-y)}{|y|}dydx = \int_{\mathbb{D}}\int_{\mathbb{R}}^3 \frac{f(x-y)}{|y|}dy = \int_{\mathbb{D}} \frac{1}{|y|}\|f\|_1 \, dx = \|f\|_1 \cdot C.$$

Since $\int_{\mathbb{D}} \frac{1}{|y|}dy$ is finite via a change to spherical coordinates. □

Part (b)

Proof. To prove that φ is locally integrable we fix a bounded measurable set $E \subseteq \mathbb{R}^3$. We can write

$$\int_E \left|\int_{\mathbb{R}^3} \frac{f(x-y)}{|y|}dy\right| dx = \underbrace{\int_E \left(\int_{\mathbb{D}} \frac{f(x-y)}{|y|}dy + \int_{\mathbb{D}^c} \frac{f(x-y)}{|y|}dy\right)}_{<\infty}$$

By a similar argument as Part (a). □

(2005 #7) In this problem you can use the fact that simple functions are dense in $L^2([0,1])$.

(a) Show that $C([0,1])$ is dense in $L^2([0,1])$ (in the L^2 norm).

(b) Show that there is an orthonormal basis $\{f_n\}_{n=0}^\infty$ for $L^2([0,1])$ such that f_n is a polynomial of degree n for all n.

Part (a)

Proof. We know that simple functions are dense in L^p. Hence it is enough to approximate $\mathbb{1}_E$ where E is Borel with finite measure. By regularity we know for every $\epsilon > 0$ there exists compact $K \subseteq E$ and open $\mathcal{O} \supset E$ such that $m(\mathcal{O} \setminus K) < \epsilon$. By Urysohn's lemma there exists f which is continuous and $\mathbb{1}_K \leq f \leq \mathbb{1}_{\mathcal{O}}$. It follows that

$$\|f - \mathbb{1}_E\|_p^p \leq \int_{\mathcal{O}\setminus K} 1^p d\mu < \epsilon.$$

□

Part (b)

Proof. We know that the polynomials are dense in the uniform norm in $C[0,1]$ (this follows from Stone-Weirstrauss since 1 belongs to the family and the family separates points) therefore the set $E := \text{Span}\{1, x^2, x^3, \ldots\}$ is dense in $C[0,1]$ in the L^2 norm (true since $\|f - f_n\|_u = \|f - f_n\|_2$ when $X = [0,1]$. Part (a) implies this is dense in $L^2[0,1]$. Since each x^n satisfies $x^n \notin E \setminus \{x^n\}$ we know that the set $\{1, x^2, \ldots\}$ is linearly independent and thus a basis for E. The Graham-Schmidt process constructs from $\{1, x^2, \ldots\}$ an orthonormal basis $\{f_k\}$ by taking $f_k = \sum_{j=1}^k a_j x^j$ with $a_k \neq 0$. By the same reasoning for which $\{1, x^2, \ldots\}$ forms a basis we conclude that $\{f_k\}$ contains precisely one polynomial of degree k for each $k \in \mathbb{N}$. □

(2005 #8) Show that for all nonnegative measurable functions f and g on $[0,\infty)$,

$$\int_0^\infty \int_0^x \frac{f(t)g(x)}{x} dtdx \le 2\left[\int_0^\infty f(t)^2 dt \int_0^\infty g(x)^2 dx\right]^{1/2}.$$

(*Hint: Write* $\frac{f(t)g(x)}{x} = \frac{f(t)t^{1/4}}{x^{3/4}} \cdot \frac{g(x)}{t^{1/4}x^{1/4}}$.)

Conclude that the operator T defined by $Th(x) = x^{-1}\int_0^x h(t)dt$ is bounded on $L^2([0,\infty))$.

Part (a)

Proof. As the hint suggests we write $\frac{f(t)g(x)}{x} = \frac{f(t)t^{1/4}}{x^{3/4}} \cdot \frac{g(x)}{t^{1/4}x^{1/4}} := \varphi(x,t)\cdot\psi(x,t)$. Let Ω be the domain of integration (it looks like an infinite triangle). We apply Cauchy-Schwarz

$$\int_\Omega \varphi(x,t)\psi(x,t)dtdx \le \underbrace{\left(\int_\Omega \varphi(x,t)^2 dtdx\right)^{1/2}}_{I}\underbrace{\left(\int_\Omega \psi(x,t)^2 dtdx\right)^{1/2}}_{II}.$$

We can compute I, II separately.

$\boxed{I}$ We can write

$$\int_0^\infty \int_0^x \frac{f(x)^2\sqrt{t}}{x^{3/2}} dtdx \overset{*}{=} \int_0^\infty \int_t^\infty \frac{f(t)^2\sqrt{t}}{x^{3/2}} dxdt = \int_0^\infty f(t)^2\sqrt{t}\left(\int_t^\infty x^{-3/2}\right) = 2\|f\|_2^2.$$

$\boxed{II}$ We have

$$\int_0^\infty \int_0^x \frac{g(x)^2}{t^{1/2}x^{3/2}} dtdx = \int_0^\infty g(x)^2 x^{-3/2}\left(\int_0^x \sqrt{t}dt\right)dx = \sqrt{2}\|g\|_2.$$

□

Part (b)

Proof. By Part (a) we know that $|(Th,g)| \le 2\|g\|\|g\|$. So if $g = Th$ then we have $\|th\|^2 \le 2\|h\|\|Th\|$. Therefore $\|Th\| \le 2\|h\|$. □

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(2006 #1) State the open mapping theorem and closed graph theorem then derive the open mapping theorem from the closed graph theorem.

Proof.

Theorem 9 (OMT)**.** *Any bounded linear surjective operator between Banach spaces is open*

Theorem 10. *(CGT) If the graph of a linear operator T between two Banach spaces is closed, then T is bounded.*

We assume the CGT. Supose that $T : X \to Y$ is a bounded linear surjective operator between two Banach spaces. We claim that T is open. By linearity it suffices to find $\delta > 0$ such that $B(y, \delta) \subseteq T(B(x, 1))$. The idea is to construct a continuous inverse of T. Consider the quotient space $X/\ker T = \{x + \ker T : x \in X\}$. Since $\ker T$ is closed in X (as T is bounded and linear) we know that

$$\|x + \ker T\| = \inf_{z\in\ker T} \|x - z\|,$$

is a norm in $X/\ker T$. Since X is complete we know that $X/\ker T$ is complete. Now define $G : Y \to X/\ker T$ where $y = Tx$ (which we can do since T is onto). It is easy to check that G is linear.

Next we claim that the graph of G denoted by $\Gamma_G = \{(y, x + \ker T) : Tx = y\}$ is closed in $Y \times X/\ker T$. To see this suppose that $\{x_n\} \subseteq X$ and $\{y_n\} \subseteq Y$ with $x \in X, y \in Y$ such that $T(x_n) = y_n$ and $(y_n, x_n + \ker T) \to (y, x + \ker T)$. We would like to show that $G(y) = x + \ker T$ (i.e. show that $Tx = y$). Since $x_n - x + \ker T \to 0$ and

$$\inf_{z\in\ker T} \|x_n - x - z\| \to 0$$

we conclude that there exist $n_1 < n_2 < \cdots$ such that

$$\inf_{z\in\operatorname{Ker} T} \|x_{n_i} - x - z\| < \frac{1}{i}.$$

So for all i there exists $z_i \in \ker T$ such that

$$\|x_{n_i} - x - z_i\| < \frac{1}{i}.$$

Then we have $x_{n_i} - z_i \to x$ as $i \to \infty$. Since T is continuous it follows that $T(x_{n_i} - z_i) \to Tx$ and that $T(x_{n_i} - z_i) = T(x_{n_i})$. Therefore $T(x_{n_i}) \to Tx$ and $y_{n_i} \to Tx$ and thus $y_n \to y$. Hence G is closed.

We can now apply the CGT and conclude that G is bounded. So there exists $\delta > 0$ such that

$$G(B(y, \delta)) \subseteq B(x/\ker T, 1) \qquad (*).$$

And so we are done if we can prove $B(y, \delta) \subseteq T(B(x, 1))$.

To see this we let $y \in B(y, \delta)$. Since T is onto we can pick x such that $Tx = y$ and

$$x + \ker T = G(y) \in B(x/\ker T, 1)$$

by $*$. Therefore $\inf_{z\in\ker T} \|x - z\| < 1$. This means that there exists $z_0 \in \ker T$ such that $\|x - z_0\| < 1$ and $y = Tx = T(x - z_0) \in T(B(x, 1))$ So $B(y, 1) \subseteq T(B(x, 1))$.

□

(*BETTER PROOF*)

Proof. let $T\colon X \to Y$ be a continuous and surjective linear map. We will show that $T(a)$ is an open map. Let $M = T^{-1}(\{0\})$. Define $\tilde{T}\colon X/M \to Y\colon (x + M) \mapsto Tx$. This is clearly linear. We claim that $\tilde{T}$ is continuous since

$$\left\|\tilde{T}(x + M)\right\| = \|T(x + y)\| \le \|T\| \|x + y\|,$$

for $y \in M$. So take the inf over M. Notice that by construction $\tilde{T}$ is bijective. We will show that the graph $\Gamma(\tilde{T}^{-1})$ is closed. If $(y_n, x_n + M) \to (y, x + M) \in Y \times X/M$ we know that there exists $m_n \in M$ such that $x_n - x + m_n \to 0$. Therefore $T(x_n) - T(x) \to 0$ and so $y_n - T(x) \to 0$. This implies that $(y, x + M) \in \Gamma(\tilde{T}^{-1})$. Therefore $\tilde{T}$ is an isomorphism. Hence for $U \subseteq X$ an open set we know that $U + M$ is open in X/M. Therefore $T(U) = \tilde{T}(U + M)$ is open in Y.

□

(2006 #2) Let

$$\mathcal{H} = \left\{ u : \mathbb{R}^n \to \mathbb{R} \mid \; u \text{ is continuous and } \forall x \in \mathbb{R}^n, \forall r > 0 : u(x) = \frac{1}{m(\mathbb{B}(x,r))} \int_{\mathbb{B}(x,r)} u(y) dy \right\}$$

Show that if $\{u_n\}_1^\infty \subseteq \mathcal{H}$ is uniformly bounded on compact subsets of $\mathbb{R}^n$ then there exists a subsequence that converges uniformly to a function $u \in \mathcal{H}$ on compact subsets of $\mathbb{R}^n$.

Proof. When $r = 1$ we know that

$$u_n(x) = \frac{1}{|\mathbb{B}(x,1)|} \int_{\mathbb{B}(x,1)} u_n(y) dy = \frac{1}{\alpha(n)} \int_{\mathbb{R}^n} u_n(y) \mathbb{1}_{\mathbb{B}(0,1)}(x - y) dy = \int_{\mathbb{R}^n} u_n(y) \frac{\mathbb{1}_{\mathbb{B}(0,1)}(x-y)}{\alpha(n)} dy.$$

□

So we can write $u_n(x) = u_n * \varphi(x)$ where $\varphi(z) = \frac{\mathbb{1}_{\mathbb{B}(0,1)}(z)}{\alpha(n)}$. So for $B_k = \overline{\mathbb{B}(0,k)}$ there exists a constant C_k such that $\|u_n\|_{\infty, B_k} < C_k$. We can write

$$\begin{aligned} |u_n(x) - u_n(y)| &= |u_n * \varphi(x) - u_n * \varphi(y)| \\ &= \left| \int u_n(z) \varphi(x - z) - u_n(z) \varphi(y - z) dz \right| \\ &\leq \int C_{k+2} |\varphi(x - z) - \varphi(y - z)| dz \\ &\overset{*}{\leq} C_{k+2} \int |\varphi(\delta + z) - \varphi(z)| dz. \end{aligned}$$

Where at * we take y close to z that $z \sim y - z$. So the family is equicontinuous. We can then extract convergent subsequences $\{u^i_{n_k}\}$ on each ball B_i. If we diagonalize we have $\{u^i_{n_i}\}_{i=1}^\infty$ converges uniformly on compact sets.

Let u be the limit of this sequence. It remains to prove that u belongs to $\mathcal{H}$. Fix $r > 0$ and choose $x \in \mathbb{R}^n$. Let

$$f_n = u_n(y) \mathbb{1}_{\mathbb{B}(x,r)}(y).$$

Notice that $f_n \to u(y) \mathbb{1}_{\mathbb{B}(x,r)}(y)$ point wise and since they are bounded and therefore dominated we have $u \in \mathcal{H}$.

Proof. First consider $K = \{x\}$. Fix $\epsilon > 0$ and n. Let δ_n be such that

$$\begin{aligned}
|u_n(x) - u_n(y)| &= \left| \frac{1}{m(B(x,1))} \int_{B(x,1)} u_n(z)dz - \frac{1}{m(B(y,1))} \int_{B(y,1)} u_n(z)dz \right| \\
&= |\frac{1}{\omega_n \cdot 1^n} \left(\int_{B(x,1)\cap B(y,1)} u_n(z)dz - \int_{B(x,1)\setminus B(y,1)} u_n(z)dz \right) \\
&\qquad - \frac{1}{\omega_n \cdot 1^n} \left(\int_{B(x,1)\cap B(y,1)} u_n(z)dz - \int_{B(y,1)\setminus B(x,1)} u_n dz \right) | \\
&\leq \frac{1}{\omega_n} \int_{B(x,1)\Delta B(y,1)} |u_n(z)| dz \\
&\leq \frac{1}{\omega_n} \|u_n\|_{L^\infty(B(x,1+\delta)} \cdot m(B(x,1)\Delta B(y,1)) \\
&\leq \frac{1}{\omega_n} \|u_n\|_{L^\infty(B(x,1+\delta))} \cdot ((1+\delta)^n - (1-\delta)^n)\omega_n.
\end{aligned}$$

Since R^n is LCH and σ-compact we can apply the Arzela Ascoli theorem and obtain a subsequence $u_{n_k} \to u$ uniformly. It remains to prove that $u \in \mathcal{H}$. □

(2006 #4) Suppose $(X, \mathcal{M}, \mu)$ is a measure space and $\mathcal{F}$ is a family of nonnegative integrable functions on X with following properties:

(a) If $\varphi, \psi \in \mathcal{F}$ then $\varphi + \psi \in \mathcal{F}$.

(b) If $\varphi, \psi \in \mathcal{F}$ then $\max\{\varphi, \psi\} \in \mathcal{F}$.

(c) IF f is measurable, $f \geq 0$ and $\int f d\mu > 0$ then there is $\varphi \in \mathcal{F}$ such that $\varphi \leq f$ and $\int \varphi d\mu > 0$. Prove that if f is nonnegative and measurable then

$$\int f d\mu = \sup \left\{ \int \varphi d\mu \colon \varphi \in \mathcal{F}, \varphi \leq f \right\}.$$

Proof. One direction is trivial. Let $\varphi_n \in \mathcal{F}$ be a sequence which attains the supremum. Since we can define $\psi_n = \max\{\varphi_1, \ldots, \varphi_n\}$ with $\psi_n \in \mathcal{F}$ by (b), we can assume without loss of generality that $\varphi_n \leq \varphi_{n+1}$. It follows from monotone convergence that $\int \varphi_n \to \int \varphi$ for some $\varphi \leq f$. We claim that $\int \varphi = \int f$. To see this, suppose not. Therefore $\int f - \varphi > 0$. By (c) choose $\gamma \in \mathcal{F}$ such that $\gamma \leq f - \varphi$ and $\int \gamma \geq \delta > 0$. Let n be such that $\int \varphi - \varphi_n < \delta/2$. Note that (a) guarantees $\varphi_n + \gamma \in \mathcal{F}$. Since the sup is attained at φ we must have

$$\int f - (\varphi_n + \gamma) \geq \int f - \varphi.$$

Cancelling the $\int f$ terms and rearranging this implies

$$\int \gamma \leq \int \varphi - \varphi_n.$$

However, $\int \gamma \geq \delta \geq \delta/2 \geq \int \varphi - \varphi_n$. A contradiction. □

(2006 #5) Let μ be a regular Borel measure on $\mathbb{R}^n$, and let m denote the Lebesgue measure on $\mathbb{R}^n$. Assume that there exists a constant $C > 1$ such that for all $x \in \mathbb{R}^n$ and $r > 0$:

$$C^{-1}r^n \leq \mu(B(x,r)) \leq Cr^n.$$

(a) Show that m and μ are mutually absolutely continuous.

(b) Let f denote the Radon-Nikodym derivative of μ with respect to m. Show that $f, \frac{1}{f} \in L^\infty(m)$.

(Tim)

Part (a)

Proof. Suppose that $\mu(E) = 0$. Fix $\epsilon > 0$ and let $\mathcal{O} \supseteq E$ be such that $\mu(\mathcal{O}) \leq \epsilon$. Write $\mathcal{O} \bigcup B_i$. Since $\mu(B(x,r))$ controls the maximum value of r we can apply Vitali's infinite covering lemma and obtain a disjoint subcollection

$$\mathcal{O} \subseteq \bigcup_{j=1}^{\infty} 5B_{i_j}.$$

Also $m(5B_{i_j}) = 5^n m(B_{i_j})$. We now have

$$\begin{aligned} m(E) &\leq \sum_{1}^{\infty} 5^n m(B_{i_j}) \\ &= 5^n \sum_{1}^{\infty} m(B_{i_j}) \\ &= 5^n \sum_{j=1}^{\infty} \omega_n r_{i_j}^n \\ &\leq 5^n \omega_n C \sum_{j=1}^{\infty} \mu(B_{i_j}) \\ &\leq 5^n \omega_n C \mu(\mathcal{O}) \\ &\leq 5^n \omega_n C \epsilon. \end{aligned}$$

Hence $m(E) = 0$. The other direction is similar. □

Part (b)

Proof. $\boxed{\frac{1}{f} \text{ case:}}$ Suppose that for any E measurable let $\delta = \mu(E)$. Once again we can take $\mathcal{O} \supset E$ with $\mu(\mathcal{O}) \leq \delta + \epsilon$. By Part (a) we conclude that

$$m(E) \leq 5^n C\omega_n(\delta + \epsilon).$$

And so $m(E) \leq 5^n C\omega_n \mu(E)$. Hence

$$\int_E f dm := \mu(E) \geq \frac{m(E)}{5^n C\omega_n} = \frac{1}{5^n c\omega_n} \int_E dm.$$

Hence m-a.e. it holds that

$$f \geq \frac{1}{5^n C\omega_n}, \qquad \frac{1}{f} \leq 5^n C\omega_n.$$

Therefore $\|1/f)\|_\infty < \infty$.

$\boxed{f \text{ case:}}$ Let E be measurable and let $\delta = m(E)$. Choose $\mathcal{O} \supset E$ with $m(\mathcal{O}) \leq \delta + \epsilon$. We now have

$$\mu(E) \leq C5^n\omega_n^{-1}\delta + \epsilon,$$

taking $\epsilon \to 0$ yields

$$\mu(E) \leq C5^n\omega_n^{-1}\delta = C5^n\omega_n^{-1}m(E).$$

Hence

$$\int_E f dm = \mu(E) \leq \frac{5^n C}{\omega_n} m(E) \leq \frac{5^n C}{\omega_n}\int_E dm.$$

Hence m-a.e. we have $f \leq \frac{5^n C}{\omega_n}$. □

(2006 #6) Let $(X, \mathcal{M}, \mu)$ be a finite measure space and let $K : X \times X \to \mathbb{R}$ be a measurable function satisfying

$$\int\int |K(x,y)|^2 d\mu(x)d\mu(y) < \infty.$$

Let $\{f_n\} \subseteq L^(\mu)$ satisfy

$$\sup_n \|f_n\|_2 \leq 1.$$

Assume there exists $f \in L^2(\mu)$ so that f_n converges weakly to f in $L^2(\mu)$ i.e. for all $g \in L^2(\mu)$

$$\lim_{n\to\infty}\int f_n g = \int fg.$$

For $h \in L^2(\mu)$ define

$$Kh(x) = \int K(x,y)h(y)d\mu(y).$$

Show that Kf_n converges strongly to Kf in $L^2(\mu)$.

Proof. We will use Minkowski's integral inequality. First we let $G(y) := \left(\int |K(x,y)|^2 d\mu(x)\right)^{1/2}$ and appeal to Fubini which tells us that G belongs to $L^2(\mu)$.

$$\begin{aligned}
\|Kf - Kf_n\|_2 &= \left(\int |Kf(x)d\mu(x) - Kf(y)|^2 d\mu(x) \right)^{1/2} \\
&= \left(\int \left| \int K(x,y)f(y)d\mu(y)d\mu(x) - \int K(x,y)f(y)d\mu(y) \right|^2 d\mu(x) \right)^{1/2} \\
&\overset{!}{=} \left(\int \left| \int K(x,y)\left[f(y) - f_n(y)\right] d\mu(y) \right|^2 d\mu(x) \right)^{1/2} \\
&\leq \left(\int \left(\int |K(x,y)| \, |f(y) - f_n(y)| \, d\mu(y) \right)^2 d\mu(x) \right)^{1/2} \\
&\leq \int \left(\int |K(x,y)|^2 \, |f(y) - f_n(y)|^2 \, d\mu(x) \right)^{1/2} d\mu(y) \\
&= \int |f(y) - f_n(y)| \left(\int |K(x,y)|^2 \, d\mu(x) \right)^{1/2} d\mu(y) \\
&= \int |f(y) - f_n(y)| \, G(y) d\mu(y).
\end{aligned}$$

So it suffices to show that $\int |f(y) - f_n(y)| G(y) d\mu(y) \to 0$. Let $A = \{y : f_n(y) < f(y), \ \forall n\}$ and let $B = A^c$. Define the function

$$H(y) := \mathbb{1}_A(y)G(y) - \mathbb{1}_B(y)G(y).$$

Notice that $H \in L^2(\mu)$ since

$$\begin{aligned}
\int |H(y)|^2 d\mu(y) &= \int |\mathbb{1}_A(y)G(y) - \mathbb{1}_B(y)G(y)|^2 \, d\mu(y) \\
&= \int_A |G(y)|^2 \, d\mu(y) - \int_B |G(y)|^2 d\mu(y) \\
&\leq 2 \int |G(y)|^2 d\mu(y) \\
&\leq 2 \, \|K\|_2^2 < \infty.
\end{aligned}$$

It follows from our construction that

$$\begin{aligned}
\int |f(y) - f_n(y)| \, G(y) d\mu(y) &= \int_A [f(y) - f_n(y)] G(y) d\mu(y) - \int_B [f(y) - f_n(y)] G(y d\mu(y) \\
&= \int [f(y) - f_n(y)] H(y) d\mu(y).
\end{aligned}$$

Which goes to zero by the hypothesis that $f_n \rightharpoonup f$.

Alternatively we could have proceeded up to (!) and then defined the function

$$H_n(x) := \int K(x,y)[f(y) - f_n(y)] dy.$$

Notice that the H_n are integrable since by Cauchy-Schwarz

$$\|H_n\|_2 \leq \|K(x)\|_2 \, \|f(y) - f_n(y)\|_2 \leq G(y) 3 \sup \|f_n\|_2 = A \cdot G(y).$$

Also $H_n(x) \to 0$ pointwise because $f_n \rightharpoonup f$. Since $G \in L^2$ (because $G \in L^2 \subseteq L^1$ by finiteness). It follows from dominated convergence that $H_n \to^{L^2} 0$.

□

(2006 #7)

(a) Let X be a topological space and $\mathcal{O} \subseteq X$ an open set. Prove that the points of discontinuity of the characteristic function $\mathbb{1}_{\mathcal{O}}$ form a nowhere dense subset of X. Recall that $A \subseteq X$ is nowhere dense if int $\overline{A} = \emptyset$.

(b) Let X be a complete metric space and $\{U_i\}$ a countable collection of open sets. Show that there exists $x \in X$ such that $\mathbb{1}_{U_i}$ is continuous at x for each i.

Part (a)

Proof. Let A denote the set of discontinuities of $\mathbb{1}_{\mathcal{O}}$. Notice that if $x \in A$ it follows that for all neighborhoods $U \ni x$ we have $U \cap \mathcal{O} \neq \emptyset$ and $U \cap \mathcal{O}^c \neq \emptyset$. Therefore $A \subseteq \partial\mathcal{O}$. Since by definition $\partial\mathcal{O}$ is has empty interior, it follows that A is nowhere dense. □

Part (b)

Proof. B_i denote the set of continuous points of $\mathbb{1}_{U_i}$. To show a contradiction suppose that $\bigcap_1^\infty B_i = \emptyset$. But this says that $\bigcup_1^\infty (B_i)^c = X$. Since $B_i^c \subseteq \overline{(B_i)^c}$ and $\overline{(B_i)^c}$ is nowhere dense from Part (a) we conclude that a countable metric space is a countable union of nowhere dense closed sets, which directly contradicts the Baire category theorem.

□

(2006 #8) Let $(X, \mathcal{M}, \mu)$ be a finite measure space and let $1 \leq p < \infty$. Let $\{f_n\}_1^\infty$ be a sequence in $L^p(\mu)$ and let f be a $\mathcal{M}$-measurable function such that f is finite μ-a.e. and $f_n \to f$ μ-a.e. Prove that $f \in L^p(\mu)$ and $\|f_n - f\|_p \to 0$ if and only if for each $\epsilon > 0$ there exists $\delta > 0$ such that

$$\sup_n \int_E |f_n|^p d\mu < \epsilon \text{ whenever } \mu(E) < \delta \text{ and } E \in \mathcal{M}.$$

Proof. $\boxed{\Rightarrow}$ Suppose that $f \in L^p(\mu)$ and that $\|f_n - f\|_p \to 0$. Fix $\epsilon > 0$. Choose N such that for $n > N$ it holds that $\|f_n - f\|_p < (\epsilon/2)^{1/p}$. Choose $\delta_0 > 0$ such that for $E \in \mathcal{M}$ satisfying $\mu(E) < \delta_0$

it holds that $\int_E |f|^p d\mu < \epsilon/2$. This is possible since $f \in L^p$. So for $n > N$ and $\mu(E) < \delta$ we have

$$\begin{aligned}
\int_E |f_n|^p d\mu &= \int |f_n|^p |\mathbb{1}_E|^p d\mu \\
&= \|f_n \mathbb{1}_E\|_p^p \\
&= \|f_n \mathbb{1}_E - f\mathbb{1}_E + f\mathbb{1}_E\|_p^p \\
&\overset{*}{\leq} \|f_n \mathbb{1}_E - f\mathbb{1}_E\|_p^p + \|f\mathbb{1}_E\|_p^p \\
&\leq \|f_n - f\|_p^p + \int_E |f|^p d\mu \\
&< ((\epsilon/2)^{1/p})^p + \epsilon/2 \\
&= \epsilon.
\end{aligned}$$

Where at * we use Minkowski's inequality. We wrap it up by taking each $n \leq N$ an appropriate δ_n such that for $\mu(E) < \delta_n$ it holds that $\int_E |f_n|^p d\mu < \epsilon$. We then let $\delta = \min_{i=0}^N \{\delta_i\}$.

$\boxed{\Leftarrow}$ Suppose that the consequent holds. Fix $\epsilon > 0$. Let $\delta > 0$ be as guaranteed by the hypothesis. Also, since $f_n \to f$ μ-a.e. and X is a finite measure space we can apply Egorov's theorem and let $A \subseteq X$ be such that $m(X \setminus A) < \delta$ and $f_n \to f$ uniformly on A. It follows that there exists N such that for $n > N$ it holds that $|f(x) - f_n(x)|^p < \epsilon$ for all $x \in A$. We can now write

$$\begin{aligned}
\int |f|^p d\mu &= \int_A |f|^p d\mu + \int_{X\setminus A} |f|^p d\mu \\
&\leq \int_A |f - f_n|^p d\mu + \int_A |f_n|^p d\mu + \int_{X\setminus A} |f|^p d\mu \\
&\leq \mu(A)\epsilon + \|f_n\|_p^p + \liminf_{n\to\infty} \int_{X\setminus A} |f_n|^p d\mu \\
&\overset{*}{\leq} \mu(A)\epsilon + \|f_n\|_p^p + \epsilon \\
&< \infty.
\end{aligned}$$

Where at $*$ we use Fatou's lemma where we assume without loss of generality that $f_n \to f$ on $X \setminus A$ (which we can do since the convergence is μ-a.e. and we are taking an integral). Therefore $f \in L^p(\mu)$.

Lastly we prove that $f_n \to f$ in $L^p(\mu)$. Letting ϵ and A be as before we can write

$$\int |f_n - f|^p d\mu = \int_A |f_n - f|^p d\mu + \int_{X\setminus A} |f_n - f|^p d\mu.$$

Let $n > N$ so that $|f_n - f| < \epsilon$ on A. We then have

$$\int |f_n - f|^p < \epsilon^p \mu(x) + \int_{X\setminus A} |f_n - f|^p d\mu.$$

By Minkowski's inequality and then Fatou's lemma we have

$$\begin{aligned}
\left\|(f_n - f)\mathbb{1}_{X\setminus A}\right\|_p &\leq \left\|f\mathbb{1}_{X\setminus A}\right\|_p + \left\|f\mathbb{1}_{X\setminus A}\right\|_p \\
&\leq \left\|f_n\mathbb{1}_{X\setminus A}\right\|_p + \liminf_{n\to\infty} \left\|f_n\mathbb{1}_{X\setminus A}\right\|_p \\
&< \epsilon + \epsilon.
\end{aligned}$$

Therefore $\|f_n - f\|_p$ is bounded by an expression involving only ϵ and constants, which goes to zero.

□

17 2007

(2007 #1) Let E, F be complex Banach spaces and let $T : E \to F$ be a surjective, continuous linear functional. Let $T^* : F^* \to E^*$ denote the dual adjoint given by $T^*f(x) = f(T(x))$. Show there exists a constant $c > 0$ such that $\|T^*(f)\| \geq c\|f\|$ for all $f \in F$.

(Hon's Solution)

Proof. By the OMT we know that T is an open map. Hence there exists $c > 0$ such that $\mathbb{D}(0, c) \subseteq T(\mathbb{D})$. Notice that

$$\begin{aligned}
\|T^*f\| &= \sup_{\|x\|\leq 1} |(T^*f)(x)| \\
&= \sup_{\|x\|\leq 1} |f(Tx)| \\
&\geq \sup_{\|y\|\leq c} |f(y)| \\
&= \sup_{\|z\|\leq 1} |f(zc)| \\
&= c\|f\|.
\end{aligned}$$

□

(My Incorrect Solution)

Proof. To show a contradiction suppose that for each $c_n = 1/n$ there exists an $f_n \in F^*$ such that $\|T^*f_n\| < c_n\|f_n\|$. In particular we have for $f_n' = \frac{f_n}{\|f_n\|}$ it must hold that $\|f_n'\| = 1$ and

$$\left\|T^*f'\right\| = \frac{1}{\|f_n\|}T^*(f_n) < c_n \qquad (1).$$

Hence we have a sequence $f_n \in F^*$ such that $\|f_n\| = 1$ and $\|T^*f_n\| \to 0$. The open mapping theorem guarantees that T is an open map, therefore $T(\mathbb{D}) \subseteq F$ is an open set containing zero. Let $\Delta = \mathbb{D}(0, r) \subseteq T(\mathbb{D})$ for some $r > 0$. Line (1) says that

$$\sup_{y\in\Delta} |f_n(y)| = \frac{1}{r}\|f_n\| \to 0.$$

However, this contradicts the fact that $\|f_n\| = 1$.

□

(2007 #2)

(a) Let f be a nonnegative Lebesgue measurable function on $[0,\infty)$ such that $\int_0^\infty f(x)dx < \infty$. Show that there exists a positive, strictly increasing measurable function a on $[0,\infty)$ with $\lim_{x\to\infty} a(x) = \infty$ and such that $\int_0^\infty a(x)f(x)dx < \infty$.

(b) Let $\{f_n\}$ be a sequence of nonnegative Lebesgue measurable functions on $[0,\infty)$ such that $\lim_{n\to\infty} \int_0^\infty f_n(x)dx = 0$. Show that there exists a positive, strictly increasing measurable function b on $[0,\infty)$ with $\lim_{x\to\infty} b(x) = \infty$ and such that $\lim_{n\to\infty} \int_0^\infty b(x)f_n(x)dx = 0$.

Part (a)

Proof. As $f \in L^1((0,\infty))$, let $x_n \in (0,\infty)$ be a strictly increasing sequence such that

$$\int_{x_n}^{x_{n+1}} f(x)dx < \frac{1}{2^n}.$$

Let $a(x_n) = n$ and $a(0) = 0$. Define $a(x)$ everywhere by letting it take value on the line segment connecting (x_n, n) and $(x_{n+1}, n+1)$

$$a(x) = \left(\frac{1}{x_{n+1} - x_n}\right)(x - x_n) + n, \qquad \text{when } x \in (x_n, x_{n+1}).$$

Since the intervals (x_n, x_{n+1}) are disjoint we conclude that a is positive and strictly increasing. Also, our construction guarantees that on $[x_n, x_{n+1}]$ we have $a(x) \leq n+1$. Moreover,

$$\int_{x_1}^\infty a(x)f(x)dx = \sum_1^\infty \int_{x_n}^{x_{n+1}} a(x)f(x)dx \leq \sum_1^\infty \frac{n+1}{2^n} < \infty.$$

□

Part (b)

Proof. Without loss of generality assume that all of the f_n are integrable (since all but finitely many must be integrable). We claim there exists a strictly increasing sequence $\{y_k\}$ such that for all n we have

$$\int_{y_k}^\infty f_n(y)dy < \frac{1}{2^k} \tag{1}$$

This is true since we know that there exists N such that for all $n > N$ it holds that

$$\int_0^\infty f_n(y)dy < \frac{1}{2^k}.$$

Since the remaining f_j with $j \leq N$ are integrable there exists a y_k^j, $j = 1, \ldots, N$ such that

$$\int_{y_k^j}^\infty f_j(y)dy < \frac{1}{2^k}.$$

Let $y_k = \max_{j=1,\ldots,N} y_k^j$. This construction guarantees that (1) holds. Moreover, (1) implies the weaker conclusion that

$$\int_{y_k}^{y_{k+1}} f_n(y)dy < \frac{1}{2^k}, \qquad \forall n.$$

Let $b(x)$ be the function with $b(y_k) = k$ and $b(y)$ is on the line segment connecting (y_k, k) and $(y_{k+1}, k+1)$ with $y \in [y_k, y_{k+1}]$ as in Part (a). Since $b(y) \leq k+1$ for $y \in [y_k, y_{k+1}]$, we can now write (taking $y_0 = 0$) for fixed n,

$$\begin{aligned}\int_0^\infty b(y) f_n(y) dy &= \sum_{k=0}^\infty \int_{y_k}^{y_{k+1}} b(y) f_n(y) dy \\ &= \sum_{k=0}^\infty \frac{(k+1)}{2^k} \\ &< M < \infty.\end{aligned}$$

We can apply Hölder's inequality to write

$$\int \sqrt{b} f_n = \int \sqrt{b}\sqrt{f_n}\sqrt{f_n} \leq \sqrt{\left(\int b f_n\right)\left(\int f_n\right)}.$$

Since $\int b f_n$ is bounded independent n and $\int f_n \to 0$ we have $\sqrt{b}$ is our function (it still has the desired properties.) ☐

(2007 #3) Let $f : [0,1] \to M$ with M a compact metric space with metric $d(x,y)$. Let $\Omega = \{f \colon [0,1] \to M \text{ such that } d(f(s), f(t)) \leq |s-t|\}$. Let $f, g \in \Omega$ and define $\rho(f,g) = \sup_{t \in [0,1]} d(f(t), g(t))$. Show that (Ω, ρ) is sequentially compact.

Proof. The easy way is to use the following version of Arzela-Ascoli

Theorem 11 (Arzela-Ascoli)**.** *Let $f_\alpha \in BC(X,Y)$ where X is a compact metric space and Y is a metric space. TFAE*

(i) $\{f_\alpha\}_{\alpha \in A}$ is precompact in $BC(X,Y)$.

(ii) $\{f_\alpha\}_{\alpha \in A}$ is pointwise precompact and equicontinuous.

It follows quickly form this. But this isn't in Folland. So consider the following . Let $\{f_n\} \in \Omega$ and take $\{q_i\}$ to be an enumeration of the rationals in $[0,1]$. For each i have a subsequence $f_{i,n}(q_i) \to f(q_i)$ where we define $f(q_i)$ to be the value of the limit. By constructing a diagonal sequence we then have a sequence $g_i = f_{ii}$ which converges on each rational. We will show that the limit exists for all $s \in [0,1]$. To do this we prove g is Cauchy. Fix $s \in [0,1]$. We have

$$\begin{aligned}d(g_m(s), g_n(s)) &\leq d(g_m(s), g_m(q)) + d(g_m(q), g_n(q)) + d(g_n(q), g_n(s)) \\ &\leq 2|s-q| + d(g_m(q), g_n(q)).\end{aligned}$$

So if we let $q \to s$ with $q \in \mathbb{Q}$ we have $d(g_m(s), g_n(s)) \to 0$. By completeness we now have g_i converges for all $s \in [0,1]$. Next, we prove that $g_i \to f$ uniformly. Make a δ-ball cover of $[0,1]$ with $B_1, B_2, \ldots, B_n$ with radius δ. Choose $q_j \in B_j$ and find N such that $n \geq N$ implies $d(g_n(q_j), f(q_j)) < \epsilon$ for $1 \leq j \leq n$. We then have

$$\begin{aligned}d(g_n(s), f(s)) &\leq d(g_n(s), g_n(q_j)) + d(g_n(q_k, g_m(q_j))) + d(g_m(q_j), f(s)) + d(g_m(s), f(s)) \\ &< 2\delta_d(g_n(q_j), g_m(q_j)) + d(g_m(s), f(s)).\end{aligned}$$

As $m \to \infty$ this goes to zero. It remains to prove the space is closed. This is a consequence of the fact that the unit ball in the Lipschitz norm is closed in the uniform norm on $C[0,1]$.

$\square$

(2007 #4) Let f be a strictly positive Borel measurable function on $\mathbb{R}$ and let E be a Borel measurable subset of $\mathbb{R}$ with strictly positive measure. For every $t \in \mathbb{R}$ define

$$\varphi(t) = \int_E f(t+x)dx.$$

(a) Prove that φ is Borel measurable from $\mathbb{R}$ to $[0,\infty]$.

(b) Suppose that $\varphi \in L^1(\mathbb{R})$. Prove that E has finite Lebesgue measure and that $f \in L^1(\mathbb{R})$.

Part (a)

Proof. An elementary way to deduce the measurability of φ is to appeal to Tonelli's theorem. When we consider the integral $\int_{\mathbb{R}}\int_{\mathbb{R}} f(x+t)\mathbb{1}_E(t)dxdt$ notice that f and $\mathbb{1}_E$ are positive functions. Additionally, since f is a Borel function and $(x,t) \mapsto x+t$ is also Borel we conclude that the composition $f(x+t)$ is Borel. Also $\mathbb{1}_E$ is certainly Borel (since E is Borel). Because a product of Borel functions is also Borel we satisfy the hypotheses of Tonelli's theorem and conclude that the function

$$x \mapsto \int_{\mathbb{R}} f(x+t)\mathbb{1}_E(t)dt = \int_E f(x+t)dt = \varphi(t)$$

is measurable.

$\square$

Part (b)

Proof. Since $\|\varphi\|_1 < \infty$ and $\varphi \geq 0$ we can write

$$\int_{\mathbb{R}} \varphi(t)dt = \int_{\mathbb{R}}\int_E f(x+t)dxdt < \infty.$$

We can therefore apply Tonelli's theorem to interchange the order of integration. Also we use the fact that the L^1 norm is translation invariant and so $f_x(t) = f(x+t)$ satisfies $\|f_x\|_1 = \|f\|_1$. These two facts allow us to rewrite the integral as

$$\int_E\int_{\mathbb{R}} f(x+t)dtdx = \int_E\int_{\mathbb{R}} f_x(t)dtdx = m(E)\,\|f\|_1 < \infty$$

$\square$

(2007 #5) Let $1 \leq p < \infty$ and let ℓ^p be the Banach space of the p^{th} power summable complex valued sequences, so that $x = \{x_n\} \in \ell^p$ provided that

$$\|x\|_p = \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} < \infty.$$

Let E be a closed subset of ℓ^p. Prove that E is compact in the norm topology of ℓ^p if and only if E satisfies the following conditions:

(i) There exists a constant $C > 0$ such that $\|x\|_p \leq C$ for all $x \in E$.

(ii) For every $\epsilon > 0$ there is $n_0 \in \mathbb{N}$ such that for all $x \in E$ we have $\sum_{n=n_0}^\infty |x_n|^p < \epsilon$.

Proof. First observe that in a metric space we have a closed subset is compact if and only if it is complete and totally bounded. Since E is closed it suffices to prove the following

E is totally bounded if and only if (i) and (ii) hold.

($\Rightarrow$) Suppose that E is totally bounded. As E is totally bounded certainly (i) holds. To prove (ii) fix $\epsilon > 0$ and let $\{x^{(1)}, \ldots, x^{(k)}\} \subseteq E$ be an ϵ-net. As $\sum_{n=1}^\infty |x_n^{(i)}|$ converges, let $n_1, \ldots, n_k$ be such that $\sum_{n=n_i}^\infty |x_n^{(i)}|^p < \epsilon$. Let $n_0 = \max\{n_1, \ldots, n_k\}$. Letting $\|x\|^{n_0} = \sum_{n=n_0}^\infty |x_n|^p$, it now holds for any $x \in E$ we must have for some $x^{(i)}$ in our net that $\left\|x - x^{(i)}\right\| < \epsilon$. Applying the triangle inequality yields

$$\|x\|^{n_0} \leq \left\|x - x^{(i)}\right\|^{n_0} + \left\|x^{(i)}\right\|^{n_0} \leq \left\|x - x^{(i)}\right\| + \epsilon < 2\epsilon.$$

Thus, (ii) is proven.

($\Leftarrow$) Suppose that (i) and (ii) hold. Fix $\epsilon > 0$, we will construct an ϵ-net on E. For $x \in E$ let $x_{[n]} = (x_1, \ldots, x_n)$. Let n_0 correspond to $(\epsilon/2)^p$ as in (ii). Consider the projection $\pi : E \to E_{n_0} : x \mapsto x_{[n_0]}$. This maps E into a closed and bounded (by (i)) subset of $\mathbb{R}^{n_0}$. Since $\mathbb{R}^{n_0}$ is finite dimensional we know that E_{n_0} is compact and so there exists an $\epsilon/2$-net $\{x^{(1)}, \ldots, x^{(N)}\}$. Let $z^{(1)} = (x^{(1)}, 0, 0, \ldots)$, $z^{(2)} = (x^{(2)}, 0, 0, \ldots)$, and so on up to $z^{(k)} = (x^{(k)}, 0, 0, \ldots)$. Consider any $x \in E$. Suppose that $\pi(x) \in \mathbb{B}(x^{(i)}, \epsilon)$. It follows that

$$\begin{aligned}\left\|x - z^{(i)}\right\| &\leq \left\|\pi(x) - x^{(i)}\right\| + \|(0, 0, \ldots, x_{n_0+1}, x_{n_0+2}, \ldots\| \\ &\leq \epsilon + \left(\sum_{n_0}^\infty |x_n|^p\right)^{1/p} \\ &= \epsilon/2 + ((\epsilon/2)^p)^{1/p} \\ &= \epsilon.\end{aligned}$$

So the $z^{(i)}$ form and ϵ-net.

□

(2007 #6) Let X and Y be compact metric spaces. Suppose that $\varphi : X \to Y$ is continuous and surjective. Let

$$D := \{f \in C(X) : f(x) = f(x') \text{ whenever } \varphi(x) = \varphi(x')\}.$$

(a) Show that D is a closed subspace of $C(X)$ and that

$$D = \{g \circ \varphi : g \in C(Y)\}.$$

(b) Let ν be a finite positive Borel measure on Y. Prove that there is a finite positive Borel measure μ on X such that $\mu(\varphi^{-1}(F)) = \nu(F)$ for all Borel subsets $F \subseteq Y$.

Part (a)

Proof. Suppose that $f_n \to f$ in the sup norm with $f_n \in D$. We then have by continuity whenever $\varphi(x) = \varphi(x')$ it holds that

$$f(x) = \lim f_n(x) = \lim f_n(x') = f(x').$$

It follows that $f_n \to f$ point wise. Therefore $f \in D$. Let $f \in D$. Define $g : Y \to \mathbb{R}$ by $g(y) := f(x)$ where $x \in \varphi^{-1}(y)$. Suppose $x, x' \in \varphi^{-1}(y)$, then $f(x) = f(x')$, so the map is well defined. We would like to prove that g is continuous. Let $\varphi(x_n) =: y_n \to y$. Since we are in a compact metric space there exists a sequence $x_{n_k} \to x \in X$. Since φ is continuos we know that

$$y_{n_k} := \varphi(x_{n_k}) \to \varphi(x).$$

Therefore $y = \varphi(x)$ and thus $f(x) = g(y)$. Using a subsequence of subsequence convergence argument we have g is continuous. □

Part (b)

Lemma 4. *Suppose $M \subseteq C(X)$ is a subspace with X compact. Moreover, suppose there exists $h \in M$ with $h(x) \geq c_0 > 0$ for all $x \in X$. Then any positive linear functional $\ell_0 \in M^*$ can be extended to a positive linear functional $\ell \in C(X)^*$.*

Proof. Let $p(f) = \inf\{\ell_0(g) \colon g \in M, g \geq f\}$. We have p is positive and subadditive and $p(g) \geq \ell_0(g)$ for all $g \in M$. By Hahn-Banach we have ℓ_0 extends to ℓ with $\ell(f) \leq p(f)$ for all $f \in C(X)$. If $f \leq 0$ then

$$\ell(f) \leq p(f) \leq p(0) = \ell_0(0) = 0.$$

□

Proof of Problem Statement

Proof. We are given an object in $\nu \in C(Y)^*$. We want to show that given $f \in D$ there exists a unique $g \in C(Y)$ such that $f = g \circ \varphi$. Define a linear functional on $f = g \circ \varphi \in D$ by

$$Lf = \int g d\nu.$$

This is a bounded linear functional on the closed set D. By Hahn-Banach we can extend to $C(X)$. Lastly, by Reisz Representation there exists a positive μ such that $Lf = \int f d\mu = \int g d\nu$. Hence

$$\int g \circ \varphi d\mu = \int g d\nu.$$

Let the g converge to a characteristic function of a Borel subset $F \subseteq Y$ with $K \subseteq F \subseteq U$ where K is compact and U is open. Urysohn guarantees for any $\epsilon > 0$ there exists sets K, U and a continuous function ψ such that $\mathbb{1}_K \leq \psi \leq \mathbb{1}_U$. We now have

$$\nu(K) \leq \int \psi d\nu = \int \psi \circ \varphi d\mu \leq \nu(U).$$

Since $\psi \to \mathbb{1}_F$ as $\epsilon \to 0$ we can apply DCT to obtain

$$\nu(F) = \int \mathbb{1}_C \circ \varphi d\mu - \mu(\varphi^{-1}(F)).$$

Since ν is positive we know that μ is positive. ☐

(2007 #7) Let X be a compact metric space and μ be a finite positive Borel measure on X. Suppose that $\mu(\{x\}) = 0$ for every $x \in X$. Prove that for every $\epsilon > 0$ there is a $\delta > 0$ such that if E is any Borel subset of X having diameter less than δ, then $\mu(E) < \epsilon$.

Proof. Around each point in X consider $\mathbb{B}(x, 1/n)$. We have that from our singletons hypothesis that $\bigcap_n \mathbb{B}(x, /1n) = x$. Also $B(x, 1/n) \subseteq B(x, 1/(n-1))$. So we have

$$0 = \mu(\{x\}) = \lim_{n\to\infty} \mu(\mathbb{B}(x, 1/n)).$$

Hence, for each $\epsilon > 0$ and $x \in X$ there exists $N_x \in \mathbb{N}$ such that $\mu(\mathbb{B}(x, 1/n)) < \epsilon$ for $n \geq N_x$. We have $\{B(x, \frac{1}{2N_x}\}_{x\in X}$ is an open cover of X and so has a finite subcover $\{B_i\}_1^n$. Let $N = \max\{N_{x_i}\}$ and suppose that E is a nonempty Borel subset of X with diameter less than $\delta = \frac{1}{2N}$. et $e_0 \in E$. Then there exist x_i such that $e_0 \in B_i$ and for any $e \in E$ it holds that $d(e_0, e) \leq \delta$. By the triangle inequality

$$d(x_i, e) \leq d(x_i, e_0) + d(e_0, e) \leq \frac{1}{2N_i} + \frac{1}{2N} \leq \frac{1}{2N_i} + \frac{1}{2N_i} = \frac{1}{N_i}.$$

Hence $e \in B_i$ and so $\mu(E) \leq \mu(B_i) < \epsilon$. ☐

(2007 #8) Let T be the triangle $\{(x, y) \in \mathbb{R}^2 : 0 \leq |x| \leq y \leq 1\}$ and μ be the restriction of the planar Lebesgue measure to T. Suppose that $f \in L^2(T, \mu)$. Prove that

$$\liminf_{y\to 0^+} \int_{-y}^{y} |f(x, y)| dx = 0.$$

Proof. Let $\varphi(y) = \int_{-y}^{y} |f(x, y)| dy$ and let $\alpha = \inf_{y\to 0^+} \varphi(y)$. Clearly, $\alpha \geq 0$. Suppose to show a contradiction that $\alpha > 0$. Hence there exists $\epsilon, \delta > 0$ such that for all $y < \delta$ it holds that $\varphi(y) \geq \alpha + \epsilon$. Let $T_\delta = \{0 \leq |x| \leq \delta\}$. We have

$$\begin{aligned}
\delta(\alpha + \epsilon) &= \int_0^\delta (\alpha + \epsilon) dy \\
&\leq \int_0^\delta \varphi(y) dy \\
&= \int_{T_\delta} |f(x, y)| dy \\
&\leq \sqrt{\mu(T_\delta)} \left\| \mathbb{1}_{T_\delta} f \right\|_2 .
\end{aligned}$$

Notice that $\mu(T_\delta) = \frac{1}{2}(\text{base})(\text{width}) = \frac{1}{2}(2\delta)(\delta) = \delta^2$. Therefore

$$\delta(\alpha + \epsilon) \leq \delta \left\| \mathbb{1}_{T_\delta} f \right\|_2 .$$

Canceling the δ gives

$$\alpha + \epsilon \leq \|\mathbb{1}_{T_\delta} f\|_2 .$$

This is a contradiction since $\|\mathbb{1}_{T_\delta} f\|_2 \to 0$ as $\delta \to 0$.

□

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(2008 #1) Let $g \in L^2(\mathbb{R})$ and set $f(x) = \int_0^x g(t)dt$ for $x \in \mathbb{R}$.

(a) Show by example that f need not be differentiable at 0.

(b) Must f have any points of differentiability? Explain.

(c) Let $\varphi(x) = f(x)^2$. Show that φ is differentiable at 0 and find $\varphi'(0)$.

Part (a)

Proof. Let $g = \frac{1}{\sqrt{3}} \cdot \mathbb{1}_{[0,1]} x^{-1/3}$. First off $g \in L^2$ since

$$\int_{\mathbb{R}} |g|^2 = \int_0^1 \frac{1}{3} x^{-2/3} dx = \sqrt[3]{x}\Big|_0^1 = \sqrt[3]{1} - \sqrt[3]{0} = 1.$$

We claim that $f'(0)$ does not exist, we turn to the limit definition of the derivative.

$$f'(0) = \lim_{h\to 0} \frac{1}{h} \int_0^h \frac{1}{\sqrt{3}} \cdot \mathbb{1}_{[0,1]} x^{-1/3} dx = \frac{1}{\sqrt{3}} \lim_{h \to 0} \frac{1}{h} \sqrt[3]{h} = \frac{1}{\sqrt{3}} \lim_{h\to 0} \frac{1}{h^{2/3}}.$$

As the final limit does not exist, we conclude that $f'(0)$ does not exist. □

Part (b)

Proof. Yes. In fact f must be almost everywhere differentiable. This follows from the Lebesgue differentiation theorem. Letting $I_{h,x} = (x, h)$ we can write $f'(x)$ as

$$f'(x) = \lim_{h\to 0} \frac{1}{h} \int_x^h g(t)dt = \lim_{m(I_{h,x})\to 0} \frac{1}{m(I_h, x)} \int_{I_{h,x}} g(t)dt.$$

The above line says that the set of all x for which $f'(x)$ exists is precisely the Lebesgue set of f. Since $g \in L^2(\mathbb{R})$ we know that $g \in L^1_{loc}(\mathbb{R})$ and we have now written f' in the necessary form to apply the Lebesgue differentiation theorem. Hence f' is defined for every set in the Lebesgue set of f which is a.e. □

Part (c)

Proof. We will prove that $\varphi'(0) = 0$ using the definition of the derivative and then Hölder's inequality.

$$\begin{aligned}
\varphi'(0) &= \lim_{h\to 0} \frac{1}{h}\left(\int_0^h g(t)dt\right)^2 \\
&= \lim_{h\to 0} \frac{1}{h}\left|\int_0^h g(t)dt\right|^2 \\
&\le \lim_{h\to 0} \frac{1}{h}\left(\int_0^h |g(t)|\, dt\right)^2 \\
&= \lim_{h\to 0} \frac{1}{h}\left(\int_0^h |g(t)|\, |1| dt\right)^2 \\
&\overset{*}{\le} \lim_{h\to 0} \frac{1}{h}\left(\int_0^h |g(t)|^2\, dt\right)\left(\int_0^h |1|^2 dt\right) \\
&= \lim_{h\to 0} \frac{1}{h}\left(\int_0^h |g(t)|^2\, dt\right)\cdot h \\
&= \lim_{h\to 0} \left(\int_0^h |g(t)|^2\, dt\right) \\
&\overset{**}{=} 0
\end{aligned}$$

Where at * we think of $\left(\int_0^h |g(t)|\,|1|dt\right)^2 = \langle g, 1\rangle^2 \le \|g\|^2_{L^2([0,h])}\,\|1\|^2_{L^2([0,h])}$ as an inner product on the space $L^2([0,h])$ and apply Cauchy's theorem. This is acceptable since both $g, 1 \in L^2([0,h])$. At ** we use the fact that $g \in L^2(\mathbb{R})$ and thus the limit goes to zero (this can be shown using dominated convergence by considering the sequence of functions $g_n(x) = \mathbb{1}_{[0,1/n]}(x)g(x)$ and taking a limit). □

(2008 #2) Evaluate $\lim_{n\to\infty} n^{3/2}\int_0^1 \frac{x^2}{(1+x^2)^n}dx$.

Proof. Let $y = n^{1/2}x$ so that $dy = n^{1/2}dx$ we have

$$\lim_{n\to\infty}\int_0^{n^{1/2}} \frac{y^2}{(1+\frac{y^2}{n})^n}dy.$$

Since $(1+\frac{y^2}{n})^n \nearrow e^{y^2}$ we can dominate by $\frac{y^2}{(1+\frac{y^2}{2})^2}$ which is integrable and apply DCT to write the integral as

$$\lim_{n\to\infty}\int_0^{n^{1/2}} \frac{y^2}{(1+\frac{y^2}{n})^n}dy. = \int_0^\infty \frac{y^2}{e^{y^2}}dy.$$

We can evaluate this using integration by parts with $u = y$ and $dv = ye^{-y^2}$. To get the answer $\int_0^\infty e^{-y^2}dy = \sqrt{\pi}/2$.

□

(2008 #3) Let X be a compact Hausdorff space. Suppose that $\{f_n, n \geq 1\}$ is a sequence of continuous real-valued functions on X such that $f_{n+1}(x) \leq f_n(x)$ for every $n \geq 1$ and $x \in X$. Suppose also that $f(x) = \lim_{n\to\infty} f_n(x)$ is finite for every $x \in X$. Show that if the limiting function f is continuous on X then f_n converges to f uniformly on X.

Proof. Suppose that $g_n = f_n - f$ and that $g_n \geq 0$ with $g_n \geq g_{n+1}$, also suppose that g_n is continuous. Fix $\epsilon > 0$. Let $K_n = \{x \in X \mid g_n(x) \geq \epsilon\}$. Notice that K_n is closed. Since X is compact this implies that K_n is compact. Since the g_n are decreasing we have $K_{n+1} \subseteq K_n$. For all $x \in X$ there exists N_x such that $g_n(x) < \epsilon$ for all $n > N_x$. This implies that $x \notin K_{N_x}$. Therefore $x \notin \bigcap_1^\infty K_n$, but this says $\bigcap_1^\infty K_n = \emptyset$. By finite intersection property and compactness we know that there is an integer M such that $\bigcap_1^N K_n = \emptyset$. Hence $g_n \to 0$ uniformly on x which implies that $f_n \to f$ uniformly. □

(ALTERNATE PROOF)

Proof. Alternatively we could have defined $U_n = \{x \mid f_n(x) - f(x) < \epsilon\}$. Since U_n is the preimage of $(-\epsilon, \epsilon)$ we have U_n is open by continuity and $U_n \subseteq U_{n+1}$ since the f_n are decreasing. Also $X = \bigcup_n U_n$ because $f_n \to f$ pointwise. By compactness we have a finite subcover $X = \bigcup_1^N U_n$. Therefore $X = U_{n_0}$ for some n_0. This means that $|f_k - f| < \epsilon$ for all $k \geq n_0$ and therefore the convergence is uniform. □

(2008 #4) Suppose $\mathcal{H}$ is a separable real Hilbert space with an orthonormal basis $\{e_k, k \geq 1\}$ and with inner product denoted by $(\cdot,\cdot)$. Let $\{y_k\} \subseteq \mathcal{H}$. Prove that the following two statements are equivalent.

(i) $\lim_{k\to\infty}(x, y_k) = 0$ for every $x \in \mathcal{H}$.

(ii) $\sup_{k\geq 1} \|y_k\| < \infty$ and $\lim_{k\to\infty}(e_n, y_k) = 0$ for every $n \geq 1$.

Proof. (i) ⇒ (ii) The second half is immediate. Since for all x we know that $\lim(x, y_k) = 0$ we have for each fixed x it holds that

$$\sup_{k\geq 1} |(x, y_k)| < \infty.$$

And for each fixed y_k, the functional given by $(\cdot, y_k)$ is a bounded linear functional. By the uniform boundedness principle it follows that

$$\sup_{k\geq 1} \sup_{\|x\|=1} |(x, y_k)| < \infty.$$

In particular,

$$\sup_{k\geq 1} \left| \left(\frac{y_k}{\|y_k\|}, y_k \right) \right| < \infty.$$

Hence

$$\sup_{k\geq 1} \|y_k\| < \infty.$$

(ii) $\Leftarrow$ (i) Let $M = \sup_{k\geq 1} \|y_k\|$ and suppose $x \in \mathcal{H}$. Using our basis, let us write $x = \sum_1^\infty (x, e_n)e_n$. For every $\epsilon > 0$ there exists N such that

$$\left\| x - \sum_1^N (x, e_n)e_n \right\| < \frac{\epsilon}{2M}.$$

For any y_k we have by linearity of the inner product

$$\begin{aligned} |(x, y_k)| &= \left| \left(x - \sum_1^N (x, e_n)e_n, y_k \right) + \left(\sum_1^N (x, e_n)e_n, y_k \right) \right| \\ &\leq \left\| x - \sum_1^N (x, e_n)e_n \right\| \|y_k\| + \sum_1^N |(x, e_n)(e_n, y_k)|. \end{aligned}$$

Since $\lim_{k\to\infty}(e_n, y_k) = 0$ for all n we have for each $n \in N$ there exists $K_n \in \mathbb{N}$ such that

$$|(e_n, y_k)| < \frac{\epsilon}{2N|(x, e_n) + 1|}, \qquad \forall k \geq K_n.$$

Let $K = \max_{n\leq N}\{K_n\}$. We now have for all $k \geq K$ it holds that

$$|(x, e_n)(e_n, y_k)| < |(x, e_n)| \frac{\epsilon}{2N|(x, e_n) + 1|} < \frac{\epsilon}{2N}.$$

We have now shown that

$$|(x, r_k)| < \frac{\epsilon}{2M} M + \sum_1^N \frac{\epsilon}{2N} = \epsilon.$$

□

(2008 #5) Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space. Let $1 < p < \infty$ and let $f_n, f \in L^p(X, \mu)$. Suppose that $\sup_n \|f_n\|_p < \infty$ and that $f_n \to f$ a.e. Show that $f_n \rightharpoonup f$ in L^p.

Proof. Fix $g \in L^q$. Let $\{X_n\} \subseteq X$ be such that $\mu(X_n) < \infty$, $X_n \subseteq X_{n+1}$ and $X = \bigcup X_n$. Choose N sufficiently large such that

$$\left(\int_{X\setminus X_N} |g|^q d\mu \right)^{1/q} < \frac{\epsilon}{2M}$$

where $M = \|f\|_p + \sup_n \|f_n\|_p$. Next we may choose $\delta > 0$ such that if $A \in \mathcal{F}$ and $\mu(A) < \delta$ then $\left(\int_A |g|^q d\mu\right)^{1/q} < \frac{\epsilon}{2M}$ by absolute continuity of μ. By Egorov's theorem there exists $E \subseteq X_N$ such that $\mu(E) < \delta$ and $f_n \to f$ uniformly on $X_N \setminus E$. Hence

$$\begin{aligned} \left| \int_X (f_n - f) g d\mu \right| &\leq \int_{X\setminus X_N} |f_n - f)|g| d\mu + \int_{X_N\setminus E} |f_n - f||g| d\mu + \int_E |f_n - f||g| d\mu \\ &\leq \|f_n - f\|_p \left(\int_{X\setminus X_N} |g|^q d\mu \right)^{1/q} + \|g\|_q \left(\int_{X_N\setminus E} |f_n - E|^p d\mu \right)^{1/p} + \|f_n - f\|_p \left(\int_E |g|^q d\mu \right)^{1/q} \\ &\to 0 \end{aligned}$$

Which is the definition of $f_n \rightharpoonup f$. □

(2008 #6) Let (X, d) be a metric space. Let $f : X \to X$ be such that $d(x, y) = d(f(x), f(y))$ for all $x, y \in X$. Prove that if (X, d) is compact then f is surjective.

Proof. Suppose f is not surjective. Let $x \notin f(X)$ and let $\epsilon = \text{dist}(x, f(X))$. Since f is continuous (it is Lipschitz) we know that $f(X)$ is compact and so $\epsilon > 0$. Again, because $f(X)$ is a compact set it can be covered by a finite collection of balls $B_k = B(x_k, \epsilon)$. Let $\{B_k\}_1^N$ be a minimal collection of such balls. Let $D_k = f^{-1}(B_k)$. Since f is an isometry we know that each $D_k = \mathbb{B}(f^{-1}(x_k), \epsilon)$ and thus $\{D_k\}_1^N$ is a covering of X be ϵ-balls. Without loss of generality suppose that $x \in D_N$. Since $\text{dist}(x, f(X)) = \epsilon$ it follows that $\{D_k\}_1^{N-1}$ still covers $f(X)$, contradicting minimality. □

(2008 #7) Let E be the set of real numbers in $[0, 1]$ whose decimal expansion contains an infinite number of 7's. Show that E is Lebesgue measurable and determine the Lebesgue measure of E.

Proof. We can kill two birds with one stone by proving the $m^*(E^c) = 0$. Let

$$E_k = \{x \in [0, 1] : x \text{ contains } k \text{ 7's}\}.$$

Notice that $E^c = \bigcup_{k=0}^\infty E_k$. By countable subadditivity it suffices to prove that $m^*(E_k) = 0$ for all $k \geq 0$. We proceed by induction.

Base Case: Define the sets

$$F_n^0 = \{.a_0 a_1 \dots a_n \dots : a_j \neq 7 \text{ when } j \leq n\},$$

Observe that $F_n \subseteq F_{n+1}$ and also notice we can write $E_0 = \bigcap_{n=0}^\infty F_n^0$. We claim that $m(F_n^0) = \left(\frac{9}{10}\right)^n$, which is easy to see since $(F_n^0)^c = \{.77\dots 7a_{n+1}a_{n+2}\dots\}$ has measure $\left(\frac{1}{10}\right)^n$ by a simple counting argument. It follows that $m(E_0) = \lim_{n\to\infty} m(F_n^0) = 0$.

Inductive Step: Suppose that $m(E_k) = 0$. Define the sets

$$E_k^n = \{.b_0 b_1 \dots b_{n-1} 7 a_0 a_1 \dots \mid .a_0 a_1 \dots \in E_k \text{ and } b_j \neq 7\}.$$

Notice that $E_{k+1} = \bigcup_{n=0}^\infty E_k^n$. Also, we can write

$$E_k^n = \bigcup_{b_1,\dots,b_{n-1}\neq 7} \left\{ \frac{1}{10^n} x + .b_1 \dots b_{n-1} 700 \dots \mid x \in E_n \right\}.$$

By translation and scaling invariance it follows that

$$m(E_k^n) = \frac{1}{10^n} \sum_{b_1,\dots,b_{n-1}\neq 7} m(E_k) = 0.$$

Where we use the fact that the sum is finite.

□

(2008 #8) Let f be a positive, continuously differentiable function on $(0,\infty)$ satisfying $f'(x) > 0$ for all $x \in (0,\infty)$. Suppose that for some constant $C > 0$

$$f(x) \leq Cx^2, \qquad x \geq 1.$$

Show that

$$\int_0^\infty \frac{1}{f'(x)} dx = \infty.$$

Hint: Establish and use the fact that if $\int_0^\infty \frac{1}{f'(x)} dx < \infty$ *then* $\lim_{a\to\infty} \int_a^\infty \frac{1}{f'(x)} dx = 0$.

Proof. (Hon) It holds for any $g \in L^1((0,\infty))$ with $g > 0$ that $\lim_{a\to\infty} \int_a^\infty g(x)dx = 0$. This follows from the monotone convergence theorem.

Suppose to show a contradiction that $\int_0^\infty \frac{1}{f'(x)} dx < \infty$. By the previous discussion this implies that

$$\lim_{a\to\infty} \int_a^\infty \frac{1}{f'(x)} dx = 0.$$

This is equivalent to

$$\lim_{a\to\infty} \lim_{b\to\infty} \int_a^b \frac{1}{f'(x)} dx.$$

Let $a > 1$. For any $b > a$ we have

$$\begin{aligned}(b-a)^2 &= \left(\int_a^b \frac{1}{\sqrt{f'(x)}} \sqrt{f'(x)} dx \right)^2 \\ &\leq \left(\int_a^b \frac{1}{f'(x)} dx \right) \left(\int_a^b f'(x) dx \right) \\ &= \left(\int_a^b \frac{1}{f'(x)} dx \right) (f(b) - f(a))\end{aligned}$$

As $f(b) > f(a)$ (since $f' > 0$) we use the fact that $f(x) \geq Cx^2$ to conclude that

$$\int_a^b \frac{1}{f'(x)} dx \geq \frac{(b-a)^2}{f(b) - f(a)} \geq \frac{(b-a)^2}{C(b^2 - a^2)}.$$

Now when we take $b \to \infty$ we obtain

$$\lim_{b\to\infty} \int_a^b \frac{1}{f'(x)} dx \geq \frac{1}{C}.$$

We now arrive at the contradiction

$$\lim_{a\to\infty} \int_a^\infty \frac{1}{f'(x)} dx \geq \frac{1}{C} > 0.$$

□

19 2009

(2009 #1) Suppose that $A \subseteq [0,1]\times[0,1]$ is Borel and let

$$A_x = \{r \in [0,1] : (x,y) \in A\}, \qquad A^y = \{x \in [0,1] : (x,y) \in A\},$$

$$B = \{x \in [0,1] : m(A_x) = \frac{1}{3}\}.$$

Suppose that $m(B) = \frac{1}{5}$. Prove that there exists $y \in [0,1]$ such that $m(A^y) \leq 13/15$.

Proof. Suppose for all $y \in [0,1]$ we have $m(A^y) > 13/15$. Since A is Borel and of finite measure, we can apply Tonelli and write

$$m(A) = \int_0^1 \int_0^1 \mathbb{1}_{A^y}(x)dxdy = \int_0^1 m(A^y)dy > 13/15. \qquad (1)$$

Alternatively, we can write

$$m(A) = \int_0^1 \int_B m(A_x)dx + \int_0^1 \int_{B^c} m(A_x)dx \overset{*}{=} \frac{1}{15} + \int_0^1 \int_{B^c} m(A_x)dx.$$

Where at $*$ we use the fact that for $x \in B$ we have $m(A_x) = 1/3$ and that $m(B) = 1/5$. Worst case scenario would be that for all $x \in B^c$ it holds that $m(A_x) = 1$. Because $m(B^c) = \frac{4}{5}$ we conclude

$$m(A) \leq \frac{1}{15} + \frac{4}{5}\cdot 1 = \frac{13}{15}.$$

This contradicts (1). □

(2009 #2) Let (X,μ) be a measure space. Suppose $1 < p < 2$ and $1/p + 1/q = 1$. We denote $L^p = L^p(\mu)$.

(a) Show that $L^p \cap L^q$ is complete with respect to the norm $\|f\| = \|f\|_p + \|f\|_q$.

(b) Show that if $f \in L^p \cap L^q$ then $f \in L^2$ and $\|f\|_2 \leq \sqrt{\|f\|_p \|f\|_q}$.

(c) Show that $L^p \cap L^q$ is dense in L^2 with respect to the L^2 norm.

(d) Since $\sqrt{ab} \leq (1/2)(a+b)$ for $a, b \geq 0$, part (b) implies that $\|f\|_2 \leq (1/2)\|f\|$. Show that if there is a constant $c > 0$ such that $\|f\|_2 \geq c\|f\|_{p,q}$ for all $f \in L^p \cap L^q$ then $L^p \cap L^q = L^2$. Give an example of a space (X,μ) where this is the case.

Part (a)

Proof. Let $\{f_n\}$ be a Cauchy sequence in $L^{p,q}$. Hence $\{f_n\}$ is Cauchy in L^p and L^q. Since L^p and L^q are complete we know that $f_n \overset{L^p}{\to} f$ and $f_n \overset{L^q}{\to} f'$ for $f \in L^p$ and some $f' \in L^q$. Since L^r convergence implies convergence in measure we know that

$$f_n \overset{\mu}{\to} f, \qquad f_n \overset{\mu}{\to} f'.$$

But since μ-convergence has a unique limit* we conclude that $f = f'$ a.e. and thus $f_n \overset{L^{p,q}}{\to} f + f = 2f$.

(*Another way to proceed would be to use the fact that L^r convergence implies that given any subsequence $\{f_{n_j}\}$ there exists a pointwise convergent sub-subsequence. Accordingly, ever subsequence has a subsequence converging to f and another converging to f'. Therefore $f = f'$ a.e.)

□

Part (b)

Proof. We apply Hölder's inequality to $|f \cdot f| = |f|^2$.

$$\int |f|^2 = \int |f \cdot f| \\ \leq \|f\|_p \|f\|_q$$

It follows that $\|f\|_2 \leq \sqrt{\|f\|_p \|f\|_q}$.

□

Part (c)

Proof. Since the set of all simple functions $SS = \{\sum_1^n a_k \mathbb{1}_{E_k}| : \mu(E_k) < \infty\}$ belongs to both L^p and L^q and is dense in all of the L^r we know that $L^p \cap L^q$ is dense in L^2. □

Part (d)

Proof. We proved in part (b) that $L^p \cap L^q \subseteq L^2$. It remains to prove the other containment. Let $f \in L^2$. Let $\{g_k\} \subseteq L^p \cap L^q$ be such that $\|f - g_k\|_2 \to 0$. Such functions g_k exist by Part (c). We claim that $\{g_k\}$ is Cauchy. This follows since

$$\|g_k - g_j\|_{p,q} \leq \frac{1}{c} \|g_k - g_j\|_2 \leq \frac{1}{c} \|g_k - f\|_2 + \frac{1}{c} \|g_k - f\|_2 \to 0.$$

Since $L^{p,q}$ is complete we now have $g_k \overset{L^{p,q}}{\to} g$ for some $g \in L^{p,q}$. However, since $g_k \overset{L^2}{\to} f$ we know (from the argument in part (a)) that $g = f$ a.e. and thus $f \in L^{p,q}$.

For our example consider the single point space $(\{p\}, \mu)$ with μ the counting measure. In $(\{p\}, \mu)$ we have $L^p = L^r$ for all $p, r \geq 1$ and thus $L^p \cap L^q = L^2$.

□

(2009 #3) The Fourier cosine transform of a function $f \in L^1(\mathbb{R})$ is the function $\hat{f}$ on $\mathbb{R}$ defined by

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} \cos(\omega t) f(t) dt.$$

Prove that if $f \in L^1(\mathbb{R})$ then $\hat{f}$ is continuous and vanishes at infinity. You can assume that the functions of class C^1 that vanish outside a bounded interval are dense in $L^1(\mathbb{R})$.

Proof. Let $\omega_n \to \omega_0$, we wish to prove that $\hat{f}(\omega_n) \to \hat{f}(\omega_0)$. Since $|\cos(\omega_n t) f(t)| \leq |f(t)|$. As $f \in L^1(\mathbb{R})$, we can apply DCT and write

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} \cos(\omega_n t) f(t) dt \to \int_{-\infty}^{\infty} \cos(\omega_0 t) f(t) dt.$$

Next, let $f \in C_C^1$ and have compact support, with $\operatorname{supp} f \subseteq [-N, N]$. We can use integration by parts to write

$$\hat{f}(\omega) = \frac{\sin(\omega t)}{\omega} f(t)\Big|_{-\infty}^{\infty} - \frac{1}{\omega}\int_{-\infty}^{\infty} \sin(\omega t) f'(t) dt.$$

As $f \in C_C^1$ we know that the $f' \in C_C$ and that the left term is zero. Letting $M = \sup_{[-N,N]} |f'(t)|$ we can bound the right term by a constant, and letting $\omega \to \infty$ makes the whole integral tend to zero since

$$\left| \frac{1}{\omega} \int_{-\infty}^{\infty} \sin(\omega t) f'(t) dt \right| \leq \frac{2NM}{\omega} \to 0.$$

We can now use density of C_C^1. Fix $\epsilon > 0$ and let $f \in L^1(\mathbb{R})$ and $f_n \xrightarrow{L^1} f$ with $\{f_n\} \subseteq C_C(\mathbb{R})$. Let K be such that for all $n > K$ it holds that $\|f_n(\omega) - f(\omega)\|_1 < \epsilon$. We then have

$$|\hat{f}_n(\omega) - \hat{f}(\omega)| = \left| \int \cos(\omega t)(f_n(t) - f(t)) dt \right| \leq \int_{-\infty}^{\infty} |f_n(t) - f(t)| dt = \|f_n - f\|_1 \to 0.$$

So $\hat{f}(\omega) \to 0$ as $\omega \to \infty$.

□

(2009 #4) Let K be a continuous function on $[0,1] \times [0,1]$. For $f \in L^2([0,1])$ and $x \in [0,1]$ define $Tf(x) = \int_0^1 K(x,y) f(y) dy$.

(a) Show that $\|Tf\|_u \leq \|K\|_u \|f\|_2$ for all $x \in [0,1]$ and that Tf is continuous.

(b) Show that if $\{f_n\}$ is a bounded sequence in $L^2([0,1])$ the sequence $\{Tf_n\}$ contains a uniformly convergent subsequence.

(c) Assume that T is injective. Show that T does not map $L^2([0,1])$ onto $C([0,1])$.

Part (a)

Proof. Fix $x \in [0,1]$. We have

$$\begin{aligned} |Tf(x)| &\leq \int |K(x,y) f(y)| dy \\ &\leq \|K_x\|_2 \|f\|_2 \\ &\leq \|K\|_u \|f\|_2 . \end{aligned}$$

Also, we have for $|h|$ small

$$|Tf(x) - Tf(x+h)| \leq \int |K(x,y) - K(x+h,y)||f(y)| dy \leq \|K(x,y) - K(x+h,y)\|_u \|f\|_2 \to 0.$$

Which goes to zero since $K \in C([0,1] \times [0,1])$ and is therefore uniformly continuous. Therefore Tf is continuous.

□

Part (b)

Proof. Since our bound obtained in (a) does not depend on x and can be bounded by $\sup_n \|f_n\|_2$ we conclude that the family $\{Tf_n\}$ is bounded and equicontinuous. The desired result follows from Arzela-Ascoli. □

Part (c)

Proof. To show a contradiction suppose that T is onto. The open mapping theorem implies that T^{-1} is continuous and thus $\left\|T^{-1}\varphi\right\| \leq c\|\varphi\|$ for all $\varphi \in C([0,1])$. Therefore any bounded sequence in $C([0,1])$ maps under T^{-1} to a bounded sequence in $L^2([0,1])$. Consider the collection $\varphi_n(x) = e^{inx}$. Notice that $\|\varphi_n\| = 1$ and also that the collection is linearly independent and thus contains no convergent subsequence. This however contradicts b which says that $\{T^{-1}(\varphi_n)\}$ contains a convergent subsequence and by continuity of T we must have $T(\varphi_{n_k}) =$

(#2) Let $\mathbb{D} \subseteq L^2$ be the unit ball. By the previous part we know any subsequence $\{f_n\} \subseteq \mathbb{D}$ has a convergent subsequence $\{f_{n_k}\}$. Hence $Tf_{n_k} \to g \in C([0,1])$. Since T is bijective we would have $T^{-1}(Tf_{n_k}) \to T^{-1}g \in \mathbb{D}$. Hence $\mathbb{D}$ is sequentially compact, a contradiction. □

(2009 #5) We say that $f_n \rightharpoonup f$ if for every bounded linear functional T on $C([0,1])$ it holds that $Tf_n \to Tf$.

(a) Show that $f_n \rightharpoonup f$ in $C([0,1])$ if and only if $f_n \to f$ pointwise and there is a constant C such that $\|f_n\|_u \leq C$ for all $n \geq 1$.

(b) Show that if $f_n \rightharpoonup f$ in $C([0,1])$ then $f_n \rightharpoonup f$ in L^p with respect to lebesgue measure for all $p \in [1,\infty)$.

Part (a)

Proof. $\boxed{\Rightarrow}$ Let $X = C([0,1])$. Suppose that $f_n \rightharpoonup f$. We know that $\|f\| = \sup_{\|\ell\|=1} |\ell(f)| < \infty$. Also, since to each f_n we can associate an element $\hat{f}_n \in X^{**}$ via $(\hat{f}_n\ell)(x) = \ell(f_n)(x)$. Also, the $\hat{f}_n$ satisfy $\|f_n\| = \|\hat{f}_n\|$. Since $\|f_n\| = \sup_{\|\ell\|=1} |\ell(f_n)|$ and $\ell(f_n) \to \ell(f)$ it follows that $\|f_n\| \to \|f\| < \infty$ and thus $\sup_n \|f_n\| < \infty$.

To see pointwise convergence we note that the point-mass measures $d\delta_x$ belong to X^* via the map $\delta_x(f) = \int_0^1 f(y)d\delta_x$. Recall that $\delta_x(y) = \begin{cases} 1, & y = x \\ 0, & y \neq x \end{cases}$. Pointwise convergence follows immediately from weak convergence.

$\boxed{\Leftarrow}$ We use the Riesz representation theorem which states that $X^* = \{\text{radon measures on } [0,1]\}$. Fix $\mu \in X^*$. It suffices to prove that

$$\int (f - f_n)d\mu \to 0.$$

Since for all x it holds that $|f(x) - f_n(x)| \leq 2C$ and by definition we have $\mu([0,1]) < \infty$, it follows from bounded convergence (and the fact that $f_n \to f$ pointwise) that $\lim_{n\to\infty} \int f_n \to \int f d\mu$. □

Part (b)

Proof. We know that $|f_n(x) - f(x)|^p \to 0$ for a.e. x. We can make the bound

$$|f_n(x) - f(x)|^p \leq 2^{p-1}[|f_n(x)|^p + |f(x)|^p] \leq 2^{p-1}[M^p + |f(x)|^p] \leq 2^p M^p.$$

Take $(2M)^p$ to be the dominating function and apply DCT.

□

(ALTERNATE PROOF)

Proof. Fix $\epsilon > 0$. We demonstrated in part (a) that since $f_n \rightharpoonup f$ it follows that $\sup \|f_n\| = C < \infty$ and that $f_n \to f$ pointwise. Fix $p \in [1, \infty)$ and let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. As guaranteed by Egorov's theorem (since $m([0,1]) < \infty$), let $A \subseteq [0,1]$ be a measurable set such that $f_n \to f$ uniformly and $m(A^c) < \epsilon$. Choose n large enough so that for $x \in A$ it holds that $|f(x) - f_n(x)| < \frac{\sqrt[p]{\epsilon}}{m(A)}$.

$$\begin{aligned}\int |f(x) - f_n(x)|^p dm &= \int_A |f(x) - f_n(x)|^p dm + \int_{A^c} |f(x) - f_n(x)|^p dm \\ &< \qquad \epsilon \qquad + \int |f(x) - f_n(x)|^p \mathbb{1}_{A^c} dm \\ &\leq \qquad \epsilon \qquad + \|(f - f_n)^p\|_2 \, \|\mathbb{1}_{A^c}\|_2 \\ &\leq \epsilon + (2C)^p \sqrt{\epsilon} \\ &\to 0.\end{aligned}$$

□

> **(2009 #6)** Let $\{q_n\}$ be the set of rational numbers in $[0,1]$, ordered in some way. Prove that the series
> $$\sum_1^\infty (-1)^n n^{-3/2} |x - q_n|^{-q_n + \frac{1}{2}}$$
> converges to a finite limit for a.e. $x \in [0,1]$ (Lebesgue measure).

Proof. It is a consequence of the monotone convergence theorem that if $a_k(x) \geq 0$ is a measurable family of functions then

$$\int \sum_1^\infty a_k(x) dx = \sum_1^\infty \int a_k(x) dx.$$

So we define $a_n(x) = n^{-3/2} |x - q_n|^{1/2 - q_n} \geq 0$ then

$$\int_0^1 a_n(x) dx = n^{-3/2} \left[\int_0^{q_n} (q_n - x)^{1/2 - q_n} dx + \int_{q_n}^1 (x - q_n)^{1/2 - q_n} dx \right] = n^{-3/2} \left[\frac{q_n^{3/2 - q_n} + (1 - q_n)^{3/2 - q_n}}{\frac{3}{2} - q_n} \right]$$

It suffices to prove that the integral

$$\int_1^\infty (-1)^y y^{-3/2} |x - q_n|^{-q_n + \frac{1}{2}} dy$$

converges for a.e. $x \in [0,1]$. Notice that $|x - q_n| \leq 1$ and so $|x - q_n|^{-q_n + (1/2)} \leq 1$. We now have

$$\left| \int_1^\infty (-1)^y y^{-3/2} |x - q_n|^{-q_n + \frac{1}{2}} dy \right| \leq \int_1^\infty |y^{-3/2}||x - q_n|^{-q_n + \frac{1}{2}} dy$$
$$\leq \int_1^\infty y^{-3/2} dy$$
$$< \infty$$

Since $q_n \in [0,1]$ we know that $3/2 - q_n \geq 1/2$ and thus the above sum is bounded by $4n^{-3/2}$.

We can then conclude by comparisson

$$\sum_1^\infty \int_0^1 a_n(x) dx < \infty.$$

We wrap it up by interchanging the sum and the integral and concluding that

$$\int_0^1 \sum_1^\infty a_n(x) dx < \infty,$$

and thus the sum is finite for a.e. $x \in [0,1]$. We completely finish by noting that since the above converges we must have

$$\sum_1^\infty (-1)^n a_n(x) dx < \infty,$$

□

(2009 #7) Let $\mathcal{K}$ be the family of all nonempty compact subsets of $\mathbb{R}$. For $A, B \in \mathcal{K}$ define

$$d(A,B) = \sup_{x \in A} \inf_{y \in B} |x - y| + \sup_{y \in B} \inf x \in A|x - y|.$$

(a) Prove that $(\mathcal{K}, d)$ is a metric space.

(b) Prove that $(\mathcal{K}, d)$ is separable.

Part (a)

Proof. We check the three properties of a metric space.

Clearly $d(A,B) = d(B,A)$.

(ii) Next we show that $d(A,B) = 0 \iff A = B$. If $A = B$ this is trivial. When $d(A,B) = 0$ we extract a sequence considering each supremum separately and use the fact that A and B are closed and it is easy to see that $A \subseteq B \subseteq A$.

(iii) Lastly we check the triangle inequality. Fix sets A, B, C and let $a, b, c \in A, B, C$. We know that

$$|a - b| \leq |a - c| + |c - b|.$$

Taking appropriate inf and sup we have

$$\begin{aligned}\text{dist}(a, B) &\leq |a - c| + \text{dist}(c, B)\\ &\leq \text{dist}(a, C) + \inf_{c' \in C} \text{dist}(c', B)\end{aligned}$$

□

Part (b)

Proof. We will show that the collection of finite sets of rationals is dense. Let A be compact and cover A with balls $B_1, \ldots, B_n$ where each B_j has radius ϵ¿ Pick $q_j \in B_j$ and let $Q = \{q_1, \ldots, q_n\}$. It follows that $\text{dist}(A, B) \leq 2\epsilon + \epsilon$. □

(2009 #8) Let X be a compact Hausdorff space. A funciton $f : X \to \mathbb{R}$ is called upper semicontinuous if for every $x \in X$ and every $\epsilon > 0$ there is a neighborhood U of x such that $f(y) < f(x) + \epsilon$ for all $y \in U$.

(a) Show that f is upper semi-continuous if and only if $\{x \colon f(x) < a\}$ is open in X for every $a \in \mathbb{R}$.

(b) Show that if f is upper semi-continuous there exists $K \in \mathbb{R}$ such that $f(x) < K$ for all $x \in X$.

(c) Show that if f is upper semi-continuous then

$$f(x) = \inf\{g(x) : g \in C(X) \text{ and } g(y) > f(y) \text{ for all } y \in X\}.$$

Part (a)

Proof. ($\Rightarrow$) Suppose that f is upper semicontinuous. Suppose that $f(x) < a$. For all $\epsilon > 0$ there exists an open neighborhood U about x such that $f(y) < f(x) + \epsilon$. If $f(x) < a$ take $0 < \epsilon < a - f(x)$ and so $U \subseteq \{y \colon f(y) < a\}$ which proves U is open.

($\Leftarrow$) Suppose that $\{x \colon f(x) < a\}$ is open. Similar argument.

□

Part (b)

Proof. For each x let $U_x = \{y \in X \colon f(y) < f(x) + 1\}$. This is an open cover, so refine to a finite subcover $\{U_i\}_1^m$. Take $K = \max_i f(x_i) + 1$.

□

Part (c)

Proof. Use Urysohn's lemma. We let $Y = \{x_0\}$ and $Z = f^{-1}([f(x_0), +\epsilon, \infty)]$. These are both closed sets. So there exists $\varphi \in C(X, [0, 1])$ such that $\varphi(x_0) = 0$ and $\varphi \mid Z \equiv 1$. We then consider the function $K\varphi + f(x_0) + \epsilon$ with K as in Part (b)and let $\epsilon \to 0$.

□

20 2010

(2010 #1) Prove for all $t > 0$ that

$$\int_0^\infty \frac{e^{-x} - e^{-xt}}{x} dx = \ln t.$$

Proof. Let $f(x,s) = e^{-xs} \cdot \mathbb{1}_{[1,t]}(s)$ for fixed $t > 0$. We know that $f \in L^+$. By Tonelli we can then write

$$\int_0^\infty \int_0^\infty f(x,s) dx ds = \int_1^t \int_0^\infty e^{-xs} dx ds = \int_1^t -\frac{1}{s} e^{-xs}\Big|_0^\infty ds = \int_1^t \frac{1}{s} ds = \ln t.$$

Moreover, by Fubini we know that

$$\int_0^\infty \int_0^\infty e^{-xs} \mathbb{1}_{[1,t]}(s) dx ds = \int_0^\infty \int_1^t e^{-xs} ds dx = \int_0^\infty \frac{e^{-x} - e^{-xt}}{x} dx = \ln t.$$

□

ALTERNATE PROOF

Proof. We will use differentiation under the integral. In order to apply this we need two things

(i) $\partial_t f(x,t)$ exists and is bounded in absolute value by some $g(x) \in L^1$.

(ii) $f(x,t) \in L^1$ for all t.

Let $f(x,t) = \frac{e^{-x} - e^{-xt}}{x}$. Note that L'Hospital gives f is continuous on $[0, \infty)$. Letting $M_t = \sup_{[0,1]} |f(x,t)|$ Then

$$\begin{aligned} \int_0^\infty |f(x,t)| dx &= \int_0^1 |f(x,t)| dx + \int_1^\infty |f(x,t)| dt \\ &\le M_t + \int_1^\infty |e^{-x} - e^{-xt}| dx \\ &< \infty. \end{aligned}$$

Consider a compact interval $[a,b] \ni t$ on which $e^{-xt} \le e^{-ax} \in L^1$. Call this $g(x) = e^{-ax}$. We can now differentiate and write

$$\partial_t \int_0^\infty \frac{e^{-x} - e^{-xt}}{x} dx = \int_0^\infty e^{-xt} dx = \frac{1}{t}.$$

Now you can directly verify by asserting they are equal and computing the constant at say $t = 1$ or using the FTC. □

(2010 #2) $(\Omega, \mathcal{F}, \mu)$ is a measure space with $\mu(\Omega) = 1$. Let $\varphi \colon \mathbb{R} \to \mathbb{R}$ be continuous with

$$\varphi\left(\int f(x) d\mu(x)\right) \le \int \varphi(f(x)) d\mu(x).$$

for all real bounded measurable functions f. Suppose there exists a measurable set A with $\mu(A) = \frac{1}{2}$. Prove that φ is convex.

Proof. First we claim that φ is midpoint continuous and so for all $s, t \in \Omega$ it holds that

$$\varphi\left(\frac{s+t}{2}\right) \leq \frac{\varphi(s)}{2} + \frac{\varphi(t)}{2}.$$

To do this we define $f(x) = \begin{cases} s, & x \in A \\ t, & x \in A^c \end{cases}$.

Next we claim that midpoint convex and continuous implies convexity. To see this we fix s, t and first show inductively that if $0 \neq \lambda = \sum_{k=1}^{n} a_k 2^{-k}$ with the $a_k \in \{0,1\}$ and $a_n = 1$ then

$$\varphi(\lambda s + (1-\lambda)t) \leq \lambda\varphi(s) + (1-\lambda)\varphi(t).$$

By assumption this holds for $n = 1$. Suppose true for $n = m$ and we are given $\alpha = \sum_{k=1}^{m+1} a_k 2^{-k}$ we let $\lambda = \sum_{k=2}^{m+1} a_k 2^{-k+1}$. So we then have

$$\alpha = \frac{1}{2}a_1 + \frac{1}{2}\lambda.$$

Therefore

$$\alpha s + (1-\alpha)t + \frac{1}{2}a_i s + \frac{1}{2}\lambda s + \frac{1}{2}(1-a_1)t + \frac{1}{2}(1-lam)t$$

and by our inductive hypothesis we know that

$$\lambda\varphi(s) + (1-\lambda)\varphi(t) \geq \varphi(\lambda s + (1-\lambda)t)$$

Writing this out gives

$$\begin{aligned}\frac{1}{2}(\lambda\varphi(s) + (1-\lambda)\varphi(t)) + \frac{1}{2}(a_1\varphi(s) + (1-a_1)\varphi(t)) &\geq \frac{1}{2}\varphi(\lambda s + (1-\lambda)t) + \frac{1}{2}\varphi(a_1 s + (1-a_1)t) \\ &\geq \varphi\left(\frac{\lambda s + (1-\lambda)t + a_1 s + (1-a_1)t}{2}\right).\end{aligned}$$

Since the collection of λ is dense in $[0,1]$ we can use continuity to extend. □

(2010 # 3) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. Let $\{[f_n]\}$ be a sequence in $L^p(\mu)$ with $1 < p < \infty$. Suppose f_n converges to f a.e. and $\{\int f_n g d\mu\}$ is bounded for all $[g] \in L^q(\mu)$. Prove that $[f] \in L^p(\mu)$.

Proof. Consider the family of operators $\{\hat{f}_n\} \subseteq (L^q)^*$ with $\hat{f}_n g = \int f_n g d\mu$. By hypothesis we know that $\{\hat{f}_n\}$ is pointwise bounded. Hence, by the uniform boundedness principle we conclude that $\sup_n \|\hat{f}_n\| = M < \infty$. Since $(L^q)^* = (L^p)^{**} = L^p$, we conclude that $\sup_n \|f_n\| = M$. We can now apply Fatou's lemma

$$\int |f|^p d\mu \leq \liminf_n \int |f_n|^p d\mu \leq M^p.$$

□

(2010 #4) Suppose $1 < p < \infty$. Let $f \in L^p(0,\infty)$ relative to the Lebesgue measure, and set

$$F(x) = \frac{1}{x}\int_0^x f(t)dt, \quad 0 < x < \infty.$$

Prove Hardy's inequality

$$\|F\|_p \leq \frac{p}{p-1}\|f\|_p .$$

Hint: First explain why it suffices to consider $f \geq 0$. Then start with the case when f is continuous, nonnegative and compactly supported in the open interval $(0,\infty)$ and apply integration by parts to the integral $\int_0^\infty F^p(x)dx$.

Proof. Define $g : (0,\infty) \times (0,1) \to \mathbb{C}$ by $g(x,t) = f(xt)$. We can then write

$$F(x) = \frac{1}{x}\int_0^x f(t)dt = \int_0^1 f(xt)dt = \int_0^1 g(x,t)dt.$$

First we check that $g(x,\cdot) \in L^p$. This is true since

$$\int_0^\infty |g(x,t)|^p d = \int_0^\infty |f(xt)|^p dx = \frac{1}{t}\|f\|_p^p$$

We also need $t \mapsto \|g(\cdot,t)\|_p$ is in L^1. We compute

$$\int_0^1 \|g(\cdot,t)\|_p dt = \int_0^1 t^{-1/p}\|f\|_p dt = \frac{p}{p-1}\|f\|_p .$$

This then immediately follows from the Minkowski inequality.

□

Proof. Since we could write $f = f^+ - f^-$ we need only consider $f \geq 0$. Next, suppose that $f \in C_c((0,\infty))$ with $\operatorname{supp} f \subseteq [\delta_0, M_0]$. We can write

$$\int_0^\infty f(x)^p dx = \int_0^\infty x^{-p}\left(\int_0^x f(t)dt\right)^p dx.$$

Let $u = \left(\int_0^x f(t)dt\right)^p$ and $dv = x^{-p}dx$. By the fundamental theorem of calculus we have $du = p\left(\int_0^x f(t)dt\right)^{p-1} f(x)dx$. Let δ and M be arbitrary, applying integration by parts yields

$$\begin{aligned}\int_\delta^M F(x)^p dx &= \frac{x^{-p+1}}{-p+1}\left(\int_0^x f(t)dt\right)^p\Bigg|_\delta^M + \frac{p}{p-1}\int_\delta^M x^{-p+1}f(x)\left(\int_0^x f(t)dt\right)^{p-1}dx \\ &= \frac{M^{-p+1}}{-p+1}\left(\int_0^M f(t)dt\right)^p - \frac{\delta^{-p+1}}{-p+1}\left(\int_0^\delta f(t)dt\right)^p .\end{aligned}$$

Since $\operatorname{supp} f \subseteq [\delta_0, M_0]$ if $0 < \delta < \delta_0$ then $\int_0^\delta f(t)dt = 0$. Also, for $M > M_0$ we have

$$\int_0^M f(t)dt = \int_0^{M_0} f(t)dt = A < \infty.$$

Therefore
$$\lim_{M\to\infty} \frac{M^{-p+1}}{-p+1}\left(\int_0^M f(t)dt\right) \le \lim_{M\to\infty} \frac{M^{-p+1}}{-p+1} A = 0.$$

We now have
$$\begin{aligned}\int_0^\infty F(x)^p dx &= \frac{p}{p-1}\int_\delta^M f(x)\left(\frac{1}{x}\int_0^x f(t)dt\right)^{p-1} dx \\ &= \frac{p}{p-1}\int_\delta^M f(x)\cdot f(x)^{p-1}dx \\ &\le \frac{p}{p-1}\|f\|_p\left(\int |f^{p-1}|^q\right)^{1/q} \\ &= \frac{p}{p-1}\|f\|_p\left(\int |f|^p\right)^{1-\frac{1}{p}}.\end{aligned}$$

Dividing by $\left(\int |f|^p\right)^{1-\frac{1}{p}}$ gives
$$\|F\|_p \le \frac{p}{p-1}\|f\|_p, \qquad \forall f \in C_c((0,\infty)).$$

Now let $f \in L^p$ and choose a sequence $f_n \xrightarrow{L^p} f$ with $\{f_n\} \subseteq C_c((0,\infty))$. This implies there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ a.e. WLOG relabel f_{n_k} to be f_n. Define
$$F_n(x) := \frac{1}{x}\int_0^x f_n(t)dt.$$

Since $f_n \to f$ a.e. and we can assume that the $f_n \ge 0$, it follows from Fatou's lemma that
$$F(x) \le \liminf_n F_n(x).$$

Because $F \ge 0$ we can conclude that $F(x)^p \le \liminf F_n(x)^p$. Hence
$$\int F(x)^p dx \le \int \liminf F_n(x)^p dx.$$

Again since $F_n \ge 0$, Fatou's lemma implies that
$$\int_0^\infty \liminf F_n(x)dx \le \liminf \int_0^\infty F_n(x)^p dx.$$

As $f_n \in C_c((0,\infty))$ we know that
$$\|F_n\|^p \le \left(\frac{p}{p-1}\right)^p \|f_n\|_p^p.$$

Finally we obtain
$$\liminf \|F_n\|_p^p \le \left(\frac{p}{p-1}\right)^p \liminf \|f_n\|_p^p.$$

And because $\|f_n - f\|_p \to 0$ and $\|f_n\|_p \to \|f\|_p$ it follows that
$$\|f\|_p^p \le \left(\frac{p}{p-1}\right)^p \|f\|_p^p.$$

□

(2010 #5) Suppose μ is a finite signed Borel measure on $[0,1]$ such that

$$\int_0^1 e^{-nx} d\mu(x) = 0, \qquad \forall n = 0,1,2,\ldots.$$

Prove that $\mu = 0$.

SOLUTION 1:

Proof. Recall that the Stone-Weierstrass theorem states that

Theorem 12 (Stone). *Let X be a compact Hausdorff space and $C(X)$ be the space of continuous real valued functions on X. Suppose that $\mathcal{A} \subseteq C(X)$ is a subalgebra which separates points. If there exists $x_0 \in X$ such that $f(x_0) = 0$ for all $f \in \mathcal{A}$, then $\mathcal{A}$ is dense in $\{f \in C(X) : f(x_0) = 0\}$. Otherwise $\mathcal{A}$ is dense in $C(X)$.*

Theorem 13 (Riesz). *Let X be a compact metric space. For any bounded linear functional ℓ on $C(X)$ there exists a unique finite signed Borel measure ν on X such that $\ell(f) = \int_X f d\nu$, for all $f \in C(X)$.*

Define $I_\mu : C[0,1] \to \mathbb{R}$ by $I_\mu(f) = \int f d\mu$. Notice that I_μ is linear and that

$$I_\mu(f) \leq \int |f| d|\mu| \leq \|f\|_u \, |\mu|([0,1]).$$

Let $f_n(x) = e^{nx}$. By assumption we know that $I_\mu(f_n) = 0$ for all $n = 0,1,2,\ldots$. Let $\mathcal{A} = \text{span}\{f_n\}$. This is a subalgebra since $e^{-nx}e^{-mx} = e^{-(n+m)x}$. For all $x, y \in [0,1]$ with $x \neq y$ we know that $e^{-x} \neq e^{-y}$ and thus $\mathcal{A}$ separates points. Also since $e^0 = 1 \in \mathcal{A}$ we know that $\mathcal{A}$ must be dense in all of $C([0,1])$.

We now claim that $I_\mu(f) = 0$ for all $f \in C(X)$. Fix $f \in C(X)$ and let $g \in \mathcal{A}$ be such that $\|f - g\|_u < \epsilon$. As g is a finite linear combination of the f_n we know that $I_\mu(g) = 0$. We can write

$$|I_\mu(f)| = |I_\mu(f - g) + I_\mu(g)| \leq \int |f - g| d|\mu| \leq \epsilon |\mu|([0,1]).$$

Therefore $I_\mu(f) = 0$, and thus $I_\mu = I_0$. By uniqueness of Riesz we conclude that $\mu = 0$. □

SOLUTION 2:

Proof. Since the set of μ-measurable sets is Borel, it suffices to prove that $\mu(F) = 0$ for all closed sets F. The Hahn decomposition theorem guarantee that there exist two positive measures, η, λ such that $\mu(E) = \int_E d\eta - \int_E d\lambda$. Also, since the subalgebra of $\mathcal{F} \subseteq C([0,1])$ generated by $\{e^{-nx}\}$ contains 1 and separates points the Stone-Weierstrass theorem ensures that $\mathcal{F}$ is dense in $C([0,1])$. So, if we fix $g \in C([0,1])$ and let $f_\epsilon \in \mathcal{F}$ be such that $\|g - f_\epsilon\|_u < \epsilon$ we have $\int f + \epsilon d\mu = 0$ and thus

$$\left| \int g d\mu \right| = \left| \int g d\mu - \int f_\epsilon d\mu \right| \leq \int |g - f_\epsilon| |d\mu| \leq \epsilon \cdot |\mu|([0,1]) \to 0.$$

Now for any closed set F we let $f_k \to \mathbb{1}_F$ with f_k continuous bump functions bounded by $\mathbb{1}_F$. It follows from dominated convergence that

$$\mu(F) = \int \mathbb{1}_F d\mu = \lim \int f_n d\mu = 0.$$

□

(2010 #6) Let $(X, \mathcal{M})$ be a measurable space on which two probability measures μ and ν exist (i.e. $\mu(X) = 1 = \nu(X)$.) Define the total variation metric between the two measures as

$$\|\mu - \nu\| = \sup_{A \in \mathcal{M}} |\mu(A) - \nu(A)|.$$

Suppose that λ is a positive σ-finite measure on $\mathcal{M}$ such that $\mu \ll \lambda$ and $\nu \ll \lambda$. Prove that

$$\|\mu - \nu\| = \frac{1}{2} \int \left| \frac{d\mu}{d\lambda} - \frac{d\nu}{d\lambda} \right| d\lambda.$$

Here $\frac{d\mu}{d\lambda}$ and $\frac{d\nu}{d\lambda}$ are the Radon-Nikodym derivatives.

Proof. Define the signed measure $\eta := \mu - \nu$.

The Hahn Decomposition Theorem (3.3 in Folland) tells us that there exists a set $B \in M$ such that η is positive on all subsets of B and negative on all subsets of B^c (up to a set of measure zero). It follows that the total variation $|\eta|$ is given by

$$|\eta|(X) = |\eta(B)| + |\eta(B^c)| \overset{*}{=} \sup_{A \in \mathcal{M}} |\eta(A)| + |\eta(A^c)|. \tag{2}$$

Where the equality at (*) just says that B is a set on which the supremum is attained.

Since μ, ν are probability measures they both satisfy for $A \in \mathcal{M}$ it holds that $\mu(A^c) = 1 - \mu(A)$ and $\nu(A^c) = 1 - \nu(A)$. It follows that for $A \in \mathcal{M}$

$$|\eta(A^c)| = |\mu(A^c) - \nu(A^c)| = |1 - \mu(A) - (1 - \nu(A))| = |\nu(A) - \mu(A)| = |\eta(A)|.$$

Therefore we can rewrite (2) as

$$|\eta|(X) = 2 \sup_{A \in \mathcal{M}} |\eta(A)| \tag{3}$$

Writing $\eta = \int \left(\frac{d\mu}{d\lambda} - \frac{d\nu}{d\lambda} \right) d\lambda$ we know that an equivalent definition of total variation on X is

$$|\eta|(X) = \int_X \left| \frac{d\mu}{d\lambda} - \frac{d\nu}{d\lambda} \right| d\lambda. \tag{4}$$

Setting (3) and (4) equal and dividing by 2 gives the desired equality. □

(2010 #7) Suppose X is a Hausdorff space, (Y, d) is a metric space and $\{f_n\}$ is an equicontinuous sequence of functions from X to Y. Prove that

$$C = \{x \in X : \{f_n(x)\} \text{ is a Cauchy sequence in } Y\}$$

is closed in X.

Proof. First recall that since X is Hausdorff limits are unique. Also, a set, A, is closed if and only if $A' \subseteq A$ (here A' denotes the set of all limit points of A).

Assume that $p \in \overline{C}$ and thus every neighborhood of p intersects C. Choose a neighborhood $U \ni p$ such that if $x \in U$ then $d(f_n(x), f_n(p)) \leq \frac{\epsilon}{3}$. Choose any $\overline{x} \in U \cap C$. Then since $\overline{x} \in C$ we have for all m, n sufficiently large

$$d(f_n(\overline{x}), f_m(\overline{x})) < \frac{\epsilon}{3}.$$

We now have

$$d(f_n(p), f_m(p)) \leq d(f_n(p), f_n(\overline{x})) + d(f_n(\overline{x}), f_m(\overline{x})) + d(d_m(\overline{x}), f_m(p)) < \epsilon.$$

Hence $p \in C$ and C is closed. □

(2010 #8) Let $C^1[0,1]$ be the set of functions with continuous first derivative. Equip $C^1[0,1]$ with the inner product

$$(f,g)_1 = \int_0^1 f(t)g(t)dt + \int_0^1 f'(t)g'(t)dt.$$

The inner product, in a natural way, induces a norm and a metric on $C^1[0,1]$. Prove that any Cauchy sequence from $C^1[0,1]$ converges (in an L^2 sense) to a continuous function. In other words the completion of $C^1[0,1]$ in this metric can be taken to be a subset of $C[0,1]$.

Proof. Let $\{f_n\}$ be Cauchy. Fix $\epsilon > 0$ and let N be such that for all $n, m > N$ it holds that

$$\|f_n - f_m\|_2 + \left\|f_n' - f_m'\right\|_2 < \epsilon.$$

As both terms are positive we know that $\|f_n - f_m m\|_2 < \epsilon$ and $\|f_n' - f_m'\|_2 < \epsilon$. It follows that $f_n \to f$ in L^2 and $f_n' \to g$ in L^2. As f_n' is convergent in L^2 it is bounded, let $M = \sup_n \|f_n'\|_2$. Notice that

$$\begin{aligned}
|f_n(t) - f_n(t_0)| &= \left| \int_{t_0}^t f_n'(s)dx \right| \\
&\leq \int_0^1 |f_n'(s)| \mathbb{1}_{[t,t_0]}(s)dx \\
&\leq \left\|f_n'\right\|_2 \cdot \sqrt{|t - t_0|} \\
&< M\sqrt{|t - t_0|}
\end{aligned}$$

and therefore the collection $\{f_n\}$ is equicontinuous. Next we use the fact that $f_n \to f$ in L^2 to extract a pointwise a.e. convergent subsequence $f_{n_k} \to f$. As f is finite a.e. there exists $t_0 \in [0,1]$ such that $|f(t_0)| < \infty$ and $f_{n_k}(t_0) \to f(t_0)$. So we have

$$|f_{n_k}(t) \ \leq |f_{n_k}(t) - f_{n_k}(t_0)| + |f_{n_k}(t_0)| \leq M\sqrt{t - t_0} + C.$$

Hence the collection is pointwise bounded. Applying Arzelá- Ascoli we conclude that $f_{n_k'} \to h$ uniformly for some continuous function h. We also have $f_{n_k'} \to f$ in L^2. It follows that

$$\left(\int |f - h|^2 \right)^{1/2} \leq \left\| f - f_{n_k'} \right\|_2 + \left\| f_{n_k'} - h \right\|_2 \to 0.$$

□

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(2011 #1) Let $A \subseteq \mathbb{R}$ be a measurable set with $m(A) > 0$. Show that if $0 < b < m(A)$ then there exists $B \subseteq A$ a compact set with $m(B) = b$.

Proof. Let K be a compact subset of A with $m(K) > b$ (by inner-regularity). Define the function $f(x) = m(K \cap [-x, x])$. We claim f is continuous since (wlog assume $x > y$)

$$|f(x)-f(y)| = |m(K\cap[-x,x])-m(K\cap[-y,y]) = m(K\cap[-x,x]\backslash(K\cap[-y,y]) \le m([-x,x]\backslash[-y,y]) = 2(x-y) \to 0.$$

Since $f(0) = 0$ and since $K \subseteq [-N, N]$ for some N we have $f(N) = m(K) > b$ we apply the intermediate value theorem and conclude there exists $0 < t < \infty$ with $f(t) = b$. Therefore, $B := K \cap [-t, t]$ has $m(B) = b$ and B is compact. □

(2011 #2) Use the fact that $\int_0^\infty e^{-st} t^{n+\frac{1}{2}} dt = \frac{1}{2} \cdot \frac{3}{2} \cdots (n + \frac{1}{2})\sqrt{\pi} s^{-n-\frac{3}{2}}$ to show that

$$\int_0^\infty e^{-st} \sin\sqrt{t} dt = \frac{\sqrt{\pi}}{2s^{3/2}} e^{-\frac{1}{4s}}.$$

Proof. Recall that if (X, μ) is a measure space and $\{f_n\}$ is a sequence of measurable functions with $\sum_1^\infty \int_X |f_n| < \infty$, then $f(x) = \sum f_n$ converges for a.e. x and $\sum_1^\infty \int_X f_n = \int_X \sum_1^\infty f_n$.

We can write

$$\sin\sqrt{t} = \sum_0^\infty \frac{(-1)^n}{(2n+1)!} t^{n+\frac{1}{2}}.$$

So we have

$$\int_0^\infty e^{-st} \sin\sqrt{t} dt = \int_0^\infty \sum_0^\infty e^{-st} \frac{(-1)^n}{(2n+1)!} t^{n+\frac{1}{2}}.$$

We need to check that $\sum_0^\infty \int_0^\infty e^{-st} \frac{1}{(2n+1)!} t^{n+\frac{1}{2}} < \infty$. We prove the sum is finite by computing

$$\begin{aligned}
\sum_0^\infty \int_0^\infty e^{-st} \frac{(-1)^n}{(2n+1)!} t^{n+\frac{1}{2}} &= \frac{\sqrt{\pi}}{2s^{3/2}} \sum_0^\infty \frac{1}{(2n+1)!} \frac{3}{2} \frac{2n+1}{2} \frac{1}{s^n} \\
&= \frac{\sqrt{\pi}}{2s^{3/2}} \sum_0^\infty \frac{1}{(2 \cdot 4 \cdots (2n))} \frac{1}{2^n} \frac{1}{s^n} \\
&= \frac{\sqrt{\pi}}{2s^{3/2}} \sum_0^\infty \frac{1}{n!} \frac{1}{4^n} \frac{1}{s^n} \\
&= \frac{\sqrt{\pi}}{2s^{3/2}} e^{\frac{1}{4s}}.
\end{aligned}$$

When we get rid of the absolute value signs we pickup a $(-1)^n$ and then get $e^{\frac{-1}{4s}}$.

(Another way is we could use contours via the substitution $u = \sqrt{t}$, looking at a real part and completing the square. Unrelated, we can always dominate $\sin\sqrt{t}$ by $e^{\sqrt{t}}$.) □

(2011 #3) Let $f : [0,1] \to \mathbb{R}$. Suppose that the one-sided derivatives

$$D_- f(x) = \lim_{h<0, h\to 0} \frac{f(x+h) - f(x)}{h} \qquad (0 < x \leq 1)$$

$$D_+ f(x) = \lim_{h>0, h\to 0} \frac{f(x+h) - f(x)}{h} \qquad (0 < x \leq 1)$$

exist for all x in the indicated ranges and are bouned in absolute value by a constant $K < \infty$. Prove that the derivative $f'(x)$ exists for almost every $x \in (0,1)$.

Proof. We will prove that f is Lipschitz. Suppose not, and so there existx, y and $\epsilon > 0$ such that

$$|f(x_1) - f(x_2)| > (K + \epsilon)|x_1 - y_2|.$$

Split the interval in half and choose the next two points $[x_3, x_4] \subseteq [x_1, x_2]$ so that

$$|f(x_3) - f(x_4)| > (K + \epsilon)|x_3 - x_4|.$$

We obtain a nested sequence of intervals $I_{j+1} \subseteq I_j = [a_j, b_j]$ with $m(I_{j+1}) = \frac{1}{2} m(I_j)$. We have $\bigcap I_j = \{x\}$ by Cantor Intersection. We claim that the contradiction $D_+ f(x), D_- f(x) \geq K + \epsilon$. We have just proven that

$$|f(y_j) - f(a_j)| \geq (K + \epsilon)(b_j - a_j)$$

It follows from the triangle inequality that

$$|f(x) - f(a_j)| \geq (K + \epsilon)|x - a_j| \qquad \text{or } |f(b_j) - f(x)| \geq (K + \epsilon)|b_j - x|.$$

(Another way is to use a Rolles theorem approach. Suppose there exists x, y so that $g(x, y) = \frac{f(y) - f(x)}{y - x} = K + \epsilon$. If $f \equiv K + \epsilon$ we are done. Suppose not then we must have an extrema (wlog a max). At this point the left derivative is positive and the right derivative is negative. This implies (with some work) that one of the left or right derivatives is larger than K.) ☐

(2011 #4) Suppose that f is a nonnegative measurable function on $(0,1]$ such that $\int_0^1 t^3 f(t)^4 dt < \infty$. Show that

$$\frac{\int_x^1 f(t)dt}{|\log x|^{3/4}} \to 0 \qquad \text{as } x \to 0.$$

Hint: First show that $\int_x^1 f(t)dt \leq C|\log x|^{3/4}$ *then refine the argument by writing* $\int_x^1 = \int_x^\delta + \int_\delta^1$ *for a suitably chosen* δ.

Proof. Let $C = \left\| t^3 f(t) \right\|_1$. Fix $x \in (0,1)$. Via Hölder's inequality we have

$$\begin{aligned}
\int_x^1 f(t)dt &= \int t^{-3/4} t^{3/4} f(t) \mathbb{1}_{[x,1]} dt \\
&\leq \left\| t^{-3/4} f(t) \right\|_4 \left\| t^{-3/4} \mathbb{1}_{[x,1]} \right\|_{4/3} \\
&= C \left| \int_x^1 t^{-1} dt \right|^{3/4} \\
&= C \left|\log x\right|^{3/4}.
\end{aligned}$$

Fix $0 < \epsilon < 1$. Observe that by the previous inequality we have

$$\int_{x^\epsilon}^1 f(t)dt \leq C|\log(x^\epsilon)|^{3/4} = C|\epsilon \log x|^{3/4},$$

and also

$$\int_x^{x^\epsilon} f(t) \leq C \ \ |\epsilon \log x - \log x|^{3/4} = (\epsilon - 1)^{3/4} |\log x|^{3/4} \left(\int_0^{x^\epsilon} t^3 f(t)^4 dt \right)^{1/4}.$$

Therefore

$$\frac{\int_x^1 f(t)dt}{|\log x|^{3/4}} \leq \frac{C\epsilon^{3/4}|\log x|^{3/4} + (\epsilon-1)^{3/4}|\log x|^{3/4}\left(\int_0^{x^\epsilon} t^3 f(t)^4 dt\right)^{1/4}}{|\log x|^{3/4}} = C\epsilon^{3/4} + (\epsilon - 1)^{3/4}\left(\int_0^{x^\epsilon} t^3 f(t)^4 dt\right)^{1/4}$$

Fix $\delta > 0$ and let ϵ be such that $C\epsilon^{3/4} < \epsilon/2$. There exist $\eta > 0$ such that

$$\left(\int_0^\eta t^3 f(t)^4 dt\right)^{1/4} < \frac{\epsilon}{2(1-\epsilon)^{3/4}}.$$

Pick x such that $x^\epsilon < \eta$ and we're done.

□

(2011 #5) Suppose $f \in L^1([0,1])$. Show that if $\int_0^1 f(x)(\sin x)^n dx = 0$ for all $n = 1, 2, \ldots$ then $f = 0$ a.e.

Proof. Let $\varphi_n = \sin^n(x)$ and let $\mathcal{F} \subseteq C([0,1])$ be the sub algebra generated by $\{\varphi_n\}_1^\infty$. Notice that $\mathcal{F}$ separates points since $\sin x$ is injective on $[0,1]$. Also, since $\varphi_n(0) = 0$ for all n it follows from Stone-Weierstrass that $\mathcal{F}$ is dense in $\Gamma_0 := \{f \in C([0,1]) : f(0) = 0\}$. Notice that Γ_0 is dense in $C_C((0,1))$.

Consider the measure $\mu(A) = \int_A f(x)dx$. Since μ

□

Proof. Let $\varphi_n = \sin^n(x)$ and let $\mathcal{F} \subseteq C([0,1])$ be the sub algebra generated by $\{\varphi_n\}_1^\infty$. Notice that $\mathcal{F}$ separates points since $\sin x$ is injective on $[0,1]$. Also, since $\varphi_n(0) = 0$ for all n it follows from Stone-Weierstrass that $\mathcal{F}$ is dense in $\Gamma_0 := \{f \in C([0,1]) : f(0) = 0\}$. Let $g \in \Gamma_0$, we claim that $\int fg = 0$. Let $g_\epsilon \in \mathcal{F}$ be such that $\|g - g_\epsilon\|_u < \epsilon$. We then have

$$\left| \int fg - \int fg_\epsilon \right| = \int |f||g - g_\epsilon| \leq \epsilon \|f\|_1 \to 0.$$

Define the measure $\mu(E) = \int_E f(x)dx$. It then suffices to prove that $\mu(F) = 0$ for all closed sets $F \subseteq [0,1]$. So, fix a closed set $F \subseteq [0,1]$ and $\delta > 0$. Let $g_n \in \Gamma_0$ be such that $\|g_\delta - \mathbb{1}_F\|_u < 1/n$, and $g_n(x) \leq \mathbb{1}_F(x)$ for all $x \in [0,1]$. Since $f \in L^1([0,1])$ we know that $f\mathbb{1}_F \in L^1([0,1])$. Using the fact that have $g_n \to \mathbb{1}_F$ we apply dominated convergence to write

$$0 = \left| \int fg_n \right| \to \left| \int f\mathbb{1}_F \right| = \int_F f = \mu(F).$$

□

(2011 #6) Let Λ be the set of all functions $f : \mathbb{R} \to \mathbb{R}$. Let $\mathcal{G}$ be the family of all sets $A \subseteq \Lambda$ of the form

$$A = \{f \in \Lambda : (f(x_1), f(x_2), \ldots, f(x_n)) \in B\}$$

for some finite set $\{x_1, \ldots, x_n\} \subseteq \mathbb{R}$ with $(n = 1, 2, \ldots)$ and some open set $B \subseteq \mathbb{R}^n$ in the usual Euclidean topology on $\mathbb{R}^n$. Let $\mathcal{T}$ be the topology generated by $\mathcal{G}$, that is the weakest topology on Λ such that $\mathcal{G} \subseteq \mathcal{T}$.

(a) Show that a sequence $\{f_n\} \subseteq \Lambda$ converges to f with respect to $\mathcal{T}$ if and only if $f_n \to f$ pointwise.

(b) Show that the continuous functions are dense in λ with respect to $\mathcal{T}$ if and only if $f_n \to f$ pointwise.

(c) Show that $\mathcal{T}$ is not metrizable - that is, for any metric ρ on Λ, the weakest topology $\mathcal{T}_\rho$ which contains all open balls $\{y \in \Lambda : \rho(x, y) < a\}$ with $x \in \Lambda$ and $a > 0$ is different from $\mathcal{T}$.

(2011 #7) Let X, Y be Banach spaces and $T : X \to Y$ be an injective bounded linear map whose range $T(X)$ is closed in Y.

(a) Show that for each bounded linear functional φ on X there is a bounded linear functional ψ on Y such that $\varphi = \psi \circ T$

(b) There is a constant C (independent of φ) such that ψ can be chosen to satisfy $\|\psi\| \leq C \|\varphi\|$.

Proof. The map $\ell : T(X) \to \mathbb{R} : T(x) \mapsto \varphi(x)$ is linear on $T(x)$ and $|\ell(T(x))| \leq |\varphi(x)|$. Since $|\varphi(x)|$ is a semi-norm we can extend ℓ to a linear functional ψ on all of Y which satisfies $\psi(y) \leq$

□

Part (a)

More Thorough Solution:

Proof. We know that $T(X)$ is a Banach space hence $T : X \to T(X)$ is a bounded bijection between two Banach spaces and by the open mapping theorem T^{-1} is a bounded linear function. Define $\tilde{\psi} \in T(X)^*$ by $\tilde{\psi} = \varphi \circ T^{-1}$. For any $y \in T(X)$ we have

$$|\tilde{\psi}(y)| \leq \|\varphi\| \left\|T^{-1}\right\| \|y\|.$$

Define $p(y) = \|\varphi\| \left\|T^{-1}\right\| \|y\|$. By Hahn-Banach there exists an extension ψ on Y which satisfies the desired properties.

□

LESS THOROUGH SOLUTION:

Proof. Since T is injective we know that the adjoint $T^* : Y^* \to X^*$ given by $T^*\psi(x) = \psi(T(x))$ is surjective. This is true since given $\varphi \in X$ for $y \in T(X)$ we let $\psi(y) = \varphi(T^{-1}y)$. This gives a linear map on $T(X)$. Since $T(X)$ is closed, by Hahn-Banach ψ extends to all of Y and also we obtain the inequality

$$\|\psi\| \le \left\|T^{-1}\right\| \varphi.$$

Recall that the Hahn-Banach theorem states that:

Theorem 14 (Hahn-Banach)**.** *Let Y be a Banach space, let $M \subseteq Y$ be a closed subspace of Y. Let $\tilde{T}$ be a linear functional on M and let $p : Y \to [0, \infty)$ be a seminorm on Y such that $|\tilde{T}(y)| \le p(y)$ for $t \in M$. Then there exists an extension $T \in Y^*$ such that $T \equiv T$ on M and $|T(y)| \le p(y)$ for all $y \in Y$.*

□

Part (b)

Proof. To show a contradiction suppose that for k there exists φ_k such that for all $\psi_k \in (T^*)^{-1}\varphi_k$ it holds that

$$\|\psi_k\| > k \|\varphi_k\| .$$

Rearranging slightly and writing $\varphi_k = T^*\psi_k$, this can be written as

$$\|T^*\psi_k\| < (1/k) \|\psi_k\| , \qquad \forall \psi_k \mapsto \varphi_k.$$

This says that $\frac{\|T^*\psi_k\|}{\|\psi_k\|} \to 0$. By linearity this implies that $\|T^*\| = 0$. Hence there exists a sequence $\{\psi_k\}$ such that $T^*\psi_k \to 0$ and $\|\psi_k\| = 1$. Accordingly there exists a sequence $\{x_k\} \subseteq \partial\mathbb{D} \subseteq X$ with $T^*\psi_k(x_k) = \psi_k(T(x_k)) \to 0$, and thus $T(x_k) \to 0$. Since $\partial\mathbb{D}$ is weak* compact we choose a weakly convergent sequence $x_{k_j} \rightharpoonup x \in \partial\mathbb{D}$. This says that $Tx_{k_j} \to Tx$, but since $Tx_{k_j} \to 0$ we conclude that $T(x) = 0$, contradicting the fact that T is injective.

□

(2011 #8) Let $B(x, r) \subseteq \mathbb{R}^2$ denote the open disc with center x and radius r and let $S(x, r)$ be the boundary of $B(x, r)$. Let $\mathbb{D} = \mathbb{B}((0, 0), 1)$ be the open unit disc, and let $\mathcal{H}$ be the family of all bounded Borel measurable functions $f : \mathbb{D} \to \mathbb{R}$. Let

$$\mathcal{A} = \left\{ f \in \mathcal{H} : f(x) = \frac{1}{2\pi r} \int_{S(x,r)} f(y) d\sigma(y), \qquad \text{for all circles } S(x, y) \subseteq \mathbb{D} \right\}$$

where $d\sigma(y)$ denotes the arc length measure on $S(x, r)$ and

$$\mathcal{B} = \left\{ f \in \mathcal{H} : f(x) = \frac{1}{2\pi r} \int_{\mathbb{B}(x,r)} f(y) dy, \qquad \text{for all discs } \mathbb{B}(x, y) \subseteq \mathbb{D} \right\},$$

where dy denotes 2-dimensional Lebesgue measure. Prove that $\mathcal{A} = \mathcal{B}$

Proof. $\boxed{\subseteq}$ Let $f \in \mathcal{A}$ and let $\mathbb{B}(x, r) \subseteq \mathbb{D}$. We have

$$\frac{1}{\pi r^2} \int_{\mathbb{B}(x,r)} f(y) dy = \frac{1}{\pi r^2} \int_0^r \int_{S(x,s)} f(y) d\sigma(y) ds.$$

Since $f \in \mathcal{A}$ we have $S(x,s) \subseteq \mathbb{B}(x,r) \subseteq \mathbb{D}$ and thus

$$\int_{S(x,s)} f(y)d\sigma(y) = 2\pi s f(x).$$

We then obtain

$$\frac{1}{\pi r^2}\int_{\mathbb{B}(x,r)} f(y)dy = \frac{1}{\pi r^2}\int_0^r 2\pi s f(x)ds = f(x)ds.$$

This says that $f \in \mathcal{B}$.

$\boxed{\supseteq}$ Let $f \in \mathbb{B}$ with $|f| < M$. First we prove that f is continuous. To see this, fix $x \in \mathbb{D}$ and $\mathbb{B}(x,r) \subseteq \mathbb{D}$. Given $\epsilon > 0$ we can take $\delta < r$ such that $1 - \frac{(r-\delta)^2}{r^2} < \frac{\epsilon}{2M}$. Let $x' \subseteq \mathbb{B}(x,\delta)$. Notice that $\mathbb{B}(x', r-\delta) \subseteq \mathbb{B}(x,r)$. It follows that

$$\begin{aligned}
|f(x') - f(x)| &= \left| \frac{1}{\pi r^2}\int_{\mathbb{B}(x,r)} f(y)dy - \frac{1}{\pi(r-\delta)^2}\int_{\mathbb{B}(x,r-\delta)} f(y)dy \right| \\
&= \left| \frac{1}{\pi r^2}\int_{\mathbb{B}(x,r)\setminus\mathbb{B}(x',r-\delta} f(y)dy - \left(\frac{1}{\pi r^2} - \frac{1}{\pi(r-\delta)^2}\right)\int_{\mathbb{B}(x',r-\delta} f(y)dy \right| \\
&\leq \left| \frac{1}{\pi r^2} - \frac{1}{\pi(r-\delta)^2} \right| \int_{\mathbb{B}(x,r-\delta)} |f|dy + \frac{1}{\pi r^2}\int_{\mathbb{B}(x,r)\setminus\mathbb{B}(x,r-\delta)} |f|dy \\
&\leq \left| \frac{1}{\pi r^2} - \frac{1}{\pi(r-\delta)^2} \right| M\pi(r-\delta) + \frac{1}{\pi r^2} M(\pi r^2 - \pi(r-\delta)^2) \\
&< \epsilon.
\end{aligned}$$

Next we compute

$$\lim_{h\to 0} \frac{\int_{\mathbb{B}(x,r)} f(y)dy - \int_{\mathbb{B}(x,r+h)} f(y)dy}{-h} = \lim_{h\to 0}\frac{\pi r^2 f(x) - \pi(r+h)^2 f(x)}{-h} = 2\pi r f(x).$$

Since $\int_{\mathbb{B}(x,r)} f(y)dy = \int_0^r \int_{S(x,s)} f(y)d\sigma(y)ds$, we can differentiate with respect to r and conclude since f is continuous from the Fundamental theorem of calculus that

$$2\pi r f(x) = \int_{S(x,s)} f(y)d\sigma(y)$$

Therefore, $f \in \mathcal{A}$.

□

22 2012

(2012 #2) Let (X,d) be a complete and separable metric space. Let μ be a finite Borel measure. Show that μ is tight (i.e. for all $\epsilon > 0$ there exists a compact $K \subseteq X$ such that $\mu(K) > \mu(X) - \epsilon$.)

Proof. Recall that $K \subseteq X$ is compact if and only if it is closed and totally bounded (in a complete metric space). Fix $\epsilon > 0$ and let Q denote our countable set. For $K, N \in \mathbb{N}$ define

$$A_{K,N} = \bigcup_{j=1}^{N} \mathbb{B}(q_j, \frac{1}{k}).$$

Since $\overline{Q} = X$ we have

$$X = \bigcup_{1}^{\infty} \mathbb{B}(q_j, \frac{1}{k}), \qquad \forall K \in \mathbb{N}.$$

For each K we have $A_{K,N} \nearrow X$ as $N \to \infty$. By continuity from below we know there exists $N(K,\epsilon)$ such that

$$\mu(X) - \mu(A_{K,N(K,\epsilon)}) < \frac{\epsilon}{2^k}\mu(X).$$

Define

$$K = \bigcap_{k=1}^{\infty} A_{K,N(K,\epsilon)}.$$

K is pre compact because it is totally bounded. To see this fix $\delta > 0$. There exists $K \in \mathbb{N}$ such that $\frac{1}{k} < \frac{\delta}{2}$. We see that by our construction

$$\{\mathbb{B}(q, \frac{1}{k}) \colon 1 \leq j \leq M(K,\epsilon)\}$$

is a cover of K and so $\{\mathbb{B}(q,\delta)\}$ covers $\overline{K}$. Therefore $\overline{K}$ is compact. Lastly,

$$\begin{aligned}
\mu(\overline{K}^c) &\leq \mu(K^c) \\
&= \mu\left(\left(\bigcap_{k\geq 1} A_{K,N(K,\epsilon)}\right)^c\right) \\
&\leq \sum_{k=1}^{\infty} \frac{\epsilon}{2^k}\mu(X) \\
&= \epsilon\mu(X).
\end{aligned}$$

□

> **(2012 #3)** Let $f_n : (0,1) \to \mathbb{R}$ with $f_n \to f$ point wise and f_n, f are convex, differentiable functions. Prove that $\lim_{n\to\infty} f_n'(x) = f'(x)$. *Hint: Try to argue by contradiction. The convexity assumption is crucial.*

Proof. We are not going to argue by contradiction and instead do it directly. Let $x < z < y < w$. Convexity guarantees that

$$\frac{f_n(z) - f_n(x)}{z - x} \leq f_n'(z) \leq \frac{f_n(y) - f_n(z)}{y - z}. \qquad (*)$$

Taking lim sup of $(*)$ we have

$$\frac{f(z) - f(x)}{z - x} \leq \limsup f_n'(z) \leq \frac{f(y) - f(z)}{y - z},$$

and letting $y \to z, x \to z$ we have $f'(z) \leq \limsup f_n'(z) \leq f'(z)$. Similarly for the liminf.

□

(2012 #4) Let m denote Lebesgue measure on $\mathbb{R}$ and let $\mathcal{F}$ be the σ-field of Lebesgue measurable subsets of $\mathbb{R}$. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ is a Λ-Lipschitz function. Show that if $A \in \mathcal{F}$ then $f(A) \in \mathcal{F}$ and $m(f(A)) \leq \Lambda m(A)$.

Proof. We know that f takes measure zero sets to measure zero sets. Let $A \in \mathcal{F}$. Write $A = Z \cup \bigcup K_n$ with the K_n compact and $m(Z) = 0$. We then have $f(A) = f(Z) \cup \bigcup f(K_n) = Z' \cup K'$ with $m(Z') = 0$ and K' an F_σ, which is measurable.

Next, notice that the inequality holds for intervals I_j and therefore for open sets $\mathcal{O} = \bigcup_1^\infty I_j$ (since $m(f(\mathcal{O})) = m(f(\cup_1^\infty I_j)) \leq \Lambda m(\mathcal{O}))$. So, if we consider $A \in \mathcal{F}$ and take $\mathcal{O}_n \searrow A$ we have

$$m(f(A)) \leq m(f(\mathcal{O}_n)) \leq \Lambda m(\mathcal{O}_n) \to \Lambda m(A).$$

□

(2012 #5) Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. For some $1 < p < \infty$ consider the L^p space corresponding to the measure space $(\Omega, \mathcal{F}, \mu)$. Let f, g be two functions in L^p. For $u \in \mathbb{R}$, define the function $F(u) = \int_\Omega |f + ug|^p d\mu$. Show that this function is differentiable at zero and

$$F'(0) = p \int_\Omega |f|^{p-2} fg d\mu.$$

Hint: Reduce the problem to justifying taking a limit under the integral.

Proof. We will show that

$$\left| \frac{F(u) - F(0)}{u} - p \int |f|^{p-2} fg d\mu \right| \to 0$$

as $|u| \to 0$. We can expand the above

$$\left| \frac{F(u) - F(0)}{u} - p \int |f|^{p-2} fg d\mu \right| = \left| \int_\Omega \frac{|f + ug|^p - |f|^p}{u} - p|f|^{p-2} fg d\mu \right|.$$

Call the second function $\phi(x, u) = \left| \int_\Omega \frac{|f+ug|^p - |f|^p}{u} - p|f|^{p-2} fg du \right|$. We know that when $g = 0$ we have $\phi = 0$ and when $f = 0$ we have $\phi \to 0$ as $n \to \infty$. Since $\frac{d}{dy}|y|^p = p|y|^{p-1}\text{sign}(y) = p|y|^{p-2}|y$ we can conclude that $\frac{|f+ug|^p - |f|^p}{u} - p|f|^{p-2} fg$ converges to 0. Now we wish to apply dominated convergence theorem

$$\left| \frac{|f + ug|^p - |f|^p}{u} - p|f|^{p-2} fg \right| \leq \underbrace{\left| \frac{|f + ug|^p - |f|^p}{u} \right|}_{I} + \underbrace{|p|f|^{p-1} g|}_{II}.$$

We can bound (I) by using the MVT to the function $g(u) = |f + ug|^p$ to obtain $0 < v < u$ such that

$$\left| \frac{|f + ug|^p - |f|^p}{u} \right| \leq p|f + vg|^{p-1}|g| \leq p|f + vg|^{p-1}|f + vg|$$

$$\left|\frac{|f+\mu g|^p-|f|}{u}\right| \leq p|f|^{p-2}|f|g| \leq \left|\frac{|f+\mu g|^p-|f|}{u}\right| + p(|f|+|g|)^{p-2}(|f|+g|)|g|$$

(*Need to finish the argument... uses MVT*)

□

(2012 #6) Let $p \geq 1$ and let $f_n, f \in L^p(\mathbb{R})$. Show the following are equivalent.

(i) $\sup \|f_n\|_p < \infty$ and $\lim_{n\to\infty} \int_0^x f_n dx = \int_0^x f dx$.

(ii) $f_n \rightharpoonup f$ in L^p.

Proof. ((i) $\implies$ (ii)) We can rewrite the second condition of (i) as $\lim \int f_n \mathbb{1}_{[a,b]} dx \to \int f \mathbb{1}_{[a,b]}$ for any $a < b$. We will show that weak convergence holds for all step functions. Fix $g \in L^q$ and consider a step function $\varphi = \sum_1^n \mathbb{1}_{I_j}$ with $\|g - \varphi\|_q < \epsilon$. We can apply Hölder and write

$$\left|\int (f-f_n)g - \int (f-f_n)\varphi\right| = \left|\int (f-f_n)(g-\varphi)\right| \leq \|f-f_n\| \, \|g-\varphi\|_q \leq M\epsilon \to 0.$$

((ii) $\implies$ (i)) Suppose that $f_n \rightharpoonup f$ and so $\tilde{g}(f_n) \to \tilde{g}(f)$ in L^p for all $\tilde{g} \in (L^p)^*$. So, we associate $\tilde{g}$ to some $g \in L^q$ which satisfies $\|\tilde{g}\| = \|g\|_q$. Since $\tilde{f_n}(g) \to \tilde{f}(g)$ for all g we know from the Uniform Boundedness principle that $\sup \|f_n\| < \infty$. The second property of (i) follows from considering $g_x = \mathbb{1}_{[0,x]}$.

□

(2012 #8) Let$C$ be a nonempty, closed and convex subset of a hilbert space $\mathcal{H}$. Show that for all $x \in \mathcal{H}$ there exists a unique $y \in C$ such that $\|x - y\| \leq \|x - z\|$ for all $z \in C$.

Proof. First we prove existence. Let $x \in \mathcal{H}$ and note that if $x \in C$ we take $y = x$. Otherwise there exists a sequence $\{z_n\} \subseteq C$ such that $\|x - z_n\| \searrow \text{dist}(x, C) = d$. We claim that $\{z_n\}$ is Cauchy. Let $\epsilon > 0$. There exists N such that for all $n \geq N$ it holds that that $\|x - z_n\| < d + \epsilon$. Look at z_n, z_m with $n, m \geq N$. By the parallelogram law we know that

$$2\|z_n - x\|^2 + 2\|z_m - x\|^2 = \|z_n - z_m\|^2 + \|z_n + z_m - 2x\|^2.$$

Rearranging we obtain

$$4(d+\epsilon)^2 - \|z_n - z_m\|^2 > 4\left\|\frac{1}{2}(z_n + z_m) - x\right\|^2 > 4d^2.$$

Where the last inequality follows since if it did not hold we would have $\left\|\frac{1}{2}(z_n + z_m) - x\right\| < d$ which is a contradiction of the definition of d since C is convex. Therefore $\{z_n\}$ is Cauchy and by completeness and the fact that C is closed converges to some $y \in C$.

Uniqueness follows by taking $\epsilon = 0$ in the above inequalities. □

23 Misc

23.1 Practice Exam

1. Let $A, B \subseteq [0,1] \times [0,1]$.

   (a) Suppose for any line $L_c = \{(c, y) : y \in [0,1]\}$ the cardinality of $A \cap L_c$ is countable. What can be said about $m(A)$?

   (b) Let $L^c = \{(c, y) : y = x + c\}$ for $-1 \le c \le 1$. Suppose $B \cap L^c$ is countable for every c. What can be said about $m(B)$?

2. For $f \in L^1(\mathbb{R})$ prove that

$$\lim_{h \to 0} \frac{1}{h} \int_{|y-x|<h} |f(y) - f(x)| dx = 0.$$

3. Let $T : V \to W$ be a linear map of Banach spaces. Suppose that whenever $v_i \in V$ is a sequence of vectors sic that $v_i \to 0$ in V and $T(v_i) \to w$ for some $w \in W$ then $w = 0$. Show that T is continuous.

4. Let $X = C^1(\mathbb{R})$. Suppose that $f_n \in X$ for $n \ge 1$ and $f_n(x) = 0$ for $|x| > 1$ and all $n \ge 1$. Assume that there is a $p > 1$ such that $\|f_n'\|_p \le 1$ for all $n \in \mathbb{N}$. Prove that there is a subsequence of the f_n' which converges uniformly on X.

5. Let μ_k and μ be locally finite Borel measures on $\mathbb{R}^n$ such that $\int f d\mu_k \to \int f d\mu$ for all $f \in C_c(\mathbb{R}^n)$.

   (a) Show that if $U \subseteq \mathbb{R}^n$ is open then $\mu(U) \le \liminf_{k\to\infty} \mu_k(U)$.

   (b) Show that if S is a bounded Borel set and $\mu(\partial S) = 0$ then $\mu(S) = \lim_{k\to\infty} \mu_k(S)$.

6. Show that if $K \subseteq C([0,1])$ is compact then the set of functions $f \in K$ is equicontinuous. Then show that $C([0,1])$ has no non-empty open sets with compact closure.

7. Let $f_n \in L^2(0,1)$ converge in measure to f.

   (a) Show that $f \in L^2(0,1)$ and $\|f\|_2 \le \liminf \|f_n\|_2$.

   (b) Show that$\|f_n\|_2 \to \|f\|_2$ if and only if $f_n \to f$ in L^2.

8. Let f_k be an orthogonal set of functions in $L^2(0,1)$ with $|f_k(x) \le M$ for all k and a.e. $x \in (0,1)$. Define $\sigma_n = \frac{1}{n}\sum_{k=1}^n f_k$.

   (a) Show that $\sum_{k=1}^\infty \|\sigma_{k^2}\|_2^2 < \infty$.

   (b) Show that $\sigma_n(x) \to 0$ for a.e. $x \in (0,1)$.

23.1.1 Solutions

1. Follows easily from Fubini. Part (b) follows from rotation invariance of Lebesgue measure and Fubini.

2. For each $r \in \mathbb{Q}$ let E_r be the complement of the Lebesgue set of $F_r(x) = |f(x) - r|$. Since $f \in L^1$ we know that $m(E_r) = 0$ so $E = \bigcup_{r \in \mathbb{Q}} E_r$ has $m(E) = 0$. For $x \in E^c$ we can apply Lebesgue differentiation to $|f|$. Fix $\epsilon > 0$ and let $|f(x) - r| < \epsilon$. Also, let $H > 0$ be such that for all $h < H$ and
$$\frac{1}{h} \int_{\mathbb{B}(x,h)} |f(y) - r| dy < \epsilon$$
We can write
$$\begin{aligned} \frac{1}{h} \int |f(y) - f(x)| dy &\leq \frac{1}{h} \int_{\mathbb{B}(x,h)} |f(y) - r| dy + \frac{1}{h} \int_{\mathbb{B}(x,h)} |r - f(x)| dy \\ &= |f(x) - r| + \epsilon \\ &< 2\epsilon. \end{aligned}$$

3. Follows quickly from Closed Graph Theorem with $(v_i, T(v_i)) \to (v, w)$ and since $(v_i - v, T(v_i - v)) \to (0, 0)$ we know from linearity that $T(v_i) \to w$.

4. Equicontinuity and pointwise boundedness follow from FTC and Holder.

5. Fix U. We can approximate U from inside with compact sets and apply Urysohn to obtain $f_n \in C_c(\mathbb{R}^n)$ such that $f_n \to \mathbb{1}_U$ and $0 \leq f_n \leq \mathbb{1}_U$. We can now write
$$\mu_k(U) = \int \mathbb{1}_U d\mu_k \geq \int f_n d\mu_k.$$
If we take a $\liminf_{k \to \infty}$ on both sides
$$\liminf_k \mu_k(U) \geq \liminf_k \int f_n d\mu_k \stackrel{!}{=} \int f_n d\mu.$$
Where at ! we use weak* convergence of the μ_k. Conclude by taking $\lim_{n \to \infty}$ of the RHS.

23.1.2 Other Exam

1. Let $f \in L^1([0,1])$ Show that for all $\epsilon > 0$ there exists a continuous function g such that $\|f - g\|_1 < \epsilon$.

   Proof. We prove that we can approximate simple functions by continuous functions. Since the simple functions are dense this is sufficient. Let $\varphi = \sum_1^N a_n \mathbb{1}_{A_n}$. It suffices to prove we can approximate $\mathbb{1}_{A_n}$. This follows by inner regularity and Urysohn, since we can approximate with a compact set from the inside and an open set outside and obtain an appropriate continuous function.

   □

2. Let $\mathcal{H}$ be a Hilbert space with orthonormal basis $\{e_i\}$. Assume $\{x_i\}$ is a sequence of vectors such that
$$\sum \|x_n - e_n\|^2 < 1.$$
Prove that $\{x_i\}$ is dense in $\mathcal{H}$.

Proof. Let $\mathcal{A} = \overline{\{x_i\}}$ and suppose $\mathcal{A} \subsetneq \mathcal{H}$. Let $z \in \mathcal{A}^\perp$. It follows from the reverse triangle inequality, the Parseval's identity then the Cauchy-Schwarz inequality that

$$\begin{aligned}
0 &= \sqrt{\sum_1^\infty |(z, x_n)|^2} \\
&= \sqrt{\sum_1^\infty |(z, e_n) - (z, x_n - e_n)|^2} \\
&\geq \sqrt{\sum_1^\infty \Big| |(z, e_n)| - |(z, x_n - e_n)| \Big|^2} \\
&= \left\| \sum_1^\infty (|(z, e_n)| - |(z, x_n - e_n)|) e_n \right\| \\
&= \left\| \sum_1^\infty |(z, e_n)| e_i - \sum_1^\infty |(z, x_n - e_n)| e_n \right\| \\
&\geq \left| \|z\| - \sum_1^\infty \|z\| \, \|x_n - e_n\| \right| \\
&= \|z\| \, (1 - \sum_1^\infty \|x_n - e_n\|) \\
&> 0.
\end{aligned}$$

A contradiction. □

3. Suppose $f_n \in L^1[0, 1]$. Prove or provide a counterexample to the following.
   (a) $f_i \to f$ in measure implies $f_i \to f$ in L^1.
   (b) $f_i \to f$ in measure implies $f_i \to f$ a.e.
   (c) $f_i \to f$ a.e. implies $f_i \to f$ in measure.

4. Let $E \subseteq [0, 1]$ be the set of $.a_1 a_2 \ldots \in [0, 1]$ such that $a_{n_2} \notin \{a_n, a_{n+1}\}$. What is $m(E)$? Prove it!

5. Let $f_n \in L^2$ be such that $f_n \to 0$ weakly. Prove there exists a subsequence such that the sequence of means $F_{n_k} = \frac{1}{n_k} \sum_1^{n_k} f_{n_k}$ converges to 0 in L^2.

6. Let (X, μ) be a measure space and $f \in L^1(X, \mu)$. Prove that for all $\epsilon > 0$ there exists a $\delta > 0$ such that if A is measurable with $\mu(A) < \delta$ then

$$\int_A |f| d\mu < \epsilon.$$

Proof. Fix $\epsilon > 0$. Define the sets $A_m = \{x \colon |f| < m\}$. Notice that $\mathbb{1}_{A_m} |f| \nearrow |f|$ and so by MCT

$$\int_{A_m} |f| \to \int_X |f|.$$

Let M be such that $\int_{X \setminus A_M} |f| < \epsilon$. Now, let $\delta = \frac{\epsilon}{M}$. For any set A with $\mu(A) < \delta$ it holds that

$$\begin{aligned}\int_A |f| &= \int_{A \cap A_M} |f| + \int_{A \cap A_M^c} |f| \\ &\leq m\mu(A) + \int_{A_M^c} |f| \\ &\leq 2\epsilon.\end{aligned}$$

□

7. Let $\{r_k\}$ be an enumeration of $\mathbb{Q} \cap [0,1]$. Define $f : [0,1] \to \mathbb{R}$ by $f(x) = \sum_1^\infty \frac{|x - r_k|}{3^k}$. You may use the fact that f is continuous on $[0,1]$. Prove that f is differentiable at every irrational point in $(0,1)$.

8. Let $f_n \in L^2[0,1]$ be a sequence of nonzero elements. Show there exists $g \in L^2$ such that for all $n \geq 1$ we have
$$\int_0^1 g(x) f_n(x) dx \neq 0.$$

   Proof. Each f_n is a bounded linear functional on L^2, and so the set $K_n = \ker\{f_n\}$ is closed and one dimensional. Let $E = \bigcup_1^\infty K_n$. Note that Baire Category Theorem guarantees that $E \neq L^2$ since otherwise we would have L^2 (a complete metric space) is a union of closed, nowhere dense sets.

   □

23.2 Other Schools

(Stanford Reals Exam)

(a) Let X be a banach space and $e_n \rightharpoonup x$ with $\|e_n\| = 1$. Prove that $\|x\| \leq 1$.

(b) Let $\mathcal{H}$ be a Hilbert space with countable basis. Let $x \in \mathcal{H}$ with $\|x\| \leq 1$. Construct a sequence $x_n \rightharpoonup x$ with $\|x_n\| = 1$ for all n.

Part (a)

Proof. Suppose that $x \notin \overline{\mathbb{D}}$. Recall that the geometric Hahn-Banach theorem states that since $\{x\}$ is compact and $\overline{\mathbb{D}}$ is closed, with both sets convex and disjoint we can find a linear function ℓ and $\alpha > 0$ such that $\ell(x) \leq \alpha - \epsilon$ and for $y \in \overline{\mathbb{D}}$ we have $\ell(y) \geq \alpha + \epsilon$. This contradicts the fact that $\ell(y_n) \to \ell(y)$.

□

Part (b)

Proof. WLOG we can assume that $\mathcal{H} = \ell^2$. Fix $x = (x_1, \dots,)$ and let $x^n = (x_1, x_2, \dots, x_n, \sqrt{c_{n+1}}, 0, 0, \dots)$ with $c_{n+1} = 1 - \sum_1^n |x_i|^2$. Notice that since $\|x\| \le 1$ the quantity $\sqrt{c_{n+1}}$ is well defined and decreases to $c_{n+1} \searrow 1 - \|x\|$. Clearly the x^n satisfy $\|x^n\| = 1$. Moreover, for any $y \in \mathcal{H}$ it holds that

$$(x^n, y) = c_{n+1} y_{n+1} + \sum_1^n x_i y_i \to \sum_1^\infty x_i y_i = (x, y).$$

□

(Folland 5.1.12) Let $\mathcal{H}$ be a normed vector space and $\mathcal{M} \subseteq \mathcal{H}$ a closed subspace.

(a) $\|x + \mathcal{M}\| = \inf\{\|x + y\| : y \in \mathcal{M}\}$ is a norm on $\mathcal{H}/\mathcal{M}$, called the quotient norm.

(b) For any $\epsilon > 0$ there exists $x \in \mathcal{H}$ such that $\|x\| = 1$ and $\|x + \mathcal{M}\| \ge 1 - \epsilon$.

(c) The projection $\pi(x) = x + \mathcal{M}$ from $\mathcal{H}$ to $\mathcal{H}/\mathcal{M}$ has norm 1.

(d) If $\mathcal{H}$ is complete, so is $\mathcal{H}/\mathcal{M}$ (*hint:* use theorem 5.1).

Part (a)

Proof. We check the obligatory things:

1. Clearly $\|x + \mathcal{M}\| \ge 0$ for all x and since $0 \in \mathcal{M}$ we know that $\|x + \mathcal{M}\| = 0 \iff x = 0$.
2. Also, for $\alpha \in \mathbb{C}$ we have have $\|\alpha x + \mathcal{M}\| = \inf\{\|\alpha x + y\| : y \in \mathcal{M}\} = \inf\{\|\alpha x + \alpha y\| : y \in \mathcal{M}\} = \alpha \|x + \mathcal{M}\|$. Where we use the fact that $\mathcal{M}$ is closed and thus $y \in \mathcal{M} \iff \alpha y \in \mathcal{M}$.
3. The triangle inequality holds in this norm since

$$\begin{aligned}\|(x+y) + \mathcal{M}\| &= \inf\{\|(x+y) + z\| : z \in \mathcal{M}\} \\ &\le \inf\{\|x + (1/2)z\| : z \in \mathcal{M}\} + \inf\{\|y + (1/2)z\| : z \in \mathcal{M}\} \\ &= \|x + \mathcal{M}\| + \|y + \mathcal{M}\|.\end{aligned}$$

□

Part (b)

Proof. Fix $\epsilon > 0$. Let $\delta = \sup_{\|x\|=1} d(x, \mathcal{M})$ and let x_0 be such that $\|x_0\| = 1$ and $d(x, \mathcal{M}) > \delta - \epsilon$. Since $y \in \mathcal{M}$ implies that $-y \in \mathcal{M}$ we have

$$\|x + \mathcal{M}\| = \inf\{\|x - y\| : y \in \mathcal{M}\} = d(x, \mathcal{M}) > \delta - \epsilon.$$

□

24 Outline

There are roughly three topics.

(i) Lebesgue Measure on $\mathbb{R}$

  (a) Definition of measure
  (b) Outer measure
  (c) Measurable functions
  (d) Integrals
  (e) Three Key Convergence Theorems
    (1) Monotone
    (2) Fatou
    (3) Lebesgue Dominated Convergence
  (f) Differentiation Theorems
    (1) Functions of Bounded Variation $\implies f'$ exists a.e.
    (2) Absolute continuity, lets us write $f = \int f'$
    (3) $L^p(\mathbb{R})$
    (4) Minkowski and Hölder inequalities
    (5) Riesz (completeness)
    (6) $C_c(\mathbb{R})$ is dense in L^p
    (7) Riesz representation

(ii) Topology, Banach Spaces, Functional Anaylsis

(iii) Everything Else

$$\sum_{k=1}^{\infty} \frac{\sin(kx)}{k^2+1} = \int_0^x \sum_{k=1}^{\infty} \frac{\cos(kt)}{k(k^2+1)} dt$$