Math 230, Exam 2 Name: ____________________

November 30, 2018

Instructions.

- You are allowed one side of handwritten notes
- No calculators.
- Leave your answers unsimplified. For example, it is preferable for you to write $13C_{4,4}C_{48,1}$ rather than 624.
- There are 6 problems on 5 pages. Make sure your exam is complete.

Run LaTeX again to produce the table

Useful facts

(General Poisson Approximation) If $p_1 + \cdots + p_n = \lambda$ are the probabilities of independent events $A_1, \ldots, A_n$ and N is the number of the A_i that occur, and $X = \text{Poisson}(\lambda)$ then $|P(N = k) - P(X = k)| \leq \sum_{i=1}^n p_i^2$ for all k.

(Poisson thinning) If $X = \text{Poisson}(\lambda)$ and $(X_1, X_2, \ldots, X_n)$ are the groupings of X after multinomial thinning with $p_1, p_2, \ldots, p_n$ then the X_i are independent and $X_i = \text{Poisson}(p_i\lambda)$.

(Union bound) For any events $A_1, \ldots, A_n$ it holds that $P(\cup_{i=1}^n A_i) \leq \sum_{i=1}^n P(A_i)$.

[4 points] 1. Eight students, 4 boys and 4 girls, line up for a photograph. What is the probability the boys and girls alternate?

[5 points] 2. You pick 8 cards out of a 52-card deck. Let A be the event you have exactly 3 of at least one type. Let A_i be the event you have exactly 3 of type i (there are 13 types). Use the inclusion-exclusion formula to determine $P(A)$. (*Hint: You do not need all 13 formula terms.*)

3. 13 friends vote between going to a concert or going to a movie. They do so by secretly placing a pebble in one of two jars. The concert gets 7 pebbles and the movie gets 6. Suddenly, 5 people decide they cannot go. Rather than re-vote or ask the people who left what they voted for, the remaining people decide to randomly remove 5 pebbles from the jars. A tie will not change their decision.

[1 point] (a) How many pebbles must be removed from the concert jar to change their decision?

[5 points] (b) What is the probability randomly removing these 5 pebbles changes their decision?

4. There are one hundred 20, 30, and 40 year olds in a study on car accidents. So, three hundred people in total. The probability, p_x, that an x-year old gets in an accident the next year are: $p_{20} = 5/1000$, $p_{30} = 3/1000$ and $p_{40} = 2/1000$.

[3 points] (a) Write the mean of a Poisson random variable that predicts the total number of car accidents for these 300 people over the next year. Simplify as much as you can.

[2 points] (b) Use Poisson approximation to estimate the probability there is at least one accident in the next year.

[3 points] (c) A car accident is fatal with probability 10^{-2}. Use Poisson thinning and Poisson approximation to estimate the probability that no one in the study is in a fatal car accident.

[2 points] (d) Use the error bound from the cover to estimate how close your approximation in Part (b) is. You do not need to simplify.

5. The government decides to issue a tax rebate. All taxpayers fall in one of three tax brackets: high, middle, and low. They will send H, M and L dollars to each bracket, respectively.

[3 points] (a) Suppose that 2^{10} hundred dollar bills are going to be rebated, and that $p_h = 2^{-2}, p_m = 2^{-1}$, and $p_\ell = 2^{-2}$ are the proportion of taxes paid into the government from each group. The government decides H, M and L according to a multinomial distribution with these three parameters. What are the distributions of H, M and L?

$H =$ $M =$ $L =$

[3 points] (b) Suppose instead the government decides to distribute Poisson(2^{10}) hundred dollar bills with the same random allocation rule. What are the distributions of H, M, and L?

$H =$ $M =$ $L =$

[3 points] (c) There are 2^4 people in the high bracket, 2^8 people in the middle bracket, and 2^8 people in the low bracket. Suppose we use the scheme in (b), and then disperse each hundred dollar bill from H, M and L uniformly randomly among the people in the corresponding tax bracket. Write the distribution for the amount of hundred dollar bills a person in the high, middle, and low tax bracket receives.

high =

middle =

low =

[5 points] (d) Let A be the event that everyone gets at least one hundred dollar bill. Find $P(A)$.

6. A skin cell reproduces by splitting into two new cells. Each of the two cells independent of one another move one unit to the left, or one unit to the right of the parent cell. The 2 new cells also split and the two new cells randomly displace left or right in the same fashion. Cells keep splitting like this indefinitely. Suppose left displacement has probability 1/5 and right has probability 4/5.

[2 points] (a) The number of cells in the nth generation is 2^n. Let $A_{i,n}$ be the event that the ith cell of these is at $-n$. What is $P(A_{i,n})$?

[3 points] (b) Are the events $A_{i,n}$ independent? Give an intuitive explanation in words with no calculation.

[3 points] (c) Let E_n be the event that there is are any cells from the nth generation at $-n$. Use a union bound to estimate $P(E_n)$. Please simplify your answer so there is no summation. (*Hint: there are* 2^n *cells in the* n*th generation.*)

[3 points] (d) Use another union bound to estimate the probability that there exists a time $n \geq 1$ with a cell at $-n$. You may leave your answer as a sum.

(e) The answer to the previous part is an overestimate. For 1 point extra credit, what is the true probability of the event in (d)?