#2: Velocity, Secant Lines, Tangents and Limits
Chapters 2.1, 2.2

Contents

1 Velocity

Life is easy when velocity is constant, like in the parametric examples from last time, but in real life velocity often varies. Consider the following example.

Example 1. *Suppose that Matt runs a race so that his position function is given by* $p(t) = e^{2t}$

Diagram

The distance traveled in t seconds is

$$\mathit{Distance} = p(t) - p(0)$$

Notice in the first second I travel

$$p(1) - p(0) = e^2 - 1 \approx 6.289$$

And in the second second ($1 \leq t \leq 2$), I travel

$$p(2) - p(1) = e^4 - e^2 \approx 47.209$$

So, my velocity is definitely not constant.

GOAL Given a position function try to find a way to describe velocity at any time.

1.1 Average Velocity

The intuition is that

$$\text{Average Velocity} = \frac{\text{distance traveled}}{\text{time elapsed}}.$$

So, it only makes sense to talk about average velocity if we are given a time interval. Let's take a look at the formula.

Definition: If $p(t)$ is a position function, the **average velocity** on a time interval $[a, b]$ is given by

$$\mathcal{A}_{[a,b]}(p(t)) = \frac{p(b) - p(a)}{b - a} = \frac{\text{distance traveled}}{\text{time elapsed}}.$$

Example:

Move around the room and time myself.

Example:

$p(t) = e^{2t}$ then $\mathcal{A}_{[0,1]}(p(t)) \approx 6.38$.

Computation

! We are assuming all motion is on a fixed track. So it makes sense to talk about things like net distance, negative distance and negative velocity. We will deal with more complicated motion later, i.e. parametric equations.

Diagram

Example:

(Signed Velocity:) Let $p(t) = \cos t$.

Computation

$$\mathcal{A}_{[0,2\pi]}(p(t)) = 0$$

Diagram

WRABD: (fyi WRABD stands for What's This Really About Breakdown.)

Definition: The **secant line** to a curve at points P and Q is the unique line going through P and Q.

If we look at $p(t) = e^{2t}$ at the time $t = 0$ and $t = 1$ we can draw a *secant line*

Diagram

Now if we look at a general curve when we compute the slope of this secant we get.

Diagram

The slope of our line, L, is

$$\text{slope of } L = \frac{y_2 - y_1}{x_2 - x_1} = \frac{p(b) - p(a)}{b - a} = \mathcal{A}_{[0,1]}(p(t)).$$

Fact: The slope of the secant line is the average velocity.

1.2 Hearkening Back to Our Goal

Remember that we are trying to find a way to define the velocity at a given instant.

The **genius idea of differential calculus** is to look at really small time intervals to 'approximate the velocity in a given instant.

Example:

Let's use this idea to approximate the velocity of $p(t) = e^{2t}$ at time $t = 0$.

Interval	$\mathcal{A}_{[0,t]}(p(t))$
[0,1]	6.3891
$[0, 1/10]$	2.214
$[0, 1/100]$	2.020
$[0, 1/1000]$	2.002
$[0, 1/10000]$	2.0002

So $v(0) = 0$.

Diagram

Notice that it looks like we are approaching a tangent line.

Whole Point of Calculus Fact The slope of the tangent line to a position function is the velocity and we find this by taking 'limits' of slopes of secant lines.

Unofficial Formula Given a position function $p(t)$, the velocity at time t_0 is given by

$$v(t_0) = \lim_{t \to t_0} \mathcal{A}_{[t,t_0]}(p(t)) = \lim_{t \to t_0} \frac{p(t_0) - p(t)}{t_0 - t}.$$

And so by the WRABD we know that for a function $f(x)$ we have

$$\begin{aligned} \text{slope of tangent at } x_0 &= \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x_0 - x} \\ &= \text{Limit of Slopes of Secant Lines Through } (x_0, f(x_0)) \text{ and } (x, f(x)) \end{aligned}$$

But this raises an important question. **What is a limit!?**

2 Limits

2.1 Definition

Definition: When we write $\lim_{x \to a} f(x) = L$ this means that we can make values of $f(x)$ as close as we want to L by looking at *all* values of x sufficiently close to but *not equal* to a.

Diagram

Example:

(a) $f(x) = 2x^2$, we then have $\lim_{x \to 2} f(x) = 2(2^2) = 8$

Diagram

Example:

(b) $f(x) = \begin{cases} 0, & x = 2 \\ 2x^2, & x \neq 2 \end{cases}$. Then $\lim_{x\to 2} f(x) = 8$ still because when $x \neq 2$ we can still make $f(x)$ close to 8.

Diagram

! Notice how we aren't concerned with the case $x = 2$. (Look at the definition to see why this is okay.)

Example:

(c) $f(x) = \sin(\pi/x)$.

Diagram

Notice that it doesn't look promising that the limit exists. We can be a little more careful and look at values near zero

x	$f(x)$
1	0
1/100	0
$1/10^{10}$	0

But the limit definitely isn't zero.

! Notice how in the last example we needed *all* values of x close to 0.

Definition: When the limit doesn't exist we write **DNE**.

Fact: If $\lim_{x\to a} f(x) = L$ then L is unique. In other words there can be only one limit.

2.2 Left and Right Limits

You will see that there are a lot of strange functions out there.

Example:

$$g(x) = \begin{cases} -1, & x < 0 \\ 1, & x \geq 0 \end{cases}.$$

Diagram

Although the limit as $x \to 0$ doesn't exist (since there are two limits) it seems like we can make sense of breaking it up into left and right limits.

Definition:

(i) $\lim_{x \to a^+} f(x) = L^+$, the **limit as** x **goes to** a **from the right of** $f(x)$ **equals** L^+, means that $f(x)$ can be made close to L^+ for all x close to but not equal to a with $x > a$.

(ii) $\lim_{x \to a^-} f(x) = L^-$, the **limit as** x **goes to** a **from the left of** $f(x)$ **equals** L^-, means that $f(x)$ can be made close to L^- for all x close to but not equal to a with $x < a$.

Best way to remember:

$$Left \to Linus \to Minus$$

$$Right \to Rositive \to Positive$$

Example:

$$\lim_{x \to 0^+} g(x) = 1, \qquad \lim_{x \to 0^-} g(x) = -1$$

Fact: $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x)$.

So $\lim_{x \to 0} g(x)$ DNE.

2.3 'To ∞ and... that's far enough.'

So far we have seen that a limit can exist and equal L (i.e. $\lim_{x \to a} f(x) = L$, or the limit doesn't exist and we have DNE. There is a third possibility.

(a) $\lim_{x \to a} f(x) = L$

(b) $\lim_{x \to a} f(x) = DNE$

(c) $\lim_{x \to a} f(x) = +\infty$

(d) $\lim_{x \to a} f(x) = -\infty$

Definition:

(i) We say that $\lim_{x \to a} f(x) = +\infty$ if for all x as x gets closer to but not equal to a the function $f(x)$ gets arbitrarily large.

(ii) We say that $\lim_{x \to a} f(x) = -\infty$ if for all x as x gets closer to but not equal to a the function $f(x)$ gets arbitrarily small.

Example:

$h(x) = \frac{1}{x}$

Diagram

$\lim_{x \to 0^+} h(x) = \infty$, $\lim_{x \to 0^-} = -\infty$ so by the previous fact: $\lim_{x \to 0} = DNE$

Definition: If $\lim_{x \to a^\pm} f(x) = \pm\infty$ then we say that f has a **vertical asymptote** at a.

Example:

Diagram