Extra Practice: **Chapter 4:** 12, 13, 24 **Chapter 5:** 3, 5, 17

1. Suppose that X and Y are independent uniform(0,1) random variables. Recall they have distribution function $F(x) = P(X \leq x) = x$.

   (a) What is the distribution function of $\max(X, Y)$?

   Solution: Let $Z = \max(X, Y)$. We have $P(Z \leq x) = P(X \leq x, Y \leq x) = P(X \leq x)P(Y \leq x) = F(x)F(x) = x^2$.

   (b) What is the distribution function of $\min(X, Y)$?

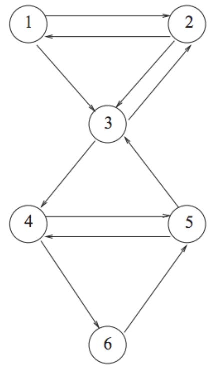
   Solution: Let $W = \min(X, Y)$. We have $P(W \leq x) = 1 - P(W > x) = 1 - P(X > x, Y > x) = 1 - (1 - P(X > x))(1 - P(Y > x)) = 1 - (1 - F(x))^2 = 1 - (1 - x)^2$.

   (c) Suppose we connect the line between X and Y. What is the probability a is on this line? (Hint: Condition on what X is.)

   Solution:

$$
\begin{aligned}
P(a \in [X, Y]) &= P(a \in [X, Y] \mid X < a)P(X < a) + P(a \in [X, Y] \mid X > a)P(X > a) \\
&= aP(Y > a) + (1 - a)P(Y < a) = 2a(1 - a).
\end{aligned}
$$

2. A website consists of six pages linked to one another as depicted below:



Arrows indicate which pages have a link pointing to others.

(a) Suppose a person clicks links randomly. Write the transition matrix, p, for this Markov chain.

Solution:

$$p = \begin{pmatrix} 0.0 & 0.5 & 0.5 & 0.0 & 0.0 & 0.0 \\ 0.5 & 0.0 & 0.5 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.5 & 0.0 & 0.5 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.5 & 0.5 \\ 0.0 & 0.0 & 0.5 & 0.5 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0 & 0.0 \end{pmatrix}$$

(b) Here is p^{10}:

$$p^{10} = \begin{pmatrix} 0.076 & 0.151 & 0.225 & 0.22 & 0.22 & 0.108 \\ 0.075 & 0.152 & 0.225 & 0.22 & 0.22 & 0.108 \\ 0.076 & 0.148 & 0.223 & 0.225 & 0.217 & 0.111 \\ 0.073 & 0.146 & 0.217 & 0.22 & 0.228 & 0.116 \\ 0.073 & 0.149 & 0.225 & 0.228 & 0.22 & 0.108 \\ 0.076 & 0.141 & 0.223 & 0.217 & 0.232 & 0.111 \end{pmatrix}.$$

Suppose the clicker starts at page 3. What is the probability the random clicker is at page 5 after 10 clicks?

Solution:
$p^{10}(3, 5) = .217.$

(c) The vector

$$\pi = \left[\frac{2}{27}, \frac{4}{27}, \frac{6}{27}, \frac{6}{27}, \frac{6}{27}, \frac{3}{27}\right]$$

satisfies $\pi p = \pi$. (This was found using a computer.) Which site is least likely to be occupied after 1000 clicks?

Solution: Site 1.

(d) Which sites are most likely to be occupied?

Solution: Sites 3,4,5

3. Gary is gambling and starts with \$1. He is flipping a fair coin and receives \$1 if he flips heads and loses \$1 if he flips tails. He stops playing when he has no money. Let X_n be the amount of flips for which he has n dollars.

(a) Explain why $EX_1 = 2$.

Solution: It is a geometric random variable with parameter 1/2.

(b) What is $P(X_n > 0)$? (Hint: Gambler's ruin.)

Solution: $\frac{1}{n}$.

(c) Suppose Gary has reached n dollars. What is the probability, p, he goes broke before visiting n again?

Solution: He has to flip tails, then not return to n. Again this is gambler's ruin but this time the picture is reversed. So we have $\frac{1}{2}\frac{1}{n}$. The 1/2 is for flipping the first tails.

(d) You have just shown that when Gary reaches n he visits it a Y = geometric(p) number of times. What is EX_n?

Solution:

$$EX_n = E[X_n \mid X_n > 0]P(X_n > 0) = E[X_n \mid X_n > 0]\frac{1}{n} = \frac{EY}{n} = 2.$$

Answers to Homework 6

Chapter 4

5. (b) A has probability 3/4, while B and C have probability 1/8 each. The probability of B at time 3 is 13/32.

9. The Markov chain predicts 51.2% in 1980, 56.8% in 1990, and 60.9% in 2000.

11. (a) .55 (b) .575 (c) .6

15. .38, .25, .24

21. 7/13

Chapter 5

2. 3/4

4. 81/36, 243/45, .3375

9. (a) $x^2/4$ for $0 \le x \le 2$ (b) 1/4 (c) 7/16

20. $f(y) = y^{(1/n)-1}/n$ for $0 < y < 1$ and 0 otherwise. Notice that $P(Y \le y) = P(X \le y^{1/n}) = y^{1/n}$. The fundamental theorem of calculus ensures that $f(y) = \frac{d}{dy}F(y)$ so we differentiate $y^{1/n}$ to obtain the answer.

Extra practice problem answers

12. (a) .415 (b) .46 (c) 2/3

13. .782, .727, 8/11

24. .96 Binomial(30,.04)

3. 1/2, .3, .05

5. 15/28, 127/49, 2.304

17. Recall that the distribution function for an exponential(λ) is $F(x) = P(X \le x) = 1 - e^{-\lambda x}$. If $Y = \min(X_, \ldots, X_n)$ then

$$\begin{aligned} P(Y \le x) &= 1 - P(Y > x) \\ &= 1 - P(X_1, \ldots, X_n > x) \\ &= 1 - P(X_1 > x)P(X_2 > x) \cdots P(X_n > x) \\ &= 1 - P(X_1 > x)^n \\ &= 1 - (1 - P(X_1 \le x)^n \\ &= 1 - (1 - (1 - e^{-\lambda x}))^n \\ &= 1 - e^{-\lambda n x}. \end{aligned}$$

This is the distribution function of an exponential(λn), thus Y has this distribution.

Material on exam

Chapter 4

1. Writing the transition matrix p for a larger markov chain.
2. Interpreting p^m.
3. For $q = [q(1) \cdots q(m)]$ a probability distribution on the initial state interpreting qp and qp^m.
4. Interpreting the stationary distribution π.
5. Finding π for the 2-state case.
6. The symmetric gambler's ruin. You do not need to know the asymmetric case.

Chapter 5

1. Understanding what a density function is. Knowing that $\int_{-\infty}^{\infty} f(x)dx = 1$.
2. Understanding a distribution function $F(a) = P(X \leq a) = \int_{-\infty}^{a} f(x)dx$.
3. Computing $EX = \int xf(x)dx$, $EX^2 = \int x^2 f(x)dx$ and $\text{var}(X) = EX^2 - (EX)^2$. (Only need to know $\int cx^a dx = \frac{c}{a+1}x^{a+1}$ and $\int e^{cx}dx = \frac{1}{c}e^{cx}$.)
4. Good comfort with exponential(λ) and uniform(a, b) random variables. For example, dealing with minima and maxima, or line segments connecting the two.

Description of exam: At least two of the MAPs problems will appear, but slightly modified. There will be a 2×2 Markov chain question where you find the stationary distribution, and maybe a multistep probability. You will likely need to interpret the meaning of qp. For the continuous case, you will be asked to make a function into a density by finding the appropriate constant, then asked to compute the variance.