Contents

1 Topics to Know According to Marshal

1.1 Residues

1.1.1 Steps to Computing an Integral

(i) First decide what contour to use.

(ii) Check how the Residue theorem applies.

(iii) Compute the residue.

$$\operatorname{Res}(f,a) = \lim_{z\to a} \frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}(z-a)^n f(z).$$

(a) (simple pole) When we have a Laurent series for the function $f = \sum_{-\infty}^{\infty} a_n(z-a)^n$ on a punctured neighborhood of $\Delta \setminus \{a\}$ every term will integrate to zero except for $\frac{a-1}{z-a}$. So look at $\lim_{z\to a}(z-a)f(z)$.

(b) If simple pole and $f = \frac{g}{h}$ then $\operatorname{Res}(f,a) = \frac{g(a)}{h'(a)}$.

(c) If f has a pole of higher order. For example, $f(z) = \frac{g(z)}{(z-a)^3}$ then let

$$g(z) = g(a) + (z-a)g'(z) + \frac{(z-a)^2}{2}g''(a) + \cdots$$

and we have $\operatorname{Res}_a f = \frac{g''(a)}{2}$.

(d) Sometimes it helps to write

$$(z-a)f(z) = \frac{z-a}{1/f(z)} \to \frac{1}{(1/f)'(a)} = \frac{1}{\frac{\frac{1}{f(z)} - \frac{1}{f}(a)}{z-a}}.$$

For example $(z-a)\cot z = \frac{z-a}{\tan z} \to \frac{1}{\sec^2 a}$.

(iv) Estimate the contour. Most likely with the estimate

$$\left| \int_{\gamma_1} f(z)dz \right| \le \sup_{\gamma_1} |f| \cdot |\gamma|.$$

1.1.2 Types of Contours

- Wedge Root of Unity Integrals
- Keyhole integrals.
  - i $\frac{x}{x^\alpha(x+1)}$
  - ii $\frac{\log x}{1+x}$
- Take the real part half circle integrals.
- Rectangles.

1.2 Construct a Function

TOOLBOX

- Runge's Theorem

  Theorem (Runge). *If $D \subseteq \mathbb{C}$ is open and $A \subseteq \mathbb{C}^* \setminus D$ intersects every component of $\mathbb{C}^* \setminus D$. If $f \in H(D)$ then there exists a sequence of rational functions $\{R_j\}$ such that $R_j \to f$ uniformly on compact subsets of D.*

- Mittag-Leffler
- Weirstrauss Factorization

$$E_p(z) := \begin{cases} (1-z)e^{\sum_1^p \frac{z^k}{k}}, & p \geq 1 \\ (1-z), & p = 0 \end{cases}$$

  We then take

$$\prod_1^\infty E_{p_n}\left(\frac{z}{a_n}\right)$$

  Theorem (Specifying Zeros). *Let $\{a_n\}$ be a sequence of not necessarily distinct nonzero complex numbers such that $\lim_{n\to\infty} a_n = \infty$. Let $\{p_n\}$ be a sequence of non negative integers such that for any $R > 0$ it holds that*

$$\sum_1^\infty \left(\frac{R}{|a_n|}\right)^{1+p_n} < \infty$$

  Then the product

$$f(z) = \prod_1^\infty E_{p_n}\left(\frac{z}{a_n}\right)$$

  is an entire function which has zeros at each a_n and no other zeroes in $\mathbb{C}$. (If a_n appears m-times then its multiplicity is m).

  Theorem (Weierstrass Factorization Theoerem). *Let f be entire with $f \not\equiv 0$. Let $a_1, a_2, \ldots$ be the nonzero zeros of f listed according to mulitplicity. Then there exists $g \in H(\mathbb{C})$ and a sequence $\{p_n\}$ of non-negative integers such that*

$$f(z) = z^k \prod_1^\infty E_{p_n}(z/a_n)e^{g(z)} \tag{2}$$

  where k is the order of the zero at 0.

- Blashke Products

1.2.1 Conformal Maps

We have the following maps in our toolbox.

TOOLBOX

- *LFT*
  1. Remember that any LFT is determined by three points.
  2. Disks to Disks and Lines to Lines. (Or perhaps the switch).
  3. Schwarz reflection extends across lines of symmetry. For example if you want to map a line, L, to the disk choose a point b on one side of the line and let b_R be the refleciton. By reflection we know that the map $\frac{z-b}{z-b_R}$ does the trick. For example, $\frac{z-i}{z-(-i)} : UHP \to \mathbb{D}$.
- z^α
- $\log z$
- $z + \frac{1}{z}$
- e^z
- $\sin z$ (maps $\{z : \operatorname{Re} z \geq 0, \operatorname{Im} z \in [\frac{-}{\pi}/2, \pi/2]\} \to \mathbb{H}$ taking rectangles to half ellipses and takes the whole half strip to $\mathbb{C}$ minus the rays on the real line starting at -1 and 1.

1.3 Growth of Functions

TOOLBOX

- **Maximum Principle**
- **Schwartz Lemma:**

  Theorem (Theorem 12)**.** *(Schwarz's Lemma) Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic and $f(0) = 0$, then*

  (a) $|f(z)| \leq |z|$ for all $z \in \mathbb{D}$.
  (b) $|f'(0)| \leq 1$

  Moreover, if $|f(z)| = |z|$ for $z \in \mathbb{D} \setminus \{0\}$ or if $|f'(0)| = 1$ then there exists $\alpha \in \mathbb{R}$ so that
  $$f(z) = e^{i\alpha} z$$

- Poisson Integral Formula
  $$P_r(\theta) = \frac{1-r^2}{1-2r\cos\theta + r^2} = \operatorname{Re}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right).$$

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t) f(e^{it})dt, \qquad f \in C(\partial\mathbb{D}).$$

$$u(x+iy) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-t)^2+y^2} f(t)dt, \qquad f \in C(\mathbb{R}).$$

- Herglotz Integral Formula: $f(z) + ic = \int_0^{2\pi} \frac{e^{i\theta}+\frac{z}{R}}{e^{i\theta}-\frac{z}{R}} u(Re^{i\theta})\frac{d\theta}{2\pi}$. with $z \in \mathbb{D}(0,R)$.
- $\sum a_n z^n$.
- Cauchy Integral Formula and Cauchy Derivative Formula.

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{\partial\Delta} f(\zeta)\frac{d\zeta}{(\zeta - z)^{k+1}}$$

- $f(z) = a_0 + a_1(z-a) + \lambda(z)$. With $\lambda(z) \in \mathcal{O}(z^2)$.
- Proof of Riemann Mapping Theorem Type Argument

1.4 Prove f Looks Like.

TOOLBOX

- Schwarz Lemma
- Schwarz Reflection Principle
- Laurent Series Expansion
- Poisson Integral Formula
- Proof of Riemann Mapping Theorem Type Argument

1.5 Normal Families

TOOLBOX

- **Montel's Theorem**: $\mathcal{F}$ locally bounded family of analytic functions implies $\mathcal{F}$ is normal.
- **Hurwitz Theorem**

  Theorem (Hurwitz's Theorem)**.** *Let $\{f_k(z)\} \in H(D)$ and $f_k \to f$ locally uniformly in D. Suppose also that f has a zero of order N at $z_0 \in D$. Then there exists $\rho > 0$ such that for all large k it holds that $f_k(z)$ has N zeroes in $\{|z - z_0| < \rho\}$ and all these zeros converge to z_0 as $k \to \infty$.*

- $\mathcal{F}$ is normal iff $\mathcal{F}_n = \{f^{(n)} : f \in \mathcal{F}\}$ is normal iff $\mathcal{F}^n = \{f^n : f \in \mathcal{F}\}$ is normal.

1.6 Zeros, Poles, Winding Numbers

TOOLBOX

- Rouche's Theorem
- Argument Principle
- Jensen's Formula

Definition 1. *Define the function*

$$n(r, 0, f) = n(r) := \text{ \# of zeros of } f \text{ in } \overline{\mathbb{D}}(0, r) \text{ counted with multiplicity.}$$

Note that n is a step function which is continuous from the right. Also, define the function

$$N(r, 0, f) = N(r) := \sum_{0 \le |z_j| \le r} \log \frac{r}{|z_j|}, \qquad z_j \text{ zeros of } f$$

Assume f has no zero at the origin. Then $n(t) = 0$ in a neighborhood of 0. We can write

$$\begin{aligned} N(r) &= \int_{[0,r^-]} \log \frac{r}{t} dn(t) \\ &= (\log \frac{rt}{n}(t)|_{t=0^+}^{t=r^+} - \int_0^r n(t) \frac{d}{dt} n(t) dt \\ &= \int_0^r n(t) \frac{dt}{t} \end{aligned}$$

If f has a zero at 0 then $N(r) := \int_0^r \frac{n(t)-n(0)}{t} dt$.

Theorem (Jensen's Formula)**.** *Let $f \in H(\mathbb{D}(0, R))$ and $f \not\equiv 0, r \in (0, R)$. Then*

(a) If $f(0) \neq 0$ then

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(re^{i\theta})| d\theta = N(r, 0, f) + \log |f(0)|$$

(b) $f(z) = cz^k g(z)$ where $k \ge 1, g(0) = 1, c \neq 0$. We have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(re^{i\theta})| d\theta = N(r, 0, f) + k \log r + \log |c|$$

- Hurwitz Theorem

1.7 Harmonic Functions

TOOLBOX

- **Harnack's Inequality**
- **Harnack's Theorem**
- **Poisson Integral Formula**
- **Mean Value Property**

2 Other Schools Prelims

SPRING 2003 COMPLEX ANALYSIS QUALIFYING EXAM

Please attempt all the problems and show all your work. In the following, "holomorphic" is synonymous with "analytic." Also, Δ will denote the open unit disk in $\mathbb{C}$.

(1) (a) Let $f : \mathbb{C} \to \mathbb{C}$ be meromorphic with a pole at infinity. Show that f must be a rational function.

(b) Use the above to prove the following: if $f : \Delta \to \mathbb{C}$ is holomorphic with a continuous extension to the boundary of Δ such that $|f(z)| = 1$ for all $|z| = 1$, then $f(z)$ is the restriction of a rational function.

(2) Let $f : \Delta \to \Delta$ be a holomorphic function with $f(0) = 0$ and $|f'(0)| = M$. If $0 \neq w \in \Delta$ is any other zero of $f(z)$, show that:

$$\frac{M}{1+M} \leq |w| \ .$$

(3) Let C be the closed curve defined by two pieces: the first piece is given by the set of all z satisfying $|z-1| = 3$ and $\mathrm{Re}(z-1) \geq 0$. The second piece is the straight line segment from $1+3i$ to $1-3i$. Orient C in the counterclockwise direction, and let Ω be the region enclosed by C. Suppose f is holomorphic in a neighborhood of $\overline{\Omega}$ with no zeros on C. Suppose also that:

$$\frac{1}{2\pi i}\int_C \frac{zf'(z)}{f(z)}dz = 3 \qquad \text{and} \qquad \frac{1}{2\pi i}\int_C \frac{z^2 f'(z)}{f(z)}dz = \frac{5}{2} \ .$$

Determine all the zeros of f in Ω explicitly.

(4) (a) State Rouché's Theorem.

(b) Let $\varphi : \Omega \to \mathbb{C}$ be holomorphic on an open convex set Ω. Show that for $z, w \in \Omega$

$$|\varphi(z) - \varphi(w)| \leq \max_{\xi \in L} |\varphi'(\xi)||z-w| \ ,$$

where L is the straight line segment from z to w.

(c) Use the above to prove the following: suppose

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

where

$$\sum_{n=2}^{\infty} n|a_n| \leq 1 \ .$$

Show that $f(z)$ is a 1-1 holomorphic function on Δ.

2.1 John Hopkins 2003

1. (a) *Proof.* Since the set of singularities of f cannot have a limit point and C^* is compact, we know that f has finitely many poles, $a_1, \dots, a_k \in \mathbb{C}$ each of order l_j. Consider the function

$$h(z) = \left(\prod_{j=1}^{k} (z - a_j)^{l_j} \right) \cdot f(z)$$

Notice that we have eliminated all of the poles of f on $\mathbb{C}$ and then must have $h \in H(\mathbb{C})$. Suppose h has a pole of order l at ∞. This implies that $h(z)$ is a polynomial since $h \in H(\mathbb{C})$ has a Taylor expansion which satisfies $\frac{1}{z^l} h(z)$ is bounded. So, if we let $h(z) = P(z)$ with P a polynomial and let $Q(z) = \prod_{j=1}^{k} (z - a_j)^{l_j}$ then we have the rational function

$$f(z) = \frac{h(z)}{\prod_{j=1}^{k} (z - a_j)^{l_j}} = \frac{P(z)}{Q(z)}$$

□

(b) *Proof.* Since $|f(z)| = 1$ on $\partial\mathbb{D}$ we apply Schwarz reflection to extend to an entire function $f(z) = \begin{cases} \frac{1}{f(\frac{1}{\bar{z}})}, & z \in \mathbb{D}^c \\ f(z), & z \in \mathbb{D} \end{cases}$. Notice that since f has finitely many zeros in $\overline{\mathbb{D}}$ we must have f is meromorphic with a pole at ∞ of the same order as the zero of f at $z = 0$. By part (a) we can extend to rational function. We conclude by applying uniqueness. □

2. Consider the function $g(z) = \frac{f(z)}{z}$. Since $g(0) = 0$ we know that g extends to be analytic on $\mathbb{D}$ with $g(0) = f'(0)$. Moreover, a simple application of Schwarz lemma tells us that $|f(z)| \leq |z|$ and therefore $g : \mathbb{D} \to \mathbb{D}$. Recall that the Schwarz-Pick lemma states that for any $w \in \mathbb{D}$ we have

$$\left| \frac{g(w) - g(0)}{1 - \overline{g(0)} g(w)} \right| \leq \left| \frac{w - 0}{z - \bar{0} \cdot w} \right| = |w|.$$

An application of reverse triangle inequality and the fact that $|g(w)|$ is positive gives

$$\begin{aligned} |w| &\geq \left| \frac{g(w) - g(0)}{1 - \overline{g(0)} g(w)} \right| \\ &\geq \frac{||g(w)| - |g(0)||}{|1 - \overline{g(0)} g(w)|} \\ &\geq \frac{|g(0)|}{1 + |\overline{g(0)}||g(w)|} \\ &\geq \frac{M}{1 + M} \end{aligned}$$

3. Let $Z = \{z \in \int C : f(z) = 0\}$. Notice that $\frac{f'}{f}$ has only simple poles. Moreover, since our curve avoid $z = 0$ we know that $\frac{zf'}{f}$ has the same poles as $\frac{f'}{f}$, except the residue is multiplied by z. We can use the residue theorem to write

$$3 = \frac{1}{2\pi i}\int_C \frac{zf'(z)}{f(z)}dz = \sum_{z_j} z_j \operatorname{Res}_{z_j}(f'/f) = \sum z_j n_j.$$

Where n_j is the order of the zero at z_j. We can further deduce that there are precisely two zeroes, z_1, z_2, since $|z_j| > 1$. We can conclude now that

$$z_1 + z_2 = 3, \qquad z_1^2 + z_2^2 = \frac{5}{2}.$$

Solving we get $z_1 = \frac{3}{2} + i$ and $z_2 = \frac{3}{2} - i$.

4. (a) Let $f, g \in H(\Omega)$ and $\gamma \subseteq \Omega$. If for $z \in \gamma$ it holds that $|f(z) - g(z)| < |g(z)|$ then f and g have the same number of zeros in γ.

   (b) We can write

$$\begin{aligned}|\varphi(z) - \varphi(w)| &= \left|\int_z^w \varphi'(\zeta)d\zeta\right| \\ &\leq \int_z^w |\varphi'(\zeta)|d\zeta \\ &\leq |z - w| \sup_{z \in L_{z,w}} |\varphi'(\zeta)|.\end{aligned}$$

   (c) First off, we know that $f \in H(\mathbb{D})$ since Let $\varphi(z) = f(z) - z$. Fix $w \in \mathbb{D}$. We apply Rouche's theorem to the function $F(z) = f(z) - f(w)$ and $G(z) = z - w$. Notice that

$$\begin{aligned}|F(z) - G(z)| &= |\varphi(z) - \varphi(w)| \\ &\leq \sup_{\zeta \in L_{z,w}} |\varphi'(\zeta)||z - w| \\ &\overset{*}{<} (1)|z - w| \\ &= |G(z)|.\end{aligned}$$

   Where at * we use the triangle inequality and the assumption that $\sum na_n \leq 1$ to conclude that $|\varphi'(\zeta)| < \sum na_n \leq 1$.

Complex Analysis Qualifying Examination

August 2011

1. Suppose $u(x, y)$ is a (real-valued) harmonic function on a simply connected domain in $\mathbb{C}$. Show that $u(x, y)$ can be written in the form $f(x + iy) + g(x - iy)$, where f and g are holomorphic functions.

2. An *inversion* is a function on the extended complex numbers of the form $z \mapsto \dfrac{1}{z - z_0}$, where z_0 is some complex constant. Show that the dilation $z \mapsto 4z$ can be obtained by composing three inversions.

3. Determine, with proof, the set of all biholomorphic self-mappings of $\mathbb{C} \setminus \{0\}$, the punctured plane.

4. Suppose f is a continuous function on $\{ z \in \mathbb{C} : |z| \leq 1 \}$, the closed unit disk, and f is holomorphic on the open unit disk. Prove that if $f(z)$ is real when $|z| = 1$, then f is a constant function.

5. Suppose that g is a bounded, continuous function on the real axis. Show that the improper integral $\int_0^\infty e^{-zt} g(t)\, dt$ (the Laplace transform) represents a holomorphic function of z in the half-plane where $\operatorname{Re} z > 0$.

6. Use the residue theorem to prove that $\displaystyle\int_0^\infty \frac{x^2}{1 + x^5}\, dx = \frac{\pi/5}{\sin(2\pi/5)}$.

7. Find the general form of an entire function f satisfying the property that
$$\frac{f(w) - f(z)}{w - z} = f'\left(\frac{w + z}{2}\right)$$
for all distinct complex numbers w and z.

8. Let $\{f_n\}_{n=1}^\infty$ be the sequence of iterates of the sine function: namely, $f_1(z) = \sin(z)$, and $f_{n+1}(z) = \sin(f_n(z))$ when $n \geq 1$. Show that this sequence $\{f_n\}$ is not locally bounded in any neighborhood of the origin.

9. Suppose that f is holomorphic on $\{ z \in \mathbb{C} : 0 < |z| < 1 \}$ (the punctured unit disk), and f has no zeroes. Show that there exist an integer m and a function g holomorphic on the punctured disk such that $f(z) = z^m e^{g(z)}$ for all z in the punctured disk.

10. State and prove *one* of the following theorems: the Riemann mapping theorem, Runge's theorem about polynomial approximation, or the Schwarz reflection principle.

2.2 Texas A&M 2011

1. Not sure. Maybe use conjugate to $u(x, -y)$.

2. Use the fact that an LFT $\frac{az+b}{cz+d}$ can be represented by a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. So we want to solve the system of equations
$$\begin{bmatrix} 1 & 0 \\ 1 & a \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & b \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & c \end{bmatrix} = \begin{bmatrix} 4\lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

3. Rotations, dilations and inversions. The reason being the LFT's are the automorphisms of the sphere, so we want all automorphisms that either fix or swap 0 and ∞, i.e. the north and south poles.

4. Let $g : UHP \to \mathbb{D} : z \mapsto \frac{z-i}{z+i}$. We have $f \circ g : UHP \to \mathbb{C}$ with $\lim_{y\to 0} \operatorname{Im}(f \circ g)(x + iy) = 0$, therefore we can apply reflection and extend to an entire function $G : \mathbb{C} \to \mathbb{C}$ with $G(z) = (f \circ g)(z)$ for $z \in UHP$ and $G(z) = \overline{(f \circ g)(\overline{z})}$ for $z \in LHP$. Since f is continuous on $\overline{\mathbb{D}}$ we know that the image of G is bounded. By Liouville, this implies that G is constant and therefore f is constant.

5. We seek to apply Morera's theorem. Fix a rectangle $R \subseteq RHP$. We can write
$$\int_{\partial R} \int_0^\infty e^{-zt} g(t) dt dz = \int_0^\infty g(t) \int_{\partial R} e^{-zt} dz dt = 0.$$
Where we use the fact that e^{-zt} is holomorphic and we also apply Fubini to switch the order of integration as we can bound the integral
$$\int_{\partial R} \int_0^\infty |e^{-zt}||g(t)| dt dz \leq M \int_{\partial R} \int_0^\infty |e^{-\operatorname{Re}(z)t}| dt dz < \infty.$$

6. Use wedge contour through $e^{2\pi i/5}$.

7. Nope.

8. To show a contradiction suppose that $\{f_n\}$ is bounded by M in some neighborhood of the origin which contains the disk $\mathbb{D}(0, r)$. Montel's theorem implies that some subsequence $f_{n_k} \to f$ locally uniformly for some $f : \mathbb{D}(0, r) \to \mathbb{C}$. Let $\epsilon < \pi/2 \in \mathbb{R}$ be such that $0 < \epsilon < r$. We claim that for each n_k we have $|f'_{n_k}(\epsilon)| < 1$. For any f_{n_k} we have
$$\frac{d}{dz} f_{n_k}(\epsilon) = f'_{n_k}(f_{n_k-1}(\epsilon) f'_{n_k-1}(\epsilon).$$

3 Gamelin

(Gamelin) Let $\mathcal{F} = \{f \in H(\mathbb{D}) : f \text{ is } 1-1, f(0) = 0, f'(0) = 1\}$. Show that $\mathcal{F}$ is normal.

Proof. Use Zalcman's Lemma. □

4 1988

(1988 #3) Describe a brach of the function $f(z) = \sqrt{z^2 - 1}$ that is continuous on the counterclockwise circle C of radius 2 around the origin, and compute $\int_C f(z)dz$.

Proof. We can rewrite $f(z) = i\sqrt{(1-z)(1+z)}$, so we must make a branch cut for $f_1(z) := \sqrt{1+z}$ and $f_2(z) := \sqrt{1+z}$. For f_1 we make a cut on $(-1, \infty)$ and let $\arg(1+z)$ take values $[0, 2\pi)$. For f_2 we make a cut from $(1, \infty)$ and let $\arg(1-z)$ range from $[2\pi, 0)$ (notice that because of the negative sign when we traverse a circle counter-clockwise around 1 the argument is a decreasing function). This ensures that for $x > 1$ the argument of $\arg(1-x) + \arg(1+x) = 2\pi$ regardless of whether we take a limit to x from the upper or lower half plane.

Now we can make sense of the integral $I := \int_{|z|=2} f(z)dz$. Since f is analytic on $\mathbb{C} \setminus [-1, 1]$ we notice that $I = \int_R f(z)dz$ for any rectangle R enclosing $[-1, 1]$. So, consider the rectangle $R_\epsilon = L_1 + W_1 + L_2 + W_2$ where $L_1 = \{x - i\epsilon : x \in (-1-\epsilon, 1+\epsilon)\}$, $W_1 = \{1 + \epsilon + iy : y \in (-\epsilon, \epsilon)\}$, $L_2 = \{x + i\epsilon : x \in (1+\epsilon, -1-\epsilon)\}$, and $W_2 = \{-1 - \epsilon + iy : y \in (\epsilon, -\epsilon)\}$. Notice that for $k = 1, 2$ we have

$$\int_{W_k} f(z)dz \leq 2\epsilon \sup_{z \in W_k} |f(z)| \to 0.$$

Next we notice that

$$\int_{L_1} i\sqrt{1-z}1 + z = i\int_{L_1} e^{\frac{1}{2}(\log|1-z| + i\arg|1-z|)}.$$

$\square$

5 1989

(1989 #6) Evaluate the integral

$$\int_{|z|=10} \frac{2z^{12} + z^5 + 5z^3 + 1}{z^{13} + z^7 + z^6 + z} dx.$$

Proof. A straightforward application of Rouche tells us that all of the poles of $p(z) = \frac{2z^{12}+z^5+5z^3+1}{z^{13}+z^7+z^6+z}$ lie inside the disk of radius 10. It follows from the residue theorem that

$$\int_{|z|=10} p(z)dz = \lim_{R\to\infty} \int_{|z|=R} p(z).$$

We claim that the above limit is equal to $\lim_{R\to\infty} \int_{|z|=R} \frac{2}{z}$. To see this notice that $p(z) \sim \mathcal{O}(\frac{2}{z})$ and thus for large R we can exchange the limit and the integrand below to write

$$\begin{aligned}
\left| \int_{|z|=R} \frac{2z^{12} + z^5 + 5z^3 + 1}{z^{13} + z^7 + z^6 + z} dz - \int_{|z|=R} \frac{2}{z} dz \right| &= \left| \int_{|z|=R} \frac{2z^{12} + z^5 + 5z^3 + 1}{z^{13} + z^7 + z^6 + z} - \frac{2}{z} dz \right| \\
&\leq \int_{|z|=R} \left| \frac{2z^{12} + z^5 + 5z^3 + 1}{z^{13} + z^7 + z^6 + z} - \frac{2}{z} \right| dz \\
&= \int_{|z|=R} \left| \frac{z(2z^{12} + z^5 + 5z^3 + 1) - 2(z^{13} + z^7 + z^6 + z)}{z(z^{13} + z^7 + z^6 + z)} \right| dz \\
&\leq 2\pi R \sup_{|z|=R} \left| \frac{-2z^7 - z^6 + 5z^3 - 2z^3 + 1}{z^{14} + z^8 + z^7 + z^2} \right| \\
&\to 0.
\end{aligned}$$

Lastly we use the residue theorem to compute $\text{Res}(\frac{2}{z}, 0) = 2$ and therefore $\int_{|z|=R} \frac{2}{z} dz = 4\pi i = \int_{|z|=10} p(z)dz$. □

(1989 #7) Prove the following generalization of the maximum modulus theorem:

Let $\Omega \subseteq \mathbb{C}$ be an open connected region and f_k, $k = 1, \ldots, n$ be holomorphic functions on Ω. If the function $|vp = \sum_1^n |f_k|$ has an interior maximum then it is constant.

(In fact, if φ is constant then the functions f_k are all constant but you do not have to prove this).

Proof. (Short Way) It follows from the triangle inequality and Cauchy's integral formula that φ is a subharmonic function and therefore obeys the maximum principle.

(Long Way) Without loss of generality let us suppose that φ attains a maximum at the interior point $0 \in \Omega$. Let $\alpha_1^1, \ldots, \alpha_n^1 \in [0, 2\pi)$ be such that $g_k^1(z) = e^{i\alpha_k^1 z} f_k(z)$ satisfies $g_k(0) \in \mathbb{R}^+$. Let $\psi = \sum_1^n g_k$ and let $h_1(z) = \sum_1^n g_k$. Observe that h_1 is analytic on Ω and by our construction satisfies $|h_1(z)| \leq \psi(z)$ for all z. Since $h_1(0) = \psi(0)$ is a maximum, it follows from the maximum modulus principle that $h_1 \equiv c_1$ for some constant $c_1 \in \mathbb{C}$. Notice that if we let $g_j^j = e^{i(\alpha_j+\pi)}$ so that the function $h_j = g_1^j + \cdots + g_{j-1}^j - g_j^j + g_{j+1}^j + \cdots g_n$ attains a max at $z = 0$ then we would have $h_j \equiv c_j$ for $j = 1, \ldots, n$. This gives a linearly independent system of n-equations of the form

$$\begin{aligned}(e^{i\alpha_1}, \ldots, e^{i\alpha_n}) \cdot (f_1, \ldots, f_n) &\equiv c_1\\ (e^{i\alpha_1}, e^{i(\alpha_2+\pi)}, \ldots, e^{i\alpha_n}) \cdot (f_1, \ldots, f_n) &\equiv c_2\\ &\vdots\\ (e^{i\alpha_1}, \ldots, e^{i(\alpha_n+\pi)}) \cdot (f_1, \ldots, f_n) &\equiv c_n.\end{aligned}$$

Therefore we can write f_n as a linear combination of the (this idea should work but I don't feel like writing out the details. it is a lot easier to check the case where $n = 2$.)

□

(2000 #4) Let f be analytic from $\mathbb{D}$ to $\{z : \operatorname{Re} z > 0\}$ such that $f(0) = 1$. Prove that $|f(z)| \leq \frac{1+|z|}{1-|z|}$ for all $z \in \mathbb{D}$.

(1999 #5) Let u be a bounded harmonic function in $G = \mathbb{D} \setminus \{0\}$. (a) Show that $u(w) \leq \sup_{\zeta \in \partial \mathbb{D}} \limsup_{z \to \zeta, z \in G} u(z)$ for all $w \in F$, by considering the function $u(z) + c \log |z|$ for appropriate choices of c. (b) Show that u has a harmonic extension to $\mathbb{D}$.

(1997 #8) Let f be analytic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. ssume that$f(z) \neq 0$ for $z \in \overline{\mathbb{D}}$ and $|f(z)| \leq 1$ for $z \in \mathbb{D}$. Prove that

$$\inf_{|z| \leq \frac{1}{7}} |f(z)| \geq \sup_{|z| \leq \frac{1}{5}} |f(z)|^2.$$

Proof. Let $u = \log \frac{1}{|f|}$. We apply Harnack's inequality to the log modulus and take a limit on the boundaries.

□

(2000 #2) Let $\Omega \subseteq \mathbb{C}$ be the region given by

$$\Omega := \{z : |z - 1| < 1 \text{ and } |z - i| < 1\}.$$

Find a conformal map $f : \Omega \to \mathbb{D}$.

(2002 #4) Find a conformal map of the half strip $\{z : \operatorname{Re} z > 0,\ 0 < \operatorname{Im} z < \pi\}$ onto the $UHP = \{z : \operatorname{Im} z > 0\}$.

(2003 #4) Let f be an analytic function from $\mathbb{D}$ to $\mathbb{D}$ such that $|f(z)| < c < 1$ for all $z \in \mathbb{D}$. Prove that f has at least one fixed point.

Proof. Let $g(z) = f(z) - z$. Notice that on the circle $|z| = c$ we have $|f(z)| < |z| = c$ and thus $|g(z) - z| = |f(z)| < c = |z|$. So, $g(z)$ and z have the same number of zeros in $\mathbb{D}(0, c)$, which says that f has a fixed point. □

(2003 #5) Let $p(x)$ be a monic polynomial and $f(z)$ a meromorphic function on $\mathbb{C}$ which satisfies the identity $p(f(z)) = e^z$ for all $z \in \mathbb{C}$. Prove that $f(z)$ is entire and that $p(x) = (x-a)^n$ for some $a \in \mathbb{C}$ and $n \in \mathbb{Z}^+$.

Proof. Since e^z is nonconstant then we must have p is also nonconstant. To see that f has no poles, suppose to show a contradiction that f has a pole at z_0. Any nonconstant polynomial, q, satisfies $\lim_{z\to\infty} q(z) \to \infty$, therefore we must have $\lim_{z\to z_0} p(f(z)) \to \infty$. However, this contradicts the fact that $|e^z|$ is finite for all $z \in \mathbb{C}$.

To prove that $p(z)$ has only one root (and therefore has the form $p(x) = (x-a)^n$) we assume, to show a contradiction, that p has two distinct roots $a_0 \neq a_1$. Because e^z is never zero we conclude that f must omit both a_0 and a_1. This however would imply (by little Picard's theorem) that f is constant, a contradiction (since e^z is nonconstant).

□

(2002 #6) Let $f(z)$ be entire with $f : \mathbb{R} \to \mathbb{R}$ and $f : UHP \to UHP$.

(a) Show that $f(z)$ has at most one zero. (*Hint: Look at* $\arg f(z)$*, first in UHP then in LHP*).

(b) Hence show that $f(z) = \alpha z + \beta$ where α and β are constants, $\alpha > 0$ and $\beta \in \mathbb{R}$.

Part a.

Proof. Let $\Gamma_R = \{|z| = R\}$ and let $\gamma_R : [0,1] \to \mathbb{C}$ be the curve which traverses Γ_R once. Since $f : \mathbb{R} \to \mathbb{R}$ we know that $f(\gamma_R(0)), f(\gamma_R(1)) \in \mathbb{R}$. Next, since f fixes the UHP we know that (without loss of generality) there exists a value t_0 such that $F(\gamma_R((0,t_0))) \subseteq UHP$, $f(\gamma_R(t_0)) \in \mathbb{R}$ and $f(\gamma_R((t_0,1))) \subseteq LHP$ (here LHP denotes the *lower* half plane). It follows that

$$\int_{\gamma_R} d\arg(f(z))dz = \int_0^{t_0} d\arg(f \circ \gamma_R(t))\gamma_R'(t)dt + \int_{t_0}^1 d\arg(f \circ \gamma_R(t))\gamma_R'(t)dt \overset{*}{=} \pi i + \pi i = 2\pi i.$$

Where at * we use the fact that $f(\gamma_R((0,t_0))) \subseteq UHP$ and thus the argument will change by exactly π (and similarly on the LHP). The argument principle tells us that inside of γ_R

$$\frac{1}{2\pi i}(\#\text{ zeros } + \#\text{ poles }) = \int_{\gamma_R} d\arg(f(z))dz = 2\pi i,$$

and since f is analytic we know it has no poles. Therefore f has exactly one zero, regardless of our radius R. This means that f has only one zero in all of $\mathbb{C}$. □

Part b.

Proof. Let $c \in \mathbb{R}$. Notice that the argument above can be repeated for the function $f_c(z) = f(z) - c$ and so we can conclude that f_c has only one zero. This means that f attains each point in $\mathbb{C}$ exactly once, which can only happen if f is linear (becuase if f is not a polynomial it has an essential singularity at ∞ and attains all but one value in the plane infinitely often and if f is a non-linear polynomial then by the fundamental theorem of calculus f must have at least two zeros). □

(2002 #7) Suppose f is entire and suppose there exists $R_0 > 0$ such that $u = \operatorname{Re} f$ satisfies

$$|u(z)| \le M|z|^k$$

when $|z| > R_0$. Show that f is a polynomial of degree at most k.

Proof. Write $f = u + iv$ and fix $R > R_0$. As f is entire we can apply the Schwarz integral formula and write for $z \in \mathbb{D}(0, R)$

$$f(z) = -iv(0) + \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta} + \frac{z}{R}}{e^{i\theta} - \frac{z}{R}} u(Re^{i\theta}) d\theta.$$

Taking the absolute value signs inside the integral and dividing by z^k then applying an ML estimate gives

$$|f(z) + iv(0)|/|z|^k \le \frac{1}{2\pi}\int_0^{2\pi} \left| \frac{e^{i\theta} + \frac{z}{R}}{e^{i\theta} - \frac{z}{R}} \right| \left| \frac{u(Re^{i\theta})}{z^k} \right| d\theta \le C \cdot M.$$

So, for sufficiently large R and $|z| \le R/2$. Therefore

$$|f(z)| \le M_1 |z|^k.$$

It follows that f has a removable singularity at ∞ and must therefore be a polynomial. Clearly, $\deg f \le k$. □

(2001 #3) Let $f(z)$ be entire with $|f(z)| \le a|z|^m + b$ for $a > 0, b > 0$. Prove that f is a polynomial of degree less than or equal to m.

Proof. Dividing by $|z|^m$ we have $\left|\frac{f(z)}{z^m}\right| \le a + \frac{b}{|z|^m}$. It follows that $\lim_{z\to\infty} \left|\frac{f(z)}{z^m}\right| \le a$. In particular, $\frac{f(z)}{z^m}$ has a removable singularity at $z = \infty$. This immediately implies that $f(z)$ is a polynomial (since if f were an entire non-polynomial then f would have an essential singularity at ∞, which could not be resolved by dividing by $|z|^m$). This also implies that $\deg f \le m$ since otherwise if $\deg f = n > m$ we would have $\lim_{z\to\infty} \frac{f(z)}{z^m} \to \infty$, which is contrary to what we just proved. □

(2001 #4) Let $f(z)$ be analytic in $\mathbb{D}$ except for finitely many poles. Suppose that $\lim_{z\to e^{i\theta}} |f(z)| = 1$ for every θ. Prove that f is a rational function.

Proof. Since $f(S^1) = S^1$ (with S^1 the unit circle) the Schwarz reflection principle guarantees that we can extend the domain of f by reflecting $\mathbb{D}$ across its boundary, S^1, where each inner circle $\Gamma_r \subseteq \mathbb{D}$ of radius $0 < r < 1$ is reflected to the circle $\{|z| = \frac{1}{r}\}$ via the mapping $z \mapsto \frac{1}{\bar{z}} = \frac{z}{|z|^2}$. We then extend f to the map $F : \mathbb{C} \to \hat{\mathbb{C}}$ via the map

$$F(z) = \begin{cases} f(z), & z \in \overline{\mathbb{D}} \\ \frac{1}{\overline{f(\frac{1}{\bar{z}})}}, & z \in \mathbb{C} \setminus \overline{\mathbb{D}} \end{cases}.$$

Since f is analytic in a bounded set, we know that f has finitely many zeros $a_1, \ldots, a_m$. These zeros will map to poles of the function F at the points $\frac{a_k}{|a_k|^2}$. Letting $b_1, \ldots, b_n$ denote the poles of f in $\mathbb{D}$ these will map to zeros of the function F at $\frac{b_j}{|b_j|^2}$. Since we have an explicit map for F in terms of f we know that these are the only zeros and poles. We now have F takes the form

$$F(z) = \frac{(z - a_1) \cdots (z - a_m)(z - \frac{b_1}{|b_1|^2}) \cdots (z - \frac{b_n}{|b_n|^2)})}{(z - b_1) \cdots (z - b_m)(z - \frac{a_1}{|a_1|^2}) \cdots (z - \frac{a_n}{|a_n|^2)})},$$

and by definition is a rational function. □

(2001 #4) Prove that if f and g are zero-free entire functions with $f + g = 1$ then f and g are constant.

Proof. Dividing by g gives $g(\frac{f}{g} + 1) = 1$. Since f is never zero we know that $\frac{f}{g}$ omits 0. Also, since $\frac{f}{g} + 1$ is never 0, it also omits -1. Lastly, because g is never zero we know that $\frac{f(z)}{g(z)} < \infty$ for all $z \in \mathbb{C}$, and thus $\frac{f}{g}$ has no poles and omits ∞. To summarize we have proven that $\frac{f}{g}$ omits $\{0, -1, \infty\}$, by Big Picard we conclude that $\frac{f}{g}$ is constant and therefore $f = cg$ for some $c \in \mathbb{C}$. We would then have $1 = f + g = cg + g = g(c + 1)$ and so both f and g are constant. □

(1999 #3) Show that there is no entire function f with $f\left(\frac{i}{n}\right) = \frac{1}{n+1}$ for all $n = 1, 2, 3, \ldots$.

Proof. Consider the function $g(z) = \frac{1}{\frac{i}{z}+1} = \frac{z}{i+z}$. Notice that g is analytic in $\mathbb{C} \setminus \mathbb{D}(-i, \epsilon)$ for any $\epsilon > 0$ and has a pole at $z = -i$ and satisfies $g\left(\frac{i}{n}\right) = \frac{1}{n+1}$ for all $n \in \mathbb{N}$. Hence if there existed an entire function f with the property in the hypothesis we would have $f\left(\frac{i}{n}\right) = g\left(\frac{i}{n}\right)$ for all $n \in \mathbb{N}$. Notice that $\{\frac{i}{n} : n \in \mathbb{N}\}$ has a limit point at zero. By the uniqueness theorem it follows that $f(z) = g(z)$ for all $z \in \mathbb{C} \setminus \mathbb{D}(-i, \epsilon)$ and therefore must agree on all of $\mathbb{C}$. This immediately contradicts the fact that f is entire since it would have a pole at $-i$.

□

(2001 #6) Let $\{D_n\}$ be a sequence of simply connected domains contained in the open unit disc with $0 \in D_n$, with $\overline{D}_n \subseteq D_{n+1}$ and with $\bigcup_1^\infty D_n = \mathbb{D}$. Let $\varphi_n : \mathbb{D} \to D_n$ be the Riemann map with $\varphi_n(0) = 0$ and $\varphi_n'(0) > 0$. Prove that $\varphi_n(z) \to z$ uniformly on compact subsets of U.

(2000 #8) Let $p(z)$ be a polynomial and z_0 a point such that $p(z_0) = z_0$. Define

$$p_n = \underbrace{p \circ p \circ \cdots \circ p}_{n}$$

to be the n^{th} iterate of p. Prove that if the family $\{p_n\}_{n\geq 1}$ is normal in a neighborhood of z_0 then $|p'(z_0)| \leq 1$.

Proof. Since $\{p_n\}$ is normal we know that the family $\{p_n'\}$ is normal and relates to $p'(z_0)$ by taking powers. Therefore we must have $|p'(z_0)| \leq 1$ otherwise it diverges to ∞. □

(2000 #6) Let $p(z)$ be a polynomial of degree $N \geq 1$, define $L = |p(z)|^{-1}(1)$. Prove that $\mathbb{C} \setminus L$ has at most $N + 1$ components.

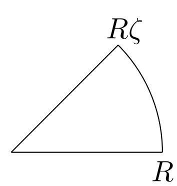
(1997 #3) Let $a \geq 1$. Prove that $\tan z - az$ has only real zeros. *Hint: Consider the behavior of* $\tan z$ *and* az *on the square with corners* $\pm N\pi \pm N\pi i$*, where* N *is a large positive integer.*

Proof. Recall that $\tan z = \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}$. □

6 Solomyak Practice Problems

(S−1) Compute $\int_0^\infty \sin(x^2)dx$ and $\int_0^\infty \cos(x^2)$.

Proof. Let $f(z) = e^{iz^2} = \cos(z^2) + i\sin(z^2)$. Also let $\zeta = e^{i\pi/4}$ and let Γ_R be the curve through $[0, R] \cup [0, R\zeta] \cup \mathbb{C}_R$ where C_R is the semicircle joining the endpoints.



The function $f(z)$ has no poles inside this contour and therefore

$$\int_{\Gamma_R} f(z)dz = 0.$$

We know that □

(S0) Let $\mathcal{F}$ be the family of univalent functions on $\mathbb{D}$. Prove that $\mathcal{F}' = \{f' \text{ such that } f \in \mathcal{F}\}$ is normal.

Lemma 1. *The family,* $\mathcal{G}$*, of univalent functions holomorphic on* $\mathbb{D}$ *which omit* 0 *is normal.*

Proof. Since $\mathbb{D}$ is simply connected and each $g \in \mathcal{G}$ is never zero, we can for each g choose some branch of the logarithm so that $h(z) = e^{\frac{1}{2}\log g(z)}$ is analytic on $\mathbb{D}$. Therefore $h(z)^2 = g(z)$ for all $z \in \mathbb{D}$. Moreover, we claim that each h can be chosen so that $-1 \notin h(\mathbb{D})$. To do this we prove that $h(\mathbb{D})$ and $-h(\mathbb{D})$ are disjoint and therefore one of the sets must not contain -1 (and that will be the branch we choose). To see they are disjoint suppose that $h(z_1) = -h(z_2)$ squaring we then have $g(z_1) = g(z_2)$ and by univalent $z_1 = z_2$. We now know that the family $\sqrt{\mathcal{G}} = \{\sqrt{g}\}$ with appropriate branch cuts omits $0, -1, \infty$ and so by Montel it follows that $\sqrt{\mathcal{G}}$ is normal. It is then easy to see that $\mathcal{G}$ is normal since if $\{g_n\} \subseteq \mathcal{G}$ then there exist $h_{n_k} \subseteq \sqrt{G}$ such that $h_{n_k}^2 = g_{n_k}$ and $h_{n_j} \overset{l.u.}{\to} h$. Therefore $g_{n_k} \to h^2$ locally uniformly. □

Proof of Problem Statement

Proof. (NEEDS SOME FIXING, BUT BASICALLY THE RIGHT IDEA) Fix a sequence $\{f_n\} \subseteq \mathcal{F}$. Consider the functions $F_n = \frac{f_n(z) - f_n(0)}{f_n'(0)}$. Notice that each F_n is univalent and moreover $F_n(0) = 0$ and $|F_n'(0)| = 1$. Also, $F_n(\mathbb{D})$ cannot be contain the closed unit disk since the restriction of the inverse $F_n^{-1}\big|_{\mathbb{D}} : \mathbb{D} \subseteq F_n(\mathbb{D}) \to \mathbb{D}$ would contradict the hypothesis of Schwarz's lemma. So for each F_n there exists $c_n \in \overline{\mathbb{D}}$ such that $F_n - c_n$ is zero free. Without loss of generality let us suppose that the c_n converge to $c \in \overline{\mathbb{D}}$. It follows from the lemma that the family $\{F_n - c_n\}$ contains a convergent subsequence $F_{n_k} - c_{n_k} \to F - c$ with F analytic. We can take the derivative and conclude that $F_{n_k}' \to F'$ where $F_{n_k}'(z) = \frac{1}{f_{n_k}'(0)} f_n'(z)$. Note it is possible that $f_{n_k}' \to \infty$ or to 0, but in either case we still have convergence.

□

> **(S0)** Let $\mathcal{F}$ be the set of univalent functions on $\mathbb{D}$ such that $f(0) = 0$ and $|f'(0)| = 1$ show there exists $\kappa > 0$ such that $\mathbb{D}(0, \kappa) \subseteq f(\mathbb{D})$ for all $f \in \mathcal{F}$.

Part b.

Proof. Suppose no such κ exists. Let $f_n \in \mathcal{F}$ be such that f_n omits a_n with $|a_n| < \frac{1}{n}$. Notice the $|a_n| \searrow 0$. Since $\mathcal{F}$ is a normal family there exists a subsequence of $\{f_n - a_n\}$ which converges locally uniformly to some f.

□

> **(S1)** Prove that all of the zeros of the polynomial
>
> $$p(z) = z^n + c_{n-1}z^{n-1} + ... + c_1 z + c_0$$
>
> lie in the disc centered at 0 with radius
>
> $$R = \sqrt{1 + |c_{n-1}|^2 + ... + |c_1|^2 + |c_0|^2}.$$

Proof. We show that $p(z)$ has the same number of zeroes as z^n inside the disc $D(0, R)$. By Rouche's theorem, this is true if we can show

$$|p(z) - z^n| < |z^n|$$

on $C(0,R)$. Notice that by Cauchy-Schwartz inequality,

$$LHS \leq \sqrt{|z|^{2n-2}+...+1}\sqrt{|c_{n-1}|^2+...+|c_1|^2+|c_0|^2} = \sqrt{\frac{|z|^{2n}-1}{|z|^2-1}}\sqrt{R^2-1}.$$

Notice that on $C(0,R)$, the last expression equals

$$\sqrt{R^{2n}-1} < R^n = |z|^n = |z^n|.$$

z^n has n zeroes in $D(0,R)$. Since $p(z)$ has degree n, it has n zeroes, all of which must then lie in $D(0,R)$. □

(S2) Let $f:\mathbb{D}\to G$ be a conformal map with $f(0)=0$. Define $g:\mathbb{D}\to\mathbb{C}$ by $g(z) = z\sqrt{f(z^2)/z^2}$. Prove that g is a well-defined analytic function and a conformal map

Proof. Let $f(z) = a_1 z + a_2 z^2 + \cdots$. It follows that

$$f(z^2)/z^2 = a_1 + a_2 z^2 + \cdots .$$

Since f is conformal and $f(0)=0$ it follows that $f(z^2)/z^2$ is never zero and so the square root is well-defined. Therefore $\sqrt{f(z^2)/z}$ is analytic and since multiplication of analytic functions is analytic it follows that g is analytic.

To see that g is injective suppose that $g(a)=g(b)$. Squaring both sides implies that

$$a^2(f(a^2)/a^2) = b^2(f(b^2)/b^2),$$

and since f is injective it follows that $a^2=b^2$, and so $a=\pm b$ (note that $a,b\neq 0$). We claim that $a\neq -b$. Suppose that $a=-b$, the fact that $g(a)=g(b)$ implies the contradiction that

$$-b\sqrt{f(b^2)/b^2} = b\sqrt{f(b^2)/b^2}.$$

Hence g is conformal.

□

(S3) Prove that if a sequence of analytic polynomials converges uniformly on a region Ω then the sequence converges uniformly on a simply connected region containing Ω.

Proof. Let $p_n\to f$ uniformly on Ω. First we claim that $p_n\to f$ uniformly on $\overline{\Omega}$. To see this we fix $\epsilon>0$ and let N be such that for all $n>N$ it holds that $|p_n(z)-f(z)|<\epsilon$. It follows that for any sequence $a_k\to a$ with $a_k\in\Omega$ and $a\in\partial\Omega$ we have $|p_n(a_k)-f(a_k)|<\epsilon$. Letting $k\to\infty$, it follows that $|p_n(a)-f(a)|<\epsilon$, and so the uniform convergence extends to the boundary.

Consider the set $W=\overline{\Omega}^c$. Notice that W is open and therefore can be written as $W=\bigcup_1^\infty A_j$ with A_1 an unbounded, open component and the A_j bounded, open components. We claim that we can extend the convergence of the p_n on each A_j with $(j>1)$. By our construction we know that $p_n\to f$ uniformly on ∂A_j. This says that $|p_n(z)-f(z)|\to 0$ uniformly for $z\in\partial A_j$, by the maximum principle we must have $|p_n-f|\to 0$ for all $z\in A_j$. It follows that $p_n\to f$ uniformly for all $z\in A_1^c$. This is a connected set since A_1 is a component (maximal connected set).

□

(S4) Does there exist a function f analytic on $\overline{\mathbb{D}}(0,300)$ with $f(0)=1$, with 10 zeros in $\overline{\mathbb{D}}(0,100)$ and satisfying $|f(z)|<1024$ when $|z|=300$? Produce such a function or prove it does not exist.

Proof. No. Suppose such a function existed. By scaling there would be a function $f:\mathbb{D}\to\mathbb{D}$ with $f(0)=\frac{1}{2^{10}}$, $|f(\partial\mathbb{D})|<1$ and 10 zeros in $z_1,\dots,z_{10}\in\overline{\mathbb{D}}(0,1/3)$. Consider a Blashke product g with zeros at the z_j. Notice that $|g(0)|=\prod|z_j|=\frac{1}{3^{10}}$. Next, since g is entire and f extends to be analytic on a slightly larger disk than $\mathbb{D}$ we can apply the maximum principle on $\partial\mathbb{D}$ to the function $\frac{f}{g}$ (which is analytic by construction) and conclude that $\frac{f}{g}:\mathbb{D}\to\mathbb{D}$. However, $\left|\frac{f(0)}{g(0)}\right|\geq\frac{\frac{1}{2^{10}}}{\frac{1}{3^{10}}}=\frac{3^{10}}{2^{10}}>1$, a contradiction. □

(S5) Suppose f is an entire function with $|f(z)|=1$ for $|z|=1$. Show that $f(z)=e^{i\theta}z^n$ for some θ and $n\in\mathbb{N}$.

Proof. Let n denote the order of the zero at $f(0)$ (note that n can be 0). We can then write $f(z)=z^ng(z)$ with g an entire function satisfying $g(0)\neq 0$. Notice that $g(z)=\frac{f(z)}{z^n}$ and since $|f(z)/z^n|=1$ when $|z|=1$ it follows from the maximum principle that $g:\mathbb{D}\to\mathbb{D}$ with $g:\partial\mathbb{D}\to\partial\mathbb{D}$. We claim that g is a bounded and so by Liouville is constant. To see this we us the Schwarz reflection principle to conclude that for $|z|>1$ we have

$$g(z)=\frac{1}{\overline{g}(\frac{1}{\overline{z}})}.$$

This says that $\lim_{z\to\infty}g(z)=\frac{1}{\overline{g}(0)}$, which is bounded since $g(0)\neq 0$. Therefore g is bounded and must be constant. Lastly, since we know $|g(z)|=1$ for $|z|=1$ it must be the case that $g(z)\equiv e^{i\theta}$ for some θ. We now have $f(z)=z^ne^{i\theta}$, as desired. □

(S6) Prove that there do not exist entire functions f and g such that $e^f+e^g=1$ for all $z\in\mathbb{C}$.

Proof. Suppose such functions existed. It follows that

$$e^g(e^{f-g}+1)=1.$$

Therefore $f(z)-g(z)\neq n\pi i$ for all $n\in\mathbb{Z}$. By little Picard this implies that $f(z)-g(z)=c$ a constant. We now have

$$e^g(e^{f-g}+1)=e^g(e^c+1)=1.$$

Therefore $e^g=c'$ and $g=c''$ for some constants c',c''. Lastly, this says that $e^f=c'''$ and therefore $f=c''''$. Hence f and g are constant. □

(S7) Let $\{f_n\}$ be a collection of analytic functions on a region Ω with $|f_n|\leq 1$ on $|\Omega$. Let K be a compact set contained in Ω. Suppose that $\{f_n\}$ on verges at infinitely many points in K. Then is it true or false that $\{f_n\}$ necessarily converges at every point of Ω?

Proof. Let $f_n \to f$ on $\{a_k\} \subseteq K$. Without loss of generality suppose that $a_k \to a \in K$ (follows from compactness). Take a subsequence f_{n_k}, since the $\{f_{n_k}\}$ are bounded it follows from Montel's theorem that there exists a convergent subsequence $f_{n_{k_l}} \to f^l$. Notice that $f_{n_{k_l}}(a_k) \to f(a_k)$ for all $a_k \in K$. By the uniqueness theorem we know that $f^l \equiv f$. It follows that every subsequence has a convergent subsequence converging locally uniformly to f. Therefore $f_n \to f$ locally uniformly on all of Ω.

□

(S8) Let $\mathcal{F} = \{f \in H(\mathbb{D}) \cap C(\overline{\mathbb{D}}) : \int_0^{2\pi} |f(e^{i\theta})| d\theta \leq M\}$. Prove that $\mathcal{F}$ is normal with respect to the Euclidean metric.

Proof. Let $f \in \mathcal{F}$. Write $f = \sum_0^\infty \frac{f_n^{(k)}(0)}{k!} z^k = \sum_0^\infty a_k z^k$. The Cauchy-Derivative formula tells us that for $r < 1$ we have

$$|a_k| = \left| \int_{|\zeta|=r} \frac{f(\zeta)}{\zeta^k} d\zeta \right| \leq \int_{|\zeta|=r} \left| \frac{f(\zeta)}{\zeta^k} \right| d\zeta \leq \frac{1}{r^k} \int_{|\zeta|=r} |f(\zeta)| d\zeta.$$

Letting $r \to 1$ we conclude that $|a_k| \leq M$. It follows that every f in $\mathcal{F}$ is bounded by

$$f(z) \leq M \sum_0^\infty |z|^k = \frac{M}{1 - |z|}.$$

Since the left-hand function is bounded on compact sets we conclude that $\mathcal{F}$ is locally uniformly bounded and therefore normal. □

7 Practice Exam

7.1 Misc Practice Problems

(Harvard) For $a \in \mathbb{R}$ show that

$$\int_0^\infty \frac{\sin(ax)}{\sinh(x)} dx = \frac{\pi}{2} \tanh \frac{a\pi}{2}.$$

Recall $\sinh(t) = \frac{e^t - e^{-t}}{2}$ and $\tanh(t) = \frac{e^t - e^{-t}}{e^t + e^{-t}}$.

Proof. Let $f(z) = \frac{\sin(az)}{\sinh(z)}$. Let $I = \int_0^\infty \frac{sin(ax)}{\sinh(x)} dx$ and $J = \int_0^\infty \frac{\cos(ax)}{\cosh(x)} dx$. Note that f has a removable singularity at $z = 0$ (look at the power series), f is even and also f has a simple pole at $i\pi$. We will compute the integral over two contours, Γ_R and Θ_R. Where Γ_R is the rectangle with height $3\pi/2$ and width $2R$ centered at 0 with its base on the real axis and Θ_R is the rectangle with height $\pi/2$ and width $2R$ centered at zero and base on the real axis. Let $A = \text{Res}_{z=i\pi}(f)$. The residue theorem guarantees that for $R > 0$ we have

$$\int_{\Gamma_R} f(z) dz = 2\pi i A$$

$$\int_{\Theta_R} f(z)dz = 0.$$

Write $\Gamma_R = L_1 + L_2 + L_3 + L_4$, with L_1, L_3 the horizontal sides and L_2, L_4 the vertical sides. We claim that as $R \to \infty$ it holds that for $i = 2, 4$ the integral over L_i goes to zero. We show this explicitly for L_2

$$\begin{aligned}
\left|\int_{L_2} f(z)dz\right| &\leq \text{length}(L_2) \sup_{z\in L_2} |f(z)| \\
&= (3\pi/2) \sup_{t\in[0,3\pi/2]} |f(R+it)| \\
&= (3\pi/2) \sup_{t\in[0,3\pi/2]} \left|\frac{\sin(a(R+it))}{\sinh(R+it)}\right| \\
&= \frac{3\pi}{2} \sup_{t\in[0,\frac{3\pi}{2}]} \left|\frac{e^{ia(R+it)} - e^{-ia(R+it)}}{e^{R+it} - e^{-R-it}}\right| \\
&\leq \frac{3\pi}{2} \sup_{t\in[0,\frac{3\pi}{2}]} \frac{e^{-t} + e^{t}}{e^{R} - e^{-R}} \\
&\leq \frac{C}{e^R - 1} \\
&\to 0.
\end{aligned}$$

And similarly for L_4. Next we parametrize L_3 via $z = t + \frac{3\pi}{2}i$ with $t \in [-R, R]$. This lets us rewrite $\int_{L_3} f(z)dz$ as a real integral

$$\begin{aligned}
\int_{L_3} f(z)dz &= \int_R^{-R} f(t + \frac{3\pi}{2}i)dt \\
&= -\int_R^{-R} \frac{e^{ia(t+\frac{3\pi}{2}i)} - e^{-ia(t+\frac{3\pi}{2}i)}}{e^{t+\frac{3\pi}{2}i} - e^{-(t+\frac{3\pi}{2}i)}}dt \\
&= -\int_{-R}^{R} \frac{e^{iat}e^{\frac{3\pi}{2}i} - e^{-iat}e^{\frac{-3\pi}{2}i}}{e^{t}e^{\frac{3\pi}{2}i} - e^{-t}e^{\frac{3\pi}{2}i}}dt \\
&= -\int_{-R}^{R} \frac{e^{iat}(-i) - e^{-iat}i}{e^{t}(-i) - e^{-t}i}dt \\
&= \int_{-R}^{R} \frac{e^{iat} + e^{-iat}}{e^{t} + e^{-t}}dt \\
&= \int_{-R}^{R} \frac{\cos(at)}{\cosh(t)}dt \\
&\to \int_{-\infty}^{\infty} \frac{\cos(at)}{\cosh(t)}dt \\
&= 2J.
\end{aligned}$$

So we now as $R \to \infty$ it holds that $2I + 2J = 2\pi iA$. So

$$I + J = \pi iA.$$

We now prove that I and J are related by integrating over Θ_R and letting $R \to \infty$. Let $\Theta_R = W_1 + W_2 + W_3 + W_4$ as with Γ_R. A similar argument shows that the integral over W_2 and W_4 tends to zero. We parametrize W_3 via $z = t + \frac{\pi}{2}i$ and again obtain a relation

$$\begin{aligned}
\int_{W_3} f(z)dz &= \int_R^{-R} f(t + \frac{\pi}{2}i)dt \\
&= -\int_R^{-R} \frac{e^{ia(t+\frac{\pi}{2}i)} - e^{-ia(t+\frac{\pi}{2}i)}}{e^{t+\frac{\pi}{2}i} - e^{-(t+\frac{\pi}{2}i)}}dt \\
&= -\int_{-R}^{R} \frac{e^{iat}e^{\frac{\pi}{2}i} - e^{-iat}e^{\frac{-\pi}{2}i}}{e^t e^{\frac{\pi}{2}i} - e^{-t}e^{\frac{\pi}{2}i}}dt \\
&= -\int_{-R}^{R} \frac{e^{iat}(i) - e^{-iat}(-i)}{e^t i - e^{-t}(-i)}dt \\
&= -\int_{-R}^{R} \frac{e^{iat} + e^{-iat}}{e^t + e^{-t}}dt \\
&= -\int_{-R}^{R} \frac{\cos(at)}{\cosh(t)}dt \\
&\to -\int_{-\infty}^{\infty} \frac{\cos(at)}{\cosh(t)}dt \\
&= -2J.
\end{aligned}$$

Therefore $2I - 2J = 0$ and so $I = J$. We can now conclude that

$$I = \frac{\pi i}{2}A.$$

It just remains to compute the residue A.

$$\begin{aligned}
A &= \text{Res}_{z=\frac{3\pi i}{2}}(f) \\
&= \frac{\sin(3a\pi i/2)}{\cosh(\frac{3\pi i}{2})}.
\end{aligned}$$

□

(Stanford) Suppose f is entire and satisfies the relation

$$p_n(z)(f(z))^n + p_{n-1}(z)f(z)^{n-1} + \cdots + p_0(z) = 0$$

where $p_0, \ldots, p_n$, not all trivial, are rational functions on $\mathbb{C}$. Show that f must be a polynomial.

Proof. By combining denominators we can assume that the $p_n(z)$ are polynomials. Consider the two variable polynomial $Q : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$: $(z, w) \mapsto \sum_1^n p_n(z)w^n$. Notice that for each fixed z_0 there are by the FTA at most n zeros to the polynomial $Q(z_0, w)$. The assumption $Q(z, f(z)) \equiv 0$ then implies that for each z_0 there are at most n elements in $f^{-1}(z_0)$. Hence f cannot have an essential singularity (otherwise it would assume some value in the plane infinitely many times).

□

(Stanford) Let $p_k(z)$ be a polynomial of degree k with $p_k(0) = 1$ and assume that p_k has no zeros in $\overline{\mathbb{D}(0, k^3)}$. Show that $\prod_1^\infty p_k(z)$ converges locally uniformly in $\mathbb{C}$.

Proof. Let $a_{1,k}, \dots, a_{k,k}$ be the zeros of p_k with $|a_{j,k}| > k^3$ for all j, k. If A_k is the top degree coefficient of p_k then we can write each $p_k(z) = A_k \prod_1^k (a_{j,k} - z) = A_k \prod_1^k a_{j,k}(1 - \frac{z}{a_{j,k}})$ and since $p_k(0) = 1$ we know that $A_k = (\prod a_{j,k})^{-1}$ and in particular

$$|A_k| \le \frac{1}{(k^3)^k} \le \frac{1}{k^3}$$

To prove convergence of $\prod p_k$ it suffices to prove the log of the product converges absolutely.

$$\log \prod_1^\infty p_k = \sum \log p_k = \sum_{k=1}^\infty \left(A_k + \sum_1^k \log(1 - \frac{z}{a_{j,k}}) \right).$$

Since $|A_k| < \frac{1}{k^3}$ we know that $\sum_1^\infty |A_k| < \infty$. It remains to prove that

$$\sum_{k=1}^\infty \sum_1^k |\log(1 - \frac{z}{a_{j,k}})|$$

converges on compact sets. To see this fix a disk $\Delta_R = \overline{\mathbb{D}(0, R)}$ and let $k \ge R$. We claim there exists C such that for $z \in \Delta_R$ we have $\sum_1^k |\log(1 - \frac{z}{a_{j,k}})| < \frac{C}{k^2}$. This follows from the bound

$$\begin{aligned}
|\log(1 - \frac{z}{a_{j,k}})| &= \left| \sum_1^\infty \frac{z^n}{n a_{j,k}^n} \right| \\
&\le \sum_{n=1}^\infty \frac{|z|^n}{n |a_{j,k}|^n} \\
&\le \sum_{n=1}^\infty \frac{R^n}{k^{3n}} \\
&= \frac{R}{k^3} + \frac{R}{k^3} \sum_{n=1}^\infty \frac{R^n}{k^{3n}} \\
&\le \frac{R}{k^3} + \frac{R}{k^3} \sum_{n=1}^\infty \frac{R^n}{R^{3n}} \\
&\le \frac{R}{k^3} + \frac{A \cdot R}{k^3} \\
&\le \frac{R + A \cdot R}{k^3}.
\end{aligned}$$

Let $C = R + A \cdot R$ where A is the sum of the series $\sum_1^\infty R^{-2n}$. We have just proven that for $k \ge R$ it holds that $|\log(1 - \frac{z}{a_{j,k}}| \le \frac{C}{k^3}$. It follows that

$$\sum_1^k |\log(1 - \frac{z}{a_{j,k}}| \le \frac{kC}{k^3} = \frac{C}{k^2}.$$

Therefore for $k \geq R$

$$\sum_{k=1}^{\infty} \sum_{1}^{k} |\log(1 - \frac{z}{a_{j,k}})| \leq C \sum_{k=1}^{\infty} \frac{1}{k}^2 < \infty.$$

It follows that the tail of the product $\prod_R^\infty p_k(z)$ converges uniformly on Δ_R. □

7.2 Hoffman's Exam

> **(Practice #1)** Show that the function $f(z) = e^{\pi z} - e^{-\pi z}$ assumes any value w with positive real part once and only once in the half strip $\mathrm{Re}(z) > 0, -1 < \mathrm{Im}(z) < 1$.

Proof #2. Let $g(z) = e^{\pi z}$ and let $h(z) = z - \frac{1}{z}$. Notice that we can write $f(z)$ as the composition $h(g(z))$. Since $g(z)$ maps the half strip conformally onto the set $\Omega = \overline{\mathbb{D}}^c \cap \{\mathrm{Re}\, z > 0\}$, it suffices to prove that

(i) $h(\Omega) \subseteq \{\mathrm{Re}\, z > 0\}$.

(ii) h is injective.

(iii) h is surjective.

If we write $z = x + iy$ a simple calculation lets us decompose h into real and imaginary parts. Let $r^2 = x^2 + y^2$,

$$h(x + iy) = \left(\frac{r^2 - 1}{r^2}\right) x + i \left(\frac{r^2 + 1}{r^2}\right) y.$$

Since any $z \in \Omega$ satisfies $r^2 - 1 > 1$ property (i) follows immediately. Next notice that for any $w \in \{\mathrm{Re}\, z > 0\}$ we have

$$z - \frac{1}{z} = \frac{z^2 - 1}{z} = w,$$

gives the quadratic $z^2 - wz - 1 = 0$. Which has solutions

$$z_1 = \frac{w + \sqrt{w^2 - 4}}{2}, \qquad z_2 = \frac{w - \sqrt{w^2 - 4}}{2}.$$

Since $\mathrm{Re}\, w > 0$ we know that $z_1 \in \Omega$ and $z_2 \notin \Omega$. Hence (i) and (ii) hold (because we characterized $h^{-1}(w)$).

(Alternate Techniques are to use the fact that sin takes lines to ellipses and use winding numbers to proof injective).

□

> **(Practice #2)** Let Ω be the region $-1 < \mathrm{Re}(z) < 1$ and let F be the collection of all analytic functions $f(z)$ defined on Ω such that $f(0) = 0$ and $|f(z)| < 1$ for all $z \subset \Omega$. Find
>
> $$\sup_{f \in F} |f(1/2)|.$$

Proof. Let $\varphi : \mathbb{D} \to \Omega : z \mapsto e^{-i\frac{\pi}{2}} \log\left(\frac{1-z}{1+z}\right)$. Notice that $\varphi(0) = 0$. Let $w_0 = \varphi^{-1}(1/2)$. By Schwarz'a lemma we have for any $f \in \mathcal{F}$ it holds that $f \circ \varphi : \mathbb{D} \to \mathbb{D} : 0 \mapsto 0$. Therefore

$$|f(\varphi(w_0))| \leq |w_0|, \qquad \forall f \in \mathcal{F}.$$

□

(Practice #3) Let Ω be a connected open subset of $\mathbb{C}$.

(a) Let $h(z)$ be a non-trivial analytic function defined on Ω. Let $\{a_n\}_{n\geq 1}$ be all the (distinct) zeros of $h(z)$ and let $\{c_n\}_{n\geq 1}$ be a sequence of complex numbers. Show that there is an analytic function $H(z)$ defined on Ω such that $H(a_n) = c_n$ for all n.

(b) Let $f(z)$ and $g(z)$ be two analytic functions defined on Ω with no common zeros in Ω. Assume that both $f(z)$ and $g(z)$ have only simple zeros. Prove that there are analytic functions $F(z)$ and $G(z)$ defined on Ω such that over Ω,

$$F(z)f(z) + G(z)g(z) = 1.$$

Part a.

Proof. The uniqueness theorem guarantees that in a neighborhood of each a_n we can write $h(z) = (z - a_n)^{m_n} k_n(z)$ with $k(a_n) \neq 0$. By Mittag-Leffler (since $\{a_n\}$ has no limit points, let $f(z)$ be a meromorphic function with a pole of order m_n at each a_n and principle part $\frac{c_n}{k_n(z)}$ at each a_n. Consider the function $H(z) = f(z)h(z)$. Notice that H is analytic on $\Omega \setminus \{a_n\}$ since f and h are analytic on this set. Moreover, our construction guarantees that

$$H(a_n) = f(a_n)h(a_n) = c_n(k_n(a_n))\frac{c_n}{k_n(a_n)} = c_n.$$

Therefore H is analytic on all of Ω.

□

Part b.

Proof. Let $\{a_n\}, \{b_n\}$ be the zero sets of f and$g$. If there is a limit point the problem is trivial. Define a function $F \in H(\Omega)$ such that $F(b_n) = \frac{1}{f(b_n)}$. Let $G(z) = \frac{1-F(z)f(z)}{g(z)}$. Since the zeros of g are simple we know that G extends to be analytic on Ω. □

(Practice #4)

(a) Write down an entire function $f : \mathbb{C} \to \mathbb{C}$ in the form of an infinite product so that its set of zeros equals $\{\log(n) : n = 2, 3, 4, ...\}$.

(b) An entire function f is said to be of finite order if there exist numbers $a, r > 0$ such that

$$|f(z)| \leq e^{|z|^a}$$

for all $|z| > r$. Is there an entire function of finite order whose zero set is $\{\log(n) : n = 2, 3, 4, ...\}$? Prove the existence or the non-existence of such functions.

Part a.

Proof. Consider the function $f(z) = \prod_{n=2}^{\infty}(1 - \frac{z}{\log n}) \exp\left(\sum_{k=1}^{n-1} \frac{z^k}{k(\log n)^k}\right)$. We know that an infinite product, $g(z) = \prod h_k(z)$, with the h_k linear terms, satisfies $g(a) = 0$ if and only if some $h_k(a) = 0$. Since the exp term is never zero we know that f is zero on precisely the desired set. It remains to prove that f converges for all $z \in \mathbb{C}$. Fortunately, this follows from the Weierstrauss factorization theorem which states that

Theorem. *Let $\{a_n\}$ be a sequence of not necessarily distinct nonzero complex numbers such that $\lim_{n\to\infty} a_n = \infty$. Let $\{p_n\}$ be a sequence of non negative integers such that for any $R > 0$ it holds that*

$$\sum_{1}^{\infty} \left(\frac{R}{|a_n|}\right)^{1+p_n} < \infty$$

Then the product

$$f(z) = \prod_{1}^{\infty} E_{p_n}\left(\frac{z}{a_n}\right)$$

is an entire function which has zeros at each a_n and no other zeroes in $\mathbb{C}$. (If a_n appears m-times then its multiplicity is m).

Here we are taking $\{a_n\} = \{\log n\}_2^{\infty}$ and $p_n = n - 1$. We claim that for any R we have $\sum_2^{\infty} \frac{R^n}{(\log n)^n} < \infty$. To see this we compute the radius of convergence ρ_R:

$$\frac{1}{\rho_R} = \limsup_{n\to\infty} \left(\frac{R^n}{(\log n)^n}\right)^{1/n} = \limsup_{n\to\infty} \frac{R}{\log n} \to 0.$$

Hence $\rho_R = \infty$ for all R. It follows that f is the sought after function.

□

Part b.

Proof. We will make use of Jensen's formula. Suppose such an f exists. We then have

$$\sum_{\text{zeros} \le r} \log\left(\frac{r}{\log n}\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(re^{i\theta})|d\theta \le |r|^a.$$

Notice on the disc $\mathbb{D}(0, r/2)$ we have $e^{r/2}$ zeros in this disc. Hence

$$\log\left(\frac{r}{\log n}\right) \ge \log\left(\frac{r}{(r/2)}\right) = \log 2.$$

So we obtain the contradiction

$$e^{r/2}\log 2 \le |r|^a.$$

□

(Practice #5) Let $K \subset \mathbb{C}$ be a compact *perfect* set, i.e. a compact set such that every point in K is a limit of some sequence of distinct points in K. Suppose $f(z)$ is a continuous function on K which is *locally analytic* on K in the sense that for each point in K, there is a function analytic in a disk about the point which agrees with f on the disk intersected with K. If $\mathbb{C} \setminus K$ is connected, prove that f is the uniform limit on K of a sequence of polynomials.

Proof. Assume that K is connected. Cover K with disks $\bigcup \mathbb{D}(z_k, \epsilon_k)$. Choose $z_{i,j} \in \mathbb{D}(z_i, \epsilon_i) \cap \mathbb{D}(z_j, \epsilon_j)$. Let $\{z_n^{i,j}\} \to z_{i,j}$. This guarantees that there are infinitely many $z_{i,j}$ in $\mathbb{D}(z_i, \epsilon_i) \cap \mathbb{D}(z_j, \epsilon_j) \cap \mathbb{D}(z_{i,j}, \epsilon_{i,j})$. Since $f = g_{i,j}$, this implies that f is analytic on K. Applying Runge's theorem guarantees the convergence we want.

If K is not connected, we consider the components $K = \bigcup_\alpha K_\alpha$. Since the components are isolated choose neighborhoods $U_\alpha \supset K_\alpha$ with $U_\alpha \cap U_\beta = \emptyset$ for $\alpha \neq \beta$. By taking $U_\alpha \cap \bigcup_{z \in U_\alpha} \mathbb{D}(z, \epsilon)$ we get $W_\alpha \supset K_\alpha$ and $f_\alpha \in H(W_\alpha)$. This pieces together to give the function $f : \bigcup_\alpha W_\alpha \to \mathbb{C}$. Since f is locally holomorphic we know that $f \in H\left(\bigcup_\alpha W_\alpha\right)$. Now we apply Runge's theorem and have the result. □

(Practice #6) Let $f(t)$ be a real-valued, infinitely differentiable function on $[0, 1]$. Define $g(z)$, for $\Omega = \{\text{Re}(z) > -1\}$, by

$$g(z) = \int_0^1 t^z f(t) dt.$$

Show that (a) g is holomorphic on Ω and (b) can be continued analytically to a meromorphic function on the entire complex plane.

Part a.

Proof. We seek to apply Morera's theorem. As g is an integral we know that g is continuous. It remains to prove that for any rectangle $R \subseteq \Omega$ we have $\int_R g(z)dz = 0$. Define the functions

$$g_\epsilon(z) := \int_\epsilon^1 t^z f(t) dt.$$

Since everything is bounded we can write

$$\int_R g_\epsilon(z) dz = \int_\epsilon^1 f(t) \int_R t^z dz dt = \int_\epsilon^1 f(t) \cdot 0 dt = 0.$$

Since the $g_\epsilon \to g$ locally uniformly we have $\int_R g(z)dz = 0$.

□

Part b.

Proof. We make use of integration by parts.

$$\int f(t) t^z dt = -\int f'(t) \frac{t^{z+1}}{z+1} dt + \left. \frac{f(t) t^{z+1}}{z+1} \right|_0^1.$$

□

(Practice #7) Suppose f_n is a sequence of holomorphic functions in a domain R for which $u(z) = \lim \mathrm{Re}(f_n(z))$ exists uniformly on every compact subset of R. Also assume there is a point $z_0 \in R$ for which $v(z_o) = \lim \mathrm{Im}(f_n(z_0)) = l$ exists. Prove f_n converges uniformly on every compact subset of R.

Proof. Write $u_n = \mathrm{Re}\, f_n$. Without loss of generality assume that $z_0 = 0$ and $\overline{\mathbb{D}} \subseteq R$. By Herglotz (Schwarz) formula we know that

$$f_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z} u_n(e^{i\theta})d\theta + iv_n(0).$$

We claim that f_n is locally uniformly bounded on $\overline{\mathbb{D}}$. Fix $0 < r < 1$. Let $V = \sup_n |v_n(0)|$ and $U_r = \sup_{|z|\le r,n} |u_n(z)|$. For $|z| \le r$ we have

$$|f_n(z)| \le V + U_r \cdot \max_{|z|\le r} \left| \frac{1+e^{-i\theta z}}{1-e^{-i\theta}z} \right| \le V + U_r \left| \frac{1+R}{1-R} \right|.$$

Hence $\{f_n\}$ is locally bounded and (by Montel) contains a locally uniformly convergent subsequence $f_{n_j} \to f$. We must have $\mathrm{Re}\, f = u$ and $\mathrm{Im}\, f$ is the harmonic conjugate of u such that $v(0) = \frac{l-u(0)}{i}$. Since we can repeat this for any subsequence it follows that $f_n \to f$ locally uniformly on the disk.

We have now prove that for each $z_i \in R$ it holds that if $\lim f_n(z_i)$ exists then $f_n \to \tilde{f}$ locally uniformly on the disk $\mathbb{D}(z, \rho)$. Take a compact set $K \subseteq R$. Without loss of generality assume that K is connected. Choose $z_0 \in K$. We can cover K with finitely many disks $D_0, \dots, D_m$. Relabel so that $z_0 \in D_0$. Choose a dsik D_1 such that $D_0 \cap D_1 \ne \emptyset$. Continue so that

$$D_j \cap (C_0 \cup \cdots \cup D_{j-1}) \ne \emptyset.$$

Notice that if $D_0 \cap D_1 \ne \emptyset$ then we have $f_n \to f$ locally uniformly on D_0 implies that $f_n \to f$ locally uniformly on D_1. Proceeding in this manner we can extend the local uniform convergence to all of K.

(another idea is to prove that {points where $v_n(p) \to v(p)$ converges} is open and closed and thus all of R.) □

(Practice #8) Let $u(z)$ be a real harmonic function on $\{z : 1 < |z| < 2\}$. Show that for some α and for some function $f(z)$ analytic in $\{z : 1 < |z| < 2\}$, $u(z) = \alpha \log r + \mathrm{Re}\, f(z)$.

Proof. We define $h = U_y + iU_x$. It can be proven that h is holomorphic.

□

8 2003

(2003 #1) Let $a \in \mathbb{C}$ with $|a| > e$. Let n be a positive integer. How many solutions (counted with multiplicity) does $e^z = az^n$ have in the half plane $\{z = x + iy : x < 1\}$.

Proof. No solutions. To see this we apply Rouche on rectangles, R, contained in the half plane. Notice that for any $z \in \partial R$ we have $\operatorname{Re} z < 1$ and $|z| > 1$. It follows that for $z \in \partial R$

$$|e^z - az^n - (-az^n)| = |e^z| = e^{\operatorname{Re} z} < e < |az^n|.$$

So by Rouche e^z has the same number of zeros as az^n in each rectangle. But az^n has all of its zeros at $z = 0$. Therefore $e^z - az^n$ has no solutions. □

(2003 #2) Find all entire functions which satisfy $f(f(z)) = f(z)$ (denoted by *).

Proof. The only such functions are the constant functions and $f(z) = z$. Suppose that g satisfies *, is non constant and not the identity function. Picard's theorem implies that the image of g omits at most one point. First let us assume that g subjects onto $\mathbb{C}$. Fix $a \in \mathbb{C}$ and let $g(a) = b$. Since g satisfies $*$ we know that

$$g(g(a)) = g(a) \implies g(b) = b.$$

Since g is surjective, it follows that for every $b \in \mathbb{C}$ we have $g(b) = b$ and so g is the identity map.

If g omits one point the same argument implies that g is the identity on $\mathbb{C} \setminus \{c\}$ for some $c \in \mathbb{C}$. By uniqueness we know that g extends to be the identity map on all of $\mathbb{C}$. □

(2003 #3) Let f be analytic in $\mathbb{D} \setminus \{0\}$. Show that f has a removable singularity at 0 if and only if

$$\lim_{\epsilon \to 0^+} \int_{A_\epsilon} |f(z)| dxdy < \infty, \qquad A_\epsilon := \{z : \epsilon < |z| < \frac{1}{2}\}.$$

Proof. Suppose that f has a removable singularity at 0. By definition f extends to be analytic at 0. It follows from continuity that $\sup_{z \in A_0} |f(z)| = M < \infty$ and so

$$\int_{A_0} |f(z)| dz \leq 2\pi M.$$

Let's suppose that $\lim_{\epsilon \to 0} \int_{A_\epsilon} |f(z)| dz = M < \infty$.
we can make the substitution $z = \rho e^{i\theta}$ and so

$$\int_{\mathbb{D}} |f(z)| dz = \int_0^{2\pi} \int_0^1 |f(\rho e^{i\theta})| \rho \, d\rho d\theta$$

□

(2003 #4) Find an explicit formula for a function $u \in Ha(\mathbb{D})$ such that

$$\lim_{z \to e^{i\theta}, z \in \mathbb{D}} u(z) = \begin{cases} 1, & 0 < \theta < \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < \theta < 2\pi \end{cases}.$$

(2003, #3) Let $p(z)$ be a polynomial with complex coefficients and of degree $N \geq 1$. Let $\|p\|_\infty = \sup\{|p(z)| : |z| = 1\}$. Prove that $|p(w)| \leq \|p\|_\infty \max\{1, |w|^N\}$ for all $w \in \mathbb{C}$.

Proof. By the maximum principle the inequality is clear for $|w| \leq 1$. For any $\epsilon > 0$, we can find $R > 1$ such that $|p(z)/z^N| < c_N + \epsilon$ for all $|z| = R$. Since $g(z) = p(z)/z^N$ is holomorphic away from zero, by the maximum principle it follows that g attains its max on the annulus $\mathcal{A}_R = \{1 < |z| < R\}$ at the boundary. This is bounded by $\max\{\|p\|_\infty, c_N + \epsilon\}$. We claim that $c_N \leq \|p\|_\infty$. Suppose not. If we let $q(z) = p(z) - c_N z^N$ and $g(z) = -c_N z^N$ we have on $\partial\mathbb{D}$

$$|q(z) - g(z)| = |p(z)| \leq \|p\|_\infty < |c_N| = |g(z)|.$$

Therefore $q(z)$ has N zeros in the disk. This contradicts the fact that $\deg q = N - 1$. The claimed inequality then follows. □

(2003 #8) Let $\mathcal{F} = \{f \mid f : \mathbb{D} \to \mathbb{D}, f \in H(\mathbb{D}), f(1/2) = f(-1/2) = 0\}$. Find $\max f \in \mathcal{F}|f(0)|$.

Proof. Let $B(z) = \left(\frac{z-\frac{1}{2}}{1-\frac{1}{2}z}\right)\left(\frac{z+\frac{1}{2}}{1+\frac{1}{2}z}\right)$. Notice that $B \in \mathcal{F}$ and that $B(0) = \frac{1}{4}$. We claim that every $f \in \mathcal{F}$ satisfies $|f(z)| \leq |B(z)|$ for $z \in \mathbb{D}$ and thus our upper bound is $\frac{1}{4}$. To see this we fix $f \in \mathcal{F}$ and consider the quotient $q(z) = \frac{f(z)}{B(z)}$.

Since $f(-1/2) = 0 = f(1/2)$ we know that the singularities of q are removable and thus $q \in H(\mathbb{D})$. We know that $|f(z)| < 1$ for all $z \in \mathbb{D}$ and so $\lim_{R\to 1} \sup_{|z|=R} |f(z)| \leq 1$. Moreover, since B fixes $\partial\mathbb{D}$ we know that $\lim_{R\to 1} \sup_{|z|=R} B(z) = 1$. It follows that

$$\lim_{R\to 1} \sup_{|z|=R} |q(z)| \leq 1.$$

By the maximum modulus principle we must then have $q : \mathbb{D} \to \mathbb{D}$ and therefore $|f(z)| \leq |B(z)|$ for all $z \in \mathbb{D}$. □

(2003 #5) Compute $\int_0^\infty \sin(x^2)dx$ by integrating e^{iz^2} over an appropriate contour. You may assume that $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$.

(2003 #6) Prove that $\prod_1^\infty (1 + z^{2^n})$ converges uniformly on compact subsets of $\mathbb{D}$ and show

$$\frac{1}{1-z^2} = \prod_1^\infty (1 + z^{2^n}).$$

(2003 #7) Let $c > 0$. Define

$$\mathcal{F}_c = \{f : f(z) = \sum_0^\infty a_n z^n, \text{ where } f \text{ is analytic on } \mathbb{D}, \text{ and } \sum_0^\infty |a_n|^2 \leq c\}.$$

Prove that $\mathcal{F}_c$ is a normal family.

(2003 #8) Let $S = \{z = x + iyL|x| < 1, |y| < 1\}$. Let f be analytic on S and continuous on the closure of S. LEt the four sides of the square, ∂S, be denoted by S_1, S_2, S_3, S_4. Suppose that $|f(z)| \leq M_i$ for all $z \in S_i, i = 1, \ldots, 4$. Prove that

$$|f(0)|^4 \leq M_1 M_2 M_3 M_4.$$

9 2004

(2003 #6) (a) Let f, g be entire with $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{C}$. Prove that $f(z) = cg(z)$.
(b) Let $\psi : [0, \infty) \to (0, \infty)$ be a continuous real valued function. Also let $f_n(z)$ and $g9z)$ be entire functions satisfying the inequality

$$|f_n(z)| \leq (1/n)\psi(|z|) + |g(z)|.$$

for all $z \in \mathbb{C}$. Prove that

$$f_n'(z)g(z) - f_n(z)g'(z) \to 0$$

uniformly on all compact subsets of $\mathbb{C}$.

Part b.

Proof. Consider $\left(\frac{f_n}{g}\right)' = \frac{f_n' g - g' f_n}{g^2}$. We know that

$$\left|\frac{f_n}{g}\right| \leq \frac{f_n \psi(|z|}{|g|} + 1.$$

Let K be a compact set which avoids the zeros of g. We conclude that $\left|\frac{f_n}{g}\right|$ is bounded on K and thus contains a compact subset. Take a sequence of such compact sets K_j and diagonalize. We obtain a sequence $f_k \to f$ which converges on $\mathbb{C}$ minus the zero of g.

Tip: If trying to prove convergence of a sequence in a normal family it suffices to prove that there is only one limit point in the family. □

(2003 #7) Let $\mathcal{F}$ be a normal family of analytic functions on $\mathbb{D}$ with $f'(0) = 1$ for all $f \in \mathcal{F}$. Prove that there exist $r > 0$ such that every $f \in \mathcal{F}$ is injective on $\{z \in \mathbb{C} : |z| < r\}$.

Proof. Suppose not. Take a sequence $f_n - c_n$ not 1-1 on $(1/n)\mathbb{D}$ with c_n a double zero of f_n. By normality we can assume (wlog) that $f_n - c_n \to f$. Because $\mathcal{F}$ is normal we know it is bounded on compact sets and thus the c_n are bounded. We conclude that $\frac{f_n'}{f_n} \to \frac{f'}{f}$ uniformly. So, $f'(0) = 1$ when we integrate we obtain the contradiction that f has a double zero at 0 and thus has $f'(0) = 0$. □

(2004 #1) Let f be holomorphic on a neighborhood of $\overline{\mathbb{D}}$.

(a) Evaluate the integrals

$$\frac{1}{2\pi i}\int_{|\zeta|=1}\frac{f(\zeta)}{\zeta - z}d\zeta \quad \text{and} \quad \frac{1}{2\pi i}\int_{|\zeta|=1}\frac{f(\zeta)}{\zeta - 1/z}d\zeta, \qquad |z|<1.$$

(b) Prove that for $0 \leq r < 1$

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{s\pi}\frac{1-r^2}{1-2r\cos(\theta-\varphi)+r^2}f(e^{i\theta})d\varphi.$$

Part a.

Proof. It follows immediately from Cauchy's formula that the first integral is z (since $z \in \mathbb{D}$) and the second integral is 0 (since $1/z \notin \mathbb{D}$). □

Part b.

Proof. Let $z = re^{i\theta}$. Using Cauchy's formula and a parametrization of $|\zeta| = 1$ we have

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{|\zeta|=1}\frac{f(\zeta)}{\zeta - z}d\zeta \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f(e^{i\varphi})e^{i\varphi}}{e^{i\varphi}-z}d\varphi \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}f(e^{i\varphi})\frac{e^{i\varphi}}{e^{i\varphi}}\frac{1}{1-ze^{-i\varphi}}d\varphi \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}f(e^{i\varphi})\frac{1}{1-ze^{-i\varphi}}d\varphi. \end{aligned}$$

Notice also that

$$\begin{aligned} \frac{1}{1-ze^{-i\varphi}} + \frac{\overline{z}e^{i\varphi}}{1-\overline{z}e^{i\varphi}} &= \frac{1-\overline{z}e^{i\varphi}+\overline{z}e^{i\varphi}-z\overline{z}}{|1-ze^{\imath\varphi}|^2} \\ &= \frac{1-r^2}{|1-ze^{-i\varphi}|^2} \\ &= \frac{1-r^2}{|1-re^{-i\varphi}|^2} \\ &= \frac{1-r^2}{1+r^2-2r\cos(\theta-\varphi)}. \end{aligned}$$

But since $1/\overline{z} \notin \mathbb{D}$ we have again by Cauchy's formula that

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi}f(e^{i\varphi})\frac{\overline{z}e^{i\varphi}}{1-\overline{z}e^{i\varphi}}d\varphi &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{i\zeta}f(\zeta)\frac{\overline{z}\zeta}{1-\overline{z}\zeta}d\zeta \\ &= \frac{1}{2\pi i}\int_{|\zeta|=1}f(\zeta)\frac{\overline{z}}{1-\overline{z}\zeta}d\zeta \\ &= 0. \end{aligned}$$

It follows that

$$\begin{aligned} f(z) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\varphi})\frac{1}{1-ze^{-i\varphi}}d\varphi \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\varphi})\left[\frac{1}{1-ze^{-i\varphi}} - \frac{1}{1-ze^{-i\varphi}}\right]d\varphi \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta-\varphi)}d\varphi. \end{aligned}$$

Changing our bounds to $[0, 2\pi]$ has no effect, and thus

$$f(z) = \frac{1}{2\pi}\int_{0}^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta-\varphi)}d\varphi.$$

□

(2004 #2)

(a) State the Weierstrass factorization theorem and the Mittag-Leffler theorem.

(b) Given a sequence $\{z_n\}$ of distinct points with no limit point in $\mathbb{C}$ and $\{c_n\}$ an arbitrary sequence in $\mathbb{C}$ show there is an entire function with $f(z_n) = c_n$ for all n.

Part a.

Theorem (Theorem 5). *(Weierstrass Factorization Theoerem) Given a sequence of points $\{a_n\}$ with no limit point, there exists an entire function with zeros at each a_n (counting multiplicity).*

Theorem (Mittag-Leffler). *Suppose that $A \subseteq \mathbb{C}$ and A has no limit point. For each $a \in A$, let there be given $m(a) \in \mathbb{Z}$ and a rational function $p(a) = \sum_{j=1}^{m(a)} c_{j,a}(z-a)^{-j}$. Then there exists an $f \in M(\mathbb{C})$ whose principle part at each $a \in A$ is P_a and f has no other poles.*

Part b.

Proof. By Weierstrass's theorem there exists a $g \in H(\mathbb{C})$ such that g has a simple zero at each z_n. By Mittag-Leffler there exists an $h \in M(\mathbb{C})$ with no poles except at z_n with principle part

$$\frac{w_n}{z-z_n}, \qquad w_n = \frac{c_n}{g'(z_n)}.$$

Let $f = gh$. For $z \in \mathbb{C} \setminus \{z_n\}$ we know that $f(z) = \frac{g(z)}{z-z_n}[h(z)(z-z_n)]$. As $z \to z_n$ we have

$$f(z) \to g'(z_n)w_n = c_n.$$

Therefore, f has a removable singularity at z_n and we define f to be w_n at z_n.

□

(2004 #3) Show that for $c > 0$

$$\int_{c-i\infty}^{c+i\infty} \frac{a^z}{z} dz = \begin{cases} 2\pi i, & a > 1 \\ 0, & 0 < a < 1 \end{cases}.$$

The path of integration of the vertical line $\operatorname{Re} z = c$ is traversed upward.

Proof. $\boxed{a > 1}$ Let $\alpha = \log a$. Notice that $a > 1 \implies \alpha > 0$. Observe that the function

$$a^z/z = e^{z \log a}/z = \sum_0^\infty z^{k-1}(\alpha)^k/k! = 1/z + \sum_1^\infty z^{k-1}(\alpha)^k/k!$$

has a single pole at $z = 0$ with residue 1. For R satisfying $\sqrt{R} > c$ let S_R denote the rectangle comprised of the four line segments

$$L_1 = [c - iR, c + iR], \quad L_2 = [c + iR, c - \sqrt{R} + iR]$$
$$L_3 = [c - \sqrt{R} + iR, c - \sqrt{R} - iR], \quad L_4 = [c\sqrt{R} - iR, c - iR].$$

First observe that $z = 0$ belongs to the interior of S_R and so

$$\int_{S_R} a^z/z dz = 2\pi i \text{Res}(a^z/z; S_R) = 2\pi i.$$

We will prove that the integral over each segment L_2, L_3, L_4 goes to zero as $R \to \infty$ and thus $\int_{L_1} a^z/z dz = 2\pi i$ as desired.

(i) First let us consider the integral over L_2. Using the ML-estimate we have

$$\int_{L_2} a^z/z dz \leq \sqrt{R} \sup_{z \in L_2} \frac{|a^z|}{|z|} \tag{1}$$

For R sufficiently large we know that $\inf_{L_2} |z| = R$. And also writing $z = x + iy$ we have

$$\sup_{L_2} |a^z| = \sup_{L_2} |e^{z\alpha}| = \sup_{L_2} |e^{x\alpha} e^{iy\alpha}| = \sup_{L_2} |e^{x\alpha}|.$$

Since $a > 1$ we know that $\alpha > 0$ and thus the above supremum is obtained when x is largest, which is at $z = c + iR$. Thus, $\sup_{L_2} |a^z| = e^{c\alpha} := C_2$. It follows that the value in (1) can be bounded by

$$\int_{L_2} a^z/z dz \leq \sqrt{R} \sup_{z \in L_2} \frac{|a^z|}{|z|} = \sqrt{R} \cdot \frac{e^{c\alpha}}{R} = \frac{C_2}{R} \to 0.$$

Which goes to 0 as $R \to \infty$.

(ii) Next let us consider the integral over L_3. As in (i) we use the ML-estimate to write

$$\int_{L_3} a^z/z dz \leq 2R \sup_{z \in L_3} \frac{|a^z|}{|z|} \tag{2}$$

In a similar calculation as (i) we obtain

$$\sup_{z \in L_3} \frac{|a^z|}{|z|} = \frac{|e^{(c-\sqrt{R}) \log a}|}{|\sqrt{R} - c|} \leq \frac{|e^{c\alpha}|}{|\sqrt{R} - c||e^{\sqrt{R}\alpha}|}.$$

Since $e^{\sqrt{R}\alpha}$ grows faster than R, it is then easy to see that $2R \sup_{L_3} |a^z|/|z| \to 0$ as $R \to \infty$.

(iii) Lastly, the calculation for L_4 is very similar to (i) and yields the same conclusion.

Thus, the desired result holds for $a > 1$.

$\boxed{0 < a < 1}$ In this case note that $\alpha = \log a < 0$. We consider the integral over the rectangle Q_R spanned by the line segments

$$T_1 = [c - iR, c + iR], \quad T_2 = [c + iR, c - \sqrt{R} + iR]$$

$$T_3 = [c - \sqrt{R} + iR, c - \sqrt{R} - iR], \quad T_4 = [c\sqrt{R} - iR, c - iR].$$

In this case we have the pole $z = 0 \notin Q_R$ and thus by the residue theorem $\int_{Q_R} a^z/zdz = 0$. We show in a similar manner as the $a > 1$ case that the integral over each segment T_2, T_3, T_4 goes to zero as $R \to 0$ and thus $\int_{T_1} a^z/zdz = 0$.

(I) Just as in (i) we can apply the ML-estimate and obtain the bound

$$\int_{T_2} a^z/zdz \le \sqrt{R} \sup_{z \in T_2} \frac{|a^z|}{|z|} \tag{3}$$

The key difference from (i) is that $\alpha < 0$. This means that $\sup_{T_2} |a^z| = \sup_{T_2} |e^{x\alpha}|$ occurs when x is smallest. Thus, $\sup_{T_2} |a^z| = e^{c\alpha}$. Also, notice that $\inf_{T_2} |z| = |c + iR| \le R$. So, we have

$$\sup_{z \in T_2} \frac{|a^z|}{|z|} \le \frac{e^{c\alpha}}{R}.$$

It follows that

$$\int_{T_2} a^z/zdz \le \sqrt{R}\frac{e^{c\alpha}}{R} \to 0.$$

Which goes to zero as $R \to \infty$.

(II) Along T_3 we obtain the bound

$$\int_{T_3} a^z/zdz \le 2R \sup_{z \in T_3} \frac{|a^z|}{|z|} \tag{4}$$

We have $\sup_{T_3} |a^z| = \sup_{T_3} |e^{x\alpha}|$ occurs at $x = c + \sqrt{R}$. So just as in (ii) we wish to minimize x. This occurs at $x = c + \sqrt{R}$ and gives

$$\sup_{T_3} |a^z| = e^{(c+\sqrt{R})\alpha}.$$

Since $\alpha < 0$ we have

$$\int_{T_3} a^z/zdz \le 2R \frac{e^{c\alpha}}{\sqrt{R}e^{-\alpha\sqrt{R}}} \to 0.$$

The above goes to 0 as $R \to \infty$.

(III) Lastly, when we integrate along T_4 it is symmetric to case (I) and gives the same result.

We conclude that the integral along each T_i is 0. □

(2004 #4) Let $f \in H(\mathbb{D})$.

(a) Show that the quantity
$$\int_{|z|=r} |f(z)|^2 \frac{dz}{iz}$$
is a convex function of r on the interval $(0,1)$. You may use the fact that a function h on (a,b) is convex if it has two continuous derivatives and $h'' > 0$ on (a,b).

(b) Prove the same result for the integrals
$$\int_{|z|=r} |f(z)|^k \frac{dz}{iz}$$
for k any even positive number.

Part a.

Proof. As $f \in H(\mathbb{D})$ we can write f as a power series $f(z) = \sum_1^\infty a_k z^k$. Since the convergence is uniform we can multiply termwise. It follows that $|f(z)|^2 = f(z)\overline{f(z)} \left(\sum_1^\infty a_k z^k\right)\left(\sum_1^\infty \overline{a}_k \overline{z}^k\right)$. Making this substitution and parametrizing the curve $|z| = r$ yields

$$\int_{|z|=r} |f(z)|^2 \frac{dz}{iz} = \int_0^{2\pi} f(re^{i\theta})\overline{f(re^{i\theta})} ire^{i\theta} \frac{d\theta}{ire^{i\theta}} = \int_0^{2\pi} \left(\sum_1^\infty a_j r e^{ij\theta}\right)\left(\sum_1^\infty \overline{a}_k r e^{-ik\theta}\right) d\theta.$$

Notice that
$$\int_0^{2\pi} r^2 a_j \overline{a}_k e^{i(j-k)\theta} d\theta = r^2 a_j \overline{a}_k \int_0^{2\pi} e^{i(j-k)\theta} d\theta = 0, \qquad \forall j \neq k.$$

Hence we only need consider the terms on the diagonal. More precisely we have shown

$$\int_{|z|=r} |f(z)|^2 \frac{dz}{iz} = \int_0^{2\pi} \sum_{n=1}^\infty a_n \overline{a}_n r^2 d\theta = 2\pi \sum_{n=1}^\infty |a_n|^2 r^2 > 0.$$

Letting $I(r) = \int_{|z|=r} |f(z)|^2 \frac{dz}{iz}$ we now have

$$I'(r) = 4\pi \sum_{n=1}^\infty |a_n|^2 r, \qquad I''(r) = 4\pi \sum_{n=1}^\infty |a_n|^2.$$

Notice that both derivatives exists and are continuous (since we can differentiate termwise) and also that $I'' > 0$. □

Part b.

Proof. As k is even, write $k = 2m$. Since $f \in H(\mathbb{D})$, so is the function $g = f^m$. And because $|g|^2 = |f^m|^2 = |f|^{2m} = |f|^k$ we can apply Part (a) and conclude that $J(r) = \int_{|z|=r} |g(z)|^2 \frac{dz}{iz}$ satisfies the conclusions of Part (a).

□

(2004 #5) Let P be a polynomial with complex coefficients, p not identically zero.

(a) Prove that the series
$$\sum_0^\infty P(n)z^n$$
converges in $\mathbb{D}$ and in no larger open set.

(b) Show that if f is the sum of this series, then f continues into the whole Riemann sphere as a meromorphic function.

Part a.

Proof. Let $f(z) = \sum_0^\infty P(n)z^n$ and let $\deg P = d$. We first prove that the series $\sum_{n=1}^\infty n^d z^n$ has radius of convergence precisely $R = 1$. This follows from the ratio test since

$$\lim_{n\to\infty} \left| \frac{(n+1)^d z^{n+1}}{n^d z^n} \right| = \lim_{n\to\infty} |z| \left| \frac{(n+1)^d}{n^d} \right| = |z|.$$

It follows that $R = 1$ and thus $\mathbb{D}$ is the largest open set on which the series $\sum n^d z^n$ converges. Notice that for each term of $P(z)$, $a_j z^j$, with $j \leq d$ the same argument implies the radius of convergence is 1. Therefore the sum $\sum P(n)z^n$ converges.

□

Part b.

Proof. Notice that $\frac{1}{1-z} = \sum z^n$ and that

$$\frac{d^k}{dz^k}\left(\frac{1}{1-z}\right) = \frac{k!}{(1-z)^{k+1}} = \sum_{n=k}^\infty (n\cdot(n-1)\cdots(n-k+1))z^{n-k}.$$

Hence we can write $f(z) = \sum P(n)z^n$ as a linear combination of the derivatives of $1/(1-z)$. Since each derivative of $1/(1-z)$ is a rational function it follows that f is a finite sum of rational functions and thus meromorphic on all of $\hat{\mathbb{C}}$.

□

(2004 #6) Show that there is an entire function f with the property that for every entire function g, for every $\epsilon > 0$ and for every compact set $K \subseteq \mathbb{C}$, there is $c > 0$ such that

$$\sup_{z\in K} |g(z) - f(z+c)| < \epsilon.$$

Proof. Let $\{P_j\}$ be a countable dense sequence of polynomials in which each P_j occurs infinitely ofter. Also, let $\{D_j\}$ be a countable collection of disjoint closed discs, each with radius j such that the centers c_j form an increasing sequence on the positive real axis. Let $E_j = \overline{\mathbb{D}}(0, r_j)$ be a collection discs such that each $D_j \subseteq E_j$ and $E_{j+1} \cap D_j = \emptyset$. Let $Q_1 = P_1$. By Runge's theorem there is a polynomial Q_2 such that $\sup_{E_1} |Q_2| < 1/2$ and such that

$$|Q_2(z) - (P_2(z - c_2) - Q_1(z))| < 1/2, \qquad z \in D_2.$$

Next, choose a polynomial Q_3 such that $\sup_{E_2} |Q_3| < (1/2^2)$ and such that

$$|Q_3(z) - (P_3(z - c_2) - Q_1(z) - Q_2(z))| < 1/2, \qquad z \in D_3.$$

Continue inductively so that Q_n is a polynomial satisfying $\sup_{E_{n-1}} Q_n| < (1/2^{n-1})$ and so that

$$\left| Q_n(z) - \left[P_n(z - c_n) - \sum_1^{n-1} Q_i(z) \right] \right| < \frac{1}{2^{n-1}}, \qquad z \in D_n.$$

Consider the function $f = \sum_1^\infty Q_n$. Our construction guarantees that f is entire. It remains to prove that for any $g \in H(\mathbb{C})$, compact set $K \subseteq \mathbb{C}$ and $\epsilon > 0$ there is a j such that $|g(z) - f(z + c_j)| < e\epsilon$ for $z \in K$. So, fix g, K, ϵ. As K is compact it is bounded. Let R be such that $K \subseteq \mathbb{D}(0, R)$. It then suffices to prove the statement on the set $\overline{\mathbb{D}}(0, R)$. Moreover, since $\{P_j\}$ is dense in $H(\mathbb{C})$ it suffices to prove the statement for $g = P \in \{P_k\}$. Since if we let $\|g - P\| < \epsilon/2$ the desired inequality would follow from the triangle inequality along with an $\epsilon/2$ argument.

To summarize, let P, ϵ, R be fixed, we seek c_j such that $|P(z) - f(z + c_j)| < \epsilon$. Since P occurs infinitely often in $\{P_k\}$, say $P = \{P_{k_1}, P_{k_2}, \ldots\}$ we can choose k_i large enough so that $\mathbb{D}(0, R) \subseteq E_{k_i}$,

$$\sup_{E_{k_i - 1}} \left| f - \sum_1^{k_i} Q_i \right| < \epsilon/2$$

and

$$\left| \sum_1^{k_i} Q_i(z) - P(z - c_{k_i}) \right| < \epsilon/2.$$

Hence for all $z \in D_k$ we have

$$|P(z - c_k) - f(z)| < \epsilon.$$

We finish by letting $w = z - c_k$ and thus

$$|P(w) - f(w + c_k)| < \epsilon.$$

□

(2004 #7) Give explicitly a function that maps the set

$$G = \{z = x + iy : |z| < 1 \text{ and either } x > 0 \text{ or } y^2 > 2x^2\}$$

conformally onto the unit disc.

Proof. The region G is the open subset of $\mathbb{D}$ that is missing the region between the rays with angles $2\pi/3$ and $4\pi/3$. Thus we are deleting the region spanned by a slice of angle $2\pi/3$. The slice of the disk included in G is has angular measure $4\pi/3$. We define the map $f_1 : G \to \mathbb{D} \cap \{z : \operatorname{Re} z > 0\}$ via the map $f_1(z) = z^{\frac{3}{4}}$. Call the set $\mathbb{D} \cap \{z : \operatorname{Re} z > 0\} =: G_1$.

Next, we claim that the map $f_2(z) = \frac{1}{z-i}$ maps $G_1 \to \{z : \operatorname{Re} z, \operatorname{Im} z > i/2\}] : G_2$. This is easy to see since the imaginary axis is mapped to the imaginary axis and $f_2(i) = i/2$.

Next, we translate down by $(i/2)$ via the map $f_3(z) = z - (i/2)$. Notice that $f_3 : G_2 \to URQ$ with URQ the upper-positive-quadrant. Squaring this (i.e. $f_4(z) = z^2$) gives a map from URQ to the whole right-half plane. We conclude by taking the map $f_5(z) = \frac{z+1}{z-1} : RHP \to \mathbb{D}$. To summarize we let $f = (f_5 \circ f_4 \circ f_3 \circ f_2 \circ f_1)$. Our map $f : G \to \mathbb{D}$ via the mappings

$$z \mapsto z^{3/4} \mapsto \frac{1}{z-i} \mapsto z - (i/2) \mapsto z^2 \mapsto \frac{1-z}{1+z}.$$

□

(2004 #8) Let w_1 and w_2 be complex numbers that are linearly independent over $\mathbb{R}$ so that neither is a nonzero real multiple of the other.

(a) Prove that the only entire functions f that satisfy the functional equations $f(z+w_1) = f(z)$ and $f(z+w_2) = f(z)$ for all $z \in \mathbb{C}$ are the constants.

(b) Set $w_{m,n} = m + in$ for all integers m and n. Define f by

$$f(z) = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left[\frac{1}{(z - w_{m,n})^2} - \frac{1}{w_{m,n}^2} \right].$$

Show that f is a meromorphic function on $\mathbb{C}$ that satisfies the equations $f(z+i) = f(z) = f(z+1)$ and has a pole of order two at the origin.

Part a.

Proof. First off, notice that the condition $f(z+w_1) = f(z)$ implies that $f(z+mw_1) = f(z)$ for any integer m, and similarly $f(z+nw_2) = f(z)$. If follows for all $n, m \in \mathbb{Z}$ we have

$$f(z + mw_1 + nw_2) = f(z), \qquad \forall n, m \in \mathbb{Z} \tag{5}$$

As w_1, w_2 are linearly independent, let $z = tw_1 + uw_2$ with $t, u \in \mathbb{R}$ and let $m, n \in \mathbb{Z}$ be such that

$$0 \le t - m < 1, \qquad 0 \le u - n < 1. \tag{6}$$

We can write

$$z = tw_1 + uw_2 = (t-m)w_1 + (u-n)w_2 + mw_1 + nw_2.$$

It follows from (5) that

$$f(z) = f(tw_1 + uw_2) = f((t-m)w_1 + (u-n)w_2 + mw_1 + nw_2) \overset{(5)}{=} f((t-m)w_1 + (u-n)w_2).$$

So the range of f is determined by the values $s, r \in [0, 1)$. We then have

$$\sup_{\mathbb{C}} |f(z)| \le \sup_{r,s\in[0,1]} |f(rw_1 + sw_2)|.$$

As $[0,1] \times [0,1]$ is compact we know that the supremum is obtained and finite, therefore f is bounded. Liouville's theorem implies that f is constant.

□

Part b.

Proof. To show that f is meromorphic, we prove that f is absolutely convergent away from its poles. First we combine denominators

$$\begin{aligned} f(z) &= \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left[\frac{1}{(z-w_{m,n})^2} - \frac{1}{w_{m,n}^2}\right] \\ &= \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left[\frac{w_{n,m}^2 - (z-w_{m,n})^2}{(z-w_{m,n})^2 w_{m,n}^2}\right]. \end{aligned}$$

Since the poles $w_{m,n}$ form an integer-spaced lattice on $\mathbb{C}$ it is easy to see that we can cover the poles with arbitrarily small disks $\Delta_{m,n} = \mathbb{D}(w_{m,n}, 2^{-m-n})$. So on the set $\Omega = \mathbb{C} \setminus \bigcup_{m,n} \Delta_{m,n}$ we can consider the sum

$$\sum_{(m,n)\neq(0,0)} \left|\frac{w_{n,m}^2 - (z-w_{m,n})^2}{(z-w_{m,n})^2 w_{m,n}^2}\right|$$

The denominator is $\mathcal{O}(w_{m,n}^4)$ and the numerator is $\mathcal{O}(w_{m,n}^2)$. It follows by comparison to the harmonic series that the sum converges on Ω and thus f is meromorphic. It remains to prove the required periodicity statements.

We check that $f(z+i) = f(z)$ by noting that $i - w_{m,n} = -w_{m,n+1}$ and rearranging the terms.

$$\begin{aligned} f(z+i) &= \frac{1}{(z+i)^2} + \sum_{(m,n)\neq(0,0)} \left[\frac{1}{((z+i)-w_{m,n})^2} - \frac{1}{w_{m,n}^2}\right] \\ &= \frac{1}{(z+w_{0,1})^2} + \sum_{(m,n)\neq(0,0)} \left[\frac{1}{(z-w_{m,n+1})^2} - \frac{1}{w_{m,n}^2}\right] \\ &= \frac{1}{(z+w_{0,1})^2} + \frac{1}{z^2} + \sum_{(m,n)\neq(0,0),(0,-1)} \left[\frac{1}{(z-w_{m,n+1})^2} - \frac{1}{w_{m,n}^2}\right] \\ &= \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left[\frac{1}{(z-w_{m,n})^2} - \frac{1}{w_{m,n}^2}\right] \\ &= f(z). \end{aligned}$$

Similarly we use the fact that $1 - w_{m_n} = w_{m+1,n}$ to obtain $f(z+1) = f(z)$

$$\begin{aligned} f(z+i) &= \frac{1}{(z+1)^2} + \sum_{(m,n)\neq(0,0)} \left[\frac{1}{((z+1)-w_{m,n})^2} - \frac{1}{w_{m,n}^2}\right] \\ &= \frac{1}{(z+w_{1,0})^2} + \sum_{(m,n)\neq(0,0)} \left[\frac{1}{(z-w_{m+1,n})^2} - \frac{1}{w_{m,n}^2}\right] \\ &= \frac{1}{(z+w_{1,0})^2} + \frac{1}{z^2} + \sum_{(m,n)\neq(0,0),(-1,0)} \left[\frac{1}{(z-w_{m+1,n})^2} - \frac{1}{w_{m,n}^2}\right] \\ &= \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left[\frac{1}{(z-w_{m,n})^2} - \frac{1}{w_{m,n}^2}\right] \\ &= f(z). \end{aligned}$$

□

10 2005

(2005 #1) Compute $\int_0^\infty \frac{dx}{1+x^n}$ where n is an even, positive integer. *Hint: Find a contour that surrounds only one pole.*

Proof. Let $f(z) = \frac{1}{1+z^n}$. Let $I(n) = \int_0^\infty \frac{dx}{1+x^n}$. Let L_R be the line segment connecting 0 and $Re^{(2\pi/n)i}$, and let Γ_R be the partial circle connecting R and $Re^{(2\pi/n)i}$. Notice that $\zeta_1 = e^{(\pi/n)i}$ is the only pole of $f(z) = \frac{1}{1+z^n}$ inside the contour $\Gamma = L_R + \Gamma_R + [0, R]$. Also, we have

$$\text{Res}(f, e^{(\pi/n)i}) = \lim_{z\to\zeta_1} \frac{d}{dz} f(z) = \frac{e^{-\pi i/n}}{n}.$$

Let $I_\epsilon(n) = \int_\Gamma f(z)dz$. By the residue theorem we then have

$$n\zeta_1^{n-1} = \frac{1}{2\pi i}\int_\Gamma f(z)dz.$$

We split the integral up into each component and make an estimate on Γ_R:

$$\frac{1}{2\pi i}\int_{\Gamma_R} f(z)dz = \frac{1}{2\pi i}\int_{\Gamma_R} \frac{iRe^{i\theta}}{1+R^n e^{in\theta}} d\theta \le \frac{R^2}{n+\epsilon}\frac{1}{|1-R^n|} \to 0.$$

And if we parametrize L_R, we can write

$$\int_{L_R} f(z)dz = \int_0^R \frac{e^{(2\pi/n)i}}{1+t^n e^{2\pi i}} dt \to e^{(2\pi/n)i} I(n), \qquad R\to\infty.$$

Accounting for orientation and applying the Residue Theorem it follows that,

$$(1 - e^{2\pi i/n})I = 2\pi i \,\text{Res}(e^{i\pi/n}, f).$$

Solving gives

$$I = -\frac{2\pi i}{n}\frac{e^{\pi i/n}}{1-e^{2\pi i/n}} = \frac{\pi}{n}\frac{2i}{e^{\pi i/n} - e^{-\pi i/n}} = \frac{\pi}{n}\frac{1}{\sin(\pi/n)}.$$

□

(2005 #2) Find all Taylor and Laurent series of $\frac{1}{(z-a)(z-b)}$ centered at $z = 0$, where $0 < |a| < |b|$.

Proof. We can write $\frac{1}{z-a} - \frac{1}{z-b} = \frac{a-b}{(z-a)(z-b)}$. Hence

$$f(z) = \frac{1}{a-b}\left(\frac{1}{z-a} - \frac{1}{z-b}\right).$$

So, we write out two Laurent series. First we assume z is small ($|z| < |a|$).

$$\begin{aligned}\frac{1}{z-a} &= -\frac{1}{a}\cdot\frac{1}{-\frac{z}{a}+1} \\ &= -\frac{1}{a}\left(1 + \frac{z}{a} + \left(\frac{z}{a}\right)^2 + \cdots\right)\end{aligned}$$

When $|z| > |a|$ we have

$$\frac{1}{z-a} = -\frac{1}{z} \cdot \frac{1}{-\frac{a}{z}+1}$$
$$= -\frac{1}{z}\left(1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \cdots\right)$$

Similarly for b. Now we have to combine for the various domains... □

(2005 #3) Let G be a domain and $f_n : G \to \mathbb{C}$ a sequence of analytic functions such that $f_n \to f$ converges for every $z \in G$. Suppose there are analytic functions g_n with $|g_n| \le 1$ and $|f_n - g_n| \ge 1$ in G. Prove that f_n converges uniformly on compact subsets of G.

Proof. Since $|g_n| \le 1$ we know from Montel's theorem that there exists a convergent subsequence $g_{n_k} \to g$, so it suffices to consider the case $|f_n - g| \ge 1$ with g analytic in G. Let $\psi_n = \frac{1}{f_n - g}$. Then $|\psi_n| \le 1$ and so ψ_n is analytic on G. Moreover, because the ψ_n are uniformly bounded we can apply Montel's theorem and extract a locally uniformly convergent subsequence $\psi_{n_k} \to \psi$.

Since $|f_n - g_n| \ge 1$ we know that the ψ_{n_k} and ψ are zero free. Hence on a compact set K it holds that $\frac{1}{\psi}, \frac{1}{\psi_{n_k}} < M_K$. We now have $f_{n_k} = g + \frac{1}{\psi_{n_k}}$ is bounded on K by $M_K + \sup_K g(z)$ independent of k. It follows that the collection $\{f_{n_k}\}$ is locally uniformly bounded and thus normal. We can then extract a locally uniformly convergent subsequence $f_{n_{k_l}} \to f$. Since we can repeat this and obtain a convergent subsequence of every subsequence, it follows that $f_n \to f$ locally uniformly.

Though we can just cite this fact, we prove it by supposing that $f_n \not\to f$ locally uniformly. Hence there exists K a compact set and $\epsilon > 0$ such that for every N there exists $n_N > N$ such that $\|f_{n_N} - f\|_{\infty,K} > \epsilon$. A contradiction since this subsequence has no convergent subsequences.

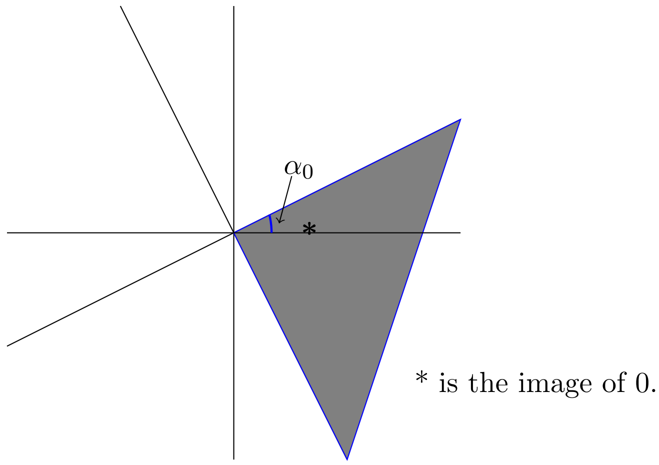
□

(2005 #4) Let $U = \{|z-1| < 2\} \cap \{|z+1| < 2\}$. Find a conformal map from $U \to \mathbb{D}$.

Proof. Let $\alpha_0 = \frac{\pi}{2} - \arctan(1/2)$. We claim the composition

$$z \mapsto \frac{\sqrt{3}i - z}{\sqrt{3}i + z} \mapsto e^{i\alpha_0 z} \mapsto z^2 \mapsto \frac{i-z}{i+z}$$

is a map from $U \to \mathbb{D}$. To see this note that the two circles intersect at $\pm\sqrt{3}i$. It is easy to check that the tangent lines at these points are orthogonal. As this is a Möbius transformation it follows that the circles are mapped to orthogonal lines which intersect at the origin and ∞. The first map takes $2i + 1 \mapsto -(1/5)(2i+1)$ and $\arg(-(1/5)(2i+1)) = \arctan(1/2) = \alpha_0$. Lastly since $0 \mapsto 1$ we conclude that U is mapped to



Now that we know the angle α_0, the other maps do the cleanup and get us to the disk. □

(2005 #5) Suppose that f is analytic in $H = \{x > 0\}$. Also, suppose that for $x + iy \in H$ it holds that $0 \leq \operatorname{Re} f(x + iy) \leq Mx$ for some $M > 0$. Show that $f(z) = mz + ic$ for constants $0 \leq m \leq M$ and $c \in \mathbb{R}$.

Proof. Our bound tells us that u is continuous on $\overline{H}$, so we can reflect to extend to an entire function f which satisfies $|u(x + iy)| \leq M|x|$ on all of $\mathbb{C}$. Next, pick z with $|z| \geq |v(0)|$ and let $r = 2|z|$. We can use the Hurglotz integral formula to write

$$\begin{aligned}
|f(z)| &= \left| \frac{1}{2\pi i} \int_{|\zeta|=r} \frac{\zeta + z}{\zeta - z} u(\zeta) \frac{d\zeta}{\zeta} + iv(0) \right| \\
&\leq \frac{1}{2\pi} \int_{|\zeta|=r} \frac{|\zeta| + |z|}{|\zeta| - |z|} |u(\zeta)| \frac{d|\zeta|}{|\zeta|} + |v(0)| \\
&\leq \frac{3}{2\pi r} \int_{|\zeta|=r} |u(\zeta)| d|\zeta| + |v(0)| \\
&\leq 3Mr + |v(0)|.
\end{aligned}$$

Where $|u(\zeta)| \leq M|\operatorname{Re}\zeta| \leq M|\zeta|$. We conclude that

$$\begin{aligned}
|f(z)| &\leq 3Mr + |v(0)| \\
&\leq 3M2|z| + |z| \\
&\leq (6M + 1)|z|.
\end{aligned}$$

Therefore $f(z) = mz + b$. The last thing we do is check when $z = 0$ we have $\operatorname{Re} b = 0$ and so $b = ic$, when $z = i$ we have $\operatorname{Im} m = 0$ and so $m \in \mathbb{R}$ and when $z = 1$ we have $0 \leq m \leq M$. □

ALTERNATE PROOF

Proof. It suffices to prove the result on the domain $U = \{x + iy : y > 0\}$ where $\operatorname{Im} f(x + iy) \leq My$. For any sequence $z_n \subseteq U$ with $z_n \to \mathbb{R}$ we know that $f(z_n) \to 0$. We can apply Schwarz refection and extend to an entire function F given by $F(\overline{z}) = \overline{f(z)}$ for $z \in U$. Notice that for any $x - iy \in U^c$ we have

$$\operatorname{Im} F(x - iy) = \operatorname{Im} \overline{f(x + iy)} = -\operatorname{Im} f(x + iy).$$

Therefore $0 \geq \operatorname{Im} F(x - iy) \geq My$. It follows that F is an analytic function which satisfies

$$|\operatorname{Im} F(x + iy)| \leq M|y| \tag{7}$$

Write $F(z) = \sum_0^\infty a_k z^k$. Recall that Caratheodory's Inequality guarantees for any $0 < r < R$ we have

$$\begin{aligned} |f(z)| &\leq \frac{2r}{R-r} \sup_{|z|<R} \operatorname{Im} f(z) + \frac{R+r}{R-r}|f(0)| \\ &\leq \frac{2r}{R-r} MR + \frac{R+r}{R-r}|f(0)|. \end{aligned}$$

Writing $a_k = \frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{k+1}} dz$ we can apply this bound

$$|a_k| \leq \frac{1}{r^k} \left[\frac{2r}{R-r} MR + \frac{R+r}{R-r}|f(0)| \right].$$

Let $R = 2r$. The above becomes

$$|a_k| \leq \frac{1}{r^k} \left[4Mr + 3|f(0)| \right].$$

For any $k > 1$ the above goes to zero as $r \to \infty$. Therefore F has the form $F(z) = mz + ic$. Lastly, note that $0 \leq m \leq M$ otherwise we would have a quick contradiction of the inequality at (7).

□

(2005 #6) Let f be an analytic function in $\mathbb{D}(0, r)$ and let $\gamma : [a, b] \to \mathbb{C}$ be a curve with $\gamma(a) \in \mathbb{D}(0, r)$.

(a) Write down the definition that f has an analytic along γ.

(b) Now suppose that f has an analytic extension along every line segment through 0. Prove that f has an analytic extension to all of $\mathbb{C}$.

Proof. f has an **analytic continuation along** γ if there exists a partition of $a = t_0 < t_1 < \cdots < t_n = b$ and a sequence $r_0, \ldots, r_n > 0$ such that f extends to an analytic function f_i on each disk $\Delta_i = \mathbb{D}(\gamma(t_i), r_i)$ and $f_i = f_j$ on $\Delta_i \cap \Delta_j$.

Suppose f has an analytic continuation along every line segment through 0. Let L_z be the line segment through 0 and z and let $\gamma_z : [0, 1] \to L_z$. Define $F(z)$ to be the analytic continuation $f(\gamma_z(1))$. This guarantees that F has a power series expansion at each z. However, we need to check that F is well defined. Fix a point $z_0 \in \mathbb{C}$ and let $\Delta_0 = \mathbb{D}(z_0, r_0)$ be a neighborhood of z_0. We call $\tilde{F}(z) = \sum_{n=0}^\infty a_n z^n$ the power series expansion about z_0 in Δ_0 and need to show that for any $w \in \Delta_0$ it holds that $F(w) = \tilde{F}(w)$. Let $\sigma_{z_0,w}$ be a path following the straight line connecting z_0 and w. We then have $L_{z_0} + \sigma_{z_0,w}$ is a path from 0 to w. Since $\sigma_{z_0,w} \in \Delta_0$ we can extend f analytically along this path and the value is $\tilde{F}(w)$. Also, the paths L_w and $L_{z_0} + \sigma_{z_0,w}$ are homotopic. It follows from the monodromy theorem that $\tilde{F}(w) = F(w)$. Therefore F is well defined and an analytic extension to all of $\mathbb{C}$. □

(2005 #7) Write down an infinite product that converges to an entire function $f(z)$ with zeros of order 1 at the points $z_n = \sqrt{n}$ with $n = 1, 2, \ldots$. Prove the convergence of the product.

Proof. Consider the product

$$f(z) = \prod_{n=1}^{\infty}(1 - \frac{z}{\sqrt{n}} \exp\left(\frac{z}{\sqrt{n}} + \frac{z^2}{n}\right).$$

To prove that the product converges we use the fact that $\prod a_n$ converges if and only if $\sum \log a_n$ converges (locally uniformly). Fix a closed disk $\overline{\mathbb{D}}(0,R)$ and let $a_n = (1 - \frac{z}{\sqrt{n}}) \exp\left(\frac{z}{\sqrt{n}} + \frac{z^2}{n}\right)$. We have

$$\begin{aligned}
|\log a_n| &= |\log(1 - \frac{z}{\sqrt{n}}) + \frac{z}{\sqrt{n}} + \frac{z^2}{n}| \\
&= |-(z/\sqrt{n} + z^2/n + z^3/\sqrt{n}^3 + \cdots) + z/\sqrt{n} + z^2/n^2| \\
&= |z^3/\sqrt{n}^3 + z^4/\sqrt{n}^4 + \cdots| \\
&\sim \mathcal{O}\left(\frac{z^3}{n^{3/2}}\right).
\end{aligned}$$

We make the final conclusion by letting $b_k = \frac{1}{n^{k/2}}$ an noticing that for $k > 3$ we have

$$\lim_{n\to\infty} \frac{b_k}{\frac{1}{n^3/2}} = \lim_{n\to\infty} \frac{1}{n^{k-\frac{k}{2}}} = 0.$$

Noting that on $\overline{\mathbb{D}}(0,R)$ with $R > 1$ we can take $n > R^3$ and dominate each term $b_k z^k$ by $\frac{R^k}{(R^3)^{k/2}} = \left(\frac{1}{\sqrt{R}}\right)^k$. Since $\frac{1}{R} < 1$ we know that $\sum \left(\frac{1}{\sqrt{R}}\right)^k$ converges and so by dominated convergence we have the sum $\sum_{k=3}^{\infty} b_k z^k$ converges to $\frac{z^2}{n^{3/2}}$.

We have thus shown that $\sum \log a_n \sim \sum \frac{z^n}{n^{3/2}} < \infty$, as desired. $\square$

Proof. Consider the function f given by

$$f(z) = \prod_{n=1}^{\infty}(1 - \frac{z}{\sqrt{n}}) \exp\left(\frac{z}{\sqrt{n}} + \frac{z^2}{2n}\right).$$

We will show that this product is convergent for $|z| < R$ for any fixed R. Consider the tail

$$\prod_{\sqrt{n}>2R}(1 - \frac{z}{\sqrt{n}}) \exp\left(\frac{z}{\sqrt{n}} + \frac{z^2}{2n}\right).$$

Since none of the terms are zero, we can prove that the log of the product is convergent.

$$\sum_{\sqrt{n}>2R}\left|\log\left((1 - \frac{z}{\sqrt{n}}) \exp\left(\frac{z}{\sqrt{n}} + \frac{z^2}{2n}\right)\right)\right| = \sum_{\sqrt{n}>2R} |\log\left(1 - \frac{z}{\sqrt{n}}\right) + \frac{z}{\sqrt{n}} + \frac{z^2}{2n}|.$$

Consider $\log(1-w)+w+\frac{w^2}{2} = -\sum_{d=3}^{\infty}\frac{1}{d}w^d$. Since the power series of $\log(1-w)$ is analytic on $\mathbb{D}$ we know that

$$g(w) = \frac{\log(1-w)+w+\frac{w^2}{2}}{w^3}$$

is bounded on $\{|w| \le \frac{1}{2}\}$ and therefore bounded by C. By the maximum modulus theorem we know that $|g(w)|$ attains C at some $|w_0| = \frac{1}{2}$. Since $\sqrt{n} > 2R$ we know that $|z/\sqrt{n}| < \frac{1}{2}$ and therefore

$$\left|\log(1-z/\sqrt{n}) + z/\sqrt{n} + z^2/(2n)\right| \le C\left|\frac{z}{\sqrt{n}}\right|^3 \le C\left|\frac{R^3}{n^{3/2}}\right|.$$

We can then write

$$\sum_{\sqrt{n}>2R}\left|\log(1-z/\sqrt{n}) + z/\sqrt{n} + \frac{z^2}{2n}\right| \le \sum_{\sqrt{n}>2R} CR^3\frac{1}{n^{3/2}}M\infty.$$

It follows that

$$\prod_{\sqrt{n}>2R}(1-\frac{z}{\sqrt{n}})\exp(z/\sqrt{n}+z^2/(2n))$$

converges absolutely and uniformly on $|z| < R$ and so by the M-test this means the product from $n=1$ to ∞ also converges and is therefore analytic on $|z| < R$. Since this holds for any R we have an entire function. □

(2005 #8) Let A denote the annulus $\{r_1 < |z| < r_2\}$.

(a) Construct a harmonic function h on A such that H is continuous up to the boundary and $h = 0$ on $\{|z| = r_1\}$ and $h = 1$ on $\{|z| = r_2\}$.

(b) Let u be harmonic in A and continuous up to the boundary and set $m_j = \max_{\{|z|=r_j\}} u(z)$ for $j = 1, 2$. Find the best possible upper bound for $u(z)$ in terms of m_1, m_2, r_1, r_2 and z.

Proof. **Part a.**

Let $h(z) = \frac{\log|z/r_1|}{\log|r_2/r_1|}$. This is harmonic since $\log|f|$ is harmonic on $\{f \neq 0\}$. □

Part b.

Let $h(z)$ be as in part (a). Define $g(z) = \frac{\log|z/r_1|}{\log|r_2/r_1|}$. Consider the function $f(z) = m_2h(z) + m_1g(z)$. Notice that $|f| = m_j$ on $|z| = r_j$. By the maximum principle we have $u(z) - f(z) \le 0$ on the boundary and therefore it holds on ∂A. Since f is harmonic we have a sharp bound.

□

□

11 2006

(2006 #1) Prove that all of the zeros of the polynomial $p(z) = z^n + c_{n-1}z^{n-1} + \cdots + c_1 z + c_0$ lie in the disc $\mathbb{D}(0, R)$ with $R = \sqrt{1 + |c_{n-1}|^2 + \cdots + |c_1|^2 + |c_0|^2}$.

Proof. Proof. Proof. We claim that for $|z| = R$ it holds that $|p(z) - z^n| < |z^n|$. It would then follow from Rouche's theorem and the Fundamental Theorem of Algebra that all of the zeros of p lie inside $\mathbb{D}(0, R)$. To prove the claim we note that

$$|p(z) - z^n| = |c_{n-1}z^{n-1} + \cdots + c_0| \leq |c_{n-1}||z^{n-1}| + \cdots |c_0| = \sum_{k=0}^{n-1} |c_k| R^k.$$

Hence it suffices to prove that $\sum_{k=0}^{n-1} |c_k| R^k < R^n = |z^n|$. Let $\mathbf{c} = (|c_0|, \ldots, |c_{n-1}|) \in \mathbb{R}^n$ and let $\mathbf{R} = (1, R, R^2, \ldots, R^{n-1})$. Notice that $\sqrt{1 - R^2} = |\mathbf{c}|$. Also, we can write

$$\sum_{k=0}^{n-1} |c_k| R^k = \mathbf{c} \cdot \mathbf{R}.$$

The Cauchy-Schwarz inequality guarantees that

$$|\mathbf{c} \cdot \mathbf{R}| \leq |\mathbf{c}||\mathbf{R}| = \sqrt{1 - R^2}|\mathbf{R}|.$$

Lastly we use the geometric series summation formula to make the calculation,

$$|\mathbf{R}| = \sqrt{\sum_{k=0}^{n-1} R^{2k}} = \sqrt{\frac{1 - R^{2n}}{1 - R^2}}.$$

It follows that $|\mathbf{c}||\mathbf{R}| < |R|^n$. □

(2006 #2) Prove that there exists a sequence of polynomials p_k such that

$$\lim_{k\to\infty p_k(z)} = \begin{cases} 1, & \operatorname{Re} z > 0 \\ 0, & \operatorname{Re} z = 0 \\ -1, & \operatorname{Re} z < 0 \end{cases}.$$

Proof. Recall that Runge's theorem states that

Theorem (Runge). *If $D \subseteq \mathbb{C}$ is open and $A \subseteq \mathbb{C}^* \setminus D$ intersects every component of $\mathbb{C}^* \setminus D$. If $f \in H(D)$ then there exists a sequence of rational functions $\{R_j\}$ such that $R_j \to f$ uniformly on compact subsets of D.*

That said, we let $K_n = K_n^+ \cup K_n^0 \cup K_n^-$ where K_n^+ is the closed rectangle with vertices $\pm n + \frac{i}{n}, \pm n + in$, K_0 is the segment $[-n, n] \subseteq \mathbb{R}$ and K_n^- is the closed rectangle with vertices $\pm n - \frac{i}{n}, \pm n - in$. Let $f_n : K_n \to \mathbb{C}$ be given by

$$f_n(z) = \begin{cases} 1, & z \in K_n^+ \\ 0, & z \in K_n^0 \\ -1, & z \in K_n^- \end{cases}.$$

Since $\mathbb{C}^* \setminus K_n$ is connected and f_n is holomorphic in a neighborhood of K_n Runge's theorem guarantees that there exists a polynomial p_n such that

$$|f(z) - p_n(z)| < 1/n, \qquad z \in K_n,$$

which says that $|1 - p_n| < 1/n$ on K_n^+, $|p_n| < 1/n$ on K_n^0 and that $|-1-p_n| < 1/n$ on K_n^-. But if $\operatorname{Re} z > 0$ then there exists N such that $z \in K_n^+$ for all $n > N$; therefore $p_n(z) \to 1$. Similarly, if $\operatorname{Re} z = 0$ then $z \in K_n^0$ for large n and so $p_n(z) \to 0$. Lastly, if $\operatorname{Re} z < 0$ then $p_n(z) \to -1$. □

(2006 #3) If $f : \mathbb{D} \to \mathbb{D}$ has $f \in H(\mathbb{D})$, $f(0) = 0$ and $f'(0) = \frac{1}{2}$ show $\mathbb{D}(0, 7 - 4\sqrt{3}) \subseteq f(\mathbb{D})$. (*Hint: Proof of Riemann mapping theorem.*)

Proof. Assume that there exists p such that $|p| < 7 - 4\sqrt{3}$ and $p \notin f(\mathbb{D})$. Note that $p \neq 0$. Define

$$T_p(z) = \frac{z - p}{1 - \bar{p}z} : p \mapsto 0.$$

We see that $T_p \circ f$ is zero free, so we can take a square root. Let g be such that $g(z)^2 = (T_p \circ f)(z)$ and define $g(0) = b$. We get that $b^2 = -p$ hence $|b|^2 = |p|$. Define the map $T_b(z) = \frac{z-b}{1-\bar{b}z}$ and $\varphi = T_b \circ g$. Notice that $\varphi : \mathbb{D} \to D : 0 \mapsto 0$ and so by Schwarz's lemma $|\varphi'(0)| \leq 1$. We have

$$|\varphi'(0)| = \frac{1}{1 - |b|^2} \cdot \frac{(1 - |p|^2)\frac{1}{2}}{2|b|} = \frac{1}{1 - |p|} \frac{(1 - |p|^2)\frac{1}{2}}{2|b|} = \frac{1 + |p|}{4|b|} = \frac{1 + |p|}{4\sqrt{|p|}} \leq 1.$$

Hence $1 + |p| - 4\sqrt{|p|} \leq 0$, solving gives $|p| \geq 2 - \sqrt{3}$ and so $|p| \geq 7 - 4\sqrt{3}$.

□

(2006 #4) Let $\Omega \subseteq \mathbb{C}$ be an open domain. We say that u is strictly subharmonic on Ω if

$$u(z_0) < \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + \rho e^{i\theta}) d\theta, \qquad \forall z_0 \text{ such that } \{|z - z_0| \leq \rho\} \subseteq \Omega.$$

Show that if f is a nonconstant analytic function on Ω then $|f(z)|$ is strictly subharmonic.

Proof. The Cauchy integral formula guarantees that for any $z \in \Omega$ we have

$$|f(z)| = \left| \frac{1}{2\pi} \int_0^{2\pi i} f(z + \rho e^{i\theta}) d\theta \right| \leq \frac{1}{2\pi} \int_0^{2\pi i} |f(z + \rho e^{i\theta})| d\theta.$$

We know that

Lemma 2. $\left| \int_a^b f(x)dx \right| = \int_a^b |f(x)|dx$ *implies that there exists* $\alpha \in \mathbb{C}$ *such that* $f(x)/\alpha \in [0, \infty)$.

Hence, taking the absolute value sign inside the integral yields a strict inequality only if the argument of f is nonconstant. So we write $f/\alpha = u + iv$ and so $v \equiv 0$ on $\partial\mathbb{D}(z_0, \rho)$. Since v is harmonic we can apply the maximum and minimum principle and conclude that $v \equiv 0$ on $\overline{\mathbb{D}(z_0, \rho)}$. By the CR equations we have $u \equiv c$ with $c \in \mathbb{C}$. So f is constant on Ω. □

(2006 #5) Let $f(z)$ be an entire function with only finitely many zeros. Define $m(r) = \min_{|z|=r}|f(z)|$. Show that if f is not a polynomial then $m(r) \to 0$ as $r \to \infty$.

Proof. Let $\Delta = \overline{\mathbb{D}(0,R)}^c$ be such that f is zero free on Δ. Suppose that $m(r) \not\to 0$ as $r \to \infty$. Hence there exists $\epsilon > 0$ and a strictly increasing sequence $r_k \to \infty$ such that $m(r_k) > \epsilon$ and $r_1 > R$. Let $A_{j,k} = \{z : r_j \leq |z| \leq r_k, j < k\}$. Since f is zero free on all of the $A_{j,k}$ we can apply the minimum-modulus theorem and conclude that

$$\inf_{z \in A_{j,k}} |f(z)| = \inf_{|z|=r_j \text{ or } |z|=r_k} |f(z)| > \epsilon.$$

Since the collection $A_{j,k}$ covers $\Delta_2 := \mathbb{C}\backslash\overline{\mathbb{D}(0,r_1)}$ it follows that $\mathbb{D}(0,\epsilon)\cap f(\Delta_2) = \emptyset$. This contradicts Big Picard which necessitates that $f(\Delta_2)$ omits at most one point. □

(2006 #6) Let $\Omega = \mathbb{D}(0,2)$ and let $\Omega' = \Omega \setminus [0,1]$.

(a) Prove that if $f \in C(\Omega)$ and f is analytic on Ω' then $f \in H(\Omega)$.

(b) Give an explicit example of function which is bounded and analytic on Ω' which cannot be extended to an analytic function on Ω.

Part a.

Proof. By Morera's theorem we need only show that $\int_{\partial R} f(z)dz = 0$ for all rectangles $R \subseteq \Omega$. So we let $R = \{z : \text{Re}(z) \in (a,b), \text{Im}(z) \in (c,d)\}$. We consider several cases:

(i) If $c > 0$ we know from Cauchy's theorem that $\int_{\partial R} = 0$.

(ii) If $c = 0$, let $\epsilon > 0$. Define $R_\epsilon^+ = \{\text{Re}\, z \in (a,b), \text{Im}\, z \in (\epsilon, d)\}$ and $R_\epsilon^0 = R \setminus R_\epsilon^+$. It follows that

$$\int_{\partial R} f(z)dz = \int_{\partial R_\epsilon^+} f(z)dz + \int_{\partial R_\epsilon^0} f(z)dz = \int_{\partial R_\epsilon^0} f(z)dz.$$

We would like to show that the leftmost integral tends to 0 as $\epsilon \to 0$. Let $M = \sup_{z\ in \overline{R}} |f(z)|$. We can use an ML-estimate on the vertical sides of the rectangle

$$\left|\int_{\partial R} f(z)dz\right| \leq 2M\epsilon + \int_a^b |f(t) - f(t+i\epsilon)|dt$$

By absolute continuity the leftmost integral goes to 0 as $\epsilon \to 0$.

(iii) All of the other cases are analogous to the (i) and (ii).

□

Part b.

Proof. Consider the function $f(z) = \sqrt{z(z-1)}$. We let $g(z) = \sqrt{z}$ be analytic on $\mathbb{C} \setminus (-\infty, 0]$ and $h(z) = \sqrt{z-1}$ analytic on $\mathbb{C} \setminus (-\infty, 1]$. Since f is the product of these functions it is analytic on $\mathbb{C} \setminus (-\infty, 1]]$. Moreover, we claim that f is continuous on $\Omega \setminus [0, 1]$. To prove this we choose z_0 near $[-1, 0]$ and show that $f(z_0)$ is small.

A alternative would be to solve the Dirichlet problem on the half circle of radius $\sqrt{2}$ taking the boundary data to be 0 on the circle union $[-\sqrt{2}, -1] \cup [0, \sqrt{2}]$ and smoothly nonzero on $[-1, 0]$. We take the harmonic conjugate so the function is bounded and analytic on the desired domain.

Another alternative is to let $f(z) = z(\log z + \log(1 - z))$, with both $\log z$ and $\log(1 - z)$ having branch cuts on $(-\infty, 0]$. □

> **(2006 #7)** Let $K \subseteq \mathbb{C}$ be countable and closed. Prove that if f is bounded and analytic on $\mathbb{C} \setminus K$ then f is constant.

Proof. We mimic the proof of Liouville's theorem. Since K is countable we know that there exists arbitrarily large $R > 0$ such that $C_R = \partial\mathbb{D}(0, R)$ satisfies $C_R \cap K = \emptyset$. Given such an R there exists δ such that the annulus $\{R - \delta < |z| < R\} \cap K = \emptyset$. By compactness of $\overline{\mathbb{D}(0, R - \delta)}$ we can obtain a finite cover of the points in K by discs. Let $z_0 \in \mathbb{C} \setminus K$ and let $\Delta_R = \mathbb{D}(z_0, R)$. Let Γ be the combined circles and single circles boundaries. Since $C_R - \Gamma$ is a curve homologous to zero (it doesn't wind any points in K). Let $|f(z)| < M$ for $z \in \mathbb{C} \setminus K$. Hence f is analytic on the region just defined. We can apply the Cauchy integral formula and write for any $z \in \mathbb{D}(0, R) \setminus K$ that

$$f(z) = \int_{C_R} \frac{f(\zeta)}{\zeta - z} d\zeta + \int_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Now that we are only deleting finitely many disks, we can apply Cauchy's derivative formula and the ML estimate to obtain

$$|f'(z_0)| = \left| \frac{1}{2\pi} \int_{\partial\Delta_R} \frac{f(z)}{(z - z_0)^2} dz \right| \leq \frac{M}{R}.$$

Letting $R \to \infty$ we have $|f'(z_0)| = 0$ and thus f is constant on $\mathbb{C} \setminus K$. But the rightmost derivative goes to zero as $\epsilon \to 0$.

□

> **(2006 #8)** Let $\Omega \subseteq \mathbb{C}$ be a connected open set with $0 \in \Omega$. Suppose that $u_n(z)$ is a sequence of positive harmonic functions on Ω and $\lim_{n\to\infty} u_n(0) = 0$. Prove that u_n converges uniformly to 0 on any compact $K \subseteq \Omega$.

Proof. We claim that $\{u_n\}$ is locally uniformly bounded and thus a normal family (by Montel's theorem). Fix K a compact set and a number $\epsilon > 0$. Without loss of generality suppose that $0 \in K$. Let N be such that for all $n > N$ it holds that $u_n(0) < \epsilon$. It follows from Harnack's inequality that there exists a constant C_K such that for all $n > N$ we have

$$\sup_{z\in K} u_n(z) \leq C_K \inf_{z\in K} u_n(z) \leq C_K u_n(0) < C_K \epsilon.$$

Let $A_K = \sup_{z \in K, j \leq N} u_j(z)$. Notice that $A_K < \infty$ since K is compact and the collection $j \leq N$ is finite. We now have for all n

$$\sup_{z \in K} u_n(z) \leq \max\{C_K \epsilon, A_K\},$$

and so $\{u_n\}$ is locally bounded and thus normal.

By normality, any subsequence $\{u_{n_k}\}$ contains a convergent subsequence $u_{n_{k_l}} \to u$. Since $\lim_{l \to \infty} u_{n_{k_l}}(0) = 0$ we conclude that $u(0) = 0$ and by Harnack's inequality must be identically zero on K. Since every subsequence contains a convergent subsequence to 0 we conclude that u_n converges locally uniformly to zero.

□

12 2007

(2007 #1) Calculate $\int_0^\infty \frac{\cos(ax)}{(1+x^2)^2} dx$ for $a > 0$.

Proof. Let $I(a) = \int_0^\infty \frac{\cos(ax)}{(1+x^2)^2} dx$. We compute a contour over the upper semi-circle. Let $C_R := \{z : |z| = R, \operatorname{Im} z \geq 0\}$, and let $\Gamma_R = C_R \cup [-R, R]$. Consider the function $f(z) = \frac{e^{aiz}}{(1+z^2)^2}$ and let $J_R(a) = \int_{\Gamma_R} f(z)dz$. Notice that f has a double pole at i with residue

$$\begin{aligned}
\operatorname{Res}(f, i) &= \lim_{z \to i} \frac{d}{dz} (z-i)^2 f(z) \\
&= \lim_{z \to i} \frac{d}{dz} \frac{e^{aiz}}{(z+i)^2} \\
&= \lim_{z \to i} \frac{aie^{az}(z+i)^2 - 2(z+i)e^{aiz}}{(z+i)^4} \\
&= \frac{aie^{-a}(2i)^2 - 2(2i)e^{-a}}{(2i)^4} \\
&= -i \frac{e^{-a}}{4}[a+1].
\end{aligned}$$

We can write

$$J_R(a) = \int_{C_R} f(z)dz + \int_{-R}^0 f(z)dz + \int_0^R f(z).$$

Applying the ML-estimate to the first integral gives

$$\left| \int_{C_R} f(z)dz \right| \leq \pi R \sup_{z \in C_R} \frac{|e^{iz}|}{|1+z^2|^2} \overset{*}{\leq} \frac{\pi R^2}{|1-R^2|^2} \to 0, \qquad R \to \infty.$$

Where the inequality at * follows from the triangle inquality for the denominator and the numerator bound arises the fact that

$$\sup_{z \in C_R} |e^{iz}| = \sup_{\theta \in [-\pi, \pi]} |e^{iRe^{i\theta}}| = \sup_{\theta \in [-\pi, \pi]} \operatorname{Re}(|e^{iR\cos\theta - R\sin\theta}|) \leq e^{-R} \leq R, \qquad (R \text{ suff. large}).$$

We conclude then that

$$2\pi \frac{e^{-a}}{4} = \operatorname{Re}(\lim_{R \to \infty} J_R(a)) = \int_{-\infty}^0 \operatorname{Re}(f(z))dz + \int_0^\infty \operatorname{Re}(f(z))dz \overset{**}{=} 2I(a).$$

Where the equality at ** follows from the fact that $\text{Re}(f(z))$ is an even function. It follows that

$$I(a) = \frac{\pi e^{-a}}{4}[a+1].$$

$\square$

(2007 #2) Find all conformal maps from the open unit disk $\mathbb{D}$ onto the region

$$U := \{z : |z| < 1 \text{ and } |x - (1/2)| > 1/2\}.$$

Proof. First we find one map.

$$z \mapsto z - 1 \mapsto -z \mapsto \frac{1}{z} \mapsto \pi e^{-i\pi}(z - i) \mapsto e^z \mapsto \frac{z-i}{z+i}.$$

$f_1(z) = z - 1$ translates. $f_2(z) = -z$ reflects across the imaginary axis. $f_3(z) = \frac{1}{z}$ takes us to the vertical strip $\{z : 1/2 < \text{Re}\, z < 1\}$. $f_4(z) = \pi e^{-i\pi}(z - i)$ takes us to the strip $\{z : 0 < \text{Im}\, z < \pi\}$. Next $f_5(z) = e^z$ takes us to the UHP, and we finish with $f_6(z) = \frac{z-i}{z+i}$. Let g be the composition of all of the inverses.

To construct all of the maps, suppose there existed a different conformal map $f : \mathbb{D} \to U$. Then consider $h = g^{-1} \circ f^{-1} : \mathbb{D} \to \mathbb{D} : z \mapsto e^{i\alpha} \frac{z-a}{1-\bar{a}z}$. Hence $f(z) = g\left(\frac{z-a}{1-\bar{a}z}\right)$.

***Note. I think the composition $z \mapsto \frac{z}{z-1} \mapsto e^{i\frac{\pi}{4}} z \mapsto z^2 \mapsto \frac{1-z}{1+z}$ also works. Also, I know the mapping $z \mapsto \frac{1+z}{1-z} \mapsto e^{i\pi z} \mapsto \frac{z-i}{z+i}$ works. $\square$

(2007 #3) Let $p(z) = az^4 + bz + 1$ for $a, b \in \mathbb{R}$. Find the maximum number of roots of p in the annulus $\mathcal{A} := \{z : 1 < |z| < 2\}$ provided that $a \in [1, \pi]$ and $\in [2\pi - 2, 7]$.

Proof. First notice that on $|z| = 1$ we have $|az^4| = a \leq \pi$ and $|bz + 1| \geq 2\pi - 1$. Since $2\pi - 1 > \pi$, it follows from the maximum modulus principle that p has no zeros on $|z| = 1$. Moreover, from Rouche's theorem we can conclude that $az^4 + bz + 1$ has the same number of zeros in $|z| \leq 1$ as $bz + 1$. Since $b > 1$, we know $bz + 1$ has exactly 1 zero in $\mathbb{D}$ and thus p has one zero in $\mathbb{D}$.

On $|z| = 2$ we have $|az^4| = 16a \geq 16$ and that $|bz + 1| \leq 2b + 1 \leq 14 + 1 = 15$. Once again it follows that p is not zero on $|z| = 2$ and that p has the same number of zeros as az^4 on $|z| = 2$ (four zeros). It follows that p has three zeros on the annulus. $\square$

(2007 #4) Is there a 1-1 analytic map from the annulus $\Omega_1 := \{z : \frac{1}{2} < |z| < 1\}$ onto the punctured disk $\Omega_2 := \{0 < |z| < 1\}$?

Proof. No. Suppose that $\varphi : \Omega_2 \to \Omega_1$. Since $\varphi(\Omega_1)$ is bounded it follows that φ extends to be analytic at 0 and thus $\varphi : \mathbb{D} \to \Omega_1 \cup \{\varphi(0)\}$ is analytic. The open mapping theorem guarantees that $\varphi(0) \in \Omega_2$.Moreover, φ is injective since if there existed $p \in \Omega_1$ with $\varphi(p) = \varphi(0)$ then we would have disjoint neighborhoods $U_0, U_p \subseteq \mathbb{D}$ with $\varphi(U_0 \setminus \{\varphi(0)\}) \cap \varphi(U_p \setminus \{\varphi(p)\}) \neq \emptyset$. This contradicts injectivity of φ.

We now have $\varphi : \mathbb{D} \to \Omega_1$ bijectively and $\varphi : \mathbb{D} \setminus \{0\} \to \Omega_2$ bijectively, a contradiction. $\square$

(2007 #5)

(a) State and prove Schwarz's Lemma, including the case of equality.

(b) Given $f : \mathbb{D} \to \mathbb{D}$ an analytic function, show that

$$\frac{|f'(z)|}{1-|f(z)|^2} \leq \frac{1}{1-|z|^2}, \qquad \forall z \in \mathbb{D}.$$

Part a.

Proof.

Theorem. *Let $f : \mathbb{D} \to \mathbb{D}$ with $f(0) = 0$, then $|f(z)| \leq z$ and $|f'(0)| \leq 1$. Equality holds if and only if $f(z) = e^{i\theta}z$.*

Let $g(z) = \begin{cases} \frac{f(z)}{z}, & z \neq 0 \\ f'(0), & z = 0 \end{cases}$. Since $f(0) = 0$ we know that g has a removable singularity at $z = 0$ and thus extends to be analytic on $\mathbb{D}$. Notice that on $\partial\mathbb{D}(0,r)$ with $r < 1$ we have $|g(z)| \leq \frac{1}{r}$ by the maximum modulus principle. Letting $r \to 1$ implies that $|g(z)| \leq 1$ and so $|f(z)| \leq |z|$ and $|g(0)| = |f'(0)| \leq 1$.

If at some point $z_0 \in \mathbb{D}$ it holds that $|f(z_0)| = |z_0|$ or if $|f'(0)| = 1$ then g achieves its max in $\mathbb{D}$ and is constant by the maximum modulus principle. Hence $\frac{f(z)}{z} = e^{i\theta}$, completing the argument. □

Part b.

Proof. Fix $z_1 \in \mathbb{D}$. Let $T(z) = \frac{a_1 - z}{1-\overline{a}_1 z}$ and $U(z) = \frac{z_1 - z}{1-\overline{z}_1 z}$. Notice that $T \circ f \circ U : \mathbb{D} \to \mathbb{D} : 0 \mapsto 0$ and thus

$$|T \circ f \circ U| \leq |z|.$$

Composing with U^{-1} gives

$$\left| \frac{f(z_1) - f(z_2)}{1 - \overline{f(z_1)} f(z_2)} \right| \leq \left| \frac{z_1 - z_2}{1 - \overline{z}_1 z_2} \right|,$$

and dividing gives

$$\left| \frac{f(z_1) - f(z_2)}{(z_1 - z_2)} \right| \left| \frac{1}{1 - \overline{f(z_1)} f(z_2)} \right| \leq \left| \frac{1}{1 - \overline{z}_1 z_2} \right|,$$

Letting $z_2 \to z_1$ yields the desired result. □

(2007 #6) Let M_n, for all $n = 0, 1, 2, \ldots$ be a sequence of positive numbers and define the family of functions

$$\mathcal{F} := \{f : f \in H(\mathbb{D}) \text{ and } |f^{(n)}(0)| \leq M_n, \quad \forall n = 0, 1, 2, \ldots\}.$$

Show that $\mathcal{F}$ is a normal family if and only if $\sum_0^\infty \frac{M_n z^n}{n!}$ converges on $\mathbb{D}$.

Proof. ($\Rightarrow$) Suppose that $\mathcal{F}$ is normal. Let $f_k = \sum_0^k \frac{M_n}{n!} z^n$. Notice that $f_k \in \mathcal{F}$ since $|f_k^{(n)}(0)| = M_n$ for all $n \leq k$ and $|f_k^{(n)}(0)| = 0$ when $n \geq k$. Since $\mathcal{F}$ is normal there exists a convergent subsequence $\{f_{k_j}\} \to g$. Since the $f_k \in H(\mathbb{D})$ it follows that $g \in H(\mathbb{D})$. Moreover, our construction guarantees that $g(z) = \sum_1^\infty \frac{M_n}{n!} z^n$. Since $g \in H(\mathbb{D})$ it follows that the sum converges in $\mathbb{D}$.

($\Leftarrow$) Suppose that $\sum_0^\infty \frac{M_n}{n!} z^n$ converges in $\mathbb{D}$. Fix $K \subseteq \mathbb{D}$ a compact set. Let $f \in \mathcal{F}$. Writing the Taylor expansion yields

$$|f(z)| = \left| \sum_0^\infty \frac{f^{(n)}(0)}{n!} z^n \right| \leq \sum_0^\infty \frac{|f^{(n)}(0)|}{n!} |z|^n \leq \sum \frac{M_n}{n!} |z|^n \leq \sup_{z \in K} \sum \frac{M_n}{n!} |z|^n < \infty.$$

Therefore $\mathcal{F}$ is locally bounded and normal by Montel's theorem. □

(2007 #7) Prove that the function $f(z) = \sin z - z^2$ has infinitely many complex zeros.

Proof. Suppose, to show a contradiction, that f has finitely many zeros. Accordingly let $\Delta = \mathbb{D}(0, R)$, $R > 0$ be a sufficiently large disk such that f has no zeros on Δ^c. For $z \in \Delta^c$ we can write (since $0 \notin \Delta^c$)

$$f(z) = z^2 \left(\frac{\sin z}{z^2} - 1 \right).$$

Since f is never zero on Δ^c it follows that the function $\frac{\sin z}{z^2}$ never takes the value 1. Additionally, because $\sin z/z^2$ is odd it can never assume the value -1. Notice that $\sin z/z^2$ has an essential singularity at ∞. This contradicts Big Picard. □

(2007 #8) Given a subharmonic function $u : \mathbb{C} \to \mathbb{R}$, recall that

$$u(z) \leq \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta}) d\theta,$$

for any z and for any $r > 0$. Let u be a subharmonic function and let $M(r) = \sup_{|z|=r} u(z)$.

(a) Prove that for any $0 < r_1 \leq |z| \leq r_2$,

$$u(z) \leq \frac{\log r_2 - \log |z|}{\log r_2 - \log r_1} M(r_1) + \frac{\log |z| - \log r_1}{\log r_2 - \log r_1} M(r_2).$$

(b) Show that $\lim_{r \to \infty} \frac{M(r)}{\log r}$ exists (possibly infinite).

Part a.

Proof. Notice that $v(z) := \frac{\log r_2 - \log |z|}{\log r_2 - \log r_1} M(r_1) + \frac{\log |z| - \log r_1}{\log r_2 - \log r_1} M(r_2)$ is a harmonic function and thus $u(z) - v(z)$ is subharmonic and does not have any local extrema. Notice that for $|z_i| = r_i$ with $i = 1, 2$ we have $v(z_i) = M(r_i)$ and thus

$$u(z_i) - v(z_i) = u(z_i) - M(r_i) \leq 0.$$

Since the subharmonic function must attain it's maximum on the boundary it follows that $u(z) - v(z) \leq 0$ for all z.

□

Part b.

Proof. Fix r_1. Without loss of generality let us suppose that $M(r_1) = 0$ (which is fine since we can translate without effecting the existence of the limit). Also, by scaling assume $r_1 = 1$. We then have from part (a) that

$$M(|z|) \leq \frac{\log |z|}{\log r_2} M(r_2).$$

Dividing by $\log |z|$ gives

$$\frac{M(|z|)|}{\log |z|} \leq \frac{M(r_2)}{\log r_2}.$$

Since $r_2 > 1$, this says that the left side is an increasing function and thus has a limit. □

13 2008

(2008 #1) Compute $\int_0^\infty \frac{\log x}{(1+x^2)^2} dx$.

Proof. Let $f(z) = \frac{\log^2 z}{(1+z^2)^2}$. We will make a branch cut along the negative real axis and use a keyhole contour which omits the negative real axis. Let $\Gamma = C_R \cup C_r \cup L_1 \cup L_2$ and orient counter-clockwise. The residue theorem guarantees that

$$2\pi i[\text{Res}(f,i) + \text{Res}(f,-i)] = \int_\Gamma f(z)dz = \left(\int_{C_R} + \int_{C_\epsilon} + \int_{L_1} + \int_{L_2}\right) f(z)dz.$$

It isn't a lot of fun but by using the formula for a double pole $\text{Res}(f, z_k) = \lim_{z\to z_k} \frac{d}{dz}(z - z_k)^2 f(z)$ we obtain

$$2\pi i[\text{Res}(f,i) + \text{Res}(f,-i)] = 2\pi i \left[\left(-\frac{\pi}{4} + \frac{i\pi^2}{16}\right) - \left(\frac{\pi}{4} - \frac{i\pi^2}{16}\right)\right] = -i\pi^2.$$

Two simple ML estimates show that the integrals over C_R and C_r go to zero as $R \to \infty$ and $r \to 0$.

$$\left|\int_{C_R} f(z)dz\right| \le 2\pi R \sup_{C_R} \frac{|\log z|^2}{|1+z^2|^2} \le \frac{\log^2 R + \pi^2}{(R^2-1)^2} \to 0$$

and

$$\left|\int_{C_r} f(z)dz\right| \le 2\pi r \sup_{C_r} \frac{|\log z|^2}{|1+z^2|^2} \le \frac{\log^2 r + \pi^2}{(r^2-1)^2} \to 0$$

Next, we parametrize L_1 and L_2 via $z = -t \pm i\epsilon$; $dt = -dz$,

$$\int_{L_1} f(z)dz = -\int_\infty^0 \frac{\log^2(-t+i\epsilon)}{(1+(t+i\epsilon)^2)^2} dt \to \int_0^\infty \frac{(\log t + i\pi)^2}{(1+t^2)^2} dt$$

and also

$$\int_{L_2} f(z)dz = -\int_0^\infty \frac{\log^2(t-i\epsilon)}{(1+(t-i\epsilon)^2)^2} dt \to -\int_0^\infty \frac{(\log t - i\pi)^2}{(1+t^2)^2} dt.$$

It follows that as $\epsilon \to 0$

$$\left(\int_{L_1} + \int_{L_2}\right) f(z)dz = \int_0^\infty \frac{(\log t + i\pi)^2 - (\log t - i\pi)^2}{(1+t^2)^2} dt = \int_0^\infty \frac{4\pi i \log t}{(1+t^2)^2} dt.$$

It follows that $4\pi i \int_0^\infty \frac{\log t}{(1+t^2)^2} dt = -i\pi^2$. Therefore the answer is $-\pi/4$. □

(2008 #2) Suppose that $f\colon \mathbb{C} \to \mathbb{C}$ is an entire, non-constant function satisfying

$$|f(z)| \le e^{\sqrt{|z|}}, \qquad \forall z \in \mathbb{C}.$$

For $R > 0$ let $n(R)$ denote the cardinality of the set $\{|z| \le R : f(z) = 0\}$. Prove there exist constants $A, B > 0$ such that for all $R > 0$ it holds that

$$n(R) \le A + B\sqrt{R}.$$

Proof. Let $Z = \{z \in \mathbb{C} : f(z) = 0\}$. It suffices to prove that there exists an r_0 such that for all $R > r_0$ it holds that $n(R) - n(r_0) \leq B\sqrt{r}$ for some constant B. This is sufficient since if such an r_0 exists we could take $A = n(r_0)$ and obtain the result.

Recall that Jensens formula states that for $N(r) = \sum_1^{n(r)} \log\left(\frac{r}{|z_k|}\right)$ with $|z_k| < r$ and $f(z_k) = 0$ we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(re^{i\theta})|d\theta = N(r) - \log|f(0)|.$$

Let $b = \log|f(0)|$, and without loss of generality suppose that $f(0) \neq 0$ (this is okay since if $f(0) = 0$ we instead consider $\frac{1}{z^k}f(z)$). The bound in the hypothesis tells us that

$$N(r) + \log|f(0)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \sqrt{r}d\theta = \sqrt{r}. \tag{1}$$

We seek to bound $N(r) = \sum_1^{n(r)} \log\frac{r}{|z_k|}$. Let $\epsilon \in (0,1)$. We can decompose $N(r)$ as

$$\begin{aligned} N(r) &= \sum_{|z_k|>\epsilon r} \log\frac{r}{|z_k|} + \sum_{|z_k|<\epsilon r} \log\frac{r}{|z_k|} \\ &\geq \sum_{|z_k|<\epsilon r} \log\frac{r}{|z_k|} \\ &\geq n(\epsilon r)\log(\frac{r}{r\epsilon}). \end{aligned}$$

Let $\epsilon = \frac{1}{e}$, the above becomes

$$N(r) \geq n\left(\frac{r}{e}\right).$$

Substituting into (1) we have

$$n\left(\frac{r}{e}\right) \leq \sqrt{r} - \log|f(0)|.$$

Let $u = \frac{r}{e}$ and we get the desired inequality.

□

(2008 #3) Let $\mathcal{H} = \{f \in H(\mathbb{D}) : f(0) = 0, f'(0) = 1, |f(z)| \leq 100\}$. Prove that there is a constant $r > 0$ such that for any $f \in \mathcal{H}$ the image of the unit disk under f contains the disk $\mathbb{D}(0,r)$.

SOLUTION 1:

Proof. Let $\mathcal{F} = \{\frac{f}{100} : f \in \mathbb{H}\}$. Notice that every $f \in \mathcal{F}$ satisfies

(i) $f : \mathbb{D} \to \mathbb{D}$.

(ii) $f(0) = 0$.

(iii) $|f'(0)| = \frac{1}{100}$.

It suffices to prove that there exists $r > 0$ such that for all $f \in \mathcal{F}$ it holds that $\mathbb{D}(0, r) \subseteq f(\mathbb{D})$. This is sufficient since our construction would than guarantee that for all $h \in \mathbb{H}$ we have $\mathbb{D}(0, 100r) \subseteq h(\mathbb{D})$.

To show a contradictions suppose that there exists a sequence $\{a_k\} \subseteq \mathbb{D}$ with $|a_k| < \frac{1}{k}$ and for each a_k there exists $f_k \in \mathcal{F}$ with $a_k \notin f(\mathbb{D})$. For each k let $T_k : \mathbb{D} \to \mathbb{D} : z \mapsto \frac{z-a_k}{1-\bar{a}_k z}$. Notice that $T_k : a_k \mapsto 0$ and thus $T_k \circ f$ is never zero on $\mathbb{D}$. Hence there exists a function $g : \mathbb{D} \to \mathbb{D}$ such that $g_k^2(z) = (T_k \circ f)(z)$. Let $g_k(0) = b_k$, notice that $g_k^2(0) = b_k^2 = T_k(f(0)) = -a_k$. Hence $|b_k|^2 = |a_k|$. Consider the map $U_k : \mathbb{D} \to \mathbb{D} : z \mapsto \frac{z-b_k}{1-\bar{b}_k z}$. We now have

$$U_k \circ g_k : \mathbb{D} \to \mathbb{D} : 0 \mapsto 0.$$

Applying the Schwarz lemma we know that $|U_k(g_k(0))'| \leq 1$. We need to compute two derivatives. First we have

$$\begin{aligned} \frac{d}{dz} U_k(g_k(z)) &= U_k'(g_k(z)) g_k'(z) \\ &= \frac{1-|b_k|^2}{(\bar{b}_k g_k(z) - 1)^2} g_k'(z). \end{aligned}$$

Next, since $g_k^2(z) = T_k(f_k(z))$ we know that $2g_k(0)g_k'(0) = T_k'(f(0))f'(0)$ and by substituting in the values we know and using the fact that $|f'(0)| = 1/100$ we have

$$|g_k'(0)| = \left| \frac{1-|a_k|^2}{200 b_k} \right|.$$

Using this, the fact that $g_k(0) = b_k$ and that $|b_k|^2 = |a_k|$ it follows that

$$\begin{aligned} \left| \frac{d}{dz} U_k(g_k(0)) \right| &= \left| \frac{1-|b_k|^2}{(|b_k|^2 - 1)^2} g_k'(0) \right| \\ &= \frac{1-|a_k|^2}{(200 b_k)(1-|b_k|^2)} \\ &= \left| \frac{1-|a_k|^2}{(200 b_k)(1-|a_k|)} \right| \\ &= \frac{1+|a_k|}{200\sqrt{|a_k|}} \\ &\leq 1. \end{aligned}$$

Hence we have

$$1 + |a_k| - 200\sqrt{|a_k|} \leq 0.$$

Making the substitution $x = \sqrt{|a_k|}$ we obtain the quadratic

$$x^2 - 200x + 1 \leq 0.$$

which has solutions $100 \pm 3\sqrt{1111}$, hence for arbitrary k we have $|a_k| \geq (100 - 3\sqrt{1111})^2 > 0$, but this contradicts the fact that the $a_k \to 0$.

□

SOLUTION 2:

Proof. Suppose not. Hence for each n there exist $f_n \in \mathbb{H}$ such that there is $|p_n| < 1/n$ with $p_n \notin f_n(\mathbb{D})$. Therefore $f_n(z) - p_n$ is zero free on $\mathbb{D}$. Since the image of f is bounded we can assume without loss of generality that $f_n \to f$ locally uniformly. Since $p_n \to 0$ we also have $g_n \to f$ locally uniformly. Notice that f is nonconstant since$f'(0) = 1$. We know that the number of zeros of g_n in $\mathbb{D}(0, \rho)$ for some $\rho > 0$ is zero, but by Hurwitz we know that g_n has the same number of zeros as f. A contradiction since $f_n(0) - p_n \to 0$ and so we have $f(0) = 0$.

□

SOLUTION 3:

Proof. Write $f(z) = z + z^2 \cdot r(z)$ where r is analytic on $\mathbb{D}$. We know that $|r(z)| \leq 101$. □

(2008 #4) Suppose $f \in H(U)$ with $U \subseteq \mathbb{C}$, and is continuous on ∂U with $|f(z)| = 1$ for $z \in \partial U$. Prove that either f is constant or $f(U) = \mathbb{D}$.

[Solution 1 (credit to Clanaghan)]

Proof. Suppose that $0 \notin f(U)$. The minimum modulus principle then implies that f attains its min on ∂U and therefore f is constant. Next, suppose that f is nonconstant and that there exists a point $b \in \mathbb{D}$ such that $b \notin f(U)$. Let $g \in \operatorname{Aut}\mathbb{D}$ with $g(b) = 0$. Notice that the composition

$$g \circ f : \overline{U} \to \overline{\mathbb{D}} \to \overline{\mathbb{D}}$$

fixes the boundaries and must satisfy $|g(f(z))| = 1$ for $z \in \partial U$. Also, since f omits b we conclude that $g \circ f$ is never zero, which by the minimum modulus theorem says that $g \circ f$ is constant and thus f is constant, a contradiction.

□

[Solution 2]

Proof. Assume that f is nonconstant. We know that $f : U \to \overline{\mathbb{D}}$ since f is analytic. We will prove that $f(U)$ is closed in $\mathbb{D}$. Pick $w \in \mathbb{D} \setminus f(U)$. Consider $g(z) = \frac{1}{f(z)-w}$ which is analytic in U and continuous in $\overline{U}$. By the maximum principle we know that for all $z \in U$ it holds that

$$|g(z)| \leq \max_{\partial U} \frac{1}{|f(z) - w|} \leq \max_{\partial U} \frac{1}{||f(z)| - |w||} = \frac{1}{1 - |w|}.$$

Therefore $|w - f(z)| \geq 1 - |w|$ for all $z \in U$ and so the complement is open as $\mathbb{D}(w, 1-|w|) \subseteq \mathbb{D} \setminus f(U)$. By the open mapping theorem we then have $f(U)$ is open in $\mathbb{D}$. Therefore $f(U) = \mathbb{D}$.

□

(2008 #5) Let $f_n \to f$ with the $f_n \in H(G)$ and injective on G for some region G. Prove that $f \in H(G)$ and that f is either injective or constant on G.

Proof. First off, f is continuous on G since the f_n are continuous. For any rectangle $R \subseteq G$ we have

$$\begin{aligned}\left|\int_R f(z)dz\right| &= \left|\int_R f(z) - f_n(z)dz + \int_R f_n(z)dz\right| \\ &\leq \int_R |f(z) - f_n(z)|dz + \left|\int_R f_n(z)dz\right| \\ &\stackrel{*}{=} \int_R |f(z) - f_n(z)|dz + 0 \\ &\stackrel{**}{\to} 0.\end{aligned}$$

Where at * we use the fact that the f_n are analytic and by Cauchy's theorem integrate to zero over any rectangle. The conclusion at ** follows from an ML estimate which uses the fact that R is compact and thus $\sup_{z\in R}|f(z) - f_n(z)| \to 0$ as $n \to \infty$. It follows from Morera's theorem that $f \in H(G)$.

To show a contradiction, suppose that f is not injective. That is assume $f(a) = f(b)$ for $a, b \in G$ with $a \neq b$. Let $g_n(z) := f_n(z) - f_n(b)$. Observe that $g_n \to f(z) - f(a) =: g(z)$. Let $\mathbb{D}(a, r)$ be such that $b \notin \mathbb{D}(0, r)$. Hurwitz's theorem guarantees that for sufficiently large f it holds that f_n and f have the same number of zeros inside $\mathbb{D}(0, r)$. Notice that g has at least one zero inside of $\mathbb{D}(0, r)$ (in particular at $z = f(a)$). This implies that there is some n and some point $c \in \mathbb{D}(0, r)$ with $g_n(c) = 0$. But this says that $f_n(c) = f_n(b)$ with $c \neq b$, contradicting injectivity of f_n. □

(2008 #6) Suppose U is a bounded region in $\mathbb{C}$ and $w_0 \in U$. Set

$$\mathcal{F} = \{f : \mathbb{D} \to U \mid f \text{ is analytic, one to one and } f(0) = w_0\}.$$

Set

$$M = \sup_{f\in\mathcal{F}} |f'(0)|.$$

Prove that (a) there exists an $f \in \mathcal{F}$ for which $|f'(0)| = M$ and that (b)for any such f is $f(\mathbb{D}) \subseteq V \subseteq U$ and V is a simply connected reqion, then $f(\mathbb{D}) = V$.

Part a.

Proof. Let $\{f_k\} \subseteq \mathcal{F}$ be a subsequence such that $\lim_{k\to\infty} |f_k'(0)| = M$. Since U is bounded it follows that the family $\{f_k\}$ is uniformly bounded. By Montel's theorem we conclude that there exists a locally uniformly convergent subsequence to some function f. Since the convergence is uniform we know that $f_{k_j}'(0) \to f'(0)$ and thus $|f'(0)| = M$. Since f is analytic this implies that $M < \infty$. Moreover, problem 5 guarantees that f is injective.

□

Part b.

Proof. Let $f \in \mathcal{F}$ satisfy $|f'(0)| = M$. Let $\varphi : V \to \mathbb{D} : f(0) \mapsto 0$ be the conformal and onto map guaranteed by RMT. Notice that $\varphi \circ f : \mathbb{D} \to \mathbb{D} : 0 \mapsto 0$. Since we know from the proof of Riemann mapping theorem that φ has maximal derivative in the family $\mathcal{G} = \{\psi : V \to \mathbb{D}$ such that ψ is conformal$\}$ it follows that $\varphi \circ f$ has maximal derivative in the family of maps which take $\mathbb{D} \to \mathbb{D}$. It follows from Schwarz's Lemma that $\varphi \circ f(z) = e^{i\theta} z$ and therefore f must be onto V. □

(ALTERNATE PROOF)

Proof. Suppose that $f(\mathbb{D}) \neq V$ and thus misses the point $z_0 \in V$. As guaranteed by the Riemann mapping theorem let $\varphi : V \to \mathbb{D}$ with $\varphi(f(0)) = 0$ and $\varphi'(f(0)) > 0$. We will prove that the map $g = \varphi \circ f \in \mathcal{F}$ with $|g'(0)| > M$. By the chain rule it suffices to prove that $|\varphi'(z_0)| > 1$.

Let $a = \varphi(z_0)$ and let $T_a(z) = \frac{z-a}{1-\bar{a}z}$. Since $a \notin g(\mathbb{D})$ it follows that $T_a \circ g$ is never zero and thus there exists a function $h \in H(\mathbb{D})$ so that

$$h^2(z) = (T_a \circ g)(z), \qquad \forall z \in \mathbb{D}.$$

Let $b = h(0)$. Notice that $|b|^2 = a$. Let $\psi = T_b \circ h$. Notice that $\psi : \mathbb{D} \to \mathbb{D} : 0 \mapsto 0$. From Schwarz's lemma we have $|\psi'(0)| \leq 1$. Using the fact that

$$|T_a'(z)| = \frac{1-|a|^2}{|1-\bar{a}z|^2} = \frac{1-|T_a(z)|}{1-|z|^2},$$

we can write

$$\left|\psi'(0)\right| = \frac{1}{1-|b|^2} \cdot \frac{1-|a|^2}{2|b|} = \frac{1+|b|^2}{2|b|} > 1$$

by the estimate $2AB < A^2 + B^2$

□

(2008 #7) Show that the function $w = \log\left(\frac{z+1}{z-1}\right) + \frac{2z}{z^2+1}$ maps $\mathbb{D}$ bijectively to the full w-plane with four half-lines deleted. Find the locations of the four end points of the four half-lines. You will need to choose a branch of logarithm.

(2008 #8) Suppose $U \subseteq \mathbb{C}$ is a region and $z_0 \in U$. Let $\{f_n\} = \{u_n(z) + iv_n(z)\} \in H(U)$ be a sequence satisfying

(a) $u_n(z) \leq u_{n+1}(z)$ for all $z \in U$ and $n \in \mathbb{N}$.

(b) $\lim_{n\to\infty} f_n(z_0)$ exists in $\mathbb{C}$.

Prove that $\{f_n\}$ converges uniformly on compact subsets of U.

Proof. Harnack's theorem guarantees that any increasing sequence of harmonic functions either converges locally uniformly or diverges to infinity for all $z \in U$. However, condition (b) guarantees that $u_n(z_0) \not\to \infty$ and thus u_n converge locally uniformly to some harmonic function $u \in Ha(U)$. Letting $a = \lim_{n\to\infty} v_n(z_0)$, the Schwarz integral formula states that on any disk $\Delta = \mathbb{D}(z,r) \subseteq U$ we have

$$f_n(z) = \int_{\partial\Delta} \frac{\zeta+z}{\zeta-z} u_n(z) \frac{d\zeta}{\zeta} + iv_n(z_0).$$

Since the convergence $u_n \to u$ is locally uniform we obtain the following local uniform convergence:

$$\lim_{n\to\infty} f_n(z) = \lim_{n\to\infty} \left[\int_{\partial\Delta} \frac{\zeta+z}{\zeta-z} u_n(z) \frac{d\zeta}{\zeta} + iv_n(z_0)\right] = \int_{\partial\Delta} \frac{\zeta+z}{\zeta-z} u(z) \frac{d\zeta}{\zeta} + ia.$$

Hence $f_n \to f$ where $f(z) = \int_{\partial\Delta} \frac{\zeta+z}{\zeta-z} u(z) \frac{d\zeta}{\zeta} + ia$ which is analytic in U. □

14 2009

(2009 #1) Let $\operatorname{Re} z > 0$. Use an appropriate contour to show that

$$\int_0^\infty \frac{e^{-t} - e^{-tz}}{t} dt = \log z.$$

Proof. Look at the contour which has a partial δ-circle around zero which stops at the real line (positive) and the line segment through 0 and z. We consider $g(w) = \frac{e^{-w}}{w}$ and by making bounds conclude that as C shrinks and expands we get the answer. □

(2009 #2) If $a > 1$ prove that $z + e^{-z}$ assumes the value a exactly once in RHP.

Proof. Let $f(z) = z + e^{-z} - a$. By the argument principle we know that the number of zeros is equal to the winding number $n(f \circ \gamma, 0)$ With γ a rectangle in the right half plane with vertical side on $i\mathbb{R}$ denoted by L_1 with adjacent side L_3 and horizontal sides L_2, L_4. We can write

$$\operatorname{Re} f = x + e^{-x} \cos y - a$$

$$\operatorname{Im} f = y + e^{-x} \sin y.$$

So on L_1 we have $\operatorname{Re} f = \cos y - a$ and $\operatorname{Im} f = y + \sin y$. Also on L_3 we have $\operatorname{Re} f = R - a$. Similarly, if we check on L_2 and L_4 we end up getting a squiggly rectangle that surround zero and has winding number 1. □

Proof. We seek to apply Rouche. Let R_M be the rectangle centered at A with horizontal width $2M$ and height $4M$. Consider the function $f(z) = e^{-z} + z - a$ and $g(z) = z - a$. If we look at

$$|f(z) - g(z)| = |e^{-z}| = e^{-\operatorname{Re} z} \le 1, \qquad z \in \partial R_M.$$

Also, we have $|g(z)| = |z - a| \ge a$ on ∂R_M. Since $a > 1$ we conclude that $|f(z) - g(z)| \le 1 < a < |g(z)|$ on ∂R_M. So, by Rouche f has a lone zero on all of the right half plane.

□

(2009 #3) Let f be an entire function with the property that for every z there is an n such that the n-th derivative $f^{(n)}(z) = 0$. Show that f is a polynomial.

Proof. Let $E_n := \{z \, in\overline{\mathbb{D}} : f^{(n)}(z) = 0\}$. This implies that $\bigcup_1^\infty E_n = \overline{\mathbb{D}}$. Hence there exists N such that E_N has uncountably many points in $\overline{\mathbb{D}}$. As $\overline{\mathbb{D}}$ is compact we know that E_N has a limit point. Therefore (by uniqueness) $f^{(N)} \equiv 0$ on $\mathbb{D}$ and therefore f is a polynomial on $\mathbb{D}$ and by uniqueness a polynomial on all of $\mathbb{C}$.

□

(2009 #4) If D and D' are open discs with $\overline{D}' \subseteq D$ show that there is $R > 1$ and a linear fractional transformation f that maps $D \setminus \overline{D}'$ conformally onto $\{1 < |z| < R\}$.

Proof. Without loss of generality assume that D_1 and D_2 are symmetric about $\mathbb{R}$. There exists a third circle Γ also centered on $\mathbb{R}$ which intersects both D_1 and D_2 orthogonally. Let A be the leftmost point of Γ and B be the rightmost x-coordinate of Γ. Let T be the LFT which maps

$$B \mapsto 0$$

$$A \mapsto \infty$$

$$-1 \mapsto -1.$$

Notice that $\varphi : \mathbb{R} \to \mathbb{R}$. Also, we have $\varphi(\partial\Gamma)$ intersects $\mathbb{R}$ orthogonally since it is a line that contains 0 it must be the imaginary axis. So $\varphi(\partial D_2)$ and $\varphi(\partial D_1)$ intersect $\mathbb{R}$ and $i\mathbb{R}$ orthogonally and there are circles. Therefore the centers of $\varphi(D_1)$ and $\varphi(D_2)$ are the same. So, if we scale we are done. □

Proof. Without loss of generality suppose that $D = \mathbb{D}$. Let $\psi(z) = -i\left(\frac{z+1}{z-1}\right)$. Notice that ψ takes circles to circles. If we translate and dilate $\psi(D')$ so that it is centered at the imaginary axis then take ψ^{-1} we obtain a composition of linear fractional transformations and are done. □

(2009 #5) Let $W = \mathbb{D}(0, 2)$ and $I = [-1, 1]$.

(a) Show that every bounded harmonic function in $W \setminus \{0\}$ extends continuously to W.

(b) Construct a bounded harmonic function on $W \setminus I$ that does not extend continuously to W.

Proof. (a) By subtracting the solution to the Dirichlet problem on $\partial\mathbb{D}$ we can assume that u has value 0 on $\partial\mathbb{D}$. Moreover, the harmonic function

$$v_\epsilon(z) = u(z) + \epsilon \log|z| - \epsilon$$

is negative on $\partial\mathbb{D}$ and also, because u is bounded, is negative near 0. If we let $\epsilon \to 0$ we conclude from the MMP that $u(z) \leq 0$ on $\mathbb{D} \setminus \{0\}$. Similarly, $-u(z) \leq 0$ on $\mathbb{D} \setminus \{0\}$. It follows that u can be extended to be 0 on all of $\mathbb{D}$.

(b) The map $f(z) = \frac{z-1}{z+1}$ takes W to a subset of $\mathbb{C}$ with $[0, \infty)$ deleted. Take the standard branch cut of *arg* on $f(W)$. The function $arg \circ f$ is clearly harmonic, bounded and does not have an extension since *arg* has no continuous extension across $[0, \infty)$. □

(2009 #6) (a) If $f : \mathbb{D} \to \mathbb{H}$ is analytic show that $|f'(0| \leq 2 \operatorname{Re} f(0)$. (b) If $f : \mathbb{D} \to \mathbb{D} \setminus \{0\}$ is analytic, show that $|f'(0)| \leq \frac{2}{e}$.

Part a.

Proof. The function $g(w) = \frac{w-1}{w+1}$ take $RHP \to \mathbb{D}$. Let $h(z) = \frac{f(z)-1}{f(z)+1}$ takes $\mathbb{D} \to \mathbb{D}$. By Schwarz-Pick we know that

$$\frac{|h'(z)|}{1 - |h(z)|^2} \leq \frac{1}{1 - |z|^2}.$$

One can compute $h'(0) = \frac{2f'(0)}{(1+f(0))^2}$ and $h(0) = \frac{f(0)-1}{f(0)+1}$. So at $z = 0$ we have

$$|h'(0)| \leq 1 - |h(0)|^2$$

and therefore

$$2|f'(0)| \leq |1 + f(0)|^2 - |f(0) - 1|^2$$
$$= 4\operatorname{Re} f(0).$$

□

Part b.

Proof. As f is nonzero, we can write $f = e^{-g}$ for some $g : \mathbb{D} \to RHP$. We have

$$|f'(0)| = |e^{-g(0)}||-g'(0)| = |f(0)||g'(0)|,$$

with

$$|g'(0)| \leq 2\operatorname{Re} g(0) = -2\log|f(0)|.$$

Therefore

$$|f'(0)| \leq -2\log|f(0)||f(0)|.$$

Consider $x \mapsto -x\log x$. This has a maximum at $x = e^{-1}$. So we obtain

$$|f'(0)| \leq -2(-1)e^{-1} = \frac{2}{e}.$$

□

(2009 #7)

(a) Prove that $\prod(1 + \frac{i}{k})$ diverges but that $\prod|1 + \frac{i}{k}|$ converges.

(b) Construct an infinite product that is an entire function with simple zeroes precisely at the positive integers and prove convergence from first principles.

Part a.

Proof. $\boxed{|1 + \frac{1}{k}| \text{ converges}}$ Let $P_n = \prod_1^n |n + \frac{i}{k}|$. Taking the log and noting that $\log P_n = \frac{1}{2}\log P_n^2 = \sum_1^\infty \frac{1}{2}\log\left(1 + \frac{1}{k^2}\right)$, it suffices to prove that

$$\sum_1^\infty \frac{1}{2}\log\left(1 + \frac{1}{k^2}\right) < \infty.$$

Notice that $\log(1 + \frac{1}{k^2}) \sim \frac{1}{k^2}$ therefore $\log P_n$ converges.

$\boxed{(1 + \frac{1}{k}) \text{ diverges}}$ Let $Q_n = \prod_1^n (n + \frac{i}{k})$. Suppose $Q_n \to Q \neq 0$. We then have $\arg Q_n = \sum_1^n \arg(1 + \frac{i}{k})$ and therefore

$$\arg(1 + \frac{i}{k}) \sim \frac{1}{k}, \qquad k \to \infty.$$

□

Proof. We take the log and consider the sequence

$$\log\left(\prod_{n=1}^{\infty}\left(1+\frac{i}{n}\right)\right)=\sum_{1}^{\infty}\log\left(1+\frac{i}{n}\right)$$

We can write the Taylor expansion of $\log|1+i/n|$ as

$$\log|1+i/n|=\sum_{k=1}^{\infty}(-1)^{k+1}\frac{(i/n)^k}{k}=i/n+O(1/n^2)$$

Therefore by the comparison test we have

$$\sum_{1}^{\infty}\log(1+i/n)\longrightarrow\sum_{1}^{\infty}i/n=i\sum_{1}^{\infty}1/n=\infty$$

Since the log of the product does not converge, the product certainly does not converge.

On the other hand we have

$$\log\left(\prod_{n=1}^{\infty}(|1+i/n|)\right)=\sum_{1}^{\infty}\log|1+i/n|$$

Since $|1+i/n|>1$ we know that

$$0<\log|1+i/n|$$

Moreover, since $|1-i/n|=|1+i/n|$ and $(1+i/n)(1-i/n)=(1+1/n^2)$ we can write

$$\log|1+i/n|=\log\sqrt{1+1/n^2}=\frac{1}{2}\log(1+1/n^2)<1/n^2$$

By comparison to the series $\sum 1/n^2$ we then have $\sum_1^\infty \log|1+i/n|$ converges and thus so does the product $\prod_1^\infty|1+i/n|$

□

Part b.

Proof. Let $E_p=\begin{cases}(1-z)\exp\left(\sum_1^p\frac{z^k}{k}\right), & p\geq 1\\ (1-z), & p=0\end{cases}$. The explicit function we seek is

$$f(z)=\prod_{n=1}^{\infty}E_1\left(\frac{z}{n}\right).$$

To see that f converges we notice that $\log(1-\frac{z}{n})+\frac{z}{n}=\mathcal{O}(z^2/n^2)$ which converges on compact sets. □

(2009 #8) Let D be bounded and $f:D\to D$ be analytic. If f has a fixed point $f(z_0)=z_0$ and $|f'(z_0)|<1$ show that the sequence of iterates $f_n=f\circ\cdots\circ f$ (n times) converges uniformly on compact subsets of D.

Proof. Let $U = \{z - z_0 : z \in D\}$ and let $g(z) = f(z + z_0)$. Notice that $g(0) = 0$ and that $|g'(0)| = |f'(z_0)| > 1$. It suffices to prove that the sequence g_n converges locally uniformly. As U is bounded we know (from Montel's theorem) that given a subsequence g_{n_k}, it contains a locally convergent subsequence $g_{n_{k_j}} \to h_0$. Similarly, any other subsequence g_{m_k} contains a convergent subsequence $g_{m_{k_j}} \to h_1$. We would like to prove that $h_0 \equiv h_1$ on U. We will actually prove that both functions are identically zero on D.

Let $\Delta = \overline{\mathbb{D}}(0, r) \subseteq D$. For $a \in \Delta$ we have for sufficiently large n $|f_n(a)| = |af_n'(0) + a^2 \cdot h| = |a||f_n'(0) + a\lambda(a)| < |a|$. Where we use the fact that $|f_n'(0)|t\o 0$ and $a^2\lambda(a) \to 0$.Therefore for $a \in \Delta$ we have $h_0(a) = h_1(a) = 0$. By uniqueness we must have $h_0 \equiv h_1 \equiv 0$ on all D.

□

15 2010

(2010 #1) Compute $\int_0^\infty \frac{dx}{x^c(x+1)}$ for each $c \in (0, 1)$.

Proof. Let $I(c) = \int_0^\infty \frac{dx}{x^c(x+1)}$ and let $f(z) = \frac{1}{z^c(z+1)}$ with a branch cut along the positive real axis. Notice that $\text{Res}(f; -1) = e^{-i\pi c}$. Consider the standard keyhole contour (traversed counter-clockwise) $\Gamma = \Gamma_R + \Gamma_\epsilon + L_1 + L_2$. First we have

$$\left|\frac{1}{2\pi}\int_{\Gamma_R} f(z)dz\right| \leq \frac{R}{\inf_{z\in\Gamma_R} |z|^c|z+1|} = \frac{R}{R^c(R-1)} \to 0.$$

Also, since $c \in (0, 1)$ we have

$$\left|\frac{1}{2\pi}\int_{\Gamma_\epsilon} f(z)dz\right| \leq \frac{\epsilon}{\epsilon^c(1-\epsilon)} \to 0.$$

Since we are taking a branch cut along the positive real axis we know that as $\delta \to 0$ we must have for $z \in L_2$ it holds that

$$z^c = e^{c(\log|z|+i\arg z+2\pi i)} = e^{2\pi ci}z^c.$$

Taking orientation into account we have

$$\int_{L_1} f(z)dz + \int_{L_2} f(z)dz \to I(c) - e^{2\pi ci}I(c) = I(c)[1 - e^{2\pi ci}].$$

Applying the residue theorem tells us that

$$2\pi i\,\text{Res}(f; -1) = \int_\Gamma f(z)dz = [1 - e^{2\pi ci}]I(c).$$

Therefore

$$I(c) = \frac{2\pi i}{e^{\pi ci}(1 - e^{2\pi ci})} = \frac{2\pi i}{e^{\pi ci} - e^{-\pi ci}} = \frac{\pi}{\sin(c\pi)}.$$

□

(2010 #2)

(a) Construct a conformal map from UHP to $\Omega = \{z : \arg z \in (-\pi/4, \pi/4)\} \setminus [0,1]$.

(b) Suppose f is a bounded harmonic function on Ω such that f is continuous on $\partial\Omega \setminus \{0\}$ and

$$f(z) = \begin{cases} 0, & \arg z = \pi/4 \\ 1, & \arg z = -\pi/4 \\ 3, & z \in [0,1] \end{cases}.$$

Find an expression for $f(3)$.

Part a.

Proof. The composition

$$z \overset{f_1}{\mapsto} z^2 \overset{f_2}{\mapsto} z + 1 \overset{f_3}{\mapsto} z^{1/4}$$

does the trick.

Alternatively, the composition

$$z \mapsto \sqrt{z} \mapsto \frac{z-1}{z+1} \mapsto z^2 \mapsto \frac{1+z}{1-z} \mapsto \frac{1}{z}$$

□

Part b.

Proof. Let $\varphi : \Omega \to UHP : z \mapsto i\sqrt{z^4 - i}$ be the conformal map which takes $0 \mapsto -1$ and $0 \mapsto 1$ and $1 \mapsto 0$. We know that

$$\frac{1}{\pi} \arg(\varphi(z) + 1)$$

takes boundary values (reading top to bottom on $\partial\Omega$) $(1, 0, 0)$. And

$$\frac{1}{\pi} \arg(\varphi(z) - 1)$$

takes boundary values $(1, 1, 0)$. We then choose

$$f(z) = 1 + 2\frac{\arg(\varphi(z) - 1}{\pi} + 3\frac{\arg(\varphi(z) + 1)}{\pi}.$$

□

(2010 #3) Let $f \in H(\overline{\mathbb{D}})$ with $f(-\log 2) = 0$ and $|f(z)| \leq |e^z|$ for all $|z| = 1$. Give a sharp upper bound for $f(\log 2)$.

Proof. Let $\varphi(z) = \frac{z - \log 2}{1 - \log 2z}$. Notice that $\varphi(0) = -\log 2$ and therefore $(f \circ \varphi)(0) = 0$. The max modulus principle guarantees that $|f(z)| \leq e$ for all $z \in \overline{\mathbb{D}}$. It follows that the map

$$\frac{1}{e} f(\varphi(z)) : \mathbb{D} \to \mathbb{D} : 0 \mapsto 0.$$

By Schwarz's lemma we must have $|f(\varphi(z))| \leq |z| \cdot |e^{\varphi(z)}|$. Notice that $\varphi^{-1}(\log 2) = \frac{2\log 2}{1-(\log 2)^2}$. Therefore

$$|f(\log 2)| \leq |e^{\log 2}| \cdot \frac{2\log 2}{1-(\log 2)^2}.$$

To see this we take

$$f(z) = \frac{z+\log 2}{1+z\log 2}e^z.$$

□

> **(2010 #4)** how many roots of the equation $p(z) = z^4 + 8z^3 + 3z^2 + 8z + 3 = 0$ lie in the right half-plane?

Proof. First we check the zeros on the real line. Notice that $f(0) > 0$ and $f(-1) = -9$ and $f(-3) > 0$. So f has precisely two real roots and both are negative. By symmetry it must be the case that there are either 2 or zero roots in the RHP.

Let $\Gamma_R = \Gamma_1 + \Gamma_2 + \Gamma_4$ be the contour, traversed counter clockwise starting at 0, that is the quarter circle sitting on the positive real and imaginary axes. We have $d\theta(\Gamma_1) = 0$ and $d\theta(\Gamma_2) \approx \frac{\pi}{2} \cdot 4 = 2\pi$ (since the z^4 term eventually dominates). To compute $d\theta(\Gamma_3)$ we plug in

$$f(it) = (t^4 - 4t^2 + 3) + 8(t - t^3)i.$$

Since the real part is always positive we know that $f(\Gamma_3) \subseteq RHP$. Notice that as $R \to \infty$ we have $\arg(f(R)) \to 0$ and $\arg(f(0)) = 0$. It follows that $d\theta(\Gamma_3) \approx 0$. Summing these gives the total change in argument in the first quadrant is 2π. Therefore there is one root in the upper quadrant and one in the lower quadrant.

□

> **(2010 #5)** Let $\mathbb{N} = \{1, 2, \ldots\}$. Find an explicit series representation for a meromorphic function on $\mathbb{C}$ which is holomorphic on $\mathbb{C} \setminus \mathbb{N}$ and has at each $n \in \mathbb{N}$ a simple pole with residue n.

Proof. Let $F(z) = z\sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{1}{1-\frac{z}{k}} - 1 - \frac{z}{k}\right)$. First notice that $\text{Res}(F, k) = k$. Next, the series converges locally uniformly since on a compact set $K \subseteq \mathbb{C}$ we let $M_k = \sup_{z\in K} |z| \left|\sum_{j=2}^{\infty} \frac{z^j}{k^j}\right|$ and take $M = \sup_k M_k$. Notice that M is finite since the M_k are decreasing (so we actually take $M = M_1$).

$$\frac{1}{1-\frac{z}{k}} - 1 - \frac{z}{k} = 1 + \frac{z}{k} + \frac{z^2}{k^2} + \cdots - 1 - \frac{z}{k} = \frac{z^2}{k^2} + \frac{z^3}{k^3} + \cdots$$

Therefore, for $z \in K$ we have

$$|z|\sum_{k=1}^{\infty}\left|\frac{1}{k}\left(\frac{1}{1-\frac{z}{k}} - 1 - \frac{z}{k}\right)\right| = |z|\sum_{k=1}^{\infty}\left|\frac{1}{k}\left(\frac{z^2}{k^2} + \frac{z^3}{k^3} + \cdots\right)\right| \leq \sum_{k=1}^{\infty}\frac{M}{k^2} < \infty.$$

By the M-test we conclude that F is analytic. □

(2010 #6) Let $g(z) = u + iv$ be holomorphic in the domain $\Omega = \{z : |z| < 1, y > 0\}$ and continuous in $\overline{\Omega}$. Also, assume that

(i) $u = 0$ for $y = 0$ and $x > 0$.

(ii) $v = 0$ for $y = 0$ and $x < 0$.

Show that $\frac{|g(z)|}{|z|^{1/2}}$ is bounded in Ω.

Proof. By making an appropriate branch cut and considering $h(z) = g(z^2)$ then using Schwartz reflection we can extend to a function $\frac{h(z)}{z} \in H(\mathbb{D})$ which is continuous up to the boundary and therefore bounded. Next compose with $z \mapsto \sqrt{z}$ and we have $|g(z)/z^{1/2}|$ is bounded. □

(2010 #7) Let $\mathcal{F}$ be the family of functions f holomorphic in the unit disc $\mathbb{D}$ such that

$$\int\int_{|z|<1} |f(x+iy)|^2 dxdy \leq 1.$$

Prove that $\mathcal{F}$ is a normal family.

Proof. We will prove that $\mathcal{F}$ is locally bounded. Let $a \in \mathbb{D}(a, r) \subseteq \mathbb{D}$. Since $|f(z)|^2$ is subharmonic we can write $|f(a)|^2 \leq \frac{1}{2\pi}\int_0^{2\pi} |f(a + re^{i\theta})|^2 d\theta$. We can integrate with respect to ρ and write

$$\frac{|f(a)|^2}{2} r^2 = \frac{1}{2\pi}\int_0^r |f(a)|^2 \rho d\rho \leq \frac{1}{2\pi}\int_0^r \int_0^{2\pi} |f(a + \rho e^{i\theta})|^2 \rho d\rho = \frac{1}{2\pi}\int\int_{|z|<r} |f(z)|^2 dz.$$

Therefore

$$|f(a)| \leq \frac{1}{\pi r^2}\int\int_{|z-a|<r} |f(x+iy)|^2 dxdy \leq \frac{1}{\pi r^2}\int\int_{|z|<1} |f(x+iy)|^2 dxdy \leq \frac{1}{\pi r^2}.$$

Since r is fixed, it follows that $\mathcal{F}$ is locally uniformly bounded. Applying Montel's theorem gives the result. □

(2010 #8) Let $F(z) = \int_0^\infty x^{z-1} e^{-x^2} dx$ for $\operatorname{Re} z > 0$.

(a) Prove that F is analytic on the right half plane.

(b) Prove that F extends to be meromorphic on the whole plane.

(c) Find all poles of F and find the singular parts of F at these poles.

TIPS: (a) Use Morera with rectangles which avoid the bad point. Apply Fubini and obtain a convergent sequence. (b) Use integration by parts and Taylor expansion (c) Check the right stuff.

Part a.

Proof. As F is the integral of a continuous function, we know that F is continuous. We seek to apply Morera's theorem. Accordingly, fix a rectangle $R \subseteq RHP$. Let $F_n(z) = \int_{1/n}^\infty x^{z-1}e^{-x^2}dx$. Since x^{z-1} is an analytic function away from zero we know know that $\int_{\partial R} x^{z-1}dz = 0$. By Fubini's theorem we then have

$$\int_{\partial R} F_n(z)dz = \int_{1/n}^\infty e^{-x^2} \int_{\partial R} x^{z-1}dzdx = \int_{1/n}^\infty 0\ dx = 0.$$

Therefore the F_n are analytic functions on RHP. Next, we claim that some subsequence $F_{n_k} \to F$ locally uniformly. This holds since for $z \in K$ with $K \subseteq RHP$ a compact set we have

$$\sup_n |F_n(z)| \le \int_0^\infty |x^{z-1}|e^{-x^2}dx \le \int_0^\infty (1+|x|^{|z-1|})e^{-x^2} = M < \infty.$$

By Montel's theorem we have $\{F_n\}$ is locally uniformly bounded and therefore contains a convergent subsequence. It follows that F is a local uniform limit of analytic functions and therefore analytic. □

Part b.

Proof. We write $F(z) = \int_0^1 x^{z-1}e^{-x^2}dx + \int_1^\infty x^{z-1}e^{-x^2}dx =: F_0(z) + F_1(z)$. Notice that F_1 converges for all $z \in \mathbb{C}$ and therefore defines an entire function. Secondly, we use the Taylor expansion of e^{-x^2} to write

$$F_0(z) = \int_0^1 x^{z-1}\sum_{k=0}^\infty (-1)^k \frac{x^{2k}}{k!} = \sum_0^\infty \frac{(-1)^k}{k!}\int_0^1 x^{z+2k-1}dx = \sum_0^\infty \frac{(-1)^k}{k!}\left(\frac{1}{z+2k}\right).$$

Hence $F(z) = F_1(z) + \sum_0^\infty \frac{(-1)^k}{k!}\left(\frac{1}{z+2k}\right)$. To show that F is meromorphic it suffices to prove that the righthand sum converges locally uniformly away from it's poles.

Fix a compact set $K \subseteq \mathbb{C}\{0,-2,-4,\dots\}$. Let $M = \sup_{z\in K}\frac{1}{z}$ we have for $z \in K$ it holds that $\left|\frac{1}{k!(z+2k)}\right| \le \frac{M}{k!}$. Hence by the M-test the sum in F converges. □

ALTERNATIVELY

Proof. We can integrate by parts with $u = x^{z-2}$ and $dv = xe^{-x^2}$ to get

$$F(z) = (\frac{z}{2}-1)F(z-2), \qquad \operatorname{Re} z > 2.$$

If we map $z \mapsto z+2$ we can conclude $F(z) = \frac{2}{z}F(z+2)$. the argument to extend to a meromorphic function with poles at $0,-2,-4,\dots$. □

Part c.

Proof. We see from part (b) that F has simple poles at $\{0,-2,-4,\dots\}$ and singular part $\frac{1}{z+2k}$. The principal part at $z=-2n$ is $\frac{(-1)^n/n!}{z+2n}$ which follows by an inductive argument (that is messy!). □

16 2011

(2011 #2) For what $\alpha \in \mathbb{C}$ does $\prod \cos \frac{1}{n^\alpha}$ converge absolutely?

Proof. Recall that a product $\prod(1+c_n)$ converges absolutely if $\prod(1+|c_n|)$ converges. We can rewrite our product as

$$\prod(1+(\cos\frac{1}{n^\alpha}-1)).$$

Suppose that $\operatorname{Re}\alpha > 0$ so that $\frac{1}{n^\alpha} \to 0$. We would like to show that

$$\prod(1+|\cos\frac{1}{n^\alpha}-1|) < \infty.$$

Taking the log gives the equivalent condition

$$\sum \log(1+|\cos(1/n^\alpha)-1|) < \infty.$$

Since $\log(1+|\cos(1/n^\alpha)-1|) \sim |\cos(1/n^\alpha)| \sim \frac{1}{2}|1/n^\alpha|$, the product converges if $2\operatorname{Re}\alpha > 1$ and diverges if $0 < \operatorname{Re}\alpha \le \frac{1}{2}$. □

(2011 #3) Suppose f is an entire function such that $|f(x)| = 1$ for all $x \in \mathbb{R}$. Prove there exists an entire function g such that $f(z) = e^{g(z)}$.

Proof. Notice that $f \not\equiv 0$ and there f has countably many zeros $\{a_n\}$. Also, for any sequence z_n such that $\operatorname{Im} z_n \to 0$ we have $|f(z_n)| \to 0$, by Schwarz reflection we can define the funciton

$$F(z) = \begin{cases} f(z), & z \in UHP \cup \mathbb{R} \\ \frac{1}{\overline{f(\overline{z})}}, & \text{otherwise} \end{cases}.$$

Since f has finitely many zeros we know that F is meromorphic, but by uniquenesss $F \equiv f$ on all of $\mathbb{C}$. Since f is entire we conclude that f is zero free and thus such a g exists. □

(2011 #4) Suppose that $f \in H(\mathbb{D})$ and f is injective on $\mathcal{A}(01/2,1)$. Prove that f is injective on all of $\mathbb{D}$.

Proof. Let $R \in (1/2, 1)$. Then f is 1-1 on $\Delta = \partial\mathbb{D}(0,R)$ and therefore maps Δ to a simple closed curve. Let $c \notin f(\Delta)$, then $f - c$ is zero free on $f(\Delta)$. By the argument principle $f-c$ has at most one zero inside $\mathbb{D}(0,R)$. As f is 1-1 on $\mathcal{A}(0,1/2,1)$ for all $c \in \mathbb{C}$ we can find a sequence $R_n \to 1$ such that $f-c$ is zero free on Δ_n. We then have $f-c$ has at most 1 zero in $\mathbb{D}$ for all $c \in \mathbb{C}$. □

(2011 #5) If u is bounded and harmonic on $\mathbb{D}(z_0,R) \setminus \{z_0\}$ prove that u has a harmonic extension to $\mathbb{D}(z_0,R)$.

Proof. WLOG suppose our domain is $\mathbb{D} \setminus \{0\}$. Let v be the solution to the dirichlet problem on $\partial\mathbb{D}(0, 1/2)$ with boundary values $v(z) = u(z)$ when $|z| = 1/2$. Fix $0 < r < 1/2$. There exists $\epsilon > 0$ such that

$$u - v - \epsilon \log |z|) \leq 0, \qquad |z| = r,$$

and therefore

$$u - v + \epsilon \log |1/2| = \epsilon \log |1/2| \leq 0, \qquad |z| = 1/2.$$

It follows that $u - v + \epsilon \log |z| \leq 0$ on $r < |z| < 1/2$. Taking $r \to 0$ we must also have $\epsilon \to 0$. In the limit we now have

$$u - v \leq 0, \qquad z \in \mathbb{D}(1/2.0) \setminus \{0\}.$$

We can repeat this argument for the difference $v - u$ by considering $(v + \epsilon \log |z|) - u$ and obtain $u - v \leq 0$ on $\mathbb{D} \setminus \{0\}$. We then have $u = v$ on $\mathbb{D} \setminus \{0\}$ and by uniqueness must be equal everywhere. □

(2011 #6) Give an explicit example of a nonconstant bounded analytic function on $\mathbb{D}$ such that each point of $\partial\mathbb{D}$ is a limit point of zeros of f. Prove any convergence.

Proof. First a claim: Suppose there is a sequence $\{a_n\} \subseteq \mathbb{D} \setminus \{0\}$ such that $\sum(1 - |a_n|) < \infty$, then the function

$$f(z) = \prod -\frac{\overline{a}_n}{|a_n|} \frac{z - a_n}{1 - \overline{a}_n z}$$

converges locally uniformly for all $z \in \mathbb{D}$. This follows from the M-Test on compact sets. Also f is bounded since it maps the disk to the disk.

We now need to construct a sequence. The sequence $a_n = 1 - \frac{1}{n^2} + i \sum_1^n \frac{1}{k}$ does the trick.

□

(2011 #7)

Part a.

Proof. Consider the map $z \mapsto z^2; (x, y) \mapsto (x^2 - y^2, 2xy)$ and notice that the image of $\operatorname{Re} z = 1$ is mapped to the parabola $1 - \frac{1}{2}y^2$. Since z^2 is injective on $\Omega = \{\operatorname{Re} z \geq 1\}$ we conclude that Ω is mapped to the outside of the parabola. If we rotate and scale via the map

$$z \mapsto \frac{i(z - 1)}{3\sqrt{2}},$$

we obtain what we want. □

Part b.

Proof. □

(2011 #8) We say that f is a Bloch function if $f \in \mathbb{H}(\mathbb{D})$ and

$$\|f\|_{\mathcal{B}} = \sup_{\mathbb{D}} (1-|z|^2)|f'(z)| < \infty.$$

Prove that if $f = \sum a_k z^k$ is a Bloch function then there exists C independent of f such that $\sup |a_k| < C\|f\|_{\mathcal{B}}$.

Proof. Let $\|f\|_{\mathcal{B}} = M$ so that $|f'(z)| \leq M/(1-|z|^2)$. It follows from the Cauchy's theorem applied to $f'(z) = \sum n a_n z^{n-1}$ that

$$|a_n| = \frac{1}{2\pi n}\left|\int_{\partial\mathbb{D}(0,r)} \frac{f'(\zeta)}{\zeta^n} d\zeta\right|.$$

We can now make some estimates

$$\begin{aligned}
|a_n| &= \frac{1}{2\pi n}\left|\int_{\partial\mathbb{D}(0,r)} \frac{f'(\zeta)}{\zeta^n} d\zeta\right| \\
&\leq \frac{1}{2\pi n}\int_{\partial\mathbb{D}(0,r)} \left|\frac{f'(\zeta)}{\zeta^n}\right| d|\zeta| \\
&\leq \frac{1}{2\pi n}\int_{\partial\mathbb{D}(0,r)} \left|\frac{M}{(1-|\zeta|^2)|\zeta|^n}\right| d|\zeta| \\
&= \frac{M 2\pi r}{2\pi n(1-r^2)r^n} \\
&= \frac{M}{n(1-r^2)r^{n-1}}.
\end{aligned}$$

By taking the derivative it can be shown that the max of $h_n(r) = (1-r^2)r^{n-1}$ is at $r = \sqrt{\frac{n-1}{n+1}}$. Plugging this in we have

$$|a_n| \leq \frac{M}{n(1-\frac{n-1}{n+1})(\frac{n-1}{n+1})^{n/2}} = \frac{M}{\left(\frac{2n}{n+1}\right)\left(1-\frac{2}{n+1}\right)^{\frac{(n+1)}{2}}\left(1-\frac{2}{n+1}\right)^{-1}} \to \frac{e}{2}.$$

We conclude that $\sup_n a)n < \frac{Me}{2} < \infty$ independent of f for $n \geq 2$. We need to treat a_1 separately but we can compute this since $|f'(0)| \leq \|f\|_{\mathcal{B}}$. This gives the desired bound. □

17 2012

(2012 #1) Is there a conformal map from $D = \{x+iy\colon -1 < y < 1\} \setminus \{x+iy\colon x \leq 0, y = 0\}$ onto $\mathbb{D}$.

Proof. Yes. Consider the composition

$$z \mapsto \pi z \mapsto e^z \mapsto z-1 \mapsto \sqrt{z} \mapsto \frac{1-z}{1+z}.$$

□

(2012 #2) Let $f(z) = u(z) + iv(z)$ be a non-constant entire function.

(a) Use the Poisson integral formula to show that for all $|z| < r$

$$f(z) = iv(0) + \frac{1}{2\pi}\int_0^{2\pi} \frac{re^{it} + z}{re^{it} - z} u(re^{it})dt.$$

(b) Show that if $u(z) \leq |z|$ for all $z \in \mathbb{C}$ then f is linear.

(c) Show that there are $a > 0, C > 0$ such that for all $r \geq 0$

$$\max_{|z|=r} |e^{f(z)}| \geq Ce^{ar}.$$

Part a.

Proof. Fix $r > 0$. Let $g(z) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{it})\frac{re^{it}+z}{re^{it}-z}dz$. We claim that g is analytic. The Poisson integral formula guarantees that $u(z) = \text{Re}(g(z))$. By uniqueness we must have $f = g + ic$ for some $c \in \mathbb{R}$. The correct constant is easily seen to be $c = v(0)$. This gives the desired formula. □

Part b.

(Prof. Toro's proof)

Proof. Write $u = u^+ - u^-$. WLOG assume $u(0) = 0$. By the mean value property we have

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(re^{it})dt = \frac{1}{2\pi}\left[\int u^+ - \int u^-\right].$$

Since $0 \leq \frac{1}{2\pi}\int_0^{2\pi} u^+(re^{it})dt \leq r$ and similarly for the integral of u^- we conclude that

$$\frac{1}{2\pi}\int_0^{2\pi} |u(re^{it})|dt \leq 2r.$$

If we take $|z| < \frac{r}{2}$ we have

$$\begin{aligned} f''(z) &= \frac{1}{2\pi}\int_0^{2\pi} \frac{d^2}{dz^2}\left(\frac{re^{it}+}{re^{it} - z}\right) u(re^{it})dt. \\ &\leq C\int_0^{2\pi} \left|\frac{re^{it}}{(re^{it} - z)^3}\right| |u(reit)|dt \\ &\leq \frac{C}{r^3}\int_0^{2\pi} |u(reit)|dt \\ &\leq \frac{C'}{r}. \end{aligned}$$

□

(Kristen's Proof)

Proof. Let $u(z) = \operatorname{Re} f(z)$. Recall Harnack's theorem states that if $h(z)$ is a positive harmonic function on $\mathbb{D}$ then for $|z| \leq r < 1$

$$\frac{1-r}{1+r}h(0) \leq h(z) \leq \frac{1+r}{1-r}h(0).$$

Fix $R > 0$. On $|z| < R$ we have $R - u(z) > 0$ and so for the function $u_R(w) = u(Rw)$ it holds that $R - u_R(w)$ is a positive harmonic function on $\mathbb{D}$. Let $|w| = \frac{1}{2}$. By Harnack we then have

$$\frac{1}{3}(R - u_R(0)) \leq R - u_R(w) \leq 3(R - u_R(0)).$$

We can rearrange and write

$$3u_R(0) - 2R \leq u_R(w) \leq \frac{1}{3}u_R(0) + \frac{2}{3}R.$$

So if $z = Rw$ on $|z| = \frac{R}{2}$ we have

$$3u(0) - 2R \leq u(z)$$

$$3u(0) - 4|z| \leq u(z).$$

This holds for any $R > 0$ and in fact holds if $R = 0$. So $3u(0) - 4|z| \leq u(z) \leq |z|$. This gives the bound

$$|u(z)| \leq 3|u(0)| + 4|z| \qquad (*).$$

Now we can use (*) and the Herglotz Integral from part (a) to make a bound. We have for $|z| < \frac{r}{2}$ it holds that

$$\begin{aligned}|f(z)| &\leq |v(0)| + \frac{1}{2\pi}\int_0^{2\pi} \left| \frac{re^{it}+z}{re^{it}-z} |u(re^{it})| \right| dt \\ &\leq A + B|z|.\end{aligned}$$

So now if we consider $\frac{f(z)}{z}$ this is a meromorphic function bounded in a neighborhood of ∞ and by Big Picard and has removable singularities and therefore extends to be entire. By Liouville's theorem we know that $\frac{f(z)}{z}$ is constant □

Part c.

(Toro's Proof)

Proof. Assume not. Therefore there exist $\rho_n \to \infty$ such that there exists z_n such that $|z_n| = \rho_n$ and

$$\max_{|z| \leq \rho_n} e^{u(z)} \leq \max_{|z| = \rho_n} e^{u(z)} \leq \frac{1}{n}e^{\rho_n}.$$

So when $|z| \leq \rho_n$ e have $u(z) \leq \log \frac{1}{n} + \rho_n$. If we let $|z| \leq \frac{\rho_n}{2}$ then

$$|f''(z)| \leq C\rho_n \to 0.$$

This implies the contradiction that f is linear.

□

(Kristen's Proof)

Proof. WLOG assume $u(0) < 0$. Since $|e^f| = e^u$ it suffices to prove that there exists $a > 0$ and $b \in \mathbb{R}$ such that

$$M_r(u) \geq ar + b,$$

where $M_r(u) = \max_{|z|=r} |u(z)|$. If there are no such a, b then for all a, b there exists r such that $M_r(u) < ar + b$. If $a = 1$ and $b = -n$ then we get a sequence $r_n \to \infty$ for which the condition fails. So $M_{r_n}(u) < r_n - n < r_n$. We claim that $u(z) \leq D|z|$ for some constant $D > 0$. Assume this is false. Then for all $k > 0$ there exists a sequence $\{s_k\}$ such that $M_{s_k}(u) > kS_k$. Choose some $s_k = S$ and $r_n > s_k$ and call $r_n = R$. For $|z| < R$ we have $u(z) < R$. So $R - u(z)$ is a positive harmonic function and therefore $R - u_R(w)$ is positive and harmonic on $\mathbb{D}$with $|w| = S/R$. Applying Harnack's inequality we then have

$$\frac{1 - \frac{S}{R}}{1 + \frac{S}{R}}(R - u(0)) \leq R - u_R(w) \leq \frac{1 + \frac{S}{R}}{1 - \frac{S}{R}}(R - u(0)).$$

Rearranging, if $z = Rw$ on $|z| = 2$ then

$$kS < u(z) \leq \frac{R - S}{R + S}(u(0) - R) + R.$$

In particular we have

$$(k-2)S < (k-1)S\left(\frac{R+S}{R-S}\right) - S < u(0)$$

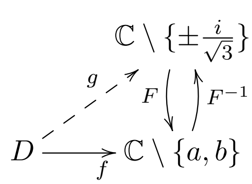
which is a contradiction since this says $(k-2)S < u(0)$ but we assumed $u(0) < 0$. Therefore, by part (b) we know that $f(z) = az + b$. After some calculus by looking at $z = x + iy$ we obtain $M_r(u) = r|a| + \text{Re}(b)$.

□

(2012 #5) Let $D \subseteq \mathbb{C}$ be a simply connected domain. Find a points $a, b \in \mathbb{C}$ such that for any analytic function $f : D \to \mathbb{C} \setminus \{a, b\}$ there exists an analytic function $g \in H(D)$ such that $f(z) = g(z)^3 + g(z)$.

Proof. Let $F(z) = z^3 + z$. This has derivative $F'(z) = 3z^2 + 1$ which is nonzero so long as $z \neq \pm\frac{i}{\sqrt{3}}$. Accordingly, let $a = F(\frac{i}{\sqrt{3}})$ and $b = F(-\frac{i}{\sqrt{3}})$. It follows from the inverse function theorem that $F^{-1} : \mathbb{C} \to \mathbb{C} \setminus \{a, b\}$ is a locally analytic function. Accordingly, we can locally define a function $g(z) = F^{-1}(f(z))$ which satisfies

$$g(z)^3 + g(z) = F(g(z)) = F(F^{-1}(f(z))) = f(z).$$



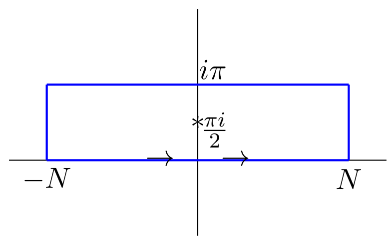
We claim that we can extend g to be analytic on all of D. Fix $z_0, z_1 \in D$. Since D is simply connected it (by the monodromy theorem) suffices to show that for any simple curve $\gamma \subseteq D$ with $\gamma(0) = z_0$ and $\gamma(1) = z_1$ we can analytically extend $g(z)$ along γ.

Let $z_t = \gamma(t)$ and let Δ_t be the maximal neighborhood of z_t such that $F^{-1} : f(\Delta_t) \to \mathbb{C} \setminus \{a, b\}$ is defined and has three distinct components. Reduce to a finite subcover and include Δ_0. Make a choice of one of the components of $F^{-1}(f(\Delta_0))$ and from there we inductively build the extension by choosing the component which intersects $F^{-1}(f(\Delta_{n-1}))$.

□

(2012 #6) Show that $\int_{-\infty}^{\infty} \frac{\cos x}{e^x + e^{-x}} = \frac{\pi}{e^{\pi/2} + e^{-\pi/2}}$.

Proof. Let $f(z) = \frac{e^{iz}}{e^z + e^{-z}}$. This has a simple pole at $z = \frac{\pi i}{2}$. Let $A = \int_0^\infty f(z)dz$. Consider the contour along a rectangle, R_N,



It is easy to prove that on the vertical sections of the rectangle the integral of $f(z)$ goes to zero as $N \to \infty$. Using the relation $f(t + i\pi) = e^{-\pi} f(t)$ we use the residue theorem to write

$$\int_{R_N} f(z)dz = A + e^{-\pi} A = 2\pi i \operatorname{Res}(f, \pi i/2).$$

We can compute the residue $\operatorname{Res}(f, \pi i/2) = \frac{e^{-\pi/2}}{2i}$. Plugging in and simplifiying gives the formula.

□

(2012 #7) Let $\tau \in \mathbb{C} \setminus \mathbb{R}$ and $\Lambda = \{a + b\tau : a, b \in \mathbb{C}\}$. Suppose that f is a non constant meromorphic function with $f(z + \omega) = f(z)$ for all $z \in \mathbb{C}$ and $\omega \in \Lambda$. The fundamental parallelogram for $a \in \mathbb{C}$ is $P_a = \{a + t + s\tau : 0 \le t, s < 1\}$.

(a) For each $a \in \mathbb{C}$ prove that f assumes each value $c \in \mathbb{C}^*$ an equal number of times (counting multiplicities) in P_a.

(b) Show that the degree f is at least 2.

Proof. (a) First suppose that f has no zeroes or poles on ∂P_a. We can integrate and use the fact that f and f' are periodic to cancel along the differently oriented sides of the integral

$$\int_{\partial P_a} \frac{f'(z)}{f(z)} dz = 0 = \#\{\text{zeros in } P_a\} - \#\{\text{Poles in } P_a\}$$

Fix $c \in \mathbb{C}^*$. Notice that the same argument applies to $f_c(z) = f(z) - c$. But $f_c(z)$ has the same number of poles as f. Therefore f_c and f have the same number of zeros. It follows that f attains each c an equal number of times.

If there are zeros or poles on ∂P_a then we can avoid them with small ϵ disks and still obtain the same counting result.

(b) We can write

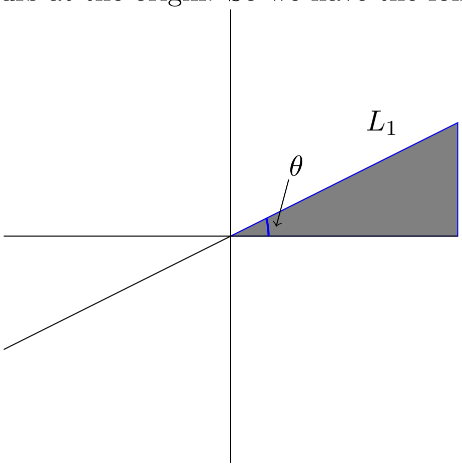
$$int_{\partial P_a} f(z)dz = 0 = 2\pi i \sum \text{Res}_{z\in P_a} f(z).$$

If f has only a simple pole at z_0 then $\text{Res}\, z_0 f(z) \neq 0$. Therefore f has either two poles a a double pole. So f has degree at least 2.

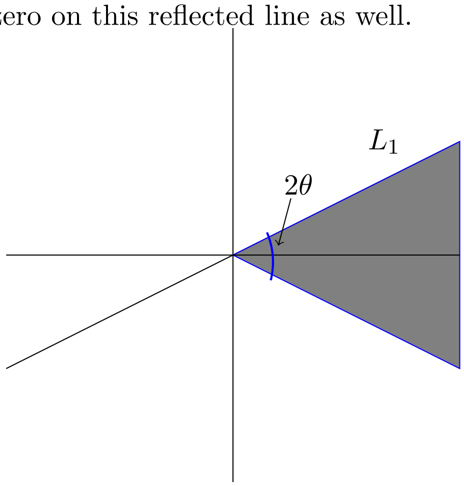
□

(2012 #8) Let u be a nonzero harmonic function on $\mathbb{C}$. Show that if u vanishes on two intersecting lines then the angle between the lines is a rational multiple of π.

Proof. By translating and rotating we can assume that one line is the real axis and the intersection occurs at the origin. So we have the following



Let u be the imaginary part of some entire function $f(z) = v(z)+iu(z)$. Since $\text{Im}(f(z)) \to 0$ as $z \to \mathbb{R}^+$ we can reflect across $\mathbb{R}^+$ with the extension $f(\overline{z}) = \overline{f(z)}$. The reflection of L_1 is the line with angle $-\theta$ through the origin. The extension guarantees that u is zero on this reflected line as well.



We can then rotate by θ and reflect again to obtain a new line on which u is zero. If θ is not a rational multiple of π then repeating this process countably many times will give a dense set of lines on which u vanishes. This implies the contradiction that $u \equiv 0$. □