Instructions.

- Turn your answers in to your Sakai Dropbox by 10pm on Saturday.
- Please submit as a typed text file, latexed pdf, or word document converted to a pdf. Written answers will not be accepted. It should be short, and roughly look like this (with NAME replaced by your name, and '#' an actual number or expression):

  NAME

  1. #
  2. exact # $\approx$ decimal #
  3. (a) # (b) #
  4. (a) # (b) #
  5. #
  6. (a) # (b) #
  7. #
  8. formula involving a
  9. (a) # (b) #
  10. (a) # (b) row vector with 6 #s
- No justification or shown-work is needed for any problem. Just write a number or exact expression that could be evaluated.
- You may write a short explanation next to your answer if you want to have a better chance at partial credit.
- If it comes up, please type $C_{n,k}$ as $C(n,k)$.
- You may use the textbook, your notes from class, content on the course website, and wolfram alpha.
- No time limit.
- Do not discuss the exam questions with others, or use the internet resources beyond those listed above.
- There are 10 problems. Some are easier than others, but each is worth 4 points. Stay cool, but this means that each point is worth 1% of your course grade. If you are close you might get partial credit. Please carefully check your answers.

1. Daryl is flipping a fair coin. What is the probability she gets 3 heads before flipping 4 tails?

Solution: $1/8 + C_{3,1}/16 + C_{4,2}/32 + C_{5,3}/64 = 21/32$.

2. Poi(36.5) people walk into a room. Assume that each of them is equally likely to have any of the 365 non-leap year birthdays. What is the probability that at least two people have the same birthday? Please write an exact expression and a decimal.

Hint: let N_i be the number of people with birthday i. Poisson thinning can be used to exactly compute the probability that $N_i \leq 1$ for all $i = 1, 2, \dots, 365$.

Solution: Let $n = 36.5$. If N_i is the number of people with birthday i then we are interested in the complement of the event $B = \cap_{i=1}^{365}\{N_i \leq 1\}$. By Poisson thinning $N_i = \text{Poi}(n/365)$ and so $P(N_i \leq 1) = e^{-n/365} + e^{-n/365}(n/365) = e^{-n/365}(1 + n/365) = e^{-36.5}(11/10)^{365}$. Using the fact that the N_i are independent we have

$$P(B) = P(N_1 \leq 1)^{365} = (e^{-n/365}(1 + n/365))^{365} = e^{-n}(1 + n/365)^{365} = e^{-36.5}(1 + 1/10)^{365}$$

This is equal to .1805. Thus the answers are $1 - e^{-36.5}(11/10)^{365}$ and $1 - e^{-36.5}(11/10)^{365} \approx .8195$.

3. Suppose that $X_1, \dots, X_{10}$ are independent geometric random variables with success parameter 1/2. Let $X = \max_{i \leq 10} X_i$ be the maximum.

(a) Use a union bound (*i.e.* $P(\cup_1^n A_i) \leq \sum_1^n P(A_i)$) to provide an upper bound on $P(X > 5)$.

Solution: A union bound gives

$$P(X > 5) = P(\cup_1^{10}\{X_i > 20\}) \leq 10P(X_1 > 5) = 10(1/2)^5 = 10/32 \approx .3125.$$

(b) Find $P(X > 5)$ exactly.

Solution:
The exact solution is $1 - P(X_1 \leq 5)^{10} = 1 - (1 - (1/2)^5)^{10} = .272$.

4. You are driving a bus with 10 people on it. The bus only stops if someone asks to get off. Each person is equally likely to ask to get off at each of the 8 stops. Let X be the number of stops the bus makes.

(a) What is EX?

Solution: Let X_i be 1 if the bus stops at stop i. Notice that $P(X_i = 1) = 1 - (7/8)^{10} = .7369 = p$. Since $X = \sum_1^5 X_i$ is the number of stops we have $EX = 8p \approx 5.9$.

(b) What is var(X)?
Hint: If $A_i = \{stop\ at\ i\}$ then $P(A_1 \cap A_2) = P(A_1) + P(A_2) - P(A_1 \cup A_2) = 2P(A_1) - (1 - P(A_1^c \cap A_2^c))$.

Solution: Let $A_i = \{X_i = 1\}$. $\text{cov}(X_1, X_2) = P(A_1 \cap A_2) - p^2 = P(A_1) + P(A_2) - P(A_1 \cup A_2) - p^2 = 2p - (1 - P(X_1, X_2 = 0) - p^2 = 2p - 1 + (6/8)^{10} - p^2 \approx -.01289 = q$. So

$$\text{var}(X) = 8p(1-p) + 2C_{8,2}q \approx .829.$$

5. A marathon runner has observed that her mile splits for the first 16 miles of a marathon are i.i.d. with mean 7 minutes and standard deviation 0.5 minutes. Use normal approximation to estimate the probability she runs the first 16 miles in less than ('<') 108 minutes. Actually lookup Φ in the table.

Solution: Let T_i be the duration of her ith mile. The time to run 16 miles is $T = \sum_1^{16} T_i$. Notice that $\text{var}(T) = 16 * .25 = 4$ and so $\sigma(T) = 2$. We can write

$$P(T \leq 120) \approx P(X \leq \frac{108 - 112}{.2}) = P(X \leq -2) = 1 - \Phi(2).$$

6. The length of human pregnancies is approximately normal with mean 266 days and standard deviation 16 days. Assume days are discrete and we take the floor (i.e. a 244.63 day pregnancy is a 244 day pregnancy).
   (a) What is the probability that a pregnancy lasts less than ('<') 240 days (about 8 months)? Write your answer in terms of $\Phi(x) = P(N \leq x)$. You do not need to look anything up on the table, or do any arithmetic.

   Solution: $Y = N(266, 16^2)$. Because of the floor condition we do not need to histogram correct. For example, $X = 228$ if the pregnancy is anywhere in $[228, 229)$. So, $\{X < 240\} \approx \{Y < 240\}$.

   $$P(X < 240) \approx P\Big(N \leq \frac{240 - 266}{16}\Big) = P(N \leq -1.625) = 1 - \Phi(1.625).$$

   Okay to leave as $\Phi(-1.625)$.

   (b) Find c so that there is a 99% probability that a pregnancy lasts longer than c days.

   Solution: We want $P(X \geq c) = .99$. This means that $P\Big(Y \geq \frac{c-266}{16}\Big) = .99$. Looking at the table we need

   $$\frac{c - 266}{16} = -2.33.$$

   This has solution $c = 228.72$. Since we are taking the floor we need $c = 228$.

7. The alpha fetal protein test is meant to detect spina bifida in unborn babies, a condition that affects 1 out of 1000 children who are born. The literature on the test indicates that 5% of the time a healthy baby will cause a positive reaction. We will assume that the test is positive 100% of the time when spina bifida is present. Your doctor has just told you that your baby's alpha fetal protein test was positive. What is the probability that your baby has spina bifida?

Solution:

Let A = "a positive reaction," and let B = "the baby has spina bifida." We want to calculate $P(B|A)$. By the definition of conditional probability,

$$P(B|A) = P(B \cap A)/P(A)$$

To evaluate the numerator we use the multiplication rule

$$P(B \cap A) = P(B)P(A|B) = 0.001 \cdot 1 = 0.001$$

Similarly,

$$P(B^c \cap A) = P(B^c)P(A|B^c) = 0.999 \cdot 0.05 \approx 0.05$$

Now $P(A) = P(B \cap A) + P(B^c \cap A)$ so

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{0.001}{0.001 + 0.050} = \frac{1}{51}$$

Thus the probability of spina bifida given the positive reaction is only about 2%.

8. Suppose X and Y are independent uniform(0,1) random variables. Let $0 \le a \le 1$. What is the probability that a is in the line segment connecting X and Y. Your answer will be a formula involving a.

   Hint: condition on the value of X relative to a.

Solution:

$$\begin{aligned} P(a \in [X,Y]) &= P(a \in [X,Y] \mid X < a)P(X < a) + P(a \in [X,Y] \mid X > a)P(X > a) \\ &= aP(Y > a) + (1-a)P(Y < a) = 2a(1-a). \end{aligned}$$

9. A car rental company has rental has lots at both Kennedy and LaGuardia airports. Assume that a car rented at one airport must be returned to one of the two airports. If the car was rented at LaGuardia the probability it will be returned there is $\frac{3}{4}$; for Kennedy the probability is $\frac{1}{2}$. Suppose that each week all of the cars are rented once and that we start with 1/3 of the cars at LaGaurdia.

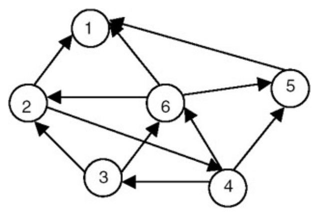
   (a) What is the fraction of cars at LaGuardia after the first week?

   Solution: $qp = [\frac{1}{4} + \frac{1}{3} \quad \frac{1}{12} + \frac{1}{3}] = [\frac{7}{12} \quad \frac{5}{12}]$. So the answer is $\frac{7}{12}$.

   (b) What is the long term fraction of cars at Kennedy?

   Solution: The stationary distribution is $\pi = [\frac{1/2}{3/4} \quad \frac{1/4}{3/4}] = [\frac{2}{3} \quad \frac{1}{3}]$. So in the long run $\frac{1}{3}$ of the cars will be at Kennedy.

10. A website has the following link structure:



(a) A site-surfer randomly clicks around the website. This markov chain has a transition matrix p (which you do not need to find). Given that

$$p^3 = \begin{pmatrix} 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.666 & 0.139 & 0.0 & 0.055 & 0.055 & 0.083 \\ 0.5 & 0.0 & 0.139 & 0.083 & 0.139 & 0.139 \\ 0.583 & 0.055 & 0.037 & 0.194 & 0.092 & 0.037 \\ 1.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.61 & 0.092 & 0.055 & 0.037 & 0.092 & 0.111 \end{pmatrix}.$$

What is the probability a site-surfer starting at page 6 is at page 4 after three clicks?

Solution: $p^3(6,4) = .037$.

(b) Without doing any calculations find the stationary distribution of p. Write your answer as a row vector $\pi = (\pi(1), \pi(2), \ldots, \pi(6))$.

Hint: Look at the link structure. What must happen in the long term?

Solution: $\pi = [1\ 0\ 0\ 0\ 0\ 0]$ since Page 1 is an absorbing state.