NUMERICAL WEAK APPROXIMATION OF STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY LÉVY PROCESSES

by

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Notation

$$
\begin{aligned}
\mathbb{N}^0 &= \mathbb{N} \cup \{0\} \\
\mathbb{R}^d_0 &= \mathbb{R}^d \backslash \{0\} \\
\mathbb{R}^+ &= (0, \infty) \\
H &= [0, T] \times \mathbb{R}^d \\
(x, y) &= \sum_{i=1}^{d} x_i y_i, \forall x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d, y = (y_1, y_2, \ldots, y_d) \in \mathbb{R}^d \\
|x| &= (x, x)^{\frac{1}{2}} \\
|B| &= \sum_{i=1}^{d} |B^{ii}|, \forall B \in \mathbb{R}^{d \times d} \\
C_0^\infty(\Gamma) &= \{\text{infinitely differential functions on } \Gamma \text{ with compact support}\} \\
\partial_0 u(t, x) &= \partial_t u(t, x) = \frac{\partial}{\partial t} u(t, x) \\
\partial_i u(t, x) &= \frac{\partial}{\partial x_i} u(t, x), i = 1, \ldots, d \\
\partial_{ij} u(t, x) &= \frac{\partial^2}{\partial x_i x_j} u(t, x), i, j = 1, \ldots, d \\
\partial_x u(t, x) &= \nabla u(t, x) = \big(\partial_1 u(t, x), \ldots, \partial_d u(t, x)\big) \\
\partial_x^2 u(t, x) &= \Delta u(t, x) = \sum_{i=1}^{d} \partial_{ii} u(t, x) \\
\partial_x^\gamma u(t, x) &= \frac{\partial^{|\gamma|}}{\partial^{\gamma_1} x_1 \ldots \partial^{\gamma_d} x_d} u(t, x), \text{ where } \gamma = (\gamma_1, \ldots, \gamma_d) \text{ is a multiindex}
\end{aligned}
$$

$$
\begin{aligned}
\mathscr{F}u(t,\xi) &= \int e^{-i(x,\xi)}u(t,x)dx \\
\mathscr{F}^{-1}u(t,x) &= (2\pi)^{-d}\int e^{i(x,\xi)}u(t,\xi)d\xi
\end{aligned}
$$

Abstract

Lévy processes are the simplest generic class of processes having a.s. continuous paths interspersed with jumps of arbitrary sizes occurring at random times, which makes them useful tools in a variety of fields including mathematics, physics, engineering, and finance.

In stochastic analysis, it is frequently necessary to evaluate functionals of the process modeling the system of interest. In general, the law of the process is unknown and a closed-form solution is unrealistic. An alternative possibility is to numerically approximate the functionals by discrete time Monte-Carlo simulation, which is widely applied in practice. The simplest scheme for Monte-Carlo simulation is the weak Euler approximation. In such numerical treatment of stochastic differential equations, it is of theoretical and practical importance to estimate the rate of convergence of the discrete time approximation.

In this dissertation, the weak Euler approximation for stochastic differential equations driven by Lévy processes is studied. The model under consideration is in a more general form but with weaker assumptions than those in existence. Hence, it is applicable to a broader range of processes arising from various fields. In order to investigate the convergence of the weak Euler approximation to the process considered, the existence of a unique solution to the corresponding integro-differential equation in Hölder space is first proved. It is then identified that the Euler scheme yields a positive weak order of convergence,

provided that the coefficients of the stochastic differential equation are Hölder-continuous and the test function is continuously differentiable to some positive order. In particular, if the coefficients are slightly more than twice differentiable and the test function has up to the fourth order derivative, then first weak order convergence is guaranteed.

Chapter 1

Introduction

1.1 Motivation

1.1.1 From Wiener Process to Lévy Process

Since its introduction in the early years of the twentieth century as a model for the physical phenomenon of Brownian motion by Einstein [30] and Smoluchowski [79] and as a description of the dynamical evolution of stock prices by Bachelier [5], the Wiener process has been the most intensively studied stochastic process.

Two important properties of the Wiener process are continuity of sample paths and scale invariance, while many phenomena that were first described by the Wiener process do not exhibit those two properties. For example, the classical Black-Scholes model [19] assumes that a stock price follows a geometric Brownian motion. However, stock prices change by units and opening prices are often not exactly the same as the closing prices of the previous trading day. In addition, external shocks occur regularly, which are either reasonably predictable or wholly inaccessible. Predictable external shocks include earnings announcements, going ex-dividend, scheduled meetings of the central bank to adjust interests rates, and so on. Inaccessible ones include unexpected events such as wars, political assassinations, terrorist attacks, currency collapses, and natural disasters. Hence, stock prices move essentially by jumps, including both regularly reasonable small ones and occasionally unpredictable large ones, at intraday scales, and only over longer time scales does their behavior resemble Brownian motion. In addition, empirical studies of stock prices indicate distributions with heavy tails and skewness, which are incompatible with models based on the Wiener process. Therefore, a more realistic model is desirable for stock market behavior. Analogous considerations apply in other arenas such as foreign exchange currency markets and government security markets, where alone there are often substantial jumps related either to central bank intervention or to the release of significant macroeconomic information.

One alternative option is the jump-diffusion process [57], which is capable of modeling large and sudden changes and naturally exhibits high skewness and leptokurtosis levels typically observed in financial time series. Hence, it has gradually become a standard modeling tool in various markets including equity, foreign exchange, fixed income, commodity, and energy derivatives [6, 7, 14, 15, 20, 21, 31, 32, 39, 45, 46, 47, 56, 70]. In particular, when the jump-diffusion process is a Lévy process such as in Merton's and Kou's models [52, 57], the characteristic function of the process can be obtained in a closed form, which is essential, for example, in pricing European options efficiently by inverting the Fourier transform using the fast Fourier transform algorithm [23, 29].

For this as well as other theoretical and practical reasons [10], there has been a renaissance of interest in Lévy processes [33, 82] in recent years. Lévy processes are the simplest generic class of processes having a.s. continuous paths interspersed with random jumps of arbitrary sizes occurring at random times, which makes them useful tools in a broad variety of fields from mathematics including differential geometry and extreme value theory to physics including Burgers' turbulence and quantum theory. Other applications can be found in engineering and sciences [16, 24, 69, 75, 85, 92], and particularly, in finance [11, 26, 80], from portfolio and risk management to option and bond pricing and hedging.

1.1.2 From Analytic Solution to Numerical Approximation

Let $X = \{X_t\}_{t\in[0,T]}, T \in \mathbb{R}^+$ be the process modeling the system of interest. In stochastic analysis, it is frequently necessary to evaluate functionals of the system, such as moments, functional integrals, invariant measures, and Lyapunov exponents. Specifically, for a given test function g, the problem of computing the expectation $\mathrm{E}[g(X_T)]$ arises from various applications. In random mechanics, given a random dynamical system with white noise, it is important to find the first two moments of the response or the probability that the response reaches a certain level. In telecommunications, $\mathrm{E}[g(X_T)]$ represents the average energy of the system at time T, which is critical to the design and maintenance of telephone lines. In finance and insurance, it is extremely common to evaluate $\mathrm{E}[g(X_T)]$. For instance, in the standard Black-Scholes model, a stock price X is assumed to follow a diffusion process, the solution to a stochastic differential equation driven by the Wiener process. An option is then priced as $\mathrm{E}[g(X_T)]$, for a known convex function g.

When both the model and the test function g are sufficiently simple, there is a closed-form expression for the expectation. However, the real world is much more complicated. Take equity markets as an example. Security prices usually have jumps, as mentioned in Section 1.1.1. There are serious problems in the loss of completeness, the martingale representation property mathematically, for models with jumps. Nevertheless, arbitrage-free models can still be constructed for a theory of option pricing in the same spirit as the

Black-Scholes model, which again leads to the problem of evaluating $\mathrm{E}[g(X_T)]$, where X is the solution to a stochastic differential equation driven by a Lévy process with jumps.

In special cases where g is sufficiently smooth and the increments of the driving Lévy process can be simulated, a closed-form solution in analogy to the Black-Scholes paradigm may exist. For instance,

$$X_t = X_0(1+c)^{N_t} \exp\left[\left(a - \frac{1}{2}b^2\right)t + bW_t\right]$$

is the solution to

$$X_t = X_0 + a\int_0^t X_{s-}ds + b\int_0^t X_{s-}dW_s + c\int_0^t X_{s-}dN_s, \forall t \in [0,T],$$

where $W = \{W_t\}_{t\in[0,T]}$ is a scalar Wiener process, $N = \{N_t\}_{t\in[0,T]}$ is a scalar Poisson process [26, 35], and X_{t-} denotes the left hand limit of X at time t. In general, the laws of X_T and $g(X_T)$ as well as their means are all unknown, so a closed-form solution is unrealistic.

One alternative possibility is then to numerically approximate $\mathrm{E}[g(X_T)]$ by a discrete time Monte-Carlo simulation of the stochastic process X. This approach has been widely applied [1, 2, 22, 27, 38, 40, 90, 94] and is very popular in practice. The simplest discrete time approximation of X that can be used for such Monte-Carlo methods is the weak Euler approximation.

1.2 Objective

It is of theoretical and practical importance to estimate the rate of convergence of a discrete time approximation. The Euler approximation Y of a stochastic process X is said to converge with a weak order $\kappa > 0$ if for each smooth function g, there exists a constant K, depending only on g, such that

$$|\mathrm{E}[g(Y_T)] - \mathrm{E}[g(X_T)]| < K\delta^{\kappa},$$

where $\delta > 0$ is the maximum step size of the time discretization.

Platen and Kloeden gave an extensive list of papers that deal with the discrete time approximations for Itô processes by Euler and higher order schemes [48, 73]. Milstein was among those who first studied the order of weak convergence, in particular first-order convergence, of discrete-time approximations for diffusion processes [66, 67, 68]. Talay and Hu investigated a class of second weak-order approximations for diffusion processes [41, 86, 87, 88]. Talay and Tubaro, as well as Kloeden, Platen, and Hofmann, developed the related extrapolation techniques [49, 89]. Schurz, with Saito and Mitsui, investigated stability of numerical schemes for stochastic diffusions to control error propagation [77, 81]. For the discrete time approximations of Itô processes with jump components, Mikulevičius and Platen showed first weak-order convergence in the case in which the coefficient functions possess fourth-order continuous differentiability [58]. Protter, Talay, Jacod, and Rubenthaler presented similar results for Lévy-driven stochastic differential equations [42, 75, 76].

In practice, the coefficient functions and the test function do not always have the smoothness properties assumed in the papers cited above, as studied by Bally, Talay, and

Guyon [8, 9, 37]. Mikulevičius and Platen first proved that there is still some weak-order convergence of the Euler approximation for diffusion processes under Hölder conditions on the coefficient and test functions [59]. Kubilius and Platen generalized the result to diffusion processes with jumps [54]. The goal of this dissertation is to prove that the convergence still holds for stochastic processes driven by Lévy motions.

The dissertation is organized as follows. Chapter 2 briefly reviews Lévy processes and stochastic differential equations, and specifies the model considered. Chapter 3 discusses solutions to integro-differential equations in Hölder space, the result of which is invoked in the proof of the main theorem [64]. Chapter 4 defines the Euler approximation with the basic time discretization and presents the main theorem as well as the proof. Chapter 5 generalizes the result obtained in Chapter 4 to a general equation. Chapter 6 outlines future work.

Chapter 2

Stochastic Differential Equations Driven by Lévy Processes

%addcontentslinetocchapterChapter 2 Stochastic Differential Equations Driven by Lévy Processes

In this chapter, the definition, examples, and properties of Lévy processes are first discussed. The stochastic differential equation considered is then introduced.

2.1 Lévy Process

2.1.1 Definition

Let $T \in \mathbb{R}^+$ denote the time horizon and $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ be a filtered probability space, where $\mathcal{F} = \mathcal{F}_T$ and the filtration $\mathbf{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$, an increasing family of sub σ-algebras of $\mathcal{F}$, satisfies the usual hypotheses [28]:

- Completeness: $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$, and
- Right Continuity: $\mathcal{F}_t = \mathcal{F}_{t+}$, where $\mathcal{F}_{t+} = \bigcap_{\varepsilon>0} \mathcal{F}_{t+\varepsilon}$.

A stochastic process [44] is a family of random variables $X = \{X_t\}_{t\in[0,T]}$ indexed by time. The parameter t may be either discrete or continuous. For each realization of the randomness ω, the trajectory $X(\omega) : t \to X_t(\omega)$ defines a function of time, called a sample path of the process. Thus, stochastic processes can also be viewed as random functions: random variables taking values in function spaces. A stochastic process can also be seen as a function X of both time t and the randomness ω on $[0, T] \times \Omega$.

Definition 2.1. *A stochastic process* $X = \{X_t\}_{t\in[0,T]}$ *is adapted to the given filtration* $\mathbf{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ *if each* X_t *is* $\mathcal{F}_t$*-measurable.*

An $\mathcal{F}_t$-adapted process is also called a nonanticipating process, whose value at time t is revealed by the information $\mathcal{F}_t$.

Definition 2.2. *A function* $f : [0, T] \mapsto \mathbb{R}^d$ *is said to be càdlàg if it is continue à droite et limité à gauche, that is, right continuous with left limits.*

Specifically, a function f is càdlàg if for each $t \in [0, T]$, the two limits $f_{t-} = \lim_{s\uparrow t} f_s$ and $f_{t+} = \lim_{s\downarrow t} f_s$ exist, and $f_t = f_{t+}$.

Define $\Delta f_t = f_t - f_{t-}$. If f is càdlàg, it has only jump discontinuities and $\{t : \Delta f_t \neq 0, t \in [0, T]\}$ is at most countable. In addition, $\{t : \Delta f_t > \varepsilon, t \in [0, T]\}$, $\forall\varepsilon > 0$ is finite. Hence, a càdlàg function on $[0, T]$ has a finite number of "large jumps" and a possibly infinite but countable number of "small jumps".

A Lévy process [3, 18, 78] is essentially a stochastic process with stationary and independent increments.

Definition 2.3. *A càdlàg, adapted, real-valued process* $L = \{L_t\}_{t\in[0,T]}$ *with* $L_0 = 0$ *a.s. is called a Lévy process if it has*

(L1) *Independent Increments:* $L_t - L_s \perp \mathcal{F}_s$, $0 \le s < t \le T$;

(L2) *Stationary Increments:* $L_t - L_s \overset{d}{=} L_{t-s}$, $0 \le s < t \le T$; *and*

(L3) *Stochastic Continuity:* $\lim_{s \to t} P(|L_t - L_s| > \varepsilon) = 0$, $\forall \varepsilon > 0, \forall t \in [0, T]$.

2.1.2 Examples

Common examples of Lévy processes include linear drift, the Wiener process, the Poisson process, the compound Poisson process, the Lévy jump-diffusion process, and the stable process.

Wiener Process

The Wiener process is the only non-deterministic Lévy process with continuous sample paths. In one dimension, a càdlàg, adapted, real-valued process $W = \{W_t\}_{t \in [0,T]}$ is call a *Wiener process* with variance σ^2 if it satisfies

- $W_0 = 0$;
- $W_t - W_s \perp \mathcal{F}_s$, $0 \le s < t \le T$;
- $W_t - W_s \sim \text{Normal}\big(0, \sigma^2(t - s)\big)$, $0 \le s < t \le T$; and
- W has continuous sample paths.

If $\sigma^2 = 1$, $W = \{W_t\}_{t \in [0,T]}$ is called a *standard Wiener process*.

The following are a number of useful properties of the Wiener process.

- The Wiener process is locally Hölder continuous with exponent $\alpha \in (0, \frac{1}{2})$, that is, there exists a constant $K = K(T, \omega)$ such that

$$|W_t(\omega) - W_s(\omega)| \le K|t - s|^{\alpha}, 0 \le s < t \le T, \forall \omega \in \Omega.$$

- The sample path $W(\omega) : t \to W_t(\omega), \forall \omega \in \Omega$ is almost surely nowhere differentiable.

- For any sequence $\{t_n\}_{n\in\mathbb{N}}$ with $t_n \uparrow \infty$,

$$\liminf_{n\to\infty} W_{t_n} = -\infty \quad \text{a.s.} \quad \text{and} \quad \limsup_{n\to\infty} W_{t_n} = \infty \quad \text{a.s.}$$

Compound Poisson Process

A càdlàg and adapted stochastic process $N = \{N_t\}_{t\in[0,T]}$ is called a *Poisson process* with intensity $\lambda \in \mathbb{R}^+$ if it satisfies

- $N_0 = 0$;
- $N_t - N_s \perp \mathcal{F}_s$, $0 \le s < t \le T$; and
- $N_t - N_s \sim \text{Poisson}\big(\lambda(t-s)\big)$, $0 \le s < t \le T$.

Let N be a Poisson process with intensity λ. The process $\tilde{N} = \{\tilde{N}_t\}_{t\in[0,T]}$, where $\tilde{N}_t = N_t - \lambda t$, is called a *compensated Poisson process*, which satisfies $\mathrm{E}[\tilde{N}_t] = 0$ and $\mathrm{E}[\tilde{N}_t^2] = \lambda t$, $\forall t \in [0,T]$.

Definition 2.4. *An adapted process $X = \{X_t\}_{t\in[0,T]}$ is called a martingale if it satisfies*

$$\mathrm{E}[|X_t|] < \infty, \forall t \in [0,T]$$

and

$$\mathrm{E}[X_t|\mathcal{F}_s] = X_s \quad \text{a.s.}, \quad \forall t \in (s,T], \forall s \in [0,T].$$

Proposition 2.5. *The compensated Poisson process is a martingale.*

Define nonnegative random variables $\{T_n\}_{n\in\mathbb{N}^0}$ by

$$T_0 = 0 \quad \text{and} \quad T_n = \inf\{t : N_t = n, t \in [0,T]\}, n \in \mathbb{N}. \tag{2.1}$$

The T_n's are gamma distributed. In addition, the inter-arrival times $T_n - T_{n-1}, n \in \mathbb{N}$ are i.i.d. exponential random variables with mean $\frac{1}{\lambda}$. The sample paths of N are piecewise constant with jump discontinuities of size 1 at each of the random times $\{T_n\}_{n\in\mathbb{N}}$.

A *compound Poisson process* is defined as

$$L_t = \sum_{k=1}^{N_t} J_k,$$

where $N = \{N_t\}_{t\in[0,T]}$ is a Poisson process with intensity λ and $J = \{J_k\}_{k\in\mathbb{N}}$ is a sequence of i.i.d. random variables with probability distribution function F and $\mathrm{E}[J] = \kappa < \infty$. Hence, jumps arrive according to the Poisson process and F describes the distribution of jump sizes. By conditioning and independence, the characteristic function of L_t is

$$\begin{aligned}
\mathrm{E}\big[e^{iuL_t}\big] &= \mathrm{E}\Big[\exp\Big(iu\big(\sum_{k=1}^{N_t} J_k\big)\Big)\Big] \\
&= \sum_{n=0}^{\infty} \mathrm{E}\Big[\exp\Big(iu\big(\sum_{k=1}^{N_t} J_k\big)\Big)\Big|N_t = n\Big] P(N_t = n) \\
&= \sum_{n=0}^{\infty} \Big[\int e^{iux} F(dx)\Big]^n e^{-\lambda t} \frac{(\lambda t)^n}{n!} \\
&= \exp\Big[\lambda t \int \big(e^{iux} - 1\big) F(dx)\Big].
\end{aligned}$$

A sample path of a compound Poisson process is piecewise constant with jump discontinuities at the random times $\{T_n\}_{n\in\mathbb{N}}$ as defined in (2.1) and the sizes of the jumps themselves are random. The jumps can be any value in the range of the random variables J_k's.

Lévy Jump-Diffusion Process

In one dimension, a *Lévy jump-diffusion process* $L = \{L_t\}_{t\in[0,T]}$ is the sum of a linear drift, a Wiener process, and a compensated compound Poisson process, that is,

$$L_t = bt + \sigma W_t + \Big(\sum_{k=1}^{N_t} J_k - \lambda t\kappa\Big), \forall t \in [0,T], \tag{2.2}$$

where $b \in \mathbb{R}$, $\sigma \in \mathbb{R}^+$, $W = \{W_t\}_{t\in[0,T]}$ is a standard Wiener process, $N = \{N_t\}_{t\in[0,T]}$ is a Poisson process with intensity λ, and $J = \{J_k\}_{k\in\mathbb{N}}$ is a sequence of i.i.d. random variables with probability distribution function F and $\mathrm{E}[J] = \kappa < \infty$. All sources of randomness are mutually independent. By independence, the characteristic function of L_t is

$$\begin{aligned}
\mathrm{E}\big[e^{iuL_t}\big] &= \mathrm{E}\Big[\exp\Big(iu\big(bt + \sigma W_t + \sum_{k=1}^{N_t} J_k - \lambda t\kappa\big)\Big)\Big] \\
&= e^{iubt}\mathrm{E}\big[e^{iu\sigma W_t}\big]\mathrm{E}\Big[\exp\Big(iu\big(\sum_{k=1}^{N_t} J_k - \lambda t\kappa\big)\Big)\Big] \\
&= e^{iubt}\exp\big[-\frac{1}{2}u^2\sigma^2 t\big]\exp\Big[\lambda t\Big(\int \big(e^{iux} - 1\big)F(dx) - iu\kappa\Big)\Big] \\
&= \exp\Big[t\Big(iub - \frac{1}{2}u^2\sigma^2 + \int \big(e^{iux} - 1 - iux\big)\lambda F(dx)\Big)\Big].
\end{aligned}$$

Since both the Wiener process and the compensated compound Poisson process are martingales, L is a martingale if and only if $b = 0$. If the compensated compound Poisson process in (2.2) is replaced by a compound Poisson process, that is,

$$L_t = bt + \sigma W_t + \sum_{k=1}^{N_t} J_k,$$

then, the process is called an *interlacing process*.

Stable Process

A random variable X is said to be *stable* if there exist two real-valued sequences $\{c_n\}_{n\in\mathbb{R}}$ and $\{d_n\}_{n\in\mathbb{R}}$ with $c_n > 0, \forall n$ such that

$$c_n X + d_n \stackrel{d}{=} X_1 + X_2 + \cdots + X_n,$$

where $X_1, \ldots, X_n$ are independent copies of X. X is said to be *strictly stable* if $d_n = 0$.

The only possible choice for c_n is $c_n = \sigma n^{\frac{1}{\alpha}}$, where $\alpha \in (0, 2]$ and $\sigma \in \mathbb{R}^+$. The parameter α plays a key role and is called the *index of stability*, a measure of concentration. It determines the rate at which the tails of the distribution taper off. $\mathrm{E}[X^2] < \infty$ if and only if $\alpha = 2$ and $\mathrm{E}[|X|] < \infty$ if and only if $\alpha \in (1, 2]$. In general, the p-th moment of a stable random variable is finite if and only if $p < \alpha$.

Proposition 2.6. *If X is a stable real-valued random variable, then its characteristic triplet (b, c, ν) must take one of the two forms:*

- *$\nu = 0$ and $X \sim \mathrm{Normal}(b, c)$, when $\alpha = 2$; or*
- *$c = 0$ and $\nu(dx) = \dfrac{c_1}{x^{1+\alpha}} \mathbf{1}_{\{x>0\}}(x)dx + \dfrac{c_2}{x^{1+\alpha}} \mathbf{1}_{\{x<0\}}(x)dx$, where $c_1, c_2 \geq 0$ and $c_1 + c_2 > 0$, when $\alpha \neq 2$.*

Three important examples of stable distributions whose density functions are in closed forms include the normal distribution ($\alpha = 2$), the Cauchy distribution ($\alpha = 1$ and $\beta = 0$), and the Lévy distribution ($\alpha = \frac{1}{2}$ and $\beta = 1$). In general, the density function of a stable distribution can be expressed only in a series form but cannot be written analytically.

However, the general characteristic function, which determines the probability distribution, is given by Proposition 2.7.

Proposition 2.7. *A real-valued random variable X is stable if and only if there exist $b \in \mathbb{R}$, $\sigma \in \mathbb{R}^+$, and $\beta \in [-1,1]$ such that for all $u \in \mathbb{R}$,*

$$\varphi_X(u) = \exp\left[ibu - |\sigma u|^\alpha \Big(1 - i\beta \mathrm{sgn}(u)\big(\mathbf{1}_{\{\alpha\neq 1\}} \tan(\frac{\pi\alpha}{2}) - \mathbf{1}_{\{\alpha=1\}}\frac{2}{\pi}\log(|u|)\big)\Big)\right].$$

In Proposition 2.7, b is the location parameter, a measure of centrality, σ is the scale parameter, a measure of dispersion, and β is the skewness parameter, a measure of asymmetry. The distribution is skewed to the right when $\beta > 0$, skewed to the left when $\beta < 0$, and symmetric when $\beta = 0$. Since a stable random variable X is characterized by four parameters, it is denoted by $X \sim S_\alpha(\sigma, \beta, b)$. If X is symmetric, written as $X \sim S\alpha S$, then Proposition 2.7 yields

$$\varphi_X(u) = \exp\left[ibu - |\sigma u|^\alpha\right], \alpha \in (0, 2].$$

One reason that stable distributions are important in applications is the nice decay property of tails, except when $\alpha = 2$, in which case tails decay exponentially. The relatively slow decay of tails for non-Gaussian stable distributions, the property of leptokurtic distributions with heavy tails, makes them ideally suitable for modeling a wide range of interesting phenomena.

A *stable process* is a Lévy process $Z = \{Z_t\}_{t\in[0,T]}$ whose Lévy symbol is given by Proposition 2.6. Hence, each Z_t is a stable random variable. A process is called a rotationally invariant stable process if the Lévy symbol is given by

$$\psi(u) = -\sigma^\alpha |u|^\alpha,$$

where α is the index of stability. This class of processes is important in applications because they display self-similarity. A stochastic process $Y = \{Y_t\}_{t\in[0,T]}$ is *self-similar* with a Hurst index $H \in \mathbb{R}^+$ if the two processes $\{Y_{at}\}_{t\in[0,T]}$ and $\{a^H Y_t\}_{t\in[0,T]}$ have the same distribution for all $a \in \mathbb{R}^+$. A Lévy process Z is self-similar if and only if each Z_t is strictly stable. A rotationally invariant stable process is self-similar with a Hurst index $H = \frac{1}{\alpha}$, and so, for example, the Wiener process is self-similar with $H = \frac{1}{2}$.

2.1.3 Properties

Definition 2.8. *The law p_X of a random variable X is infinitely divisible if for all $n \in \mathbb{N}$, there exists a random variable $X^{(1/n)}$ such that*

$$\varphi_X(u) = \left(\varphi_{X^{(1/n)}}(u)\right)^n,$$

where $\varphi_X(u) = \displaystyle\int e^{i(u,x)} p_X(dx)$ is the characteristic function of X.

Examples of infinitely-divisible distributions include normal distributions, Poisson distributions, compound Poisson distributions, exponential distributions, Γ-distributions, geometric distributions, negative binomial distributions, Cauchy distributions, and strictly stable distributions. Counter-examples are uniform distributions and binomial distributions.

A random variable is infinitely divisible if its law is infinitely divisible. The product UV of random variables is infinitely divisible if U is arbitrary but nonnegative, V is exponentially distributed, and U and V are independent [36].

Theorem 2.9. (**Lévy-Khintchine Formula**) *The law p_X of a random variable X is infinitely divisible if and only if there exists a triplet (b, c, ν), with c nonnegative, $\nu(\{0\}) = 0$, and $\int (1 \wedge |x|^2)\nu(dx) < \infty$, such that*

$$\varphi_X(u) = \mathrm{E}\big[e^{i(u,X)}\big] = \exp\Big[i(u,b) - \frac{1}{2}(u,cu) + \int \big(e^{i(u,x)} - 1 - i(u,x)\mathbf{1}_{\{|x|<1\}}\big)\nu(dx)\Big].$$

Here, b is the *drift coefficient*, c the *diffusion coefficient*, and ν the *Lévy measure*. They represent linear drift, the Wiener process, and jumps, respectively.

The triplet (b, c, ν) is called the *Lévy* or *characteristic triplet* and the exponent

$$\psi(u) = i(u,b) - \frac{1}{2}(u,cu) + \int \big(e^{i(u,x)} - 1 - i(u,x)\mathbf{1}_{\{|x|<1\}}\big)\nu(dx)$$

is called the *Lévy symbol* or *characteristic exponent*.

Consider a Lévy process $L = \{L_t\}_{t\in[0,T]}$. By the fact that

$$L_t = L_{\frac{t}{n}} + \big(L_{\frac{2t}{n}} - L_{\frac{t}{n}}\big) + \cdots + \big(L_t - L_{\frac{(n-1)t}{n}}\big), \forall n \in \mathbb{N}, \forall t \in (0,T]$$

and the independence and stationarity of increments, it follows that the random variable L_t is infinitely divisible.

Proposition 2.10. *If $L = \{L_t\}_{t\in[0,T]}$ is a Lévy process, then L_t is infinitely divisible and*

$$\mathrm{E}\big[e^{i(u,L_t)}\big] = e^{t\psi(u)}, \forall t \in [0,T],$$

where $\psi(u) = i(u,b) - \frac{1}{2}(u,cu) + \int \left(e^{i(u,x)} - 1 - i(u,x)\mathbf{1}_{\{|x|<1\}}\right)\nu(dx)$ *is the Lévy symbol of* L_1*, a random variable whose law is infinitely divisible.*

Every Lévy process can be associated with the law of an infinitely divisible distribution, and given an infinitely divisible random variable X, a Lévy process $L = \{L_t\}_{t\in[0,T]}$ can be constructed such that $L_1 \stackrel{d}{=} X$, with the introduction of Poisson random measures.

For a Lévy process $L = \{L_t\}_{t\in[0,T]}$, $\Delta L = \{\Delta L_t\}_{t\in[0,T]}$ is called the *jump process* associated with L.

Proposition 2.11. *Let* $L = \{L_t\}_{t\in[0,T]}$ *be a Lévy process. For fixed* $t \in \mathbb{R}^+$*,* $\Delta L_t = 0$ *a.s.*

Proof. Let $\{t_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^+$ with $t_n \uparrow t$ as $n \to \infty$. On one hand, since L has càdlàg paths, $\lim_{n\to\infty} L_{t_n} = L_{t-}$. On the other hand, the sequence $\{L_{t_n}\}_{n\in\mathbb{N}}$ converges in probability to L_t by stochastic continuity, and so there exists a subsequence which converges almost surely to L_t. The result then follows by uniqueness of limits. □

For $t \in [0,T]$, in general it is possible that $\sum_{0\le s\le t} |\Delta L_s| = \infty$ a.s., but it always holds that $\sum_{0\le s\le t} |\Delta L_s|^2 < \infty$ a.s.

Let $A \in \mathcal{B}(\mathbb{R}_0^d)$ be a set which is *bounded below*, that is, $0 \notin \bar{A}$. The *random measure of jumps* of a Lévy process $L = \{L_t\}_{t\in[0,T]}$ is defined as

$$\mu^L(\omega; t, A) = \#\{s : \Delta L_s(\omega) \in A, s \in [0,t]\} = \sum_{0\le s\le t} \mathbf{1}_{\{\Delta L_s(\omega)\in A\}}, \forall t \in [0,T].$$

Proposition 2.12. *Let* L *be a Lévy process. If* A *is bounded below, then* $\mu^L(t,A) < \infty$ *a.s.,* $\forall t \in [0,T]$.

Proof. Define a sequence of stopping times $\{T_n^A\}_{n\in\mathbb{N}}$ by $T_1^A = \inf\{t : \Delta L_t \in A, t \geq 0\}$ and $T_n^A = \inf\{t : \Delta L_t \in A, t > T_{n-1}^A\}, \forall n > 1$. Since L has càdlàg paths, $T_1^A > 0$ a.s. and $\lim_{n\to\infty} T_n^A = \infty$ a.s. Hence, $\mu^L(t, A) = \sum_{n\in\mathbb{N}} \mathbf{1}_{\{T_n^A \leq t\}} < \infty$ a.s., $\forall t \in [0, T]$. □

The measure $\mu^L(\omega; t, A)$ has the property that $\mu^L(t, A) - \mu^L(s, A) \in \sigma(\{L_u - L_v, s \leq v < u \leq t\})$, and so $\mu^L(t, A) - \mu^L(s, A)$ is independent of $\mathcal{F}_s$, that is, $\mu^L(\cdot, A)$ has independent increments. In addition, $\mu^L(s+u, A) - \mu^L(s, A)$ equals the number of jumps of $L_{s+u} - L_s$ in A for $u \in [0, t-s]$. Then, by the stationarity of the increments of L, $\mu^L(t, A) - \mu^L(s, A) \overset{d}{=} \mu^L(t-s, A)$, that is, $\mu^L(\cdot, A)$ has stationary increments. Therefore, $\mu^L(\cdot, A)$ is a Poisson process and μ^L is a Poisson random measure. The intensity of the process $\nu(A) = \mathrm{E}[\mu^L(1, A)]$ is called the *Lévy measure* of L.

Let $A \in \mathcal{B}(\mathbb{R}_0^d)$ be a set bounded below and f be a finite Borel measurable function on A. The *Poisson integral* of f with respect to a Poisson random measure is defined as

$$\int_A f(x)\mu^L(\omega; t, dx) = \sum_{x\in A} f(x)\mu^L(\omega; t, \{x\}) = \sum_{0\leq s\leq t} f(\Delta L_s)\mathbf{1}_{\{\Delta L_s(\omega)\in A\}}.$$

Each $\int_A f(x)\mu^L(t, dx)$ is a real-valued random variable and generates a càdlàg stochastic process.

Theorem 2.13. *Let $A \in \mathcal{B}(\mathbb{R}_0^d)$ be a set bounded below and f be a finite Borel measurable function on A.*

(i) *The process $\big(\int_0^t \int_A f(x)\mu^L(ds, dx)\big)_{t\in[0,T]}$ is a compound Poisson process with characteristic function*

$$\mathrm{E}\big[\exp\big(iu\int_0^t \int_A f(x)\mu^L(ds, dx)\big)\big] = \exp\big(t\int_A (e^{iuf(x)} - 1)\nu(dx)\big);$$

(ii) *If* $f \in L^1(A)$*, then*

$$\mathrm{E}\big[\int_0^t \int_A f(x)\mu^L(ds, dx)\big] = t \int_A f(x)\nu(dx);$$

(iii) *If* $f \in L^2(A)$*, then*

$$\mathrm{Var}\big(\big|\int_0^t \int_A f(x)\mu^L(ds, dx)\big|\big) = t \int_A |f(x)|^2\nu(dx).$$

The Poisson integral may fail to have a finite mean if $f \notin L^1(A)$. For each $f \in L^1(A), t \in [0, T]$, the *compensated Poisson integral*

$$\int_0^t \int_A f(x)\tilde{\mu}^L(ds, dx) = \int_0^t \int_A f(x)\mu^L(ds, dx) - t \int_A f(x)\nu(dx)$$

is a martingale with the properties that

$$\mathrm{E}\big[\exp\big(iu \int_0^t \int_A f(x)\tilde{\mu}^L(ds, dx)\big)\big] = \exp\big(t \int_A (e^{iuf(x)} - 1 - iuf(x))\nu(dx)\big), \forall u \in \mathbb{R}$$

and

$$\mathrm{Var}\big(\big|\int_0^t \int_A f(x)\tilde{\mu}^L(ds, dx)\big|\big) = t \int_A |f(x)|^2\nu(dx), \forall f \in L^2(A).$$

Theorem 2.14. (**Lévy-Itô Decomposition**) *If* $L = \{L_t\}_{t\in[0,T]}$ *is a Lévy process, then there exists a triplet* (b, c, ν)*, with* c *nonnegative,* $\nu\big(\{0\}\big) = 0$*, and* $\int \big(1 \wedge |x|^2\big)\nu(dx) < \infty$*,* *such that L can be decomposed into four independent processes, that is,*

$$L_t = bt + c^{\frac{1}{2}} W_t + \int_0^t \int_{|x|\geq 1} x\mu^L(ds, dx) + \int_0^t \int_{|x|<1} x(\mu^L - \nu^L)(ds, dx), \forall t \in [0, T],$$

where $\mu^L(t, A) = \#\{s : \Delta L_s \in A, s \in [0, t]\}$ and $\nu^L(dt, dx) = \nu(dx)dt$.

In Theorem 2.14, the first part corresponds to a deterministic linear drift process with parameter b, the second to a Wiener process with covariance c, the third to a compound Poisson process with jump magnitude $F(dx) = \dfrac{\nu(dx)}{\nu(\mathbb{R}^d \backslash (-1, 1))} \mathbf{1}_{\{|x|\geq 1\}}$ and intensity $\lambda = \nu(\mathbb{R}^d \backslash (-1, 1))$, and the fourth to the compensated sum of small jumps, a square-integrable pure jump martingale. All the four terms except the third one have finite moments to all orders, so if a Lévy process fails to have a moment, it is due entirely to the large-jump part, which has finite activities.

Proposition 2.15. *Let $L = \{L_t\}_{t\in[0,T]}$ be a Lévy process with characteristic triplet (b, c, ν).*

(i) *If $\int \nu(dx) < \infty$, almost all paths of L have a finite number of jumps on every compact interval, that is, L has finite activities; If $\int \nu(dx) = \infty$, almost all paths of L have an infinite number of jumps on every compact interval, that is, L has infinite activities;*

(ii) *If $c = 0$ and $\int_{|x|\leq 1} |x|\nu(dx) < \infty$, almost all paths of L have finite variation; If $c \neq 0$ or $\int_{|x|\leq 1} |x|\nu(dx) = \infty$, almost all paths of L have infinite variation;*

(iii) L_t *has a finite pth moment for* $p \in \mathbb{R}^+$, *that is,* $\mathrm{E}|L_t|^p < \infty$, *if and only if* $\int_{|x|\geq 1} |x|^p \nu(dx) < \infty$; L_t *has a finite pth exponential moment for* $p \in \mathbb{R}$, *that is,* $\mathrm{E}\left[e^{pL_t}\right] < \infty$, *if and only if* $\int_{|x|\geq 1} e^{px}\nu(dx) < \infty$.

Theorem 2.16. (**Itô's Formula**) *Let* $L = \{L_t\}_{t\in[0,T]}$ *be a Lévy process with characteristic triplet* (b, c, ν) *and* $f : H \mapsto \mathbb{R}$ *be a function in* $C^{1,2}$. *Then,*

$$\begin{aligned} f(t, L_t) &= f(0, L_0) + \int_0^t \partial_0 f(s, L_s) ds \\ &\quad + \int_0^t \big(\partial_x f(s, L_{s-}), dL_s\big) + \frac{1}{2}\int_0^t \sum_{i,j=1}^d c^{ij}\partial_{ij} f(s, L_s) ds \\ &\quad + \sum_{0\leq s\leq t} \Big(f(s, L_{s-} + \Delta L_s) - f(s, L_{s-}) - \big(\partial_x f(s, L_{s-}), \Delta L_s\big)\Big), \forall t \in [0, T]. \end{aligned}$$

2.2 Stochastic Differential Equations

2.2.1 Diffusion Process

Let $W = \{W_t\}_{t\in[0,T]}$ be a standard Wiener process. The classical SDE [72] with respect to the Wiener process is

$$X_t = X_0 + \int_0^t a(X_s)ds + \int_0^t b(X_s)dW_s, \forall t \in [0, T]. \tag{2.3}$$

A sufficient condition for (2.3) to have a unique solution is bounded variation of b [71] or finite quadratic variation of b [55].

In the case where the coefficients are time inhomogeneous, that is,

$$X_t = X_0 + \int_0^t a(s, X_s)ds + \int_0^t b(s, X_s)dW_s, \forall t \in [0, T],$$

where $a(t,x)$ is a d-dimensional vector, $b(t,x)$ a $d \times d$-dimensional matrix, and $W = \{W_t\}_{t\in[0,T]}$ a d-dimensional standard Wiener process, there also exists a unique solution by Picard iteration, provided that the coefficients a and b are smooth [89].

2.2.2 Stochastic Process with Jumps

The simplest analogue to (2.3) in the jump case is

$$X_t = X_0 + \int_0^t c(X_{s-})dZ_s, \forall t \in [0,T], \tag{2.4}$$

where $Z = \{Z_t\}_{t\in[0,T]}$ is a stochastic process with jumps.

When the driving motion is a Wiener process with a Poisson random measure, that is,

$$\begin{aligned} X_t \quad = \quad & X_0 + \int_0^t a(X_{s-})ds + \int_0^t b(X_{s-})dW_s \\ & + \int_0^t \int_{|y|>1} f(X_{s-},y)N(ds,dy) + \int_0^t \int_{|y|\le 1} g(X_{s-},y)\tilde{N}(ds,dy), \forall t \in [0,T], \end{aligned}$$

where $W = \{W_t\}_{t\in[0,T]}$ is a d-dimensional standard Wiener process and $N(dt,dy)$ a Poisson random measure with compensator $\tilde{N}(dt,dy) = N(dt,dy) - \nu(dy)dt$, there exists a unique solution by Picard iteration, provided that the following three conditions are satisfied [3, 74, 83]:

- Lipschitz Condition: there exists a constant $K \in \mathbb{R}^+$ such that for all $x, \tilde{x} \in \mathbb{R}^d$,

$$|a(x) - a(\tilde{x})|^2 + |\tilde{B}(x,\tilde{x})| + \int_{|y|\le 1} |g(x,y) - g(\tilde{x},y)|^2 \nu(dy) \le K|x - \tilde{x}|^2;$$

- Growth Condition: there exists a constant $K \in \mathbb{R}^+$ such that for all $x \in \mathbb{R}^d$,

$$|a(x)|^2 + |B(x,x)| + \int_{|y|\leq 1} |g(x,y)|^2 \nu(dy) \leq K(1+|x|^2);$$

- Big Jumps Condition: f is jointly measurable and $x \to f(x,y)$, $\forall y \in \{y : |y| > 1\}$ is continuous.

Here $B(x,\tilde{x}) = b(x)^T b(\tilde{x})$ and $\tilde{B}(x,\tilde{x}) = B(x,x) - 2B(x,\tilde{x}) + B(\tilde{x},\tilde{x})$.

Another common case is that Z is a one-dimensional symmetric stable process with index $\alpha \in (0,2)$. By Picard iteration, there exists a pathwise unique solution to (2.4) if c satisfies the Lipschitz condition [13]. If c is bounded and has a modulus of continuity ρ, that is, $|c(x) - c(\tilde{x})| \leq \rho(|x-\tilde{x}|), \forall x, \tilde{x}$, where ρ satisfies

$$\int_0^{\varepsilon} \frac{1}{\rho(x)^{\alpha}} dx = \infty, \forall \varepsilon > 0, \tag{2.5}$$

then (2.4) admits a strong pathwise unique solution [12]. Condition (2.5) is satisfied, for instance, if c is Hölder-continuous of order $\frac{1}{\alpha}$. The condition is the exact analogue to the Yamada-Watanabe condition [93] for stochastic differential equations driven by Wiener processes [50].

2.2.3 Stochastic Process Driven by Lévy Motion

Under appropriate conditions on c, the solution to (2.4) is a strong Markov process if and only if the driving motion Z is a Lévy process [43], in which case the resulting system is a stochastic process driven by a Lévy process.

Let $T \in \mathbb{R}^+$ and $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with filtrations $\mathbf{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ and $\tilde{\mathbf{F}} = \{\tilde{\mathcal{F}}_t\}_{t\in[0,T]}$, satisfying the usual conditions. The process X under consideration is $\mathbf{F}$-adapted and the Euler approximation of X is $\tilde{\mathbf{F}}$-adapted.

In this dissertation, for $\alpha \in (0, 2]$, the following stochastic differential equation is considered,

$$\begin{aligned} X_t &= X_0 + \int_0^t a^{(\alpha)}(X_{s-})ds + \int_0^t b^{(\alpha)}(X_{s-})dW_s \\ &\quad + \int_0^t \int_{|y|>1} yp^X(ds, dy) + \int_0^t \int_{|y|\le 1} yq^X(ds, dy), t \in [0, T], \end{aligned} \tag{2.6}$$

where

$$\begin{aligned} a^{(\alpha)}(x) &= \mathbf{1}_{\{\alpha\in(0,1)\}}\big(\int_{|y|\le 1} ym^{(\alpha)}(x, y)\frac{dy}{|y|^{d+\alpha}} + \int_{|y|\le 1} y\rho^{(\alpha)}(x, y)\nu^{(\alpha)}(dy)\big) \\ &\quad +\mathbf{1}_{\{\alpha=1\}}\big(a(x) + \int_{|y|\le 1} y\rho^{(\alpha)}(x, y)\nu^{(\alpha)}(dy)\big) \\ &\quad +\mathbf{1}_{\{\alpha\in(1,2]\}}\big(c(x) - \int_{|y|>1} ym^{(\alpha)}(x, y)\frac{dy}{|y|^{d+\alpha}}\big), \\ b^{(\alpha)}(x) &= \mathbf{1}_{\{\alpha=2\}}b(x), \end{aligned}$$

X_0 is the $\mathbf{F}_0$-measurable initial value, $W = \{W_t\}_{t\in[0,T]}$ is an $\mathbf{F}$-adapted d-dimensional standard Wiener process, p^X is the jump measure of X_t with

$$p^X\big([0, t] \times A\big) = \sum_{s\in(0,t]} \mathbf{1}_A(\Delta X_s),$$

and q^X is an $(\mathbf{F}, \mathbb{P})$-martingale measure with

$$q^X(dt, dy) = p^X(dt, dy) - m^{(\alpha)}(X_{t-}, y)\frac{dy}{|y|^{d+\alpha}}dt - \rho^{(\alpha)}(X_{t-}, y)\nu^{(\alpha)}(dy)dt.$$

In the stochastic equation (2.6), a, c, b, $m^{(\alpha)}$, and $\rho^{(\alpha)}$ are measurable functions, with a and c d-dimensional vectors, b a $d \times d$-dimensional symmetric nonnegative definite matrix, $m^{(\alpha)}$ and $\rho^{(\alpha)}$ nonnegative functions, and $\nu^{(\alpha)}$ a nonnegative measure on $\mathbb{R}_0^d$. In addition, $m^{(\alpha)}(x,y)$ and its partial derivatives $\partial_y^\gamma m^{(\alpha)}(x,y), \gamma \in \left\{\gamma : |\gamma| \leq d_0 = \left[\frac{d}{2}\right] + 1\right\}$ are continuous in (x,y), $m^{(\alpha)}(x,y)$ is homogeneous in y with index zero, and

$$\int_{S^{d-1}} y m^{(1)}(\cdot, y) \mu_{d-1}(dy) = 0, \quad m^{(2)} \equiv 0,$$

where S^{d-1} is the unit sphere in $\mathbb{R}^d$ and μ_{d-1} is the Lebesgue measure on it.

For $\beta = [\beta]^- + \left\{\beta\right\}^+ > 0$, where $[\beta]^- \in \mathbb{N}$ and $\left\{\beta\right\}^+ \in (0,1]$, let $C^\beta(H)$ denote the space of measurable functions u on H such that the norm

$$\begin{aligned}
|u|_\beta \quad = \quad & \sum_{|\gamma| \leq [\beta]^-} \sup_{t,x} |\partial_x^\gamma u(t,x)| + \mathbf{1}_{\{\{\beta\}^+ < 1\}} \sup_{\substack{|\gamma| = [\beta]^-, \\ t,x,h \neq 0}} \frac{|\partial_x^\gamma u(t,x+h) - \partial_x^\gamma u(t,x)|}{|h|^{\{\beta\}^+}} \\
& + \mathbf{1}_{\{\{\beta\}^+ = 1\}} \sup_{\substack{|\gamma| = [\beta]^-, \\ t,x,h \neq 0}} \frac{|\partial_x^\gamma u(t,x+h) - 2\partial_x^\gamma u(t,x) + \partial_x^\gamma u(t,x-h)|}{|h|^{\{\beta\}^+}}
\end{aligned}$$

is finite. Accordingly, $C^\beta(\mathbb{R}^d)$ denotes the corresponding space of functions on $\mathbb{R}^d$.

Throughout this dissertation, it is assumed that the following conditions are satisfied by the stochastic differential equation considered:

(A1) There exists a constant $\mu > 0$ such that for all $x \in \mathbb{R}^d$ and $|\xi| = 1$,

$$\begin{aligned}
& (B(x)\xi, \xi) \geq \mu, \text{ where } B(x) = b(x)^T b(x), \\
& \int_{S^{d-1}} |(w,\xi)|^\alpha m^{(\alpha)}(x,w) d\xi \geq \mu, \alpha \in (0,2), \\
& \lim_{\delta \downarrow 0} \sup_x \int_{|y| \leq \delta} |y|^\alpha \rho^{(\alpha)}(x,y) \nu^{(\alpha)}(dy) = 0; \qquad (2.7)
\end{aligned}$$

(A2) For $\beta \in \mathbb{R}^+$,

$$M_\beta^{(\alpha)} + N_\beta^{(\alpha)} < \infty,$$

where

$$M_\beta^{(\alpha)} = \mathbf{1}_{\{\alpha=1\}}|a|_\beta + \mathbf{1}_{\{\alpha=2\}}|B|_\beta + \mathbf{1}_{\{\alpha\in(0,2)\}} \sup_{\substack{|\gamma|\leq d_0,\\ |y|=1}} |\partial_y^\gamma m^{(\alpha)}(\cdot, y)|_\beta$$

and

$$\begin{aligned} N_\beta^{(\alpha)} &= \mathbf{1}_{\{\alpha\in(1,2]\}}|c|_\beta + \sup_{|\gamma|=[\beta],x} \int_{\mathbb{R}_0^d} \big(|y|^\alpha \wedge 1\big)\big[\big|\rho^{(\alpha)}(x,y)\big| + \big|\partial_x^\gamma \rho^{(\alpha)}(x,y)\big|\big]\nu^{(\alpha)}(dy) \\ &\quad + \sup_{\substack{|\gamma|=[\beta],\\ x,h\neq 0}} \frac{1}{|h|^{\beta-[\beta]}} \int_{\mathbb{R}_0^d} \big(|y|^\alpha \wedge 1\big)\big|\partial_x^\gamma \rho^{(\alpha)}(x+h,y) - \partial_x^\gamma \rho^{(\alpha)}(x,y)\big|\nu^{(\alpha)}(dy); \end{aligned}$$

Remark 2.17. *Under the above assumptions, for any* $\beta \in \mathbb{R}^+$, *there exists a unique weak solution to equation* (2.6) [61].

Let $X = \{X_t\}_{t\in[0,T]}$ be the weak solution to (2.6) and $Y = \{Y_t\}_{t\in[0,T]}$ be the Euler approximation of X. The convergence rate of Y to X is investigated in this dissertation.

Remark 2.18. *Compared with* (2.4), (2.6) *is a more general model. Each equation of the form* (2.4) *can be transformed into its counterpart of the form* (2.6).

For example, in the case where

$$Z_t = \int_0^t \int y\big(p(ds,dy) - \frac{dy}{|y|^{d+\alpha}}ds\big), \alpha \in (1,2], \tag{2.8}$$

$$
\begin{aligned}
\sum_{s\in[0,t]} \mathbf{1}_A(\Delta X_s) &= \sum_{s\in[0,t]} \mathbf{1}_A\big(c(X_{s-})\Delta Z_s\big) \\
&= \int_0^t \int \mathbf{1}_A\big(c(X_{s-})y\big)\frac{dy}{|y|^{d+\alpha}}ds \\
&= \int_0^t \int \mathbf{1}_A\big(\bar{y}\big)\frac{\big|\det c(X_{s-})^{-1}\big|}{\frac{\left|c(X_{s-})^{-1}\bar{y}\right|^{d+\alpha}}{|\bar{y}|^{d+\alpha}}}\cdot\frac{d\bar{y}}{|\bar{y}|^{d+\alpha}}ds \\
&= \int_0^t \int \mathbf{1}_A\big(y\big)\frac{\big|\det c(X_{s-})^{-1}\big|}{\frac{\left|c(X_{s-})^{-1}y\right|^{d+\alpha}}{|y|^{d+\alpha}}}\cdot\frac{dy}{|y|^{d+\alpha}}ds \\
&= \int_0^t \int \mathbf{1}_A\big(y\big)\frac{\big|\det c(X_{s-})^{-1}\big|}{\left|c(X_{s-})^{-1}\frac{y}{|y|}\right|^{d+\alpha}}\cdot\frac{dy}{|y|^{d+\alpha}}ds.
\end{aligned}
$$

Hence, let

$$
m^{(\alpha)}(X_{s-},y) = \frac{\big|\det c(X_{s-})^{-1}\big|}{\left|c(X_{s-})^{-1}\frac{y}{|y|}\right|^{d+\alpha}},
$$

(2.4) is then of the form (2.6). That is,

$$
X_t = X_0 + \int_0^t \int y q^X(ds,dy), \forall t \in [0,T],
$$

where

$$
p^X\big([0,t]\times A\big) = \sum_{s\in[0,t]} \mathbf{1}_A(\Delta X_s) \quad \text{and} \quad q^X(dt,dy) = p^X(dt,dy) - m^{(\alpha)}(X_{t-},y)\frac{dy}{|y|^{d+\alpha}}dt.
$$

In addition, assumptions outlined above are satisfied provided that c is non-degenerate, that is, $|\det c(\cdot)| \geq \varepsilon > 0$.

Chapter 3

Solution to Integro-Differential Equations in Hölder Space

In order to prove the main theorem in Chapter 4, some auxiliary results are presented in this chapter. Specifically, an associated integro-differential equation in Hölder space is solved to determine the rate of convergence.

3.1 Lévy Operators in Hölder Space

For $u \in C^{\alpha+\beta}(H)$, denote

$$A_y^{(\alpha)}u(t,x) = u(t,x+y) - u(t,x) - \big(\mathbf{1}_{\{|y|\le 1\}}\mathbf{1}_{\{\alpha=1\}} + \mathbf{1}_{\{\alpha\in(1,2)\}}\big)\big(\partial_x u(t,x), y\big)$$

and

$$B_y^{(\alpha)}u(t,x) = u(t,x+y) - u(t,x) - \big(\mathbf{1}_{\{|y|\le 1\}}\mathbf{1}_{\{\alpha\in(1,2]\}}\big)\big(\partial_x u(t,x), y\big).$$

Let

$$\mathcal{A}_z^{(\alpha)}u(t,x) \quad = \quad \mathbf{1}_{\{\alpha=1\}}\big(a(z), \partial_x u(t,x)\big) + \frac{1}{2}\mathbf{1}_{\{\alpha=2\}}\sum_{i,j=1}^{d} B^{ij}(z)\partial_{ij}u(t,x)$$

$$
\begin{aligned}
& \quad + \int_{\mathbb{R}_0^d} A_y^{(\alpha)} u(t,x) m^{(\alpha)}(z,y) \frac{dy}{|y|^{d+\alpha}}, \\
\mathcal{A}^{(\alpha)} u(t,x) &= \mathcal{A}_x^{(\alpha)} u(t,x) = \mathcal{A}_z^{(\alpha)} u(t,x)|_{z=x},
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{B}_z^{(\alpha)} u(t,x) &= \mathbf{1}_{\{\alpha \in (1,2]\}} \big(c(z), \partial_x u(t,x)\big) + \int_{\mathbb{R}_0^d} B_y^{(\alpha)} u(t,x) \rho^{(\alpha)}(z,y) \nu^{(\alpha)}(dy), \\
\mathcal{B}^{(\alpha)} u(t,x) &= \mathcal{B}_x^{(\alpha)} u(t,x) = \mathcal{B}_z^{(\alpha)} u(t,x)|_{z=x}.
\end{aligned}
$$

Note that for $u \in C^{\alpha+\beta}(H)$, $\beta \in \mathbb{R}^+$,

$$
\mathcal{A}_z^{(\alpha)} u(t,x) = \mathscr{F}^{-1}\big[\psi^{(\alpha)}(z,\xi) \mathscr{F} u(t,\xi)\big](x)
$$

with $\psi^{(\alpha)}(z,\xi)$ being the logarithm of the characteristic function of a stable distribution with parameter α, which is given by

$$
\begin{aligned}
\psi^{(\alpha)}(z,\xi) &= -K \int_{S^{d-1}} |(\omega,\xi)|^\alpha \big[1 - i\big(\mathbf{1}_{\{\alpha \neq 1\}} \tan \frac{\alpha\pi}{2} \operatorname{sgn}(\omega,\xi) \\
&\quad - \frac{2}{\pi} \mathbf{1}_{\{\alpha=1\}} \operatorname{sgn}(\omega,\xi) \ln |(\omega,\xi)|\big)\big] m^{(\alpha)}(z,\omega) \mu_{d-1}(d\omega) \\
&\quad - i \mathbf{1}_{\{\alpha=1\}} \big(a(z),\xi\big) - \frac{1}{2} \mathbf{1}_{\{\alpha=2\}} \big(B(z)\xi,\xi\big),
\end{aligned}
$$

where $K = K(\alpha)$ is a constant, S^{d-1} is the unit sphere in $\mathbb{R}^d$, and μ_{d-1} is the Lebesgue measure on it.

The operator $\mathcal{L}^{(\alpha)} = \mathcal{A}^{(\alpha)} + \mathcal{B}^{(\alpha)}$ is the generator of X_t defined in (2.6). $\mathcal{A}^{(\alpha)}$ is the principal part and $\mathcal{B}^{(\alpha)}$ is the lower order or subordinated part of $\mathcal{L}^{(\alpha)}$.

Accordingly, for u not depending on t, $\mathcal{A}_z^{(\alpha)} u(x)$, $\mathcal{A}^{(\alpha)} u(x)$, $\mathcal{B}_z^{(\alpha)} u(x)$, and $\mathcal{B}^{(\alpha)} u(x)$ denote analogous operators.

Remark 3.1. *The stochastic process*

$$u(X_t) - \int_0^t \big(\mathcal{A}^{(\alpha)} + \mathcal{B}^{(\alpha)}\big) u(X_s) ds, \forall u \in C^{\alpha+\beta}(\mathbb{R}^d)$$

is a martingale [61], *where* $X = \{X_t\}_{t\in[0,T]}$ *is the unique weak solution to equation* (2.6).

For $f \in C^{\alpha}(\mathbb{R}^d)$, denote

$$\partial^{\alpha} f(x) = \mathscr{F}^{-1}\big[|\xi|^{\alpha} \mathscr{F} f(\xi)\big](x) = \begin{cases} K \displaystyle\int A_y^{(\alpha)} f(x) \frac{dy}{|y|^{d+\alpha}}, & \alpha \in (0,2), \\ \displaystyle\sum_{i=1}^{d} \partial_{ii} f(x) = \Delta f(x), & \alpha = 2, \end{cases}$$

where $K = K(\alpha)$ is a constant.

The following result on $\partial^{\alpha} f$ [51, 53] is used in the proof of the main theorem.

Lemma 3.2. *For* $\delta \in (0,1)$ *and* $f \in C^{\delta}(\mathbb{R}^d)$,

$$f\big(x+h\big) - f(x) = K(\delta, d) \int k^{(\delta)}(h,y) \partial^{\delta} f(x-y) dy,$$

where for a constant K, $k^{(\delta)}(h,y) = |y+h|^{-d+\delta} - |y|^{-d+\delta}$ *with*

$$\int |k^{(\delta)}(h,y)| dy \le K|h|^{\delta}.$$

3.2 Equivalent Norms in Hölder Space

By Lemma 6.1.7 [17], there exists a nonnegative function $\phi \in C_0^\infty(\mathbb{R}^d)$ such that supp$\phi = \{\xi : \frac{1}{2} \leq |\xi| \leq 2\}$ and

$$\sum_{j=-\infty}^{\infty} \phi(2^{-j}\xi) = 1, \forall \xi \neq 0.$$

Define $\varphi_k \in \mathcal{S}(\mathbb{R}^d)$, $k = 0, \pm 1, \ldots$ by

$$\mathscr{F}\varphi_k = \phi(2^{-k}\xi) \tag{3.1}$$

and $\psi \in \mathcal{S}(\mathbb{R}^d)$ by

$$\mathscr{F}\psi = 1 - \sum_{k \geq 1} \mathscr{F}\varphi_k(\xi), \tag{3.2}$$

where $\mathcal{S}(\mathbb{R}^d)$ is the Schwartz space of rapidly decreasing, infinitely differentiable functions on $\mathbb{R}^d$.

By Lemma 12 [63],

$$|u|_\beta \sim \sup_x |\psi * u(x)| + \sup_{k \geq 1} 2^{\beta k} \sup_x |\varphi_k * u(x)|$$

and for $\alpha \in (0, 2]$, $\beta \in \mathbb{R}^+$, $|u|_{\alpha+\beta}$ is equivalent to the norm $|u|_{\alpha,\beta} = |u|_0 + |\partial^\alpha u|_\beta$.

3.3 Solution to Cauchy Problem

For the proof of Theorem 3.8, Lemma 3.3 [25, 91], Lemma 3.4, Lemma 3.6, and Lemma 3.7 are called. For functions possessing the same properties as a, c, b, $m^{(\alpha)}$, and $\rho^{(\alpha)}$, the corresponding norms and operators are defined as follows.

Let $\bar{a}$, $\bar{c}$, $\bar{b}$, $\bar{m}^{(\alpha)}$, and $\bar{\rho}^{(\alpha)}$ be measurable functions, with $\bar{a}$ and $\bar{c}$ d-dimensional vectors, $\bar{b}$ a $d \times d$-dimensional symmetric nonnegative definite matrix, and $\bar{m}^{(\alpha)}$ and $\bar{\rho}^{(\alpha)}$ nonnegative functions. In addition, assume that $\bar{m}^{(\alpha)}(x,y)$ and its partial derivatives $\partial_y^\gamma \bar{m}^{(\alpha)}(x,y), \gamma \in \left\{\gamma : |\gamma| \leq d_0 = \left[\frac{d}{2}\right] + 1\right\}$ are continuous in (x,y), $\bar{m}^{(\alpha)}(x,y)$ is homogeneous in y with index zero, and

$$\int_{S^{d-1}} y\bar{m}^{(1)}(\cdot,y)\mu_{d-1}(dy) = 0, \quad \bar{m}^{(2)} \equiv 0,$$

where S^{d-1} is the unit sphere in $\mathbb{R}^d$ and μ_{d-1} is the Lebesgue measure on it.

For $\beta \in \mathbb{R}^+$ and $\bar{B}(x) = \bar{b}(x)^T \bar{b}(x), x \in \mathbb{R}^d$, define

$$\begin{aligned} \bar{M}_\beta^{(\alpha)} &= \mathbf{1}_{\{\alpha=1\}}|\bar{a}|_\beta + \mathbf{1}_{\{\alpha=2\}}|\bar{B}|_\beta + \mathbf{1}_{\{\alpha\in(0,2)\}} \sup_{\substack{|\gamma|\leq d_0,\\ |y|=1}} |\partial_y^\gamma \bar{m}^{(\alpha)}(\cdot,y)|_\beta, \\ \bar{N}_\beta^{(\alpha)} &= \mathbf{1}_{\{\alpha\in(1,2]\}}|\bar{c}|_\beta + \sup_{|\gamma|=[\beta],x} \int_{\mathbb{R}_0^d} \left(|y|^\alpha \wedge 1\right)\left[|\bar{\rho}^{(\alpha)}(x,y)| + |\partial_x^\gamma \bar{\rho}^{(\alpha)}(x,y)|\right]\nu^{(\alpha)}(dy) \\ &\quad + \sup_{\substack{|\gamma|=[\beta],\\ x,h\neq 0}} \frac{1}{|h|^{\beta-[\beta]}} \int_{\mathbb{R}_0^d} \left(|y|^\alpha \wedge 1\right)|\partial_x^\gamma \bar{\rho}^{(\alpha)}(x+h,y) - \partial_x^\gamma \bar{\rho}^{(\alpha)}(x,y)|\nu^{(\alpha)}(dy), \end{aligned}$$

and

$$\begin{aligned} \bar{\psi}^{(\alpha)}(z,\xi) &= -K\int_{S^{d-1}} |(\omega,\xi)|^\alpha \big[1 - i\big(\mathbf{1}_{\{\alpha\neq 1\}} \tan\frac{\alpha\pi}{2}\mathrm{sgn}(\omega,\xi) \\ &\quad -\frac{2}{\pi}\mathbf{1}_{\{\alpha=1\}}\mathrm{sgn}(\omega,\xi)\ln|(\omega,\xi)|\big)\big]\bar{m}^{(\alpha)}(z,\omega)\mu_{d-1}(d\omega) \end{aligned}$$

$$-i\mathbf{1}_{\{\alpha=1\}}\big(\bar a(z),\xi\big)-\frac{1}{2}\mathbf{1}_{\{\alpha=2\}}\big(\bar B(z)\xi,\xi\big).$$

For $f\in C^{\alpha+\beta}(\mathbb{R}^d)$, let

$$\begin{aligned}
\bar{\mathcal{A}}_z^{(\alpha)}f(x) &= \mathbf{1}_{\{\alpha=1\}}\big(\bar a(z),\partial_x f(x)\big)+\frac{1}{2}\mathbf{1}_{\{\alpha=2\}}\sum_{i,j=1}^d \bar B^{ij}(z)\partial_{ij}f(x)\\
&\quad+\int_{\mathbb{R}_0^d} A_y^{(\alpha)}f(x)\bar m^{(\alpha)}(z,y)\frac{dy}{|y|^{d+\alpha}},\\
\bar{\mathcal{A}}^{(\alpha)}f(x) &= \bar{\mathcal{A}}_x^{(\alpha)}f(x)=\bar{\mathcal{A}}_z^{(\alpha)}f(x)|_{z=x},
\end{aligned}$$

and

$$\begin{aligned}
\bar{\mathcal{B}}_z^{(\alpha)}f(x) &= \mathbf{1}_{\{\alpha\in(1,2]\}}\big(\bar c(z),\partial_x f(x)\big)+\int_{\mathbb{R}_0^d} B_y^{(\alpha)}f(x)\bar\rho^{(\alpha)}(z,y)\nu^{(\alpha)}(dy),\\
\bar{\mathcal{B}}^{(\alpha)}f(x) &= \bar{\mathcal{B}}_x^{(\alpha)}f(x)=\bar{\mathcal{B}}_z^{(\alpha)}f(x)|_{z=x}.
\end{aligned}$$

In addition, denote

$$\begin{aligned}
\bar M^{(\alpha)} &= \mathbf{1}_{\{\alpha=1\}}|\bar a|_\infty+\mathbf{1}_{\{\alpha=2\}}|\bar B|_\infty+\mathbf{1}_{\{\alpha\in(0,2)\}}\sup_{\substack{|\gamma|\le d_0,\\ |y|=1}}|\partial_y^\gamma \bar m^{(\alpha)}(\cdot,y)|_\infty,\\
\bar N^{(\alpha)} &= \mathbf{1}_{\{\alpha\in(1,2]\}}|\bar c|_\infty+\sup_{|\gamma|=[\beta],x}\int_{\mathbb{R}_0^d}\big(|y|^\alpha\wedge 1\big)\big[|\bar\rho^{(\alpha)}(x,y)|+|\partial_x^\gamma\bar\rho^{(\alpha)}(x,y)|\big]\nu^{(\alpha)}(dy)\\
&\quad+\sup_{\substack{|\gamma|=[\beta],\\ x,h\neq 0}}\frac{1}{|h|^{\beta-[\beta]}}\int_{\mathbb{R}_0^d}\big(|y|^\alpha\wedge 1\big)|\partial_x^\gamma\bar\rho^{(\alpha)}(x+h,y)-\partial_x^\gamma\bar\rho^{(\alpha)}(x,y)|\nu^{(\alpha)}(dy),\\
\bar\phi(z,\xi) &= \bar\psi^{(\alpha)}(z,\xi)(1+|\xi|^\alpha)^{-1},\\
\bar\Phi f(z,x) &= \mathscr{F}^{-1}\big[\bar\phi(z,\xi)\mathscr{F}f(\xi)\big](x),\\
\tilde\Phi f(x) &= \bar\Phi f(x,x),\\
\Lambda_j f(x) &= \mathscr{F}^{-1}\big[i\xi_j|\xi|^{\alpha-1}(1+|\xi|^\alpha)^{-1}\mathscr{F}f(\xi)\big](x), j=1,\ldots,d,\alpha\ge 1,\\
\Lambda_{ij} f(x) &= \mathscr{F}^{-1}\big[\xi_i\xi_j(1+|\xi|^2)^{-1}\mathscr{F}f(\xi)\big](x), i,j=1,\ldots,d.
\end{aligned}$$

For any multiindex $\gamma \in \left\{\gamma : |\gamma| \leq d_0 = \left[\frac{d}{2}\right] + 1\right\}$ and any $\xi \in \mathbb{R}^d$, the following inequalities hold,

$$\begin{aligned}
\left|\partial_\xi^\gamma \bar{\phi}(\cdot,\xi)\right|_\beta &\leq K\bar{M}_\beta^{(\alpha)}|\xi|^{-|\gamma|}, \beta \in \mathbb{R}^+, \\
\left|\partial_\xi^\gamma \left[\xi_j|\xi|^{\alpha-1}(1+|\xi|^\alpha)^{-1}\right]\right| &\leq K|\xi|^{-|\gamma|}, j = 1,\ldots,d, \\
\left|\partial_\xi^\gamma \left[\xi_j\xi_j(1+|\xi|^2)^{-1}\right]\right| &\leq K|\xi|^{-|\gamma|}, j = 1,\ldots,d,
\end{aligned} \tag{3.3}$$

where K is a constant.

Lemma 3.3. *Let $\beta \in \mathbb{R}^+$, $h \in C^\infty(\mathbb{R}^d)$, and K_0 be a constant such that $|\partial^\gamma h(\xi)| \leq K_0(1+|\xi|)^{-|\gamma|}$, $\forall \xi \in \mathbb{R}^d$, $\forall \gamma \in \left\{\gamma : |\gamma| \leq d_0 = \left[\frac{d}{2}\right] + 1\right\}$. Then for all $f \in C^\beta(\mathbb{R}^d)$, there exists a constant K such that*

$$|\mathscr{F}^{-1}(h\mathscr{F}f)|_\beta \leq KK_0|f|_\beta.$$

Proof. See 2.6.1 in [91]. □

Lemma 3.4. *Let $\beta \in \mathbb{R}^+$, $\beta \notin \mathbb{N}$, and $\bar{\beta} \in (0,\beta]$. Assume $\bar{M}_\beta^{(\alpha)} < \infty$. Then for all $f \in C^\beta(\mathbb{R}^d)$, there exists a constant K such that*

$$\begin{aligned}
|\bar{\Phi}f(z,\cdot)|_\beta &\leq K\bar{M}^{(\alpha)}|f|_\beta, \forall z \in \mathbb{R}^d, \\
|\bar{\Phi}f(\cdot,x)|_\beta &\leq K\bar{M}_\beta^{(\alpha)}|f|_{\bar{\beta}}, \forall x \in \mathbb{R}^d, \\
|\tilde{\Phi}f|_\beta &\leq K\bar{M}_\beta^{(\alpha)}|f|_\beta, \\
|\Lambda_j f|_\beta &\leq K|f|_\beta, j = 1,\ldots,d, \\
|\Lambda_{ij} f|_\beta &\leq K|f|_\beta, i,j = 1,\ldots,d.
\end{aligned}$$

Proof. Let $\zeta_1 \in C_0^\infty(\mathbb{R}^d)$ and $\zeta_2 = 1 - \zeta_1$ with $\zeta_1 \in [0,1]$ and $\zeta_1(x) = 1$ if $|x| \leq 1$. Then,

$$\bar{\Phi} f(z,x) = \bar{\Phi}_1 f(z,x) + \bar{\Phi}_2 f(z,x)$$

and

$$\tilde{\Phi} f(x) = \tilde{\Phi}_1 f(x) + \tilde{\Phi}_2 f(x),$$

where

$$\bar{\Phi}_k f(z,x) = \mathscr{F}^{-1}\big[\bar{\phi}(z,\xi)\zeta_k(\xi)\mathscr{F} f(\xi)\big](x) = \varphi * \eta_k(z,x) * f = \tilde{\eta}_k(z,x) * f$$

and

$$\tilde{\Phi}_k f(x) = \bar{\Phi}_k f(x,x), k = 1,2,$$

with $\varphi = \mathscr{F}^{-1}\big[(1+|\xi|^\alpha)^{-1}\big]$ and $\eta_k(z,x) = \mathscr{F}^{-1}\big[\bar{\psi}^{(\alpha)}(z,\xi)\zeta_k(\xi)\big](x)$.

Let $\tilde{u} = \mathscr{F}^{-1}\zeta_1$, then

$$\begin{aligned}
\eta_1(z,x) &= \mathscr{F}^{-1}\big[\bar{\psi}^{(\alpha)}(z,\xi)\zeta_1(\xi)\big](x) \\
&= \mathscr{F}^{-1}\big[\bar{\psi}^{(\alpha)}(z,\xi)\mathscr{F}\tilde{u}(\xi)\big](x) \\
&= \mathbf{1}_{\{\alpha=1\}}\big(\bar{a}(z), \partial_x \tilde{u}(t,x)\big) + \frac{1}{2}\mathbf{1}_{\{\alpha=2\}} \sum_{i,j=1}^d \bar{B}^{ij}(z)\partial_{ij}\tilde{u}(t,x) \\
&\quad + \int A_y^{(\alpha)} \tilde{u}(t,x)\bar{m}^{(\alpha)}(z,y)\frac{dy}{|y|^{d+\alpha}},
\end{aligned}$$

and so

$$\int |\eta_1(\cdot, x)|_\beta dx \le K\bar{M}_\beta^{(\alpha)}.$$

It then follows by equivalence between norms [84] that

$$\int |\tilde{\eta}_1(\cdot, x)|_\beta dx \le K\bar{M}_\beta^{(\alpha)}.$$

Also,

$$\int |\tilde{\eta}_1(z, \cdot)|_\beta dz \le K\bar{M}^{(\alpha)}.$$

Hence,

$$|\bar{\Phi}_1 f(\cdot, x)|_\beta \le K\bar{M}_\beta^{(\alpha)}|f|_\infty,$$

$$|\bar{\Phi}_1 f(z, \cdot)|_\beta \le K\bar{M}^{(\alpha)}|f|_\beta,$$

and

$$|\tilde{\bar{\Phi}}_1 f|_\beta \le K\big(\bar{M}_\beta^{(\alpha)}|f|_\infty + \bar{M}^{(\alpha)}|f|_\beta\big).$$

Applying (3.3) and Lemma 3.3 gives

$$|\bar{\Phi}_2 f(\cdot, x)|_\beta \le K\bar{M}_\beta^{(\alpha)}|f|_{\bar{\beta}},$$

$$|\bar{\Phi}_2 f(z,\cdot)|_\beta \le K\bar{M}^{(\alpha)}|f|_\beta,$$

and

$$|\tilde{\Phi}_2 f|_\beta \le K\big(\bar{M}^{(\alpha)}_\beta |f|_{\bar{\beta}} + \bar{M}^{(\alpha)}|f|_\beta\big) \le K\bar{M}^{(\alpha)}_\beta |f|_\beta.$$

For example, in Lemma 3.3, let $\mathscr{F}^{-1}(h\mathscr{F}f)(\cdot) = \tilde{\Phi}_2 f(\cdot)$ and $h(\cdot) = \bar{\phi}(z,\cdot)\zeta_2(\cdot)$, then

$$|\partial^\gamma h(\xi)| \le K_0(1+|\xi|)^{-|\gamma|}, \forall \xi \in \mathbb{R}^d, \forall \gamma \in \big\{\gamma : |\gamma| \le d_0 = [\frac{d}{2}] + 1\big\}$$

and by Lemma 3.3,

$$|\tilde{\Phi}_2 f|_\beta \le K\bar{M}^{(\alpha)}_\beta |f|_\beta.$$

Hence,

$$|\bar{\Phi} f(z,\cdot)|_\beta \le |\bar{\Phi}_1 f(z,\cdot)|_\beta + |\bar{\Phi}_2 f(z,\cdot)|_\beta \le K\bar{M}^{(\alpha)}|f|_\beta, \forall z \in \mathbb{R}^d,$$

$$|\bar{\Phi} f(\cdot,x)|_\beta \le |\bar{\Phi}_1 f(\cdot,x)|_\beta + |\bar{\Phi}_2 f(\cdot,x)|_\beta \le K\bar{M}^{(\alpha)}_\beta |f|_{\bar{\beta}}, \forall x \in \mathbb{R}^d,$$

and thus

$$|\tilde{\Phi} f|_\beta \le |\tilde{\Phi}_1 f|_\beta + |\tilde{\Phi}_2 f|_\beta \le K\bar{M}^{(\alpha)}_\beta |f|_\beta.$$

Similarly, it follows from (3.3) and Lemma 3.3 that

$$|\Lambda_j f|_\beta \le K|f|_\beta, j = 1, \ldots, d$$

and

$$|\Lambda_{ij} f|_\beta \le K|f|_\beta, i, j = 1, \ldots, d.$$

□

Remark 3.5. *For $v \in C^{\alpha+\beta}(\mathbb{R}^d)$, denote $f = (1+\partial^\alpha)v$. Then, $f \in C^\beta(\mathbb{R}^d)$ and*

$$\begin{aligned}
\bar{\Phi} f(z,x) &= \mathscr{F}^{-1}\big[\bar{\phi}(z,\xi)\mathscr{F} f(\xi)\big](x) \\
&= \mathscr{F}^{-1}\big[\bar{\psi}^{(\alpha)}(z,\xi)(1+|\xi|^\alpha)^{-1}\mathscr{F} f(\xi)\big](x) \\
&= \mathscr{F}^{-1}\Big[\bar{\psi}^{(\alpha)}(z,\xi)\mathscr{F}\mathscr{F}^{-1}\big[(1+|\xi|^\alpha)^{-1}\mathscr{F} f(\xi)\big]\Big](x) \\
&= \mathscr{F}^{-1}\Big[\bar{\psi}^{(\alpha)}(z,\xi)\mathscr{F}\big[(1+\partial^\alpha)^{-1} f(\xi)\big]\Big](x) \\
&= \mathscr{F}^{-1}\big[\bar{\psi}^{(\alpha)}(z,\xi)\mathscr{F} v(\xi)\big](x) \\
&= \bar{\mathcal{A}}_z^{(\alpha)} v(x), \forall z, x \in \mathbb{R}^d.
\end{aligned}$$

Hence, by Lemma 3.4*,*

$$|\bar{\mathcal{A}}^{(\alpha)} v|_\beta = |\bar{\Phi} f|_\beta \le K|f|_\beta = K|v+\partial^\alpha v|_\beta = K|v|_{\alpha+\beta}.$$

Lemma 3.6. *Let $\beta \in \mathbb{R}^+$, $\beta \notin \mathbb{N}$, and $\bar{\beta} \in (0,\beta]$. Assume $\bar{N}_\beta^{(\alpha)} < \infty$. Then for all $f \in C^{\alpha+\beta}(\mathbb{R}^d)$, there exists a constant K such that*

$$\begin{aligned}
|\bar{\mathcal{B}}_z^{(\alpha)} f(\cdot)|_\beta &\le K\bar{N}^{(\alpha)}|f|_{\alpha+\beta}, z \in \mathbb{R}^d, \\
|\bar{\mathcal{B}}_\cdot^{(\alpha)} f(x)|_\beta &\le K\bar{N}_\beta^{(\alpha)}|f|_{\alpha+\bar{\beta}}, x \in \mathbb{R}^d, \\
|\bar{\mathcal{B}}^{(\alpha)} f|_\beta &\le K\bar{N}_\beta^{(\alpha)}|f|_{\alpha+\beta}.
\end{aligned}$$

The statement is proved by induction.

Proof. By Lemma 3.2, for $\alpha \in (0,1)$,

$$f(x+y) - f(x) = K \int k^{(\alpha)}(y,\bar{y}) \partial^{\alpha} f(x-\bar{y}) d\bar{y}$$

and for $\alpha \in (1,2)$,

$$\begin{aligned} f(x+y) - f(x) - \big(\partial_x f(x), y\big) &= \int_0^1 \big(\partial_x f(x+sy), y\big) ds - \int_0^1 \big(\partial_x f(x), y\big) ds \\ &= \int_0^1 \big(K \int k^{(\alpha-1)}(sy,\bar{y}) \partial^{\alpha-1} \partial_x f(x-\bar{y}) d\bar{y}, y\big) ds. \end{aligned}$$

Then,

$$\begin{aligned} \bar{\mathcal{B}}_z^{(\alpha)} f(x) &= \mathbf{1}_{\{\alpha \in (1,2]\}} \big(\bar{c}(z), \partial_x f(x)\big) + \int_{|y|>1} [f(x+y) - f(x)] \bar{\rho}^{(\alpha)}(z,y) \nu^{(\alpha)}(dy) \\ &\quad + \mathbf{1}_{\{\alpha \in (0,1]\}} \int_{|y|\le 1} [f(x+y) - f(x)] \bar{\rho}^{(\alpha)}(z,y) \nu^{(\alpha)}(dy) \\ &\quad + \mathbf{1}_{\{\alpha \in (1,2]\}} \int_{|y|\le 1} \big[f(x+y) - f(x) - \big(\partial_x f(x), y\big)\big] \bar{\rho}^{(\alpha)}(z,y) \nu^{(\alpha)}(dy) \\ &= \mathbf{1}_{\{\alpha \in (1,2]\}} \big(\bar{c}(z), \partial_x f(x)\big) + \int_{|y|>1} [f(x+y) - f(x)] \bar{\rho}^{(\alpha)}(z,y) \nu^{(\alpha)}(dy) \\ &\quad + \mathbf{1}_{\{\alpha \in (0,1]\}} K \int_{|y|\le 1} \int k^{(\alpha)}(y,\bar{y}) \partial^{\alpha} f(x-\bar{y}) \bar{\rho}^{(\alpha)}(z,y) d\bar{y} \nu^{(\alpha)}(dy) \\ &\quad + \mathbf{1}_{\{\alpha \in (1,2)\}} K \int_{|y|\le 1} \int_0^1 \big(\int k^{(\alpha-1)}(sy,\bar{y}) \partial^{\alpha-1} \partial_x f(x-\bar{y}) d\bar{y}, y\big) \\ &\qquad \bar{\rho}^{(\alpha)}(z,y) ds \nu^{(\alpha)}(dy) \\ &\quad + \mathbf{1}_{\{\alpha=2\}} \int_{|y|\le 1} \int_0^1 \int_0^s \partial_{ij} f(x+ry) y^i y^j \bar{\rho}^{(\alpha)}(z,y) dr ds \nu^{(\alpha)}(dy). \end{aligned}$$

For $\beta \in (0,1)$, by Lemma 3.2,

$$|f(x+y) - f(x)| = |K \int k^{(\alpha)}(y,\bar{y}) \partial^{\alpha} f(x-\bar{y}) d\bar{y}| \le K |\partial^{\alpha} f|_{\infty} |y|^{\alpha}, \alpha \in (0,1)$$

and

$$
\begin{aligned}
|f(x+y)-f(x)-\big(\partial_x f(x),y\big)| &= \Big|\int_0^1 \big(K\int k^{(\alpha-1)}(sy,\bar y)\partial^{\alpha-1}\partial_x f(x-\bar y)d\bar y, y\big)ds\Big| \\
&\le \int_0^1 K|\partial^{\alpha-1}\partial_x f|_\infty(|sy|^{\alpha-1},y)ds \\
&\le K|\partial^{\alpha-1}\partial_x f|_\infty |y|^\alpha, \alpha\in(1,2).
\end{aligned}
$$

Then,

$$
\begin{aligned}
|\bar{\mathcal{B}}_z^{(\alpha)} f(\cdot)|_\beta^0 &= \sup_x |\bar{\mathcal{B}}_z^{(\alpha)} f(x)| \\
&\le \mathbf{1}_{\{\alpha\in(1,2]\}}|\bar c|_\infty|\partial_x f|_\infty + 2|f|_\infty\int_{|y|>1}|\bar\rho^{(\alpha)}(z,y)|\nu^{(\alpha)}(dy) \\
&\quad +\mathbf{1}_{\{\alpha\in(0,1]\}}K|\partial^\alpha f|_\infty\int_{|y|\le 1}|y|^\alpha|\bar\rho^{(\alpha)}(z,y)|\nu^{(\alpha)}(dy) \\
&\quad +\mathbf{1}_{\{\alpha\in(1,2]\}}K|\partial^{\alpha-1}\partial_x f|_\infty\int_{|y|\le 1}|y|^\alpha|\bar\rho^{(\alpha)}(z,y)|\nu^{(\alpha)}(dy) \\
&\le K\bar N^{(\alpha)}|f|_\alpha
\end{aligned}
$$

and

$$
\begin{aligned}
|\bar{\mathcal{B}}_z^{(\alpha)} f(\cdot)|_\beta^1 &= \sup_{x,h\neq 0}\frac{|\bar{\mathcal{B}}_z^{(\alpha)} f(x+h)-\bar{\mathcal{B}}_z^{(\alpha)} f(x)|}{|h|^\beta} \\
&\le \mathbf{1}_{\{\alpha\in(1,2]\}}|\bar c|_\infty|\partial_x f|_\beta + 2|f|_\beta\int_{|y|>1}|\bar\rho^{(\alpha)}(z,y)|\nu^{(\alpha)}(dy) \\
&\quad +\mathbf{1}_{\{\alpha\in(0,1]\}}K|\partial^\alpha f|_\beta\int_{|y|\le 1}|y|^\alpha|\bar\rho^{(\alpha)}(z,y)|\nu^{(\alpha)}(dy) \\
&\quad +\mathbf{1}_{\{\alpha\in(1,2]\}}K|\partial^{\alpha-1}\partial_x f|_\beta\int_{|y|\le 1}|y|^\alpha|\bar\rho^{(\alpha)}(z,y)|\nu^{(\alpha)}(dy) \\
&\le K\bar N^{(\alpha)}|f|_{\alpha+\beta},
\end{aligned}
$$

Hence,

$$
|\bar{\mathcal{B}}_z^{(\alpha)} f(\cdot)|_\beta = |\bar{\mathcal{B}}_z^{(\alpha)} f(\cdot)|_\beta^0 + |\bar{\mathcal{B}}_z^{(\alpha)} f(\cdot)|_\beta^1 \le K\bar N^{(\alpha)}|f|_{\alpha+\beta}.
$$

Similarly,

$$
\begin{aligned}
|\bar{\mathcal{B}}^{(\alpha)}_{\cdot} f(x)|^0_\beta &= \sup_z |\bar{\mathcal{B}}^{(\alpha)}_z f(x)| \\
&\leq \mathbf{1}_{\{\alpha\in(1,2]\}}|\bar{c}|_\infty|\partial_x f|_\infty + 2|f|_\infty \sup_z \int_{|y|>1} |\bar{\rho}^{(\alpha)}(z,y)|\nu^{(\alpha)}(dy) \\
&\quad +\mathbf{1}_{\{\alpha\in(0,1]\}}K|\partial^\alpha f|_\infty \sup_z \int_{|y|\leq 1} |y|^\alpha |\bar{\rho}^{(\alpha)}(z,y)|\nu^{(\alpha)}(dy) \\
&\quad +\mathbf{1}_{\{\alpha\in(1,2]\}}K|\partial^{\alpha-1}\partial_x f|_\infty \sup_z \int_{|y|\leq 1} |y|^\alpha |\bar{\rho}^{(\alpha)}(z,y)|\nu^{(\alpha)}(dy) \\
&\leq K\bar{N}^{(\alpha)}|f|_\alpha
\end{aligned}
$$

and

$$
\begin{aligned}
|\bar{\mathcal{B}}^{(\alpha)}_{\cdot} f(x)|^1_\beta &= \sup_{z,h\neq 0} \frac{|\bar{\mathcal{B}}^{(\alpha)}_{z+h} f(x) - \bar{\mathcal{B}}^{(\alpha)}_z f(x)|}{|h|^\beta} \\
&\leq \mathbf{1}_{\{\alpha\in(1,2]\}}|\bar{c}|_\beta|\partial_x f|_\infty + 2|f|_\infty \sup_{z,h\neq 0} \int_{|y|>1} \frac{|\bar{\rho}^{(\alpha)}(z+h,y) - \bar{\rho}^{(\alpha)}(z,y)|}{|h|^\beta}\nu^{(\alpha)}(dy) \\
&\quad +\mathbf{1}_{\{\alpha\in(0,1]\}}K|\partial^\alpha f|_\infty \sup_{z,h\neq 0} \int_{|y|\leq 1} |y|^\alpha \frac{|\bar{\rho}^{(\alpha)}(z+h,y) - \bar{\rho}^{(\alpha)}(z,y)|}{|h|^\beta}\nu^{(\alpha)}(dy) \\
&\quad +\mathbf{1}_{\{\alpha\in(1,2]\}}K|\partial^{\alpha-1}\partial_x f|_\infty \sup_{z,h\neq 0} \int_{|y|\leq 1} |y|^\alpha \frac{|\bar{\rho}^{(\alpha)}(z+h,y) - \bar{\rho}^{(\alpha)}(z,y)|}{|h|^\beta}\nu^{(\alpha)}(dy) \\
&\leq K\bar{N}^{(\alpha)}_\beta |f|_\alpha,
\end{aligned}
$$

Hence,

$$
|\bar{\mathcal{B}}^{(\alpha)}_{\cdot} f(x)|_\beta = |\bar{\mathcal{B}}^{(\alpha)}_{\cdot} f(x)|^0_\beta + |\bar{\mathcal{B}}^{(\alpha)}_{\cdot} f(x)|^1_\beta \leq K\bar{N}^{(\alpha)}_\beta |f|_{\alpha+\bar{\beta}}.
$$

Also,

$$
\begin{aligned}
|\bar{\mathcal{B}}^{(\alpha)} f|^0_\beta &= \sup_x |\bar{\mathcal{B}}^{(\alpha)}_x f(x)| \\
&\leq \mathbf{1}_{\{\alpha\in(1,2]\}}|\bar{c}|_\infty|\partial_x f|_\infty + 2|f|_\infty \sup_x \int_{|y|>1} |\bar{\rho}^{(\alpha)}(x,y)|\nu^{(\alpha)}(dy)
\end{aligned}
$$

$$
\begin{aligned}
& +\mathbf{1}_{\{\alpha\in(0,1]\}} K|\partial^\alpha f|_\infty \sup_x \int_{|y|\le 1} |y|^\alpha |\bar\rho^{(\alpha)}(x,y)|\nu^{(\alpha)}(dy) \\
& +\mathbf{1}_{\{\alpha\in(1,2]\}} K|\partial^{\alpha-1}\partial_x f|_\infty \sup_x \int_{|y|\le 1} |y|^\alpha |\bar\rho^{(\alpha)}(x,y)|\nu^{(\alpha)}(dy) \\
\le \; & K\bar N^{(\alpha)}|f|_\alpha
\end{aligned}
$$

and

$$
\begin{aligned}
|\bar{\mathcal{B}}^{(\alpha)} f|_\beta^1 &= \sup_{x,h\neq 0} \frac{|\bar{\mathcal{B}}^{(\alpha)}_{x+h} f(x+h) - \bar{\mathcal{B}}^{(\alpha)}_x f(x)|}{|h|^\beta} \\
&\le \sup_{x,h\neq 0} \frac{|\bar{\mathcal{B}}^{(\alpha)}_{x+h} f(x+h) - \bar{\mathcal{B}}^{(\alpha)}_{x+h} f(x)|}{|h|^\beta} + \sup_{x\in\mathbb{R}^d,h\neq 0} \frac{|\bar{\mathcal{B}}^{(\alpha)}_{x+h} f(x) - \bar{\mathcal{B}}^{(\alpha)}_x f(x)|}{|h|^\beta} \\
&\le K\big(|\bar{\mathcal{B}}^{(\alpha)}_{x+h} f(\cdot)|_\beta^1 + |\bar{\mathcal{B}}^{(\alpha)}_\cdot f(x)|_\beta^1\big) \\
&\le K\big(\bar N^{(\alpha)}|f|_{\alpha+\beta} + \bar N^{(\alpha)}_\beta |f|_\alpha\big) \\
&\le K\bar N^{(\alpha)}_\beta |f|_{\alpha+\beta}.
\end{aligned}
$$

Hence,

$$
|\bar{\mathcal{B}}^{(\alpha)} f|_\beta = |\bar{\mathcal{B}}^{(\alpha)} f|_\beta^0 + |\bar{\mathcal{B}}^{(\alpha)} f|_\beta^1 \le K\bar N^{(\alpha)}_\beta |f|_{\alpha+\beta}.
$$

Assume the inequalities hold for $\beta \in \bigcup_{l=0}^{n-1}(l,l+1)$, $n\in\mathbb{N}$. For $\beta\in(n,n+1)$ and $f\in C^{\alpha+\beta}(\mathbb{R}^d)$, $\beta-1\in(n-1,n)$ and $\partial_x f\in C^{\alpha+\beta-1}(\mathbb{R}^d)$. Hence,

$$
|\bar{\mathcal{B}}^{(\alpha)}_z \partial_x f(\cdot)|_{\beta-1} \le K\bar N^{(\alpha)}|\partial_x f|_{\alpha+\beta-1} \le K\bar N^{(\alpha)}|f|_{\alpha+\beta}.
$$

Similarly,

$$
|\partial_z \bar{\mathcal{B}}^{(\alpha)}_\cdot f(x)|_{\beta-1} \le K\bar N^{(\alpha)}_\beta |f|_\alpha \le K\bar N^{(\alpha)}_\beta |f|_{\alpha+\bar\beta}.
$$

Since for $|\gamma| \leq [\beta]^-$,

$$\partial^\gamma(\bar{\mathcal{B}}^{(\alpha)} f) = \sum_{\kappa+\mu=\gamma} \partial_z^\kappa \bar{\mathcal{B}}_z^{(\alpha)} (\partial^\mu f)(x)|_{z=x}.$$

Then,

$$\begin{aligned}
|\bar{\mathcal{B}}_z^{(\alpha)} f(\cdot)|_\beta &= \sum_{|\gamma|\leq[\beta]^-} \sup_x |\partial_x^\gamma \bar{\mathcal{B}}_z^{(\alpha)} f(x)| + \sup_{\substack{|\gamma|=[\beta]^-,\\ x,h\neq 0}} \frac{|\partial_x^\gamma \bar{\mathcal{B}}_z^{(\alpha)} f(x+h) - \partial_x^\gamma \bar{\mathcal{B}}_z^{(\alpha)} f(x)|}{|h|^{\{\beta\}^+}} \\
&= \sup_x |\bar{\mathcal{B}}_z^{(\alpha)} f(x)| + \sum_{|\gamma|\leq[\beta-1]^-} \sup_x |\partial_x^\gamma \bar{\mathcal{B}}_z^{(\alpha)} \partial_x f(x)| \\
&\quad + \sup_{\substack{|\gamma|=[\beta-1]^-,\\ x,h\neq 0}} \frac{|\partial_x^\gamma \bar{\mathcal{B}}_z^{(\alpha)} \partial_x f(x+h) - \partial_x^\gamma \bar{\mathcal{B}}_z^{(\alpha)} \partial_x f(x)|}{|h|^{\{\beta\}^+}} \\
&= \sup_x |\bar{\mathcal{B}}_z^{(\alpha)} f(x)| + |\bar{\mathcal{B}}_z^{(\alpha)} \partial_x f(\cdot)|_{\beta-1} \\
&\leq K\big(\bar{N}^{(\alpha)} |f|_\alpha + \bar{N}^{(\alpha)} |\partial_x f|_{\alpha+\beta-1}\big) \\
&\leq K\bar{N}^{(\alpha)} |f|_{\alpha+\beta},
\end{aligned}$$

$$\begin{aligned}
|\bar{\mathcal{B}}_\cdot^{(\alpha)} f(x)|_\beta &= \sum_{|\gamma|\leq[\beta]^-} \sup_z |\partial_z^\gamma \bar{\mathcal{B}}_z^{(\alpha)} f(x)| + \sup_{\substack{|\gamma|=[\beta]^-,\\ z,h\neq 0}} \frac{|\partial_z^\gamma \bar{\mathcal{B}}_{z+h}^{(\alpha)} f(x) - \partial_z^\gamma \bar{\mathcal{B}}_z^{(\alpha)} f(x)|}{|h|^{\{\beta\}^+}} \\
&= \sup_z |\bar{\mathcal{B}}_z^{(\alpha)} f(x)| + \sum_{|\gamma|\leq[\beta-1]^-} \sup_z |\partial_z^\gamma \partial_z \bar{\mathcal{B}}_z^{(\alpha)} f(x)| \\
&\quad + \sup_{\substack{|\gamma|=[\beta-1]^-,\\ z,h\neq 0}} \frac{|\partial_z^\gamma \partial_z \bar{\mathcal{B}}_z^{(\alpha)} f(x+h) - \partial_z^\gamma \partial_z \bar{\mathcal{B}}_z^{(\alpha)} f(x)|}{|h|^{\{\beta\}^+}} \\
&= \sup_z |\bar{\mathcal{B}}_z^{(\alpha)} f(x)| + |\partial_z \bar{\mathcal{B}}_\cdot^{(\alpha)} f(x)|_{\beta-1} \\
&\leq K\big(\bar{N}^{(\alpha)} |f|_\alpha + \bar{N}_\beta^{(\alpha)} |f|_{\alpha+\bar{\beta}}\big) \\
&\leq K\bar{N}_\beta^{(\alpha)} |f|_{\alpha+\bar{\beta}},
\end{aligned}$$

and

$$
\begin{aligned}
|\bar{\mathcal{B}}^{(\alpha)} f|_\beta &= \sum_{|\gamma|\leq[\beta]^-} \sup_x |\partial_x^\gamma \bar{\mathcal{B}}_x^{(\alpha)} f(x)| + \sup_{\substack{|\gamma|=[\beta]^-,\\ x,h\neq 0}} \frac{|\partial_x^\gamma \bar{\mathcal{B}}_{x+h}^{(\alpha)} f(x+h) - \partial_x^\gamma \bar{\mathcal{B}}_x^{(\alpha)} f(x)|}{|h|^{\{\beta\}^+}} \\
&\leq \sum_{|\gamma|\leq[\beta]^-} \sup_x |\partial_x^\gamma \bar{\mathcal{B}}_x^{(\alpha)} f(x)| + \sup_{\substack{|\gamma|=[\beta]^-,\\ x,h\neq 0}} \frac{|\partial_x^\gamma \bar{\mathcal{B}}_{x+h}^{(\alpha)} f(x+h) - \partial_x^\gamma \bar{\mathcal{B}}_{x+h}^{(\alpha)} f(x)|}{|h|^{\{\beta\}^+}} \\
&\quad + \sup_{\substack{|\gamma|=[\beta]^-,\\ x,h\neq 0}} \frac{|\partial_x^\gamma \bar{\mathcal{B}}_{x+h}^{(\alpha)} f(x) - \partial_x^\gamma \bar{\mathcal{B}}_x^{(\alpha)} f(x)|}{|h|^{\{\beta\}^+}} \\
&\leq |\bar{\mathcal{B}}_z^{(\alpha)} f(\cdot)|_\beta + |\bar{\mathcal{B}}_\cdot^{(\alpha)} f(x)|_\beta \\
&\leq K\big(\bar{N}^{(\alpha)}|f|_{\alpha+\beta} + \bar{N}_\beta^{(\alpha)}|f|_{\alpha+\bar{\beta}}\big) \\
&\leq K\bar{N}_\beta^{(\alpha)}|f|_{\alpha+\beta}.
\end{aligned}
$$

The statement then follows. □

Lemma 3.7. *Assume $u_n \in C^\beta(\mathbb{R}^d), \forall n \in \mathbb{N}$ with $\sup_n |u_n|_\beta < \infty$ and $u_n \to u$ uniformly on compact subsets. Then $u \in C^\beta$, $|u|_\beta \leq \sup_n |u_n|_\beta$, and*

$$\partial^\delta \partial^\gamma u_n \to \partial^\delta \partial^\gamma u, |\gamma| \leq [\beta]^-$$

uniformly on compact subsets as $n \to \infty$ for any $\delta \in [0,1)$ such that $[\beta]^- + \delta < \beta$.

Proof. Let $\{\varphi_k\}_{k=0,\pm1,\ldots}$ and ψ be as defined in (3.1) and (3.2). If $u_n \to u$ uniformly on compact sets, then

$$|\psi * u(x)| = \lim_n |\psi * u_n(x)| \leq \sup_n \sup_y |\psi * u_n(y)|, \forall x \in \mathbb{R}^d$$

and

$$2^{\beta k}|\varphi_k * u(x)| = 2^{\beta k}\lim_n |\varphi_k * u_n(x)| \leq \sup_n \sup_k 2^{\beta k}\sup_y |\varphi_k * u_n(y)|, \forall x \in \mathbb{R}^d.$$

Hence, $|u|_\beta \leq \sup_n |u_n|_\beta < \infty$.

Assume $\sup_n |u_n|_\beta < \infty$, $\beta > 1$. Then, $\sup_n |\partial_x u_n|_{\beta-1} < \infty$ and $\partial_x u_n$ satisfies the Arzelà-Ascoli theorem. Hence, for each $\{\partial_x u_n\}$, there exist a subsequence $\{\partial_x u_{n_k}\}$ and a function v such that $\partial_x u_{n_k}(x) \to v(x)$ uniformly on compact subsets as $n_k \to \infty$. Since

$$u_{n_k}(x+h) - u_{n_k}(x) = h\int_0^1 \partial_x u_{n_k}(x+sh)ds,$$

passing to the limit on both sides yields

$$u(x+h) - u(x) = h\int_0^1 v(x+sh)ds.$$

That is,

$$\frac{u(x+h)-u(x)}{h} = \int_0^1 \partial_x u(x+sh)ds = \int_0^1 v(x+sh)ds,$$

and so $\partial_x u = v$. By induction, $\partial^\gamma u_n \to \partial^\gamma u$, $\forall \gamma \in \{\gamma : |\gamma| \leq [\beta]^-\}$ uniformly on compact subsets.

Also, by Lemma 3.2, for $\delta \in [0,1)$ such that $[\beta]^- + \delta < \beta$ and $|\mu| \leq [\beta]^-$,

$$\partial^\delta \partial^\mu u_n(x) = \int \left[\partial^\mu u_n(x+y) - \partial^\mu u_n(x)\right]\frac{dy}{|y|^{d+\delta}}.$$

Passing to the limit yields that for $\delta \in [0,1)$ such that $[\beta]^- + \delta < \beta$ and $|\mu| \le [\beta]^-$,

$$\partial^\delta \partial^\mu u_n \to \partial^\delta \partial^\mu u$$

uniformly on compact subsets as $n \to \infty$. $\square$

Theorem 3.8. *Let $\alpha \in (0,2]$ and $\beta \in \mathbb{R}^+$, $\beta \notin \mathbb{N}$. Assume* (A1) *and* (A2) *hold. Then for $f \in C^\beta(H)$, there exist a unique solution $u \in C^{\alpha+\beta}(H)$ to the Cauchy problem*

$$\begin{aligned} \big(\partial_t + \mathcal{A}_x^{(\alpha)} + \mathcal{B}_x^{(\alpha)}\big)u(t,x) &= f(t,x), \\ u(T,x) &= 0 \end{aligned} \tag{3.4}$$

and a constant K independent of f such that $|u|_{\alpha+\beta} \le K|f|_\beta$.

For u being defined on H, $h \in \mathbb{R}$ being a given nonzero number, $\{e_k\}_{k=1,\dots,d}$ being the canonical basis in $\mathbb{R}^d$, and $k = 1,\dots,d$, denote

$$u_k^h(t,x) = \frac{u(t,x+he_k) - u(t,x)}{h},$$

$$\nabla_y^1 u(t,x) = u(t,x+y) - u(t,x),$$

$$\nabla_y^2 u(t,x) = u(t,x+y) - u(t,x) - \big(\partial_x u(t,x), y\big),$$

$$\mathcal{A}_{z,k}^{(\alpha),h} u(t,x) = \frac{1}{h}\big(\mathcal{A}_{z+he_k}^{(\alpha)} - \mathcal{A}_z^{(\alpha)}\big)u(t,x),$$

and

$$\mathcal{B}_{z,k}^{(\alpha),h} u(t,x) = \frac{1}{h}\big(\mathcal{B}_{z+he_k}^{(\alpha)} - \mathcal{B}_z^{(\alpha)}\big)u(t,x).$$

The statement is proved by induction.

Proof. For $\alpha \in (0,2]$ and $\beta \in (0,1)$, given $f \in C^{\beta}(H)$, there exists a unique solution $u \in C^{\alpha+\beta}(H)$ to the Cauchy problem (3.4) and $|u|_{\alpha+\beta} \leq K|f|_{\beta}$ [60], where K is a constant independent of f.

Assume the result holds for $\beta \in \bigcup_{l=0}^{n-1}(l,l+1)$, $n \in \mathbb{N}$.

Let $\beta \in (n,n+1)$ and $f \in C^{\beta}$. Then $\beta - 1 \in (n-1,n)$, $f \in C^{\beta-1}(H)$, and there exists a unique solution $u \in C^{\alpha+\beta-1}(H)$, $\alpha \in (0,2]$ to the Cauchy problem (3.4) with $|u|_{\alpha+\beta-1} \leq C|f|_{\beta-1}$.

In addition, for a given nonzero number $h \in \mathbb{R}$, u satisfies

$$\begin{aligned} \big(\partial_t + \mathcal{A}^{(\alpha)}_{x+he_k} + \mathcal{B}^{(\alpha)}_{x+he_k}\big)u(t,x+he_k) &= f(t,x+he_k), \\ u(T,x+he_k) &= 0, \quad k = 1,\ldots,d. \end{aligned} \tag{3.5}$$

Subtracting (3.4) from (3.5) and dividing the difference by h yields

$$\begin{aligned} \big(\partial_t + \mathcal{A}^{(\alpha)}_{x} + \mathcal{B}^{(\alpha)}_{x}\big)u^h_k(t,x) &= f^h_k(t,x) - \mathcal{A}^{(\alpha),h}_{x,k}u(t,x+he_k) - \mathcal{B}^{(\alpha),h}_{x,k}u(t,x+he_k), \\ u^h_k(T,x) &= 0, k = 1,\ldots,d. \end{aligned} \tag{3.6}$$

Since $f \in C^{\beta}(H)$ and

$$f^h_k(t,x) = \frac{f(t,x+he_k) - f(t,x)}{h} = \int_0^1 \partial_k f(t,x+he_k s)ds, \forall h \neq 0,$$

then $|f^h_k|_{\beta-1} \leq K|\partial_x f|_{\beta-1} \leq K|f|_{\beta}$ with K independent of h.

Since $u \in C^{\alpha+\beta-1}(H)$, then $(1+\partial^{\alpha})u \in C^{\beta-1}(H)$ and by Lemma 3.4,

$$|\mathcal{A}^{(\alpha)}_x u|_{\beta-1} = |\mathcal{A}^{(\alpha)}_x (1+\partial^{\alpha})^{-1}(1+\partial^{\alpha})u|_{\beta-1}$$

$$
\begin{aligned}
&= |\bar{\Phi}(1+\partial^\alpha)u|_{\beta-1} \\
&\le KM_\beta^{(\alpha)}|(1+\partial^\alpha)u|_{\beta-1} \\
&\le KM_\beta^{(\alpha)}|u|_{\alpha+\beta-1} \\
&\le KM_\beta^{(\alpha)}|f|_{\beta-1}.
\end{aligned}
$$

Since $a, B, m^{(\alpha)} \in C^\beta$, then $\partial_x a, \partial_x B, \partial_x m^{(\alpha)} \in C^{\beta-1}$ and so for

$$
\begin{aligned}
\bar{a}_{h,k}(x) &= \int_0^1 \partial_k a(x+he_k s)ds, \\
\bar{B}_{h,k}^{ij}(x) &= \int_0^1 \partial_k B^{ij}(x+he_k s)ds, \\
\bar{m}_{h,k}^{(\alpha)}(x,y) &= \int_0^1 \partial_k m^{(\alpha)}(x+he_k s, y)ds,
\end{aligned}
$$

$\bar{a}_{h,k}, \bar{B}_{h,k}, \bar{m}_{h,k}^{(\alpha)} \in C^{\beta-1}$. Also,

$$
\begin{aligned}
\mathcal{A}_{x,k}^{(\alpha),h}u(t,x+he_k) &= \frac{1}{h}\big(\mathcal{A}_{x+he_k}^{(\alpha)} - \mathcal{A}_x^{(\alpha)}\big)u(t,x+he_k) \\
&= \mathbf{1}_{\{\alpha=1\}}\Big(\frac{1}{h}\big(a(x+he_k)-a(x)\big), \partial_x u(t,x+he_k)\Big) \\
&\quad + \frac{1}{2}\mathbf{1}_{\{\alpha=2\}}\sum_{i,j=1}^d \frac{1}{h}\big[B^{ij}(x+he_k) - B^{ij}(x)\big]\partial_{ij}u(t,x+he_k) \\
&\quad + \int A_y^{(\alpha)}u(t,x+he_k)\frac{1}{h}\big[m^{(\alpha)}(x+he_k,y) - m^{(\alpha)}(x,y)\big]\frac{dy}{|y|^{d+\alpha}} \\
&= \mathbf{1}_{\{\alpha=1\}}\big(\int_0^1 \partial_k a(x+he_k s)ds, \partial_x u(t,x+he_k)\big) \\
&\quad + \frac{1}{2}\mathbf{1}_{\{\alpha=2\}}\sum_{i,j=1}^d \int_0^1 \partial_k B^{ij}(x+he_k s)ds\partial_{ij}u(t,x+he_k) \\
&\quad + \int A_y^{(\alpha)}u(t,x+he_k)\int_0^1 \partial_k m^{(\alpha)}(x+he_k s,y)ds\frac{dy}{|y|^{d+\alpha}} \\
&= \mathbf{1}_{\{\alpha=1\}}\big(\bar{a}_{h,k}(x), \partial_x u(t,x+he_k)\big) \\
&\quad + \frac{1}{2}\mathbf{1}_{\{\alpha=2\}}\sum_{i,j=1}^d \bar{B}_{h,k}^{ij}(x)\partial_{ij}u(t,x+he_k)
\end{aligned}
$$

$$+\int A_y^{(\alpha)}u(t,x+he_k)\bar{m}_{h,k}^{(\alpha)}(x,y)\frac{dy}{|y|^{d+\alpha}}.$$

Hence, by Lemma 3.4,

$$|\mathcal{A}_{\cdot,k}^{(\alpha),h}u|_{\beta-1}\le KM_\beta^{(\alpha)}|u|_{\alpha+\beta-1}\le KM_\beta^{(\alpha)}|f|_{\beta-1}, k=1,\ldots,d$$

with K independent of h and f.

Denote

$$\begin{aligned}\bar{c}_{h,k}(x) &= \int_0^1 \partial_k c(x+he_k s)ds, x\in\mathbb{R}^d,\\ \bar{\rho}_{h,k}^{(\alpha)}(x,y) &= \int_0^1 \partial_k\rho^{(\alpha)}(x+he_k s,y)ds, x,y\in\mathbb{R}^d.\end{aligned}$$

Then

$$\begin{aligned}\mathcal{B}_{x,k}^{(\alpha),h}u(t,x+he_k) &= \frac{1}{h}\big(\mathcal{B}_{x+he_k}^{(\alpha)}-\mathcal{B}_x^{(\alpha)}\big)u(t,x+he_k)\\
&= \mathbf{1}_{\{\alpha\in(1,2]\}}\frac{1}{h}\big(c(x+he_k)-c(x),\partial_x u(t,x+he_k)\big)\\
&\quad+\frac{1}{h}\int B_y^{(\alpha)}u(t,x+he_k)\big[\rho^{(\alpha)}(x+he_k,dy)-\rho^{(\alpha)}(x,y)\nu^{(\alpha)}(dy)\big]\\
&= \mathbf{1}_{\{\alpha\in(1,2]\}}\Big(\frac{1}{h}\big(c(x+he_k)-c(x)\big),\partial_x u(t,x+he_k)\Big)\\
&\quad+\int B_y^{(\alpha)}u(t,x+he_k)\frac{1}{h}\big(\rho^{(\alpha)}(x+he_k,y)-\rho^{(\alpha)}(x,y)\big)\nu^{(\alpha)}(dy)\\
&= \mathbf{1}_{\{\alpha\in(1,2]\}}\big(\int_0^1\partial_k c(x+he_k s)ds,\partial_x u(t,x+he_k)\big)\\
&\quad+\int B_y^{(\alpha)}u(t,x+he_k)\int_0^1\partial_k\rho^{(\alpha)}(x+he_k s,y)ds\nu^{(\alpha)}(dy)\\
&= \mathbf{1}_{\{\alpha\in(1,2]\}}\big(\int_0^1\partial_k c(x+he_k s)ds,\partial_x u(t,x+he_k)\big)\\
&\quad+\mathbf{1}_{\{\alpha\in(0,1]\}}\int_{|y|\le 1}\nabla_y^1 u(t,x+he_k)\int_0^1\partial_k\rho^{(\alpha)}(x+he_k s,y)ds\nu^{(\alpha)}(dy)\\
&\quad+\mathbf{1}_{\{\alpha\in(1,2]\}}\int_{|y|\le 1}\nabla_y^2 u(t,x+he_k)\int_0^1\partial_k\rho^{(\alpha)}(x+he_k s,y)ds\nu^{(\alpha)}(dy)\end{aligned}$$

$$
\begin{aligned}
& + \int_{|y|>1} \nabla_y^1 u(t, x + he_k) \int_0^1 \partial_k \rho^{(\alpha)}(x + he_k s, y) ds \nu^{(\alpha)}(dy) \\
= \quad & \mathbf{1}_{\{\alpha \in (1,2]\}} \big(\bar{c}_{h,k}(x), \partial_x u(t, x + he_k)\big) \\
& + \mathbf{1}_{\{\alpha \in (0,1]\}} \int_{|y| \le 1} \nabla_y^1 u(t, x + he_k) \bar{\rho}_{h,k}^{(\alpha)}(x, y) \nu^{(\alpha)}(dy) \\
& + \mathbf{1}_{\{\alpha \in (1,2]\}} \int_{|y| \le 1} \nabla_y^2 u(t, x + he_k) \bar{\rho}_{h,k}^{(\alpha)}(x, y) \nu^{(\alpha)}(dy) \\
& + \int_{|y|>1} \nabla_y^1 u(t, x + he_k) \bar{\rho}_{h,k}^{(\alpha)}(x, y) \nu^{(\alpha)}(dy).
\end{aligned}
$$

Applying Lemma 3.6 to $\bar{c} = \bar{c}_{h,k}$, $\bar{\rho}^{(\alpha)} = \bar{\rho}_{h,k}^{(\alpha)}$, $\beta = \bar{\beta} = \beta - 1$, and $f = u(t, \cdot + he_k)$ yields

$$
|\mathcal{B}_{\cdot,k}^{(\alpha),h} u(t, \cdot)|_{\beta-1} \le K N_\beta^{(\alpha)} |u(t, \cdot + he_k)|_{\alpha+\beta-1} \le K N_\beta^{(\alpha)} |f|_{\beta-1}, k = 1, \ldots, d,
$$

with a constant K independent of h and f.

Hence, $f_k^h(t, x) - \mathcal{A}_{x,k}^{(\alpha),h} u(t, x + he_k) - \mathcal{B}_{x,k}^{(\alpha),h} u(t, x + he_k) \in C^{\beta-1}(H)$ and by (3.4), $u_k^h \in C^{\alpha+\beta-1}(H)$, $\forall h \ne 0$, $k = 1, \ldots, d$, with

$$
|u_k^h|_{\alpha+\beta-1} \le K |f_k^h - \mathcal{A}_{\cdot,k}^{(\alpha),h} u - \mathcal{B}_{\cdot,k}^{(\alpha),h} u|_{\beta-1} \le K |f|_\beta,
$$

where K is a constant independent of h and f. Also by Lemma 3.6, $\mathcal{B}^{(\alpha)} u_k^h \in C^{\beta-1}(H)$ with $|\mathcal{B}^{(\alpha)} u_k^h|_{\beta-1} \le K |u_k^h|_{\alpha+\beta-1} \le K |f|_\beta$.

By (3.6),

$$
\begin{aligned}
u_k^h(t, x) - u_k^h(s, x) \quad = \quad & \int_s^t \big[f_k^h(r, x) - \mathcal{A}_{x,k}^{(\alpha),h} u(r, x + he_k) - \mathcal{B}_{x,k}^{(\alpha),h} u(r, x + he_k)\big] dr \\
& - \int_s^t \big(\mathcal{A}_x^{(\alpha)} + \mathcal{B}_x^{(\alpha)}\big) u_k^h(r, x) dr, 0 \le s < t \le T,
\end{aligned}
$$

and so

$$\begin{aligned}|u_k^h(t,x) - u_k^h(s,x)| &\le \big(|f_k^h - \mathcal{A}_{x,k}^{(\alpha),h} u - \mathcal{B}_{x,k}^{(\alpha),h} u|_{\beta-1} + |\big(\mathcal{A}_x^{(\alpha)} + \mathcal{B}_x^{(\alpha)}\big) u_k^h|_{\beta-1}\big)|t-s| \\ &\le K|t-s|.\end{aligned}$$

Hence, $u_k^h(t,x)$ is equicontinuous in (t,x) and by the Arzelà-Ascoli theorem, for each $\{h_n\}$, there exist a subsequence $\{h_{n_k}\}$ and a function $u_k(t,x)$ such that $u_k^{h_{n_k}}(t,x) \to u_k(t,x)$ uniformly on compact subsets as $h_{n_k} \to 0$. By Lemma 3.7, $u_k \in C^{\alpha+\beta-1}$.

It then follows from passing to the limit in (3.6) and the dominated convergence theorem that u_k is the unique solution to

$$\begin{aligned}\big(\partial_t + \mathcal{A}_x^{(\alpha)} + \mathcal{B}_x^{(\alpha)}\big) u_k(t,x) &= \partial_k f(t,x) - \big(\partial_k \mathcal{A}_x^{(\alpha)}\big) u(t,x) - \big(\partial_k \mathcal{B}_x^{(\alpha)}\big) u(t,x), \\ u_k(T,x) &= 0, k = 1, \dots, d\end{aligned}$$

and so

$$u_k^{h_n}(t,x) \to u_k(t,x), \forall h_n \to 0.$$

Hence,

$$u_k(t,x) = \lim_{h\to 0} u_k^h(t,x) = \lim_{h\to 0} \frac{u(t,x+he_k) - u(t,x)}{h} = \partial_k u(t,x)$$

and $\partial_k u \in C^{\alpha+\beta-1}(H), k = 1, \dots, d$. Therefore, $u \in C^{\alpha+\beta}(H)$.

It thus proves that for $\alpha \in (0,2]$ and $\beta \in (n, n+1)$, given $f \in C^\beta(H)$, there exists a unique solution $u \in C^{\alpha+\beta}(H)$ to the Cauchy problem (3.4) and $|u|_{\alpha+\beta} \le K|f|_\beta$, where K is a constant.

The statement then follows by induction. $\square$

Corollary 3.9. *Let* $\alpha \in (0,2]$ *and* $\beta \in \mathbb{R}^+$, $\beta \notin \mathbb{N}$. *Assume* (A1) *and* (A2) *hold. Then for* $f \in C^\beta(H)$ *and* $g \in C^{\alpha+\beta}(\mathbb{R}^d)$, *there exist a unique solution* $v \in C^{\alpha+\beta}(H)$ *to the Cauchy problem*

$$\begin{aligned} \big(\partial_t + \mathcal{A}_x^{(\alpha)} + \mathcal{B}_x^{(\alpha)}\big) v(t,x) &= f(t,x), \\ v(T,x) &= g(x) \end{aligned} \tag{3.7}$$

and a constant K *independent of* f *and* g *such that* $|v|_{\alpha+\beta} \leq K\big(|f|_\beta + |g|_{\alpha+\beta}\big)$.

Proof. For $g \in C^{\alpha+\beta}(\mathbb{R}^d)$, by Lemma 3.4 and Lemma 3.6, $|\mathcal{A}_x^{(\alpha)} g|_\beta \leq K|g|_{\alpha+\beta}$ and $|\mathcal{B}_x^{(\alpha)} g|_\beta \leq K|g|_{\alpha+\beta}$, with a constant K independent of f and g. It then follows from (3.4) that there exist a unique solution $\tilde{v} \in C^{\alpha+\beta}(H)$ to the Cauchy problem

$$\begin{aligned} \big(\partial_t + \mathcal{A}_x^{(\alpha)} + \mathcal{B}_x^{(\alpha)}\big) \tilde{v}(t,x) &= f(t,x) - \mathcal{A}_x^{(\alpha)} g(x) - \mathcal{B}_x^{(\alpha)} g(x), \\ \tilde{v}(T,x) &= 0 \end{aligned} \tag{3.8}$$

and a constant K independent of f and g such that $|\tilde{v}|_{\alpha+\beta} \leq K\big(|g|_{\alpha+\beta} + |f|_\beta\big)$.

Let $v(t,x) = \tilde{v}(t,x) + g(x)$, where $\tilde{v}$ is the solution to problem (3.8). Then v is the unique solution to the Cauchy problem (3.7) and $|v|_{\alpha+\beta} \leq K\big(|f|_\beta + |g|_{\alpha+\beta}\big)$, for a constant K independent of f and g. $\square$

Chapter 4

Rate of Convergence of Weak Euler Approximation

The rate of convergence of the weak Euler approximation to the stochastic process under consideration is identified and proved in this chapter.

4.1 Weak Euler Approximation

4.1.1 Time Discretization

The construction of the Euler approximation is based on a time discretization. Let the time discretization $\{\tau\}_\delta$ of the interval $[0,T]$ with maximum step size $\delta \in (0,1)$ be a partition by $\tilde{\mathbf{F}}$-stopping times $\{\tau_i\}_{i\in\mathbb{N}}$ such that

$$0 = \tau_0 < \tau_1 < \cdots < \tau_{i_T} = T,$$

and

$$\max_{i\in\{1,\dots,i_T\}} (\tau_i - \tau_{i-1}) \leq \delta.$$

4.1.2 Euler Scheme

Let Y_0 be an $\tilde{\mathcal{F}}_0$-measurable d-dimensional random vector satisfying

$$\mathrm{P}(Y_0 \in A) = \mathrm{P}(X_0 \in A), \forall A \in \mathcal{B}(\mathbb{R}^d). \tag{4.1}$$

For a fixed $\alpha \in (0, 2]$ and a given time discretization $\{\tau\}_\delta$, the $\tilde{\mathbf{F}}$-adapted Euler approximation $Y = \{Y_t\}_{t\in[0,T]}$ of X is defined by the stochastic equation

$$\begin{aligned} Y_t &= Y_0 + \int_0^t a^{(\alpha)}(Y_{\tau_{i_s}})ds + \int_0^t b^{(\alpha)}(Y_{\tau_{i_s}})d\tilde{W}_s \\ &\quad + \int_0^t \int_{|y|>1} yp^Y(ds, dy) + \int_0^t \int_{|y|\le 1} yq^Y(ds, dy), t \in [0, T], \end{aligned}$$

where $\tau_{i_s} = \tau_i$ if $s \in [\tau_i, \tau_{i+1})$, $\tilde{W} = \{\tilde{W}_t\}_{t\in[0,T]}$ is a d-dimensional $\tilde{\mathbf{F}}$-adapted standard Wiener process, p^Y is an $\tilde{\mathbf{F}}$-adapted point random measure with

$$p^Y\big([0, t] \times A\big) = \sum_{s\in[0,t]} \mathbf{1}_A(\Delta Y_s),$$

and

$$q^Y(dt, dy) = p^Y(dt, dy) - m^{(\alpha)}(Y_{\tau_{i_t}}, y)\frac{dy}{|y|^{d+\alpha}}dt - \rho^{(\alpha)}(Y_{\tau_{i_t}}, y)\nu^{(\alpha)}(dy)dt$$

is an $\tilde{\mathbf{F}}$-adapted martingale measure.

4.2 Rate of Convergence

For the proof of Theorem 4.3, Lemma 4.2, which pertains to the conditional expectation of the increments of the Euler approximation and invokes Lemma 4.1, is used.

Let $w \in C_0^\infty(\mathbb{R}^d)$ be a nonnegative smooth function with support in $\{|x| \leq 1\}$ such that $w(x) = w(|x|), x \in \mathbb{R}^d$ and $\int w(x)dx = 1$. Note that $\int x_i w(x)dx = 0, i = 1, \ldots, d$.

For $x \in \mathbb{R}^d$ and $\varepsilon \in (0,1)$, define

$$w^\varepsilon(x) = \varepsilon^{-d} w\big(\frac{x}{\varepsilon}\big)$$

and for $f \in C^\beta(\mathbb{R}^d)$, define the convolution

$$f^\varepsilon(x) = \int f(y)w^\varepsilon(x-y)dy = \int f(x-y)w^\varepsilon(y)dy, x \in \mathbb{R}^d. \tag{4.2}$$

Apparently, f^ε is smooth with respect to x.

Lemma 4.1. *Let $\alpha \in (0,2]$ and $\beta < \alpha$, $\beta \neq 1$. For $f \in C^\beta(\mathbb{R}^d)$ and $\varepsilon \in (0,1)$, let f^ε be as defined in* (4.2)*. Then*

(i) *there exists a constant K such that*

$$|f^\varepsilon(x) - f(x)| \leq K|f|_\beta \varepsilon^\beta, \forall x \in \mathbb{R}^d.$$

(ii) *there exists a constant K such that*

$$|\mathcal{A}_z^{(\alpha)} f^\varepsilon(x)| \leq K|f|_\beta \varepsilon^{-\alpha+\beta}, \forall z, x \in \mathbb{R}^d.$$

In particular,

$$|\partial^\alpha f^\varepsilon(x)| \leq K|f|_\beta \varepsilon^{-\alpha+\beta}, \forall x \in \mathbb{R}^d. \tag{4.3}$$

Also, for $\beta < 1$, $i = 1, \ldots, d$,

$$|\partial_i f^\varepsilon(x)| \le K|f|_\beta \varepsilon^{-1+\beta}, \forall x \in \mathbb{R}^d$$

and for $\beta < 2, \beta \neq 1$, $i, j = 1, \ldots, d$,

$$|\partial_{ij} f^\varepsilon(x)| \le K|f|_\beta \varepsilon^{-2+\beta}, \forall x \in \mathbb{R}^d.$$

(iii) *there exists a constant K such that*

$$|\mathcal{B}_z^{(\alpha)} f^\varepsilon(x)| \le K|f|_\beta \varepsilon^{-\alpha+\beta}, \forall z, x \in \mathbb{R}^d.$$

Proof. Let $f \in C^\beta(\mathbb{R}^d)$ and f^ε be the corresponding convolution with $\varepsilon \in (0,1)$.

(i) For $\beta \in (0,1)$, since $w(x) = w(-x), x \in \mathbb{R}^d$,

$$\begin{aligned} f^\varepsilon(x) - f(x) &= \int [f(x-y) - f(x)] w^\varepsilon(y) dy \\ &= \int [f(x+y) - f(x)] w^\varepsilon(y) dy \\ &= \int \big[K \int k^{(\beta)}(y,z) \partial^\beta f(x-z) dz\big] w^\varepsilon(y) dy, \end{aligned}$$

and by Lemma 3.2,

$$|f^\varepsilon(x) - f(x)| \le K|f|_\beta \varepsilon^\beta.$$

For $\beta \in (1,2)$, since $w(x) = w(-x), x \in \mathbb{R}^d$ and $\int x_i w(x)dx = 0, i = 1, \dots, d$,

$$\begin{aligned}
f^\varepsilon(x) - f(x) &= \int [f(x-y) - f(x)]w^\varepsilon(y)dy \\
&= \int [f(x+y) - f(x) - (\partial_x f(x), y)]w^\varepsilon(y)dy \\
&= \int \int_0^1 \big(\partial_x f(x+sy) - \partial_x f(x), y\big) ds w^\varepsilon(y)dy \\
&= \int \int_0^1 \big(K \int k^{(\beta-1)}(sy,z)\partial^{\beta-1}\partial_x f(x-z)dz, y\big) ds w^\varepsilon(y)dy
\end{aligned}$$

and by Lemma 3.2,

$$|f^\varepsilon(x) - f(x)| \le K|f|_\beta \varepsilon^\beta.$$

(ii) For $z, x \in \mathbb{R}^d$,

$$\begin{aligned}
\mathcal{A}_z^{(\alpha)} w^\varepsilon(x) &= \mathbf{1}_{\{\alpha=1\}}\big(a(z), \partial_x w^\varepsilon(x)\big) + \mathbf{1}_{\{\alpha=2\}} B^{ij}(z)\partial_{ij} w^\varepsilon(x) \\
&\quad + \int A_y^{(\alpha)} w^\varepsilon(x) m^{(\alpha)}(z,y)\frac{dy}{|y|^{d+\alpha}} \\
&= \varepsilon^{-\alpha}\big(\mathcal{A}_z^{(\alpha)} w\big)^\varepsilon(x) \\
&= \varepsilon^{-\alpha}\varepsilon^{-d}(\mathcal{A}_z^{(\alpha)} w)(\frac{x}{\varepsilon}) \qquad (4.4)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{A}_z^{(\alpha)} f^\varepsilon(x) &= \int \varepsilon^{-\alpha}\varepsilon^{-d}\mathcal{A}_z^{(\alpha)} w(\frac{x-y}{\varepsilon})f(y)dy \\
&= \int \varepsilon^{-\alpha}\varepsilon^{-d}(\mathcal{A}_z^{(\alpha)})w(\frac{y}{\varepsilon})f(x-y)dy
\end{aligned}$$

$$= \int \varepsilon^{-\alpha} \mathcal{A}_z^{(\alpha)} w(y) f(x - \varepsilon y) dy.$$

For $\beta < \alpha$ and $\beta \in (0,1)$, since

$$\int \varepsilon^{-\alpha} \mathcal{A}_z^{(\alpha)} w(y) f(x - \varepsilon y) dy = \int \varepsilon^{-\alpha} \mathcal{A}_z^{(\alpha)} w(y) \big[f(x - \varepsilon y) - f(x) \big] dy,$$

then

$$|\mathcal{A}_z^{(\alpha)} f^{\varepsilon}(x)| \leq K |f|_{\beta} \varepsilon^{-\alpha+\beta}.$$

For $\beta < \alpha$ and $\beta \in (1,2)$, since

$$\begin{aligned}
\mathcal{A}_z^{(\alpha)} w(y) &= \int \big[w(y + \bar{y}) - w(y) - \big(\partial_y w(y), \bar{y}\big) \big] m^{(\alpha)}(z, \bar{y}) \frac{d\bar{y}}{|\bar{y}|^{d+\alpha}} \\
&= \int \int_0^1 \big(\partial_y w(y + s\bar{y}) - \partial_y w(y), \bar{y} \big) ds\, m^{(\alpha)}(z, \bar{y}) \frac{d\bar{y}}{|\bar{y}|^{d+\alpha}}
\end{aligned}$$

and

$$\begin{aligned}
& \int \varepsilon^{-\alpha} \mathcal{A}_z^{(\alpha)} w(y) f(x - \varepsilon y) dy \\
= \; & \varepsilon^{-\alpha+1} \int \int_0^1 \big(w(y + s\bar{y}) - w(y) \big) \big(\partial_x f(x - \varepsilon y) - \partial_x f(x), \bar{y} \big) m^{(\alpha)}(z, \bar{y}) \frac{d\bar{y}}{|\bar{y}|^{d+\alpha}} dy,
\end{aligned}$$

then

$$|\mathcal{A}_z^{(\alpha)} f^{\varepsilon}(x)| \leq K \varepsilon^{-\alpha+1} \varepsilon^{\beta-1} |f|_{\beta} \int \int_0^1 |w(y + s\bar{y}) - w(y)| |y|^{\beta-1} \frac{d\bar{y}}{|\bar{y}|^{d+\alpha-1}} dy$$

$$\leq \quad K|f|_\beta \varepsilon^{-\alpha+\beta}.$$

Taking $m^{(\alpha)} = 1$ gives (4.3).

If $\beta \in (0,1)$,

$$\begin{aligned}
\partial_i f^\varepsilon(x) &= \varepsilon^{-1} \int \varepsilon^{-d} \partial_i w(\frac{x-y}{\varepsilon}) f(y) dy \\
&= \varepsilon^{-1} \int \varepsilon^{-d} \partial_i w(\frac{y}{\varepsilon}) f(x-y) dy \\
&= \varepsilon^{-1} \int \partial_i w(y) \big[f(x - \varepsilon y) - f(x) \big] dy
\end{aligned}$$

and

$$\begin{aligned}
\partial_{ij} f^\varepsilon(x) &= \varepsilon^{-2} \int \varepsilon^{-d} \partial_{ij} w(\frac{x-y}{\varepsilon}) f(y) dy \\
&= \varepsilon^{-2} \int \varepsilon^{-d} \partial_{ij} w(\frac{y}{\varepsilon}) f(x-y) dy \\
&= \varepsilon^{-2} \int \partial_{ij} w(y) \big[f(x - \varepsilon y) - f(x) \big] dy.
\end{aligned}$$

Hence,

$$|\partial_i f^\varepsilon(x)| \leq K|f|_\beta \varepsilon^{-1+\beta}$$

and

$$|\partial_{ij} f^\varepsilon(x)| \leq K|f|_\beta \varepsilon^{-2+\beta}.$$

Similarly, if $\beta \in (1,2)$,

$$\begin{aligned}\partial_i f^\varepsilon(x) &= \int \varepsilon^{-d} w(\frac{y}{\varepsilon})\partial_i f(x-y)dy \\ &= \int \varepsilon^{-d} w(\frac{x-y}{\varepsilon})\partial_i f(y)dy\end{aligned}$$

and

$$\begin{aligned}\partial_{ij} f^\varepsilon(x) &= \varepsilon^{-1}\int \varepsilon^{-d}\partial_j w(\frac{y}{\varepsilon})\partial_i f(x-y)dy \\ &= \varepsilon^{-1}\int \partial_j w(y)\big[\partial_i f(x-\varepsilon y)-\partial_i f(x)\big]dy.\end{aligned}$$

Hence,

$$|\partial_{ij} f^\varepsilon(x)| \le K|f|_\beta \varepsilon^{-1}\varepsilon^{\beta-1} = K|f|_\beta \varepsilon^{\beta-2}.$$

(iii) The equality (4.4) with $m^{(\alpha)}=1$ implies that

$$\partial^\alpha w^\varepsilon(x) = \varepsilon^{-\alpha}\big(\partial^\alpha w\big)^\varepsilon = \varepsilon^{-\alpha}\varepsilon^{-d}(\partial^\alpha w)(\frac{x}{\varepsilon}), \forall x \in \mathbb{R}^d.$$

If $\beta < \alpha < 1$,

$$f^\varepsilon(x+y) - f^\varepsilon(x) = \int k^{(\alpha)}(y,\bar{y})\partial^\alpha f^\varepsilon(x-\bar{y})d\bar{y}$$

and by (4.3),

$$|f^\varepsilon(x+y) - f^\varepsilon(x)| \le K(|y|^\alpha \wedge 1)|f|_\beta \varepsilon^{-\alpha+\beta}, \forall x, y \in \mathbb{R}^d.$$

If $\beta < \alpha = 1$,

$$f^\varepsilon(x+y) - f^\varepsilon(x) = \int_0^1 \big(\partial_x f^\varepsilon(x+sy), y\big) ds$$

and by (ii),

$$|f^\varepsilon(x+y) - f^\varepsilon(x)| \leq K(|y| \wedge 1)|f|_\beta \varepsilon^{-1+\beta}, \forall x, y \in \mathbb{R}^d.$$

If $\alpha \in (1,2)$,

$$f^\varepsilon(x+y) - f^\varepsilon(x) - \big(\partial_x f^\varepsilon(x), y\big) = \int_0^1 \big(\partial_x f^\varepsilon(x+sy) - \partial_x f^\varepsilon(x), y\big) ds.$$

By interpolation inequality and (ii), there exists a constant K independent of ε and f such that

$$\begin{aligned} |\partial_x f^\varepsilon(x+\bar{y}) - \partial_x f^\varepsilon(x)| &\leq K|\bar{y}|^{\alpha-1} \sup_{i,j,x} |\partial_{ij} f(x)|_\infty^{\alpha-1} \sup_x |\partial_x f(x)|^{2-\alpha} \\ &\leq K|\bar{y}|^{\alpha-1} |f|_\beta \varepsilon^{(-2+\beta)(\alpha-1)} \varepsilon^{(-1+\beta)(2-\alpha)} \\ &= K|\bar{y}|^{\alpha-1}|f|_\beta \varepsilon^{-\alpha+\beta}, \forall x, \bar{y} \in \mathbb{R}^d. \end{aligned}$$

Hence, for $\alpha \in (1,2)$ and $|y| \leq 1$,

$$|f^\varepsilon(x+y) - f^\varepsilon(x) - (\partial_x f^\varepsilon(x), y)| \leq K|y|^\alpha |f|_\beta \varepsilon^{-\alpha+\beta}, \forall x, y \in \mathbb{R}^d.$$

(iii) then follows.

□

Lemma 4.2. *Let $\alpha \in (0,2]$ and $\beta \in \mathbb{R}^+$, $\beta \notin \mathbb{N}$. Assume* (A1) *and* (A2) *hold. For a given time discretization $\{\tau\}_\delta$ with $\delta \in (0,1)$, let Y be the Euler approximation for the stochastic process X. Then for $f \in C^\beta(\mathbb{R}^d)$, there exists a constant K such that*

$$\left|\mathrm{E}\left[f(Y_{s-}) - f(Y_{\tau_{i_s}})\middle|\tilde{\mathcal{F}}_{\tau_{i_s}}\right]\right| \leq K|f|_\beta \delta^{\kappa(\alpha,\beta)}, \forall s \in [0,T],$$

where $i_s = i$ if $\tau_i \leq s < \tau_{i+1}$ and $\kappa(\alpha,\beta)$ is as defined in Theorem 4.3.

The proof of Lemma 4.2 is based on the application of Ito's formula to $f(Y_s) - f(Y_{\tau_{i_s}})$.

Proof. For $\beta < \alpha$, define f^ε as in (4.2) with $\varepsilon \in (0,1)$. By applying Itô's formula and by Lemma 4.1,

$$\begin{aligned}
\left|\mathrm{E}\left[f^\varepsilon(Y_{s-}) - f^\varepsilon(Y_{\tau_{i_s}})\middle|\tilde{\mathcal{F}}_{\tau_{i_s}}\right]\right| &= \Big|\mathrm{E}\Big[\int_{\tau_{i_s}}^s \Big(\mathbf{1}_{\{\alpha=1\}}\big(a(Y_{\tau_{i_s}}), \partial_x f^\varepsilon(Y_{r-})\big) \\
&\quad + \frac{1}{2}\mathbf{1}_{\{\alpha=2\}} \sum_{i,j=1}^d B^{ij}(Y_{\tau_{i_s}})\partial_{ij} f^\varepsilon(Y_{r-}) \\
&\quad + \int A_y^{(\alpha)} f^\varepsilon(Y_{r-}) m^{(\alpha)}(Y_{\tau_{i_s}}, y)\frac{dy}{|y|^{d+\alpha}} \\
&\quad + \mathbf{1}_{\{\alpha\in(1,2]\}}\big(c(Y_{\tau_{i_s}}), \partial_x f^\varepsilon(Y_{r-})\big) \\
&\quad + \int B_y^{(\alpha)} f^\varepsilon(Y_{r-})\nu^{(\alpha)}(Y_{\tau_{i_s}}, dy)\Big) dr\Big|\tilde{\mathcal{F}}_{\tau_{i_s}}\Big]\Big| \\
&= \Big|\mathrm{E}\Big[\int_{\tau_{i_s}}^s \big(\mathcal{A}^{(\alpha)}_{Y_{\tau_{i_s}}} f^\varepsilon(Y_{r-}) + \mathcal{B}^{(\alpha)}_{Y_{\tau_{i_s}}} f^\varepsilon(Y_{r-})\big) dr\big|\tilde{\mathcal{F}}_{\tau_{i_s}}\Big]\Big| \\
&\leq \mathrm{E}\Big[\int_{\tau_{i_s}}^s \big(\big|\mathcal{A}^{(\alpha)}_{Y_{\tau_{i_s}}} f^\varepsilon(Y_{r-})\big| + \big|\mathcal{B}^{(\alpha)}_{Y_{\tau_{i_s}}} f^\varepsilon(Y_{r-})\big|\big) dr\big|\tilde{\mathcal{F}}_{\tau_{i_s}}\Big] \\
&\leq K|f|_\beta \varepsilon^{-\alpha+\beta}\delta, \qquad (4.5)
\end{aligned}$$

where K does not depend on f, α, β, or ε.

It then follows from Lemma 4.1 and (4.5) that

$$\left|\mathrm{E}\left[f(Y_{s-}) - f(Y_{\tau_{i_s}})\middle|\tilde{\mathcal{F}}_{\tau_{i_s}}\right]\right| \leq \left|\mathrm{E}\left[(f-f^\varepsilon)(Y_{s-}) - (f-f^\varepsilon)(Y_{\tau_{i_s}})\middle|\tilde{\mathcal{F}}_{\tau_{i_s}}\right]\right|$$

$$
\begin{aligned}
&+\big|\mathrm{E}\big[f^{\varepsilon}(Y_{s-})-f^{\varepsilon}(Y_{\tau_{i_s}})\big|\tilde{\mathcal{F}}_{\tau_{i_s}}\big]\big| \\
\leq\ & 2\sup_x|f^{\varepsilon}(x)-f(x)|+\big|\mathrm{E}\big[f^{\varepsilon}(Y_{s-})-f^{\varepsilon}(Y_{\tau_{i_s}})\big|\tilde{\mathcal{F}}_{\tau_{i_s}}\big]\big| \\
\leq\ & K|f|_{\beta}\inf_{\varepsilon\in(0,1)}(\varepsilon^{\beta}+\varepsilon^{-\alpha+\beta}\delta).
\end{aligned}
$$

Minimizing $\varepsilon^{\beta}+\varepsilon^{-\alpha+\beta}\delta$ in $\varepsilon\in(0,1)$ yields

$$
\big|\mathrm{E}\big[f(Y_{s-})-f(Y_{\tau_{i_s}})\big|\tilde{\mathcal{F}}_{\tau_{i_s}}\big]\big|\leq K|f|_{\beta}\delta^{\kappa(\alpha,\beta)}.
$$

For $\beta>\alpha$,

$$
\begin{aligned}
\big|\mathrm{E}\big[f(Y_{s-})-f(Y_{\tau_{i_s}})\big|\tilde{\mathcal{F}}_{\tau_{i_s}}\big]\big| &= \Big|\mathrm{E}\Big[\int_{\tau_{i_s}}^{s}\Big(\mathbf{1}_{\{\alpha=1\}}\big(a(Y_{\tau_{i_s}}),\partial_x f(Y_{r-})\big) \\
&\quad+\frac{1}{2}\mathbf{1}_{\{\alpha=2\}}\sum_{i,j=1}^{d}B^{ij}(Y_{\tau_{i_s}})\partial_{ij}f(Y_{r-}) \\
&\quad+\int A_y^{(\alpha)}f(Y_{r-})m^{(\alpha)}(Y_{\tau_{i_s}},y)\frac{dy}{|y|^{d+\alpha}} \\
&\quad+\mathbf{1}_{\{\alpha\in(1,2]\}}\big(c(Y_{\tau_{i_s}}),\partial_x f(Y_{r-})\big) \\
&\quad+\int B_y^{(\alpha)}f(Y_{r-})\rho^{(\alpha)}(Y_{\tau_{i_s}},dy)\nu^{(\alpha)}(dy)\Big)dr\Big|\tilde{\mathcal{F}}_{\tau_{i_s}}\Big]\Big| \\
&= \big|\mathrm{E}\big[\int_{\tau_{i_s}}^{s}\big(\mathcal{A}^{(\alpha)}_{Y_{\tau_{i_s}}}f(Y_{r-})+\mathcal{B}^{(\alpha)}_{Y_{\tau_{i_s}}}f(Y_{r-})\big)dr\big|\tilde{\mathcal{F}}_{\tau_{i_s}}\big]\big|
\end{aligned}
$$

and so

$$
\big|\mathrm{E}\big[f(Y_{s-})-f(Y_{\tau_{i_s}})\big|\tilde{\mathcal{F}}_{\tau_{i_s}}\big]\big|\leq K|f|_{\beta}\delta.
$$

The assertion of Lemma 4.2 is then obtained. □

Theorem 4.3. *For $\alpha\in(0,2]$, $\beta\in\mathbb{R}^{+}$, $\beta\notin\mathbb{N}$, and a given time discretization $\{\tau\}_{\delta}$ with $\delta\in(0,1)$, let Y be the Euler approximation for the stochastic process X. Assume* (A1) *and* (A2) *hold. Then there exists a constant* K *such that*

$$|\mathrm{E}[g(Y_T)] - \mathrm{E}[g(X_T)]| \le K\delta^{\kappa(\alpha,\beta)}, \forall g \in C^{\alpha+\beta}(\mathbb{R}^d), \tag{4.6}$$

where

$$\kappa(\alpha,\beta) = \begin{cases} \dfrac{\beta}{\alpha}, & \beta < \alpha, \\ 1, & \beta > \alpha. \end{cases}$$

Proof. Let $v \in C^{\alpha+\beta}(H)$ be the unique solution to (3.7) with $f = 0$. By Itô's formula and (3.7),

$$\mathrm{E}[v(0, X_0)] = \mathrm{E}[v(T, X_T)] = \mathrm{E}[g(X_T)]. \tag{4.7}$$

By Lemma 3.4 and Lemma 3.6,

$$|\mathcal{A}_z^{(\alpha)} v(s,\cdot)|_\beta + |\mathcal{B}_z^{(\alpha)} v(s,\cdot)|_\beta \le K|v|_{\alpha+\beta} \le K|g|_{\alpha+\beta} \tag{4.8}$$

and by Corollary 3.9,

$$|\partial_t v(s,\cdot)|_\beta \le K|g|_{\alpha+\beta}. \tag{4.9}$$

Hence, $\mathcal{A}_z^{(\alpha)} v(s,\cdot), \mathcal{B}_z^{(\alpha)} v(s,\cdot), \partial_t v(s,\cdot) \in C^\beta(\mathbb{R}^d)$, $\forall z \in \mathbb{R}^d$, $\forall s \in [0,T]$.

Then, by Itô's formula and Corollary 3.9, with (4.7) and (4.1), it follows that

$$\begin{aligned} \mathrm{E}[g(Y_T)] - \mathrm{E}[g(X_T)] &= \mathrm{E}[v(T, Y_T)] - \mathrm{E}[v(0, X_0)] \\ &= \mathrm{E}[v(T, Y_T)] - \mathrm{E}[v(0, Y_0)] \end{aligned}$$

$$
\begin{aligned}
&= \mathrm{E}\Big[\int_0^T \Big(\partial_t v(s,Y_{s-}) + \mathbf{1}_{\{\alpha=1\}}\big(a(Y_{\tau_{i_s}}), \partial_x v(s,Y_{s-})\big) \\
&\quad +\frac{1}{2}\mathbf{1}_{\{\alpha=2\}}\sum_{i,j=1}^{d} B^{ij}(Y_{\tau_{i_s}})\partial_{ij} v(s,Y_{s-}) \\
&\quad +\int A_y^{(\alpha)} v(s,Y_{s-}) m^{(\alpha)}(Y_{\tau_{i_s}},y)\frac{dy}{|y|^{d+\alpha}} \\
&\quad +\mathbf{1}_{\{\alpha\in(1,2]\}}\big(c(Y_{\tau_{i_s}}), \partial_x v(s,Y_{s-})\big) \\
&\quad +\int B_y^{(\alpha)} v(s,Y_{s-})\rho^{(\alpha)}(Y_{\tau_{i_s}},y)\nu^{(\alpha)}(dy)\Big)ds \\
&\quad -\int_0^T (\partial_t + \mathcal{A}^{(\alpha)}_{Y_{\tau_{i_s}}} + \mathcal{B}^{(\alpha)}_{Y_{\tau_{i_s}}}) v(s,Y_{\tau_{i_s}})ds\Big] \\
&= \mathrm{E}\Big[\int_0^T \Big(\big(\partial_t v(s,Y_{s-}) - \partial_t v(s,Y_{\tau_{i_s}})\big) \\
&\quad +\big(\mathcal{A}^{(\alpha)}_{Y_{\tau_{i_s}}} v(s,Y_{s-}) - \mathcal{A}^{(\alpha)}_{Y_{\tau_{i_s}}} v(s,Y_{\tau_{i_s}})\big) \\
&\quad +\big(\mathcal{B}^{(\alpha)}_{Y_{\tau_{i_s}}} v(s,Y_{s-}) - \mathcal{B}^{(\alpha)}_{Y_{\tau_{i_s}}} v(s,Y_{\tau_{i_s}})\big)\Big)ds\Big] \\
&= \mathrm{E}\Big[\int_0^T \Big(\mathrm{E}\big[\partial_t v(s,Y_{s-}) - \partial_t v(s,Y_{\tau_{i_s}})|\tilde{\mathcal{F}}_{\tau_{i_s}}\big] \\
&\quad +\mathrm{E}\big[\mathcal{A}^{(\alpha)}_{Y_{\tau_{i_s}}} v(s,Y_{s-}) - \mathcal{A}^{(\alpha)}_{Y_{\tau_{i_s}}} v(s,Y_{\tau_{i_s}})|\tilde{\mathcal{F}}_{\tau_{i_s}}\big] \\
&\quad +\mathrm{E}\big[\mathcal{B}^{(\alpha)}_{Y_{\tau_{i_s}}} v(s,Y_{s-}) - \mathcal{B}^{(\alpha)}_{Y_{\tau_{i_s}}} v(s,Y_{\tau_{i_s}})|\tilde{\mathcal{F}}_{\tau_{i_s}}\big]\Big)ds\Big].
\end{aligned}
$$

Hence, by (4.8), (4.9), and Lemma 4.2, there exists a constant K independent of g such that

$$
\big|\mathrm{E}[g(Y_T)] - \mathrm{E}[g(X_T)]\big| \le K|g|_{\alpha+\beta}\delta^{\kappa(\alpha,\beta)}.
$$

This proves (4.6). □

Remark 4.4. *It can be noted from Theorem 4.3 that if a, b, and c are Hölder-continuous, and g is more than $\alpha+\beta$ continuously differentiable, the Euler scheme has already yielded a positive weak order of convergence. To obtain the first weak order convergence, only* *slightly more than α continuous differentiability of a, b, and c is needed, instead of fourth-order derivatives as in papers cited previously.*

Example 4.5. *For a given time discretization $\{\tau\}_\delta$ with $\delta \in (0,1)$, let Y be the Euler approximation of the stochastic process X defined by*

$$X_t = X_0 + \int_0^t c(X_{s-})dZ_s, \forall t \in [0,T], \tag{4.10}$$

where c is a $d \times d$-dimensional matrix and $Z = \{Z_t\}_{t\in[0,T]}$ is a symmetric α-stable d-dimensional process defined as

$$Z_t = \begin{cases} \int_0^t \int y p^Z(ds,dy), & \alpha \in (0,1), \\ \int_0^t \int_{|y|\le 1} y q^Z(ds,dy) + \int_0^t \int_{|y|>1} y p^Z(ds,dy), & \alpha = 1, \\ \int_0^t \int y q^Z(ds,dy), & \alpha \in (1,2), \end{cases}$$

with $p^Z(dt,dy)$ being the jump measure of Z and

$$q^Z(dt,dy) = p^Z(dt,dy) - \frac{dy}{|y|^{d+\alpha}}dt$$

being the corresponding martingale measure. Assume $|\det c(\cdot)| \ge \varepsilon > 0$, $c^{ij} \in C^\beta(\mathbb{R}^d)$, $g \in C^{\alpha+\beta}(\mathbb{R}^d)$, and $\beta \in \mathbb{R}^+, \beta \notin \mathbb{N}$. Then it holds that

$$|\mathrm{E}[g(Y_T)] - \mathrm{E}[g(X_T)]| \le K\delta^{\kappa(\alpha,\beta)}.$$

Indeed, in this case, $\mathcal{B}^{(\alpha)} = 0$ and

$$m^{(\alpha)}(x,y) = \frac{1}{|\det c(x)|}\Big(\frac{|y|}{|c(x)^{-1}y|}\Big)^{d+\alpha}, x \in \mathbb{R}^d, y \in \mathbb{R}^d_0.$$

Remark 4.6. *For stochastic differential equations of the form* (4.10)*, Protter and Talay showed that* $\kappa(\alpha,\beta) = 1$*, provided that* c *is smooth and the Lévy measure of* Z *has finite moments of sufficiently high order* [75]*.*

Chapter 5

Approximation of a General Equation

In this chapter, the result obtained on the stochastic differential equation (2.6) is generalized to a more general equation.

5.1 A General Stochastic Differential Equation Driven by Lévy Motion

Let $(U,\mathcal{U})$ be a measurable space with a nonnegative measure $\nu^{(\alpha)}(dv)$ on it. Assume that there is a decreasing sequence of subsets $\{U_n\}$ with $U_n \in \mathcal{U}$ such that $U = \bigcup_n U_n^c$.

For $\alpha \in (0,2]$, let the d-dimensional Itô process $X = \{X_t\}_{t\in[0,T]}$ be the weak solution to the stochastic equation

$$\begin{aligned} X_t &= X_0 + \int_0^t a^{(\alpha)}(X_{s-})ds + \int_0^t b^{(\alpha)}(X_{s-})dW_s \\ &\quad + \int_0^t \int_{|y|>1} yp^X(ds,dy) + \int_0^t \int_{|y|\le 1} yq^X(ds,dy) \\ &\quad + \int_0^t \int_{U_1^c} l(X_{s-},v)\tilde{p}^X(ds,dv) + \int_0^t \int_{U_1} l(X_{s-},v)\tilde{q}^X(ds,dv), t\in[0,T], \quad (5.1) \end{aligned}$$

where

$$a^{(\alpha)}(x) = \mathbf{1}_{\{\alpha\in(0,1)\}}\Big(\int_{|y|\le 1} ym^{(\alpha)}(x,y)\frac{dy}{|y|^{d+\alpha}} + \int_{U_1} l(x,v)\rho^{(\alpha)}(x,v)\nu^{(\alpha)}(dv)\Big)$$

$$
\begin{aligned}
& +\mathbf{1}_{\{\alpha=1\}}\big(a(x)+\int_{U_1} l(x,v)\rho^{(\alpha)}(x,v)\nu^{(\alpha)}(dv)\big) \\
& +\mathbf{1}_{\{\alpha\in(1,2]\}}\big(c(x)-\int_{|y|>1} ym^{(\alpha)}(x,y)\frac{dy}{|y|^{d+\alpha}}\big), \\
b^{(\alpha)}(x) &= \mathbf{1}_{\{\alpha=2\}}b(x),
\end{aligned}
$$

X_0 is the $\mathbf{F}_0$-measurable initial value, $W=\{W_t\}_{t\in[0,T]}$ is an $\mathbf{F}$-adapted d-dimensional standard Wiener process, $p^X(dt,dy)$ is a point measure on $[0,T]\times\mathbb{R}_0^d$, $q^X(dt,dy)$ is an $(\mathbf{F},\mathbb{P})$-martingale measure with

$$
q^X(dt,dy)=p^X(dt,dy)-m^{(\alpha)}(X_{t-},y)\frac{dy}{|y|^{d+\alpha}}dt,
$$

$\tilde{p}^X(dt,dv)$ is a point measure on $[0,T]\times U$, and

$$
\tilde{q}^X(dt,dv)=\tilde{p}^X(dt,dv)-\rho^{(\alpha)}(X_{t-},v)\nu^{(\alpha)}(dv)dt.
$$

In the stochastic equation (5.1), a, c, b, l, $m^{(\alpha)}$, and $\rho^{(\alpha)}$ are measurable functions, with a, c, and l d-dimensional vectors, b a $d\times d$-dimensional symmetric nonnegative definite matrix, $m^{(\alpha)}$ and $\rho^{(\alpha)}$ nonnegative functions. In addition, $m^{(\alpha)}(x,y)$ and its partial derivatives $\partial_y^{\gamma}m^{(\alpha)}(x,y),\gamma\in\left\{\gamma:|\gamma|\le d_0=\left[\frac{d}{2}\right]+1\right\}$ are continuous in (x,y), $m^{(\alpha)}(x,y)$ is homogeneous in y with index zero, and

$$
\int_{S^{d-1}} ym^{(1)}(\cdot,y)\mu_{d-1}(dy)=0,\quad m^{(2)}\equiv 0,
$$

where S^{d-1} is the unit sphere in $\mathbb{R}^d$ and μ_{d-1} is the Lebesgue measure on it.

Let $B(x) = b(x)^T b(x), x \in \mathbb{R}^d$. Denote

$$\begin{aligned}
\mathcal{A}_z^{(\alpha)} u(t,x) &= \mathbf{1}_{\{\alpha=1\}} \big(a(z), \partial_x u(t,x)\big) + \frac{1}{2} \mathbf{1}_{\{\alpha=2\}} \sum_{i,j=1}^{d} B^{ij}(z) \partial_{ij} u(t,x) \\
&\quad + \int_{\mathbb{R}_0^d} \big[u(t,x+y) - u(t,x) - \big(\mathbf{1}_{\{|y|\le 1\}} \mathbf{1}_{\{\alpha=1\}} + \mathbf{1}_{\{\alpha\in(1,2)\}}\big) \big(\partial_x u(t,x), y\big)\big] \\
&\quad\ m^{(\alpha)}(z,y) \frac{dy}{|y|^{d+\alpha}}, \\
\mathcal{A}^{(\alpha)} u(t,x) &= \mathcal{A}_x^{(\alpha)} u(t,x) = \mathcal{A}_z^{(\alpha)} u(t,x)|_{z=x},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{B}_z^{(\alpha)} u(t,x) &= \mathbf{1}_{\{\alpha\in(1,2]\}} \big(c(z), \partial_x u(t,x)\big) \\
&\quad + \int_U \big[u(t,x+l(z,v)) - u(t,x) - \big(\mathbf{1}_{\{v\in U_1\}} \mathbf{1}_{\{\alpha\in(1,2]\}}\big) \big(\partial_x u(t,x), l(z,v)\big)\big] \\
&\quad\ \rho^{(\alpha)}(z,v) \nu^{(\alpha)}(dv), \\
\mathcal{B}^{(\alpha)} u(t,x) &= \mathcal{B}_x^{(\alpha)} u(t,x) = \mathcal{B}_z^{(\alpha)} u(t,x)|_{z=x}.
\end{aligned}$$

The operator $\mathcal{L}^{(\alpha)} = \mathcal{A}^{(\alpha)} + \mathcal{B}^{(\alpha)}$ is the generator of X_t defined in (5.1). $\mathcal{A}^{(\alpha)}$ is the principal part and $\mathcal{B}^{(\alpha)}$ is the lower order or subordinated part of $\mathcal{L}^{(\alpha)}$.

It is assumed that the following conditions are satisfied by the stochastic differential equation defined in (5.1):

($\tilde{\text{A}}$1) There exists a constant $\mu > 0$ such that for all $x \in \mathbb{R}^d$ and $|\xi| = 1$,

$$\begin{aligned}
&(B(x)\xi, \xi) \ge \mu, \text{ where } B(x) = b(x)^T b(x), \\
&\int_{S^{d-1}} |(w,\xi)|^\alpha m^{(\alpha)}(x,w) d\xi \ge \mu, \alpha \in (0,2), \\
&\lim_{\delta\downarrow 0} \sup_x \int_{U_1^c} \mathbf{1}_{|l(x,v)|\le\delta} (|l(x,v)|^\alpha \wedge 1) |\rho^{(\alpha)}(x,v)| \nu^{(\alpha)}(dv) = 0, \alpha \in (0,1];
\end{aligned}$$

($\tilde{\text{A}}$2) For $\beta \in \mathbb{R}^+$,

$$M_\beta^{(\alpha)} + N_\beta^{(\alpha)} < \infty,$$

where

$$M_\beta^{(\alpha)} = \mathbf{1}_{\{\alpha=1\}}|a|_\beta + \mathbf{1}_{\{\alpha=2\}}|B|_\beta + \mathbf{1}_{\{\alpha\in(0,2)\}} \sup_{\substack{|\gamma|\le d_0,\\ |y|=1}} |\partial_y^\gamma m^{(\alpha)}(\cdot,y)|_\beta$$

and

$$N_\beta^{(\alpha)} = N_\beta^{(\alpha)}(I^1) + \mathbf{1}_{\{\beta>1\}} N_\beta^{(\alpha)}(I^2),$$

with

$$\begin{aligned}
& N_\beta^{(\alpha)}(I^1) \\
= \; & \mathbf{1}_{\{\alpha\in(1,2]\}} \sup_{|\gamma|=[\beta],x} |\partial_x^\gamma c(x)|_{\beta-|\gamma|} \\
& + \sup_{|\gamma|=[\beta],x,\tilde{x}} \int_{U_1} |l(x,v)|^{((\alpha+\beta)\wedge 1)\vee\alpha} |\partial_x^\gamma \rho^{(\alpha)}(\tilde{x},v)| \nu^{(\alpha)}(dv) \\
& + \sup_{|\gamma|=[\beta],x,\tilde{x}} \int_{U_1^c} (|l(x,v)|^{(\alpha+\beta)\wedge 1} \wedge 1) |\partial_x^\gamma \rho^{(\alpha)}(\tilde{x},v)| \nu^{(\alpha)}(dv) \\
& + \sup_{\substack{|\gamma|=[\beta],\\ x,\tilde{x},h\ne 0}} \frac{1}{|h|^{\beta-[\beta]}} \int_{U_1} |l(x+h,v)-l(x,v)|^{((\alpha+\beta)\wedge 1)\vee\alpha} |\partial_x^\gamma \rho^{(\alpha)}(\tilde{x}+h,v)| \nu^{(\alpha)}(dv) \\
& + \sup_{\substack{|\gamma|=[\beta],\\ x,\tilde{x},h\ne 0}} \frac{1}{|h|^{\beta-[\beta]}} \int_{U_1^c} (|l(x+h,v)-l(x,v)|^{(\alpha+\beta)\wedge 1} \wedge 1) |\partial_x^\gamma \rho^{(\alpha)}(\tilde{x}+h,v)| \nu^{(\alpha)}(dv) \\
& + \sup_{\substack{|\gamma|=[\beta],\\ x,\tilde{x},h\ne 0}} \frac{1}{|h|^{\beta-[\beta]}} \int_{U_1} |l(x,v)|^{((\alpha+\beta)\wedge 1)\vee\alpha} |\partial_x^\gamma \rho^{(\alpha)}(\tilde{x}+h,v) - \partial_x^\gamma \rho^{(\alpha)}(\tilde{x},v)| \nu^{(\alpha)}(dv)
\end{aligned}$$

$$+\sup_{\substack{|\gamma|=[\beta],\\ x,\tilde{x},h\neq 0}} \frac{1}{|h|^{\beta-[\beta]}} \int_{U_1^c} (|l(x,v)|^{(\alpha+\beta)\wedge 1}\wedge 1)|\partial_x^\gamma \rho^{(\alpha)}(\tilde{x}+h,v)-\partial_x^\gamma \rho^{(\alpha)}(\tilde{x},v)|\nu^{(\alpha)}(dv)$$

and

$$\begin{aligned}
& N_\beta^{(\alpha)}(I^2) \\
= & \sum_{|i|=0}^{[\beta]-1}\sum_{|j|=1}^{[\beta]-|i|} \sup_{x,\tilde{x}} \int |\partial_x^j l(x,v)|^{\frac{[\beta]-|i|}{|j|}} |\partial_x^i \rho^{(\alpha)}(\tilde{x},v)|\nu^{(\alpha)}(dv) \\
& + \sum_{|i|=0}^{[\beta]-1}\sum_{|j|=1}^{[\beta]-|i|} \sup_{x,\tilde{x},h\neq 0} \frac{1}{|h|^{\beta-[\beta]}} \int |\partial_x^j l(x,v)|^{\frac{[\beta]-|i|}{|j|}} |\partial_x^i \rho^{(\alpha)}(\tilde{x}+h,v)-\partial_x^i \rho^{(\alpha)}(\tilde{x},v)| \\
& \quad \nu^{(\alpha)}(dv) \\
& + \sum_{|i|=0}^{[\beta]-1}\sum_{|j|=1}^{[\beta]-|i|} \sup_{x,\tilde{x},h\neq 0} \frac{1}{|h|^{\beta-[\beta]}} \int |(\partial_x^j l(x+h,v))-(\partial_x^j l(x,v))|^{\frac{[\beta]-|i|}{|j|}} |\partial_x^i \rho^{(\alpha)}(\tilde{x},v)| \\
& \quad \nu^{(\alpha)}(dv) \\
& + \sum_{|i|=0}^{[\beta]-1}\sum_{|j|=1}^{[\beta]-|i|} \sup_{x,\tilde{x},h\neq 0} \frac{1}{|h|^{\beta-[\beta]}} \int (|l(x+h,v)-l(x,v)|^{(\alpha+\beta-1)\wedge 1}\wedge 1)|\partial_x^j l(x,v)|^{\frac{[\beta]-|i|}{|j|}} \\
& \quad |\partial_x^i \rho^{(\alpha)}(\tilde{x},v)|\nu^{(\alpha)}(dv);
\end{aligned}$$

Remark 5.1. *Under the above assumptions, for any $\beta\in\mathbb{R}^+$, there exists a unique weak solution to equation* (5.1) [63].

Remark 5.2. *In* (5.1)*, if $U=\mathbb{R}^d$, $U_n=\{v:|v|\leq \frac{1}{n}\}$, and $l(x,v)=v$, then the stochastic differential equation is of the form* (2.6)*.*

5.2 Convergence of Weak Euler Approximation

5.2.1 Weak Euler Approximation

Let Y_0 be an $\tilde{\mathcal{F}}_0$-measurable d-dimensional random vector satisfying

$$\mathrm{P}(Y_0 \in A) = \mathrm{P}(X_0 \in A), \forall A \in \mathcal{B}(\mathbb{R}^d). \tag{5.2}$$

For a fixed $\alpha \in (0,2]$ and a given time discretization $\{\tau\}_\delta$, the $\tilde{\mathbf{F}}$-adapted Euler approximation $Y = \{Y_t\}_{t\in[0,T]}$ of X described in (5.1) is defined by the stochastic equation

$$\begin{aligned} Y_t \;=\; & Y_0 + \int_0^t a^{(\alpha)}(Y_{\tau_{i_s}})ds + \int_0^t b^{(\alpha)}(Y_{\tau_{i_s}})d\tilde{W}_s \\ & + \int_0^t \int_{|y|>1} yp^Y(ds,dy) + \int_0^t \int_{|y|\le 1} yq^Y(ds,dy) \\ & + \int_0^t \int_{U_1^c} l(Y_{\tau_{i_s}},v)\tilde{p}^Y(ds,dv) + \int_0^t \int_{U_1} l(Y_{\tau_{i_s}},v)\tilde{q}^Y(ds,dv), t \in [0,T], \end{aligned}$$

where $\tau_{i_s} = \tau_i$ if $s \in [\tau_i, \tau_{i+1})$, $\tilde{W} = \{\tilde{W}_t\}_{t\in[0,T]}$ is a d-dimensional $\tilde{\mathbf{F}}$-adapted standard Wiener process, $p^Y(dt,dy)$ is an $\tilde{\mathbf{F}}$-adapted point measure on $[0,T]\times\mathbb{R}_0^d$, $q^X(dt,dy)$ is an $(\tilde{\mathbf{F}},\mathbb{P})$-martingale measure with

$$q^Y(dt,dy) = p^Y(dt,dy) - m^{(\alpha)}(Y_{\tau_{i_t}},y)\frac{dy}{|y|^{d+\alpha}}dt,$$

$\tilde{p}^Y(dt,dv)$ is a point measure on $[0,T]\times U$, and

$$\tilde{q}^Y(dt,dv) = \tilde{p}^Y(dt,dv) - \rho^{(\alpha)}(Y_{\tau_{i_t}},v)\nu^{(\alpha)}(dv)dt.$$

5.2.2 Rate of Convergence

The rate of convergence is given in Theorem 5.7. For the proof of the theorem, Lemma 5.4 is invoked. For functions possessing the same properties as a, c, b, $m^{(\alpha)}$, and $\rho^{(\alpha)}$, the corresponding norms and operators are defined as follows.

Let $\bar{a}$, $\bar{c}$, $\bar{b}$, $\bar{m}^{(\alpha)}$, and $\bar{\rho}^{(\alpha)}$ be measurable functions, with $\bar{a}$ and $\bar{c}$ d-dimensional vectors, $\bar{b}$ a $d \times d$-dimensional symmetric nonnegative definite matrix, and $\bar{m}^{(\alpha)}$ and $\bar{\rho}^{(\alpha)}$ nonnegative functions. In addition, assume that $\bar{m}^{(\alpha)}(x,y)$ and its partial derivatives $\partial_y^\gamma \bar{m}^{(\alpha)}(x,y), \gamma \in \left\{\gamma : |\gamma| \le d_0 = \left[\frac{d}{2}\right] + 1\right\}$ are continuous in (x,y), $\bar{m}^{(\alpha)}(x,y)$ is homogeneous in y with index zero, and

$$\int_{S^{d-1}} y\bar{m}^{(1)}(\cdot,y)\mu_{d-1}(dy) = 0, \quad \bar{m}^{(2)} \equiv 0,$$

where S^{d-1} is the unit sphere in $\mathbb{R}^d$ and μ_{d-1} is the Lebesgue measure on it.

For $\beta \in \mathbb{R}^+$ and $\bar{B}(x) = \bar{b}(x)^T\bar{b}(x), x \in \mathbb{R}^d$, denote

$$\begin{aligned}\bar{N}^{(\alpha)} &= \mathbf{1}_{\{\alpha\in(1,2]\}} \sup_{|\gamma|=[\beta],x} |\partial_x^\gamma \bar{c}(x)|_\infty \\ &+ \sup_{|\gamma|=[\beta],x,\tilde{x}} \int_{U_1} |l(x,v)|^{((\alpha+\beta)\wedge 1)\vee\alpha} |\partial_x^\gamma \bar{\rho}^{(\alpha)}(\tilde{x},v)|\nu^{(\alpha)}(dv) \\ &+ \sup_{|\gamma|=[\beta],x,\tilde{x}} \int_{U_1^c} (|l(x,v)|^{(\alpha+\beta)\wedge 1} \wedge 1)|\partial_x^\gamma \bar{\rho}^{(\alpha)}(\tilde{x},v)|\nu^{(\alpha)}(dv)\end{aligned}$$

and

$$\bar{N}_\beta^{(\alpha)} = \bar{N}_\beta^{(\alpha)}(I^1) + \mathbf{1}_{\{\beta>1\}}\bar{N}_\beta^{(\alpha)}(I^2),$$

where

$$
\begin{aligned}
& \bar{N}^{(\alpha)}_\beta(I^1) \\
= \; & \mathbf{1}_{\{\alpha\in(1,2]\}} \sup_{|\gamma|=[\beta],x} |\partial_x^\gamma \bar{c}(x)|_{\beta-|\gamma|} \\
& + \sup_{|\gamma|=[\beta],x,\tilde{x}} \int_{U_1} |l(x,v)|^{((\alpha+\beta)\wedge 1)\vee\alpha} |\partial_x^\gamma \bar{\rho}^{(\alpha)}(\tilde{x},v)|\nu^{(\alpha)}(dv) \\
& + \sup_{|\gamma|=[\beta],x,\tilde{x}} \int_{U_1^c} (|l(x,v)|^{(\alpha+\beta)\wedge 1} \wedge 1)|\partial_x^\gamma \bar{\rho}^{(\alpha)}(\tilde{x},v)|\nu^{(\alpha)}(dv) \\
& + \sup_{\substack{|\gamma|=[\beta],\\ x,\tilde{x},h\neq 0}} \frac{1}{|h|^{\beta-[\beta]}} \int_{U_1} |l(x+h,v)-l(x,v)|^{((\alpha+\beta)\wedge 1)\vee\alpha} |\partial_x^\gamma \bar{\rho}^{(\alpha)}(\tilde{x}+h,v)|\nu^{(\alpha)}(dv) \\
& + \sup_{\substack{|\gamma|=[\beta],\\ x,\tilde{x},h\neq 0}} \frac{1}{|h|^{\beta-[\beta]}} \int_{U_1^c} (|l(x+h,v)-l(x,v)|^{(\alpha+\beta)\wedge 1} \wedge 1)|\partial_x^\gamma \bar{\rho}^{(\alpha)}(\tilde{x}+h,v)|\nu^{(\alpha)}(dv) \\
& + \sup_{\substack{|\gamma|=[\beta],\\ x,\tilde{x},h\neq 0}} \frac{1}{|h|^{\beta-[\beta]}} \int_{U_1} |l(x,v)|^{((\alpha+\beta)\wedge 1)\vee\alpha} |\partial_x^\gamma \bar{\rho}^{(\alpha)}(\tilde{x}+h,v)-\partial_x^\gamma \bar{\rho}^{(\alpha)}(\tilde{x},v)|\nu^{(\alpha)}(dv) \\
& + \sup_{\substack{|\gamma|=[\beta],\\ x,\tilde{x},h\neq 0}} \frac{1}{|h|^{\beta-[\beta]}} \int_{U_1^c} (|l(x,v)|^{(\alpha+\beta)\wedge 1} \wedge 1)|\partial_x^\gamma \bar{\rho}^{(\alpha)}(\tilde{x}+h,v)-\partial_x^\gamma \bar{\rho}^{(\alpha)}(\tilde{x},v)|\nu^{(\alpha)}(dv)
\end{aligned}
$$

and

$$
\begin{aligned}
& \bar{N}^{(\alpha)}_\beta(I^2) \\
= \; & \sum_{|i|=0}^{[\beta]-1} \sum_{|j|=1}^{[\beta]-|i|} \sup_{x,\tilde{x}} \int |\partial_x^j l(x,v)|^{\frac{[\beta]-|i|}{|j|}} |\partial_x^i \bar{\rho}^{(\alpha)}(\tilde{x},v)|\nu^{(\alpha)}(dv) \\
& + \sum_{|i|=0}^{[\beta]-1} \sum_{|j|=1}^{[\beta]-|i|} \sup_{x,\tilde{x},h\neq 0} \frac{1}{|h|^{\beta-[\beta]}} \int |\partial_x^j l(x,v)|^{\frac{[\beta]-|i|}{|j|}} |\partial_x^i \bar{\rho}^{(\alpha)}(\tilde{x}+h,v)-\partial_x^i \bar{\rho}^{(\alpha)}(\tilde{x},v)|\nu^{(\alpha)}(dv) \\
& + \sum_{|i|=0}^{[\beta]-1} \sum_{|j|=1}^{[\beta]-|i|} \sup_{x,\tilde{x},h\neq 0} \frac{1}{|h|^{\beta-[\beta]}} \int |(\partial_x^j l(x+h,v))-(\partial_x^j l(x,v))|^{\frac{[\beta]-|i|}{|j|}} |\partial_x^i \bar{\rho}^{(\alpha)}(\tilde{x},v)|\nu^{(\alpha)}(dv) \\
& + \sum_{|i|=0}^{[\beta]-1} \sum_{|j|=1}^{[\beta]-|i|} \sup_{x,\tilde{x},h\neq 0} \frac{1}{|h|^{\beta-[\beta]}} \int (|l(x+h,v)-l(x,v)|^{(\alpha+\beta-1)\wedge 1} \wedge 1)|\partial_x^j l(x,v)|^{\frac{[\beta]-|i|}{|j|}} \\
& |\partial_x^i \bar{\rho}^{(\alpha)}(\tilde{x},v)|\nu^{(\alpha)}(dv).
\end{aligned}
$$

For $u \in C^{\alpha+\beta}(H)$, let

$$\begin{aligned}
\bar{\mathcal{A}}_z^{(\alpha)} u(t,x) &= \mathbf{1}_{\{\alpha=1\}}\big(\bar{a}(z), \partial_x u(t,x)\big) + \frac{1}{2}\mathbf{1}_{\{\alpha=2\}} \sum_{i,j=1}^{d} \bar{B}^{ij}(z)\partial_{ij} u(t,x) \\
&\quad + \int_{\mathbb{R}_0^d} \big[u(t,x+y) - u(t,x) - \big(\mathbf{1}_{\{|y|\le 1\}}\mathbf{1}_{\{\alpha=1\}} + \mathbf{1}_{\{\alpha\in(1,2)\}}\big)\big(\partial_x u(t,x), y\big)\big] \\
&\qquad \bar{m}^{(\alpha)}(z,y)\frac{dy}{|y|^{d+\alpha}}, \\
\mathcal{A}^{(\alpha)} u(t,x) &= \bar{\mathcal{A}}_x^{(\alpha)} u(t,x) = \bar{\mathcal{A}}_z^{(\alpha)} u(t,x)|_{z=x},
\end{aligned}$$

and

$$\begin{aligned}
\bar{\mathcal{B}}_z^{(\alpha)} u(t,x) &= \mathbf{1}_{\{\alpha\in(1,2]\}}\big(\bar{c}(z), \partial_x u(t,x)\big) \\
&\quad + \int_U \big[u(t,x+l(z,v)) - u(t,x) - \big(\mathbf{1}_{\{v\in U_1\}}\mathbf{1}_{\{\alpha\in(1,2]\}}\big)\big(\partial_x u(t,x), l(z,v)\big)\big] \\
&\qquad \bar{\rho}^{(\alpha)}(z,v)\nu^{(\alpha)}(dv), \\
\mathcal{B}^{(\alpha)} u(t,x) &= \bar{\mathcal{B}}_x^{(\alpha)} u(t,x) = \bar{\mathcal{B}}_z^{(\alpha)} u(t,x)|_{z=x}.
\end{aligned}$$

Lemma 5.3. *Let $\beta \in \mathbb{R}^+$, $\beta \notin \mathbb{N}$ and $f \in C^\beta(\mathbb{R}^d)$. Then,*

$$|f(x+h) - f(x)| \le K\big(|\partial^{\beta\wedge 1} f|_\infty + |f|_\infty\big)\big(|h|^{\beta\wedge 1} \wedge 1\big),$$

where K is a constant.

Proof. For $\beta < 1$,

$$|f(x+h) - f(x)| \le \mathbf{1}_{\{|h|\le 1\}}|\partial^\beta f|_\infty |h|^\beta + \mathbf{1}_{\{|h|>1\}} 2|f|_\infty$$

and for $\beta > 1$,

$$|f(x+h) - f(x)| \le \mathbf{1}_{\{|h|\le 1\}} |\partial_x f|_\infty |h| + \mathbf{1}_{\{|h|>1\}} 2|f|_\infty.$$

Hence,

$$|f(x+h) - f(x)| \le 2\big(|\partial^{\beta\wedge 1} f|_\infty + |f|_\infty\big)\big(|h|^{\beta\wedge 1} \wedge 1\big).$$

□

Lemma 5.4. *Let $\beta \in \mathbb{R}^+$, $\beta \notin \mathbb{N}$. Assume $(\tilde{A}1)$ and $(\tilde{A}2)$ hold for $a = \bar{a}$, $c = \bar{c}$, $b = \bar{b}$, $m^{(\alpha)} = \bar{m}^{(\alpha)}$, and $\rho^{(\alpha)} = \bar{\rho}^{(\alpha)}$. Then for each $\varepsilon > 0$ there exists a constant K_ε such that for any $f \in C^{\alpha+\beta}(\mathbb{R}^d)$,*

$$\begin{aligned} |\bar{\mathcal{B}}_z^{(\alpha)} f(\cdot)|_\beta &\le \varepsilon |f|_{\alpha+\beta} + K_\varepsilon |f|_\infty, z \in \mathbb{R}^d, \\ |\bar{\mathcal{B}}_\cdot^{(\alpha)} f(x)|_\beta &\le \varepsilon |f|_{\alpha+\beta} + K_\varepsilon |f|_\infty, x \in \mathbb{R}^d, \\ |\bar{\mathcal{B}}^{(\alpha)} f|_\beta &\le \varepsilon |f|_{\alpha+\beta} + K_\varepsilon |f|_\infty. \end{aligned}$$

The statement is proved by induction.

Proof. Denote

$$\nabla_y^1 f(x) = f(x+y) - f(x)$$

and

$$\nabla_y^2 f(x) = f(x+y) - f(x) - \mathbf{1}_{\{\alpha\in(1,2]\}} \big(\partial_x f(x), y\big).$$

By Lemma 5.3,

$$|\nabla^1_{l(z,v)} f(x)| \leq K\big(|\partial^{\alpha\wedge 1} f|_\infty + |f|_\infty\big)\big(|l(z,v)|^{\alpha\wedge 1} \wedge 1\big).$$

Obviously, for $\alpha = 1$,

$$|\nabla^2_{l(z,v)} f(x)| = \big|\int_0^1 \big(\partial_x f(x+sl(z,v)), l(z,v)\big) ds\big| \leq K|\partial^\alpha f|_\infty |l(z,v)|^\alpha.$$

By Lemma 3.2, for $\alpha \in (0,1)$,

$$|\nabla^2_{l(z,v)} f(x)| = \big|K \int k^{(\alpha)}(l(z,v),y)\partial^\alpha f(x-y)dy\big| \leq K|\partial^\alpha f|_\infty |l(z,v)|^\alpha$$

and for $\alpha \in (1,2)$,

$$\begin{aligned}
|\nabla^2_{l(z,v)} f(x)| &= |\int_0^1 \big(\partial_x f(x+sl(z,v)), l(z,v)\big) ds - \int_0^1 \big(\partial_x f(x), l(z,v)\big) ds| \\
&= |\int_0^1 \big(K \int k^{(\alpha-1)}(sl(z,v),y)\partial^{\alpha-1}\partial_x f(x-y)dy, l(z,v)\big) ds| \\
&\leq \int_0^1 K|\partial^{\alpha-1}\partial_x f|_\infty (|sl(z,v)|^{\alpha-1}, l(z,v)) ds \\
&\leq K|\partial^{\alpha-1}\partial_x f|_\infty |l(z,v)|^\alpha \\
&= K|\partial^\alpha f|_\infty |l(z,v)|^\alpha.
\end{aligned}$$

Also, by the interpolation theorem, if $\bar\beta \in (0,\beta)$, then for each $\varepsilon > 0$ there exists a constant K_ε such that

$$|f|_{\bar\beta} \leq \varepsilon |f|_\beta + K_\varepsilon |f|_\infty, \forall f \in C^\beta(\mathbb{R}^d)$$

and

$$|f|_{\alpha+\bar{\beta}} \leq \varepsilon|f|_{\alpha+\beta} + K_\varepsilon|f|_\infty, \forall f \in C^{\alpha+\beta}(\mathbb{R}^d).$$

For $\beta \in (0,1)$, since

$$\begin{aligned}
\bar{\mathcal{B}}_z^{(\alpha)} f(x) &= \mathbf{1}_{\{\alpha\in(1,2]\}}\big(\bar{c}(z), \partial_x f(x)\big) \\
&\quad + \int_{U_1^c} \nabla^1_{l(z,v)} f(x)\bar{\rho}^{(\alpha)}(z,v)\nu^{(\alpha)}(dv) \\
&\quad + \int_{U_1} \nabla^2_{l(z,v)} f(x)\bar{\rho}^{(\alpha)}(z,v)\nu^{(\alpha)}(dv),
\end{aligned}$$

then,

$$\begin{aligned}
|\bar{\mathcal{B}}_z^{(\alpha)} f(\cdot)|^0_\beta &= \sup_x |\bar{\mathcal{B}}_z^{(\alpha)} f(x)| \\
&\leq \mathbf{1}_{\{\alpha\in(1,2]\}}|\bar{c}|_\infty|\partial_x f|_\infty \\
&\quad +2\big(|\partial^{\alpha\wedge 1} f|_\infty + |f|_\infty\big)\int_{U_1^c}(|l(z,v)|^{\alpha\wedge 1}\wedge 1)|\bar{\rho}^{(\alpha)}(z,v)|\nu^{(\alpha)}(dv) \\
&\quad +K|\partial^\alpha f|_\infty \int_{U_1} |l(z,v)|^\alpha|\bar{\rho}^{(\alpha)}(z,v)|\nu^{(\alpha)}(dv) \\
&\leq K\bar{N}^{(\alpha)}|f|_{\alpha+\bar{\beta}} \\
&\leq \varepsilon|f|_{\alpha+\beta} + K_\varepsilon|f|_\infty
\end{aligned}$$

and for $\kappa \in (0,1)$,

$$\begin{aligned}
|\bar{\mathcal{B}}_z^{(\alpha)} f(\cdot)|^1_\beta &= \sup_{x,h\neq 0} \frac{|\bar{\mathcal{B}}_z^{(\alpha)} f(x+h) - \bar{\mathcal{B}}_z^{(\alpha)} f(x)|}{|h|^\beta} \\
&\leq \mathbf{1}_{\{\alpha\in(1,2]\}}|\bar{c}|_\infty|\partial_x f|_\beta \\
&\quad +K|\partial^{\alpha\wedge 1} f|_\beta \int_{U_1^c} \mathbf{1}_{|l(z,v)|\leq\kappa}|l(z,v)|^{\alpha\wedge 1}|\bar{\rho}^{(\alpha)}(z,v)|\nu^{(\alpha)}(dv) \\
&\quad +2|f|_\beta\kappa^{-(\alpha\wedge 1)}\int_{U_1^c} \mathbf{1}_{|l(z,v)|>\kappa}\big(|l(z,v)|^{\alpha\wedge 1}\wedge 1\big)|\bar{\rho}^{(\alpha)}(z,v)|\nu^{(\alpha)}(dv)
\end{aligned}$$

$$
\begin{aligned}
&+\mathbf{1}_{\{\alpha\in(0,1]\}}2|f|_\beta\int_{U_1\setminus U_n}|\bar\rho^{(\alpha)}(z,v)|\nu^{(\alpha)}(dv)\\
&+\mathbf{1}_{\{\alpha\in(1,2]\}}2|\partial_x f|_\beta\int_{U_1\setminus U_n}|l(z,v)||\bar\rho^{(\alpha)}(z,v)|\nu^{(\alpha)}(dv)\\
&+K|\partial^\alpha f|_\beta\int_{U_n}|l(z,v)|^\alpha|\bar\rho^{(\alpha)}(z,v)|\nu^{(\alpha)}(dv)\\
\leq\ &\varepsilon|f|_{\alpha+\beta}+K_\varepsilon|f|_\infty.
\end{aligned}
$$

Hence,

$$
|\bar{\mathcal{B}}_z^{(\alpha)}f(\cdot)|_\beta=|\bar{\mathcal{B}}_z^{(\alpha)}f(\cdot)|_\beta^0+|\bar{\mathcal{B}}_z^{(\alpha)}f(\cdot)|_\beta^1\leq\varepsilon|f|_{\alpha+\beta}+K_\varepsilon|f|_\infty.
$$

Similarly,

$$
\begin{aligned}
|\bar{\mathcal{B}}_\cdot^{(\alpha)}f(x)|_\beta^0 &= \sup_z|\bar{\mathcal{B}}_z^{(\alpha)}f(x)|\\
&\leq \mathbf{1}_{\{\alpha\in(1,2]\}}|\bar c|_\infty|\partial_x f|_\infty\\
&\quad+2\big(|\partial^{\alpha\wedge1}f|_\infty+|f|_\infty\big)\sup_z\int_{U_1^c}(|l(z,v)|^{\alpha\wedge1}\wedge1)|\bar\rho^{(\alpha)}(z,v)|\nu^{(\alpha)}(dv)\\
&\quad+K|\partial^\alpha f|_\infty\sup_z\int_{U_1}|l(z,v)|^\alpha|\bar\rho^{(\alpha)}(z,v)|\nu^{(\alpha)}(dv)\\
&\leq K\bar N^{(\alpha)}|f|_{\alpha+\bar\beta}\\
&\leq \varepsilon|f|_{\alpha+\beta}+K_\varepsilon|f|_\infty
\end{aligned}
$$

and

$$
\begin{aligned}
&|\bar{\mathcal{B}}_\cdot^{(\alpha)}f(x)|_\beta^1\\
=\ &\sup_{z,h\neq0}\frac{|\bar{\mathcal{B}}_{z+h}^{(\alpha)}f(x)-\bar{\mathcal{B}}_z^{(\alpha)}f(x)|}{|h|^\beta}\\
\leq\ &\mathbf{1}_{\{\alpha\in(1,2]\}}\sup_{z,h\neq0}\frac{1}{|h|^\beta}\big(\bar c(z+h)-\bar c(z),\partial_x f(x)\big)\\
&+\sup_{z,h\neq0}\frac{1}{|h|^\beta}\int_{U_1^c}|\nabla^1_{l(z+h,v)}f(x)-\nabla^1_{l(z,v)}f(x)||\bar\rho^{(\alpha)}(z+h,v)|\nu^{(\alpha)}(dv)
\end{aligned}
$$

$$
\begin{aligned}
&+\sup_{z,h\neq 0}\frac{1}{|h|^\beta}\int_{U_1^c}|\nabla^1_{l(z,v)}f(x)||\bar\rho^{(\alpha)}(z+h,v)-\bar\rho^{(\alpha)}(z,v)|\nu^{(\alpha)}(dv)\\
&+\sup_{z,h\neq 0}\frac{1}{|h|^\beta}\int_{U_1}|\nabla^2_{l(z+h,v)}f(x)-\nabla^2_{l(z,v)}f(x)||\bar\rho^{(\alpha)}(z+h,v)|\nu^{(\alpha)}(dv)\\
&+\sup_{z,h\neq 0}\frac{1}{|h|^\beta}\int_{U_1}|\nabla^2_{l(z,v)}f(x)||\bar\rho^{(\alpha)}(z+h,v)-\bar\rho^{(\alpha)}(z,v)|\nu^{(\alpha)}(dv)\\
\leq\;& \mathbf{1}_{\{\alpha\in(1,2]\}}|\bar c|_\beta|\partial_x f|_\infty\\
&+K|\partial^{(\alpha+\bar\beta)\wedge 1}f|_\infty\sup_{z,h\neq 0}\frac{1}{|h|^\beta}\int_{U_1^c}(|l(z+h,v)-l(z,v)|^{(\alpha+\bar\beta)\wedge 1}\wedge 1)|\bar\rho^{(\alpha)}(z+h,v)|\nu^{(\alpha)}(dv)\\
&+2\big(|\partial^{\alpha\wedge 1}f|_\infty+|f|_\infty\big)\sup_{z,h\neq 0}\frac{1}{|h|^\beta}\int_{U_1^c}(|l(z,v)|^{\alpha\wedge 1}\wedge 1)|\bar\rho^{(\alpha)}(z+h,v)-\bar\rho^{(\alpha)}(z,v)|\nu^{(\alpha)}(dv)\\
&+K|\partial^\alpha f|_\infty\sup_{z,h\neq 0}\frac{1}{|h|^\beta}\int_{U_1}|l(z+h,v)-l(z,v)|^\alpha|\bar\rho^{(\alpha)}(z+h,v)|\nu^{(\alpha)}(dv)\\
&+K|\partial^\alpha f|_\infty\sup_{z,h\neq 0}\frac{1}{|h|^\beta}\int_{U_1}|l(z,v)|^\alpha|\bar\rho^{(\alpha)}(z+h,v)-\bar\rho^{(\alpha)}(z,v)|\nu^{(\alpha)}(dv)\\
\leq\;& K\bar N^{(\alpha)}_\beta|f|_{\alpha+\bar\beta}\\
\leq\;& \varepsilon|f|_{\alpha+\beta}+K_\varepsilon|f|_\infty.
\end{aligned}
$$

Hence,

$$
|\bar{\mathcal{B}}^{(\alpha)}_\cdot f(x)|_\beta=|\bar{\mathcal{B}}^{(\alpha)}_\cdot f(x)|^0_\beta+|\bar{\mathcal{B}}^{(\alpha)}_\cdot f(x)|^1_\beta\leq K\big(\bar N^{(\alpha)}+\bar N^{(\alpha)}_\beta\big)|f|_{\alpha+\bar\beta}\leq\varepsilon|f|_{\alpha+\beta}+K_\varepsilon|f|_\infty.
$$

Also,

$$
\begin{aligned}
|\bar{\mathcal{B}}^{(\alpha)}f|^0_\beta \;=\;& \sup_x|\bar{\mathcal{B}}^{(\alpha)}_x f(x)|\\
\leq\;& \mathbf{1}_{\{\alpha\in(1,2]\}}|\bar c|_\infty|\partial_x f|_\infty\\
&+2\big(|\partial^{\alpha\wedge 1}f|_\infty+|f|_\infty\big)\sup_x\int_{U_1^c}(|l(x,v)|^{\alpha\wedge 1}\wedge 1)|\bar\rho^{(\alpha)}(x,v)|\nu^{(\alpha)}(dv)\\
&+K|\partial^\alpha f|_\infty\sup_x\int_{U_1}|l(x,v)|^\alpha|\bar\rho^{(\alpha)}(x,v)|\nu^{(\alpha)}(dv)\\
\leq\;& K\bar N^{(\alpha)}|f|_{\alpha+\bar\beta}
\end{aligned}
$$

$$\leq \quad \varepsilon|f|_{\alpha+\beta} + K_\varepsilon|f|_\infty$$

and

$$\begin{aligned}
|\bar{\mathcal{B}}^{(\alpha)}f|^1_\beta &= \sup_{x,h\neq 0}\frac{|\bar{\mathcal{B}}^{(\alpha)}_{x+h}f(x+h)-\bar{\mathcal{B}}^{(\alpha)}_x f(x)|}{|h|^\beta}\\
&\leq \sup_{x,h\neq 0}\frac{|\bar{\mathcal{B}}^{(\alpha)}_{x+h}f(x+h)-\bar{\mathcal{B}}^{(\alpha)}_{x+h} f(x)|}{|h|^\beta}+\sup_{x,h\neq 0}\frac{|\bar{\mathcal{B}}^{(\alpha)}_{x+h}f(x)-\bar{\mathcal{B}}^{(\alpha)}_x f(x)|}{|h|^\beta}\\
&\leq K\big(|\bar{\mathcal{B}}^{(\alpha)}_{x+h}f(\cdot)|^1_\beta + |\bar{\mathcal{B}}^{(\alpha)}_{\cdot}f(x)|^1_\beta\big)\\
&\leq \varepsilon|f|_{\alpha+\beta}+K_\varepsilon|f|_\infty.
\end{aligned}$$

Hence,

$$|\bar{\mathcal{B}}^{(\alpha)}f|_\beta = |\bar{\mathcal{B}}^{(\alpha)}f|^0_\beta + |\bar{\mathcal{B}}^{(\alpha)}f|^1_\beta \leq \varepsilon|f|_{\alpha+\beta}+K_\varepsilon|f|_\infty.$$

Assume the inequalities hold for $\beta \in \bigcup_{l=0}^{n-1}(l,l+1)$, $n\in\mathbb{N}$. For $\beta\in(n,n+1)$ and $f\in C^{\alpha+\beta}(\mathbb{R}^d)$, $\partial_x f\in C^{\alpha+\beta-1}(\mathbb{R}^d)$ and $\partial^\gamma_x f\in C^{\alpha+\beta-[\beta]^-}(\mathbb{R}^d)$, where $|\gamma| = [\beta]^-$. Hence,

$$|\bar{\mathcal{B}}^{(\alpha)}_z\partial_x f(\cdot)|_{\beta-1}\leq \varepsilon'|\partial_x f|_{\alpha+\beta-1}+K_{\varepsilon'}|\partial_x f|_\infty \leq \varepsilon|f|_{\alpha+\beta}+K_\varepsilon|f|_\infty$$

and

$$|\partial^\gamma_x f\big(x+h\big)-\partial^\gamma_x f(x)|\leq 2\big(|\partial^{\alpha\wedge 1}\partial^\gamma_x f|_\infty + |\partial^\gamma_x f|_\infty\big)(|h|^{\alpha\wedge 1}\wedge 1).$$

Since for $|\gamma|\leq[\beta]^-$,

$$\partial^\gamma(\bar{\mathcal{B}}^{(\alpha)}f) = \sum_{\kappa+\mu=\gamma}\partial^\kappa_z\bar{\mathcal{B}}^{(\alpha)}_z(\partial^\mu f)(x)|_{z=x}.$$

Then,

$$
\begin{aligned}
|\bar{\mathcal{B}}_z^{(\alpha)} f(\cdot)|_\beta &= \sum_{|\gamma|\le[\beta]^-} \sup_x |\partial_x^\gamma \bar{\mathcal{B}}_z^{(\alpha)} f(x)| + \sup_{\substack{|\gamma|=[\beta]^-,\\ x,h\neq 0}} \frac{|\partial_x^\gamma \bar{\mathcal{B}}_z^{(\alpha)} f(x+h) - \partial_x^\gamma \bar{\mathcal{B}}_z^{(\alpha)} f(x)|}{|h|^{\{\beta\}^+}} \\
&= \sup_x |\bar{\mathcal{B}}_z^{(\alpha)} f(x)| + \sum_{|\gamma|\le[\beta-1]^-} \sup_x |\partial_x^\gamma \bar{\mathcal{B}}_z^{(\alpha)} \partial_x f(x)| \\
&\quad + \sup_{\substack{|\gamma|=[\beta-1]^-,\\ x,h\neq 0}} \frac{|\partial_x^\gamma \bar{\mathcal{B}}_z^{(\alpha)} \partial_x f(x+h) - \partial_x^\gamma \bar{\mathcal{B}}_z^{(\alpha)} \partial_x f(x)|}{|h|^{\{\beta\}^+}} \\
&= \sup_x |\bar{\mathcal{B}}_z^{(\alpha)} f(x)| + |\bar{\mathcal{B}}_z^{(\alpha)} \partial_x f(\cdot)|_{\beta-1} \\
&\le \varepsilon |f|_{\alpha+\beta} + K_\varepsilon |f|_\infty.
\end{aligned}
$$

Since for $x \in \mathbb{R}^d$,

$$
\partial_z \bar{\mathcal{B}}_z^{(\alpha)} f(x) = I_z^1(x) + I_z^2(x),
$$

where

$$
\begin{aligned}
I_z^1(x) &= \mathbf{1}_{\{\alpha\in(1,2]\}} \big(\partial_z \bar{c}(z), \partial_x f(x)\big) \\
&\quad + \int_{U_1^c} \nabla^1_{l(z,v)} f(x) \partial_z \bar{\rho}^{(\alpha)}(z,v) \nu^{(\alpha)}(dv) \\
&\quad + \int_{U_1} \nabla^2_{l(z,v)} f(x) \partial_z \bar{\rho}^{(\alpha)}(z,v) \nu^{(\alpha)}(dv)
\end{aligned}
$$

and

$$
\begin{aligned}
I_z^2(x) &= \int_{U_1^c} \partial_x f(x + l(z,v)) \partial_z l(z,v) \bar{\rho}^{(\alpha)}(z,v) \nu^{(\alpha)}(dv) \\
&\quad + \mathbf{1}_{\{\alpha\in(0,1]\}} \int_{U_1} \partial_x f(x + l(z,v)) \partial_z l(z,v) \bar{\rho}^{(\alpha)}(z,v) \nu^{(\alpha)}(dv) \\
&\quad + \mathbf{1}_{\{\alpha\in(1,2]\}} \int_{U_1} \big(\partial_x f(x + l(z,v)) - \partial_x f(x), \partial_z l(z,v)\big) \bar{\rho}^{(\alpha)}(z,v) \nu^{(\alpha)}(dv) \\
&= \int_{U_1^c} \partial_x f(x) \partial_z l(z,v) \bar{\rho}^{(\alpha)}(z,v) \nu^{(\alpha)}(dv)
\end{aligned}
$$

$$+\mathbf{1}_{\{\alpha\in(0,1]\}}\int_{U_1}\partial_x f(x)\partial_z l(z,v)\bar{\rho}^{(\alpha)}(z,v)\nu^{(\alpha)}(dv)$$
$$+\int\big(\partial_x f(x+l(z,v))-\partial_x f(x),\partial_z l(z,v)\big)\bar{\rho}^{(\alpha)}(z,v)\nu^{(\alpha)}(dv),$$

then,

$$\begin{aligned}
|\bar{\mathcal{B}}^{(\alpha)}_{\cdot}f(x)|_\beta &= \sum_{|\gamma|\le[\beta]^-}\sup_z|\partial_z^\gamma\bar{\mathcal{B}}^{(\alpha)}_z f(x)|+\sup_{\substack{|\gamma|=[\beta]^-,\\ z,h\neq 0}}\frac{|\partial_z^\gamma\bar{\mathcal{B}}^{(\alpha)}_{z+h}f(x)-\partial_z^\gamma\bar{\mathcal{B}}^{(\alpha)}_z f(x)|}{|h|^{\{\beta\}^+}}\\
&= \sup_z|\bar{\mathcal{B}}^{(\alpha)}_z f(x)|+\sum_{|\gamma|\le[\beta-1]^-}\sup_z|\partial_z^\gamma\partial_z\bar{\mathcal{B}}^{(\alpha)}_z f(x)|\\
&\quad+\sup_{\substack{|\gamma|=[\beta-1]^-,\\ z,h\neq 0}}\frac{|\partial_z^\gamma\partial_z\bar{\mathcal{B}}^{(\alpha)}_{z+h}f(x)-\partial_z^\gamma\partial_z\bar{\mathcal{B}}^{(\alpha)}_z f(x)|}{|h|^{\{\beta\}^+}}\\
&= \sup_z|\bar{\mathcal{B}}^{(\alpha)}_z f(x)|+|\partial_z\bar{\mathcal{B}}^{(\alpha)}_{\cdot}f(x)|_{\beta-1}\\
&\le \sup_z|\bar{\mathcal{B}}^{(\alpha)}_z f(x)|+|I^1_{\cdot}(x)|_{\beta-1}+|I^2_{\cdot}(x)|_{\beta-1}.
\end{aligned}$$

For $|I^1_{\cdot}(x)|_{\beta-1}$, replace $\bar{c}$ with $\partial_z\bar{c}$ and $\bar{\rho}^{(\alpha)}$ with $\partial_z\bar{\rho}^{(\alpha)}$ in $\bar{\mathcal{B}}^{(\alpha)}_{\cdot}f(x)$, it then follows from the conclusion for the case $\beta-1\in(n-1,n)$ that

$$|I^1_{\cdot}(x)|_{\beta-1}\le K\big(\bar{N}^{(\alpha)}+\bar{N}^{(\alpha)}_{\beta}\big)|f|_{\alpha+\bar{\beta}}\le\varepsilon|f|_{\alpha+\beta}+K_\varepsilon|f|_\infty.$$

For $|I^2_{\cdot}(x)|_{\beta-1}$,

$$|I^2_{\cdot}(x)|_{\beta-1}=\sum_{|\gamma|\le[\beta-1]^-}\sup_z|\partial_z^\gamma I^2_z(x)|+\sup_{\substack{|\gamma|=[\beta-1]^-,\\ z,h\neq 0}}\frac{|\partial_z^\gamma I^2_{z+h}(x)-\partial_z^\gamma I^2_z(x)|}{|h|^{\{\beta\}^+}}.$$

When $\beta\in(1,2)$, clearly

$$\begin{aligned}
\partial_z^{[\beta-1]^-}I^2_z(x) &= I^2_z(x)\\
&= \int_{U_1^c}\partial_x f(x+l(z,v))\partial_z l(z,v)\bar{\rho}^{(\alpha)}(z,v)\nu^{(\alpha)}(dv)
\end{aligned}$$

$$+\mathbf{1}_{\{\alpha\in(0,1]\}}\int_{U_1}\partial_x f(x+l(z,v))\partial_z l(z,v)\bar\rho^{(\alpha)}(z,v)\nu^{(\alpha)}(dv)$$
$$+\mathbf{1}_{\{\alpha\in(1,2]\}}\int_{U_1}\big(\partial_x f(x+l(z,v))-\partial_x f(x),\partial_z l(z,v)\big)\bar\rho^{(\alpha)}(z,v)\nu^{(\alpha)}(dv)$$

$$=\sum_{|i|=0}^{[\beta-1]^-}\sum_{\sum_j k_j|j|=[\beta]^- -|i|}\int K(f)\prod_{|j|=1}^{[\beta]^- -|i|}\big(\partial_z^j l(z,v)\big)^{k_j}\partial_z^i\bar\rho^{(\alpha)}(z,v)\nu^{(\alpha)}(dv),$$

where for a constant K and a multiindex k with $|k|\le[\beta-1]^-$, $K(f)=K\partial_x^k f(x+l(z,v))$ or $K(f)=K\big(\partial_x f(x+l(z,v))-\partial_x f(x)\big)$. Since

$$\begin{aligned}\sup_z|I_z^2(x)| &\le |\partial_x f|_\infty\sup_z\int_{U_1^c}|\partial_z l(z,v)||\bar\rho^{(\alpha)}(z,v)|\nu^{(\alpha)}(dv)\\ &\quad+\mathbf{1}_{\{\alpha\in(0,1]\}}|\partial_x f|_\infty\sup_z\int_{U_1}|\partial_z l(z,v)||\bar\rho^{(\alpha)}(z,v)|\nu^{(\alpha)}(dv)\\ &\quad+2\big(|\partial^{\alpha\wedge1}\partial_x f|_\infty+|\partial_x f|_\infty\big)\sup_z\int(|l(z,v)|^{\alpha\wedge1}\wedge1)|\partial_z l(z,v)||\bar\rho^{(\alpha)}(z,v)|\nu^{(\alpha)}(dv)\\ &\le \varepsilon|f|_{\alpha+\beta}+K_\varepsilon|f|_\infty\end{aligned}$$

and

$$\begin{aligned}&\sup_{z,h\neq0}\frac{|I_{z+h}^2(x)-I_z^2(x)|}{|h|^{\{\beta\}^+}}\\ \le\ &\sup_{z,h\neq0}\frac{1}{|h|^{\{\beta\}^+}}\int_{U_1^c}|\partial_x f(x)\partial_z l(z+h,v)[\bar\rho^{(\alpha)}(z+h,v)-\bar\rho^{(\alpha)}(z,v)]|\nu^{(\alpha)}(dv)\\ &+\sup_{z,h\neq0}\frac{1}{|h|^{\{\beta\}^+}}\int_{U_1^c}|\partial_x f(x)[\partial_z l(z+h,v)-\partial_z l(z,v)]\bar\rho^{(\alpha)}(z,v)|\nu^{(\alpha)}(dv)\\ &+\mathbf{1}_{\{\alpha\in(0,1]\}}\sup_{z,h\neq0}\frac{1}{|h|^{\{\beta\}^+}}\int_{U_1}|\partial_x f(x)\partial_z l(z+h,v)[\bar\rho^{(\alpha)}(z+h,v)-\bar\rho^{(\alpha)}(z,v)]|\nu^{(\alpha)}(dv)\\ &+\mathbf{1}_{\{\alpha\in(0,1]\}}\sup_{z,h\neq0}\frac{1}{|h|^{\{\beta\}^+}}\int_{U_1}|\partial_x f(x)[\partial_z l(z+h,v)-\partial_z l(z,v)]\bar\rho^{(\alpha)}(z,v)|\nu^{(\alpha)}(dv)\\ &+\sup_{z,h\neq0}\frac{1}{|h|^{\{\beta\}^+}}\int|\big(\partial_x f(x+l(z+h,v))-\partial_x f(x),\partial_z l(z+h,v)\big)\\ &\quad[\bar\rho^{(\alpha)}(z+h,v)-\bar\rho^{(\alpha)}(z,v)]|\nu^{(\alpha)}(dv)\\ &+\sup_{z,h\neq0}\frac{1}{|h|^{\{\beta\}^+}}\int|\big(\partial_x f(x+l(z+h,v))-\partial_x f(x),\partial_z l(z+h,v)-\partial_z l(z,v)\big)\bar\rho^{(\alpha)}(z,v)|\end{aligned}$$

$$\nu^{(\alpha)}(dv)$$

$$+\sup_{z,h\neq 0}\frac{1}{|h|^{\{\beta\}^+}}\int|\big(\partial_x f(x+l(z+h,v))-\partial_x f(x+l(z,v)),\partial_z l(z,v)\big)\bar{\rho}^{(\alpha)}(z,v)|\nu^{(\alpha)}(dv)$$

$$\leq |\partial_x f|_\infty\sup_{z,h\neq 0}\frac{1}{|h|^{\{\beta\}^+}}\int_{U_1^c}|\partial_z l(z+h,v)||\bar{\rho}^{(\alpha)}(z+h,v)-\bar{\rho}^{(\alpha)}(z,v)|\nu^{(\alpha)}(dv)$$

$$+|\partial_x f|_\infty\sup_{z,h\neq 0}\frac{1}{|h|^{\{\beta\}^+}}\int_{U_1^c}|\partial_z l(z+h,v)-\partial_z l(z,v)||\bar{\rho}^{(\alpha)}(z+h,v)|\nu^{(\alpha)}(dv)$$

$$+\mathbf{1}_{\{\alpha\in(0,1]\}}|\partial_x f|_\infty\sup_{z,h\neq 0}\frac{1}{|h|^{\{\beta\}^+}}\int_{U_1}|\partial_z l(z+h,v)||\bar{\rho}^{(\alpha)}(z+h,v)-\bar{\rho}^{(\alpha)}(z,v)|\nu^{(\alpha)}(dv)$$

$$+\mathbf{1}_{\{\alpha\in(0,1]\}}|\partial_x f|_\infty\sup_{z,h\neq 0}\frac{1}{|h|^{\{\beta\}^+}}\int_{U_1}|\partial_z l(z+h,v)-\partial_z l(z,v)||\bar{\rho}^{(\alpha)}(z+h,v)|\nu^{(\alpha)}(dv)$$

$$+2\big(|\partial^{\alpha\wedge 1}\partial_x f|_\infty+|\partial_x f|_\infty\big)\sup_{z,h\neq 0}\frac{1}{|h|^{\{\beta\}^+}}\int(|l(z+h,v)|^{\alpha\wedge 1}\wedge 1)|\partial_z l(z,v)|$$

$$|\bar{\rho}^{(\alpha)}(z+h,v)-\bar{\rho}^{(\alpha)}(z,v)|\nu^{(\alpha)}(dv)$$

$$+2\big(|\partial^{\alpha\wedge 1}\partial_x f|_\infty+|\partial_x f|_\infty\big)\sup_{z,h\neq 0}\frac{1}{|h|^{\{\beta\}^+}}\int(|l(z+h,v)|^{\alpha\wedge 1}\wedge 1)|\partial_z l(z+h,v)-\partial_z l(z,v)|$$

$$|\bar{\rho}^{(\alpha)}(z,v)|\nu^{(\alpha)}(dv)$$

$$+2\big(|\partial^{\alpha\wedge 1}\partial_x f|_\infty+|\partial_x f|_\infty\big)\sup_{z,h\neq 0}\frac{1}{|h|^{\{\beta\}^+}}\int\big(|l(z+h,v)-l(z,v)|^{\alpha\wedge 1}\wedge 1\big)|\partial_z l(z,v)|$$

$$|\bar{\rho}^{(\alpha)}(z,v)|\nu^{(\alpha)}(dv)$$

$$\leq \varepsilon|f|_{\alpha+\beta}+K_\varepsilon|f|_\infty,$$

then, for $\beta\in(1,2)$,

$$|I^2_\cdot(x)|_{\beta-1}=\sup_z|I^2_z(x)|+\sup_{z,h\neq 0}\frac{|I^2_{z+h}(x)-I^2_z(x)|}{|h|^{\{\beta\}^+}}\leq\varepsilon|f|_{\alpha+\beta}+K_\varepsilon|f|_\infty.$$

Assume for $\beta\in(n-1,n)$, $n\in\mathbb{N}$ and $n>2$,

$$\partial_z^{[\beta-1]^-}I^2_z(x)=\sum_{|i|=0}^{[\beta-1]^-}\sum_{\sum_j k_j|j|=[\beta]^--|i|}\int K(f)\prod_{|j|=1}^{[\beta]^--|i|}\big(\partial_z^j l(z,v)\big)^{k_j}\partial_z^i\bar{\rho}^{(\alpha)}(z,v)\nu^{(\alpha)}(dv),$$

where for a constant K and a multiindex k with $|k| \leq [\beta-1]^-$, $K(f) = K\partial_x^k f(x+l(z,v))$ or $K(f) = K\big(\partial_x f(x+l(z,v)) - \partial_x f(x)\big)$, and $k_j \in \mathbb{N}^0$. That is,

$$\partial_z^{n-2} I_z^2(x) = \sum_{|i|=0}^{n-2} \sum_{\sum_j k_j |j| = n-1-|i|} \int K(f) \prod_{|j|=1}^{n-1-|i|} \big(\partial_z^j l(z,v)\big)^{k_j} \partial_z^i \bar{\rho}^{(\alpha)}(z,v)\nu^{(\alpha)}(dv),$$

where for a constant K and a multiindex k with $|k| \leq n-2$, $K(f) = K\partial_x^k f(x+l(z,v))$ or $K(f) = K\big(\partial_x f(x+l(z,v)) - \partial_x f(x)\big)$. Then for $\beta \in (n, n+1)$,

$$\begin{aligned}
\partial_z^{[\beta-1]^-} I_z^2(x) &= \partial_z^{n-1} I_z^2(x) = \partial_z \partial_z^{n-2} I_z^2(x) \\
&= \sum_{|i|=0}^{n-2} \sum_{\sum_j k_j |j| = n-1-|i|} \int K(f)\partial_z l(z,v) \prod_{|j|=1}^{n-1-|i|} \big(\partial_z^j l(z,v)\big)^{k_j} \partial_z^i \bar{\rho}^{(\alpha)}(z,v)\nu^{(\alpha)}(dv) \\
&\quad + \sum_{|i|=0}^{n-2} \sum_{\sum_j k_j |j| = n-1-|i|} \int K(f)\partial_z \Big(\prod_{|j|=1}^{n-1-|i|} \big(\partial_z^j l(z,v)\big)^{k_j} \Big) \partial_z^i \bar{\rho}^{(\alpha)}(z,v)\nu^{(\alpha)}(dv) \\
&\quad + \sum_{|i|=0}^{n-2} \sum_{\sum_j k_j |j| = n-1-|i|} \int K(f) \prod_{|j|=1}^{n-1-|i|} \big(\partial_z^j l(z,v)\big)^{k_j} \partial_z^{i+1} \bar{\rho}^{(\alpha)}(z,v)\nu^{(\alpha)}(dv) \\
&= \sum_{|i|=0}^{n-1} \sum_{\sum_j k_j |j| = n-|i|} \int K(f) \prod_{|j|=1}^{n-|i|} \big(\partial_z^j l(z,v)\big)^{k_j} \partial_z^i \bar{\rho}^{(\alpha)}(z,v)\nu^{(\alpha)}(dv) \\
&= \sum_{|i|=0}^{[\beta-1]^-} \sum_{\sum_j k_j |j| = [\beta]^- - |i|} \int K(f) \prod_{|j|=1}^{[\beta]^- - |i|} \big(\partial_z^j l(z,v)\big)^{k_j} \partial_z^i \bar{\rho}^{(\alpha)}(z,v)\nu^{(\alpha)}(dv).
\end{aligned}$$

Apply Hölder's inequality by neglecting those terms $\big(\partial_z^j l(z,v)\big)^{k_j}$ with $k_j = 0$, it then follows that for any $\bar{\beta} \in (n, \beta)$,

$$\begin{aligned}
\sup_z |\partial_z^{n-1} I_z^2(x)| &\leq \sum_{|i|=0}^{n-1} \sum_{\sum_j k_j |j| = n-|i|} \sup_z \Big| \int K(f) \prod_{|j|=1}^{n-|i|} \big(\partial_z^j l(z,v)\big)^{k_j} \partial_z^i \bar{\rho}^{(\alpha)}(z,v)\nu^{(\alpha)}(dv) \Big| \\
&\leq K|f|_{\alpha+\bar{\beta}} \sum_{|i|=0}^{n-1} \sup_z \prod_{|j|=1}^{n-|i|} \Big(\int |\partial_z^j l(z,v)|^{k_j \cdot \frac{n-|i|}{k_j |j|}} |\partial_z^i \bar{\rho}^{(\alpha)}(z,v)| \nu^{(\alpha)}(dv) \Big)^{\frac{k_j |j|}{n-|i|}}
\end{aligned}$$

$$
\begin{aligned}
&\leq K|f|_{\alpha+\bar{\beta}} \sum_{|i|=0}^{n-1} \sum_{|j|=1}^{n-|i|} \sup_z \frac{\left(\int |\partial_z^j l(z,v)|^{k_j \cdot \frac{n-|i|}{k_j|j|}} |\partial_z^i \bar{\rho}^{(\alpha)}(z,v)| \nu^{(\alpha)}(dv) \right)^{\frac{k_j|j|}{n-|i|} \cdot \frac{n-|i|}{k_j|j|}}}{\frac{n-|i|}{k_j|j|}} \\
&\leq K|f|_{\alpha+\bar{\beta}} \sum_{|i|=0}^{n-1} \sum_{|j|=1}^{n-|i|} \sup_z \int |\partial_z^j l(z,v)|^{\frac{n-|i|}{|j|}} |\partial_z^i \bar{\rho}^{(\alpha)}(z,v)| \nu^{(\alpha)}(dv) \\
&\leq \varepsilon |f|_{\alpha+\beta} + K_\varepsilon |f|_\infty
\end{aligned}
$$

and

$$
\begin{aligned}
& \sup_{z,h\neq 0} \frac{|\partial_z^{n-1} I_{z+h}^2(x) - \partial_z^{n-1} I_z^2(x)|}{|h|^{\{\beta\}^+}} \\
\leq\ & K|f|_{\alpha+\bar{\beta}} \Big(\sum_{|i|=0}^{n-1} \sum_{|j|=1}^{n-|i|} \sup_{z,h\neq 0} \frac{1}{|h|^{\beta-[\beta]}} \int |\partial_z^j l(z,v)|^{\frac{n-|i|}{|j|}} |\partial_z^i \bar{\rho}^{(\alpha)}(z+h,v) - \partial_z^i \bar{\rho}^{(\alpha)}(z,v)| \nu^{(\alpha)}(dv) \\
& + \sum_{|i|=0}^{n-1} \sum_{|j|=1}^{n-|i|} \sup_{z,h\neq 0} \frac{1}{|h|^{\beta-[\beta]}} \int |(\partial_z^j l(z+h,v)) - (\partial_z^j l(z,v))|^{\frac{n-|i|}{|j|}} |\partial_z^i \bar{\rho}^{(\alpha)}(z,v)| \nu^{(\alpha)}(dv) \\
& + \sum_{|i|=0}^{n-1} \sum_{|j|=1}^{n-|i|} \sup_{z,h\neq 0} \frac{1}{|h|^{\beta-[\beta]}} \int (|l(z+h,v) - l(z,v)|^{(\alpha+\beta-1)\wedge 1} \wedge 1) |\partial_z^j l(z,v)|^{\frac{n-|i|}{|j|}} \\
& |\partial_z^i \bar{\rho}^{(\alpha)}(z,v)| \nu^{(\alpha)}(dv) \Big) \\
\leq\ & \varepsilon |f|_{\alpha+\beta} + K_\varepsilon |f|_\infty .
\end{aligned}
$$

For example, for

$$
\begin{aligned}
\partial_z I_z^2(x) &= \int_{U_1^c} \partial_x f(x) \partial_z^2 l(z,v) \bar{\rho}^{(\alpha)}(z,v) \nu^{(\alpha)}(dv) \\
&\quad + \int_{U_1^c} \partial_x f(x) \partial_z l(z,v) \partial_z \bar{\rho}^{(\alpha)}(z,v) \nu^{(\alpha)}(dv) \\
&\quad + \mathbf{1}_{\{\alpha\in(0,1]\}} \int_{U_1} \partial_x f(x) \partial_z^2 l(z,v) \bar{\rho}^{(\alpha)}(z,v) \nu^{(\alpha)}(dv) \\
&\quad + \mathbf{1}_{\{\alpha\in(0,1]\}} \int_{U_1} \partial_x f(x) \partial_z l(z,v) \partial_z \bar{\rho}^{(\alpha)}(z,v) \nu^{(\alpha)}(dv) \\
&\quad + \int \left(\partial_x^2 f(x + l(z,v)), (\partial_z l(z,v))^2 \right) \bar{\rho}^{(\alpha)}(z,v) \nu^{(\alpha)}(dv)
\end{aligned}
$$

$$+\int \big(\partial_x f(x+l(z,v)) - \partial_x f(x), \partial_z^2 l(z,v)\big)\bar\rho^{(\alpha)}(z,v)\nu^{(\alpha)}(dv)$$
$$+\int \big(\partial_x f(x+l(z,v)) - \partial_x f(x), \partial_z l(z,v)\big)\partial_z\bar\rho^{(\alpha)}(z,v)\nu^{(\alpha)}(dv),$$

$$\begin{aligned}
\sup_z |\partial_z I_z^2(x)| \leq\ & |\partial_x f|_\infty \sup_z \int_{U_1^c} |\partial_z^2 l(z,v)||\bar\rho^{(\alpha)}(z,v)|\nu^{(\alpha)}(dv) \\
& +|\partial_x f|_\infty \sup_z \int_{U_1^c} |\partial_z l(z,v)||\partial_z\bar\rho^{(\alpha)}(z,v)|\nu^{(\alpha)}(dv) \\
& +\mathbf{1}_{\{\alpha\in(0,1]\}}|\partial_x f|_\infty \sup_z \int_{U_1} |\partial_z^2 l(z,v)||\bar\rho^{(\alpha)}(z,v)|\nu^{(\alpha)}(dv) \\
& +\mathbf{1}_{\{\alpha\in(0,1]\}}|\partial_x f|_\infty \sup_z \int_{U_1} |\partial_z l(z,v)||\partial_z\bar\rho^{(\alpha)}(z,v)|\nu^{(\alpha)}(dv) \\
& +|\partial_x^2 f|_\infty \sup_z \int |\partial_z l(z,v)|^2|\bar\rho^{(\alpha)}(z,v)|\nu^{(\alpha)}(dv) \\
& +2\big(|\partial^{\alpha\wedge 1}\partial_x f|_\infty + |\partial_x f|_\infty\big) \sup_z \int (|l(z,v)|^{\alpha\wedge 1}\wedge 1)|\partial_z^2 l(z,v)||\bar\rho^{(\alpha)}(z,v)| \\
& \quad \nu^{(\alpha)}(dv) \\
& +2\big(|\partial^{\alpha\wedge 1}\partial_x f|_\infty + |\partial_x f|_\infty\big) \sup_z \int (|l(z,v)|^{\alpha\wedge 1}\wedge 1)|\partial_z l(z,v)||\partial_z\bar\rho^{(\alpha)}(z,v)| \\
& \quad \nu^{(\alpha)}(dv) \\
\leq\ & \varepsilon|f|_{\alpha+\beta} + K_\varepsilon|f|_\infty
\end{aligned}$$

and

$$\begin{aligned}
& \sup_{z,h\neq 0} \frac{|\partial_z I_{z+h}^2(x) - \partial_z I_z^2(x)|}{|h|^{\{\beta\}^+}} \\
\leq\ & |\partial_x f|_\infty \sup_{z,h\neq 0} \frac{1}{|h|^{\{\beta\}^+}} \int_{U_1^c} |\partial_z^2 l(z+h,v)||\bar\rho^{(\alpha)}(z+h,v) - \bar\rho^{(\alpha)}(z,v)|\nu^{(\alpha)}(dv) \\
& +|\partial_x f|_\infty \sup_{z,h\neq 0} \frac{1}{|h|^{\{\beta\}^+}} \int_{U_1^c} |\partial_z^2 l(z+h,v) - \partial_z^2 l(z,v)||\bar\rho^{(\alpha)}(z,v)|\nu^{(\alpha)}(dv) \\
& +|\partial_x f|_\infty \sup_{z,h\neq 0} \frac{1}{|h|^{\{\beta\}^+}} \int_{U_1^c} |\partial_z l(z+h,v)||\partial_z\bar\rho^{(\alpha)}(z+h,v) - \partial_z\bar\rho^{(\alpha)}(z,v)|\nu^{(\alpha)}(dv) \\
& +|\partial_x f|_\infty \sup_{z,h\neq 0} \frac{1}{|h|^{\{\beta\}^+}} \int_{U_1^c} |\partial_z l(z+h,v) - \partial_z l(z,v)||\partial_z\bar\rho^{(\alpha)}(z,v)|\nu^{(\alpha)}(dv)
\end{aligned}$$

$$+\mathbf{1}_{\{\alpha\in(0,1]\}}|\partial_x f|_\infty \sup_{z,h\neq 0}\frac{1}{|h|^{\{\beta\}^+}}\int_{U_1}|\partial_z^2 l(z+h,v)||\bar\rho^{(\alpha)}(z+h,v)-\bar\rho^{(\alpha)}(z,v)|\nu^{(\alpha)}(dv)$$
$$+\mathbf{1}_{\{\alpha\in(0,1]\}}|\partial_x f|_\infty \sup_{z,h\neq 0}\frac{1}{|h|^{\{\beta\}^+}}\int_{U_1}|\partial_z^2 l(z+h,v)-\partial_z^2 l(z,v)||\bar\rho^{(\alpha)}(z,v)|\nu^{(\alpha)}(dv)$$
$$+\mathbf{1}_{\{\alpha\in(0,1]\}}|\partial_x f|_\infty \sup_{z,h\neq 0}\frac{1}{|h|^{\{\beta\}^+}}\int_{U_1}|\partial_z l(z+h,v)||\partial_z\bar\rho^{(\alpha)}(z+h,v)-\partial_z\bar\rho^{(\alpha)}(z,v)|\nu^{(\alpha)}(dv)$$
$$+\mathbf{1}_{\{\alpha\in(0,1]\}}|\partial_x f|_\infty \sup_{z,h\neq 0}\frac{1}{|h|^{\{\beta\}^+}}\int_{U_1}|\partial_z l(z+h,v)-\partial_z l(z,v)||\partial_z\bar\rho^{(\alpha)}(z,v)|\nu^{(\alpha)}(dv)$$
$$+|\partial_x^2 f|_\infty \sup_{z,h\neq 0}\frac{1}{|h|^{\{\beta\}^+}}\int|\partial_z l(z+h,v)|^2|\bar\rho^{(\alpha)}(z+h,v)-\bar\rho^{(\alpha)}(z,v)|\nu^{(\alpha)}(dv)$$
$$+|\partial_x^2 f|_\infty \sup_{z,h\neq 0}\frac{1}{|h|^{\{\beta\}^+}}\int|(\partial_z l(z+h,v))^2-(\partial_z l(z,v))^2||\bar\rho^{(\alpha)}(z,v)|\nu^{(\alpha)}(dv)$$
$$+2\big(|\partial^{\alpha\wedge 1}\partial_x^2 f|_\infty+|\partial_x^2 f|_\infty\big)\sup_{z,h\neq 0}\frac{1}{|h|^{\{\beta\}^+}}\int\big(|l(z+h,v)-l(z,v)|^{\alpha\wedge 1}\wedge 1\big)|\partial_z l(z,v)|^2$$
$$|\bar\rho^{(\alpha)}(z,v)|\nu^{(\alpha)}(dv)$$
$$+2\big(|\partial^{\alpha\wedge 1}\partial_x f|_\infty+|\partial_x f|_\infty\big)\sup_{z,h\neq 0}\frac{1}{|h|^{\{\beta\}^+}}\int(|l(z+h,v)|^{\alpha\wedge 1}\wedge 1)|\partial_z^2 l(z+h,v)|$$
$$|\bar\rho^{(\alpha)}(z+h,v)-\bar\rho^{(\alpha)}(z,v)|\nu^{(\alpha)}(dv)$$
$$+2\big(|\partial^{\alpha\wedge 1}\partial_x f|_\infty+|\partial_x f|_\infty\big)\sup_{z,h\neq 0}\frac{1}{|h|^{\{\beta\}^+}}\int(|l(z+h,v)|^{\alpha\wedge 1}\wedge 1)|\partial_z^2 l(z+h,v)-\partial_z^2 l(z,v)|$$
$$|\bar\rho^{(\alpha)}(z,v)|\nu^{(\alpha)}(dv)$$
$$+2\big(|\partial^{\alpha\wedge 1}\partial_x f|_\infty+|\partial_x f|_\infty\big)\sup_{z,h\neq 0}\frac{1}{|h|^{\{\beta\}^+}}\int\big(|l(z+h,v)-l(z,v)|^{\alpha\wedge 1}\wedge 1\big)|\partial_z^2 l(z,v)|$$
$$|\bar\rho^{(\alpha)}(z,v)|\nu^{(\alpha)}(dv)$$
$$+2\big(|\partial^{\alpha\wedge 1}\partial_x f|_\infty+|\partial_x f|_\infty\big)\sup_{z,h\neq 0}\frac{1}{|h|^{\{\beta\}^+}}\int(|l(z+h,v)|^{\alpha\wedge 1}\wedge 1)|\partial_z l(z+h,v)|$$
$$|\partial_z\bar\rho^{(\alpha)}(z+h,v)-\partial_z\bar\rho^{(\alpha)}(z,v)|\nu^{(\alpha)}(dv)$$
$$+2\big(|\partial^{\alpha\wedge 1}\partial_x f|_\infty+|\partial_x f|_\infty\big)\sup_{z,h\neq 0}\frac{1}{|h|^{\{\beta\}^+}}\int(|l(z+h,v)|^{\alpha\wedge 1}\wedge 1)|\partial_z l(z+h,v)-\partial_z l(z,v)|$$
$$|\partial_z\bar\rho^{(\alpha)}(z,v)|\nu^{(\alpha)}(dv)$$
$$+2\big(|\partial^{\alpha\wedge 1}\partial_x f|_\infty+|\partial_x f|_\infty\big)\sup_{z,h\neq 0}\frac{1}{|h|^{\{\beta\}^+}}\int\big(|l(z+h,v)-l(z,v)|^{\alpha\wedge 1}\wedge 1\big)|\partial_z l(z,v)|$$
$$|\partial_z\bar\rho^{(\alpha)}(z,v)|\nu^{(\alpha)}(dv)$$

$$\leq \quad \varepsilon|f|_{\alpha+\beta} + K_\varepsilon|f|_\infty.$$

Hence, it is proved by induction that $|I^2_\cdot(x)|_{\beta-1} \leq \varepsilon|f|_{\alpha+\beta} + K_\varepsilon|f|_\infty$, $\forall \beta > 1$ with $\beta \notin \mathbb{N}$.

Therefore,

$$\begin{aligned} |\bar{\mathcal{B}}^{(\alpha)}_\cdot f(x)|_\beta &\leq \sup_z |\bar{\mathcal{B}}^{(\alpha)}_z f(x)| + |I^1_z(x)|_{\beta-1} + |I^2_z(x)|_{\beta-1} \\ &\leq \varepsilon|f|_{\alpha+\beta} + K_\varepsilon|f|_\infty. \end{aligned}$$

It then follows that

$$\begin{aligned} |\bar{\mathcal{B}}^{(\alpha)} f|_\beta &= \sum_{|\gamma|\leq[\beta]^-} \sup_x |\partial^\gamma_x \bar{\mathcal{B}}^{(\alpha)}_x f(x)| + \sup_{\substack{|\gamma|=[\beta]^-,\\ x,h\neq 0}} \frac{|\partial^\gamma_x \bar{\mathcal{B}}^{(\alpha)}_{x+h} f(x+h) - \partial^\gamma_x \bar{\mathcal{B}}^{(\alpha)}_x f(x)|}{|h|^{\{\beta\}^+}} \\ &\leq \sum_{|\gamma|\leq[\beta]^-} \sup_x |\partial^\gamma_x \bar{\mathcal{B}}^{(\alpha)}_x f(x)| + \sup_{\substack{|\gamma|=[\beta]^-,\\ x,h\neq 0}} \frac{|\partial^\gamma_x \bar{\mathcal{B}}^{(\alpha)}_{x+h} f(x+h) - \partial^\gamma_x \bar{\mathcal{B}}^{(\alpha)}_{x+h} f(x)|}{|h|^{\{\beta\}^+}} \\ &\quad + \sup_{\substack{|\gamma|=[\beta]^-,\\ x,h\neq 0}} \frac{|\partial^\gamma_x \bar{\mathcal{B}}^{(\alpha)}_{x+h} f(x) - \partial^\gamma_x \bar{\mathcal{B}}^{(\alpha)}_x f(x)|}{|h|^{\{\beta\}^+}} \\ &\leq K\big(|\bar{\mathcal{B}}^{(\alpha)}_z f(\cdot)|_\beta + |\bar{\mathcal{B}}^{(\alpha)}_\cdot f(x)|_\beta\big) \\ &\leq \varepsilon|f|_{\alpha+\beta} + K_\varepsilon|f|_\infty. \end{aligned}$$

The statement is thus proved by induction. □

Lemma 5.5. *Let $\alpha \in (0,2]$, $\beta \in \mathbb{R}^+$, $\beta \notin \mathbb{N}$, and $\lambda \geq 0$. Assume $(\tilde{\text{A}}1)$ and $(\tilde{\text{A}}2)$ hold. Then for $f \in C^\beta(H)$, there exist a unique solution $u \in C^{\alpha+\beta}(H)$ to the Cauchy problem*

$$\begin{aligned} \left(\partial_t + \mathcal{A}^{(\alpha)}_x + \mathcal{B}^{(\alpha)}_x - \lambda\right) u(t,x) &= f(t,x), \\ u(0,x) &= 0 \end{aligned} \tag{5.3}$$

and a constant K independent of f such that $|u|_{\alpha+\beta} \le K|f|_\beta$.

Proof. Rewrite (5.3) as

$$\begin{aligned} \left(\partial_t + \mathcal{A}_x^{(\alpha)} - \lambda\right) u(t,x) &= f(t,x) - \mathcal{B}_x^{(\alpha)} u(t,x), \\ u(0,x) &= 0, \end{aligned}$$

then for $\varepsilon > 0$, there exist constant K and K_ε such that

$$|u|_{\alpha+\beta} \le K\big[|f|_\beta + |\mathcal{B}^{(\alpha)}u|_\beta\big] \le K\big[|f|_\beta + \varepsilon|u|_{\alpha+\beta} + K_\varepsilon|u|_\infty\big].$$

That is,

$$|u|_{\alpha+\beta} \le K\big[|f|_\beta + |u|_\infty\big].$$

Also, for given x_0, rewrite (5.3) as

$$\begin{aligned} \left(\partial_t + \mathcal{A}_{x_0}^{(\alpha)} - \lambda\right) u(t,x) &= f(t,x) - \mathcal{B}_x^{(\alpha)} u(t,x) + \big(\mathcal{A}_{x_0}^{(\alpha)} - \mathcal{A}_x^{(\alpha)}\big)u(t,x), \\ u(0,x) &= 0, \end{aligned}$$

then $\forall \bar{\beta} \in (0,1)$,

$$|u|_\infty \le |u|_{\bar{\beta}} \le \mu(\lambda)\big[|f|_{\bar{\beta}} + |\mathcal{B}^{(\alpha)}u|_{\bar{\beta}} + |\big(\mathcal{A}_{x_0}^{(\alpha)} - \mathcal{A}_x^{(\alpha)}\big)u|_{\bar{\beta}}\big] \le \mu(\lambda)\big[|f|_{\bar{\beta}} + |u|_{\alpha+\bar{\beta}}\big],$$

where $\mu(\lambda) \to 0$ as $\lambda \to \infty$. Hence,

$$|u|_{\alpha+\beta} \le K\big[|f|_\beta + \mu(\lambda)(|f|_\beta + |u|_{\alpha+\beta})\big]$$

and there exist $\lambda_0 > 0$ and a constant K independent of u such that $|u|_{\alpha+\beta} \leq K|f|_\beta$ if $\lambda \geq \lambda_0$. For $\lambda \leq \lambda_0$, if $u \in C^{\alpha+\beta}(H)$ is the solution to (5.3), then $\tilde{u}(t,x) = e^{-(\lambda_0-\lambda)t}u(t,x)$ is the solution to (5.3) with λ replaced by λ_0. Hence,

$$|u|_{\alpha+\beta} \leq e^{(\lambda_0-\lambda)T}|u|_{\alpha+\beta} \leq Ke^{(\lambda_0-\lambda)T}|f|_\beta.$$

The statement is then proved. □

Theorem 5.6. *Let $\alpha \in (0,2]$ and $\beta \in \mathbb{R}^+$, $\beta \notin \mathbb{N}$. Assume ($\tilde{\mathrm{A}}$1) and ($\tilde{\mathrm{A}}$2) hold. Then for $f \in C^\beta(H)$, there exist a unique solution $v \in C^{\alpha+\beta}(H)$ to the Cauchy problem* (3.4) *and a constant K independent of f such that $|v|_{\alpha+\beta} \leq K|f|_\beta$.*

Proof. For $\theta \in [0,1]$, denote

$$L_\theta u(t,x) = \big(\mathcal{A}_x^{(\alpha)} + \theta\mathcal{B}_x^{(\alpha)}\big)u(t,x).$$

Let $\hat{C}^{\alpha+\beta}(H)$ be the space of functions $u \in C^{\alpha+\beta}(H)$ such that

$$u(t,x) = \int_0^t F(s,x)ds, \mathbb{P}-\text{a.s.}, \forall (t,x) \in H,$$

where $F = \partial_t u \in C^\beta(H)$. $\hat{C}^{\alpha+\beta}(H)$ is a Banach space with respect to the norm

$$|u|_{\hat{C}^{\alpha+\beta}} = |u|_{\alpha+\beta} + |\partial_t u|_\beta.$$

Consider the mappings $T_\theta : \hat{C}^{\alpha+\beta}(H) \to C^\beta(H)$ defined by

$$u(t,x) = \int_0^t \partial_t u(s,x)ds \mapsto \partial_t u(t,x) - L_\theta u(t,x).$$

Obviously, for some constant K independent of θ,

$$|T_\theta u|_\beta \le K|u|_{\hat{C}^{\alpha+\beta}}.$$

On the other hand, since

$$u(t,x) = \int_0^t \partial_t u(s,x)ds = \int_0^t \big[L_\theta u(t,x) + \big(\partial_t u(s,x) - L_\theta u(t,x)\big)\big]ds,$$

$$|L_\theta u|_\beta = |\big(\mathcal{A}_x^{(\alpha)} + \theta\mathcal{B}_x^{(\alpha)}\big)u|_\beta \le K\big(|u|_{\alpha+\beta} + |u|_\beta\big) \le K|u|_{\alpha+\beta},$$

and by Lemma 5.5,

$$|u|_{\alpha+\beta} \le K|T_\theta u|_\beta = K|\partial_t u - L_\theta u|_\beta,$$

then there exists a constant K independent of θ such that

$$\begin{aligned}
|u|_{\hat{C}^{\alpha+\beta}} &= |u|_{\alpha+\beta} + |\partial_t u|_\beta \\
&\le |u|_{\alpha+\beta} + |\partial_t u - L_\theta u|_\beta + |L_\theta u|_\beta \\
&\le K\big(|u|_{\alpha+\beta} + |\partial_t u - L_\theta u|_\beta\big) \\
&\le K|\partial_t u - L_\theta u|_\beta \\
&= K|T_\theta u|_\beta.
\end{aligned}$$

Since T_0 is an onto map, by Theorem 5.2 [34], all T_θ's are onto maps and the statement then follows. □

Theorem 5.7. *For $\alpha \in (0,2]$, $\beta \in \mathbb{R}^+$, $\beta \notin \mathbb{N}$, and a given time discretization $\{\tau\}_\delta$ with $\delta \in (0,1)$, let Y be the Euler approximation for the stochastic process X as defined in*

(5.1). *Assume* ($\tilde{A}1$) *and* ($\tilde{A}2$) *hold. Then there exists a constant* K *such that*

$$|\mathrm{E}[g(Y_T)] - \mathrm{E}[g(X_T)]| \leq K|g|_{\alpha+\beta}\delta^{\kappa(\alpha,\beta)}, \forall g \in C^{\alpha+\beta}(\mathbb{R}^d)$$

and

$$\big|\mathrm{E}\big[\int_0^T f(s, Y_{s-})ds\big] - \mathrm{E}\big[\int_0^T f(s, X_{s-})ds\big]\big| \leq K|f|_\beta \delta^{\kappa(\alpha,\beta)}, \forall f \in C^\beta(H)$$

where

$$\kappa(\alpha, \beta) = \begin{cases} \dfrac{\beta}{\alpha}, & \beta < \alpha, \\ 1, & \beta > \alpha. \end{cases}$$

Proof. Let $v \in C^{\alpha+\beta}(H)$ be the unique solution to (3.7).

By Lemma 3.4 and Lemma 5.4,

$$|\mathcal{A}^{(\alpha)}_{\cdot} v(s,x)|_\beta + |\mathcal{B}^{(\alpha)}_{\cdot} v(s,x)|_\beta \leq K|v|_{\alpha+\beta} \leq K\big(|f|_\beta + |g|_{\alpha+\beta}\big). \tag{5.4}$$

Hence, $\mathcal{A}^{(\alpha)}_{\cdot} v(s,x), \mathcal{B}^{(\alpha)}_{\cdot} v(s,x) \in C^\beta(\mathbb{R}^d)$, $\forall x \in \mathbb{R}^d$, $\forall s \in [0,T]$.

Then, by Itô's formula, Corollary 3.9, and (5.2), it follows that

$$\begin{aligned} \mathrm{E}[g(Y_T)] - \mathrm{E}[v(0, X_0)] &= \mathrm{E}[v(T, Y_T)] - \mathrm{E}[v(0, X_0)] \\ &= \mathrm{E}[v(T, Y_T)] - \mathrm{E}[v(0, Y_0)] \\ &= \mathrm{E}\Big[\int_0^T \big(\partial_t + \mathcal{A}^{(\alpha)}_{Y_{\tau_{i_s}}} + \mathcal{B}^{(\alpha)}_{Y_{\tau_{i_s}}}\big)v(s, Y_{s-})ds \\ &\qquad - \int_0^T \big(\partial_t + \mathcal{A}^{(\alpha)}_{Y_{s-}} + \mathcal{B}^{(\alpha)}_{Y_{s-}}\big)v(s, Y_{s-})ds \end{aligned}$$

$$
\begin{aligned}
&\quad + \int_0^T \big(\partial_t + \mathcal{A}^{(\alpha)}_{Y_{s-}} + \mathcal{B}^{(\alpha)}_{Y_{s-}}\big) v(s, Y_{s-}) ds\Big] \\
&= \mathrm{E}\Big[\int_0^T \Big(\big(\mathcal{A}^{(\alpha)}_{Y_{\tau_{i_s}}} v(s, Y_{s-}) - \mathcal{A}^{(\alpha)}_{Y_{s-}} v(s, Y_{s-})\big) \\
&\quad + \big(\mathcal{B}^{(\alpha)}_{Y_{\tau_{i_s}}} v(s, Y_{s-}) - \mathcal{B}^{(\alpha)}_{Y_{s-}} v(s, Y_{s-})\big)\Big) ds + \int_0^T f(s, Y_{s-}) ds\Big] \\
&= \mathrm{E}\Big[\int_0^T \Big(\mathrm{E}\big[\mathcal{A}^{(\alpha)}_{Y_{\tau_{i_s}}} v(s, Y_{s-}) - \mathcal{A}^{(\alpha)}_{Y_{s-}} v(s, Y_{s-}) | \tilde{\mathcal{F}}_{\tau_{i_s}}\big] \\
&\quad + \mathrm{E}\big[\mathcal{B}^{(\alpha)}_{Y_{\tau_{i_s}}} v(s, Y_{s-}) - \mathcal{B}^{(\alpha)}_{Y_{s-}} v(s, Y_{s-}) | \tilde{\mathcal{F}}_{\tau_{i_s}}\big]\Big) ds + \int_0^T f(s, Y_{s-}) ds\Big].
\end{aligned}
$$

Also, by Itô's formula and (3.7),

$$
\mathrm{E}[g(X_T)] - \mathrm{E}[v(0, X_0)] = \mathrm{E}[v(T, X_T)] - \mathrm{E}[v(0, X_0)] = \mathrm{E}\big[\int_0^T f(s, X_{s-}) ds\big].
$$

Hence,

$$
\begin{aligned}
&\big(\mathrm{E}[g(Y_T)] - \mathrm{E}[g(X_T)]\big) - \big(\mathrm{E}\big[\int_0^T f(s, Y_{s-}) ds\big] - \mathrm{E}\big[\int_0^T f(s, X_{s-}) ds\big]\big) \\
&= \mathrm{E}\Big[\int_0^T \Big(\mathrm{E}\big[\mathcal{A}^{(\alpha)}_{Y_{\tau_{i_s}}} v(s, Y_{s-}) - \mathcal{A}^{(\alpha)}_{Y_{s-}} v(s, Y_{s-}) | \tilde{\mathcal{F}}_{\tau_{i_s}}\big] \\
&\quad + \mathrm{E}\big[\mathcal{B}^{(\alpha)}_{Y_{\tau_{i_s}}} v(s, Y_{s-}) - \mathcal{B}^{(\alpha)}_{Y_{s-}} v(s, Y_{s-}) | \tilde{\mathcal{F}}_{\tau_{i_s}}\big]\Big) ds\Big].
\end{aligned}
$$

Then, by (5.4) and Lemma 4.2, if $f = 0$, there exists a constant K independent of g such that

$$
\big|\mathrm{E}[g(Y_T)] - \mathrm{E}[g(X_T)]\big| \leq K |g|_{\alpha+\beta} \delta^{\kappa(\alpha,\beta)}
$$

and if $g = 0$, there exists a constant K independent of f such that

$$
\big|\mathrm{E}\big[\int_0^T f(s, Y_{s-}) ds\big] - \mathrm{E}\big[\int_0^T f(s, X_{s-}) ds\big]\big| \leq K |f|_{\beta} \delta^{\kappa(\alpha,\beta)}.
$$

This proves the results. □

Chapter 6

Conclusion and Future Work

6.1 Conclusion

The weak Euler approximation for stochastic differential equations driven by Lévy processes has been studied. The model under consideration was in a more general form than existing ones, and hence applicable to a broader range of processes arising from various fields including sciences, engineering, and finance.

In order to investigate the convergence of the weak Euler approximation to the process considered, the existence of a unique solution to the corresponding integro-differential equation in Hölder space was proved.

It was then shown that the Euler scheme yields positive weak order of convergence, provided that the coefficients of the stochastic differential equation are Hölder-continuous and the test function is continuously differentiable to some positive order. In particular, if the coefficients are slightly more than twice differentiable and the test function has at least up to the fourth order derivative, then first weak order convergence is obtained. The assumptions on the regularity of coefficient and test functions are significantly weaker than those in the existing literature, while the same rate of convergence is achieved.

The result was further generalized to general Lévy-driven stochastic differential equations, with the Lévy operator $\mathcal{B}_z^{(\alpha)} u(t,x)$ of the form

$$\begin{aligned} \mathcal{B}_z^{(\alpha)} u(t,x) &= \mathbf{1}_{\{\alpha \in (1,2]\}} \big(c(z), \partial_x u(t,x)\big) \\ &\quad + \int_U \big[u(t,x+l(z,v)) - u(t,x) - \big(\mathbf{1}_{\{v \in U_1\}} \mathbf{1}_{\{\alpha \in (1,2]\}}\big) \big(\partial_x u(t,x), l(z,v)\big)\big] \\ &\quad \rho^{(\alpha)}(z,v) \nu^{(\alpha)}(dv), \end{aligned}$$

instead of

$$\begin{aligned} \mathcal{B}_z^{(\alpha)} u(t,x) &= \mathbf{1}_{\{\alpha \in (1,2]\}} \big(c(z), \partial_x u(t,x)\big) \\ &\quad + \int_{\mathbb{R}_0^d} \big[u(t,x+y) - u(t,x) - \big(\mathbf{1}_{\{|y| \le 1\}} \mathbf{1}_{\{\alpha \in (1,2]\}}\big) \big(\partial_x u(t,x), y\big)\big] \\ &\quad \rho^{(\alpha)}(z,y) \nu^{(\alpha)}(dy). \end{aligned}$$

The same rate of convergence was verified for the general equations [65].

6.2 Future Work

A straightforward step following the proof of the rate of convergence is to demonstrate the result numerically. In particular, when the jump part is a compound Poisson process or a stable process, the increments of the corresponding Lévy process can be exactly simulated. In other cases, the increments can be approximated, for example, by neglecting the small jumps to deliver a compound Poisson process, which can usually be simulated [4].

The stochastic differential equations considered so far are associated with non-degenerate Lévy operators. A further step is to study the case with degenerate operators. That is, (A1) does not hold. The uniqueness of the solution in this case has already been proved [62] while this problem has not been systematically studied in the literature and there could be much more challenges involved. A plausible conjecture is as follows:

Conjecture 6.1. *For $\alpha \in (0,2]$, $\beta \in (\alpha, \infty)$, $\beta \notin \mathbb{N}$, and a given time discretization $\{\tau\}_\delta$ with $\delta \in (0,1)$, let Y be the Euler approximation for the stochastic process X. Assume* (A2) *holds. Then there exists a constant K such that*

$$|\mathrm{E}[g(Y_T)] - \mathrm{E}[g(X_T)]| \leq K\delta^{\kappa(\alpha,\beta)}, \forall g \in C^\beta(\mathbb{R}^d),$$

where

$$\kappa(\alpha,\beta) = \begin{cases} \dfrac{\beta}{\alpha} - 1, & \beta \in (\alpha, 2\alpha), \\ 1, & \beta \in (2\alpha, \infty). \end{cases}$$

Bibliography

[1] Amin, K. I. 1993. Jump Diffusion Option Valuation in Discrete Time. *The Journal of Finance*. 48(5). pp. 1833-1863.

[2] Andersen, L. and J. Andreasen. 2000. Jump-Diffusion Processes: Volatility Smile Fitting and Numerical Methods for Option Pricing. *Review of Derivatives Research*. 4(3). pp. 231-262.

[3] Applebaum, D. 2004. *Lévy Processes and Stochastic Calculus*. Cambridge University Press.

[4] Asmussen, S. and J. Rosiński. 2001. Approximations of Small Jumps of Lévy Processes with a View Towards Simulation. *Journal of Applied Probability*. 38(2). pp. 482-493.

[5] Bachelier, L. 1900. Théorie de la spéculation. *Annales Scientifiques de l'École Normale Supérieure*. 3(17). pp. 21-86. English translation in Cootner, P. (Editor). 1964. *The Random Character of Stock Market Prices*. The MIT Press.

[6] Bakshi, G., C. Cao, and Z. Chen. 1997. Empirical Performance of Alternative Option Pricing Models. *The Journal of Finance*. 52(5). pp. 2003-2049.

[7] Bakshi, G., C. Cao, and Z. Chen. 2000. Pricing and Hedging Long-Term Options. *Journal of Econometrics*. 94(1-2). pp. 277-318.

[8] Bally, V. and D. Talay. 1996. The Law of the Euler Scheme for Stochastic Differential Equations I. Convergence Rate of the Distribution Function. *Probability Theory and Related Fields*. 104(1). pp. 43-60. Springer Berlin / Heidelberg.

[9] Bally, V. and D. Talay. 1996. The Law of the Euler Scheme for Stochastic Differential Equations II. Convergence Rate of the Density. *Monte Carlo Methods Applicaitons*. 2(2). pp. 93-128.

[10] Barndorff-Nielsen, O. E., T. Mikosch, and S. Resnick (Editors). 2001. *Lévy Processes: Theory and Applications*. Birkhäuser Boston.

[11] Barndorff-Nielsen, O. E. and N. Shephard. 2001. Modelling by Lévy Processes for Financial Econometrics. In Barndorff-Nielsen, O. E., T. Mikosch, and S. Resnick (Editors) *Lévy Processes: Theory and Applications*. Birkhäuser Boston.

[12] Bass, R. F. 2004. Stochastic Differential Equations Driven by Symmetric Stable Processes. *Séminaire de Probabilités*. XXXVI. *Lecture Notes in Mathematics*. 1801. pp. 302-313. Springer Berlin / Heidelberg.

[13] Bass, R. F., K. Burdzy, and Z. Q. Chen. 2004. Stochastic Differential Equations Driven by Stable Processes for Which Pathwise Uniqueness Fails. *Stochastic Processes and Their Applications*. 111(1). pp. 1-15.

[14] Bates, D. S. 1996. Jumps and Stochastic Volatility: Exchange Rate Processes Implicit in Deutsche Mark Options. *The Review of Financial Studies*. 9(1). pp. 69-107. The Society for Financial Studies.

[15] Bates, D. S. 2000. Post-'87 Crash Fears in the S&P 500 Futures Option Market. *Journal of Econometrics*. 94(1-2). pp. 181-238.

[16] Berestychi, J. 2004. Exchangeable Fragmentation-Coalescence Processes and Their Equilibrium Measures. *Electronic Journal of Probability*. 9. pp. 770-824. Institute of Mathematical Statistics.

[17] Bergh, J. and J. Löfström. 1976. *Interpolation Spaces: An Introduction*. Springer-Verlag.

[18] Bertoin, J. 1996. *Lévy Processes*. Cambridge University Press.

[19] Black, F. and M. Scholes. 1973. The Pricing of Options and Corporate Liabilities. *Journal of Political Economy*. 81(3). pp. 637-654.

[20] Broadie, M. and Ö. Kaya. 2006. Exact Simulation of Stochastic Volatility and Other Affine Jump Diffusion Processes. *Operations Research*. 54(2). pp. 217-231.

[21] Brockwell, P. J. 2001. Lévy-Driven Carma Processes. *Annals of the Institute of Statistical Mathematics*. 53(1). pp. 113-124. Springer Netherlands.

[22] Carr, P. and A. Hirsa. 2003. Why Be Backward? Forward Equations for American Options. *Risk*. 16(1). pp. 103-107.

[23] Carr, P. and D. B. Madan. 1999. Option Valuation Using the Fast Fourier Transform. *The Journal of Computatoinal Finance*. 2(4). pp. 61-73.

[24] Çinlar, E. and M. Pinsky. 1972. On Dams with Additive Inputs and a General Release Rule. *Journal of Applied Probability*. 9(2). pp. 422-429.

[25] Coifman, R. R. and Y. Meyer. 1978. Au delà des opérateurs pseudo-différentiels. *Astérisque*. 57. Société Mathématique de France. Paris.

[26] Cont, R. and P. Tankov. 2004. *Financial Modelling with Jump Processes*. Chapman & Hall/CRC.

[27] Cont, R. and E. Voltchkova. 2005. A Finte Difference Scheme for Option Pricing in Jump Diffusion and Exponential Lévy Processes. *SIAM Journal on Numerical Analysis*. 43(4). pp. 1596-1626. Society for Industrial and Applied Mathematics.

[28] Dellacherie, C. and P. A. Meyer. 1975. *Probabilités et potentiel*. Hermann Paris.

[29] Duffie, D., J. Pan, and K. Singleton. 2000. Transform Analysis and Asset Pricing for Affine Jump-Diffusions. *Econometrica*. 68(6). pp. 1343-1376. The Econometric Society.

[30] Einstein, A. 1905. Über die von der molekularkinetischen Theorie der Wärme geforderte Bewegung von in ruhenden Flüssigkeiten suspendierten Teilchen. *Annalen der Physik*. 322(8). pp. 549-560.

[31] Ekström, E. and J. Tysk. 2007. Properties of Option Prices in Models with Jumps. *Mathematical Finance*. 17(3). pp. 381-397. Blackwell.

[32] Feng, L. and V. Linetsky. 2008. Pricing Options in Jump-Diffusion Models: an Extrapolation Approach. *Operations Research*. 56(2). pp. 304-325.

[33] Fristedt, B. 1974. Sample Functions of Stochastic Processes with Stationary, Independent Increments. In Ney, P. and S. Port (Editors) *Advances in Probability and Related Topics*. 3. pp. 241-396. Marcel Dekker New York.

[34] Gilbarg, D. and N. S. Trudinger. 2001. *Elliptic Partial Differential Equations of Second Order*. Springer.

[35] Glasserman, P. 2003. *Monte Carlo Methods in Financial Engineering*. In series *Stochastic Modelling and Applied Probability*. 53. Springer.

[36] Goldie, C. 1967. A Class of Infinitely Divisible Random Variables. *Mathematical Proceedings of the Cambridge Philosophical Society*. 63(4). pp. 1141-1143. Cambridge University Press.

[37] Guyon, J. 2006. Euler Scheme and Tempered Distributions. *Stochastic Processes and their Applications*. 116(6). pp. 877-904. Elsevier.

[38] d'Halluin, Y., P. A. Forsyth, and K. R. Vetzal. 2005. Robust Numerical Methods for Contingent Claims under Jump Diffusion Processes. *IMA Journal of Numerical Analysis*. 25(1). pp. 87-112.

[39] Heston, S. T. 1993. A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options. *Review of Financial Studies*. 6(2). pp. 327-343. Oxford University Press.

[40] Hirsa, A. and D. B. Madan. 2003. Pricing American Options under Variance Gamma. *The Journal of Computatoinal Finance*. 7(2). pp. 63-80.

[41] Hu, Y. Z. 1996. Strong and Weak Order of Time Discretization Schemes of Stochastic Differential Equations. *Séminaire de Probabilités*. XXX. *Lecture Notes in Mathematics*. 1626. pp. 218-227. Springer Berlin / Heidelberg.

[42] Jacod, J. 2004. The Euler Scheme for Lévy Driven Stochastic Differential Equations: Limit Theorems. *The Annals of Probability*. 32(3). pp. 1830-1872.

[43] Jacod, J. and P. Protter. 1991. Une remarque sur les équations différentielles stochastiques à solutions Markoviennes. *Séminaire de Probabilités*. XXV. *Lecture Notes in Mathematics*. 1485. pp. 138-139. Springer Berlin / Heidelberg.

[44] Jacod, J. and A. N. Shiryaev. 2003. *Limit Theorems for Stochastic Processes* (2nd edition). Springer.

[45] Jarrow, R. A. and D. Madan. 1995. Option Pricing Using the Term Structure of Interest Rates to Hedge Systematic Discontinuities in Asset Returns. *Mathematical Finance*. 5(4). pp. 311-336. Wiley Periodicals.

[46] Jarrow, R. A. and E. R. Rosenfeld. 1984. Jump Risks and the Intertemporal Capital Asset Pricing Model. *Journal of Business*. 57(3). pp. 337-351. University of Chicago Press.

[47] Johannes, M. 2004. The Statistical and Economic Role of Jumps in Continuous-Time Interest Rate Models. *The Journal of Finance*. 59(1). pp. 227-260. American Finance Association.

[48] Kloeden, P. E. and E. Platen. 2000. *Numerical Solution of Stochastic Differential Equations*. Springer.

[49] Kloeden, P. E., E. Platen, and N. Hofmann. 1995. Extrapolation Methods for the Weak Approximation of Itô Diffusions. *SIAM Journal on Numerical Analysis*. 32(5). pp. 1519-1534. Society for Industrial and Applied Mathematics.

[50] Komatsu, T. 1982. On the Pathwise Uniqueness of Solutions of One-Dimensional Stochastic Differential Equations of Jump Type. *Proceedings of the Japan Academy, Series A, Mathematical Sciences*. 58(8). pp. 353-356.

[51] Komatsu, T. 1984. On the Martingale Problem for Generators of Stable Processes with Perturbations. *Osaka Journal of Mathematics*. 22(1). pp. 113-132. Osaka University and Osaka City University.

[52] Kou, S. G. 2002. A Jump-Diffusion Model for Option Pricing. *Management Science*. 48(8). pp. 1086-1101.

[53] Kubilius, K. 1990. On the Rate of Convergence of the Distributions of Semimartingales to the Distribution of Stable Process. In Grigelionis B. (Editor) *Probability Theory and Mathematical Statistics*. 2. pp. 22-34.

[54] Kubilius, K. and E. Platen. 2001. Rate of Weak Convergence of the Euler Approximation for Diffusion Processes with Jumps. *Research Paper Series*. 54. Quantitative Finance Research Centre. University of Technology. Sydney.

[55] LeGall, J. F. 1983. Applications du temps local aux équations différentielles stochastiques unidimensionnelles. *Séminaire de Probabilités*. XVII. *Lecture Notes in Mathematics*. 986. pp. 15-32. Springer Berlin / Heidelberg.

[56] Liu, J., F. Longstaff, and J. Pan. 2003. Dynamic Asset Allocation with Event Risk. *The Journal of Finance*. 58(1). pp. 231-259. Blackwell Publishing.

[57] Merton, R. C. 1976. Option Pricing when Underlying Stock Returns are Discontinuous. *Journal of Financial Economics*. 3(1-2). pp. 125-144.

[58] Mikulevičius, R. and E. Platen. 1988. Time Discrete Taylor Approximations for Itŏ Processes with Jump Component. *Mathematische Nachrichten*. 138. pp. 93-104. Wiley.

[59] Mikulevičius, R. and E. Platen. 1991. Rate of Convergence of the Euler Approximation for Diffusion Processes. *Mathematische Nachrichten*. 151(1). pp. 233-239. Wiley.

[60] Mikulevičius, R. and H. Pragarauskas. 1992. On the Cauchy Problem for Certain Integro-Differential Operators in Sobolev and Hölder Spaces. *Lithuanian Mathematical Journal*. 32(2). pp. 238-264. Springer New York.

[61] Mikulevičius, R. and H. Pragarauskas. 1992. On the Martingale Problem Associated with Nondegenerate Lévy Operators. *Lithuanian Mathematical Journal*. 32(3). pp. 297-311. Springer New York.

[62] Mikulevičius, R. and H. Pragarauskas. 1993. On the Uniqueness of Solution to a Martingale Problem Associated with a Degenerate Lévy's Operator. *Lithuanian Mathematical Journal*. 33(4). pp. 352-367. Springer New York.

[63] Mikulevičius, R. and H. Pragarauskas. 2009. On Hölder Solutions of the Integro-Differential Zakai Equation. *Stochastic Processes and their Applications*. 119(10). pp. 3319-3355. Elsevier.

[64] Mikulevičius, R. and C. Zhang. 2010. On the Rate of Convergence of Weak Euler Approximation for Nondegenerate Itô Diffusion and Jump Processes. *arXiv*:1007.2914. Submitted to *Stochastic Processes and their Applications*. pp. 1-33. Elsevier.

[65] Mikulevičius, R. and C. Zhang. 2010. On the Rate of Convergence of Weak Euler Approximation for Non-degenerate SDEs. *arXiv*:1009.4728. pp. 1-40.

[66] Milstein, G. N. 1979. A Method of Second-Order Accuracy Integration of Stochastic Differential Equations. *Theory of Probability and its Applications*. 23(2). pp. 396-401. Society for Industrial and Applied Mathematics.

[67] Milstein, G. N. 1986. Weak Approximation of Solutions of Systems of Stochastic Differential Equations. *Theory of Probability and its Applications*. 30(4). pp. 750-766. Society for Industrial and Applied Mathematics.

[68] Milstein, G. N. 1995. *Numerical Integration of Stochastic Differential Equations*. In series *Mathematics and Its Applications*. 313. Springer.

[69] Moran, P. A. P. 1969. A Theory of Dams with Continuous Input and a General Release Rule. *Journal of Applied Probability*. 6. pp. 88-98.

[70] Naik, V. and M. Lee. 1990. General Equilibrium Pricing of Options on the Market Portfolio with Discontinuous Returns. *The Review of Financial Studies*. 3(4). pp. 493-521. Oxford University Press.

[71] Nakao, S. 1972. On the Pathwise Uniqueness of Solutions of One-Dimensional Stochastic Differential Equations. *Osaka Journal of Mathematics*. 9(3). pp. 513-518.

[72] Øksendal, B. K. 2003. *Stochastic Differential Equations: An Introduction with Applications* (6th Edition). Springer.

[73] Platen, E. 1999. An Introduction to Numerical Methods for Stochastic Differential Equations. *Acta Numerica*. 8. pp. 197-246. Cambridge University Press.

[74] Protter, P. E. 2003. *Stochastic Integration and Differential Equations* (2nd Edition). In series *Stochastic Modelling and Applied Probability*. 21. Springer.

[75] Protter, P. E. and D. Talay. 1997. The Euler Scheme for Lévy Driven Stochastic Differential Equations. *The Annals of Probability*. 25(1). pp. 393-423.

[76] Rubenthaler, S. 2003. Numerical Simulation of the Solution of a Stochastic Differential Equation Driven by a Lévy Process. *Stochastic Processes and their Applications*. 103(2). pp. 311-349. Elsevier.

[77] Saito, Y. and T. Mitsui. 1996. Stability Analysis of Numerical Schemes for Stochastic Differential Equations. *SIAM Journal on Numerical Analysis*. 33(6). pp. 2254-2267. Society for Industrial and Applied Mathematics.

[78] Sato, K. I. 1999. *Lévy Processes and Infinitely Divisible Distributions*. In series *Cambridge Studies in Advanced Mathematics*. 68. Cambridge University Press.

[79] von Smoluchowski, M. 1906. Zur kinetischen Theorie der Brownschen Molekularbewegung und der Suspensionen. *Annalen der Physik*. 326(14). pp. 756-780.

[80] Schoutens, W. 2003. *Lévy Processes in Finance: Pricing Financial Derivatives*. Wiley New York.

[81] Schurz, H. 1997. *Stability, Stationarity, and Boundedness of Some Implicit Numerical Methods for Stochastic Differential Equations and Applications*. Logos Verlag Berlin.

[82] Skorohod. A. V. 1991. *Random Processes with Independent Increments*. Springer. (Russian original: 1964. Nauka Moscow.)

[83] Sobczyk, K. 1991. *Stochastic Differential Equations with Applications to Physics and Engineering*. In series *Mathematics and its Applications*. 40. Kluwer Academic Publishers.

[84] Stein, E. M. 1971. *Singular Integrals and Differentiability Properties of Functions*. Princeton University Press.

[85] Stuck, B. W. and B. Kleiner, B. 1974. A Statistical Analysis of Telephone Noise. *The Bell System Technical Journal*. 53(7). pp. 1263-1320.

[86] Talay, D. 1984. Efficient Numerical Schemes for the Approximation of Expectations of Functionals of the Solution of a S.D.E., and Applications. *Filtering and Control of Random Processes. Lecture Notes in Control and Information Sciences*. 61. pp. 294-313. Springer Berlin / Heidelberg.

[87] Talay, D. 1986. Discretization of a Stochastic Differential Equation and Rough Estimate of the Expectations of Functionals of the Solution. *ESAIM: Mathematical Modelling and Numerical Analysis - Modélisation Mathématique et Analyse Numérique*. 20(1). pp. 141-179.

[88] Talay, D. 1995. Simulation of Stochastic Differential Systems. *Lecture Notes in Physics*. 451. pp. 54-96. Springer Berlin / Heidelberg.

[89] Talay, D. and L. Tubaro. 1990. Expansion of the Global Error for Numerical Schemes Solving Stochastic Differential Equations. *Stochastic Analysis and Applications*. 8(4). pp. 483-509.

[90] Tavella, D. and C. Randall. 2000. *Pricing Financial Instruments: The Finite Difference Method*. Wiley New York.

[91] Triebel, H. 1983. *Theory of Function Spaces*. Birkhaueser Verlag.

[92] Wolpert, R. L. and M. S. Taqqu. 2005. Fractional Ornstein-Uhlenbeck Lévy Processes and the Telecom Process: Upstairs and Downstairs. *Signal Processes*. 85(8). pp. 1523-1545. Elsevier North-Holland.

[93] Yamada, T. and S. Watanabe. 1971. On the Uniqueness of Solutions of Stochastic Differential Equations. *Journal of Mathematics of Kyoto University*. 11(1). pp. 155-167.

[94] Zhang, X. L. 1997. Numerical Analysis of American Option Pricing in a Jump-Diffusion Model. *Mathematics of Operations Research*. 22(3). pp. 668-690.

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