1. Suppose there are N total tuna in the seas surrounding Alaska. You gently capture, tag and release k of them. A week later you come back and catch r tuna. There are m of these tuna with one of your tags (and $r - m$ without).

   (a) What is the probability, p_N, of this event?

   (b) If we were to solve $p_N/p_{N-1} \geq 1$ (see Example 2.42) we would find this is true only when $N \leq kr/m$. Say in words what this means.

   (c) Notice that $k/(kr/m) = m/r$. Explain why having these two ratios equal makes sense as a guess for N.

   (d) In order to estimate the size of the population of bluegills (a species of fresh water fish) in a small lake in Missouri, a total of 250 bluegills are captured and tagged and then released. After allowing sufficient time for the tagged fish to disperse, a sample of 150 bluegills were caught. It was found that 16 bluegills in the sample were tagged. Estimate the size of the bluegill population in this lake.

2. This question is about a concept called *thinning*—where we take a random variable and make it smaller.

   (a) Fix numbers $0 < q, p < 1$ and a positive integer $n > 0$. Let $Y = \text{binomial}(n, p) = \sum_1^n X_i$ with $X_i = \text{bernouli}(p)$. Suppose $W = \text{binomial}(Y, q)$ and $X_i' = \text{bernouli}(q)$ for $i = 1, 2, \ldots, n$. Use the random variables X_i and X_i' to explain why $W = \text{binomial}(n, pq)$. *Hint: Set $\tilde{X}_i = X_i X_i'$. What is the distribution of $\tilde{X}_i$?*

   (b) You are picking blackberries on a sunny day. It is bright, so you can't see the berries that well. You pick 100 of them. Each is ripe independently with probability 9/10. Each has a bit of spiderweb on it with probability 1/9. The presence of a bit of web is independent of the ripeness and webs on other berries. What is the probability you pick at least 98 ripe berries with no spiderwebs?

   (c) Set $Z = \text{Poisson}(\lambda)$ and suppose $M = \text{binomial}(Z, q)$. We can't prove it with the material covered in this class, but an important property called *Poisson thinning* is that $M = \text{Poisson}(\lambda q)$. The United States have on average 20 shark attacks per year. The probability of a shark attack being fatal is 1/5. What is the probability of at least 3 deaths by shark attack this year in the United States?

   (d) The most amazing thing (in my opinion) about the Poisson random variable is that if we start with $Z = \text{Poisson}(\lambda)$, then apply a multinomial thinning of Z with parameters $p_1, \ldots, p_n$ then the number put in group i is distributed like $Z_i = \text{Poisson}(\lambda p_1)$. Moreover, the Z_i are **independent**. No other random variable has this independence property! Suppose we place $\text{Poisson}(n)$ balls uniformly randomly into n bins. What is the distribution for the number of balls inside of the first bin?

(e) Let Q be the number of empty bins after placing all Poisson(n) balls. Explain why Poisson thinning ensures $Q = \text{binomial}(n, p)$, and find p.

(f) Suppose n balls are placed uniformly into n bins. Find the probability, q, that the first bin is empty after this. Explain why the number of empty bins is not a binomial(n, q) random variable.

3. You keep playing a game with a 9/10 chance of losing \$1 and a 1/10 chance of winning a dollar. Let $X_i \in \{-1, 1\}$ be the amount of money you win on the ith try and $S_n = \sum_{i=1}^{n} X_i$ be your total winnings/losses.

   (a) Let E_n be the event that $S_{2n} = 0$. What is $P(E_n)$?

   (b) Let A be the event that there exists a time t such that $S_t = 0$. Write A in terms of the events E_n, and explain why.

(c) Use a union bound and the previous part to give an upper bound on $P(A)$.

(d) Use the fact that $C_{2n,n} \leq 4^n$ and that $\sum_{n=1}^{\infty} a^n = \frac{1}{1-a} - 1$ to estimate the sum in the previous part.

(e) Say in words what it means that $P(A) < 1$. Is this surprising to you?

Answers to Homework 2

4. $P_{30,5}$
9. (a) 17576000 (b) .63
16. .1
21. (a) 120 (b) 60 (c) 50400 (d) 34650
24. .054
34. .3548
39. .3297
47. .4232
56. 14/33
61. (a) .41 (b) .51 (c) .077
68. (a) .07 (b) Two seniors, two juniors and one sophomore
81. .3439