WHITE NOISE METHODS FOR ANTICIPATING STOCHASTIC DIFFERENTIAL EQUATIONS

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Table of Contents

Acknowledgments ii

Abstract v

Chapter 1 Introduction 1

Chapter 2 Background 3
- 2.1 Brownian Motion 3
- 2.2 Conditional Expectation 7
- 2.3 Martingale 8
- 2.4 White Noise Analysis 9
- 2.5 White Noise Differential Operator 14

Chapter 3 Stochastic Differential Equations 16
- 3.1 General Theory 16
- 3.2 Stochastic Integrals 17
- 3.3 Important Tools 23
- 3.4 Solution of Stochastic Differential Equations 28

Chapter 4 Anticipating Stochastic Differential Equations 31
- 4.1 The White Noise Methods 31
- 4.2 Some Examples 34
- 4.3 A Class of Linear Equations 39
- 4.4 Application 45

References 47

Appendix: Permission 49

Vita 52

Abstract

This dissertation focuses on linear stochastic differential equations of anticipating type. Owing to the lack of a theory of differentiation for random processes, the said differential equations are appropriately understood and studied as anticipating stochastic integral equations.

The unfolding work considers equations in which anticipation arises either from the initial condition or the integrand. In this regard, the techniques of white noise analysis are applied to such equations. In particular, by using the Hitsuda-Skorokhod integral which nicely extends the Itô integral to anticipating integrands, we then apply the S-transform from white noise analysis to study this new equation.

Chapter 1

Introduction

The revolutionary field of (Itô)-stochastic integration is an outgrowth of K. Itô's desire to construct diffusion processes to solve stochastic differential equations. Let $B(t)$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$ and let $\{\mathcal{F}_t; a \leq t \leq b\}$ be a filtration such that $B(t)$ is $\mathcal{F}_t$-measurable for each t and for any $s \leq t$, the random variable $B(t) - B(s)$ is independent of the σ-field $\mathcal{F}_s$. Under this setting,

$$dX(t) = f(t, X(t))dB(t) + g(t, X(t))dt$$

with $0 \leq a \leq t \leq b < \infty$, and initial condition $X(a)$, represents a linear stochastic differential equation, which is most appropriately understood as

$$X(t) = X(a) + \int_a^t f(s, X(s))dB(s) + \int_a^t g(s, X(s))ds \tag{1}$$

for $0 \leq a \leq t \leq b < \infty$, a linear stochastic integral equation, also known as an Itô process, provided the initial condition $X(a)$ and the integrand f are measurable with respect to the underlying filtration and f is in $L^2_{ad}([a, b] \times \Omega)$, that is f is a stochastic processes $f(t, \omega)$, $a \leq t \leq b$, $\omega \in \Omega$ such that $f(t)$ is $\mathcal{F}_t$-adapted and $\int_a^b E|f(t)|^2 \, dt < \infty$.

This dissertation is broken into 3 main chapters. It focuses on solutions of anticipating equations, which are the consequence of relaxing the measurability condition. First, the preliminary notions from probability theory and white noise analysis are discuseed in Chapter 2. The general theory and tools for studying stochastic differential equations are presented in Chapter 3. This chapter also includes a discussion of the motivation for considering anticipating stochastic differential equations.

The last chapter discusses the white noise methods for anticipating equations and includes the main results. It focuses on a class of linear equations in which anticpation arises from both the initial conditions and the integrand. The results presented allow us to capture more examples and solutions within the class of anticipating linear stochastic differential equations.

Chapter 2

Background

In this chapter, we present some of the foundational ideas from probability theory and white noise analysis, to be exploited later on. These notions are well developed in [14], [15] and also articulated in [16] and [18].

2.1 Brownian Motion

Definition 2.1. A collection $X = \{X(t,\omega); t \in T, \omega \in \Omega\}$ of random variables defined on a probability space $(\Omega,\mathcal{F},P)$ with index set T is known as a *stochastic process.*

A stochastic process can be regarded as a measurable function $X(t,\omega)$ defined on the product space $[0,\infty) \times \Omega$ such that

1. for fixed t, $X(t,\cdot)$ is a random variable;
2. for fixed ω, $X(\cdot,\omega)$ is a function of t.

If there is no confusion, we denote $X(t,\omega)$ by $X(t)$ or X_t, suppressing ω as needed.

Definition 2.2. A stochastic process $B(t,\omega)$, satisfying the conditions

1. $P\{\omega : B(0,\omega) = 0\} = 1$.
2. For any $0 \leq s < t$, the random variable $B(t) - B(s)$ is normally distributed with mean zero and variance $t - s$, that is, for any $a < b$,
$$P\{a \leq B(t) - B(s) \leq b\} = \frac{1}{\sqrt{2\pi(t-s)}} \int_a^b e^{-\frac{x^2}{2(t-s)}}\, dx.$$
3. $B(t,\omega)$ has independent increments, that is, for any $0 \leq t_1 < t_2 < \cdots < t_n$, the random variables $B(t_1), B(t_2) - B(t_1), \ldots, B(t_n) - B(t_{n-1})$ are independent.

4. Almost all sample paths of $B(t, \omega)$ are continuous functions, that is,

$$P\{\omega : B(\cdot, \omega) \text{ is a continuous function of } t\} = 1$$

is called a *Brownian motion*.

Example 2.3. Let $C[0,1]$ be the Banach space of real-valued continuous functions x on $[0,1]$ with $x(0) = 0$ and the norm given by $\|x\|_\infty = \sup_{0 \le t \le 1} |x(t)|$. Consider the mapping μ given by

$$\mu(A) = \int_U \prod_{i=1}^{n} \left(\frac{1}{\sqrt{2\pi(t_i - t_{i-1})}} \exp\left[-\frac{(u_i - u_{i-1})^2}{2(t_i - t_{i-1})} \right] \right) du_1 \dots du_n,$$

where $U \in \mathcal{B}(\mathbb{R}^n)$, the Borel σ-field of $\mathbb{R}^n$ and A is a set (called a *cylinder set*) of the form

$$A = \{x \in C[0,1]; \big(x(t_1), x(t_2), \dots, x(t_n)\big) \in U\},$$

where $0 < t_1 < t_2 < \cdots < t_n \le 1$. Then $\big(C[0,1], \mathcal{B}(C[0,1]), \mu\big)$ is a probability space and the stochastic process defined by

$$B(t, x) = x(t), \quad 0 \le t \le 1, \quad x \in C[0,1],$$

is a Brownian motion, a construction due to Norbert Wiener [13].

Theorem 2.4. *Let $B(t)$ be a Brownian motion. Then for any $s, t \ge 0$, we have $E[B(s)B(t)] = \min\{s, t\}$.*

By this theorem and the definition of Brownian motion, we see that a stochastic process $X(t)$, $t \ge 0$ which is normally distributed with mean zero and variance t and satisfying $E[X(s)X(t)] = \min\{s, t\}$, is a Brownian motion.

Theorem 2.5. *[3] The path of a Brownian motion is nowhere differentiable almost surely.*

Proof. We follow Breiman, in a 4-step proof:

Step 1: Fix $\beta > 0$ and suppose that a Brownian motion path $x(t)$ has a derivative $x'(t)$, where

$$|x'(s)| < \beta, \quad \text{for some} s \in [0,1].$$

By definition of the derivative, there exists n_0 such that for all $n > n_0$,

$$\left|\frac{x(t)-x(s)}{t-s}\right| \le 2\beta, \text{ if } |t-s| \le \frac{2}{n}.$$

In other words,

$$|x(t)-x(s)| \le 2\beta|t-s|, \text{ if } |t-s| \le \frac{2}{n}.$$

Step 2: With $x(t)$ a Brownian motion path on $[0,1]$, let

$$A_n = \left\{x(\cdot); \exists s \in [0,1] \text{ such that } |x(t)-x(s)| \le 2\beta|t-s|, \text{ if } |t-s| \le \frac{2}{n}\right\}.$$

We note that $A_n \subset A_{n+1} \subset \cdots$. In fact as $n \to \infty$, A_n increases to some limit set A, where A is the set of Brownian motion paths having a derivative at some $s \in [0,1]$, with $\|x'(s)\| \le \beta$.

In the next few steps, we will show that $P(A) = 0$.

Step 3: Let $x \in A_n$ and let k be the largest integer for which $\frac{k}{n} \le s$, that is $\frac{k+1}{n} > s$. Then we have

$$\begin{aligned} y_k(s) &= \max\left\{\left|x\left(\frac{k+2}{n}\right) - x\left(\frac{k+1}{n}\right)\right|, \left|x\left(\frac{k+1}{n}\right) - x\left(\frac{k}{n}\right)\right|,\right. \\ &\qquad \left.\left|x\left(\frac{k}{n}\right) - x\left(\frac{k-1}{n}\right)\right|\right\} \\ &\le \frac{6\beta}{n}, \end{aligned}$$

since

$$\begin{aligned} \left|x\left(\frac{k+2}{n}\right) - x\left(\frac{k+1}{n}\right)\right| &\le \left|x\left(\frac{k+2}{n}\right) - x(s)\right| + \left|x\left(\frac{k+1}{n}\right) - x(s)\right| \\ &\le \frac{4\beta}{n} + \frac{2\beta}{n}. \end{aligned}$$

For $k = 1, 2, 3, \ldots, n-2$, let

$$B_n = \left\{ x(\cdot)\,;\ \text{for at least one } k,\ y_k(x) \leq \frac{6\beta}{n} \right\}.$$

Then we see that $x \in A_n$ implies $x \in B_n$, that is $A_n \subset B_n$.

Step 4: We will show that $\lim_{n\to\infty} P(B_n) = 0$, whence we have $\lim_{n\to\infty} P(A_n) = 0$, that is $P(A) = 0$.

First of all, note that

$$B_n = \bigcup_{k=1}^{n-2} \left\{ x(\cdot)\ :\ y_k(x) \leq \frac{6\beta}{n} \right\}.$$

Consequently

$$\begin{aligned}
P(B_n) &\leq \sum_{k=1}^{n-2} P\left(\left\{ x(\cdot)\ :\ y_k(x) \leq \frac{6\beta}{n} \right\} \right) \\
&= \sum_{k=1}^{n-2} P\Bigg(\max\left\{ \left| x\left(\frac{k+2}{n}\right) - x\left(\frac{k+1}{n}\right) \right|, \left| x\left(\frac{k+1}{n}\right) - x\left(\frac{k}{n}\right) \right|, \right. \\
&\qquad\qquad \left. \left| x\left(\frac{k}{n}\right) - x\left(\frac{k-1}{n}\right) \right| \right\} \leq \frac{6\beta}{n} \Bigg) \\
&\leq n\, P\Bigg(\max\left\{ \left| x\left(\frac{3}{n}\right) - x\left(\frac{2}{n}\right) \right|, \left| x\left(\frac{2}{n}\right) - x\left(\frac{1}{n}\right) \right|, \right. \\
&\qquad\qquad \left. \left| x\left(\frac{1}{n}\right) - x\left(\frac{0}{n}\right) \right| \right\} \leq \frac{6\beta}{n} \Bigg) \\
&\leq n \left(P\left\{ \left| x\left(\frac{1}{n}\right) - x(0) \right| \right\} \leq \frac{6\beta}{n} \right)^3, \text{ (by independence)} \\
&= n \left[P\left| x\left(\frac{1}{n}\right) \right| \leq \frac{6\beta}{n} \right]^3, \text{ (since } x(0) = 0\text{)}
\end{aligned}$$

$$
\begin{aligned}
&= n\left[P\left|N\left(0,\frac{1}{n}\right)\right| \leq \frac{6\beta}{n}\right]^3 \\
&= n\left[\frac{1}{\sqrt{2\pi\frac{1}{n}}}\int_{-\frac{6\beta}{n}}^{\frac{6\beta}{n}} e^{-\frac{x^2}{2\frac{1}{n}}}\,dx\right]^3 \\
&= n\left[\sqrt{\frac{n}{2\pi}}\int_{-\frac{6\beta}{n}}^{\frac{6\beta}{n}} e^{\frac{nx^2}{2}}\,dx\right]^3 \\
&= n\left[\frac{1}{\sqrt{2\pi n}}\int_{-6\beta}^{6\beta} e^{-\frac{V^2}{2n}}\,dv\right]^3 \quad \text{where } V = nx \\
&\leq C\frac{1}{\sqrt{n}} \longrightarrow 0.
\end{aligned}
$$

Thus $P(B_n) = 0$, and the proof is complete. □

2.2 Conditional Expectation

Definition 2.6. Let X be an integrable random variable in a probability space $(\Omega,\mathcal{F},P)$ and let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-field of $\mathcal{F}$. The *conditional expectation of* X *given* $\mathcal{G}$ is the unique random variable Y such that

1. Y is $\mathcal{G}$-measurable.
2. For all $A \in \mathcal{G}$, $\int_A Y\,dP = \int_A X\,dP$.

We usually write $Y = E[X|\mathcal{G}]$.

Remark 2.7. The existence and uniqueness of the conditional expectation follow from the *Radon-Nikodym Theorem.*

Theorem 2.8. (Radon-Nikodym Theorem) *Suppose* $(\Omega,\mathcal{F},P)$ *is a probability space. Let* μ *be a signed measure (namely* $\mu : \Omega \to [-\infty,\infty]$ *is a* σ*-additive function on* $(\Omega,\mathcal{F})$ *such that* $\mu(\phi) = 0$ *for null set* ϕ*) which is absolutely continuous with respect to* P*. Then there exists a unique integrable function* f *such that*

$$\mu(A) = \int_A f\,dP, \quad A \in \mathcal{F}.$$

Remark 2.9. The function f is called the *density* or the *Radon-Nikodym derivative* of μ with respect to P. We write $f = \dfrac{d\mu}{dP}$.

Here are some simple properties of conditional expectation.

Theorem 2.10. *Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathcal{G}$ be a sub-σ-field of $\mathcal{F}$ and $X \in L^1(\Omega, \mathcal{F})$. Then each of the following hold almost surely:*

(a) $E\big(E[X|\mathcal{G}]\big) = EX$.

(b) *If X is $\mathcal{G}$-measurable, then* $E[X|\,\mathcal{G}] = X$.

(c) *If X and $\mathcal{G}$ are independent, then* $E[X|\,\mathcal{G}] = EX$.

(d) *If Y is $\mathcal{G}$-measurable and $E|\,XY| < \infty$, then* $E[XY|\,\mathcal{G}] = YE[X|\,\mathcal{G}]$.

(e) *(Tower Law) If $\mathcal{H}$ is a sub-σ-field of $\mathcal{G}$, then* $E\big[E[X|\mathcal{G}]\big|\mathcal{H}\big] = E[X|\mathcal{H}]$.

(f) *(Jensen's Inequality) If φ is a convex function on $\mathbb{R}$ and both $E|X|$ and $E|\varphi(X)|$ are finite with respect to P, then* $\varphi(E|X|\,\mathcal{G}]) \leq E[\varphi(X)|\,\mathcal{G}]$.

(g) *Let $X_n \geq 0$, $X_n \in L^1(\Omega)$, $n = 1, 2, \ldots$, and assume that $\liminf_{n\to\infty} X_n \in L^1(\Omega)$. Then* $E\left[\liminf_{n\to\infty} X_n \Big| \mathcal{G}\right] \leq \liminf_{n\to\infty} E[X_n|\mathcal{G}]$.

2.3 Martingale

Brownian motion exemplifies an important property of stochastic processes, the martingale property.

Definition 2.11. Let T be either $\mathbb{Z}_+$ (the set of positive integers) or an interval in $\mathbb{R}$. A *filtration* on T is an increasing family $\{\mathcal{F}_t : t \in T\}$ of σ-fields. A stochastic process $\{X_t; t \in T\}$ is said to be *adapted* to the filtration $\{\mathcal{F}_t : t \in T\}$ if for each t, the random variable X_t is $\mathcal{F}_t$-measurable.

Remark 2.12. We will assume that all σ-fields $\mathcal{F}_t$ are *complete*, that is if $A \in \mathcal{F}_t$ and $P(A) = 0$, then $B \in \mathcal{F}_t$ for any subset B of A.

Definition 2.13. For a filtration $\{\mathcal{F}_t : t \in T\}$ on a probability space $(\Omega, \mathcal{F}, P)$, we define $\mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_s$ for any $t \in T$. We say that the filtration $\{\mathcal{F}_t : t \in T\}$ is *right continuous* if $\mathcal{F}_{t+} = \mathcal{F}_t$ for every $t \in T$. In particular, if $t \in [a, b]$, a filtration $\{\mathcal{F}_t; a \leq t \leq b\}$ is said to be *right continuous* if $\mathcal{F}_t = \bigcap_{n=1}^{\infty} \mathcal{F}_{t+\frac{1}{n}}$ for all $t \in [a, b)$, where by convention $\mathcal{F}_t = \mathcal{F}_b$ when $t > b$.

Definition 2.14. Let X_t be a stochastic process adapted to a filtration $\{\mathcal{F}_t : t \in T\}$ and $E|X_t| < \infty$ for all $t \in T$. Then X_t is a *martingale* with respect to $\{\mathcal{F}_t\}$ if for any $s \leq t$ in T,

$$E\{X_t|\mathcal{F}_s\} = X_s, \qquad \text{almost surely.} \tag{2.2}$$

Remark 2.15. If the filtration is not explicitly specified, then the filtration $\{\mathcal{F}_t\}$ is understood to be the one given by $\mathcal{F}_t = \sigma\{X_s; s \leq t\}$.

Remark 2.16. If the equality in Equation 2.2 is replaced by $\geq$ (or $\leq$), then X_t is called a *submartingale* (or *supermartingale*) with respect to $\{\mathcal{F}_t\}$.

Example 2.17. As stated in the prelude, $B(t)$ is a martingale. Indeed, for $s < t$,

$$\begin{aligned} E[B(t)|\,\mathcal{F}_s] &= E[(B(t) - B(s)) + B(s)|\,\mathcal{F}_s] \\ &= E[B(t) - B(s)|\,\mathcal{F}_s] + E[B(s)|\,\mathcal{F}_s] \\ &= E[B(t) - B(s)] + B(s) \\ &= B(s). \end{aligned}$$

2.4 White Noise Analysis

Since the path of a Brownian motion is nowhere differentiable almost surely, it is obvious that $B'(t) = \dot{B}(t)$ does not exist as a function. However, $X(t) = \dot{B}(t) =$

white noise is understood to be a generalized stochastic process X_ξ with index $\xi \in \mathcal{S}(\mathbb{R})$ (test functions) such that:

1. $\xi \mapsto X_\xi$ is linear
2. $X_\xi \sim N\left(0, \int_{\mathbb{R}} \xi^2(t)dt\right)$.

We now turn our attention to basic notions from white noise analysis. For this, we follow [14]. First, we recall:

Theorem 2.18 (Minlos' Theorem)**.** *Let V be a nuclear space. A function $F : V \to \mathbb{C}$ is a characteristic function of a probability measure on V' (the dual of V) $\Longleftrightarrow$*

1. *$F(0) = 1$*
2. *F is continuous*
3. *F is positive definite .*

Let $V = \mathcal{S}(\mathbb{R})$ the Schwartz space of real-valued rapidly decreasing functions on $\mathbb{R}$, namely for any integers $n, k \geq 0$, $\lim_{|x|\to\infty} x^k f^{(n)}(x) = 0$.

That is

$$V = \left\{ f \ : \ \forall n, k \in \mathbb{Z} \ \lim_{|x|\to\infty} x^k f^{(n)}(x) = 0 \right\} = \mathcal{S}(\mathbb{R}).$$

Consider

$$F(f) = e^{-\frac{1}{2}\int_{\mathbb{R}} f(t)^2 dt} \quad , \quad f \in V.$$

By Minlos' Theorem, there is a unique probability measure μ on $\mathcal{S}'(\mathbb{R})$ such that

$$\int_{\mathcal{S}'(\mathbb{R})} e^{i\langle x, f\rangle} d\mu(x) = e^{-\frac{1}{2}|f|_0^2} \quad , \quad f \in \mathcal{S}(\mathbb{R}).$$

The pair $(\mathcal{S}'(\mathbb{R}), \mu)$ is called the *white noise space*.

More generally, let E be a separable Hilbert space with norm $|\cdot|_0$. Let A be a densely defined self-adjoint operator on E, whose eigenvalues $\{\lambda_n\}_{n\geq 1}$ satisfy the following conditions:

- $1 < \lambda_1 \leq \lambda_2 \leq \cdots,$
- $\sum_{n=1}^{\infty} \lambda_n^{-2} < \infty.$

For any $p \geq 0$, let $\mathcal{E}_p$ be the completion of E with respect to the norm $|\, f\,|_p := |\, A^p f\,|_0$. Note that $\mathcal{E}_p$ is a Hilbert space with the norm $|\cdot|_p$, and $\mathcal{E}_p \subset \mathcal{E}_q$ for all $p \geq q$. The second condition on the eigenvalues above implies that the inclusion map $i : \mathcal{E}_{p+1} \to \mathcal{E}_p$ is a Hilbert-Schmidt operator.

Let

$$\mathcal{E} = \text{projective limit of } \{\mathcal{E}_p : p \geq 0\},$$

$$\mathcal{E}' = \text{be the dual of } \mathcal{E}.$$

The space $\mathcal{E} = \cap_{p\geq 0}\mathcal{E}_p$ equipped with the topology given by the family $\{|\cdot|_p\}_{p\geq 0}$ of semi-norms is a nuclear space. Hence $\mathcal{E} \subset E \subset \mathcal{E}'$ is a Gel'fand triple with the following continuous inclusions:

$$\mathcal{E} \subset \mathcal{E}_q \subset \mathcal{E}_p \subset E \subset \mathcal{E}'_p \subset \mathcal{E}'_q \subset \mathcal{E}', \quad q \geq p \geq 0.$$

The Riesz Representation Theorem is used to identify the dual of E with itself.

It can be shown that for all $p \geq 0$, the dual space $\mathcal{E}'_p$ is isomorphic to $\mathcal{E}_{-p}$, which is the completion of the space E with respect to the norm $|f|_{-p} = |A^{-p}f|_0$.

Minlos' theorem allows us to define a unique probability measure μ on the Borel subsets of $\mathcal{E}'$ with the property that for all $f \in \mathcal{E}$, the random variable $\langle \cdot, f\rangle$ is normally distributed with mean 0 and variance $|f|_0^2$. Here $\langle\ ,\ \rangle$ denotes the duality between $\mathcal{E}'$ and $\mathcal{E}$. Consequently, the characteristic functional of μ is given by

$$\int_{\mathcal{E}'} e^{i\langle x,\xi\rangle}\, d\mu(x) = e^{-\frac{1}{2}|\mathcal{E}|_0^2}, \quad \forall \xi \in \mathcal{E}. \tag{2.3}$$

The probability space $(\mathcal{E}', \mu)$ is called the *white noise space*. The space $L^2(\mathcal{E}', \mu)$, also denoted by (L^2), is the set of functions $\varphi : \mathcal{E}' \to \mathbb{C}$ such that φ is measurable and $\int_{\mathcal{E}'} |\varphi(x)|^2 \, d\mu(x) < \infty$. If we denote by E_c the complexification of E, the Wiener-Itô Theorem allows us to associate to each $\varphi \in (L^2)$ a unique sequence $\{f_n\}_{n\geq 0}$, $f_n \in E_c^{\hat{\otimes} n}$ and express φ as $\varphi = \sum_{n=0}^{\infty} I_n(f_n)$ where $I_n(f_n)$ is a multiple Wiener integral of order n. This decomposition is similar to what is referred to as the "Fock-Space decomposition" as shown in [20].

The (L^2)-norm $\|\varphi\|_0$ of φ is given by

$$\|\varphi\|_0^2 = \left(\sum_{n=0}^{\infty} n! |f_n|_p^2 \right)^{\frac{1}{2}}.$$

Let

$$(\mathcal{E}_p) = \{\varphi \in (L^2); \|\varphi\|_p < \infty\}.$$

If $0 < p \leq q$, then $(\mathcal{E}_q) \subset (\mathcal{E}_p)$ with the property that for any $q \geq 0$, there exists $p > q$ such that the inclusion map $I_{p,q} : (\mathcal{E}_p) \hookrightarrow (\mathcal{E}_q)$ is a Hilbert-Schmidt operator and $\|I_{p,q}\|_{HS}^2 \leq (1 - \|i_{p,q}\|_{HS}^2)^{-1}$ where $i_{p,q}$ is the inclusion map from $\mathcal{E}_p$ into $\mathcal{E}_q$ as noted earlier on.

Analogous to the way $\mathcal{E}$ was defined, we also define

$$(\mathcal{E}) = \text{the projective limit of } \{(\mathcal{E}_p); p \geq 0\}$$

$$(\mathcal{E})^* = \text{the dual space of } (\mathcal{E}).$$

With the above result, $(\mathcal{E}) = \cap_{p\geq 0}(\mathcal{E}_p)$ with the topology generated by the family $\{\|\cdot\|_p\,; p \geq 0\}$ of norms. It is a nuclear space forming the infinite dimensional Gel'fand triple $(\mathcal{E}) \subset (L^2) \subset (\mathcal{E})^*$. Moreover we have the following continuous inclusions

$$(\mathcal{E}) \subset (\mathcal{E}_q) \subset (\mathcal{E}_p) \subset (L^2) \subset (\mathcal{E}_{-p}) \subset (\mathcal{E}_{-q}) \subset (\mathcal{E})^*, \quad q \geq p \geq 0.$$

The elements in $(\mathcal{E})$ are called *test functions* on $\mathcal{E}'$ while the elements in $(\mathcal{E})^*$ are called *generalized functions* on $\mathcal{E}'$. The bilinear pairing between $(\mathcal{E})$ and $(\mathcal{E})^*$ is denoted by $\langle\langle \cdot, \cdot \rangle\rangle$. If $\varphi \in (L^2)$ and $\psi \in (\mathcal{E})$ then $\langle\langle \varphi, \psi \rangle\rangle = (\varphi, \overline{\psi})$, where $(\cdot, \cdot)$ is the inner product on the complex Hilbert space (L^2).

An element $\varphi \in (\mathcal{E})$ has a unique representation as $\varphi = \sum_{n=0}^{\infty} I_n(f_n)$, $f_n \in \mathcal{E}_c^{\hat{\otimes} n}$ with the norm

$$\|\varphi\|_p^2 = \sum_{n=0}^{\infty} n! |f_n|_p^2 < \infty, \quad \forall p \geq 0.$$

Similarly, an element $\phi \in (\mathcal{E})^*$ can be written as $\phi = \sum_{n=0}^{\infty} I_n(F_n)$, $F_n \in (\mathcal{E}_c')^{\hat{\otimes} n}$ with the norm

$$\|\phi\|_{-p}^2 = \sum_{n=0}^{\infty} n! |F_n|_{-p}^2.$$

The bilinear pairing between ϕ and φ is then represented as

$$\langle\langle \phi, \varphi \rangle\rangle = \sum_{n=0}^{\infty} n! \langle F_n, f_n \rangle.$$

Kondratiev and Streit ([14], Chapter 4) constructed a wider Gel'fand triple than the one above in the following way: Let $0 \leq \beta < 1$ be a fixed number. For each $p \geq 0$, define

$$\|\varphi\|_{p,\beta} = \left(\sum_{n=0}^{\infty} (n!)^{1+\beta} |(A^p)^{\otimes n} f_n|_0^2 \right)^{1/2}$$

and let

$$(\mathcal{E}_p)_\beta = \{\varphi \in (L^2); \|\varphi\|_{p,\beta} < \infty\}.$$

It should be noted here that $(\mathcal{E}_0)_\beta \neq (L^2)$ unless $\beta = 0$. Similarly to the above setting, for any $p \geq q \geq 0$, we have $(\mathcal{E}_p)_\beta \subset (\mathcal{E}_q)_\beta$ and the inclusion map $(\mathcal{E}_{p+\frac{\alpha}{2}})_\beta \hookrightarrow (\mathcal{E}_p)_\beta$ is a Hilbert-Schmidt operator for some positive constant α.

Let

$$(\mathcal{E})_\beta = \text{the projective limit of } \{(\mathcal{E}_p)_\beta; p \geq 0\}$$

$$(\mathcal{E})_\beta^* = \text{the dual space of } (\mathcal{E})_\beta.$$

Then $(\mathcal{E})_\beta$ is a nuclear space and we have the Gel'fand triple $(\mathcal{E})_\beta \subset (L^2) \subset (\mathcal{E})^*_\beta$ with the following continuous inclusions:

$$(\mathcal{E})_\beta \subset (\mathcal{E}_p)_\beta \subset (L^2) \subset (\mathcal{E}_p)^*_\beta \subset (\mathcal{E})^*_\beta, \quad p \geq 0,$$

where the norm on $(\mathcal{E}_p)^*_\beta$ is given by

$$\|\varphi\|_{-p,-\beta} = \left(\sum_{n=0}^{\infty} (n!)^{1-\beta} |(A^{-p})^{\otimes n} f_n|_0^2 \right)^{1/2}.$$

Thus we have the following relationship with the earlier Gel'fand triple:

$$(\mathcal{E})_\beta \subset (\mathcal{E}) \subset (L^2) \subset (\mathcal{E})^* \subset (\mathcal{E})^*_\beta.$$

2.5 White Noise Differential Operator

The nondifferentiability of Brownian motion is well-established in Stochastic Analysis [3]. Within the white noise analysis setting, it is possible to define a differential operator. In fact, if $y \in \mathcal{E}'$ and $\varphi(x) = \langle : x^{\otimes n} :, f \rangle \in (\mathcal{E})_\beta$ is a test function, [14] asserts that the directional derivative

$$\begin{aligned} \lim_{t \to 0} \frac{\varphi(x+ty) - \varphi(x)}{t} &= n\langle : x^{\otimes(n-1)} :, \hat{\otimes} y, f \rangle \\ &= n\langle : x^{\otimes(n-1)} :, \langle y, f \rangle \rangle \\ &= n\langle : x^{\otimes(n-1)} :, y \hat{\otimes}_1 f \rangle, \end{aligned}$$

where $y\hat{\otimes}_1 : E_c^{\hat{\otimes} n} \to E_c^{\hat{\otimes}(n-1)}$ is the unique continuous linear map given by

$$y \hat{\otimes}_1 g^{\otimes n} = \langle y, g \rangle g^{\otimes(n-1)}, \quad g \in E_c.$$

Wherever it occurs, $: x :$ represents the renormalization of x. It follows that the function φ has a Gateaux derivative $D_y \varphi$ in any direction $y \in \mathcal{E}'$. The preceding inspires a definition of an operator D_y on $(\mathcal{E})_\beta$. Indeed, for $\varphi(x) = \sum_{n=0}^{\infty}\langle : x^{\otimes} : , f_n\rangle \in (\mathcal{E})_\beta$ with $f_n \in \mathcal{E}_c^{\hat{\otimes} n}$

$$\begin{aligned} D_y\varphi(x) &\equiv \sum_{n=1} n\langle : x^{\otimes(n-1)} :, y\hat{\otimes}_1 f_n\rangle \\ &\equiv \sum_{n=1} n\langle : x^{\otimes(n-1)} :, \langle f_n\rangle\rangle. \end{aligned}$$

The fact that D_y is a continuous linear differential operator on $(\mathcal{E})_\beta$ is establised in section 9.1 of [14]. Moreover, the duality between $(\mathcal{E})^*_\beta$ and $(\mathcal{E})_\beta$ allows one to define the adjoint operator D^*_y of D_y, namely

$$\langle\langle D^*_y\Phi, \varphi\rangle\rangle = \langle\langle \Phi, D_y\varphi\rangle\rangle, \quad \Phi \in (\mathcal{E})^*, \varphi \in (\mathcal{E}).$$

In particular, let $\mathcal{E}$ represent the Schwartz space of all real-valued infinitely differentiable functions f, such that for any $n, k \in \mathbb{N}$,

$$\sup_{x\in\mathbb{R}} \left| x^n \left(\frac{d^k}{dx^k}\right) f(x)\right| < \infty.$$

In addition, let $y = \delta_t$, the Dirac delta function at t. Then

- $\partial_t \equiv D_{\delta_t}$ is known as the *white noise* (or Hida) differential operator. It is also known as the annihilation operator;
- Its adjoint $\partial^*_t \equiv D^*_{\delta_t}$ is known as the creation operator.
- If $\varphi \in (\mathcal{S})_\beta$, then $\dot{B}(t)\varphi \equiv \partial_t\varphi + \partial^*_t\varphi$ is known as white noise multiplication.
- The adjoint operator and the S-transform have a particularly nice relationship. The definition of the S-transform is found in chapter 4, but we state the said relationship here (see Theorem 9.13 in [14]):

  Theorem 2.19. *If* $\Phi \in (\mathcal{S})^*_\beta$, *then*

  $$(S\partial^*_t\Phi)(\xi) = \xi(t)S\Phi(\xi), \quad \xi \in \mathcal{S}_c.$$

Chapter 3

Stochastic Differential Equations

This chapter involves a discussion of the general theory on stochastic differential equations. In particular, we reference some of the important tools in the study of these equations, as well as their solution.

3.1 General Theory

In the deterministic setting, an ordinary differential equation

$$\frac{dx}{dt} = f(t, x); \quad x(a) = x_a$$

may be viewed as an integral equation

$$x(t) = x_a + \int_a^t f(s, x(s))\, ds.$$

Within the stochastic framework resulting from the introduction of a "noise" term, we obtain an equation of the form

$$dX_t = f(t, X)\, dB(t) + g(t, X)\, dt. \tag{3.4}$$

Unlike its deterministic counterpart, the only proper mathematical understanding of Equation (3.4) is to interpret it as the corresponding integral equation:

$$X_t = X_a + \int_a^t f(s, X(s))\, dB(s) + \int_a^t g(s, X(s))\, ds.$$

Consequently, the study of stochastic differential equations, is in fact the study of stochastic integral equations.

Let us review the various types of stochastic integrals, as well as their construction. For details, the reader may consult the book [15]

3.2 Stochastic Integrals

1. Wiener Integral

Let f be a real-valued square integrable function on $[a,b]$, that is, $f \in L^2[a,b]$. Then the integral

$$\int_a^b f(t)dB(t,\omega), \quad f \in L^2[a,b],$$

is called a *Wiener integral*.

The integrals $\int_0^1 e^t dB(t)$, $\int_0^1 t\sin(\frac{1}{t})\, dB(t)$ and $\int_0^1 t\, dB(t)$ are examples of Wiener integrals.

Remark 3.1. Let $C[0,1]$ be the set of real-valued continuous functions $x(t)$ on the interval $[0,1]$ with $x(0)=0$. The integral on $C[0,1]$ with respect to the Wiener measure w in $C[0,1]$ is called a Wiener integral. The Wiener measure w is defined by

$$\begin{aligned} w(I) &= \frac{1}{\sqrt{(2\pi)^n t_1(t_2-t_1)\cdots(t_n-t_{n-1})}} \\ &\quad \int_E \exp\left[-\frac{1}{2}\left(\frac{u_1^2}{t_1}+\frac{(u_2-u_1)^2}{t_2-t_1}+\cdots+\frac{(u_n-u_{n-1})^2}{t_n-t_{n-1}}\right)\right] du_1du_2\cdots du_n, \end{aligned}$$

where E is a Borel subset of $\mathbb{R}^n$ and I is the cylinder set $I=\{x \in C[0,1] : (x(t_1),\ldots,x(t_n)) \in E\}$ for $0 < t_1 < t_2 < \cdots < t_n \le 1$ [13].

Let $L^2(\Omega)$ denote the Hilbert space of square integrable real-valued random variables on Ω with inner product $\langle X,Y\rangle = E(XY)$. We outline the construction of the Wiener integral $\int_a^b f(t)\, dB(t,\omega)$.

Step 1. *f is a step function*

Suppose f is a step function given by

$$f = \sum_{i=1}^n a_i 1_{[t_{i-1},t_i)},$$

where $a = t_0 < t_i < \ldots < t_n = b$. Define

$$I_{step}(f) = \sum_{i=1}^{n} a_i \left(B(t_i) - B(t_{i-1}) \right).$$

Then I_{step} is linear and the random variable $I_{step}(f)$ is Gaussian with mean zero and variance $E\left[\{I_{step}(f)\}^2\right] = \int_a^b f(t)^2 \, dt$.

Step 2. $\underline{f \in L^2[a,b]}$

Choose a sequence $\{f_n\}_{n=1}^{\infty}$ of step functions such that f_n approaches f in $L^2[a,b]$. The sequence $\{I_{step}(f_n)\}_{n=1}^{\infty}$ is Cauchy in $L^2(\Omega)$, hence it is convergent in $L^2(\Omega)$. We set

$$I(f) = \lim_{n \to \infty} I_{step}(f_n) \quad \text{in } L^2(\Omega),$$

and write

$$I(f)(\omega) = \left(\int_a^b f(t) \, dB(t) \right)(\omega), \quad \omega \in \Omega, \quad \text{almost surely.}$$

This $I(f)$ is the Wiener integral. We also denote it by $\int_a^b f(t) \, dB(t, \omega)$ or just $\int_a^b f(t) \, dB(t)$.

Theorem 3.2. *For each $f \in L^2[a,b]$, the Wiener integral $\int_a^b f(t) \, dB(t)$ is a Gaussian random variable with mean zero and variance $\|f\|^2_{L^2[a,b]} = \int_a^b |f(t)|^2 dt$.*

Example 3.3. The (Wiener) integral $\int_0^1 t^2 \, dB(t)$ is a Gaussian random variable with mean zero and variance $\int_0^1 t^4 \, dt = \frac{1}{5}$.

It is easy to check that $I : L^2[a,b] \to L^2(\Omega)$ is a linear transformation, whence we have the following:

Corollary 3.4. *If $f, g \in L^2[a,b]$, then*

$$E[I(f)I(g)] = \int_a^b f(t)g(t) \, dt.$$

2. Itô Integral

Suppose $B(t)$ is a Brownian motion, and let $\{\mathcal{F}_t; a \leq t \leq b\}$ be a filtration such that

(a) for each t, $B(t)$ is $\mathcal{F}_t$-measurable,

(b) for any $s \leq t$, the random variable $B(t) - B(s)$ is independent of the σ-field $\mathcal{F}_s$.

Let $L^2_{ad}\left([a, b] \times \Omega\right)$ denote the space of all stochastic processes $f(t, \omega)$, $a \leq t \leq b$, $\omega \in \Omega$, satisfying

(i) $f(t)$ is adapted to the filtration $\{\mathcal{F}_t\}$;

(ii) $\int_a^b E|\, f(t)|^2 dt < \infty$.

Remark 3.5. Note that by *Fubini's theorem* (see [2]) , condition (ii) can also be written as $E \int_a^b |\, f(t)|^{\,2}\, dt < \infty$.

The stochastic integral

$$\int_a^b f(t, \omega)\, dB(t, \omega), \quad f \in L^2_{ad}\left([a, b] \times \Omega\right)$$

is called an *Itô integral*. For convenience, we suppress the ω and we just write $\int_a^b f(t)\, dB(t)$. Before presenting some examples, let us consider the construction of the Itô integral.

Step 1. *f is a step stochastic process in $L^2_{ad}([a, b] \times \Omega)$*

Suppose f is a step stochastic process given by

$$f(t, \omega) = \sum_{i=1}^{n} \xi_{i-1}(\omega)\, 1_{[t_{i-1}, t_i)}(t),$$

where ξ_{i-1} is $\mathcal{F}_{i-1}$-measurable and $E\big[(\xi_{i-1})^2\big] < \infty$. Define

$$I_{step}(f) = \sum_{i=1}^{n} \xi_{i-1}\big(B(t_i) - B(t_{i-1})\big).$$

Then I_{step} is linear, $E[I_{step}(f)] = 0$ and

$$E\big[|\, I_{step}(f)|^2\big] = \int_a^b E|\, f(t)|^2 \, dt.$$

Step 2. *Approximation of $f \in L^2_{ad}([a,b] \times \Omega)$ by step stochastic processes*

Suppose $f \in L^2_{ad}([a,b] \times \Omega)$. Then there exists a sequence $\{f_n(t); n \geq 1\}$ of step stochastic processes in $L^2_{ad}([a,b] \times \Omega)$ such that

$$\lim_{n\to\infty} \int_a^b E\big[|\, f(t) - f_n(t)|^2\big] \, dt = 0,$$

i.e., $f_n \longrightarrow f$ in $L^2_{ad}([a,b] \times \Omega)$.

Step 3. $f \in L^2_{ad}([a,b] \times \Omega)$

By Steps 1 and 2, there exists a sequence $\{f_n(t,\omega); n \geq 1\}$ of adapted step stochastic processes such that

$$\lim_{n,m\to\infty} E\big(|\, I_{step}(f_n) - I_{step}(f_m)|^{\,2}\big) = 0.$$

Hence the sequence $\{I_{step}(f_n)\}$ is Cauchy in $L^2(\Omega)$. For $f \in L^2_{ad}([a,b] \times \Omega)$, define

$$I(f) = \lim_{n\to\infty} I_{step}(f_n), \quad \text{in } L^2(\Omega).$$

Then denote $I(f,\omega) = \int_a^b f(t,\omega)\, dB(t,\omega)$ for $f \in L^2_{ad}([a,b] \times \Omega)$.

Remark 3.6. For a deterministic function $f(t)$, the Itô integral $\int_a^b f(t)\, dB(t,\omega)$ agrees with the Wiener integral defined in section 3.2.

Example 3.7. Let $f(t,\omega) = B(t,\omega)$. Since $B(t)$ is adapted to the filtration $\{\mathcal{F}_t\}$, it follows that $f(t)$ is $\mathcal{F}_t$-adapted. Also

$$\int_a^b E|\, B(t)|^2 \, dt = \int_a^b t \, dt = \frac{1}{2}(b^2 - a^2) < \infty.$$

So $\int_a^b B(t)\, dB(t)$ is an Itô integral. In fact it can be shown that

$$\int_a^b B(t)\, dB(t) = \frac{1}{2}\left(B(b)^2 - B(a)^2 - (b-a)\right). \tag{3.5}$$

Example 3.8. The integral $\int_a^b e^{B(t)}\,dB(t)$ is an Itô integral because $e^{B(t)}$ is $\mathcal{F}_t$-adapted and

$$\begin{aligned} E\left|\,e^{2B(t)}\right| &= \int_{-\infty}^{\infty} e^{2x}\frac{1}{\sqrt{2\pi t}}\,e^{-\frac{x^2}{2t}}\,dx \\ &= \int_{-\infty}^{\infty} e^{2t}\frac{1}{\sqrt{2\pi t}}\,e^{-\frac{(x-2t)^2}{2t}}\,dx \\ &= e^{2t}, \end{aligned}$$

thus $\int_a^b E\left|\,e^{B(t)}\right|^2 dt = \int_a^b e^{2t}dt = \frac{1}{2}\left(e^{2b}-e^{2a}\right) < \infty$.

Theorem 3.9. *Suppose that* $f \in L^2_{ad}([a,b]\times\Omega)$. *Then the Itô integral* $I(f) = \int_a^b f(t)\,dB(t)$ *is a random variable with mean* $E[I(f)] = 0$ *and variance*

$$E\left(|\,I(f)|^{\,2}\right) = \int_a^b E|\,f(t)|^{\,2}\,dt.$$

Example 3.10. Consider $f(t) = sgn(B(t))$. Since

$$\int_a^b E|\,sgn(B(t))|^2\,dt = \int_a^b E(1)\,dt = b-a < \infty,$$

it follows that $f(t) = sgn(B(t)) \in L^2_{ad}([a,b]\times\Omega)$. By Theorem 3.9, the random variable $\int_a^b sgn(B(t))\,dB(t)$ has mean 0 and variance $\int_a^b E|\,sgn(B(t))|^2\,dt = b-a$.

Suppose that $f \in L^2_{ad}([a,b]\times\Omega)$. Then for any $t \in [a,b]$, $\int_a^t E|\,f(t)|^{\,2}\,dt \le \int_a^b E|\,f(t)|^{\,2}\,dt < \infty$. So $f \in L^2_{ad}([a,t]\times\Omega)$ and the integral $\int_a^t f(s)\,dB(s)$ is well-defined. Consider a stochastic process given by

$$X_t = \int_a^t f(s)\,dB(s), \quad a \le t \le b.$$

Note that by Theorem 3.9, we have

$$E\left(|X_t|^2\right) = E\left|\int_a^t f(s)\,dB(s)\right|^2 \le \int_a^b E|\,f(s)|^2\,ds < \infty.$$

So by the Schwartz Theorem, $E|\,X_t\,| \le \left[E\left(|\,X_t\,|^{\,2}\right)\right]^{1/2} < \infty$. Hence for each t, the random variable X_t is integrable.

The next two theorems establish the martingale and continuity properties of the Itô integral.

Theorem 3.11. *Suppose* $f \in L^2_{ad}([a,b]\times\Omega)$*. Then the stochastic process*

$$X_t = \int_a^t f(s)\,dB(s), \quad a \le t \le b$$

is a martingale with respect to the filtration $\{\mathcal{F}_t : a \le t \le b\}$*.*

Example 3.12. The stochastic processes $\int_a^t B(s)\,dB(s)$ and $\int_a^t e^{B(s)}dB(s)$ are martingales.

Theorem 3.13. *Suppose* $f \in L^2_{ad}([a,b]\times\Omega)$*. Then the stochastic process*

$$X_t = \int_a^t f(s)\,dB(s), \quad a \le t \le b$$

is continuous, that is, almost all its sample paths are continuous functions on $[a,b]$*.*

Example 3.14. Consider $f(t) = sgn(B(t))$. In Example 3.10 we showed that $f(t) = sgn(B(t)) \in L^2_{ad}([a,b]\times\Omega)$. Therefore $X_t = \int_a^t sgn(B(s))\,dB(s)$, $a \le t \le b$, is a continuous martingale by Theorems 3.11 and 3.13.

According to Theorem 3.11, if $f \in L^2_{ad}([a,b]\times\Omega)$, the stochastic process $X_t = \int_a^t f(s)\,dB(s)$, $a \le t \le b$, is a martingale with respect to the filtration $\{\mathcal{F}_t\}$. The converse is also true, namely, any $\mathcal{F}_t$-martingale can be represented as an Itô integral. In particular we have the following result due to Itô (see Theorem 4.3.3 in [21]).

Theorem 3.15. *Let* $F \in L^2(\mathcal{F}_T, P)$*, then there exists a stochastic process* $f \in L^2_{ad}([0,T]\times\Omega)$ *such that*

$$F = E[F] + \int_0^T f(t)\,dB(t).$$

3.3 Important Tools

The study of stochastic differential equations is greatly facilitated by the following facts, details of which may be found in [15].

1. The Itô Formula

In ordinary calculus, we deal with deterministic functions. One of the most important rules in differentiation is the Chain Rule, which states that for any differentiable functions f and g, the composite function $f \circ g$ is also differentiable and

$$\frac{d}{dt}(f \circ g)(t) = \frac{d}{dt}f(g(t)) = f'(g(t))g'(t).$$

In terms of the Fundamental Theorem of Calculus, we have

$$f(g(t)) - f(g(a)) = \int_a^t f'(g(s))g'(s)\, ds.$$

In Itô calculus, the focus shifts to random functions, that is, stochastic processes and we have the counterpart of the above Chain Rule. One must note that there is no differentiation theory in Itô calculus since almost all sample paths of a Brownian motion $B(t)$ are nowhere differentiable (Theorem 2.5). Nevertheless we have the integral version known as the *Itô formula* or the *change of variables formula*. In this section, we will see several versions of Itô's formula. For the proofs, the reader may refer to [15].

Let $B(t)$ be a Brownian motion. We start with the simplest form of the Itô formula.

Theorem 3.16. *Let f be a C^2-function, that is, f is twice differentiable and f'' is continuous. Then*

$$f(B(t)) - f(B(a)) = \int_a^t f'(B(s))\, dB(s) + \frac{1}{2}\int_a^t f''(B(s))\, ds. \tag{3.6}$$

Remark 3.17. The first integral on the right is an Itô integral as defined in Section 3.2 and the second integral is a Riemann integral for each sample path of $B(s)$.

Remark 3.18. The extra term $\frac{1}{2}\int_a^t f''(B(s))\,ds$ is a consequence of the nonzero quadratic variation of the Brownian motion $B(t)$. This extra term distinguishes Itô calculus from ordinary calculus.

Example 3.19. Let $f(x) = x^2$. Then by Equation 3.6, we get

$$B(t)^2 - B(a)^2 = 2\int_a^t B(s)\,dB(s) + (t-a).$$

Hence

$$\int_a^t B(s)\,dB(s) = \frac{1}{2}\left[B(t)^2 - B(a)^2 - (t-a)\right].$$

This agrees with Equation (3.5) in Example 3.7 with $b = t$.

Example 3.20. Let $f(x) = x^3$. Then by Equation 3.6,

$$B(t)^3 = 3\int_0^t B(s)^2\,dB(s) + 3\int_0^t B(s)\,ds.$$

So,

$$\int_0^t B(s)^2\,dB(s) = \frac{1}{3}B(t)^3 - \int_0^t B(s)\,ds.$$

Example 3.21. Let $f(x) = e^{x^2}$. Then by Equation 3.6,

$$e^{B(t)^2} - 1 = 2\int_0^t B(s)e^{B(s)^2}\,dB(s) - \int_0^t \left(e^{B(s)^2} + 2B(s)^2 e^{B(s)^2}\right)\,ds.$$

Now consider a function $f(t,x)$ of x and t. Set $x = B(t,\omega)$ to get a stochastic process $f(t,B(t))$. Notice that now t appears in two places: as a variable of f and in the Brownian motion $B(t)$. For the first t, we can apply ordinary calculus. For the second t in $B(t)$, we need to use Itô calculus. This leads to the second version of Itô's formula:

Theorem 3.22. *Let $f(t,x)$ be a continuous function which has continuous partial derivatives $\frac{\partial f}{\partial t}$, $\frac{\partial f}{\partial x}$ and $\frac{\partial^2 f}{\partial x^2}$. Then*

$$\begin{aligned} f(t,B(t)) &= f(a,B(a)) + \int_a^t \frac{\partial f}{\partial x}(s,B(s))\,dB(s) + \int_a^t \frac{\partial f}{\partial s}(s,B(s))\,ds \\ &\quad + \frac{1}{2}\int_a^t \frac{\partial^2 f}{\partial x^2}(s,B(s))\,ds. \end{aligned} \tag{3.7}$$

Example 3.23. Let $f(t,x) = x^2 - t$. Then by Equation 3.7,

$$\begin{aligned} B(t)^2 - t &= (B(a)^2 - a) + \int_a^t 2B(s)\,dB(s) + \int_a^t (-1)\,ds + \frac{1}{2}\int_a^t 2\,ds \\ &= B(a)^2 - a + 2\int_0^t B(s)\,dB(s) - (t-a) + (t-a) \end{aligned}$$

which gives

$$\int_a^t B(s)\,dB(s) = \frac{1}{2}\left[B(t)^2 - B(a)^2 - (t-a)\right],$$

which is the same as in Example 3.19.

Example 3.24. Let $f(t,x) = e^{x-\frac{1}{2}t}$. Then by Equation 3.7,

$$\begin{aligned} e^{B(t)-\frac{1}{2}t} &= 1 + \int_0^t e^{B(s)-\frac{1}{2}s}\,dB(s) - \frac{1}{2}\int_0^t e^{B(s)-\frac{1}{2}s}\,ds + \frac{1}{2}\int_0^t e^{B(s)-\frac{1}{2}s}\,ds \\ &= 1 + \int_0^t e^{B(s)-\frac{1}{2}s}\,dB(s). \end{aligned}$$

Note that by Theorem 3.11, $e^{B(t)-\frac{1}{2}t}$ is a martingale.

Now let $\{\mathcal{F}_t; a \le t \le b\}$ be a filtration as specified for Itô integrals in Section 3.2, namely

(a) for each t, $B(t)$ is $\mathcal{F}_t$-measurable,

(b) for any $s < t$, the random variable $B(t) - B(s)$ is independent of the σ-field $\mathcal{F}_s$.

Note that $\mathcal{L}_{ad}(\Omega, L^2[a,b])$ consists of all $\mathcal{F}_t$-adapted stochastic processes $f(t)$ such that $\int_a^b |\, f(t)|^2\, dt < \infty$ almost surely. Moreover, $\mathcal{L}_{ad}(\Omega, L^1[a,b])$ is the class of all $\mathcal{F}_t$-adapted stochastic processes $f(t)$ such that $\int_a^b |\, f(t)|\, dt < \infty$ almost surely.

Definition 3.25. An *Itô process* is a stochastic process of the form

$$X_t = X_a + \int_a^t f(s)\, dB(s) + \int_a^t g(s)\, ds, \quad a \le t \le b,$$

where X_a is $\mathcal{F}_a$-measurable, $f \in \mathcal{L}_{ad}(\Omega, L^2[a,b])$ and $g \in \mathcal{L}_{ad}(\Omega, L^1[a,b])$.

It is common to write the equation above in the "stochastic differential" form:

$$dX_t = f(t)\, dB(t) + g(t)\, dt.$$

Again, note that this "stochastic differential" form has no meaning because Brownian motion paths are nowhere differentiable.

Example 3.26. Let $f \in \mathcal{L}_{ad}(\Omega, L^2[a,b])$. Then

$$X_t = X_a + \int_a^t f(s)\, dB(s) + \int_a^t f(s)^2\, ds, \quad a \le t \le b,$$

is an Itô process. For example, let $f(t) = B(t)$ or $f(t) = e^{B(t)}$ or $f(t) = e^{B(t)^2}$.

Next is the third (more general) version of the Itô formula.

Theorem 3.27. *Let X_t be an Itô process given by*

$$X_t = X_a + \int_a^t f(s)\, dB(s) + \int_a^t g(s)\, ds, \quad a \le t \le b.$$

Suppose $\theta(t,x)$ is a continuous function with continuous partial derivatives $\frac{\partial \theta}{\partial t}$, $\frac{\partial \theta}{\partial x}$ and $\frac{\partial^2 \theta}{\partial x^2}$. Then $\theta(t, X_t)$ is also an Itô process and

$$\begin{aligned} \theta(t, X_t) &= \theta(a, X_a) + \int_a^t \frac{\partial \theta}{\partial x}(s, X_s) f(s)\, dB(s) \\ &\quad + \int_a^t \left[\frac{\partial \theta}{\partial t}(s, X_s) + \frac{\partial \theta}{\partial x}(s, X_s) g(s) + \frac{1}{2} \frac{\partial^2 \theta}{\partial x^2}(s, X_s) f(s)^2 \right] ds. \quad (3.8) \end{aligned}$$

In using Equation 3.8, the following table called the *Itô table* is very useful:

Table 1 : Itô table 1

$\times$	$dB(t)$	dt
$dB(t)$	dt	0
dt	0	0

For example, if $dX_t = f(t)\,dB(t) + g(t)\,dt$, then

$$(dX_t)^2 = f(t)^2(dB(t))^2 + 2f(t)g(t)dB(t)dt + g(t)^2\,(dt)^2 = f(t)^2\,dt$$

Example 3.28. Let $f \in \mathcal{L}_{ad}(\Omega, L^2[0,1])$. Consider the Itô process

$$X_t = \int_0^t f(s)\,dB(s) - \frac{1}{2}\int_0^t f(s)^2\,ds, \quad 0 \le t \le 1,$$

and the function $\theta(x) = e^x$. Then $dX_t = f(t)dB(t) - \frac{1}{2}f(t)^2 dt$. Apply the Taylor expansion and use Itô table 1 to get

$$\begin{aligned} d\,\theta(X_t) &= e^{X_t}\,dX_t + \frac{1}{2}e^{X_t}\,(dX_t)^2 \\ &= e^{X_t}\Big(f(t)\,dB(t) - \frac{1}{2}f(t)^2\,dt\Big) + \frac{1}{2}\,e^{X_t}f(t)^2\,dt \\ &= f(t)e^{X_t}\,dB(t). \end{aligned}$$

2. Bellman-Gronwall Inequality

Suppose $\phi \in L^1[a,b]$ satisfies

$$\phi(t) \le f(t) + \beta\int_a^t \phi(s)\,ds, \quad \forall t \in [a,b].$$

Then

$$\phi(t) \le f(t) + \beta\int_a^t f(s)\,e^{\beta(t-s)}\,ds.$$

In particular, if $f(t) = \alpha$, a constant, then

$$\phi(t) \le \alpha e^{\beta(t-a)}, \quad \forall t \in [a,b].$$

3.4 Solution of Stochastic Differential Equations

To start with, let $B(t)$ be a Brownian motion and let $\{\mathcal{F}_t : a \leq t \leq b\}$ be a filtration such that

(a) for each t, $B(t)$ is $\mathcal{F}_t$-measurable,

(b) for any $s < t$, the random variable $B(t) - B(s)$ is independent of the σ-field $\mathcal{F}_s$.

Let $f(t,x)$ and $g(t,x)$ be measurable functions of $t \in [a,b]$ and $x \in \mathbb{R}$. Consider the stochastic differential equation

$$dX_t = f(t, X_t)\, dB(t) + g(t, X_t)\, dt, \quad X(a) = X_a,$$

necessarily understood as the stochastic integral equation

$$X_t = X_a + \int_a^t f(s, X_s)\, dB(s) + \int_a^t g(s, X_s)\, ds, \quad a \leq t \leq b. \tag{3.9}$$

Under what circumstances can one assert the existence of n unique solution? What does it mean to say that a stochastic process X_t is a solution to (3.9)?

We turn to [15] for answers to these important questions.

Definition 3.29. A jointly measurable stochastic process X_t. $a \leq t \leq b$, is called a *solution* of the stochastic integral equation in Equation (3.9) if it satisfies the follwoing conditions:

(1) The stochastic process $f(t, X_t)$ belongs to $\mathcal{L}_{ad}(\Omega, L^2[a,b])$, so that the integral $\int_a^t f(s, X_s)\, dB(s)$ is an Itô integral for each $t \in [a,b]$;

(2) Almost all sample paths of the stochastic process $g(t, X_t)$ belong to $L^1[a,b]$;

(3) For each $t \in [a,b]$, Equation (3.9) holds almost surely.

Definition 3.30. A measurable function $g(t, x)$ on $[a, b] \times \mathbb{R}$ is said to satisfy the *Lipschitz condition in* x if there exists a constant $K > 0$ such that

$$|g(t, x) - g(t, y)| \leq K|x - y|, \quad \text{for all } a \leq t \leq b, \ x, y \in \mathbb{R}.$$

Definition 3.31. A measurable function $g(t, x)$ on $[a, b] \times \mathbb{R}$ is said to satisfy the *linear growth condition in* x if there exists a constant $K > 0$ such that

$$|g(t, x)| \leq K(|1 + |x|), \quad \text{for all } a \leq t \leq b, \ x \in \mathbb{R}.$$

Next, we state the relevant facts, proofs of which may be found in. [15]

Lemma 3.32. *Let $f(t, x)$ and $g(t, x)$ be measurable functions on $[a, b] \times \mathbb{R}$ satisfying the Lipschitz condition in x. Suppose ξ is an $\mathcal{F}_a$-measurable random variable with $E(\xi^2) < \infty$. Then the stochastic integral equation in Equation (3.9) has at most one continuous solution X_t.*

Theorem 3.33. *Let $f(t, x)$ and $g(t, x)$ be measurable functions on $[a, b] \times \mathbb{R}$ satisfying the Lipschitz and growth conditions in x. Suppose ξ is an $\mathcal{F}_a$-measurable random variable with $E(\xi^2) < \infty$. Then the stochastic integral equation in Equation (3.9) has a unique continuous solution X_t.*

Remark 3.34. Let $B(t)$ be a Brownian motion and consider the stochastic integral equation

$$X(t) = X(a) + \int_a^t f(s, X(s)) \, dB(s) + \int_a^t g(s, X(s)) \, ds, \tag{3.10}$$

where $t \in [a, b], a, b \in [0, \infty)$. Equation (3.10) is an Itô stochastic integral equation if $X(a)$ is measurable with respect to $\sigma\{B(s) : s \leq a\}$. This Itô stochastic integral equation has a unique continuous solution provided that f and g satisfy Lipschitz and growth conditions, that is

(i) there exists $C_1 > 0$ such that for any $t \in [a, b]$, and $x, y \in \mathbb{R}$,

$$|f(t, x) - f(t, y)| + |g(t, x) - g(t, y)| \leq C_1|x - y|,$$

and

(ii) there exists C_2 such that for any $t \in [a, b]$, and $x \in \mathbb{R}$,

$$|f(t,x)|^2 + |g(t,x)|^2 \leq C_2(1 + x^2),$$

respectively.

We note that the existence of a solution is established by applying the Picard iteration, where $X_0(t) = X(a)$ and for $n \geq 1$,

$$X_n(t) = X(a) + \int_a^t f(s, X_{n-1}(s))dB(s) + \int_a^t g(s, X_{n-1}(s))\, ds$$

and with probability 1, $X_n(t)$ converges to $X(t)$ on $[a, b]$ uniformly. Interestingly enough, if we relax the measurability requirement on $X(a)$, $f(s, X(a))$ may be anticipating. As a result $\int_a^t f(s, X(a))\, dB(s)$ is not an Itô integral, hence $X_1(t)$ is undefined as an Itô process. Moreover, Equation (3.10) would no longer be an Itô stochastic integral equation. This leads us to consider anticipating stochastic differential equations in the next chapter.

Chapter 4

Anticipating Stochastic Differential Equations*

As discussed in the previous chapter, the measurability condition is essential in the definition of Itô processes. In its absence, the resulting integral equations become anticipating. Possible sources of anticipation include initial conditions, integrands or even terminal conditions, namely:

- $X_t = \int_a^t f\left(s, X(s)\right) dB(s)$, with
$$X(a) = X_a \notin \mathcal{F}_t;$$
- $X_t = \int_a^t f\left(s, X(s)\right) dB(s)$, with
$$f \notin \mathcal{L}_{ad}\left(\Omega, L^2[a,b]\right);$$
- $X_t = \int_a^t f\left(s, X(s)\right) dB(s)$, with
$$X(b) = X_b,$$

respectively.

4.1 The White Noise Methods

With the same basic set up as outlined in Section 1.4 above, we provide a brief review of white noise theory.

Let $\langle \cdot, \cdot \rangle$ denote the duality between $\mathcal{E}'$ and $\mathcal{E}$. By Minlos theorem, there is a unique probability measure μ on the Borel subsets of $\mathcal{E}'$ such that for any $f \in \mathcal{E}$,

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the random variable $\langle \cdot, f \rangle$ is normally distributed with mean 0 and variance $|\, f|_0^2$. It follows that μ is uniquely determined by

$$\int_{\mathcal{E}'} e^{i\langle x,\xi \rangle}\, d\mu = e^{-\frac{1}{2}|\,\xi|_0^2}, \quad \forall\, \xi \in \mathcal{E}. \tag{4.11}$$

The probability space $(\mathcal{E}', \mu)$ is known as the *white noise space*. We denote the space $L^2(\mathcal{E}', \mu)$ by (L^2), observing that this space consists of all measurable functions $h : \mathcal{E}' \to \mathbb{C}$ such that

$$\int_{\mathcal{E}'} |\, h(x)|^2\, d\mu(x) < \infty.$$

Within this framework, the somewhat ubiquitous white noise tool, known as the S-transform is defined (see [14]). In fact, if $\varphi(t) \in (L^2)$, then for $\xi \in S_c$,

$$S\varphi(t)(\xi) = \int_{S'} \varphi(t)(x + \xi)\, d\mu(x).$$

Meantime, if X and Y are generalized functions, their Wick product, denoted $X \diamond Y$ is the unique generalized function such that

$$S(X \diamond Y) = (SX)(SY).$$

Clearly, an important feature of the S-transform is that much like the Fourier transform changes convolutions into products, it turns Wick products into ordinary products. It is worth noting that the Wick product plays an intrinsic role in stochastic integration, especially when one discusses situations involving anticipating initial conditions or integrands.

1. The Hitsuda-Skorokhod Integral

Let $\partial_t \equiv D_{\partial_t}$ be the *white noise differential operator* (also known as the *Hida differential operator* or the *annihilation operator*), as defined in [14]. The adjoint of ∂_t, denoted by $\partial_t^* \equiv D_{\partial_t}^*$, is called the *creation operator*.

Starting with the Gel'fand triple $S(\mathbb{R}) \subset L^2(\mathbb{R}) \subset S'(\mathbb{R})$ and following [14], one obtains the Gel'fand triple $(S)_\beta \subset (\mathcal{L}^2) \subset (S)^*_\beta$. If $\varphi : [a, b] \to (S)^*_\beta$ is Pettis integrable, then the (white noise) integral $\int_a^b \partial_t^* \varphi(t)\, dt$ is called the *Hitsuda-Skorokhod integral* of φ, provided $\int_a^b \partial_t^* \varphi(t)\, dt$ is a random variable in (L^2).

The Hitsuda-Skorokhod integral extends the Itô integral to $\varphi(t)$ which may be anticipating. In fact, if $\varphi(t)$ is nonanticipating and $\int_a^b \| \varphi(t)\|_0^2\, dt < \infty$, then

$$\int_a^b \partial_t^* \varphi(t)\, dt = \int_a^b \varphi(t)\, dB(t).$$

See [12] or [14] for details.

2. The White Noise Approach

In a bid to circumvent the challenges arising from anticipating initial conditions or integrands, one possibility is to consider

$$X(t) = X(a) + \int_a^t \partial_s^* f(s, X(s))\, ds + \int_a^t g(s, X(s))\, ds, \tag{4.12}$$

where $\int_a^t \partial_s^* f(s, X(s))\, ds$ is a Hitsuda-Skorokhod integral. Equality in (4.12) is as random variables in the complex Hilbert space $(\mathcal{L}^2) \equiv \mathcal{L}^2(\xi', \mu)$.

The white noise methods involve using the S-transform to convert Equation (4.12) into

$$SX(t)(\xi) = SX(a)(\xi) + \int_a^t \xi(s) Sf(s, X(s))(\xi)\, ds + \int_a^t Sg(s, X(s))(\xi)\, ds \tag{4.13}$$

which is an ordinary integral equation for each fixed $\xi \in \mathcal{S}_c$. Next assuming Equation (4.13) can be solved for each ξ, a solution to Equation (4.12) would be obtained by applying the inverse S-transform, provided of course that taking inverse S-transform is possible. This requires that the solution to Equation (4.13) be in the range of the S-transform of an appropriate space.

4.2 Some Examples

We now turn our attention to some interesting examples previously considered in turn by Buckdahn [4] and Kuo [14], using different techniques. In order to explain the key ideas, we describe the arguments from [14] for these examples.

Example 4.1. [4][14] Let us examine

$$X(t) = sgn(B(1)) + \int_0^t X(s)\, dB(s).$$

Since $sgn(B(1)) \notin \sigma\{B(s); s \leq 1\}$, the preceding equation corresponds to

$$X(t) = sgn(B(1)) + \int_0^t \partial_s^* X(s)\, ds, \quad t \in [0,1]. \tag{4.14}$$

So Equation (4.14) is not an Itô stochastic integral equation.

To solve Equation (4.14), using the approach in section 4.1, let $SX(t) = F(t)$ and $S[\,sgn(B(1))] = G$. Then Equation (4.14) becomes (after we take S-transform)

$$F(t)(\xi) = G(\xi) + \int_0^t \xi(s) F(s)(\xi)\, ds,$$

so that for each $\xi \in S_c$, we have, with $t \in [0,1]$,

$$F'(t) = \xi(t)F(t) \quad \text{and} \quad F(0) = G(\xi).$$

Consequently

$$\begin{aligned} F(t)(\xi) &= G(\xi)\, e^{\int_0^t \xi(s)\, ds} \\ &= G(\xi)\, e^{\langle 1_{[0,t)}, \xi\rangle} \qquad (4.15) \\ &= G(\xi)\, S\left(: e^{\langle \cdot, 1_{[0,t)}\rangle} :\right)(\xi) \\ &= S(sgn(B(1)))(\xi)\; S\left(: e^{\langle \cdot, 1_{[0,t)}\rangle} :\right)(\xi) \\ &= S\left[\{sgn(B(1))\} \diamond \left(: e^{\langle \cdot, 1_{[0,t)}\rangle} :\right)\right](\xi), \end{aligned}$$

whence we have

$$\begin{aligned} X(t) &= \{sgn(B(1))\} \diamond \left(: e^{\langle \cdot, 1_{[0,t)} \rangle} :\right) \\ &= \{sgn(B(1))\} \diamond \left(e^{B(t)-\frac{t}{2}}\right). \end{aligned}$$

It remains to show that $X(t) \in (L^2)$ for all t. To this end, consider

$$\varphi(t) = sgn(B(1) - t)\, e^{B(t)-\frac{t}{2}} = sgn(\langle \cdot, 1_{[0,1]} \rangle - t)\, e^{\langle \cdot, 1_{[0,t)} \rangle - \frac{t}{2}},$$

since $B(t) = \langle \cdot, 1_{[0,t)} \rangle$.

Next, let us determine $S\varphi(t)$. By definition,

$$\begin{aligned} S\varphi(t)(\xi) &= \int_{S'} \varphi(t)(x+\xi)\, d\mu(x) \\ &= \int_{S'} sgn\left(\langle x+\xi, 1_{[0,1]} \rangle - t\right)\; e^{\langle x+\xi, 1_{[0,t)} \rangle - \frac{t}{2}}\, d\mu(x) \\ &= \int_{S'} sgn\left(\langle y + 1_{[0,t)} + \xi, 1_{[0,1]} \rangle - t\right) e^{\langle y+1_{[0,t)}+\xi, 1_{[0,t)} \rangle - \frac{t}{2}} e^{-\langle y, 1_{[0,t)} \rangle - \frac{t}{2}} d\mu(y) \\ &= \int_{S'} sgn\left(\langle y+\xi, 1_{[0,1]\rangle}\right)\; e^{\langle 1_{[0,t)}, \xi \rangle}\, d\mu(y) \\ &= e^{\langle 1_{[0,t)}, \xi \rangle}\, S\left(sgn(B(1))(\xi)\right) \\ &= F(t)(\xi) \qquad \text{(by Equation 4.15)} \end{aligned}$$

Recall that $F(t) = SX(t)$, so we see here that $SX(t) = S\varphi(t)$, with $\varphi(t) \in (L^2)$ and since S is injective, it follows that

$$X(t) = \varphi(t) = sgn(B(1) - t)\, e^{B(t)-\frac{t}{2}}.$$

Example 4.2. [14] Consider the stochastic integral equation

$$X(t) = 1 + \int_0^t \partial_s^* X(s)\, ds + \int_0^t sgn(B(1) - s)\, e^{B(s)-\frac{s}{2}}\, ds \tag{4.16}$$

where $0 \le t \le 1$. We claim that the solution to Equation (4.16) is given by $X(t) = e^{B(t)-\frac{t}{2}} + t\varphi(t)$, where $\varphi(t) = sgn(B(1) - t)\, e^{B(t)-\frac{t}{2}}$.

Let $F(t) = S(X(t))$ and $G = S[sgn(B(1))]$. Then we recall from Example 4.1 that for

$$\begin{aligned} \varphi(t) &= sgn(B(1) - t)\, e^{B(t) - \frac{t}{2}} \\ &= sgn\left(\langle \cdot, 1_{[0,1]} \rangle - t\right)\, e^{\langle \cdot, 1_{[0,t)} \rangle - \frac{t}{2}}, \end{aligned}$$

we have $S\varphi(t)(\xi) = e^{\langle 1_{[0,t)}, \xi \rangle}\, S(sgn(B(1)))(\xi)$. Therefore applying the S-transform to (4.16), we get for $\xi \in S_c$

$$S(X(t))(\xi) = 1 + \int_0^t \xi(s) S(X(s))(\xi)\, ds + \int_0^t S(\varphi(s))(\xi)\, ds$$

so that

$$\begin{aligned} F(t)(\xi) &= 1 + \int_0^t \xi(s) F(s)(\xi)\, ds + \int_0^t e^{\langle 1_{[0,s)}, \xi \rangle} S(sgn(B(1))(\xi)\, ds \\ &= 1 + \int_0^t \xi(s) F(s)(\xi)\, ds + \int_0^t G(\xi) e^{\langle 1_{[0,s)}, \xi \rangle}\, ds, \end{aligned}$$

since $G = Ssgn(B(1))$. So

$$F(t)(\xi) = 1 + \int_0^t \xi(s) F(s)(\xi)\, ds + \int_0^t G(\xi) e^{\int_0^s \xi(u)\, du}\, ds. \tag{4.17}$$

Now for each $\xi \in S_c$, Equation (4.17) implies that $F(t)$ satisfies the ordinary differential equation

$$\begin{aligned} &F'(t) = \xi(t) F(t) + G(\xi) e^{\int_0^t \xi(s)\, ds}, \\ &F(0) = 1, \end{aligned} \tag{4.18}$$

for $t \in [0,1]$. We now seek a solution for Equation (4.18), which is just a first order linear ordinary differential equation. We have

$$F'(t) - \xi(t) F(t) = G(\xi) e^{\int_0^t \xi(s)\, ds}$$

with integrating factor $e^{-\int_0^t \xi(s)\, ds}$. Multiplying through by the integrating factor, Equation (4.18) becomes

$$e^{-\int_0^t \xi(s)\, ds} F'(t) - \xi(t) F(t) e^{-\int_0^t \xi(s)\, ds} = G(\xi),$$

which is

$$\frac{d}{dt}\left[F(t)e^{-\int_0^t \xi(s)\,ds}\right] = G(\xi).$$

Therefore

$$F(t)e^{-\int_0^t \xi(s)\,ds} = \int_0^t G(\xi)\,ds + K,$$

where K is a constant. Hence $F(t)$ is given by

$$F(t) = tG(\xi)e^{\int_0^t \xi(s)\,ds} + Ke^{\int_0^t \xi(s)\,ds},$$

and since $F(0) = 1$, we have $K = 1$. So

$$F(t) = e^{\int_0^t \xi(s)\,ds}\{1 + tG(\xi)\} \tag{4.19}$$

is the solution to Equations (4.18).

Next, we recall that

$$e^{\int_0^t \xi(s)\,ds} = e^{\langle 1_{[0,t)},\xi\rangle} = S\left(e^{B(t)-\frac{t}{2}}\right)(\xi).$$

Moreover

$$S(\varphi(t))(\xi) = S\left(sgn(B(1)-t)e^{B(t)-\frac{t}{2}}\right)(\xi) = G(\xi)e^{\int_0^t \xi(s)\,ds},$$

which now allows us to rewrite equation (4.19) as

$$S(X(t))(\xi) = S\left(e^{B(t)-\frac{t}{2}}\right)(\xi) + t\,S(\varphi(t))(\xi).$$

It then follows that

$$\begin{aligned} X(t) &= e^{B(t)-\frac{t}{2}} + t\,\varphi(t) \\ &= e^{B(t)-\frac{t}{2}} + t\,sgn(B(1)-t)e^{B(t)-\frac{t}{2}} \\ &= e^{B(t)-\frac{t}{2}}\{1 + t\,sgn(B(1)-t)\}. \end{aligned}$$

The preceding examples provide the basis for our first result:

Theorem 4.3. *Let $X(t)$ be a stochastic process such that*

$$S(X(t))(\xi) = G(\xi)e^{\int_0^t \xi(s)\,ds},$$

where $\xi \in S_c$. Then the solution to the stochastic integral equation

$$Y(t) = 1 + \int_0^t \partial_s^* Y(s)\,ds + \int_0^t X(s)\,ds \tag{4.20}$$

for $t \in [0,1]$ is given by $Y(t) = e^{B(t)-\frac{t}{2}} + t\,X(t)$.

Proof. Let $SY(t) = H(t)$. By hypothesis $S(X(t)) = G(\xi)e^{\int_0^t \xi(s)\,ds}$. Applying the S-transform to Equation (4.20), we have

$$\begin{aligned} S(Y(t))(\xi) &= 1 + \int_0^t \xi(s)S(Y(s))(\xi)\,ds + \int_0^t S(X(s))(\xi)\,ds \qquad (4.21)\\ \Rightarrow \quad H(t)(\xi) &= 1 + \int_0^t \xi(s)H(s)(\xi)\,ds + \int_0^t G(\xi)\,e^{\int_0^s \xi(u)\,du} ds \end{aligned}$$

So for each $\xi \in S_c$, we have

$$\begin{aligned} & H'(t) = \xi(t)H(t) + G(\xi)\,e^{\int_0^t \xi(u)\,du}, \quad H(0) = 1\\ \Rightarrow \quad & \frac{d}{dt}\left[H(t)\,e^{-\int_0^t \xi(u)\,du}\right] = G(\xi)\\ \Rightarrow \quad & H(t)\,e^{-\int_0^t \xi(u)\,du} = t\,G(\xi) + K\\ \Rightarrow \quad & H(t) = t\,G(\xi)\,e^{\int_0^t \xi(u)\,du} + K\,e^{\int_0^t \xi(u)\,du} \end{aligned}$$

Since $H(0) = 1$ and $K = 1$, we have

$$H(t) = e^{\int_0^t \xi(u)\,du} + t\,G(\xi)\,e^{\int_0^t \xi(u)\,du},$$

namely

$$S(Y(t))(\xi) = S\left(e^{B(t)-\frac{t}{2}}\right)(\xi) + t\,S(X(t))(\xi),$$

which implies $Y(t) = e^{B(t)-\frac{t}{2}} + t\,X(t)$, as desired. □

4.3 A Class of Linear Equations

We now consider a class of equations based on the general linear stochastic integral equation of Hitsuda-Skorokhod type, namely

$$X(t) = \varphi + \int_a^t \partial_s^*(f(s)X(s))\,ds + \int_a^t [g(s)X(s) + \psi(s)]\,ds,$$

where f, g are deterministic, φ is a random variable and ψ is a stochastic process. We will see how the S-transform can be used to solve this equation. We start by recalling a lemma (Lemma 13.32 in [14]) and a theorem (Theorem 13.33 in [14]) and close with another unifying result.

Lemma 4.4. *[14] If $f \in \mathcal{L}^2([a,b])$ and $\varphi \in \mathcal{L}^p(S')$ for some $p > 2$, then*

$$\varphi \diamond e^{[\int_a^t f(s)\,dB(s) - \frac{1}{2}\int_a^t f(s)^2\,ds]} = \left(T_{-1_{[a,t]}f}\varphi\right)\, e^{[\int_a^t f(s)\,dB(s) - \frac{1}{2}\int_a^t f(s)^2\,ds]},$$

where $\diamond$ is the Wick product and $T_h\varphi(x) = \varphi(x+h)$.

Proof. Let

$$\begin{aligned} Z(t) &= e^{[\int_a^t f(s)\,dB(s) - \frac{1}{2}\int_a^t f(s)^2\,ds]} \\ &= e^{\left[\langle \cdot, 1_{[a,t)}f\rangle - \frac{1}{2}\left|1_{[a,t)}f\right|_0^2\right]} \\ &= \,:e^{\langle \cdot, 1_{[a,t)}f\rangle}:. \end{aligned}$$

Then since $f \in \mathcal{L}^2([a,b])$, by Theorem 5.13 in [14], we see that $S\,Z(t)(\xi) = e^{\langle 1_{[a,t)}f, \xi\rangle}$, $\xi \in S_c$. Moreover, by hypothesis, we see that $\left(T_{-1_{[a,t)}f}\varphi\right) Z(t) \in (\mathcal{L}^2)$. Further, we have by definition that

$$\begin{aligned} &S\left(T_{-1_{[a,t)}f}\varphi\right) Z(t)(\xi) \\ &= \int_{S'} \varphi\left(x + \xi - 1_{[a,t)}f\right)\, e^{\left[\langle x+\xi,\, 1_{[a,t)}f\rangle - \frac{1}{2}\left|1_{[a,t)}f\right|_0^2\right]}\,d\mu(x) \\ &= \int_{S'} \varphi(y+\xi)\, e^{[\langle 1_{[a,t)}f,\, \xi\rangle]}\,d\mu(y) \\ &= e^{[\langle 1_{[a,t)}f,\, \xi\rangle]} \int_{S'} \varphi(y+\xi)d\mu(y) \\ &= (S\varphi)(\xi)\,(SZ(t))(\xi). \end{aligned}$$

W see that $S\left(\left(T_{-1_{[a,t)}f}\varphi\right)Z(t)\right)(\xi) = (S\varphi)(\xi)\,(SZ(t))(\xi)$ and by the injectivity of the S-transform, we have

$$\left(T_{-1_{[a,t)}f}\varphi\right)Z(t) = \varphi \diamond Z(t). \qquad \square$$

The next theorem shows the solution of the general Hitsuda-Skorokhod type stochastic integral equation when certain conditions are specified.

Theorem 4.5. *[14] Suppose $f(t)$, $g(t)$ are deterministic functions, φ is a random variable, and $\psi(t)$ is a stochastic process satisfying*

1. *$f, g \in \mathcal{L}^2([a,b]))$.*
2. *$\varphi \in \mathcal{L}^p(S')$ for some $p > 2$.*
3. *$\psi \in \mathcal{L}^q([a,b] \times S')$ for some $q > 2$.*

Then the stochastic integral equation

$$X(t) = \varphi + \int_a^t \partial_s^*(f(s)X(s))\,ds + \int_a^t (g(s)X(s) + \psi(s))\,ds$$

has a unique solution in $\mathcal{L}^2([a,b],(L^2))$ given by

$$\begin{aligned} X(t) &= \left(T_{-1_{[a,t)}f}\varphi\right) e^{\left[\int_a^t f(s)\,dB(s) + \int_a^t \left(g(s) - \frac{1}{2}f(s)^2\right)ds\right]} \\ &\quad + \int_a^t \left(T_{-1_{[s,t)}f}\psi(s)\right) e^{\left[\int_s^t f(r)\,dB(r) + \int_s^t \left(g(r) - \frac{1}{2}f(r)^2\right)dr\right]}\,ds. \end{aligned}$$

Proof. First, we show that the solution is unique. If $Y(t)$ is another solution to the above stochastic integral equation, let $Z(t) = X(t) - Y(t)$. Then $Z(t)$ satisfies

$$Z(t) = \int_a^t \partial_s^*(f(s)Z(s))\,ds + \int_a^t g(s)Z(s)ds, \quad t \in [a,b].$$

Next let $G(t) = SZ(t)$, then for almost all $t \in [a,b]$,

$$G(t)(\xi) = \int_a^t (\xi(s)f(s) + g(s))G(s)(\xi)\,ds, \quad \forall\, \xi \in S_c.$$

We observe that $G(t)(\xi)$ is defined only t-a.e. for any fixed ξ, therefore $G'(t)(\xi)$ is meaningless. As a remedy, let us define for each fixed $\xi \in S_c$,

$$H_\xi(t) = \int_a^t (\xi(s)f(s) + g(s))G(s)(\xi)\, ds, \quad t \in [a, b].$$

Then $G(t)(\xi) = H_\xi$ t-a.e. on $[a, b]$ and for any $\xi \in S_c$. Moreover, for each ξ, $H_\xi(t)$ is absolutely continuous, $H_\xi(a) = 0$ and H'_ξ exists t-a.e. on $[a, b]$.

In fact $H'_\xi(t) = (\xi(t)f(t) + g(t))H_\xi(t)$, t-a.e. on $[a, b]$. Solving this we get

$$H_\xi(t) = K\, e^{\int_a^t (\xi(s)f(s)+g(s))\, ds},$$

and since $H_\xi(a) = 0$, we have $K = 0$ and thus $H_\xi(t) = 0$ for any $t \in [a, b]$. It follows that for almost all $t \in [a, b]$,

$$G(t)(\xi) = H_\xi(t) = 0, \quad \forall\, \xi \in S_c,$$

i.e. $S(Z(t))(\xi) = 0$, so $Z(t) = 0$, t-a.e. on $[a, b]$ and hence $X(t) = Y(t)$, so the solution is unique.

Using the S-transform method let us seek the solution. Indeed, let $F(t) = SX(t)$, $G = S\varphi$ and $V(t) = S\psi(t)$. By taking the S-transform of the stochastic integral equation, we get

$$F(t)(\xi) = G(\xi) + \int_a^t (\xi(s)f(s) + g(s))F(s)(\xi) + \int_a^t V(s)(\xi)\, ds$$

with equality holding in the sense that for almost all $t \in [a, b]$,

$$F(t)(\xi) = G(\xi) + \int_a^t (\xi(s)f(s) + g(s))F(s)(\xi) + \int_a^t V(s)(\xi)\, ds$$

for all $\xi \in S_c$. Since $F(t)(\xi)$ is defined only t-a.e. on $[a, b]$, $F'(t)(\xi)$ is meaningless. So as in the proof of Lemma 4.4, define H_ξ on $[a, b]$ by

$$H_\xi(t) = G(\xi) + \int_a^t u(s)F(s)(\xi)\, ds + \int_a^t V(s)(\xi)\, ds$$

where $u(t) = g(t) + \xi(t)f(t)$ for simplicity. Then for almost all $t \in [a,b]$,

$$H_\xi(t) = F(t)(\xi), \quad \forall\, \xi \in S_c.$$

Also $H_\xi(t)$ is absolutely continuous for each fixed $\xi \in S_c$ and

$$H'_\xi(t) = u(t)H_\xi(t) + V(t)(\xi), \quad t\text{-a.e. on } [a,b],$$

with $H_\xi(a) = G(\xi)$. Solving the above ordinary differential equation, we have

$$\begin{aligned} H'_\xi(t) - u(t)H_\xi(t) &= V(t)(\xi) \\ \frac{d}{dt}\left[H_\xi(t)\, e^{-\int_a^t u(s)\,ds}\right] &= e^{-\int_a^t u(s)\,ds} V(t)(\xi) \\ e^{-\int_a^t u(s)\,ds} H_\xi(t) &= \int_a^t V(s)(\xi)\, e^{-\int_a^s u(r)\,dr}\, ds + K. \end{aligned}$$

Since $H_\xi(a) = G(\xi)$, we have $K = G(\xi)$ and so

$$\begin{aligned} H_\xi(t) &= G(\xi)\, e^{\int_a^t u(s)\,ds} + e^{\int_a^t u(s)\,ds} \int_a^t V(s)(\xi)\, e^{-\int_a^s u(r)\,dr}\, ds \\ &= G(\xi)\, e^{\int_a^t u(s)\,ds} + \int_a^t V(s)(\xi)\, e^{\int_s^t u(r)\,dr}\, ds. \end{aligned}$$

Now since

$$H_\xi(t) = F(t)(\xi), \quad t\text{-a.e. on } [a,b], \quad \forall\, \xi \in S_c,$$

we therefore have

$$F(t)(\xi) = G(\xi)\, e^{\int_a^t u(s)\,ds} + \int_a^t V(s)(\xi)\, e^{\int_s^t u(r)\,dr}\, ds, \quad t\text{-a.e. on } [a,b].$$

Finally, we need to find the inverse S-transform of $F(t)$. First we recall that since $f \in \mathcal{L}^2([a,b])$,

$$S\left(e^{\left[\int_a^t f(s)\,dB(s) - \frac{1}{2}\int_a^t f(s)^2\,ds\right]}\right)(\xi) = e^{\int_a^t \xi(s)f(s)\,ds}.$$

Also, since $(S\varphi)(\xi) = G(\xi)$, it follows that

$$\begin{aligned} S\left(\varphi \diamond e^{\left[\int_a^t f(s)\,dB(s) - \frac{1}{2}\int_a^t f(s)^2\,ds\right]}\right) &= (S\varphi(\xi))\left(S\, e^{\left[\int_a^t f(s)\,dB(s) - \frac{1}{2}\int_a^t f(s)^2\,ds\right]}(\xi)\right) \\ &= G(\xi)\, e^{\int_a^t \xi(s)f(s)\,ds}. \end{aligned}$$

In the same manner,

$$S\left(\psi(s)\diamond e^{\left[\int_s^t f(r)\,dB(r)-\frac{1}{2}\int_s^t f(r)^2\,dr\right]}\right)(\xi)=V(s)(\xi)\,e^{\int_s^t \xi(r)f(r)\,dr}.$$

But then $f,g\in\mathcal{L}^2([a,b])$ and $\psi(s)\in\mathcal{L}^p(S')$, for some $p>2$. Using these conditions, we see that

$$S\left(\int_a^t \psi(s)\diamond e^{\left[\int_s^t f(r)\,dB(r)-\frac{1}{2}\int_s^t f(r)^2\,dr\right]}\,ds\right)(\xi)=\int_a^t V(s)(\xi)\,e^{\int_s^t \xi(r)f(r)\,dr}\,ds.$$

Combining the above (last three) equations, we have for t-a.e. on $[a,b]$,

$$\begin{aligned} X(t) &= \varphi\diamond e^{\left[\int_a^t f(s)\,dB(s)+\int_a^t\left(g(s)-\frac{1}{2}f(s)^2\right)ds\right]} \\ &\quad+\int_a^t \psi(s)\diamond e^{\left[\int_s^t f(r)\,dB(r)+\int_s^t\left(g(r)-\frac{1}{2}f(r)^2\right)dr\right]}\,ds \\ &= \left(T_{-1_{[a,t)}f}\varphi\right)\,e^{\left[\int_a^t f(s)\,dB(s)+\int_a^t\left(g(s)-\frac{1}{2}f(s)^2\right)ds\right]} \\ &\quad+\int_a^t \left(T_{-1_{[a,t)}f}\psi(s)\right)\,e^{\left[\int_s^t f(r)\,dB(r)+\int_s^t\left(g(r)-\frac{1}{2}f(r)^2\right)dr\right]}\,ds, \end{aligned}$$

where Lemma 4.4 has been applied to obtain the last equality and

$$X(t)\in(\mathcal{L}^2),\quad t\text{-a.e. on }[a,b].$$ □

Finally, for simplicity, let

$$\mathcal{E}_f(t)=e^{\left[\int_0^t f(s)\,dB(s)-\frac{1}{2}\int_0^t f(s)^2\,ds\right]}.$$

In view of the general linear Hitsuda-Skorokhod type stochastic integral equation and the concluding result in the previous section, we have

Theorem 4.6. *If $f\in\mathcal{L}^2([0,1])$ and φ is a random variable, with $\varphi\in\mathcal{L}^p(S')$ for some $p>2$. Then the stochastic integral equation*

$$Y(t)=1+\int_0^t \partial_s^*(f(s)Y(s))\,ds+\int_0^t(\varphi\diamond\mathcal{E}_f(s))\,ds \tag{4.22}$$

has a solution given by

$$Y(t)=\mathcal{E}_f(t)+t(\varphi\diamond\mathcal{E}_f(t)),\quad t\in[0,1].$$

Proof. For simplicity, let $X(t) = \varphi \diamond \mathcal{E}_f(t)$, and let $SY(t) = F(t)$, $S\varphi = G$. Since $f \in \mathcal{L}^2([0,1])$, we have

$$S(\mathcal{E}_f(t))(\xi) = e^{\int_0^t \xi(s)f(s)\,ds}.$$

Therefore, by taking the S-transform of Equation (4.22), we have for $\xi \in S_c$,

$$F(t)(\xi) = 1 + \int_0^t \xi(s)f(s)F(s)(\xi)ds + \int_0^t G(\xi)\, e^{\int_0^s \xi(u)f(u)\,du}\, ds,$$

so that for each fixed $\xi \in S_c$,

$$F'(t) = \xi(t)f(t)F(t)(\xi) + G(\xi)\, e^{\int_0^t \xi(u)f(u)\,du}, \quad \text{with } F(0) = 1.$$

For each fixed $\xi \in S_c$, we have

$$\frac{d}{dt}\left[e^{-\int_0^t \xi(s)f(s)\,ds} F(t)\right] = G(\xi)$$

which yields

$$e^{-\int_0^t \xi(u)f(u)\,du} F(t) = t\,G(\xi) + K,$$

where K is a constant. Hence we have

$$F(t) = K\, e^{\int_0^t \xi(u)f(u)\,du} + t\,G(\xi)\, e^{\int_0^t \xi(u)f(u)\,du}.$$

Since $F(0) = 1$, we get $K = 1$ and thus

$$F(t) = e^{\int_0^t \xi(s)f(s)\,ds} + t\,G(\xi)\, e^{\int_0^t \xi(s)f(s)\,ds}.$$

The conclusion follows once we take the inverse S-transform, that is

$$Y(t) = \mathcal{E}_f(t) + t(\varphi \diamond \mathcal{E}_f(t)).$$

□

4.4 Application

One situation where anticipating initial conditions and integrands arise is in the pricing of bonds. In [22], the authors propose a framework for determining price dynamics of a bond $P(t,T)$ at time t, which matures at time T with a fixed expiration value $P(T,T) = 1$ almost surely. In effect, this involves considering a stochastic process driven by $B(t)$ which reaches a fixed value at a future time T almost surely.

The main idea involves considering a stochastic integral equation of the form

$$X_t = X_T - \int_t^T f(s)X(s)dB(s) - \int_t^T g(s)X(s)ds \tag{4.23}$$

where $f(s)$ is anticipating. As the authors point out, one must note that even if $f(s)$ and $g(s)$ are adapted, the terminal condition $X_T = 1$ makes for an anticipating equation. By introducing a time reversal operator (see [22]), Equation (4.23) becomes

$$X_t = X_0 - \int_0^t f(s)X(s)dB(s) - \int_0^t g(s)X(s)ds \tag{4.24}$$

with $x(0) = P(T,T) = x_0$.

Considering Equation (4.24), the authors assert the existence of a unique solution given by

$$x(t) = x_0\mathcal{E}_f(t)\text{exp}\ \{\int_0^t g(s)ds\}.$$

We will now derive this solution using the white noise methods discussed in the foregone. Indeed, in view of Equation (4.23), let $SX(t) = F(t), G = Sx_0$.

Then upon taking S-transforms, we have

$$F(t)(\xi) = G(\xi) + \int_0^u [\xi(s)f(s) + g(s)]F(s)(\xi)ds$$

almost surely, for all $u \in [0,t]$. Using similar arguments as in Theorem 4.5, let

$$H_\xi(u) = G(\xi) + \int_0^u v(s)F(s)(\xi)ds$$

where $v(u) = \xi(u)f(u) + g(u)$, so that u-a.s., $H_\xi(u) = F(u)(\xi), \forall \xi \in S_C$, and $H'_\xi(u) = v(u)H_\xi(u)$, u-a.s. in $[0, t]$ with $H_\xi(0) = G(\xi)$. Therefore,

$$H'_\xi(u) - v(u)H_\xi(u) = 0$$

$$\frac{d}{du}\{H_\xi(u)e^{-\int_0^u v(s)ds}\} = 0.$$

Since $H_\xi(0) = G(\xi), R = G(\xi)$ and so $H_\xi(u) = G(\xi)e^{\int_0^u v(s)ds}$.

$$F(u)(\xi) = G(\xi)e^{\int_0^u v(s)ds},$$

u-a.e. on $[0, t]$. Next, since $f \in L^2([0, t])$,

$$S(\mathcal{E}_f(u)) = e^{\int_0^u \xi(s)f(s)ds}.$$

Consequently, we have

$$S(x_0\mathcal{E}_f(u) = (Sx_0)(\xi)[S\mathcal{E}_f(u)(\xi)] = G(\xi)e^{\int_0^u \xi(s)f(s)ds}.$$

$$F(u)(\xi) = G(\xi)e^{\int_0^u \xi(s)f(s)ds} \cdot e^{\int_0^u g(s)ds} = G(\xi)S\varepsilon_f(u)(\xi)e^{\int_0^u g(s)ds}.$$

It follows upon inverting that

$$X(t) = x_0\mathcal{E}_f(u)\exp\{\int_0^u g(s)ds\},$$

as desired.

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Appendix: Permission

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Please confirm that this is ok with your company.

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Vita

Julius Njome Esunge was born in Victoria (Limbe), Republic of Cameroon, in January 1974. He finished his undergraduate and postgraduate studies in mathematics at the University of Buea, Cameroon, in December 1999. He earned a Master of Science degree in statistics from Lehigh University in Bethlehem, Pennsylvania, in May 2004. In August 2004, he came to Louisiana State University to continue graduate studies in mathematics. He earned a Master of Science degree from Louisiana State University in December 2005. He is now pursuing a doctoral degree. The degree of Doctor of Philosophy in mathematics will be awarded in August 2009.