Theories of Circular Functions: Part 3

Continuing our journey from last two posts we present some more approaches to the development of the theory of circular functions. One approach is based on the use of infinite series and requires basic knowledge of theory of infinite series. This approach is particularly well suited for treating circular functions as functions of a complex variable, but we will limit ourselves to the case of real variables only.

Before we start I should also mention in passing that it is possible to develop the theory of circular functions on the basis of infinite products also but dealing with infinite series is simpler and more popular.

Circular Functions as Infinite Series

We start with the definitions

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \tag{1}$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \tag{2}$$

Since both the series above are convergent for all values of x (check via ratio test), it follows that the functions $\sin x, \cos x$ are defined for all values of x. Moreover the above series belong to the category of power series (series of the form $\sum a_n x^n$), it follows that the functions defined above are continuous and differentiable in the interior of region of convergence of the series involved. Thus both $\sin x, \cos x$ are continuous as well as differentiable for all x. Since the power series can also be differentiated term by term it follows from the above definitions that

$$(\sin x)' = \cos x,\ (\cos x)' = -\sin x \tag{3}$$

Let's now consider the function

$$f(x) = \cos^2 x + \sin^2 x$$

We note that

$$f'(x) = -2\cos x \sin x + 2\sin x \cos x = 0$$

so that $f(x)$ is constant and therefore $f(x) = f(0) = 1$ and we get

$$\cos^2 x + \sin^2 x = 1 \tag{4}$$

for all values of x. From the above identity it also follows that both $\sin x, \cos x$ are bounded and

$$|\sin x| \leq 1,\ |\cos x| \leq 1 \tag{5}$$

for all x (a fact not obvious from the definitions $(1), (2)$).

Like we saw in the previous post it is easy to establish the addition formulas

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y \tag{6a}$$
$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y \tag{6b}$$

once the derivatives for $\sin, \cos$ are available.

In this approach based on the infinite series the real challenge is to introduce the number π. To that end let us estimate the value of $\cos 2$. Since the series for $\cos x$ is an alternating series it follows that $\cos x$ is less that the n^{th} partial sum of the series if n is odd and $\cos x$ is greater than the sum of n^{th} partial sum of the series if n is even. Thus we have

$$\cos 2 < 1 - \frac{2^2}{2!} + \frac{2^4}{4!} = 1 - 2 + \frac{2}{3} = -\frac{1}{3} < 0$$

and clearly $\cos 0 = 1 > 0$ therefore by intermediate value theorem it follows that there is a value $\xi \in (0, 2)$ for which $\cos \xi = 0$. Further there is a first value of $\xi \in [0, 2]$ for which $\cos \xi = 0$ and $\cos x > 0$ for all $x \in [0, \xi)$. This value of ξ is important in the development of the theory of circular functions and **we define $\pi = 2\xi$ where ξ is the first value of $x \in [0, 2]$ for which $\cos x = 0$.**

Thus we have $\cos(\pi/2) = 0$ and $\cos x > 0$ for all $x \in [0, \pi/2)$. Since $(\sin x)' = \cos x$ it follows that $\sin x$ is strictly increasing in $[0, \pi/2]$ and hence $\sin x > \sin 0 = 0$ for all $x \in (0, \pi/2]$. It now follows from equation (4) that $\sin(\pi/2) = 1$. Using addition formulas we can now prove that $\sin \pi = 0, \cos \pi = -1$ and further establish the formulas

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x, \sin\left(\frac{\pi}{2} - x\right) = \cos x \tag{7a}$$

$$\cos\left(\frac{\pi}{2} + x\right) = -\sin x, \sin\left(\frac{\pi}{2} + x\right) = \cos x \tag{7b}$$

$$\cos(x + \pi) = -\cos x, \sin(x + \pi) = -\sin x \tag{7c}$$

From these formulas it is easy to show that $\sin x, \cos x$ are periodic functions with period 2π. Thus we complete the development of theory of circular functions based on the infinite series. Another not so common approach to the theory of circular functions defines them as the solutions to the differential equation $y'' + y = 0$ which we discuss next.

Circular Functions as Solutions to the Equation $y'' + y = 0$

A slightly unusual way to look at circular functions $\cos x, \sin x$ is to view them as the solutions to the differential equation

$$\frac{d^2 y}{dx^2} + y = 0$$

Putting $y = f(x)$ we can rewrite the equation as

$$f''(x) + f(x) = 0 \tag{8}$$

Also let us fix the initial conditions as $f(0) = 0, f'(0) = 1$ so that we are first dealing with the solution $f(x) = \sin x$. From equation (8) it is obvious that the solution we are expecting is infinitely differentiable for all x and using initial conditions we can get the values of $f^{(n)}(0)$ for all n and using Taylor series for $f(x)$ we reach the infinite series for $\sin x$ presented earlier. However it is of interest to find the solution of differential equation (8) without the use of the infinite series.

First we show that the initial conditions uniquely determine the solution. Thus we show that if $f(x)$ satisfies the differential equation (8) and the initial conditions $f(0) = f'(0) = 0$ then $f(x) = 0$ for all values of x. The thing to note here is that the function

$$g(x) = \{f'(x)\}^2 + \{f(x)\}^2$$

has the derivative

$$g'(x) = 2f'(x)f''(x) + 2f(x)f'(x) = 2f'(x)\{f''(x) + f(x)\} = 0$$

and hence $g(x) = g(0) = 0$ for all x. Thus $f'(x) = f(x) = 0$ for all x. So we are guaranteed that if a solution to the differential equation (8) exists then it is unique and fully dependent on the initial values of $f(0)$ and $f'(0)$.

Next we solve the equation (8) with the initial conditions $f(0) = 0, f'(0) = 1$. As before if

$$g(x) = \{f'(x)\}^2 + \{f(x)\}^2$$

then $g(x) = g(0) = 1$ for all x and hence

$$f'(x) = \sqrt{1 - \{f(x)\}^2}$$

(we choose $+$ sign for square root so that the relation holds true for values of $f(0)$ and $f'(0)$) and if we invert the equation $y = f(x)$ by $x = f^{-1}(y)$ then we get

$$\frac{dy}{dx} = f'(x) = \sqrt{1 - y^2}$$

or

$$\frac{dx}{dy} = \frac{1}{\sqrt{1 - y^2}}$$

and we are led to the function

$$x = f^{-1}(y) = \int_0^y \frac{dt}{\sqrt{1 - t^2}} \tag{9}$$

and thus $y = f(x)$ is the inverse of this integral. The existence of this integral justifies the existence of the solution of the differential equation (8). Note however that the integral in equation (9) defines x as a strictly monotone function of y only in the interval $(-1, 1)$ and using improper integrals its definition is extended to the closed interval $[-1, 1]$.

We now introduce π as

$$\pi = 2\int_{0}^{1}\frac{dx}{\sqrt{1 - x^2}} \tag{10}$$

so that the function $x = f^{-1}(y)$ maps $[-1, 1]$ to $[-\pi/2, \pi/2]$. Thus the function $y = f(x)$ is strictly increasing and maps $[-\pi/2, \pi/2]$ to $[-1, 1]$. This unique solution to

$$f''(x) + f(x) = 0,\ f(0) = 0,\ f'(0) = 1$$

is denoted by $\sin x$ and its derivative $f'(x)$ (which also satisfies (8)) is denoted by $\cos x$. So far these functions have been defined only in the interval $[-\pi/2, \pi/2]$. From the differential equation we obtain the relations

$$(\sin x)' = \cos x,\ (\cos x)' = -\sin x$$

and using these derivatives it is easy to prove the addition formulas for $\sin, \cos$ provided the arguments for $\sin, \cos$ lie in the interval $[-\pi/2, \pi/2]$. It now make sense to use the addition formulas and extend the domain of definition of these functions via the following relations

$$\sin \pi = 0, \cos \pi = -1, \sin(x + \pi) = -\sin x, \cos(x + \pi) = -\cos x \tag{11}$$

The functions $\sin x, \cos x$ are now defined for all values of x and satisfy the differential equation (8) everywhere. From the equation (11) it also follows that these functions are periodic with period 2π.

Once the functions $\sin x, \cos x$ are available we can find general solution to the differential equation (8). Let

$$h(x) = f(x) - f(0)\cos x - f'(0)\sin x$$

and then we have $h(0) = h'(0) = 0$ and $h''(x) + h(x) = 0$ so that $h(x)$ also satisfies the differential equation (8) and as we proved before $h(x) = h(0) = 0$ for all x and hence the solution to the equation (8) is given by

$$f(x) = f(0)\cos x + f'(0)\sin x$$

and we clearly see that the initial values $f(0), f'(0)$ determine the solution completely.

The above result regarding solution of $y'' + y = 0$ can also be used to establish addition formulas for $\sin x, \cos x$. Consider the function $f(x) = \sin(x + a)$ which clearly satisfies the differential equation $f''(x) + f(x) = 0$ and we have $f'(x) = \cos(x + a)$. Thus we have

$$f(0) = \sin a, f'(0) = \cos a$$

By the argument in the previous paragraph we must have

$$\begin{aligned}\sin(x + a) &= f(x)\\ &= f(0)\cos x + f'(0)\sin x\\ &= \sin a\cos x + \cos a\sin x\end{aligned}$$

which is the desired addition formula. The formula for $\cos(x + a)$ can also be derived in similar manner.

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Labels: Mathematical Analysis , Trigonometry