Hyperbolic links: diagrams, volume and the colored Jones polynomial

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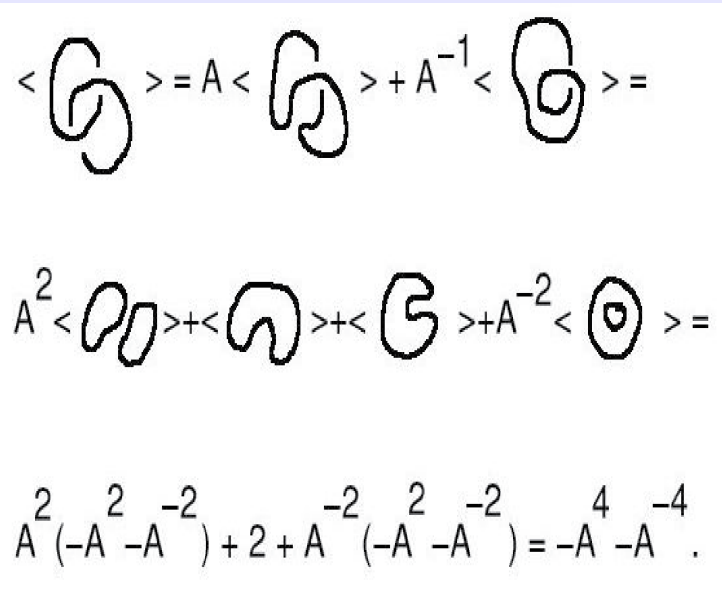
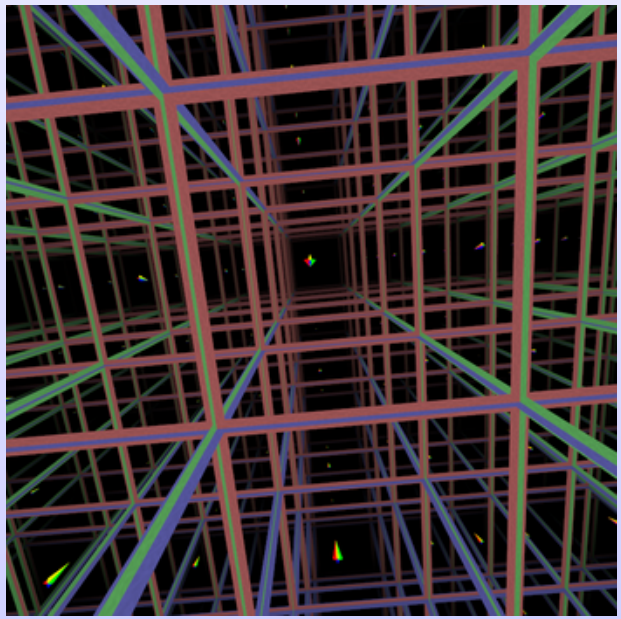
Different parts of this work are joint with Oliver Dasbach, and Morwen Thistlethwaite

Quantum invariants of the link (colored Jones polynomial) $\longleftrightarrow$ Intrinsic geometry of the link complement (hyp. volume)

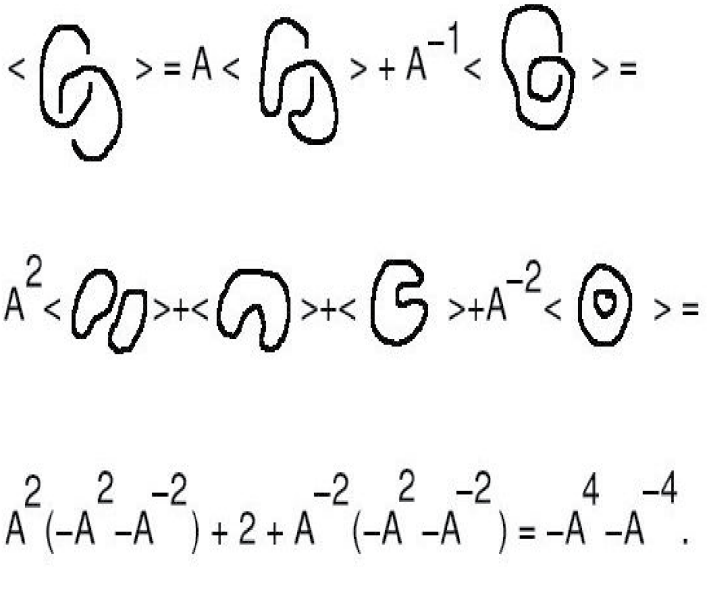


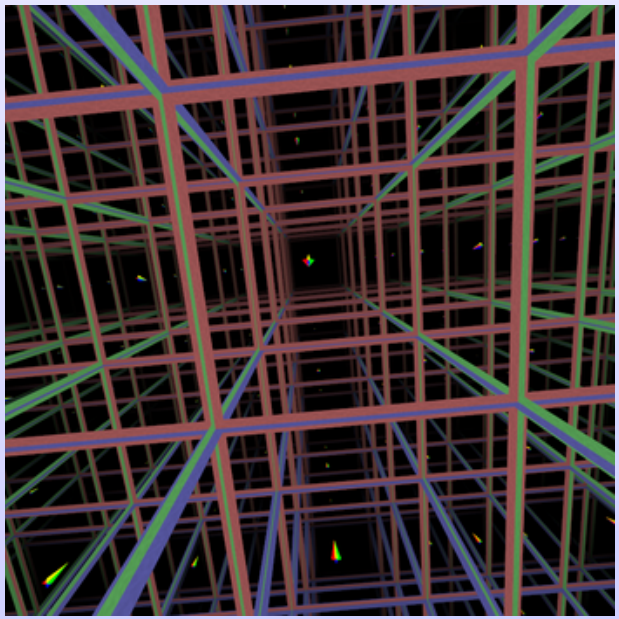


Link diagram and it properties
(combinatorial description)

Quantum invariants of the link
(colored Jones polynomial)

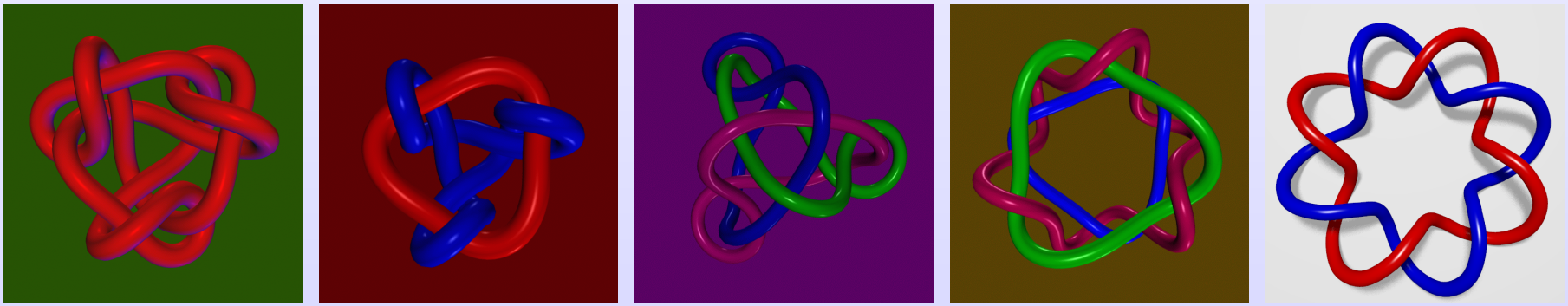
Intrinsic geometry of the link
complement (hyp. volume)





Links and their intrinsic geometry

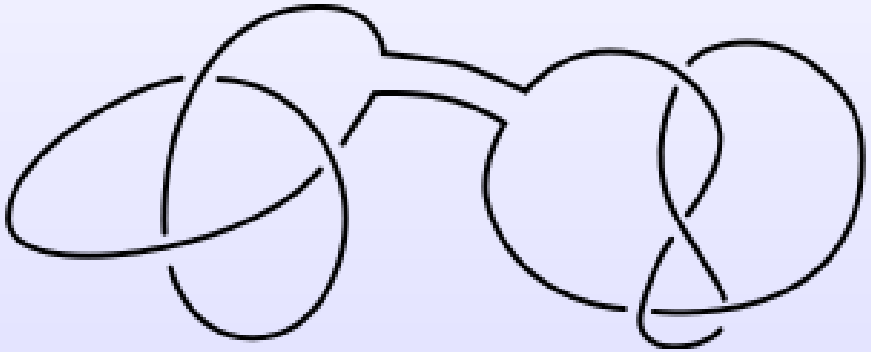
In 1976, Thurston suggested the Geometrization conjecture, and demonstrated that many 3-manifolds either have hyperbolic metric or can be decomposed into pieces with hyperbolic metric (the Hyperbolization theorem). It allowed to study manifolds from a new perspective: using geometry. By the virtue of Mostow-Prasad rigidity, for a manifold with finite volume hyperbolic metric is unique as long as it is complete, and gives rise to numerous invariants.

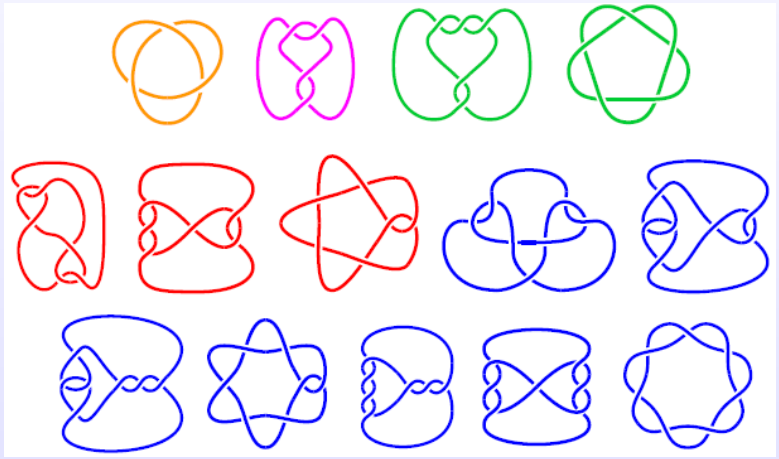


In particular, Thurston demonstrated that every link in S^3 is either a (generalized) torus link, a satellite link (*i.e.* contains it an incompressible, non-boundary parallel torus in its complement), or a hyperbolic link, and these three categories are mutually exclusive.

Why study hyperbolic knots and links?

A knot or a link is **prime** if it cannot be written as a connected sum of 2 knots.



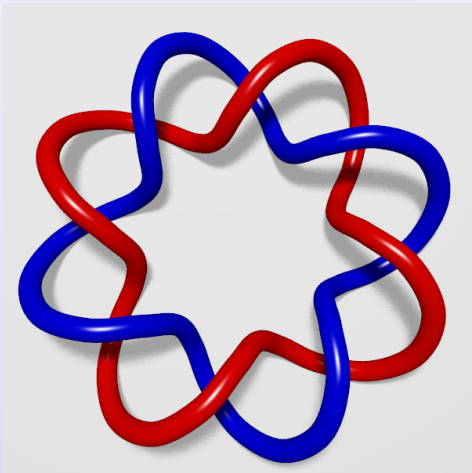


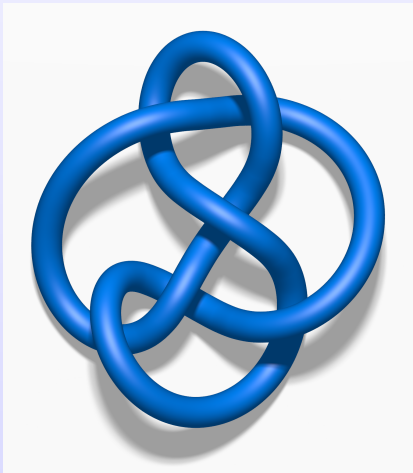
Every knot can be **uniquely decomposed** as a knot sum of **prime** knots (H. Schubert). It also true for non-split links, *i.e.* if there is no a 2–sphere in the complement separating the link (Hashizume).

Of the 14 prime knots up to 7 crossings, only 3 are non-hyperbolic.
Of the 1,701,935 prime knots up to 16 crossings, 32 are non-hyperbolic.
Of the 8,053,378 prime knots with 17 crossings, 30 are non-hyperbolic.

Why study hyperbolic knots and links?

An alternating diagram which is prime in a diagrammatic sense represents a link which is prime in the 3-dimensional sense, and the complement of such link is atoroidal (Menasco). Hence, a prime alternating diagram represents either a hyperbolic link or a torus link. The only alternating torus links are those of type $(2, n)$.



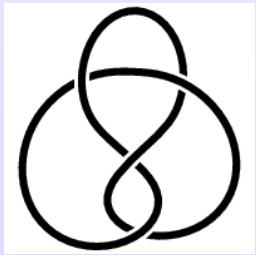
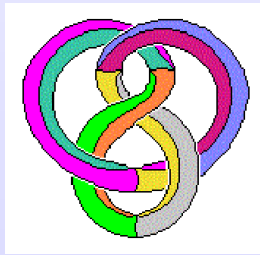
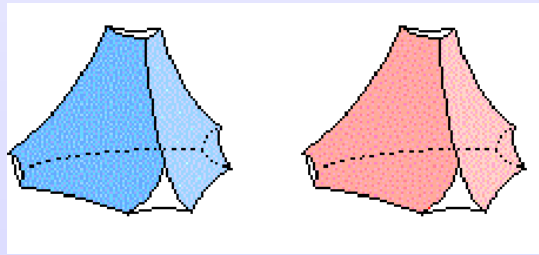


From the solution of the Tait flyping conjecture (W. Menasco, M. Thistlethwaite), each reduced alternating diagram of $(2, n)$–torus link is standard. Thus one can immediately see if a prime alternating diagram represents a hyperbolic link.

Links and their intrinsic geometry

We may view a hyperbolic 3–manifold M as the quotient of $\mathbb{H}^3$ by a discrete group of hyperbolic (fixed point free) isometries, which is isomorphic to the fundamental group of M. A hyperbolic link L in the 3–sphere S^3 is such that the complement $S^3 - L$ is a hyperbolic manifold.

To read information about the intrinsic topological and geometric properties of a link complement (e.g., about its volume) from the link diagram is among the major tasks of knot theory.

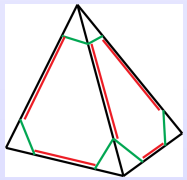




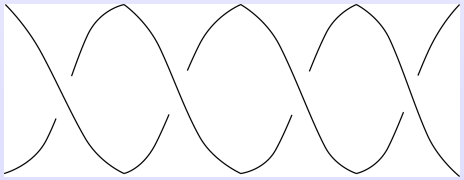
To compute the volume, one needs to triangulate the complement, and to find dihedral angles of every tetrahedron. If the angles are α_i, then the volume is $\sum_i \Lambda(\alpha_i)$, where $\Lambda(\theta) = -\int_0^\theta \log 2|\sin t|dt$ is the Lobachevsky function.

Volume bounds for hyperbolic links

There is no simple expression for volume function in general. Instead of the exact calculation, one may estimate volume from a diagram using volume bounds.

Theorem (Adams 1983; Lackenby 2004). For a hyperbolic link K with crossing number c, different from the figure-eight knot, $Vol(K) \leq (4c - 16)v_3$, where v_3 is the volume of a regular ideal hyperbolic tetrahedron.





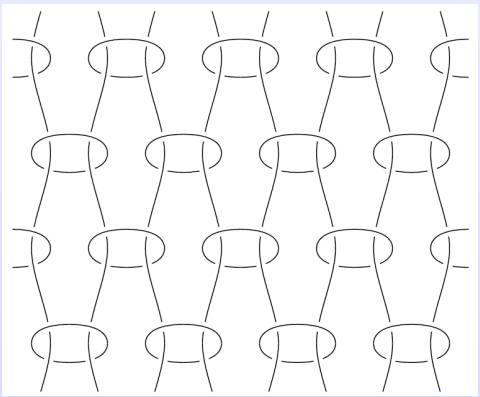
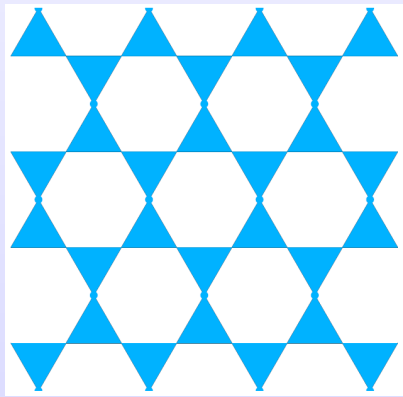
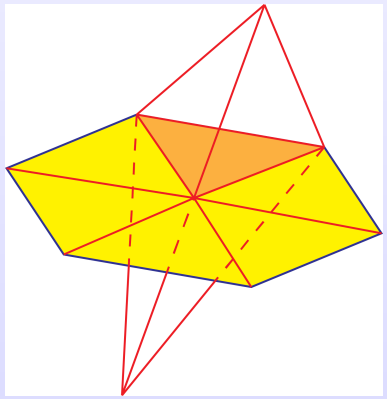
A twist is either a connected collection of bigons arranged in a row, which is not a part of a longer row of bigons, or a single crossing adjacent to no bigons. The twist number $t(D)$ is the number of twists in the diagram D.

Theorem (Lackenby 2004). For a hyperbolic link K with a prime alternating diagram D, $v_3(t(D) - 2)/2 \leq Vol(K) < v_3(16t(D) - 16)$.

Improved upper bound

Theorem (Agol and Thurston, 2004). Given a diagram D of a link K, $Vol(K) \leq 10(t(D) - 1)v_3$.

This bound is asymptotically sharp and the constant 10 cannot be improved for hyperbolic links in general. A chain fence link, which realizes the upper bound exactly, and the corresponding (infinite) polyhedron:

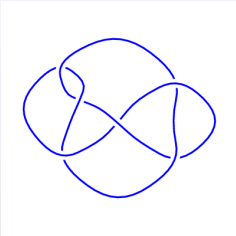




Bounds for some other families of hyperbolic links (e.g., lower bounds beyond alternating links) were obtained by Futer, Kalfagianni and Purcell.

Therefore, one may use certain properties of a link diagram to estimate the volume of the link complement.

Quantum invariants of links

In 1984, Vaughan Jones discovered the Jones polynomial for knots. To each oriented knot or link, it assigns a Laurent polynomial with integer coefficients.



$$-t^3 + 2t^2 - 2t + 3 - \frac{2}{t} + \frac{2}{t^2} - \frac{1}{t^3}$$

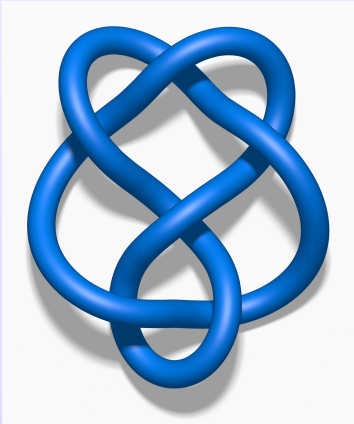
It was not the first polynomial invariant, but the discovery stimulated a development of a new field of study inspired by ideas from theoretical physics: quantum invariants of knots and 3-manifolds. Some of these invariants (to name just a few): finite type invariants (or Vassiliev invariants), Kontsevich integral, Witten-Reshetikhin-Turaev invariant, and many other ones.

Since quantum invariants were introduced into knot theory, there has been a strong interest in relating them to the intrinsic geometry of a link complement.

Colored Jones polynomial

For a link K, the colored Jones polynomial $J(K,n)(q)$ is a sequence of Laurent polynomials in q indexed by a natural number n (color), with integer coefficients. Color $n=2$ gives Jones polynomial.

$$\pm(a_n q^{k_n} - b_n q^{k_n-1} + c_n q^{k_n-2}) + \ldots \pm(\gamma_n q^{k_n-r_n+2} - \beta_n q^{k_n-r_n+1} + \alpha_n q^{k_n-r_n})$$



Polynomial coefficients

$n=2: \{1,-2,2,-2,2,-1,1\}$

$n=3: \{1,-2,0,4,...,3,-1,-1,1\}$

$n=4: \{1,-2,0,2,...,0,-1,-1,1\}$

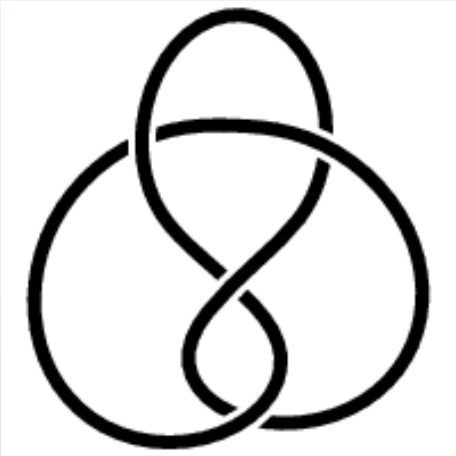
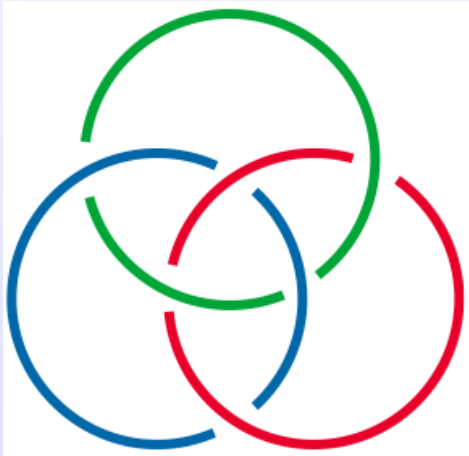
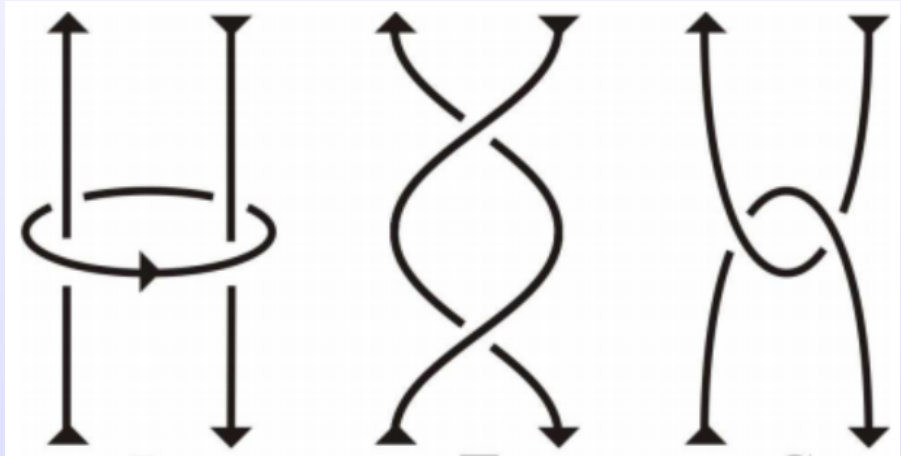
.....

There are various approaches to defining the colored Jones polynomial: e.g., via quantum groups and R-matrices (Turaev); via Kauffman bracket and the skein relation; via Chern-Simons theory (by Witten). None of these approaches provides a connection with the geometry of a link complement.

Volume Conjecture

(Kashaev 1997, J.Murakami-H.Murakami 2001).

For a hyperbolic link K, $\lim_{n\to\infty} \frac{2\pi \log |J(K,n)(e^{2\pi i/n})|}{n} = Vol(K)$.

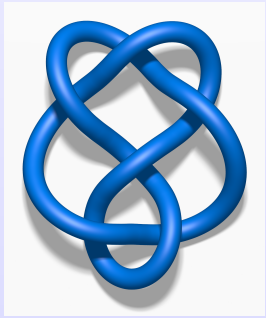




The conjecture was proved only for some hyperbolic knots and links: figure-eight knot (there are several proofs, e.g. by T. Ekholm), Borromean rings (Garoufalidis and Le), augmented octahedral links, Whitehead chains (van der Veen) and twisted Whitehead links (Zheng). Even a class of 2-bridge links remains a mystery in this regard.

Questions motivated by the Volume Conjecture

Is there a correlation of the polynomial coefficients with hyperbolic volume? And what can be said about the coefficients themselves when $n \to \infty$?

Theorem (Thistlethwaite, 1986). For a link with a connected, irreducible, alternating diagram, the first and last coefficients (i.e the coefficients of the terms of maximal and minimal degree) in Jones polynomial are ± 1.



Polynomial coefficients

$n = 2 : \{1, -2, 2, -2, 2, -1, 1\}$

$n = 3 : \{1, -2, 0, ..., -1, -1, 1\}$

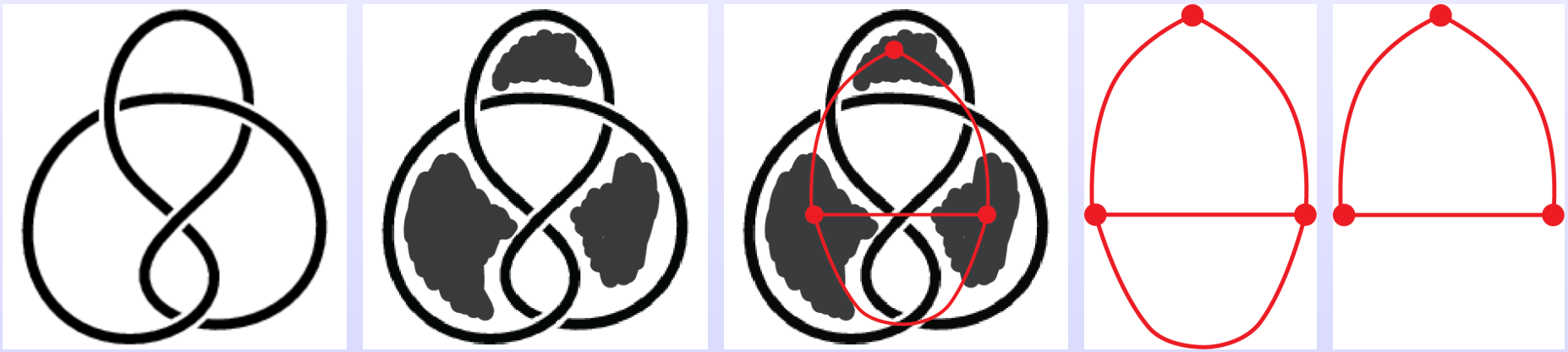
$n = 4 : \{1, -2, 0, ..., -1, -1, 1\}$

Theorem (Dasbach and Lin, 2006). For an alternating link, the absolute values of the first three and last three coefficients of the colored Jones polynomial are independent of the color n when $n \geq 3$, and second and penultimate coefficients are independent of n for $n \geq 2$. Moreover, the leading and trailing coefficients are known to be ± 1 for all n.

Coefficients of the colored Jones polynomial for an alternating link

For alternating links, the first three and the last three coefficients depend only on the reduced checkerboard graphs (Dasbach and Lin).

To obtain a black checkerboard graph, color the diagram in black and white, place a vertex inside each black region, and connect two vertices if they are adjacent to the same crossing.



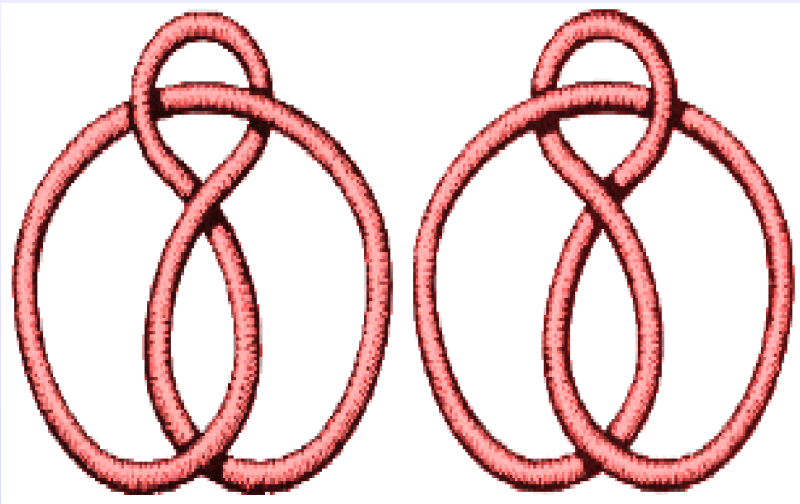
For a reduced graph, delete all multiple edges, leaving just one.

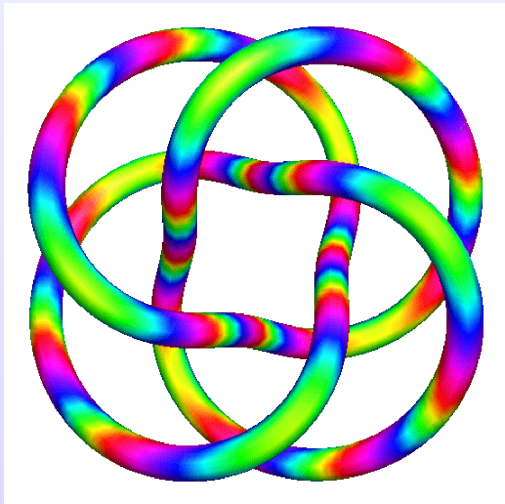
Therefore, one may use certain properties of a link diagram to compute the last and first three coefficients.

Further coefficients of the colored Jones polynomial

The tail (respectively head) of a polynomial is the sequence of its lowest (respectively highest) degree terms, up to some specified length.

For a link L and its mirror image L^*, the colored Jones polynomial $J(L, n)(q) = J(L^*, n)(1/q)$, and therefore the head of L is the tail of L^*.





But the head or tail does not always exist.

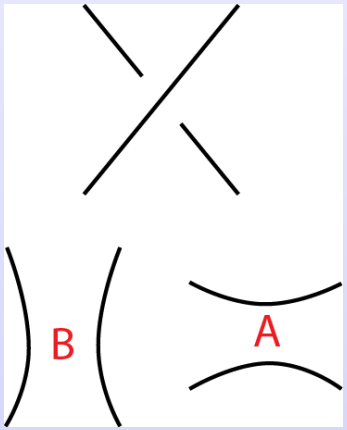
Example (Armond, Dasbach). The (4, 3)-torus knot has different tails and one head: one tail for even color n, and one tail for odd n. Its mirror image, respectively, has two heads.

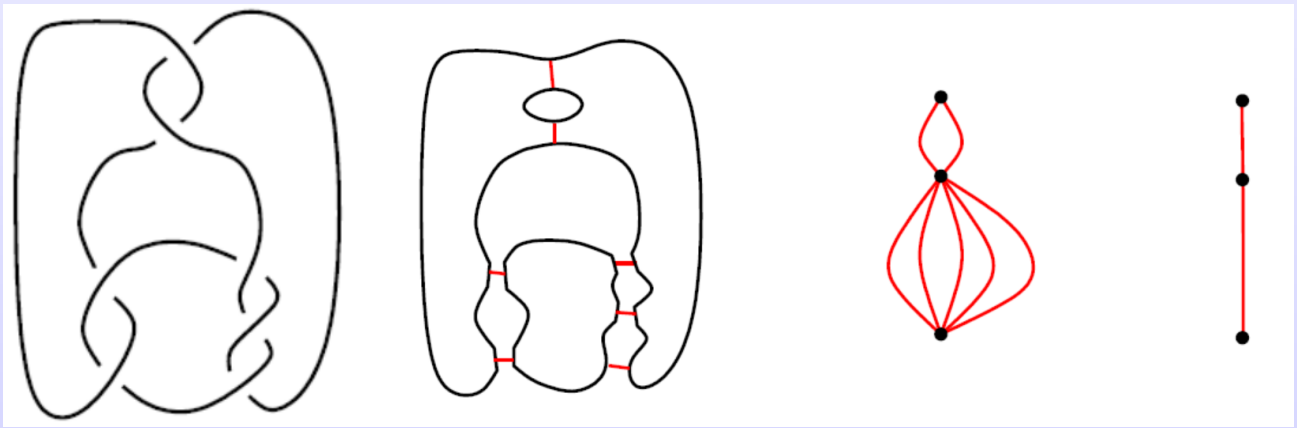
Further coefficients of the colored Jones polynomial

Conjecture (Dasbach, Lin). The head and tail exist for alternating knots.

I.e. all coefficients of $J(K, n)$ stabilize for n large enough.

For every crossing in a link diagram, choose A-resolution. The result is a collection of circles. Collapse each circle to a point, and connect the points through the letters A. If the resulting graph has no 1-edge loops, the link is called A-adequate. If it the link is A and B adequate, it is called adequate. Note: for alternating links, the resulting graphs are just black and white checkerboard graphs.





The conjecture was proved by Armond for adequate links, using skein theoretical techniques (2011). Independently with different methods it was proved by Garoufalidis and Le for alternating links (2011).

Volumish Theorems

The above results were used to establish correlation of the first two and last two coefficients of colored Jones polynomial with volume.

For color $n \geq 3$, let the colored Jones polynomial of K be
$\pm(aq^{k_n} - bq^{k_n-1} + cq^{k_n-2}) + \ldots \pm (\gamma q^{k_n-r_n+2} - \beta q^{k_n-r_n+1} + \alpha q^{k_n-r_n})$

Theorem (Dasbach and Lin, 2006). Let K be an alternating link, and b be the second coefficient of the colored Jones polynomial, and β - the penultimate coefficient. Then

$$v_8(max(|b|, |\beta|) - 1) \leq Vol(K) \leq 10v_3(|b| + |\beta| - 1),$$

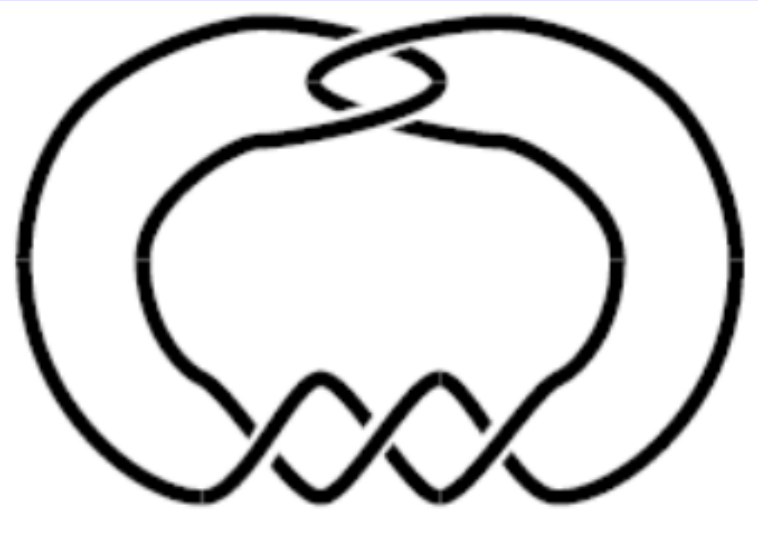
where v_8 is volume of an ideal regular hyperbolic octahedron.

Volume bounds for various families of hyperbolic links (beyond alternating) in terms of a, b, β, α were obtained by Futer, Kalfagianni, Purcell.

A refined approach to upper bound for volume

Let us look again at the upper bounds for the volume that were used to establish the correlation.

Adams, Lackenby
Work with the given diagram
(Euler characteristic argument)
4 tetrahedra per crossing
$Vol(K) \leq 4c - 16$



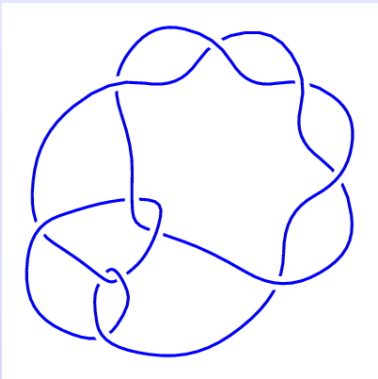
Lackenby, Agol-Thurston
Augment every twist in the diagram
(Dehn filling argument)
10 tetrahedra per twist
$Vol(K) \leq 10t - 10$

Naturally, both expressions are often serious overestimates.

A refined approach to upper bound for volume

We saw that the constant 10 in the Agol-Thurston upper bound cannot be improved in general. To show a *close correlation* of the volume with polynomial coefficients, we will refine the bound instead, involving more parameters coming from the link diagram.

Let $t_i(D)$ be the number of twists that have exactly i crossings (i.e. $i-1$ bigons), and $g_i(D)$ - the number of twists that have at least i crossings.

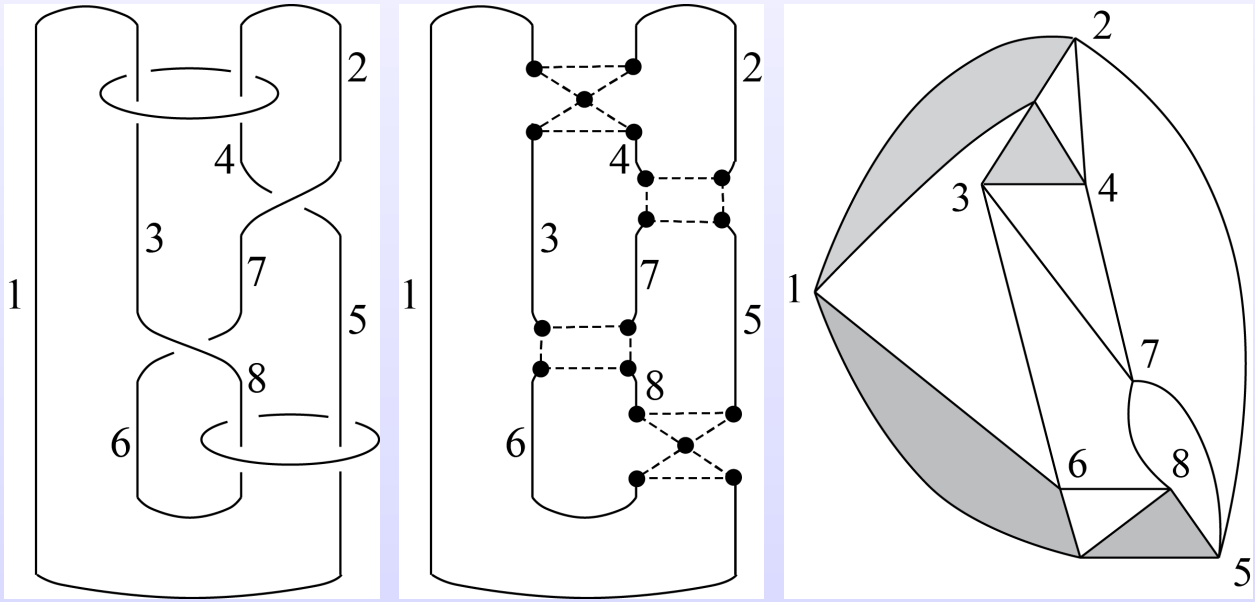


Twist number: $t = 5$
Twists with one crossing: $t_1 = 2$
Twists with two crossings: $t_2 = 2$
Twists with three crossings: $t_3 = 0$
Twists with at least four: $g_4 = 1$

Thm (Dasbach and T., 2013). For a hyperbolic alternating link K with a diagram D, $Vol(K) \leq (10g_4(D) + 8t_3(D) + 6t_2(D) + 4t_1(D) - A)v_3$, where $A = 10$ if g_4 is non-zero, $A = 7$ if t_3 is non-zero, and $A = 6$ otherwise.

Refined approach to upper bound for volume: the idea of the proof

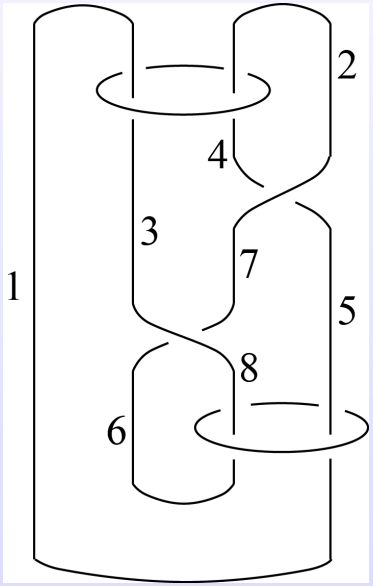
First, augment all twists of L (as Lackenby did) except for the ones with 1, 2 or 3 crossings, obtaining a new link L'. The complement of L can be obtained from that of L' by Dehn filling, and therefore $Vol(S^3 - L) \leq Vol(S^3 - L')$ (Thurston).

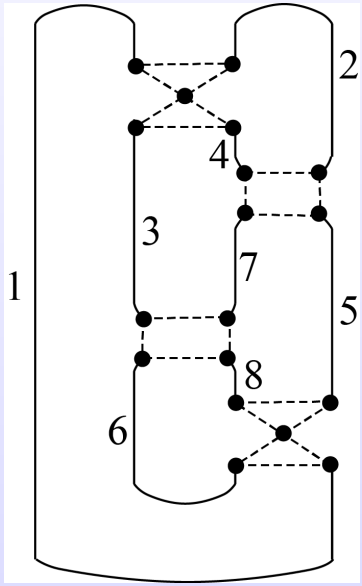


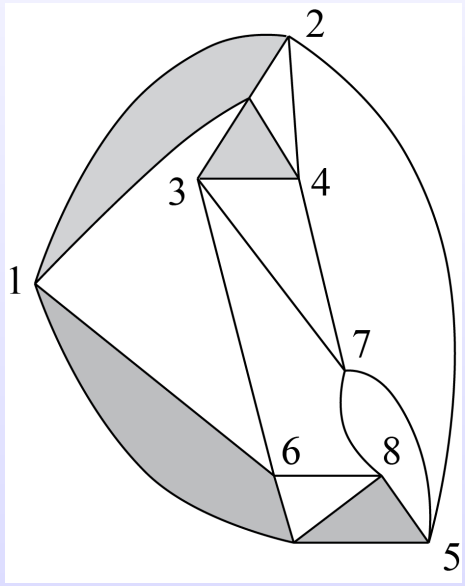
Then perform a decomposition into two polyhedra, one above the projection plane, and one below. The decomposition is a "hybrid" of Menasco's decomposition for alternating links and Agol-Thurston decomposition for fully augmented links.

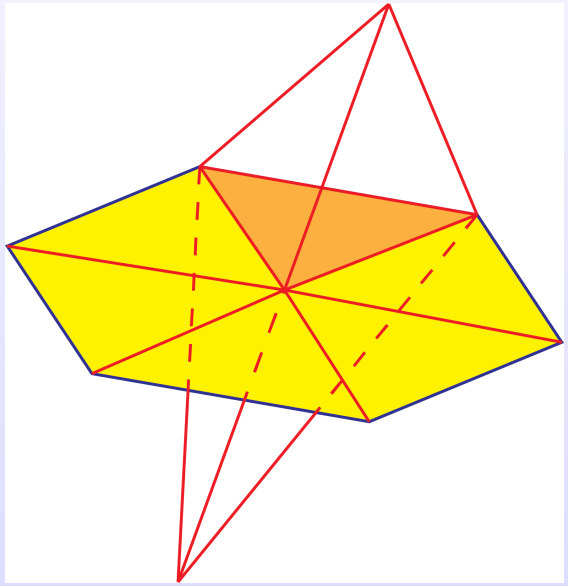
Refined approach to upper bound for volume: the idea of the proof

To obtain the complement of L', fold rectangular dotted faces, glue together two triangular dotted faces at every bow-tie, and double along the rest of the faces.









We can now perform a triangulation and count the maximal possible number of tetrahedra. Each bow-tie face yield four tetrahedra (two above and two below), and the rest of the faces yield one tetrahedron per edge (use a stellar decomposition). The faces that will be folded in the end do not add to the count.

An improved Volumish theorem for alternating links

This bound allowed to improve the Volumish theorem, showing for the first time that the first **three** and the last **three** coefficients correlate with the volume. For a link K, let the colored Jones polynomial (color n) be $\pm(a_n q^{k_n} - b_n q^{k_n-1} + c_n q^{k_n-2}) + \ldots \pm(\gamma_n q^{k_n-r_n+2} - \beta_n q^{k_n-r_n+1} + \alpha_n q^{k_n-r_n})$.

Theorem (Dasbach and T., 2013). Let K be a hyperbolic alternating link. Then $Vol(K) \leq (6((c_2+\gamma_2)-(c_3+\gamma_3)) - 2(b_2+\beta_2) - A)\, v_3$, where A is a linear constant taking values from 4 to 10 depending on the link diagram.

Note: it is not obvious from the expression, but it is an improvement over the bound in the Dasbach-Lin Volumish Theorem (can be proved using properties of $a_n, b_n, c_n, \alpha_n, \beta_n, \gamma_n$).

Example. The volume of the figure-8 is $2v_3$. The previous upper bound resulted in $10v_3$ for this link. The bound above gives $4v_3$.

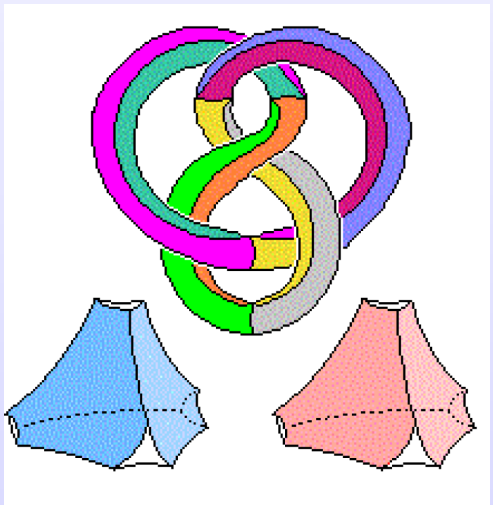
Further questions

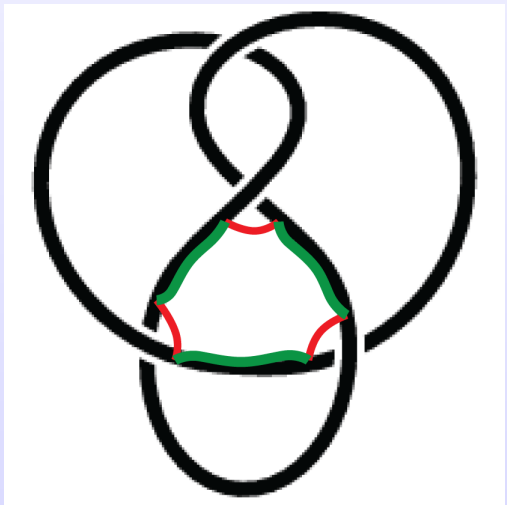
- Can one refine the **lower** bound for alternating links (by Lackenby)?
- Can one refine the volume bounds for families of links beyond alternating (by Futer, Kalfagianni, Purcell)?
- For adequate links, other coefficients of the colored Jones polynomial stabilize for a color n large enough ("the head and tail" by Dasbach-Lin, Garoufalidis-Le, Armond). How are they related to the volume? Is a further refinement possible?
- Instead using the polynomial coefficients in the volume bounds, can one show the relation of the coefficients to the exact volume?

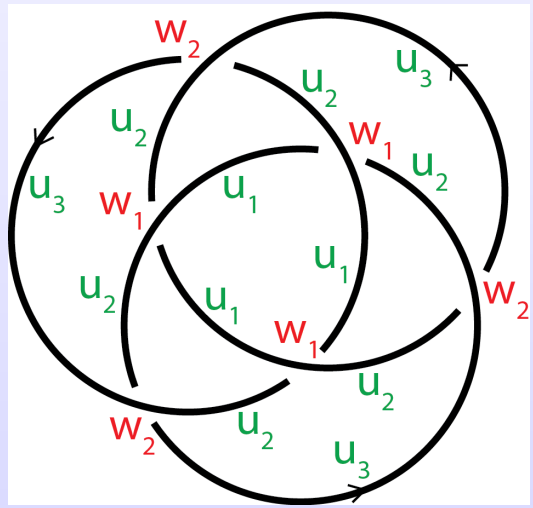
And how to compute the **exact volume**? Snappea (by Weeks, the latest version by Culler and Dunfield) produces decimal approximation, while Snap (Goodman, Hodgson, Neumann) uses the LLL algorithm for an intelligent guess of the corresponding polynomial. Our goal is rather to **construct** a polynomial and to find the volume as an analytic function of one of its roots. In what follows, we will do it for a hyperbolic 2-bridge link using its reduced alternating diagram.

Methods for computing hyperbolic structures

To compute the exact volume, we will use an alternative method (M. Thistlethwaite, A. T.) for computing hyperbolic structures of links. It is based on ideal polygons bounding the regions of a link diagram rather than decomposition of the complement into ideal tetrahedra (as compared to the classical method by Thurston, implemented in SnapPea by Weeks).



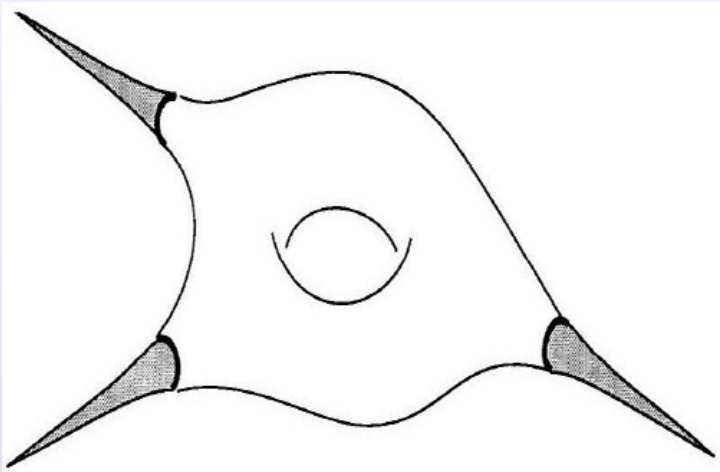


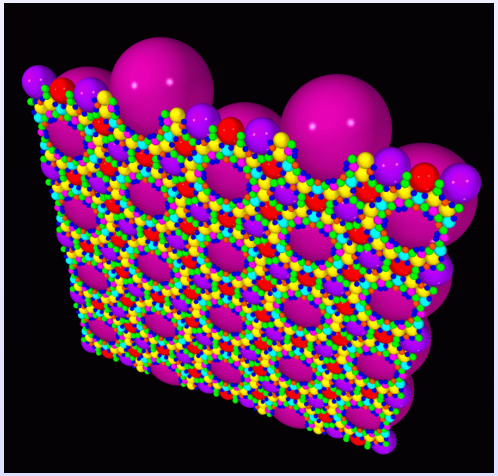


Given a diagram, we shall associate a complex number to every crossing, and a complex number to each side of every edge. We shall define these numbers so that they contain all the information about the horoball packing in $\mathbb{H}^3$. The numbers can be found from the link diagram.

The alternative method for computing hyperbolic structures

Rather than describing a triangulation, we will describe the horoball packing associated with the hyperbolic link. The preimage of a cusp in $\mathbb{H}^3$ is a set of horoballs. The horoball packing of the figure-8 knot (the upper half space model of $\mathbb{H}^3$):

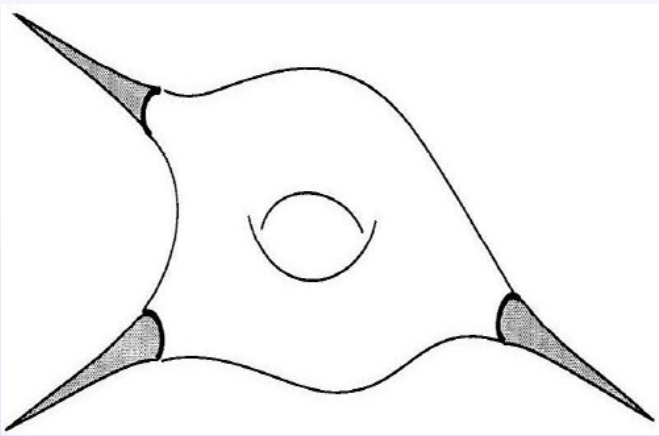


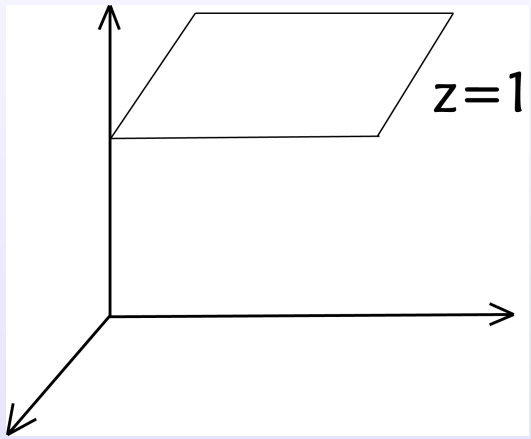


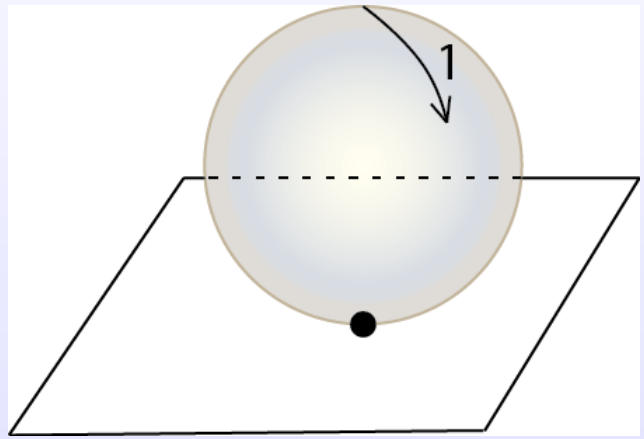
There are horoballs of an arbitrarily small Euclidean diameter and one additional horoball, the plane $z = 1$. Cusps may be chosen so that the horoballs have disjoint interiors. Their centers (*i.e.* points of tangency with the boundary of $\mathbb{H}^3$) form a dense subset of this boundary. The knot group acts transitively on the entire collection of horoballs, and for a link there is one orbit for every component.

The alternative method: preliminaries

Take horospherical cross-sections of the cusps of $S^3 - L$, so that length of a meridian on each cross-section is 1.







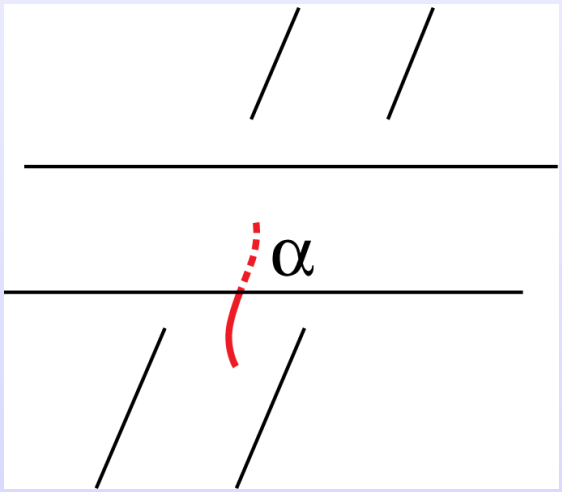
Choose the coordinates in $\mathbb{H}^3$ so that a component of the preimage of some particular cross-section is the Euclidean plane $z = 1$.

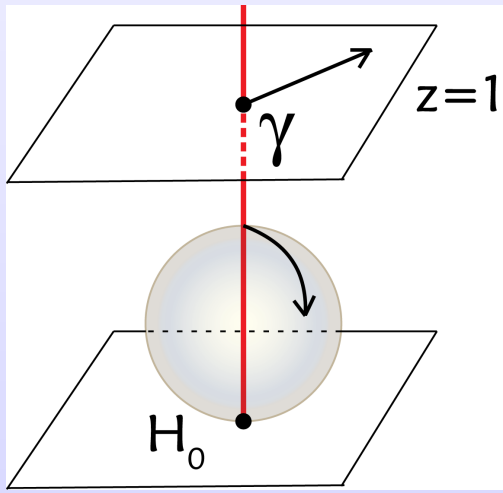
Parameterize Euclidean translations on each horosphere by complex numbers so that the meridional translation corresponds to 1.

Such choices guarantee that the horoballs in $\mathbb{H}^3$ have disjoint interiors (C. Adams).

Outline of the alternative method: complex labels

A **crossing label** contains the geometric information about a preimage of a crossing arc, traveling from an overpass to an underpass at a crossing of the diagram. Under a few diagrammatic restrictions, every such arc is properly homotopic to a unique geodesic in the link complement, which has a geodesic γ as the preimage in $\mathbb{H}^3$, joining the centers of the corresponding horospheres.

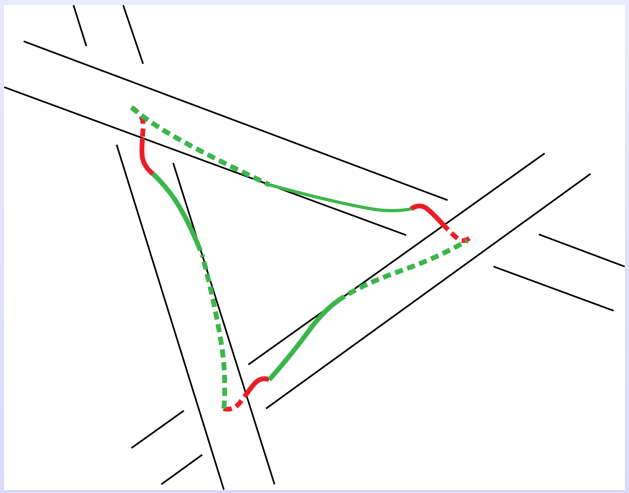


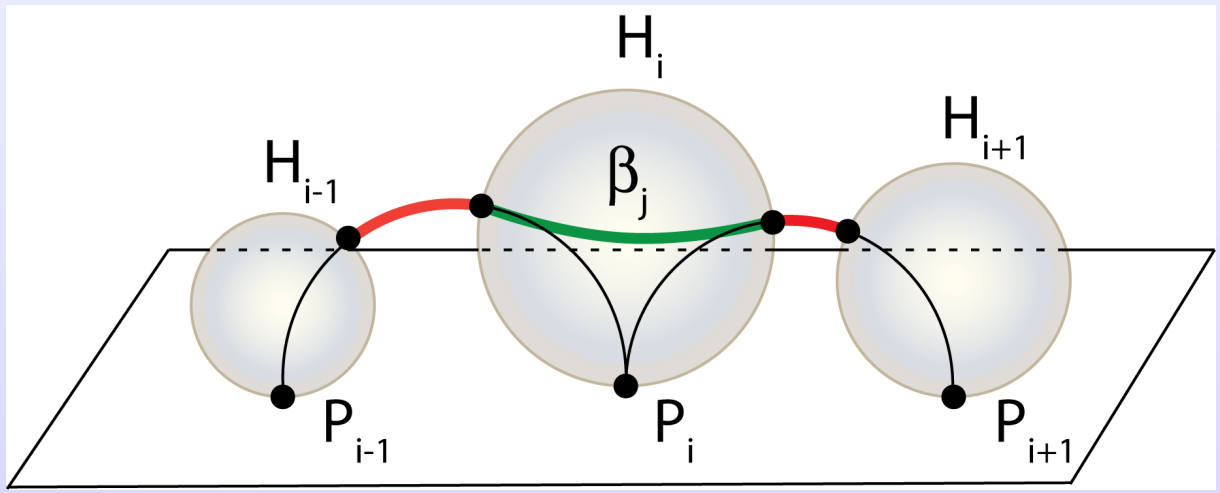


In $\mathbb{H}^3$, let the modulus of the crossing label $|w| = e^{-d}$, where d is the hyperbolic intercusp distance along the geodesic γ. Let the argument of w be the exterior dihedral angle between two half-planes, each determined by γ and the meridional translation on one of the horospheres.

Outline of the alternative method: complex labels

A k-sided **region in the link diagram** is a disk whose boundary is a union of k arcs on the boundary torus and k arcs at crossings. The preimage of this boundary in $\mathbb{H}^3$ determines a cyclic sequence of horospheres $H_1, ..., H_k$ with centers P_1 , ... , P_k . Let β_j be a sub-arc corresponding to a Euclidean line segment which joins the point where geodesics $P_{i-1}P_i$ and P_iP_{i+1} pierce H_i.

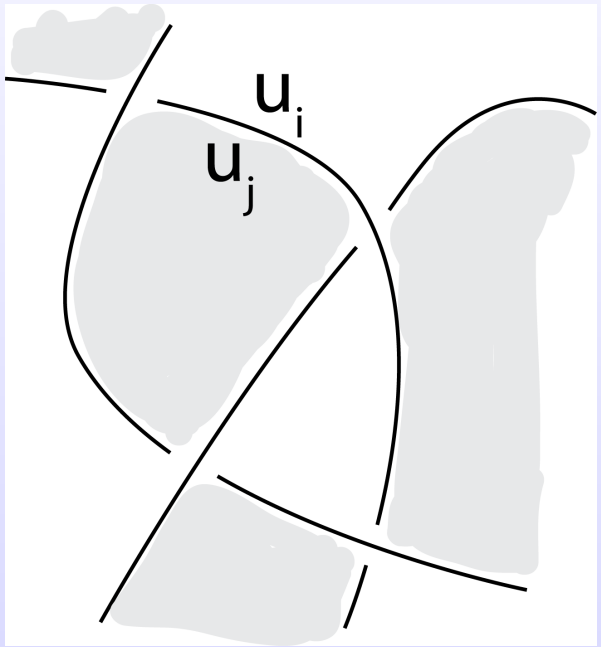


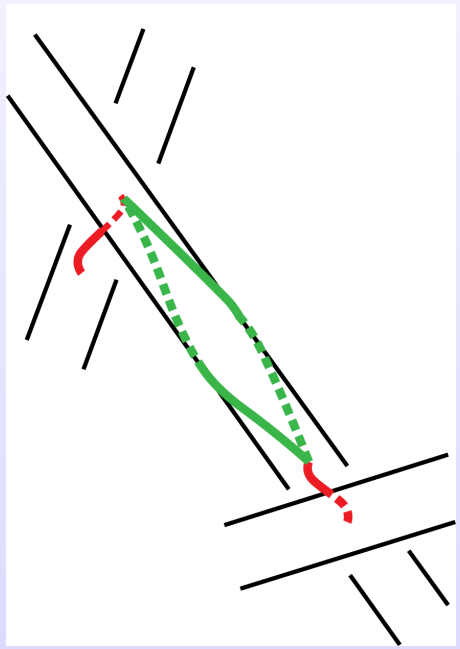


An **edge label** contains information about a preimage of the corresponding arc traveling on the boundary torus between two crossings. In $\mathbb{H}^3$, it is a complex number determining a translation along β_j (its orientation is inherited from the orientation of the link).

The alternative method: edge relations

Color the regions of the link diagram in black and white as a checkerboard. Each edge gives rise to two arcs: on the boundary of the black region, and on the boundary of the white one.

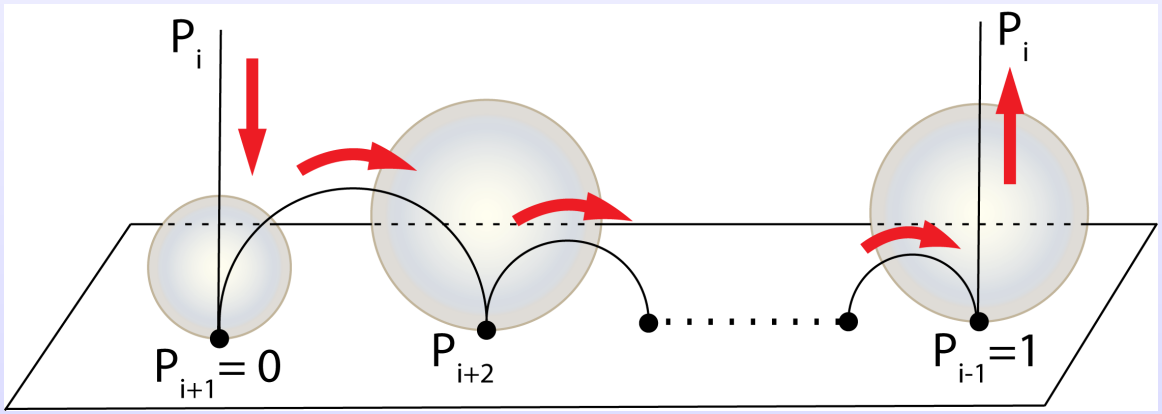


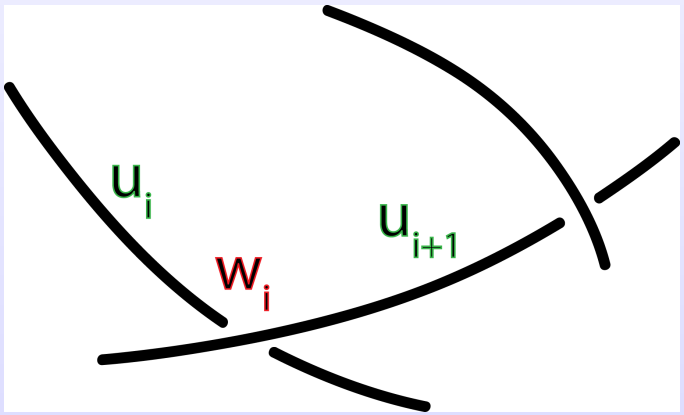


On the boundary torus, traveling along one arc and returning along another corresponds to a meridian, which is 1. In an alternating diagram, this is always the case, and $u_i - u_j = 1$ holds for every edge. In a non-alternating diagram this difference may be 1, -1 or 0.

The alternative method: region relations

Return to the ideal polygon with the vertices $P_1, ..., P_k$ in the boundary of $\mathbb{H}^3$. To every geodesic in the polygon, assign the parameter $\xi_i = \dfrac{\pm w_i}{u_i u_{i+1}}$, where the sign depends on the link orientation. Geometrically, ξ_i assigned to the geodesic $P_i P_{i+1}$, is the cross-ratio $\dfrac{(P_{i-1} - P_i)(P_{i+1} - P_{i+2})}{(P_{i-1} - P_{i+1})(P_i - P_{i+2})}$ (the "shape").





Perform an isometry to place the vertices P_{i-1}, P_i, P_{i+1} of the the polygon at $1, \infty, 0$ respectively. Then the the Möbius transformation $\rho_i : z \to \dfrac{-\xi_i}{z-1}$ determines an isometry of $\mathbb{H}^3$ which maps P_{i-1}, P_i, P_{i+1} to P_i, P_{i+1}, P_{i+2} respectively, rotating the truncated polygon forward one notch. Since every k-sided polygon closes up, the composite of k such isometries is identity.

The alternative method: region relations

If we represent the Möbius transformations by 2×2 matrices, we see that the product

$$\begin{pmatrix} 0 & -\xi_k \\ 1 & -1 \end{pmatrix} \cdots \begin{pmatrix} 0 & -\xi_1 \\ 1 & -1 \end{pmatrix}$$

is a scalar multiple of the identity matrix. From the matrix entries we read off three independent polynomial relations for every region in edge and crossing labels.

One can write a general combinatorial formula for the relations, that depends only on the number of sides in a region. E.g.,

for a 3–sided region, $\xi_i = 1$;

for a 4-sided region, $\xi_i + \xi_{i+1} = 1$;

for a 5–sided region, $1 - \xi_i - \xi_{i+1} - \xi_{i+2} + \xi_i\xi_{i+2} = 0\ , \quad 1 \le i \le 3.$

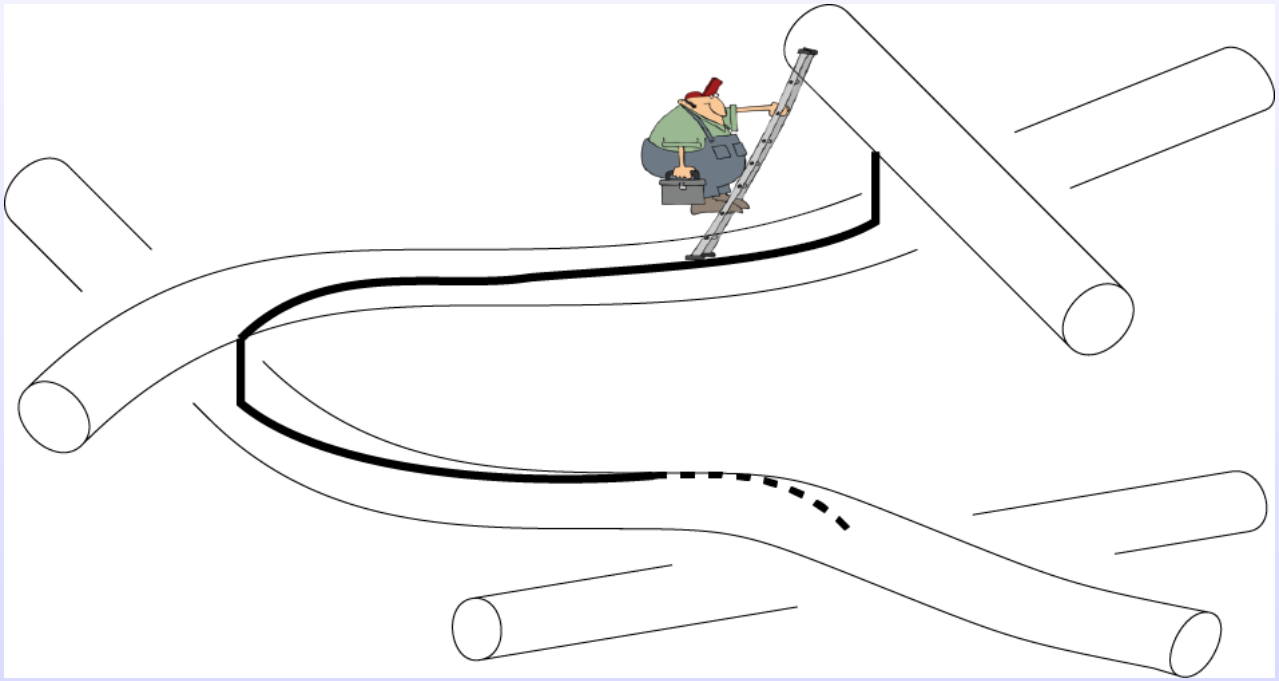
The recursive formula can be written as well:

$f_1 \equiv f_2 \equiv 0, \quad f_3 \equiv 1 - \zeta_2, \quad f_4 \equiv 1 - \zeta_2 - \zeta_3, \quad f_n \equiv f_{n-1} - \zeta_n f_{n-2} \quad (n \ge 5),$

where f_k is the relation for a k-sided region.

The holonomy representation of the link group from the labels

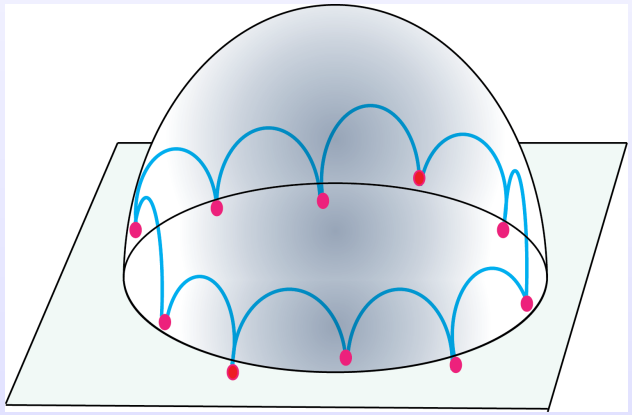
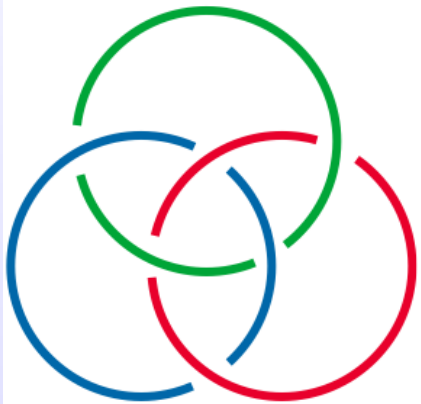
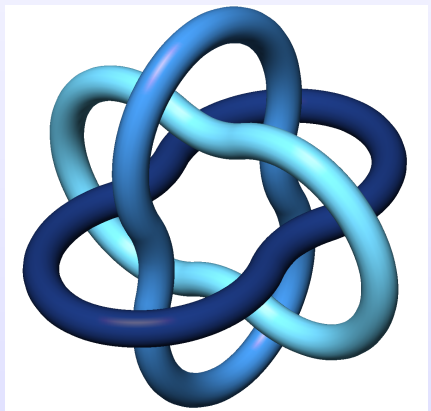
Edge and crossing labels determine the holonomy representation of the link group into $\mathrm{PSL}(2,\mathbb{C})$. To compute it, associate to every edge label u the matrix $\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$ and to every crossing label w the matrix $\begin{pmatrix} 0 & -w \\ 1 & 0 \end{pmatrix}$.



One can imagine a large playground consisting of knotted tubes, and ladders joining underpasses with overpasses. By traveling from the basepoint to one of the crossings along a loop, we obtain matrices for **Wirtinger generators** that are conjugates of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ by products of the above matrices.

The alternative method: the system and solutions

The system has several solutions. The solution with the greatest volume is the geometric one (S. Francaviglia). Empirically, for alternating links the geometric solution is the one for which the "region" ideal polygons are closest to being regular.

Example: the Borromean Rings

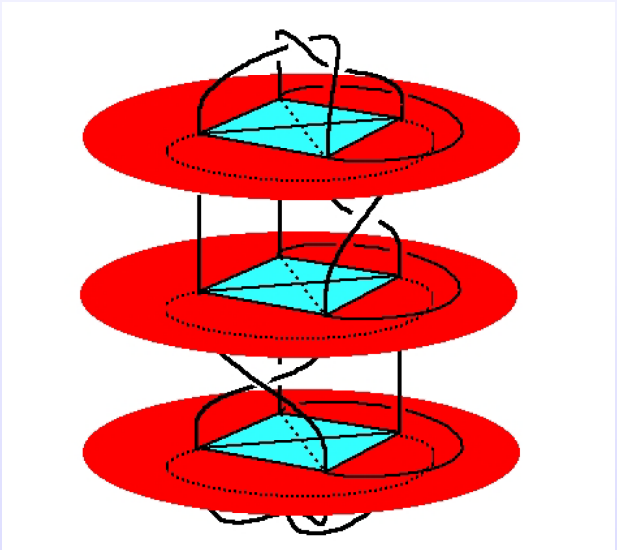
$$\frac{w_1}{u_1^2} = \frac{w_1}{u_2^2} = \frac{w_1}{-(u_1+1)(u_2+1)} = \frac{w_2}{(u_1+1)^2} = \frac{w_1}{(u_1+1)^2} = \frac{-w_2}{u_1 u_3} = 1 \ . \text{ Hence,}$$

$$u_1 = u_2 = u_3 = \frac{1}{2}(-1+i) \ , \quad w_1 = -\frac{i}{2} = -w_2 \ .$$

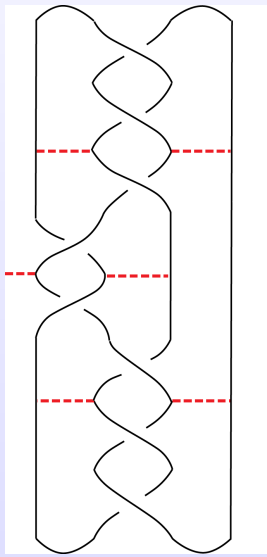
The crossing labels indicate that the overpass and underpass at each crossing are perpendicular to one another; this also transpires from the extra symmetries possessed by this link.

Exact volume of hyperbolic 2-bridge links

From a reduced alternating diagram of a hyperbolic 2-bridge link, we will construct a polynomial such that the volume can be expressed as an analytic function of one of its roots.



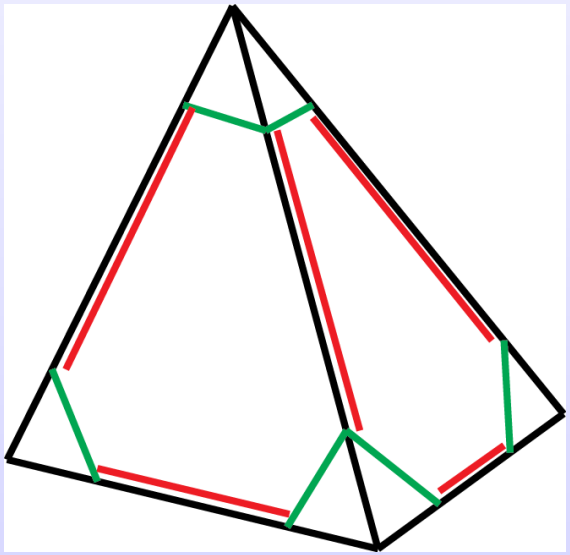
Picture by J. Weeks

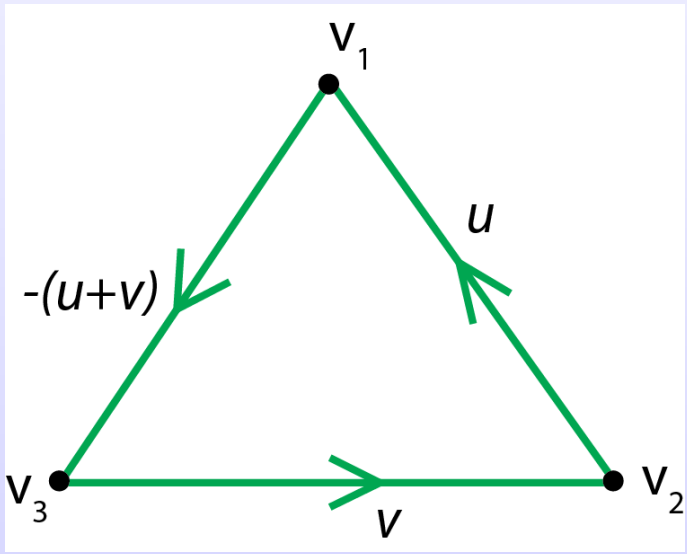


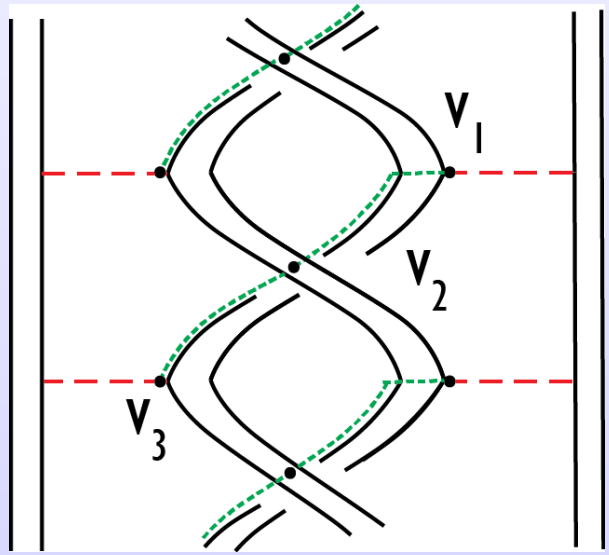
According to the Sakuma-Weeks description of the canonical cell decomposition, there are $(n-3)$ isometric pairs of tetrahedra in a hyperbolic 2-bridge link with n crossings. The red dotted lines represent geodesics in the complement that correspond to edges of the decomposition.

Exact hyperbolic volume of a hyperbolic 2-bridge link: shapes

Consider a Euclidean **cusp cross-section** of an ideal hyperbolic tetrahedron. Suppose that the (complex) translations corresponding to the sides of the cross-section are $u, v, -(u+v)$. Any of $-\frac{v}{u}$, $\frac{u}{u+v}$, $\frac{u+v}{v}$ can be taken as a shape of the tetrahedron. The volume of the link complement is then computed from the shapes of all tetrahedra using Lobachevsky function.



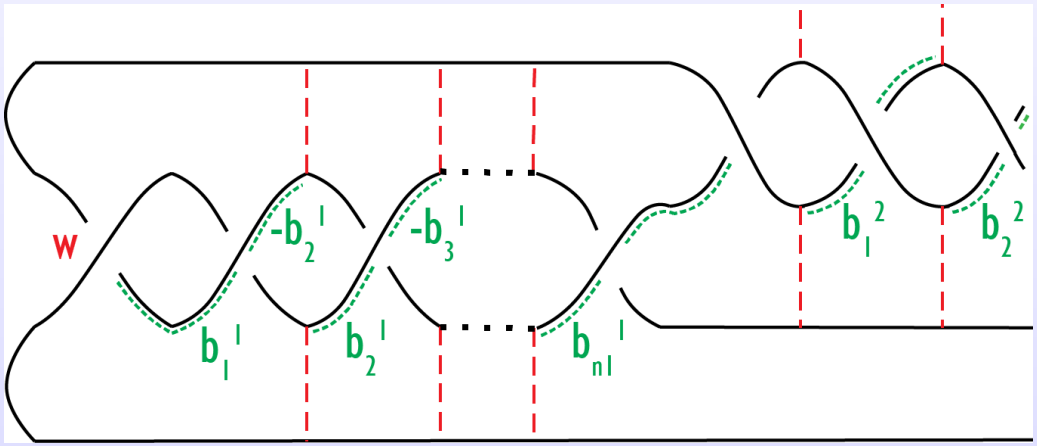




Consider a standard 2-bridge link diagram. For every pair of isometric tetrahedra, there is a Euclidean cross-section with vertices v_1, v_2, v_3 on one of (thickened) strands.

Exact volume from a link diagram

The shapes of consecutive pairs of isometric tetrahedra can be written in terms of edge labels as $-b_i/b_{i+1}$. In their turn, all the edge labels can be written recursively in terms of just one crossing label w if we proceed region by region from left to right and use the region relations. The last region yields the polynomial in this label.



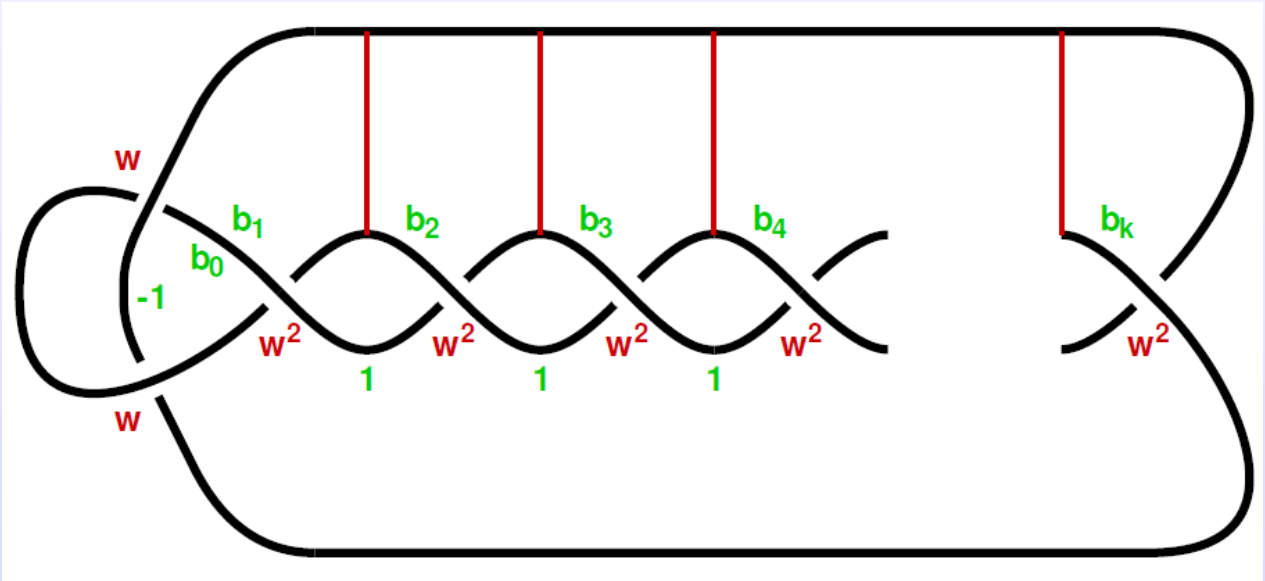
This process yields recursive formulas for the volume of a hyperbolic 2-bridge link that depend only on the number of crossings in twists.

For the formulas, see *"Exact volume of hypebolic 2-bridge links"* (A. T.).

The same idea and formulas by C. Zickert can be used to compute the **complex volume**. It is an invariant of mixed algebraic and geometric nature: its real part is the hyperbolic volume, and the imaginary part is the Chern-Simons invariant.

Example: volume of the twist knot with $k+2$ crossings

There are $k-1$ pairs of isometric tetrahedra. The shape parameters are the ratios $z_i = \dfrac{b_i}{b_{i-1}}$. All the labels can be found recursively from the crossing label w as $b_0 = (-1)^k w$, $\quad b_{i+1} = 1 + \dfrac{w^2}{b_i}$. The relation $b_k = 0$ gives a polynomial in w.



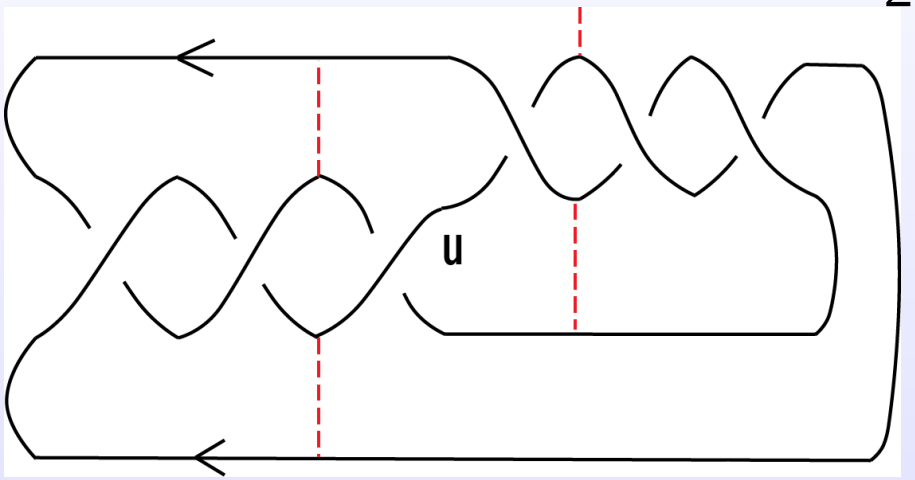
Closed formulas can often be obtained as well.
(O. Dasbach, LSU VIGRE student group, A. T.). For a twist knot with $k+2$ crossings, k even, the polynomial is

$$\sum_{j=0}^{n} \binom{2n-j}{j} w^{2j} + \sum_{j=0}^{n-1} \binom{2n-j-1}{k} w^{2j+1} = 0, \text{ where } n = k/2.$$

Example: volume of $(3,3)$ link

There are 3 pairs of isometric tetrahedra. The polynomial and the root:

$$(u^2+u-1)(u^2+u+1)=1\ , \quad u=\frac{-1+\sqrt{-3}}{2}\ .$$



The shapes of tetrahedra are then computed as rational functions of u. And the exact volume is $8\Lambda(\frac{\pi}{6})$, which is twice the volume of the figure-eight knot. Note that this is not a double-cover of the figure-eight.

Note: Sakuma and Weeks reduced the classical gluing equations to the recursive formulas for a single polynomial for 2-bridge links. This also allows to compute the shapes of tetrahedra, though the formulas are complicated.

Back to Link Diagrams and Quantum Invariants

The formulas for the head and tail of the colored Jones polynomial are known for some families of links.

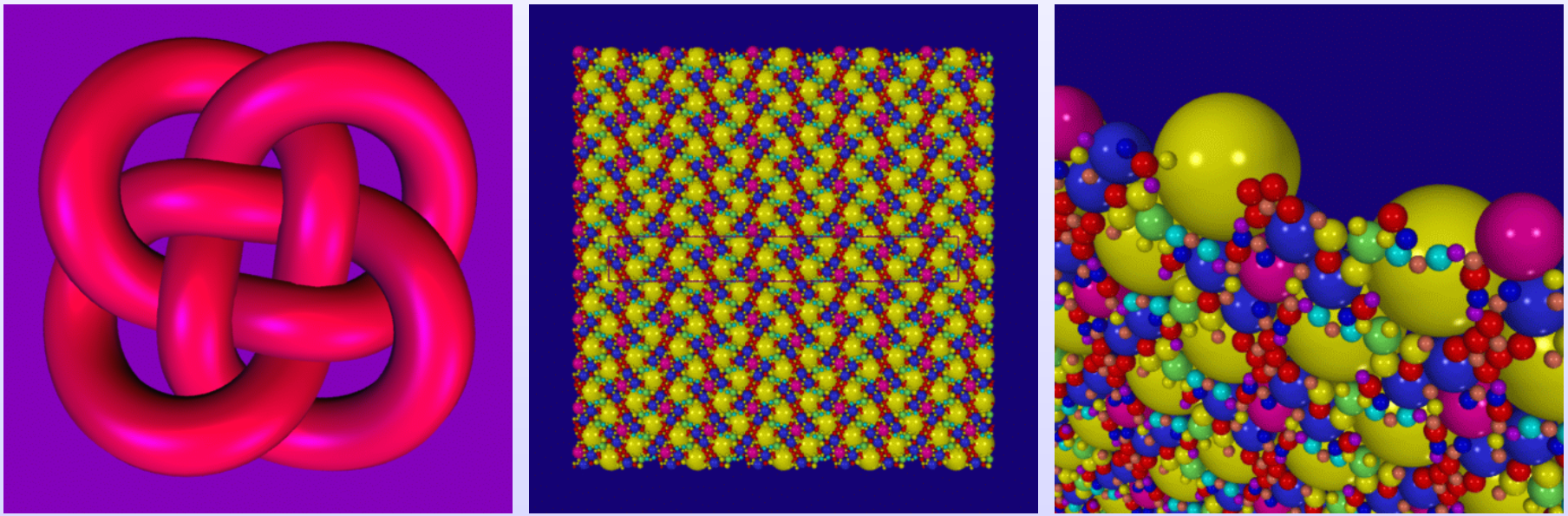
(Armond, Dasbach) For 2-bridge links, formulas for the head and tail are provided. The formulas depend only on the number of crossings in the twists of a reduced alternating diagram.

Further questions

- Can one reveal similar patterns in the formulas for the exact volume and the formulas for the coefficients of the colored Jones polynomial?
- What other families of links allow a relatively simple computation of the exact volume and of the head and tail from the link diagram?

Questions

Horoball packing of Turk's Head knot.



Pictures of horoball packings by M. Thistlethwaite