بسم الله الرحمن الرحیم

Stochastic differential equation

Solved problem

darya (masoud) khosrotash

Mehdi sourosh

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این اثر کوچک را به استادم در دوره دکتری

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در دانشگاه صنعتی شریف این علم را با پروفسور بیژن ظهوری زنگنه شروع کردم .

اما مشوق اصلی من در این علم جناب دکتر وحیدی بوده اند .

پیش گفتار

من فقط می دانم ...کوشش بی ثمری نیست جهان

و تو

چه زیبا در بر جاده تنهایی خود به جهان می نگری

این کتاب شامل هفتاد و هشت تا از سوالاتی است که در سراسر جهان پرسیده و پاسخ داده شده است . در بزرگترین سیستم آموزشی (پرسش و پاسخ) جهان که آدرس ان در ذیل آمده است ، افراد سوال مورد نظر خود را به زبان انگلیسی مطرح می کنند و کسانی دیگر از گوشه و کنار جهان بدانها پاسخ می دهند . همچنین افرادی خبره پاسخ ها را رصد می کنند و پاسخ های درست را تایید می نمایند . این افراد اساتیدی از دانشگاه های برتر در کلیه کشور ها می باشند . بنابراین وقتی سوالی با پاسخ اش تایید شد به معنی این است که تا حد بسیار زیادی قابلیت اعتماد دارد . در کتابی که روبروی شماست ، سوالاتی مربوط به معادلات دیفرانسیل تصادفی و حواشی آن برگزیده و گردآوری شده است . امید است که این مسایل بتواند گره گشایی تمرینات و مفاهیم پایه دروسی همچون آنالیز تصادفی ، معادلات دیفرانسیل تصادفی و ریاضیات مالی گردد .

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مسعود (دریا) خسروتاش
مدرس دانشگاه امام علی (ع) و دانشگاه ازاد اسلامی
amoodarya@yahoo.com
09121576134
https://telegram.me/riazidrKhosrotash
/khosrotash۱۰۴۱۷۱http://math.stackexchange.com/users/

Introduction

A **stochastic differential equation** (**SDE**) is a differential equation in which one or more of the terms is a stochastic process, resulting in a solution which is also a stochastic process. SDEs are used to model various phenomena such as unstable stock prices or physical systems subject to thermal fluctuations. Typically, SDEs contain a variable which represents random white noise calculated as the derivative of Brownian motion or the Wiener process. However, it should be noted that other types of random behaviour are possible, such as jump processes.

Stochastic calculus

Brownian motion or the Wiener process was discovered to be exceptionally complex mathematically. The Wiener process is almost surely nowhere differentiable; thus, it requires its own rules of calculus. There are two dominating versions of stochastic calculus, the Itô stochastic calculus and the Stratonovich stochastic calculus. Each of the two has advantages and disadvantages, and newcomers are often confused whether the one is more appropriate than the other in a given situation. Guidelines exist (e.g. Øksendal, 2003) and conveniently, one can readily convert an Itô SDE to an equivalent Stratonovich SDE and back again. Still, one must be careful which calculus to use when the SDE is initially written down.

Use in probability and mathematical finance

The notation used in probability theory (and in many applications of probability theory, for instance mathematical finance) is slightly different. This notation makes the exotic nature of the random function of time η_m in the physics formulation more explicit. It is also the notation used in publications on numerical methods for solving stochastic differential equations. In strict mathematical terms, η_m cannot be chosen as an ordinary function, but only as

a generalized function. The mathematical formulation treats this complication with less ambiguity than the physics formulation.

A typical equation is of the form

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dB_t,$$

where B denotes a Wiener process (Standard Brownian motion). This equation should be interpreted as an informal way of expressing the corresponding integral equation

$$X_{t+s} - X_t = \int_t^{t+s} \mu(X_u, u)du + \int_t^{t+s} \sigma(X_u, u)dB_u.$$

Existence and uniqueness of solutions

As with deterministic ordinary and partial differential equations, it is important to know whether a given SDE has a solution, and whether or not it is unique. The following is a typical existence and uniqueness theorem for Itô SDEs taking values in *n*-dimensional Euclidean space $\mathbf{R}^n$ and driven by an *m*-dimensional Brownian motion *B*; the proof may be found in Øksendal (2003, §5.2).

Let $T > 0$, and let

$$\mu : \mathbb{R}^n \times [\circ, T] \to \mathbb{R}^n;$$

$$\sigma : \mathbb{R}^n \times [\circ, T] \to \mathbb{R}^{n \times m};$$

be measurable functions for which there exist constants *C* and *D* such that

$$|\mu(x,t)| + |\sigma(x,t)| \le C(\backslash + |x|);$$

$$|\mu(x,t) - \mu(y,t)| + |\sigma(x,t) - \sigma(y,t)| \le D|x - y|;$$

for all $t \in [0, T]$ and all x and $y \in \mathbf{R}^n$, where

$$|\sigma|^{۲} = \sum_{i,j=۱}^{n} |\sigma_{ij}|^{۲} .$$

Let Z be a random variable that is independent of the σ-algebra generated by B_s, $s \geq 0$, and with finite second moment:

$$[|Z|^{۲}] < +\infty.$$

Then the stochastic differential equation/initial value problem

$$\begin{cases} dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dB_t \text{ for } t \in [۰, T] \\ X_۰ = Z \end{cases}$$

has a Pr-almost surely unique t-continuous solution $(t, \omega) \mapsto X_t(\omega)$ such that X is adapted to the filtration F_t^Z generated by Z and B_s, $s \leq t$, and

$$\left[\int_۰^T |X_t|^{۲} dt\right] < +\infty.$$

Some explicitly solvable SDEs[1]

Linear SDE: general case

$$dX_t = (a(t)X_t + c(t))dt + (b(t)X_t + d(t))dW_t$$

$$X_t = \Phi_{t,t_۰}\left(X_{t_۰} + \int_{t_۰}^{t} \Phi_{s,t_۰}^{-۱}(c(s) - b(s)d(s))ds + \int_{t_۰}^{t} \Phi_{s,t_۰}^{-۱} d(s)dW_s\right) \text{where}$$

$$\Phi_{t,t_۰} = \exp\left(\int_{t_۰}^{t}\left(a(s) - \frac{b^{۲}(s)}{۲}\right)ds + \int_{t_۰}^{t} b(s)dW_s\right)$$

Reducible SDEs: Case 1

$$dX_t = \frac{۱}{۲} f(X_t)f'(X_t)dt + f(X_t)W_t$$

[1] Kloeden 1995, page.118

for a given differentiable function f is equivalent to the Stratonovich SDE

$$dX_t = f(X_t)^{\circ} W_t$$

which has a general solution

$$X_t = h^{-1}(W_t + h(X_0))$$

Where

$$h(x) = \int^{x} \frac{ds}{f(s)}$$

Reducible SDEs: Case 2

$$dX_t = \left(\alpha f(X_t) + \frac{1}{2} f(X_t) f'(X_t)\right) dt + f(X_t) W_t$$

for a given differentiable function f is equivalent to the Stratonovich SDE

$$dX_t = \alpha f(X_t) dt + f(X_t)^{\circ} W_t$$

which is reducible to

$$dY_t = \alpha dt + dW_t$$

where $Y_t = h(X_t)$ where h is defined as before. Its general solution is

$$X_t = h^{-1}(\alpha t + W_t + h(X_0))$$

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Simple stochastic differential equation

Solve the following stochastic differential equation:

$dX_t = X_t\,dt + dW_t$. Thank you very much for help! I even don't know where to start...

Answer

The usual tool used here is Ito's Formula. To use it, you'll want to propose a possible solution, say

$$X_t = e^t\left(X_0 + \int_0^t e^{-s}\,dW_s\right),$$

and use Ito, by defining a function, say, $g(t,y)$ that is a function of deterministic t and stochastic y. For a nice introduction to SDEs I half-heartedly recommend Oksendal.

Indeed, I do not respond directly to your question. Readily available is not my intuition about generating this solution (I happened to have it "in my back pocket"), but I think it derives from intuition you might have developed in ODE (notice the exponential term, a common term in the solution to ODEs). [۲]

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[2] (Shreve: *Stochastic Calculus for Finance Volume II*, Exercise 4.8). This is a specific case of the Vasicek SDE,

$$dR_t = (\alpha - \beta R_t)dt + \sigma dW_t.$$

Use Ito's formula to compute

$$d(e^{\beta t}R_t) = \beta e^{\beta t}R_t dt + e^{\beta t}dR_t = \cdots$$

the right side will not involve R_t.

Integrate this answer to obtain the solution.

Linear Stochastic Differential Equation

Please could someone help me with the following proof:

Prove that $Y_t = e^{-2t}(Y_0 + 4\int_0^t e^{2s} dB_s)$

is the solution to the homogeneous linear stochastic differential equation

$dY_t = -2Y_t dt + 4dB_t$.

I am not at all sure how to go about this.

answer

This is a direct application of Ito lemma.

By definition of Ito integral, $Z_t = Y_0 + 4\int_0^t e^{2s} dB_s$ satisfies $dZ_t = 4e^{2t} dB_t$ with initial condition $Z_0 = Y_0$. Then we want to find the SDE for $Y_t = e^{-2t} Z_t$ and the Ito lemma is just the tool:

$$dY_t = \left(e^{-2t}\right)' Z_t dt + e^{-2t} dZ_t = \left(-2\underbrace{e^{-2t} Z_t}_{Y_t}\right) dt + 4dB_t$$ [۳]

[3] The standard stochastic differential equation (for a geometric case) is is $dS_t = \mu S_t dt + \sigma S_t dW_t$. What you have here is of the form $dS_t = \mu S_t dt + \sigma dW_t$, with $\mu = -۲$ and $\sigma = ۴$. The usual route is Ito's formula. Here it is a bit more complicated because σ is divided by S, so it decreases at lower values of S.

Solving Stochastic Differential Equations

Can anyone help me with the following SDE?

Solve the following stochastic differential equation:

$$dY_t = aY_t dt + (b(t) + cY_t)dB_t$$

with $Y_0 = 0$.

Hint: Try a solution of the form $Z_t H_t$ where $Z_t = exp(cB_t + (a - \frac{1}{2}c^2 t))$ and $dH_t = F(t)dt + G(t)dB_t$ for some adapted process F and G which need to be determined.

Please correct me if I'm wrong anywhere.

Since, $Z_t = exp(cB_t + (a - \frac{1}{2}c^2 t))$

Hence we get: $dZ_t = Z_t(cdB_t - \frac{1}{2}c^2)dt$

and we also have $dH_t = F(t)dt + G(t)dB_t$

Then we apply Ito's Lemma on $Z_t H_t$, which yields

$$d(Z_t H_t) = Z_t dH_t + H_t dZ_t + dH_t dZ_t =$$

$$Z_t(F(t)dt + G(t)dB_t) + Z_t H_t(cdB_t - \frac{1}{2}c^2 dt) + Z_t cG(t)dt =$$

$$Z_t(F(t) - \frac{1}{2}c^2 H_t + cG(t))dt + (Z_t G(t) + cZ_t H_t)dB_t$$

By letting $Y_t = Z_t H_t$, we compare between the expressions dY_t and $d(Z_t H_t)$ in the dt and dB_t terms respectively. And we get the following:

$$Y_t = Z_t H_t G(t) = \frac{Z_t}{b(t)} F(t) =$$

$$(a + \frac{1}{2}c^2)H_t - c\frac{Z_t}{b(t)}$$

Note: $F(t) = P$ and $G(t) = Q$ for your P and Q respectively.

But now, I don't really understand how would this result help me in solving the original dY_t SDE.

Answer

Here is an idea to get you started.

Note that

$$dZ_t = cZ_t dB_t + \frac{c^2}{2} Z_t dt = Z_t \left(cdB_t + \frac{c^2}{2} dt \right)$$

and let $dH_t = Pdt + QdB_t$.

Consider $A_t = Z_t H_t$. Then, by Ito's Lemma, and expansion for dZ_t,

$$A_t = Z_t dH_t + H_t dZ_t + dZ_t dH_t =$$

$$Z_t (Pdt + QdB_t) + H_t Z_t \left(cdB_t + \frac{c^2}{2} dt \right) + cQdt =$$

$$dt\left[???\right] + dB_t\left[???\right]$$

Now match this to dY_t, what can you say about P, Q ?[4]

[4] A linear SDE $dY_t = (\alpha(t) + \beta(t)Y_t)dt + (\gamma(t) + \delta(t)Y_t)dW_t$ has an explicit (strong) solution which can be found on [Wikipedia] . Here α, β, γ and δ denote deterministic functions and W a one-dimensional Wiener process. The

Long memory of stochastic differential equation

It is well known that the solution to an ordinary stochastic differential equation has the Markov property so that if one tries to model some kind of long memory process one has to instead consider SDE's driven by e.g. fractional brownian motions.

I've recently heard mentioned that one can also get this behaviour by superimposing SDE's (i.e. just adding them together, right?) intuitively how come this is possible? Does anyone have a readable source to this phenomenon.

Answer

Take $A = \mathrm{diag}(a_1,...,a_n)$, $a_i > 0$ and consider the vector SDE (n components)

$dX_t = -AX_t + \sigma dW_t$, where W_t is a vector of independent Wiener processes and $\sigma > 0$ a scalar. This is a special case of the vector OU (Ornstein-Uhlenbeck) process. Its temporal autocovariance reads

$$\langle X_t X_{t+\tau} \rangle = \mathrm{diag}(\frac{\sigma^2}{2a_1} e^{-a_1|\tau|},...,\frac{\sigma^2}{2a_n} e^{-a_n|\tau|}),$$

which means that the components of X_t are independent. Thus, if you take a linear combination

$y_t = w \quad X_t$, it will have autocovariance

---

solution, Y, is expressed in terms of the stochastic exponential, X, which is the strong solution to the linear SDE $\mathrm{dX_t = \beta(t)Y_t\,dt + \delta(t)Y_t\,dW_t}$.

$$\langle y_t y_{t+\tau} \rangle = \sum_{i=1}^{n} \frac{w_i^2 \sigma^2}{2a_i} e^{-a_i |\tau|},$$

.which allows you to mix a_i 's (inverse time-constants) of different scales. Although the tail of the autocovariance function will still decay exponentially, for processes with finite time horizon T this is not a big issue, since by incorporating a time-constant on the order of T will let the process have long memory. Moreover, you can approximate a power law quite well with a large number of exponentials.

Solving a Stochastic Differential Equation (SDE)

Question: Solve the stochastic differential equation:

$dX_t = X_t^3 dt - X_t^2 dW_t$ where:

$X_0 = 1$

My Attempt:

Using Ito's with:

$f(x) = \log(x)$ I get that:

$d\log(X_t) = dt\left(0 + \left(\frac{1}{X_T}\right)X_T^3 + \left(\frac{-1}{X_T^2}\right)X_T^4\right) + \left(\frac{1}{X_T}\right)\left(-X_T^2\right)dW_t$ w

hich gives me that:

$d\log X_t = -X_t dW_t$ and integrating from 0 to t:

$X_t = \exp\left(-\int_0^t X_s dW_s\right)$ Is this correct? I don't have too much experience with SDEs so would appreciate any hints/advice.

Answer

If $X_t = F(W_t)$, one knows that $\mathrm{d}X_t = F'(W_t)\mathrm{d}W_t + \frac{1}{2}F''(W_t)\mathrm{d}t$. If ever there exists some function F such that

$$F(0)=1, \qquad F'(w) = -F(w)^2, \qquad F''(w) = 2F(w)^3,$$

the proof is complete. Can you identify such a function F? Be aware though that there might be no solution (X_t) defined for every nonnegative t.

Solutions to stochastic differential equations

I'm in a bit of trouble with my homework and was wondering if anyone could help me find the solutions to these two stochastic differential equations. Would really appreciate it! Thanks in advance! :)

1.

$$\begin{cases} dX_t = \dfrac{b - X_t}{1-t}dt + dW_t \\ X_0 = a \in \mathbb{R} \end{cases}$$

Where b is a real constant.

2.

$$\begin{cases} dY_t = \dfrac{1}{Y_t}dt + \alpha Y_t dW_t \\ Y_0 = y \in \mathbb{R}^+ + \end{cases}$$

Where a is a real constant.

Answer

__HINT For item 1__: Use ito lemma [5] to verify that

$$\mathrm{d}\left(\frac{X_t - b}{1-t}\right) = \frac{1}{1-t}\,\mathrm{d}W_t$$

__HINT for item 2__: See if this[6] answer of mine helps. But also think if you could match the constants so that the following expression has no diffusion component:

$$\mathrm{d}\left(Y_t^2 \exp\left(\lambda t + \mu W_t\right)\right)$$

---

[5] In mathematics, **Itô's lemma** is an identity used in Itô calculus to find the differential of a time-dependent function of a stochastic process. It serves as the stochastic calculus counterpart of the chain rule. Typically, it is memorized by forming the Taylor series expansion of the function up to its second derivatives and identifying the square of an increment in the Wiener process with an increment in time. The lemma is widely employed in mathematical finance, and its best known application is in the derivation of the Black–Scholes equation for option values.Itô's lemma, which is named after Kiyosi Itô, is occasionally referred to as the Itô–Doeblin theorem in recognition of the recently discovered work of Wolfgang Doeblin.Note that while Ito's lemma was proved by Kiyosi Itô, Itô's theorem, a result in group theory, is due to Noboru Itô.

[6] Let $Y_t = e^{t-2W_t}X_t^2$. Then, applying Ito's lemma:

$dY_t = e^{t-2W_t}d(X_t^2) + X_t^2 de^{t-2W_t} + de^{t-2W_t}\cdot d(X_t^2)$ Since

$de^{t-2W_t} = e^{t-2W_t}\left(3dt - 2dW_t\right)$ and $d(X_t^2) = (X_t^2 + 2)dt + 2X_t^2 dW_t$, we get:

$dY_t = 2e^{t-2W_t}dt$ Also, initial condition is $Y_0 = e^{0-2W_0}X_0^2 = x^2$. Therefore:

$e^{t-2W_t}X_t^2 = Y_t = x^2 + \int_0^t 2e^{s-2W_s}ds$ This provides the solution:

$$X_t = \mathrm{sign}(x)\sqrt{x^2 e^{2W_t - t} + 2e^{2W_t - t}\int_0^t e^{s-2W_s}ds}$$

Understanding basic stochastic differential equations

This is from a physics course in economics, the literature provides a bare minimum of mathematical explanations. I am trying to understand how to work with stochastic differential equations given in exercises. Any explanation of how to approach would be appreciated. I am assuming this is very easy, but since the given literature is unreadable for me, I have no idea.

Assume that the time evolution of two stock prices S_1 and S_2 are

described by the two following Wiener process,

$$dS_1 = \sigma_1 \sqrt{dt}$$
$$dS_2 = \sigma_2 \sqrt{dt} + \sigma_0 \, {}_0\sqrt{dt},$$

where $\sigma_0, \sigma_1, \sigma_2$ are volatilities, ${}_0$

and are independent , normally distributed random numbers

with variance one. Furthermore, assume that $S_1(0) = S_2(0) = 0$.

1. If $\sigma_2 = 0$, what is the correlation between S_1(t) and S_2(t)?

2. Calculate the variance and the correlation between the two following portfolios

$$F_1 = S_1$$
$$F_2 = \sigma_2 S_1 - \sigma_1 S_2$$

1. Assuming $\sigma_2 = 0$ yields $dS_2 = \sigma_0 \, {}_0\sqrt{dt}$. They provide no proper definition of the correlation, but from what I have seen in an example, it seems to be given by the moment $\langle S_1 S_2 \rangle$. How is this integral derived from the given information? Do we compute

$$\langle dS_1 dS_2 \rangle = \sigma_0 \sigma_1 \langle \ _0 \ dt \rangle = \sigma_0 \sigma_1 \langle \ _0 \rangle \langle \ \rangle dt \ ?$$

Where would we go from here?

2.. Itôs formula seems to be the key. Again, they do not provide a proper definition, but I'm guessing the approach is the following. Let $f(x,t) = x$ and define $F_1 = f(S_1,t)$ and $F_2 = \sigma_2 f(S_1,t) - \sigma_1 f(S_2,t)$. We should get the following

$$dF_1 = \sigma_1 \ \sqrt{dt}$$
$$dF_2 = \sigma_2 dS_1 - \sigma_1 dS_2 = -\sigma_0 \sigma_1 \ _0 \sqrt{dt}.$$

Any suggestions? As of writing I just got my hands on a copy of Oksendal's "Stochastic differential equations" which I hope will have an approach that I am more comfortable with.

Answer

This formulation of an SDE is more suited for the discretized SDE

$$\Delta S_1 = \sigma_1 \varepsilon \sqrt{\Delta t}$$
$$\Delta S_2 = \sigma_2 \varepsilon \sqrt{\Delta t} + \sigma_0 \varepsilon_0 \sqrt{\Delta t}$$

which can be solved by simply summing up

$$S_1(n\Delta t) = \sum_{k=0}^{n-1} \sigma_1(k\Delta t)\varepsilon(k\Delta t)\sqrt{\Delta t}$$

$$S_2(n\Delta t) = \sum_{k=0}^{n-1} \left(\sigma_2(k\Delta t)\varepsilon(k\Delta t) + \sigma_0(k\Delta t)\varepsilon_0(k\Delta t)\right)\sqrt{\Delta t}$$

and all of the $\varepsilon_0(k\Delta t)$, $\varepsilon(k\Delta t)$ are independend standard-normally distributed random variables, which makes the correlation computations rather easy. [7]

The answer of the following stochastic differential equation

I want to solve the following stochastic differential equation

$$dX_t = (a(t) + b(t)X_t)dt + (c(t) + d(t)X_t)dB_t$$

where a,b,c,d are continious functions and we have the initial condition $X_0 = x$.

answer

$$\begin{cases} dX_t = (c(t) + d(t)X_t)dt + (e(t) + f(t)X_t)dW_t \\ X_0 = x_0 \end{cases}$$

---

[7] 1) Well one maybe pedantic thing I would point out is that you dont want but rather . In more complicated SDEs that would be a more substantial issue, here you can use properties of how independent Gaussian RVs sum to conclude that it's basically almost the same thing. The rest of your ideas are mostly correct, but I hope that you know what key property of and you are using to allow you to factor the expectation. Once it's factored like that what do you know about the mean of for example?

2) I wouldn't really say this even needs Ito's formula. You can just get it from the linearity of the Ito integral. If you try to compute a variance for (take , plug in what it is in terms of and and expand) you'll get terms like , and . Some of those will be zero. Some won't. Use what you know about and .

1. We start by solving the homogeneous associated SDE:

$$\begin{cases} dY_t = d(t)Y_t\,dt + f(t)Y_t\,dW_t \\ Y_0 = y_0 \end{cases}$$

In order to do so, we apply Ito's formula to $Z_t := \ln Y_t$.

$$dZ_t = [d(t) - \tfrac{1}{2}f^2(t)]dt + f(t)dW_t$$

Integrating we get

$$ln(\frac{Y_t}{Y_0}) = Z_t - Z_0 = \int_0^t \left(d(s) - \tfrac{1}{2}f^2(s)\right)\mathrm{d}s + \int_0^t f(s)\mathrm{d}W_s$$

and hence

$$Y_t = y_0 \exp\left(\int_0^t \left(d(s) - \tfrac{1}{2}f^2(s)\right)\mathrm{d}s + \int_0^t f(s)\mathrm{d}W_s\right)$$

2. Now let U_t such that U_t^{-1} solves (3.2) with initial condition $U_0 = 1$.

Then we have

$$U_t = \exp\left(-\int_0^t \left(d(s) - \tfrac{1}{2}f^2(s)\right)\mathrm{d}s - \int_0^t f(s)\mathrm{d}W_s\right)$$

Define $I_t = \int_0^t f(s)\mathrm{d}W$ or equivalently $dI_t = f(t)dW_t,\ I_0 = 0$ (so $\mu = 0, \sigma = f$)

Then we have $U_t = g(t, I_t)$ where

$$g(t,x) = \exp\left(-\int_0^t \left(d(s) - \tfrac{1}{2}f^2(s)\right)\mathrm{d}s - x\right)$$

By Ito's lemma one has

$$dU_t = dg(t, I_t) = U_t\left(\frac{\partial g}{\partial t} + \mu\frac{\partial g}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 g}{\partial x^2}\right)dt + \sigma\frac{\partial g}{\partial x}dW_t =$$

$$U_t\left[\left(f^2(t) - d(t)\right)dt - f(t)dW_t\right]$$

Finally, let $Z_t = X_t U_t$

Again Ito's formula (more precisely the product rule) gives

$$\begin{aligned}
&dZ_t = \\
&dX_t U_t + dU_t X_t + d < X, U >_t = \\
&\left[\left(c(t) + \cancel{d(t)X_t}\right)dt + \left(e(t) + \cancel{f(t)X_t}\right)dW_t\right]U_t \\
&+\left[\left(\cancel{f^2(t)} - \cancel{d(t)}\right)dt + -\cancel{f(t)dW_t}\right]X_t U_t + \\
&\left(e(t) + \cancel{f(t)X_t}\right)\left(-f(t)U_t\right)dt = \\
&\left(c(t) - e(t)f(t)\right)U_t dt + e(t)U_t dW_t
\end{aligned}$$

Integrating results in

$$\begin{aligned}
&Z_t - Z_0 = \\
&\int_0^t \left(c(s) - e(s)f(s)\right)U_s \mathrm{d}s + \int_0^t e(s)U_s \mathrm{d}W_s X_t U_t - X_0 U_0 = \\
&\int_0^t \left(c(s) - e(s)f(s)\right)U_s \mathrm{d}s + \int_0^t e(s)U_s \mathrm{d}W_s
\end{aligned}$$

and hence

$X_t = U_t^{-1}\left[x_0 + \int_0^t \left(c(s) - e(s)f(s)\right)U_s \mathrm{d}s + \int_0^t e(s)U_s \mathrm{d}W_s\right]$ is the solution for (3.1) where

$$U_t^{-1} = \frac{1}{U_t} = \exp\left(-\int_0^t \left(d(s) - \frac{1}{2}f^2(s)\right)\mathrm{d}s - \int_0^t f(s)\mathrm{d}W_s\right)$$

How to solve this stochastic differential equation?

Consider the following mean-square differential equation:

$Y_t^{'} + \alpha Y_t = X_t$ *m.s. with* $Y_0 = 0$

where X_t is a Gaussian random process with $\mu_X(t) = 0$ and $R_X(\tau) = a^2 \delta(\tau)$.

Find a partial differential equation for cross correlation function between $Y_t; t \in R$ and $X_t; t \in R$ with appropriate boundary conditions.

Is Y_t wide sense stationary?

I tried squaring both sides and taking the expectation but I'm not sure that's what the question is asking for. Any help?

Answer

There is a problem of definition here.

The only way I know to make sense of your equation is to replace it by the stochastic differential equation $\mathrm{d}Y_t = \alpha Y_t \mathrm{d}t + \mathrm{d}W_t$ where $(W_t)_{t \quad 0}$ is a standard Brownian motion. But then the autocorrelation of Y_t and R_t does not make sense since $(R_t)_{t \quad 0}$ is not defined.

So, the first thing to do is to explain what you mean by the equation $Y_t' = \alpha Y_t + R_t$, *mean square*.

Solve linear stochastic differential equation

I have to solve $dX_t = 5dt + 3X_t\, dW_t$

Let $Y_t := X_t \exp(-3W_t + \frac{9}{2}t) = X_t \cdot Z_t$

Calculating differential of Y_t we have

$$dY_t =$$
$$X_t(9Z_t\, dt - 3Z_t\, dW_t) + (5dt + 3X_t\, dW_t)Z_t +$$
$$(5dt + 3X_t\, dW_t)(9Z_t\, dt - 3Z_t\, dW_t) =$$
$$X_t(9Z_t\, dt - 3Z_t\, dW_t) + (5dt + 3X_t\, dW_t)Z_t - 9X_t Z_t\, dt =$$
$$5Z_t\, dt = 5\exp(-3W_t + \frac{9}{2}t)dt$$

$$X_t = Y_t \exp(3W_t - \frac{9}{2}t) =$$
$$\exp(3W_t - \frac{9}{2}t) + 5\left(\int_0^t \exp(-3W_s + \frac{9}{2}s)ds\right)\exp(3W_t - \frac{9}{2}t)$$
$$= \exp(3W_t - \frac{9}{2}t) + 5\int_0^t \exp(-3(W_s - W_t) + \frac{9}{2}(s-t))ds$$

It seems ok to me, but I would like to assure myself before my exams.

Answer

$\partial_t X_t = 5 - \frac{9}{2}X_t \partial_{W_t} X_t = 3X_t \partial^2_{W_t} X_t = 9X_t$ The first of the above comes from the fact

$$\frac{\partial}{\partial t}\int_0^t g(t,W_t,W_s,s)ds = g(t,W_t,W_t,t)+\int_0^t g'ds \quad / \quad (*)$$

Thus using Ito we have

$$dX_t = \left(5-\frac{9}{2}X_t+\frac{1}{2}9X_t\right)dt+3X_t dW_t$$

So you were correct as far as I can see.

Ps forgive the lazy notation when computing the integral. In step (*)

Expectation of stochastic differential equation

I have solved the nonlinear stochastic equation $dX_t = \frac{1}{2}a(a-1)X_t^{1-2/a}dt + aX_t^{1-1/a}dW_t$, by reducing it to a linear one (change of variables $Y_t = X_t^{1/a}$ and applying Ito formula). The solution reads $X_t = (X_0^{1/a}+W_t)^a$, however I'm stuck on how to compute the expectation $E[X_t]$. I am new to stochastic analysis so any kind of help is greatly appreciated! Thanks

answer

The distribution of X_0 should be given by some initial condition, typically it is a constant, but not necessarily and $W_t \sim \quad (0,t)$ (i.e. the variance is t).

Then once both distributions are known, you have to take the integral.

An alternative approach, when the solution to the SDE is not easily derivable, is to take expected value of both sides of the original SDE, which kills the stochastic part:

$$d\ [X_t] = \frac{a(a-1)}{2}\ [X_t^{1-2/a}]dt$$

which is an ordinary differential equation.

Stochastic differential equation for $Y(t) = \sqrt{X(t)}$

Assume that $X(t)$ solves the stochastic differential equation $dX(t) = \sigma(t)dW(t) + \mu(t)dt$ with $\mu(x) = bx + c$ and $\sigma^2(x) = 4x$. Assume that $X(t) \geq 0$. Find the stochastic differential equation for the process $Y(t) = \sqrt{X(t)}$.

My idea: I use Ito formula and get $df(X(t)) = \frac{1}{2\sqrt{X(t)}}dX(t) - \frac{1}{8X(t)\sqrt{X(t)}}\sigma^2(X(t))\,\mathrm{dt}$.

My problem is how to rearrange $Y(t)$ and use $\mu(x) = bx + c$.

Thank you for help.

Answer

Use $\mathrm{d}X = \sigma(X)\mathrm{d}W + \mu(X)\mathrm{d}t$, $\mathrm{d}\langle X\rangle = \sigma(X)^2\mathrm{d}t$, the explicit formulas for σ and μ , and replace every $\sqrt{X}$ by Y in Itô's formula for $\mathrm{d}Y$, to deduce after some simplifications that

$$\mathrm{d}Y = \mathrm{d}W + \tfrac{1}{2}(bY + (c-1)Y^{-1})\mathrm{d}t.$$

How to solve a linear stochastic differential equation?

I don't know how to find a solution of this stochastic differential equation:

$$dX_t = (1 + \delta\mu X_t)dt + \delta X_t dB_t$$

Where B_t is a standard Brownian motion and μ and δ are real numbers.

###Context

I've to demonstrate that

$$X_t = \int_0^t \exp[\delta(B_t - \mu t - B_s + \mu s) - \frac{1}{2}(t-s)]ds$$

I know Ito's formula $dY = \frac{\partial g}{\partial t}dt + \frac{\partial g}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}dt$ for $Y = g(t, X)$.

Answer

Hints:

First, solve the homogeneous SDE $dX_t = \delta\mu X_t dt + \delta X_t dB_t \quad / \quad (1)$

To this end, apply Itô's formula [8] to $f(x) := \log x$ and $(X_t)_{t \geq 0}$.

2. In order to solve the SDE $dX_t = (1 + \delta \mu X_t)dt + \delta X_t dB_t$, we use a variation of constants, i.e. we make the *Ansatz[9]* $X_t = Z_t X_t^0$ where

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[8] $df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dx + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} dx^2 + \cdots$

9 In physics and mathematics, an ansatz (/ˈænsæts/; German: [ˈʔanzats], meaning: "initial placement of a tool at a work piece", plural ansätze /ˈænsɛtsə/; German:[ˈʔanzɛtsə] or ansatzes) is an educated guess that is verified later by its results.

Given a set of experimental data that looks to be clustered about a line, a linear ansatz could be made to find the parameters of the line by a least squares curve fit.Variational approximation methods use ansätze and then fit the parameters.

Gershenfeld, Neil A. (1999), The Nature of Mathematical Modeling, Cambridge University Press, ISBN 0-521-57095-6

Another example could be the mass, energy, and entropy balance equations that, considered simultaneous for purposes of the elementary operations of linear algebra, are the ansatz to most basic problems of thermodynamics.

Another example of an ansatz is to suppose the solutions of a homogeneous linear differential equation and difference equation to have, respectively, exponential and power form. More generally one can guess a particular solution of a system of equations and test such an ansatz by direct substitution of the solution in the system of equations.

X_t^0 is a solution of the SDE (1) with initial value $X_0^0 = 1$. To find the stochastic differential of $(Z_t)_{t\geq 0}$, apply Itô's formula.

Literature: The monograph *Brownian Motion - An Introduction to Stochastic Processes* by René Schilling & Lothar Partzsch [1] treats (linear) SDEs.

Solution

$$\log X_t - \log X_0 = \int_0^t \frac{1}{X_s} dX_s - \frac{1}{2}\int_0^t \frac{1}{X_s^2} d\langle X\rangle_s =$$

By Itô's formula, $\int_0^t \delta\, dB_s + \int_0^t \left(\delta\mu - \frac{1}{2}\delta^2\right) ds =$

$$\delta B_t + \left(\delta\mu - \frac{1}{2}\delta^2\right)\cdot t$$

Consequently, $X_t = X_0 \exp\left(\delta B_t + \left(\delta\mu - \frac{1}{2}\delta^2\right)t\right)$.

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10 Brownian Motion: An Introduction to Stochastic Processes

by René L. Schilling, Lothar Partzsch, Björn Böttcher (Contributor) ,

Stochastic processes occur in a large number of fields in sciences and engineering, so they need to be understood by applied mathematicians, engineers and scientists alike. This work is ideal for a first course introducing the reader gently to the subject matter of stochastic processes. It uses Brownian motion since this is a stochastic process which is central to many app ...more

Set $Y_t := 1/X_t^0$, i.e. $Y_t = \exp\left(-\delta B_t - \left(\delta\mu - \frac{1}{2}\delta^2\right)t\right)$. / (2)

Applying Itô's formula yields,

$Y_t - Y_0 = -\delta\int_0^t Y_s\,dB_s + \frac{\delta^2}{2}\int_0^t Y_s\,ds + \left(\frac{\delta^2}{2} - \delta\mu\right)\int_0^t Y_s\,ds$, i.e.

$dY_t = -\delta Y_t\,dB_t + (\delta^2 - \delta\mu)Y_t\,dt$. From

$X_t \cdot Y_t = \int_0^t Y_s\,dX_s + \int_0^t X_s\,dY_s + \langle X, Y\rangle_t$ it follows that

$Z_t = X_t Y_t =$

$\int_0^t Y_s((1+\delta\mu X_s)ds + \delta X_s)dB_s) + \int_0^t X_s(-\delta Y_s dB_s + (\delta^2 - \delta\mu)Y_s\,ds)$

$-\delta^2\int_0^t Y_s X_s\,ds =$

$\int_0^t Y_s\,ds$

Note that $(Y_t)_{t\geq 0}$ is known explicitly, cf. (2). Consequently, we get

$X_t = \frac{Z_t}{Y_t} =$

$\exp\left(\delta B_t + \left(\delta\mu - \frac{\delta^2}{2}\right)t\right)\cdot\left[X_0 + \int_0^t \exp\left(-\delta B_s - \left(\delta\mu - \frac{\delta^2}{2}\right)s\right)ds\right]$

1-dimentional stochastic differential equation

I would like to solve this SDE

$$dX_t = \left(\sqrt{1+X^2} + \frac{1}{2}\right)dt + \sqrt{1+X^2}\,dB_t$$

I've tried to solve first the homogeneous equation $\$ dX_t = (\sqrt{1+X^2})dt + \sqrt{1+X^2}dB_t \$$ dividing by $\sqrt{1+X^2}$ and integrating I obtain $\sinh^{-1}(X_t) = t + B_t$ and then $X_t = \sinh(t + B_t)$.

Could me the right way? Can someone help me to continue? Thank you in advance

Answer

It seems that you missed X_t next to $\frac{1}{2}$ term. If this is the case then we want to find a solution of

$$\mathrm{d}X_t = \left(\sqrt{1+X_t^2} + \frac{1}{2}X_t\right)\mathrm{d}t + \sqrt{1+X_t^2}\,\mathrm{d}B_t.$$

We can re-write it as

$$\mathrm{d}X_t = \sqrt{1+X^2}\,\mathrm{d}t + \sqrt{1+X_t^2}\,\mathrm{d}B_t + \frac{1}{2}X_t(\mathrm{d}B_t)^2.$$

We may try the solution of the form $X_t = g(t, B_t)$, where g is twice continuously differentiable on $\mathbb{R}_+ \times \mathbb{R}$.

By using Ito formula we obtain that

$$\mathrm{d}X_t = \mathrm{d}g(B_t, t)$$

$$=\frac{\partial g}{\partial t}(t,B_t)\mathrm{d}t+\frac{\partial g}{\partial x}(t,B_t)\mathrm{d}B_t+\frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t,B_t)\cdot(\mathrm{d}B_t)^2$$

Since $\cosh(x)=\sqrt{1+\sinh^2(x)}$ we claim that $X_t=\sinh(B_t+t)$ and so $g(t,x):=\sinh(x+t)$.

Then

$\frac{\partial g}{\partial t}=\frac{\partial g}{\partial x}=\cosh(x+t)$ and

$\frac{\partial^2 g}{\partial x^2}=\sinh(x+t)$, thus

$$\mathrm{d}\sinh(B_t+t)=$$

$$\cosh(B_t+t)\mathrm{d}t+\cosh(B_t+t)\mathrm{d}B_t+\frac{1}{2}\sinh(B_t+t)\mathrm{d}t=$$

$$(\sqrt{1+\sinh^2(B_t+t)}+\frac{1}{2}\sinh(B_t+t))\mathrm{d}t+\sqrt{1+\sinh^2(B_t+t)}\mathrm{d}B_t=$$

$$\left(\sqrt{1+X_t^2}+\frac{1}{2}X_t\right)\mathrm{d}t+\sqrt{1+X_t^2}\mathrm{d}B_t$$

Stochastic differential equation problem and applying ito formula

I am given that for $b,a,\sigma>0$ and $x\in(-a,b)$ and $v\in\mathbb{R}$, I have the following stochastic differential equation:

$$dZ_t=v\,dt+\sigma\,dW_t$$

$$Z(0)=x$$

and $\tau = \inf\{t > 0 : Z(t) = b \quad \text{or} \quad Z(t) = -a\}$

and $u(x) = b \;\; [Z(\tau) = b \mid Z_0 = x] - a \;\; [Z(\tau) = -a \mid Z_0 = x]$, I want to find

$[u(Z(\tau)) \mid Z_0 = x] - u(x)$ and the differential eq that $u(x)$ satisfies.

Attempt at solution:

We know that for a starting point x, $u(x) = b \;\; [Z(\tau) = b \mid Z_0 = x] - a \;\; [Z(\tau) = -a \mid Z_0 = x]$ so then $u(Z(\tau)) = b \;\; [Z(\tau) = b \mid Z_0 = Z(\tau)] - a \;\; [Z(\tau) = -a \mid Z_0 = Z(\tau)]$ but $[Z(\tau) = b \mid Z_0 = Z(\tau)] = 1/2 = \;\; [Z(\tau) = -a \mid Z_0 = Z(\tau)]$ So then $u(Z(\tau)) = \frac{1}{2}b + \frac{1}{2}a$ But this seems wrong since it doesnt depend on time? Also how would I apply Ito to $u(x)$? It is a function of a probability of $Z(\tau)$. How would I write it as an explicit function since I cant take partials of probability?

Any help would be greatly appreciated! Thanks!

answer

After "so then", you seem to use the identity that if $v(x) = P[U = b \mid V = x]$ for every x then $v(U) = P[U = b \mid V = U]$. This is wrong, and you could have been suspicious about this, due to the fact that in the setting of the exercise, the event $U = V$ is impossible.

Instead, note that if U takes values in $\{u_k\}_k$, then $\sum_k u_k P[U = u_k \mid V = x] = E[U \mid V = x]$, by definition.

Strong solution of stochastic differential equation

Consider the stochastic differenctial equation:

$$dX_t = \frac{3}{4} X_t^2 dt - X_t^{3/2} dW_t .$$

How to find a strong solution?

answer

If we want to solve an SDE of the form

$$dX_t = \sigma(X_t) dW_t + b(X_t) dt$$

i.e. an SDE which does not depend explicitly on the time t, the transformation

$$f(x) := \int^x \frac{1}{\sigma(y)} dy$$

is always worth a try. Here, $\sigma(y) = -y^{3/2}$; hence we consider

$$f(x) = -\int_x^\infty \frac{1}{\sigma(y)} dy = \int_x^\infty y^{-3/2} dy = 2\frac{1}{\sqrt{x}}.$$

Then,

$$f'(x) = -\frac{1}{x^{3/2}} \qquad f''(x) = \frac{3}{2}\frac{1}{x^{5/2}}.$$

By Itô's formula,

$$f(X_t)-f(X_0)=$$

$$-\int_0^t \frac{1}{X_s^{3/2}}dX_s+\frac{3}{4}\int_0^t \frac{1}{X_s^{5/2}}d\langle X\rangle_s=$$

$$\int_0^t dW_s-\frac{3}{4}\int_0^t X_s^{1/2}ds+\frac{3}{4}\int_0^t X_s^{\frac{1}{2}}ds=$$

$$W_t$$

where we used that the quadratic variation $\langle X\rangle_s$ has infinitesimal variation $X_s^3 ds$. This shows that

$$X_t=f^{-1}\big(W_t+f(X_0)\big)=\frac{4}{(W_t+f(X_0))^2}=\frac{X_0}{(1+\frac{1}{2}\sqrt{X_0}\cdot W_t)^2}$$ for

every $t<T$, where

$$T=\inf\{t>0\,|\,W_t=-2/\sqrt{X_0}\}.$$

Why do people write stochastic differential equations in differential form?

I am trying to teach myself about stochastic differential equations. In several accounts I've read, the author defines an SDE as an integral equation, in which at least one integral is a stochastic integral, then writes that in practice, people usually write the SDE in differential form. My question is, why?

There are two possible reasons I can guess:

(i) If you write an SDE in differential form, it makes it easy to simulate sample paths using random walks, and you can think of the SDE as a limit

of random walks. This may provide some heuristic understanding that the integral form does not.

(ii) If you write an SDE in differential form, you can do "formal" computations with the differentials that are not necessarily rigourous, but lead to correct conclusions. Personally, I have always had trouble understanding differential forms, and from what I've read, it is difficult to define exactly what dW_t means (the differential of the Wiener process, i.e. Brownian motion). I'm not even sure if dW_t has a precise meaning.

Answer 1

As someone who does a lot of computations with SDEs, I can tell you my own personal reason that I use the differential notation: convenience.

For one thing, when using differential notation, you don't have to worry about mixing up dummy variables.

$df(X_t) = f'(X_t)dX_t + \frac{1}{2}f''(X_t)d\langle X\rangle_t$ is simple and easy to read, but if I wrote it in integral form

$$f(X_t) - f(X_0) = \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t f''(X_s)d\langle X\rangle_s$$

it both takes much longer to write and requires me to use and keep track of the dummy variable s. This makes things much easier when working with multidimensional processes especially.

Another major factor is a very handy dandy abuse of notation: $dX_t dY_t := d\langle X, Y\rangle_t$. Using this abuse of notation, I can "multiply" stochastic differentials using ordinary algebraic manipulation, for example

$$d\langle f(X)\rangle_t = df(X_t)df(X_t) =$$
$$(f'(X_t))^2 d\langle X\rangle_t +$$
$$\frac{1}{2}f'(X_t)f''(X_t)d\langle X_t,\langle X_t\rangle\rangle_t + \frac{1}{4}(f''(X_t))^2 d\langle\langle X\rangle_t\rangle_t$$
$$= (f'(X_t))^2 d\langle X\rangle_t + 0 + 0$$

which actually gives the correct answer. One should be VERY careful abusing notation like this, but if you know what hypotheses allow you to use the shortcut, it saves a great deal of time in doing computations.

I should note that I am NEVER going to assign meaning to an expression like dW_t on its own. I will ONLY use it in a context where a corresponding integral expression makes sense.

Answer 2

Intuitively, the symbol $\mathrm{d}W_t$ may be interpreted as an infinitesimal increment of a (one-dimensional) Wiener process, $\mathrm{d}W_t = W_{t+\mathrm{d}t} - W_t$

$\mathrm{d}W_t$ can be interpreted as a normal distributed random variable whose expected value is 0 and whose variance is equal to $\mathrm{d}t$. This implies that it is quite likely (probability close to 0.7) to find the random variable $\mathrm{d}W_t$ located somewhere between the numbers $-\sqrt{\mathrm{d}t}$ and $\sqrt{\mathrm{d}t}$.

As to a precise meaning of the symbol $\mathrm{d}W_t$, you have to be more specific what you mean by "precise". After all, what is the precise meaning of the symbol $\mathrm{d}t$? (Without using 1-forms or nonstandard analysis.)

Answer 3

I am somewhat inexperienced and also self taught, but from my understanding, one of the primary advantages to writing it in this form is to "abuse" notation in a similar way that people abuse multiplying by both sides with dx and canceling the dx in $\frac{dy}{dx}$. There is a lot of theory from calculating quadratic variations, but it is easier to understand and process if you just use "formulas" like $dW_t dW_t = dt$

Answer 4

My education in SDEs is all self-taught, so please excuse any inaccuracies below. In particular, my memory is fuzzy on issues of SDE vs associated PDE, and backwards versus forwards equation, for instance.

In differential equations, and in applications, we look at differentials often because that's all that we can infer in a straightforward way. Locally, we are able to make arguments about how something should behave, and we integrate to try to gather what is going on over global distances (finite, rather than infinitesimal).

The solution may depend too much on what else is going on. Initial conditions, boundary conditions. The local differential equation may be true in general, but the actual global relative values may depend greatly on the larger environment. For instance, Maxwell's equations, even in simple distributions, locally will give the same differential relation between charge and electromagnetic field, but the actual solutions of this depend greatly on the charge distribution.

In Black-Scholes-Merton, the non-arbitrage behavior is locally given by the same SDE, but the solution may vary greatly depending on for instance what sort of stock option behavior is given for the final time boundary

condition. Also, the chain rule is important in SDEs, and that should happen at the differential level.

Also, and my knowledge of this very informal, I don't consider the dW_t term to be so imprecise. It is probably often taught imprecisely, and with quite a bit of hand waving, but it is not hard to come up with convincing arguments with regards to its behavior. I have in my notes things like $dW_t = W_t\sqrt{dt}$, $dW_t = \lim_{n\to\infty}\sum_{i=1}^{n} Y(ih)\sqrt{\frac{t}{n}}$, where $Y(ih)$ is -1 or 1 with equal probability $p = 1/2$ (this is more or less from Finances text on MFE, a free online textbook). I am pretty sure I didn't write that right, but don't want to check details right now, so hopefully someone more motivated can check. Also, I don't think that sort of representation is unique, but that it suffices to use that one will work. This reminds me of how you can represent the delta function as the limit of various distributions.

Answer 5

I like your second reason which is also what I think, but the fact is that not only writing them in differential form makes the various complicated computations easier to handle, but often also they lead to solve PDEs in deterministic sense. For example the case for *Black-Scholes-Merton* SDE where with suitable terminal and boundary conditions becomes a *backward parabolic* PDE that we can solve. Therefore, it's better to write a SDE in a quite well-known fashion that we have used to even though, it's "formal" and doesn't have a specific meaning.

Solutions to a stochastic differential equation

$$dX_t = -\frac{1}{2}e^{-2X_t}\ dt + e^{-X_t}dB_t, X_0 = x_0$$

Hint: solve this equation using the substitution $X_t = u(B_t)$,

show that the solution blows up at a finite random time.

answer

Suppose $X_t = u(B_t)$.

Using Ito's lemma,

$$dX_t = u'dB_t + \frac{1}{2}u''dt$$

By coefficient matching with the given fomula $dX_t = e^{-X_t}dB_t - \frac{1}{2}e^{-2X_t}dt$, we get $u' = e^{-u}, u'' = -e^{-2u}$.

let $f = u'$, then $f^2 = -f'$.

Solve above ODE, $f(t) = \frac{1}{t+C}$, C is constant.

$$u'(t) = f(t) = \frac{1}{t+C}$$

$$\Rightarrow u(t) = \ln(t+C)$$

$$\Rightarrow X_t = u(B_t) = \ln(B_t + C)$$

$$X_0 = x_0 \Rightarrow \ln(B_0 + C) = x_0 \Rightarrow C = e^{x_0}.$$

$C = e^{x_0}$ is bounded. After finite random time, B_t hits $-e^{x_0}$ with probability one, and X_t goes to ∞.

Problem on Solving Stochastic Differential Equation

Let (Xt) be a solution to the equation

$dX_t = aX_t dt + \sqrt{(1+X_t^2)}dW_t$ where W_t is a Brownian motion process at time t

Let $Y = F(X_t)$ for a certain function F. Find F for which (Y_t) solve this equation $dY_t = f(Y_t)dt + dW_t$

where the drift term f should be explicitly determined. You may use the fact that

$$\int \frac{1}{\sqrt{1+x^2}}dx = ln(x+\sqrt{1+x^2}).$$

My attempt

$$dY_t = dF(X_t)$$

Applying ito's formula we obtain.

$$dF(X_t) = F'(X_t)dX_t + 1/2F''(X_t)(dX_t)^2$$

Substituting dX_t and $(dX_t)^2$

$$dF(X_t) = F'(X_t)(aX_t dt + \sqrt{1+X_t^2}dW_t) + \frac{1}{2}F''(X_t)(1+X_t^2)dt$$

$$dF(X_t)=F'(X_t)(aX_t+\frac{1}{2}F''(X_t)(1+X_t^2))dt+F'(X_t)\sqrt{1+X_t^2}dW_t$$

Equating coefficients for dW_t

$$F'(X_t)\sqrt{1+X_t^2}=1$$

$$F(X_t)=\int\frac{1}{\sqrt{1+X_t^2}}dx$$

$$F(X_t)=ln(X_t+\sqrt{1+X_t^2})+C$$

, where C is a constant

Now my question is; Can we find this constant? If not, is the answer above correct for $F(x)$

now after equating the coefficients of dt, the drift term f is

$$f(Y_t)=f(F(X_t))=aX_t+\frac{1}{2}F''(X_t)(1+X_t^2)$$

Question; How can I find the function f?

Answer

Your solution is correct. You find

$$F(X_t)=\ln\left(X_t+\sqrt{1+X_t^2}\right)+C,$$

for any constant C. In fact, for all such constants, (Y_t) will satisfy the expected stochastic differential equation, with initial value

$$Y_0 = \ln\left(X_0 + \sqrt{1+X_0^2} \right) + C \ .$$

Afterwards, you made a typo when writing $f(Y_t)$, it should be

$f(Y_t) = f(F(X_t)) = aX_t \mathbf{F}'(\mathbf{X_t}) + \frac{1}{2} F''(X_t)(1+X_t^2)$. Simple computations show that

$$F'(X_t) = \frac{1}{\sqrt{1+X_t^2}}, \text{ and } F''(X_t) = -X_t \left(1+X_t^2\right)^{-3/2},$$

and therefore,

$$f(F(X_t)) = \left(a - \frac{1}{2}\right) \frac{X_t}{\sqrt{1+X_t^2}}.$$

Lastly, it remains to note that $\ln\left(X_t + \sqrt{1+X_t^2} \right) = \sinh^{-1}(X_t)$, so

$$f(Y_t) = \left(a - \frac{1}{2}\right) \tanh\left(Y_t - C\right) =$$

$$\left(a - \frac{1}{2}\right) \tanh\left(Y_t - Y_0 + \ln\left(X_0 + \sqrt{1+X_0^2} \right) \right)$$

Solution to a stochastic differential equation

I could really do with some help on this question, have no idea where to start. Any advice would be much appreciated, thank u in advance.

I am given

$$\begin{cases} dx(t) &= (1+x(t))dt + x(t)dw(t) \\ x(0) &= 1 \end{cases}$$

and

$$\begin{cases} dy(t) &= y(t)dt + y(t)dw(t) \\ y(o) &= 1 \end{cases}$$

we define another stochastic process $z(t) = \int_0^t \frac{1}{y(s)} ds$

It can be shown via the multidimensional ito formula that the stochastic process $(y(t)z(t))$ satisfies:

$$d(y(t)z(t)) = z(t)dY(t) + dt.$$

Using this show that the solution $x(t)$ of the above given SDE is:

$$x(t) = y(t)(1+z(t))$$

Where $z(t)$ is defined as above, i.e. $z(t) = \int_0^t \frac{1}{y(s)} ds$

The second part of the problem is to show that x(t)-y(t) is equal in distribution to z(t)!!

answer

Recall that an SDE of the form

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t$$

has a unique solution whenenver the coefficients b, σ are globally Lipschitz continuous. In particular, we see that the SDE

$$dX_t = (1 + X_t)dt + X_t\, dW_t$$

has a unique solution. Now let $(Y_t)_t$ the (unique) solution of the SDE

$$dY_t = Y_t\, dt + Y_t\, dW_t \quad (1)$$

and set

$$Z_t := \int_0^t \frac{1}{Y_s} ds \text{ i.e. } dZ_t = \frac{1}{Y_t} dt.$$

From Itô's formula we get

$$d(Y_t \cdot Z_t) = Z_t\, dY_t + dt \qquad (2)$$

Consequently, we see that

$$d(Y_t(1 + Z_t)) = dY_t + d(Y_t \cdot Z_t) \overset{(2)}{=}$$
$$(1 + Z_t)dY_t + dt =$$

$$(1 + Z_t)Y_t\, dW_t + (1 + Z_t)Y_t\, dt + dt \overset{(1)}{=}$$
$$(1 + Z_t)Y_t\, dW_t + ((1 + Z_t)Y_t + 1)dt$$

This means that the process $A_t := (1 + Z_t)Y_t$ solves the SDE

$$dA_t = A_t\, dW_t + (A_t + 1)dt.$$

Now the claim follows from the uniqueness of the solution.

The second part works analogously.

Solving the Ornstein-Uhlenbeck Stochastic Differential Equation

I am asked to solve the following SDE:

$$dX_t = (a - bX_t)dt + cdB_t, \text{ where } X(0) = x.$$

($(B_t)_{t\geq 0}$ is a brownian motion.)

For constants a, b and c and X is a random variable independent of brownian motion.

Also, find m and σ and a condition on b so that if x is normal with mean m and variance σ then $X(t)$ is distributed as x.

Here is what I have done for solving the equation.

Let α_t be a (deterministic) process solution of

$$d\alpha_t = (a - b\alpha_t)dt, \text{ where } \alpha_0 = 1.$$

This is an ODE and solving it yields the answer $\alpha_t = \dfrac{a - e^{-bt}}{b}$.

Let us write $X_t = \alpha_t Y_t$ and search for an equation for Y_t. By the integration by parts formula (in differential form) we have :

$$dX_t = d\alpha_t Y_t + \alpha_t dY_t.$$

(The other term is zero since α has bounded variation.)

Substituting the expression for α_t we get :

$$dX_t = e^{-bt}Y_t + \alpha_t dY_t = (a - b\alpha_t)Y_t dt + \alpha_t dY_t. \quad (I)$$

On the other hand:

$$dX_t = (a - bX_t)dt + cdB = (a - b\alpha_t Y_t)dt + cdB \quad (II).$$

Equating (I) and (II) we conclude that

$$\alpha_t dY_t = cdB,$$

in other words

$$dY_t = \frac{c}{\alpha_t} dB_t.$$

With $Y_0 = \frac{X_0}{\alpha_0} = x_0$. This implies that:

$$Y_t = x_0 + c\int_0^t \frac{1}{\alpha_s} dB_s = x_0 + c\int_0^t \frac{b}{a - e^{-bt}} dB_s.$$

Therefore the solution $X_t = \alpha_t Y_t$ can be obtained by multiplying α_t by Y_t. Is this correct? the final answer obtained looks a bit messy.

Also how should approach the second part of the question? I appreciate any help.

answer

Your solution to the ODE is incorrect. Indeed,

$d\alpha_t = (a - b\alpha_t)dt, \quad \alpha_0 = 1,$ has solution

$\alpha(t) = e^{-bt} + \frac{a}{b}\left(1 - e^{-bt}\right), \quad t \geq 0.$ Additionally you made a mistake when equating (I) and (II), namely you should get

$aY_t\mathrm{d}t+\alpha_t\mathrm{d}Y_t=a\mathrm{d}t+c\mathrm{d}B_t$, so this approach does not work. (Since this S.D.E. is no easier to solve than the initial one.)

You should rather find the solution to

$\mathrm{d}\alpha_t=-b\alpha_t\mathrm{d}t,\quad \alpha_0=1,$ and apply the variation of the constant method that you propose.

Once you have done this, in order to solve question 2 start by finding the distribution of X_t when $X_0\sim\quad(m,\sigma^2)$ is independent of X_t. At this point, it should be clear what the required conditions are.

Analytic solutions to a Stochastic Differential Equation

I want to solve an SDE as follows:

$dX_t=\alpha(\beta-X_t)dt+dB_t,\quad X_0=x_0$ where α, β are positive constants and B_t is a Browian motion independent to X_t .Is there a way to solve it? Thank in advance for any help or some references to this kind of equation.

answer

Let assume this

$Y_t=X_t\mathrm{e}^{\alpha t}$ then using Ito we find

$dY_t=\alpha X_t\mathrm{e}^{\alpha t}dt+\mathrm{e}^{\alpha t}dX_t$ with some rearrangement we can find

$e^{-\alpha t} dY_t = dX_t + \alpha X_t dt$ which looks awfully like what you have as the original SDE above except we have

$dX_t = \alpha\beta dt - \alpha X_t dt + dB_t$ thus we can easily see

$e^{-\alpha t} dY_t = \alpha\beta dt + dB_t$ thus

$dY_t = \alpha\beta e^{\alpha t} dt + e^{\alpha t} dB_t$ integrating we get

$Y_t = Y_0 + \beta\left(e^{\alpha t} - 1\right) + \int_0^t e^{\alpha s} dB_s$ or

$X_t = x_0 + \beta\left(1 - e^{-\alpha t}\right) + \int_0^t e^{\alpha(s-t)} dB_s$ [11]

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[11] In mathematics, the **Ornstein–Uhlenbeck process** (named after Leonard Ornstein and George Eugene Uhlenbeck), is a stochastic process that, roughly speaking, describes the velocity of a massive Brownian particle under the influence of friction. The process is stationary, Gaussian, and Markovian, and is the only nontrivial process that satisfies these three conditions, up to allowing linear transformations of the space and time variables. Over time, the process tends to drift towards its long-term mean: such a process is called **mean-reverting**.

The process can be considered to be a modification of the random walk in continuous time, or Wiener process, in which the properties of the process have been changed so that there is a tendency of the walk to move back towards a central location, with a greater attraction when the process is further away from the centre. The Ornstein–Uhlenbeck process can also be considered as the continuous-time analogue of the discrete-time AR(1) process.

An Ornstein–Uhlenbeck process, x_t, satisfies the following stochastic differential equation:

$$dx_t = \theta(\mu - x_t)dt + \sigma dW_t$$

Stochastic differential equation: Itô's formula?

I came across a problem with SDE and need your help once again:

$$dX_t = tX_t\,dt + \exp\left(\frac{t^2}{2}\right)$$

and I'm supposed to solve this, in the way $X_t = f(t, W_t)$.

So I use Itô's formula:

$$dX_t = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dW_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}dt$$ and then get that:

$$\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{\partial x^2} = \mu(t,x) = t \cdot x$$

$$\frac{\partial f}{\partial x} = \sigma(t,x) = \exp\left(\frac{t^2}{2}\right).$$

From the second line, I'd get: $f = \exp\left(\frac{t^2}{2}\right) \cdot x$, but then the first line would become: $t \cdot \exp\left(\frac{t^2}{2}\right) \cdot x + \frac{1}{2} \cdot 0$ which is not $t \cdot x$. Where is the mistake?

---

where $\theta > \circ, \mu > \circ, \sigma > \circ$ are parameters and W_t denotes the Wiener process.The above representation can be taken as the primary definition of an Ornstein–Uhlenbeck process

Answer

Oh yes, I found the mistake:

The function is not $dX_t = txdt + (\cdots)$, but $dX_t = tX_t dt + (\cdots)$. So with $X_t = f(t,x)$, it becomes clear that $dX_t = t \cdot \exp(\frac{t^2}{2}) \cdot x + (\cdots) = t \cdot f(t,x) + (\cdots) = tX_t + (\cdots)$. Stupid me!

and then we get

$X_t = \exp(\frac{t^2}{2}) \cdot W_t$ and with $X_0 = 1$,

this is $X_t = \exp(\frac{t^2}{2}) \cdot \frac{W_t}{W_0}$

About stochastic differential equations

Consider, for all $x \in \mathbb{R}$, the process $(X_t^x)_{t \geq 0}$ unique solution of the following SDE:

$$X_t^x = x + \int_0^t \sigma\left(X_s^x\right) dB_s + \int_0^t b\left(X_s^x\right) ds, t \geq 0$$

where σ and b are lipschitziennes[12] from $\mathbb{R}$ into $\mathbb{R}$ and $\left(B_t\right)_{t \geq 0}$ is the real standard Brownian motion starting from zero.

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[12] In mathematical analysis, **Lipschitz continuity**, named after Rudolf Lipschitz, is a strong form of uniform continuity for functions. Intuitively, a Lipschitz continuous function is limited in how fast it can change: there exists a

Consider also the following stopping times:

$\forall x, y \in \mathbb{R}$, $\tau_{(x,y)} = \inf\{t \geq 0 : X_t^x = y\}$

$\forall \left]y, z\right[\subset \mathbb{R}$ bounded, $\tau_{(y,z)}(x) = \inf\{t \geq 0 : X_t^x \notin \left]y, z\right[\}$

(H1): Suppose $b \equiv 0$ and $x \in \left]y, z\right[$

1rst question:

I was asked to show that, $\forall \phi \in \overset{2}{c}(\mathbb{R}, \mathbb{R})$ we have

$$\left\{\phi\left(X_{t\wedge\tau^x}^x\right)\right\} - \phi(x) = \frac{1}{2}\left\{\int_0^{t\wedge\tau^x} a\phi''\left(X_s^x\right)ds\right\}$$

where $a(x) = \sigma^2(x)$ and $\tau^x = \tau_{(y,z)}(x)$

However, I can't show that but I can show according to Itô's lemma

---

definite real number such that, for every pair of points on the graph of this function, the absolute value of the slope of the line connecting them is not greater than this real number; this bound is called a "Lipschitz constant" of the function (or "modulus of uniform continuity"). For instance, every function that has bounded first derivatives is Lipschitz.

In the theory of differential equations, Lipschitz continuity is the central condition of the Picard–Lindelöf theorem which guarantees the existence and uniqueness of the solution to an initial value problem. A special type of Lipschitz continuity, called contraction, is used in the Banach fixed point theorem.

We have the following chain of inclusions for functions over a closed and bounded[2] subset of the real line

that

$$\phi\left(X^{x}_{t\wedge\tau^{x}}\right)-\phi(x)=\frac{1}{2}\int_{0}^{t\wedge\tau^{x}} a\phi''\left(X^{x}_{s}\right)ds+$$

$$\int_{0}^{t\wedge\tau^{x}} b\phi'\left(X^{x}_{s}\right)ds+\int_{0}^{t\wedge\tau^{x}} \sigma\phi'\left(X^{x}_{s}\right)dBs$$

then, by taking expected value for both sides, we have

$$\left\{\phi\left(X^{x}_{t\wedge\tau^{x}}\right)\right\}-\phi(x)=$$

$$\frac{1}{2}\left\{\int_{0}^{t\wedge\tau^{x}} a\phi''\left(X^{x}_{s}\right)ds\right\}+\left\{\int_{0}^{t\wedge\tau^{x}} b\phi'\left(X^{x}_{s}\right)ds\right\}$$

Where am I making a mistake ? Or must it be a mistake in the exercise?

2nd question

After all, I was asked to deduce from this result that $\left\{\tau^{x}<\infty\right\}=1$.

we can choose $\phi\in\ {}^{2}_{c}(\mathbb{R},\mathbb{R})$ such that $\phi''(x)=\dfrac{1}{a(x)}$, then by the first result , we have

$$\left\{t\wedge\tau^{x}\right\}=2\left(\left\{\phi\left(X^{x}_{t\wedge\tau^{x}}\right)\right\}-\phi(x)\right)$$

How to conclude?

Someone could help me please?

answer

1rst question: you wrote that $b \equiv 0$ under hypothesis (H1), this might solve you problem.

2nd question: I will assume that σ does not vanish. Using a cut-off function we can find $\phi \in C_c^2(\mathbb{R})$ such that $\phi'' = \frac{1}{a}$ on [y,z]. The result of the first question gives

$E[\phi(X^x_{t\wedge\tau^x})] - \phi(x) = \frac{1}{2}E[t \wedge \tau^x]$. Notice that the l.h.s. is bounded by $2\|\phi\|_\infty$ and that the r.h.s. tends to $\frac{1}{2}E[\tau^x]$ as $t \to \infty$ by Lebesgue's monotone conergence theorem. Hence we finally have

$E[\tau^x] \leq \|\phi\|_\infty < \infty$. In particular, $\tau^x < \infty$ a.s.

Stochastic differential

Im really new in the stochastic procceses please help me.

How can I solve this stochastic differential equation?

$$dX = A(t)Xdt$$

$$X(0) = X_0$$

If $A:[0,\infty] \to R$ is continous and X is a real random variable.

answer

The (deterministic) ODE

$$\frac{dx(t)}{dt} = A(t) \cdot x(t)$$

can be solved by transforming it into

$$\log x(t) - c = \int^t \frac{1}{x(s)} dx(s) = \int^t A(s) ds$$

The same approach applies to this stochastic differential equation. We use Itô's formula for $f(x) := \log x$ and the Itô process X_t and obtain

$$\log X_t - \log x_0 = \int_0^t \frac{1}{X_s} dX_s = \int_0^t A(s) ds$$

(Note that the second term in Itô's formula equals 0 .) Consequently,

$$X_t = X_0 \cdot \exp\left(\int_0^t A(s) ds \right)$$

This means that the solution of the ODE and the SDE coincide and as @user39360 already remarked that's not surprising since there is no stochastic part in your stochastic differential equation at all - and so the stochastic differential equation becomes an ordinary differential equation.

Let me finally remark that the approach described above (i.e. considering the corresponding ordinary differential equation and adapting the idea of the solution method in the determinstic case) works often quite well in a more general setting.

Use Ito's formula to determine the stochastic differential equation satisfied by Vt

A stochastic process V_t is defined by

$$V_t = \sqrt{t(t + W_t^2)}$$

W_t is the Wiener process and t denotes the time ($t > 0$).

Use Ito's formula to determine the stochastic differential equation satisfied by V_t.

I have taken the known process to be just W. Then $a(x,t) = 0$ and $b(x,t) = 1$.

The process V is:

$$V_t = h(W_t, t) \text{ where } h(x,t) = \sqrt{t(t + x^2)}$$

I then calculated the partial derivatives of $h(x,t)$ and substituted all my values into Ito's formula.

However, no matter how many times I try, I don't seem to get the correct answer and I am unsure as to why.

Answer 1

Set $f(t,x) := \sqrt{t \cdot (t + x^2)}$

By the chain rule,

$$\frac{\partial}{\partial x} f(t,x) = \frac{xt}{f(t,x)} \quad (1)$$

$$\frac{\partial^2}{\partial x^2} f(t,x) = \frac{t}{f(t,x)} - \frac{x^2 t^2}{f(t,x)^3} \quad (2)$$

$$\frac{\partial}{\partial t} f(t,x) = \frac{2t + x^2}{2f(t,x)} \quad (3)$$

In order to find the corresponding SDE, we have to express the expressions on the right-hand side in terms of $f(t,x)$ and t. From the definition of f, it follows that

$$x^2 = \frac{f(t,x)^2}{t} - t \qquad (4)$$

If we plug (4) into (2) and (3), we get

$$\frac{\partial^2}{\partial x^2} f(t,x) = \frac{t}{f(t,x)} - \left(\frac{f(t,x)^2}{t} - t \right) \frac{t^2}{f(t,x)^3} =$$

$$\frac{t^3}{f(t,x)^3} \frac{\partial}{\partial t} f(t,x) =$$

$$\frac{t}{f(t,x)} + \left(\frac{f(t,x)^2}{t} - t \right) \frac{1}{2f(t,x)} =$$

$$\frac{1}{2} \frac{t}{f(t,x)} + \frac{f(t,x)}{t}$$

Similarly,

$$\frac{\partial}{\partial x} f(t,x) =$$

$$\frac{t}{f(t,x)} \operatorname{sgn}(x) \sqrt{\frac{f(t,x)^2}{t} - t} = \operatorname{sgn}(x) \sqrt{t - \frac{t^3}{f(t,x)^2}}$$

Now if we apply Itô's formula, i.e.

$$V_t = f(t, W_t) =$$

$$\int_0^t \frac{\partial}{\partial x} f(s, W_s) dW_s + \int_0^t \left(\frac{1}{2} \frac{\partial^2}{\partial x^2} f(s, W_s) + \frac{\partial}{\partial t} f(s, W_s) \right) ds,$$

we find that

$$V_t = \int_0^t \operatorname{sgn}(W_s) \sqrt{s} \sqrt{1 - \frac{s^2}{V(s)^2}} dW_s +$$

$$\int_0^t \left(\frac{V(s)}{s} + \frac{1}{2} \frac{s}{V(s)} + \frac{1}{2} \frac{s^3}{V(s)^3} \right) ds.$$

Note that the diffusion coefficient depends on the Brownian motion $(W_t)_{t \geq 0}$. Since V_s does not contain any information whether W_s is positive or negative, we cannot expect to express $\operatorname{sgn}(W_s)$ in terms of V_s. This means that we cannot write $(V_t)_{t \geq 0}$ as the solution of a stochastic differential equation.

Answer 2

I took a slightly different route

$$S_t = V_t^2 = t(t + x^2)$$

$$x^2 = \frac{V_t^2}{t} - t$$

$$dS_t = 2V_t dV_t = \frac{\partial S_t}{\partial t} dt + \frac{\partial S_t}{\partial x} dW_t + \frac{1}{2}\frac{\partial^2 S_t}{\partial x^2} dt$$

$$\frac{\partial S_t}{\partial t} = 2t + x^2$$

$$\frac{\partial S_t}{\partial x} = 2tx$$

$$\frac{\partial^2 S_t}{\partial x^2} = 2t$$

and so

$$2V_t dV_t = \left(t + \frac{V_t^2}{t}\right) dt + 2t\left(\frac{V_t^2}{t} - t\right)^{\frac{1}{2}} dW_t + tdt =$$

$$\left(2t + \frac{V_t^2}{t}\right) dt + 2\left(tV_t^2 - t^3\right)^{\frac{1}{2}} dW_t$$

$$dV_t = \left(\frac{t}{V_t} + \frac{V_t}{2t}\right) dt + \left(t - \frac{t^3}{V_t^2}\right)^{\frac{1}{2}} dW_t$$ I'm slightly disconcerted that my dt coefficient looks so different to saz's. I'll need to do some rearranging to check whether or not they actually agree...

Condition for existence of a stochastic differential equation

With B a standard Brownian motion, write

$dX_t = f_t dt + g_t dB_t$. What are the conditions on $(f)_{t\geq 0}$ and $(g)_{t\geq 0}$ for X_t to exists?

I think $(f)_{t\geq 0}$ and $(g)_{t\geq 0}$ should be adapted to the filtration of B.

Do they have to be càdlag?

What about the integrability condition

$$E\left(\int_0^t f_s ds\right) < \infty$$

$$E\left(\int_0^t g_s^2 d[B]_s\right) < \infty ?$$

We are using several textbooks in our class, and I can't pindown the theorem that treats this.

(Kurtz, Stochastic Analysis; Oksendal; Protter)

answer

Well usually these functions depend also on the (yet to be found) solution X and must be measurable. The most used condition that guarantees a unique continuous solution is a growth condition of the form

$$|f(t,x)|+|g(t,x)|\leq C(1+|x|) \qquad x\in\mathbb{R}^n, 0\leq t\leq T$$

and a Lipschitz condition

$$|f(t,x)-f(t,y)|+|g(t,x)-g(t,y)|\leq L|x-y|$$

$$x,y\in\mathbb{R}^n, 0\leq t\leq T.$$

The proofs often use the same idea as the in ODE case (Picard-Lindelöf).[۱۳]

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[13] In mathematics, in the study of differential equations, the **Picard–Lindelöf theorem**, **Picard's existence theorem** or **Cauchy–Lipschitz theorem** is an important theorem on existence and uniqueness of solutions to first-order equations with given initial conditions.

The theorem is named after Émile Picard, Ernst Lindelöf, Rudolf Lipschitz and Augustin-Louis Cauchy.

Consider the initial value problem

$$\begin{cases} y'(t)=f(t,y(t)) \\ y(t_0)=y_0 \end{cases}$$

Suppose f is uniformly Lipschitz continuous in y (meaning the Lipschitz constant can be taken independent of t) and continuous in t. Then, for some value $\varepsilon > 0$, there exists a unique solution $y(t)$ to the initial value problem on the interval $[t_0-\varepsilon, t_0+\varepsilon]$

See : Coddington & Levinson (1955), Theorem I.3.1

Check out Theorem 5.2.1 in Oksendal!

Stochastic differential equation with trigonometric functions

I heard that the following SDE can be solved analitically by substitution:

$$\begin{cases} dX(t) = -\left[\sin(2X(t)) + \frac{1}{4}\sin(4X(t))\right]dt + \sqrt{2}\cos^2 X(t)dB(t) \\ X(0) = 1, t \in [0,\tau) \end{cases}$$

But is isn't as simple as it sounds. Any ideas? Of course is a brownian motion.

answer

Motivation: Only few stochastic differential equations (SDEs) can be solved explicitely; one of them is a linear SDE with determinstic coefficients, i.e. an SDE of the form

$$dZ_t = (\alpha(t) + \beta(t)Z_t)dB_t + (\gamma(t) + \delta(t)Z_t)dt \quad (1)$$

That's why one usually tries to transform SDEs into a linear SDE. For SDEs of the form

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t, \ (2)$$

i.e. SDEs where the coefficient do not depend on the time t, there are necessary and sufficient criterions for a transformation into a linear SDE (see e.g. *René L. Schilling/Lothar Partzsch: Brownian Motion - An Introduction to Stochastic Processes*, Section 19.4, 2nd edition).

Roughly speaking, the only two possibilities two transform an SDE (2) into a linear SDE (1) are the following two substitutions: Set

$$Z_t := f(X_t)$$

where either

$$f(x) := \gamma \int_0^x \frac{1}{\sigma(y)} dy \quad (3)$$

or

$$f(x) := \exp\left(\gamma \cdot \int_0^x \frac{1}{\sigma(y)} dy \right) \quad (4)$$

for some suitable constant $\gamma > 0$. (In fact, there are criterions in terms of the derivatives when to use which one; but they are quite messy.)

So here we go. The SDE

$$dX_t = -\left(\sin(2X_t) + \frac{1}{4}\sin(4X_t) \right) dt + \sqrt{2}\cos^2 X_t \, dB_t \quad (5)$$

is obviously of the form (5) where $\sigma(y) = \sqrt{2}\cos^2(y)$. If we try substitution (3), then

$$Z_t := f(X_t) \overset{(3)}{=} \gamma \int_0^{X_t} \frac{1}{\sqrt{2}\cos^2(y)} dy = \gamma' \tan X_t$$

where $\gamma' := \frac{\gamma}{\sqrt{2}}$. Applying Itô's formula yields

$$Z_t - Z_0 = \gamma' \int_0^t \frac{1}{\cos^2(X_s)} dX_s + \gamma \int_0^t \frac{\sin X_s}{\cos^3 X_s} d\langle X \rangle_s \overset{(5)}{=}$$

$$\sqrt{2}\gamma' \int_0^t dB_s - \gamma' \int_0^t \left(\frac{\sin(2X_s)}{\cos^2(X_s)} + \frac{\sin(4X_s)}{4\cos^2(X_s)} \right) ds +$$

$$2\gamma' \int_0^t \frac{\sin X_s}{\cos^3 X_s} \cos^4 X_s \, ds$$

In the last step we used that the quadratic variation $\langle X \rangle_s$ equals, by (5),

$$d\langle X \rangle_t = 2\cos^4 X_t \, dt.$$

Using the well-known formulas

$$\sin(2x) = 2\sin x \cdot \cos x$$

$$\sin(4x) = 8\sin x \cdot \cos^3(x) - 4\sin x \cdot \cos x$$

we get

$$Z_t - Z_0 = \sqrt{2}\gamma' \int_0^t dB_s - \gamma' \int_0^t Z_s \, ds.$$

This means that we have transformed the SDE (5) into a linear SDE of the form (1). This SDE can be solved explicitely, and therefore we find an explicit expression for the solution

$$X_t = \arctan\left(\frac{Z_t}{\gamma'} \right).$$

(Here $\gamma' > 0$ is an arbitrary constant; we may choose $\gamma' = 1$.)

Solution to the stochastic differential equation

Let $X_o = x$, $dX_t = \frac{1}{X_t}dt + X_t dW_t$, W_t is a brownian motion i am thinking of trying $Y_t = \frac{X_t^2}{2}$ and apply ito's lemma on Y_t

Answer

Let $Y_t = \mathrm{e}^{t-2W_t} X_t^2$.

Then, applying Ito's lemma:

$\mathrm{d}Y_t = \mathrm{e}^{t-2W_t}\mathrm{d}(X_t^2) + X_t^2 \mathrm{d}\mathrm{e}^{t-2W_t} + \mathrm{d}\mathrm{e}^{t-2W_t} \cdot \mathrm{d}(X_t^2)$ Since

$\mathrm{d}\mathrm{e}^{t-2W_t} = \mathrm{e}^{t-2W_t}\left(3\mathrm{d}t - 2\mathrm{d}W_t\right)$ and

$\mathrm{d}(X_t^2) = (X_t^2 + 2)\mathrm{d}t + 2X_t^2 \mathrm{d}W_t$, we get:

$$\mathrm{d}Y_t = 2\mathrm{e}^{t-2W_t}\mathrm{d}t$$

Also, initial condition is

$$Y_0 = \mathrm{e}^{0-2W_0} X_0^2 = x^2 .$$

Therefore:

$$\mathrm{e}^{t-2W_t} X_t^2 = Y_t = x^2 + \int_0^t 2\mathrm{e}^{s-2W_s}\mathrm{d}s$$

This provides the solution:

$$X_t = \mathrm{sign}(x)\sqrt{x^2 \mathrm{e}^{2W_t - t} + 2\mathrm{e}^{2W_t - t}\int_0^t \mathrm{e}^{s-2W_s}\mathrm{d}s}$$

How to solve Stochastic differential equation?

I do not have a clue on how to solve out this type of question, and how to deal with integration with a combination of brownian motion and linear function. Can anyone help me out please?

$$\begin{cases} X_t^x = B_t + \int_0^t \frac{x - X_s^x}{1-s} ds \\ t \in [0,1), x \in \mathbb{R} \end{cases}$$

Answer

HINT:

$$\int_0^t \int_0^s \frac{B_{s'}}{(1-s')^2} ds' ds = \int_0^t \int_{s'}^t \frac{B_{s'}}{(1-s')^2} ds' ds \quad (1)$$

and

$$\int_0^t \frac{B_{s'}(t-s')}{(1-s')^2} ds' = \int_0^t \frac{B_{s'}(t-1+1-s')}{(1-s'^2)} ds' \quad (2)$$

Can you complete?

We are given

$$X_t = xt + B_t - (1-t)\int_0^t \frac{B_s}{(1-s)^2} ds \quad (3)$$

and asked to show that this X_t is a solution of the following

$$X_t = B_t + \int_0^t \frac{x - X_s}{1-s} ds \quad (4)$$

Let's substitute (3) into (4) to obtain

$$X_t = xt + B_t - (1-t)\int_0^t \frac{B_s}{(1-s)^2} ds$$

$$= B_t + \int_0^t \frac{x - (xs + B_s - (1-s)\int_0^t \frac{B_{s'}}{(1-s')^2} ds')}{(1-s)} ds$$

$$= B_t + xt - \int_0^s \frac{B_s}{(1-s)} ds + \int_0^t \int_0^s \frac{B_{s'}}{(1-s')^2} ds' ds \quad (5)$$

Using Hints (1) and (2),

$$\int_0^t \int_0^s \frac{B_{s'}}{(1-s')^2} ds' ds = \int_0^t \frac{B_s}{1-s} ds - (1-t)\int_0^t \frac{B_s}{(1-s)^2} ds$$

and we are done!

Why are probabilities needed for stochastic differential equations?

A SDE has the form :

$$\mathrm{d}X_t = \mu(X_t, t)\mathrm{d}t + \sigma(X_t, t)\mathrm{d}B_t$$

By fixing w it becomes :

$$\mathrm{d}X_t(w) = \mu(X_t(w), t)\mathrm{d}t + \sigma(X_t(w), t)\mathrm{d}B_t(w)$$

So for each w we have an equation on the trajectory $X_t(w)$, we can solve it as if w was just a paramter, where are the"probabilities" needed ? It is just an equation where the unknown is a function.

I don't know if I am clear.

Answer 1

You are right. In a perhaps less canonical view of this part of mathematics, the solutions are simply measurable cocycles (or, as some groups of people usually call them, random dynamical systems).

A standard reference would be the book by Ludwig Arnold, Random Dynamical Systems, although the size usually pushes people away. Still he essentially follows the established canon of ergodic theory, in my opinion with the beautiful exception related precisely to what you ask. In particular, he discusses quite a number of examples.

Answer 2

Well the main problem stems from the fact that Brownian motion doesn't have finite variations on any interval, this implies through classical real analysis, using Banach-Steinhaus Theorem (see Protter Book Chatper 1 section 8 "Naïve Stochastic Integration Is Impossible"), that a path-by-path theory cannot be found and be totally satisfactory in the sense that it leads to a "real integral". The construction of stochastic integral via its good properties (essentially that it must fulfill the dominated convergence theorem) is used by some authors to construct such a "proper" stochastic integral, for example Protter (Stochastic integration and differential equation) or G. Lowther (in his blog "almost sure" which really is amazing and take about the same approach).

Nevertheless there are some ways to skirt this and in some cases you can build a kind "stochastic integral" (not totally satisfactory !!!) without resorting to a probability space and take a path-by-path approach, for example [Föllmer][۱۴], or [Karandikar] [۱۵]were the first to propose this kind of construction, which has been till then qualified as quasi-sure analysis unless mistaken also for an recent contribution take a look at [M. Nutz] [۱۶]which has a very general approach on the subject.

[14] http://archive.numdam.org/ARCHIVE/SPS/SPS_1981__15_/SPS_1981__15__1430/SPS_1981__15__143_0.pdf
[15] http://repository.ias.ac.in/18404/1/18404.pdf
[16] http://arxiv.org/pdf/1108.2981.pdf

Questions about existence and uniqueness theorem for stochastic differential equations in Oksendal's SDE book

In [Oksendal's SDE book], Theorem 5.2.1. (Existence and uniqueness theorem for stochastic differential equations) assumes Z is a random variable which is independent of the sigma algebra $_{\infty}^{(m)}$ generated by $B_s, s \geq 0$ and such that $E|Z|^2 < \infty$.

Why is there a need to have the solution X_t to a SDE adapted to the filtration $_t^Z$ of a random variable Z and the Brownian motion $B_s, s \leq t$, instead of the filtration of the Brownian motion $B_s, s \leq t$?

Does "(the SDE ... has) a t-continuous solution $X_t(\omega)$" means that the sample path of the process X is continuous wrt t a.s.?

Thanks and regards!

Answer

Hey in regards to your first question the filtration is generated by Z and B(s,w) for s<=t. In this case Z is the initial value and it is assumed to be a RV independent of the brownian motion.

In regards to your second questions saying it has a t-continuous solution means there exists a solution to the SDE that has continuous paths. Remember a DE (whether deterministic or stochastic) may have many solutions. However, if the conditions for the theorem are met their exists a unique t-continuous solution, ie a solution X(t,w) where for any fixed w

(except on a set of measure 0) the sample paths are continuous w.r.t time. Note: I believe the second condition is sufficient for uniqueness and the first is sufficient for existence, but this should be verified.

Hope this helped!

Variation processes and strong solutions of stochastic differential equations

Let

- $(\Omega, \quad , \mathrm{P})$ be a probability space
- be a filtration on $(\Omega, \quad)$
- τ be a -stopping time

An -adapted, real-valued stochastic process $M = (M_t)_{t \geq 0}$ on $(\Omega, \quad , \mathrm{P})$ is called **local -martingale until τ ** $:\Leftrightarrow$ There exists a sequence $(\tau_n)_{n \in \mathbb{N}}$ of -stopping times such that $\tau_n \uparrow \tau$ P-almost surely and the *stopped process* $M^{\tau_n} := (M_{\tau_n \wedge t})_{t \geq 0}$ is an uniformly integrable -martingale.

Now, let $t \mapsto H(t) = (H_{ij}(t))_{\substack{i=1,\ldots,n \\ j=1,\ldots,m}}$ be -progressively measurable, $\| \cdot \|$ be the *Frobenius norm*, $B = (B^1, \ldots, B^m)$ be a m-dimensional Brownian motion and $\mathrm{E}\left[\int_0^T H_{ij}^2 \, dt\right] < \infty$. I want to show, that $\mathrm{E}\left[\left\|\int_0^T H(t) \, dB_t\right\|^2\right] = \mathrm{E}\left[\int_0^T \|H(t)\|^2 \, dt\right]$. Therefore, let

$I_i(t) := \sum_{j=1}^{m} \int_0^t H_{ij}(s)\, dW_s^j$, for $i \in \{1,\ldots,n\}$. By fundamental facts about the *Itô integral*, one knows, that each I_i is a continuous martingale. However, I absolutely don't get, why the *variation process* of I_i is given by $\langle I_i \rangle_t := \int_0^t \sum_{j=1}^{m} H_{ij}^2(s)\, ds$ and why that implies

$$\mathrm{E}\left[\left(I_i(T) \right)^2 \right] = \mathrm{E}\left[\int_0^T \sum_{j=1}^{m} H_{ij}^2(s)\, ds \right]$$

answer

This answer takes a while. We will need several steps which I list here to guide our lines:

1)(Z simple) $I_t^M(Z)^2 - \int_0^t Z^2(s) d\langle M \rangle_s$ is a continuous local martingale

2) (X - progressively measurable) $I_t^M(X)^2 - \int_0^t X^2(s) d\langle M \rangle_s$ is a continuous local martingale

3)

$$\langle I^M(X) + I^N(Y) \rangle_t = \langle I^M(X) \rangle_t + \langle I^N(Y) \rangle_t + 2\langle I^M(X), I^N(Y) \rangle_t$$

4) $\langle I^M(X), I^N(Y) \rangle_t = \int_0^t X_u Y_u\, d\langle M, N \rangle_u$

5) note that $I_i(H) = \sum_j I^{W_j}(H_{i,j})$

6)
$$\langle I_i(H)\rangle = \langle \sum_j I^{W_j}(H_{i,j})\rangle = \sum_{j,k} \langle I^{W_j}(H_{i,j}), I^{W_j}(H_{i,k})\rangle$$
$$= \sum_j \langle I^{W_j}(H_{i,j}), I^{W_j}(H_{i,j})\rangle = \int_0^t \sum_j H_{i,j}^2(u) d\langle W_j\rangle_u$$

each of this steps is classical and can be found in (Karatazas and Shreve-Brownian motion and stochastic calculus second edition), so I will give you the pages where you may find each result

1) pgs 132 and 137, 138

2) pg 139

3) pg 32

4) pg 142

5) just use the definition of $I_t^M(X) = \int_0^t X_s dM_s$

6) pg 32

Analytic solution to stochastic differential equations

I need help to to find the analytic solution (if it exists) of the following system of SDE. Usually, I use Matlab as software but in this case I'm unable to use it in order to figure out the problem.

To begin with, (ζ_t) is assumed to revert toward zero

following an Ornstein-Uhlenbeck process; (ξ_t) is assumed to follow a Brownian motion

process

The system is as follows:

$$\begin{cases} d\zeta_t = (-\kappa\zeta_t - \varphi_\zeta - \phi_\zeta \xi_t)dt + \sigma_\zeta dz_\zeta \\ d\xi_t = (\mu_\xi - \varphi_\xi - \phi_\xi \zeta_t)dt + \sigma_\xi dz_\xi \end{cases}$$

dz_ζ and dz_ξ are correlated increments of standard Brownian

motion process with $dz_\zeta dz_\xi = \rho_{\zeta\xi} dt$.

Can someone kindly help me? Or tell me how to solve analytically this system?

Many thanks in advance

answer

Hi fortunately your system of sde is a system of linear equation under the hypothesis that :

$\phi_\zeta, \sigma_\zeta, \phi_\xi, \sigma_\xi, etc...$ are not random functions (they might depend on time) and are locally bounded.

From Karatzas and Shreve Brownian Motion and Stochastic Calculus we known(section 5.6) that there is a solution for a linear system of SDEs.

So let's be given the following system d dimensional system (written in a vectorial formulation identical to the reference) :

$$dX_t = [A(t)X_t + a(t)]dt + \sigma_t.dW_t$$

- $a(t)$ is a d dimensional vector function of time (non random measurable (in t) and locally bounded).

- $A(t)$ is a $d \times d$ matrix (non random measurable (in t) and locally bounded).

- W is a r dimensional Brownian motion.

- $\sigma(t)$ is $d \times r$ matrix (non random measurable (in t) and locally bounded).

Let's denote Φ the solution of the following ordinary matrix equation :

$\Phi(t)' = A(t)\Phi(t)$ with $\Phi(0) = Id$

Then :

$$X(t) = \Phi(t).[X_0 + \int_0^t \Phi(s)^{-1}.a(s).ds + \int_0^t \Phi(s)^{-1}.\sigma(s).dW_s]$$

Now what is left to to you is the calculation for which I have no personal interest but which seem tractable in your case.

PS: Even if the solution here is written in a framework where all component of W are independents, it's very easy to recast your framework in this setting with the proper matrix sigma (which is not diagonal).

Best regards

Does this Stochastic Differential Equation have a name?

I came across this SDE and since I am not an expert I am wondering if this SDE is known to have an closed form solution for first passage times.

The SDE is

$$dY_t = (a + be^{ct})dt + \sigma dB_t$$

How does one go about finding an explicit distribution for first passage times in this case?

Answer

Your SDE is an example of a linear SDE $dY_t = (\alpha(t) + \beta(t)Y_t)dt + (\gamma(t) + \delta(t)Y_t)dW_t$,

where α, β, γ and δ denote deterministic processes and W denotes a one-dimensional Wiener process. In your case, processes β and δ are zero processes and γ is a constant process and $\alpha(t) = a + be^{ct}$. There exists a closed form for the (strong) solution of this SDE, which can be found on Wikipedia. Since your SDE is

$$Y_t = A(t) + \sigma W_t,$$

where $A(t) = \int_0^t \alpha(s)ds$ is an exponential curve, studying first passage times of constant levels for Y is equivalent to studying first passage times of exponential curves for the Wiener process.

We want to solve:

$$dX_t = \alpha X_t dW_t + \sigma X_t dt$$

where the initial condition X_0 is given and α , σ are constant.

The solution goes as follows - We rewrite the equation as:

$$\frac{dX_t}{X_t} = \alpha dW_t + \sigma dt$$

and we can get

$$\int_0^t \frac{dX_u}{X_u} = \int_0^t \alpha dW_u + \int_0^t \sigma t = \alpha W_t + \sigma t$$

We use Itos formula and set $f(t,x) = f(x) = \ln x$. Why do we do this?

But then we get:

$$d\ln(X_t) = \frac{1}{X_t} dX_t - \frac{1}{2}\frac{1}{X_t^2}(dX_t)^2 = \frac{1}{X_t} dX_t - \frac{\alpha^2}{2} dt$$

How do get the last step i.e how is:

$$\frac{1}{2}\frac{1}{X_t^2}(dX_t)^2 = \frac{\alpha^2}{2} dt$$

By this the solution conclude that we can write

$$\ln(X_t)-\ln(X_0)=\int_0^t \frac{dX_u}{X_u}-\frac{\alpha^2 t}{2}$$

Why is this obvious?

The rest of the solution is rather trivial so i let it be out. Thanks!

answer

We use Itô's formula and set $f(t,x)=f(x)=\ln x$. Why do we do this?

Well, if we consider instead of the SDE

$$dX_t=\alpha X_t\,dW_t+\sigma X_t\,dt \quad (1)$$

the "associated" ordinary differential equation

$$dx_t=\sigma x_t\,dt$$

(i.e. we just drop the stochastic integral), then it is well-known that the substitution $y_t:=\log x_t$ leads to a very simple ODE which can be solved explicitly. So, if we want to solve the SDE (1), we simply give it a try and use the same substitution as in the deterministic case.

[....] how is $\frac{1}{2}\frac{1}{X_t^2}(dX_t)^2=\frac{\alpha^2}{2}dt$.

This follows from the heuristical (!) rules

$$(dW_t)^2=dt \qquad dt\,dW_t=0 \qquad (dt)^2=0$$

and (1) ; indeed

$$(dX_t)^2=(\alpha X_t\,dW_t+\sigma X_t\,dt)^2=\alpha^2 X_t^2\,dt.$$

By this the solution concludes that we can write

$$\ln X_t - \ln X_0 = \int_0^t \frac{dX_u}{X_u} - \frac{\alpha^2}{2} t. \quad (3)$$

We know from the above considerations that

$$d\ln(X_t) = \frac{1}{X_t} dX_t - \frac{\alpha^2}{2} dt.$$

By the very definition (!) this differential equation is to be understood as an integral equation, i.e.

$$\ln X_t - \ln X_0 = \int_0^t d\ln(X_u) = \int_0^t \frac{1}{X_u} dX_u - \int_0^t \frac{\alpha^2}{2} du.$$

how to derive the stochastic differential equation of this process

How can I derive the SDE for the vasicek model :

$$r_t = 0.1 + 0.1e^{-t} + e^{-t} \int_0^t e^s dB_s \$$$

From observation, the SDE vasicek's model is such that:

$$dr_t = b(a - r_t)dt + \sigma dB_t$$

but I do not know how to link the two.

The answer is

$$dr_t = (0.1 - r_t)dt + dB_t$$

answer

Let $X_t = \int_0^t e^s dB_s$. Then

$$r_t = f(t, X_t) = .1 + .1e^{-t} + e^{-t} X_t$$

By Ito's Lemma:

$$dr_t = \frac{\partial f}{\partial x} dX_t + (\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{dx^2} Var(X_t))dt$$

So:

$$\frac{\partial f}{\partial x} dX_t + (\frac{\partial f}{\partial t} + \frac{1}{2}\frac{\partial^2 f}{dx^2} Var(X_t))dt =$$
$$e^{-t} dX_t + (-.1e^{-t} + e^{-t} X_t)dt =$$
$$e^{-t} e^t dB_t + (-.1e^{-t} - e^{-t} X_t)dt =$$
$$dB_t + (-.1e^{-t} - e^{-t} e^t (r_t - .1 - .1e^{-t})dt =$$
$$dB_t + (.1 - r_t)dt$$

Conserved quantity for system of Stochastic Differential Equations

I'm considering the set of SDEs (in the sense of Ito)

$$\mathrm{d}x = -yx\mathrm{d}t + x^2 \mathrm{d}B_t$$
$$\mathrm{d}y = -y^2 \mathrm{d}t + xy\mathrm{d}B_t$$

where the two Brownian motions are identical (i.e there is just one noise driving the dynamics).

Now as can be seen easily,

$y \cdot \mathrm{d}x - x \cdot \mathrm{d}y = 0$ and then by Ito's formula, after defining $w = \frac{x}{y}$,

$$\mathrm{d}w = \frac{\mathrm{d}x}{y} - \frac{x\mathrm{d}y}{y^2} + \frac{1}{2} \cdot (-\frac{1}{y^2})(2\mathrm{d}x \cdot \mathrm{d}y) + \frac{1}{2}\frac{2x}{y^3}(\mathrm{d}y) = 0$$

This means that the ratio of the processes x and y is a constant.

Now the problem I actually have is modified by a factor of 2:

$$\mathrm{d}x = -\tilde{y}x\mathrm{d}t + x^2\mathrm{d}B_t$$
$$\mathrm{d}\tilde{y} = -2\tilde{y}^2\mathrm{d}t + 2x\tilde{y}\mathrm{d}B_t$$

Structurally, this new set of SDEs looks very similar to the one above, but I can't find a process like in the case above which has a similar "good" property (like the ratio being constant). Is there a trick to this I'm missing?

answer

Actually, I got it: The quantity $\frac{x}{\sqrt{y}}$ is not constant, but at least

$$\mathrm{d}(xy^{-1/2}) = \frac{1}{2}x^3y^{-1/2}\mathrm{d}t$$

which is a nice smooth quantity.

Given a process what is the stochastic differential equation it fulfils?

Given the process $X_t = (2 + t + \exp(W_t))_t$ where W_t is Brownian motion. How can I find the SDE that it fulfils.

I am actually looking for two functions σ, τ such that

$$X_t = X_0 + \int_0^t \sigma(W_s, s) dW_s + \int_0^t \tau(W_s, s) ds .$$

Is there any straight forward way to get them?

Cheers

Answer

Apply Itô's formula to $f(t,x) := 2 + t + e^x$. This gives you an expression of the form

$$X_t = X_0 + \int_0^t \sigma(s, e^{W_s}) dW_s + \int_0^t b(s, e^{W_s}) ds. \$$$

Now use that

$$\$ e^{W_s} = X_s - 2 - s.$$

Differential and Differential Equation - Difference in meaning?

I am a little confused, an exercise by a teacher has been set which says:

For $X_t = 2e^{B_t}$

Where B_t is brownian motion at time t.

a) Find the stochastic differential $d(X_t)$

b) Find the stochastic differential equation for X_t

I have completed the first exercise, but am confused as to what the difference between the two is?

for a)

$$d(2e^{B_t}) = 2e^{B_t}\,dB_t + e^{B_t}\,dt$$

Answer

The not-so-exact calculation goes as

$$d(2e^{B_t}) =$$
$$2e^{B_{t+dt}} - 2e^{B_t} =$$
$$2e^{B_t}(e^{dB_t} - 1) =$$
$$2e^{B_t}\left(1 + dB_t + \tfrac{1}{2}(dB_t)^2 + O((dB_t)^3) - 1\right)$$

and using the rules $dB_t^2 = dt$ and to ignore all higher powers results in

$d(2e^{B_t}) = 2e^{B_t}(dB_t + \frac{1}{2}dt) = e^{B_t}dt + 2e^{B_t}dB_t$ as you also computed, probably using the Ito formula.

Using X_t to get rid of the exponentials gives the SDE

$dX_t = X_t(\frac{1}{2}dt + dB_t)$ [17]

How can I obtain from a differential equation a stochastic version?

Suppose $\frac{dx}{dt} = ax + b$ and then assume that $a = c + g$ where g is a Wiener process.

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[17] The differential is a linear mapping that approximates a differentiable function in a given nieghborhood of a point. In this question, you're being asked to construct this linear mapping.

A differential equation,on the other hand, is a equation of a given function in terms of the derivative or derivatives of a given order of a given function of one or several variables with specified boundary conditions. The antiderivative-if it exists-is the function which is the solution of the equation. So a differential equation is a much more specific construction.

Ansewer

You should distinguish white noise and Brownian motion. Let ξ_t be $(0,1)$ white noise, then if

$dy_t = \xi_t dt$ then $y_t \equiv$const (from the Law of Large Numbers).

In fact it's better to obtain ODE from SDE, vice versa can be done in the following way: you write perturbed equation

$dX_t = (aX_t + b)dt + \sigma(t, X_t)dB_t$ where B_t is a Brownian motion (= Wiener process). If you put $\sigma \equiv 0$ then

$dx_t = (ax_t + b)dt$ Also, for arbitrary σ it's true for the expected value of the perturbed system $m_t = \mathrm{E}[x_t]$ we have

$dm_t = (am_t + b)dt$. You can choose any σ you like - for example constant, linear in X_t, linear in time and so on. I suggest to put σ independent of time to leave the system autonomous.

P.S. You can read in Oksendal that for the white noise ξ_t it's not interesting to consider process with increments $\xi_t dt$ because of its triviality. In fact it's more "correct" to use agreement $dB_t = \xi_t \sqrt{dt}$.

Expectation and Variance of stochastic equation

My questions is related to this question:

I.e how do you calculate the variance and expectation of $U_t = e^{-\gamma t}U_0 + \int_0^t e^{\gamma(s-t)}\sigma dX_s$?

Answer

If X is a Brownian motion then, as the function inside the stochastic integral is bounded,

the stochastic integral is centered and, if U_0 is deterministic:

$$E(U_t) = e^{-\gamma t}U_0$$

Then, using the Ito isometry:[۱۸]

[18] In mathematics, the **Itô isometry**, named after Kiyoshi Itô, is a crucial fact about Itô stochastic integrals. One of its main applications is to enable the computation of variances for stochastic processes

Let $W:[\circ,T]\times\Omega\to\mathbb{R}$ denote the canonical real-valued Wiener process defined up to time $T>\circ$, and let $X:[\circ,T]\times\Omega\to\mathbb{R}$ be a stochastic process that is adapted to the natural filtration $_*^W$ of the Wiener process.

Then $\left[\left(\int_\circ^T X_t\,dW_t\right)^۲\right] = \left[\int_\circ^T X_t^۲\,dt\right]$, where denotes expectation with respect to classical Wiener measure γ. In other words, the Itô stochastic integral, as a function, is an isometry of normed vector spaces with respect to the norms induced by the inner products

$$E\left[U_t - E(U_t)\right]^2 = E\left(\int_0^t e^{\gamma(s-t)}\sigma dX_s\right)^2 = \int_0^t e^{2\gamma(s-t)}\sigma^2 ds$$

System of Stochastic Diff Eq

How can I solve the system of stochastic differential equation

$$dX_1 = X_2 dt + a dW_1$$

$$dX_2 = -X_1 dt + b dW_1$$

Answer

Let

$X_1(t) = A(t)\sin(t) + B(t)\cos(t)\backslash\backslash X_2(t) = A(t)\cos(t) - B(t)\sin(t)$ the n thanks to the Ito fornula:

$$\begin{aligned}(X,Y)_{L^2(W)} &:= \left(\int_0^T X_t\, dW_t \int_0^T Y_t\, dW_t\right)\\ &= \int_\Omega\left(\int_0^T X_t\, dW_t \int_0^T Y_t\, dW_t\right)d\gamma(\omega)\end{aligned}$$

And

$$(A,B)_{L^2(\Omega)} := (AB) = \int_\Omega A(\omega)B(\omega)d\gamma(\omega).$$

Øksendal, Bernt K. (2003). Stochastic Differential Equations: An Introduction with Applications. Springer, Berlin. ISBN 3-540-04758-1.

$dX_1(t) =$

$dA(t)\sin(t) + A(t)\cos(t)dt + dB(t)\cos(t) - B(t)\sin(t)dt =$

$X_2(t)dt + dA(t)\sin(t) + dB(t)\cos(t)dX_2(t) =$ then you

$dA(t)\cos(t) - A(t)\sin(t)dt - dB(t)\sin(t) - B(t)\cos(t)dt =$

$-X_1(t) + dA(t)\cos(t) - dB(t)\sin(t)$

are left with

$$
\begin{aligned}
adW_1(t) &= dA(t)\sin(t) + dB(t)\cos(t) \\
bdW_1(t) &= dA(t)\cos(t) - dB(t)\sin(t) \\
\Rightarrow dA(t) &= \left[a\sin(t) + b\cos(t)\right]dW_1(t) \\
dB(t) &= \left[a\cos(t) - b\sin(t)\right]dW_1(t) \\
A(t) &= A_0 + \int_0^t \left[a\sin(s) + b\cos(s)\right]dW_1(s) \\
B(t) &= B_0 + \int_0^t \left[a\cos(s) - b\sin(s)\right]dW_1(s)
\end{aligned}
$$

How to solve $\mathrm{d}X(t) = B(t)X(t)\mathrm{d}t + B(t)X(t)\mathrm{d}B(t)$ with condition $X(0) = 1$?

aI want to solve the

stochastic differential equation $\mathrm{d}X(t) = B(t)X(t)\mathrm{d}t + B(t)X(t)\mathrm{d}B(t)$

with condition $X(0) = 1$.

answer

Solution I Dividing the SDE (formally) by X_t yields

$$\frac{dX_t}{X_t} = B(t)dt + B(t)dB_t.$$

This equation suggests using an approach similar to the *separation-by-parts*-approach for ordinary differential equations: We set $Z_t := \log X_t$ and apply Itô's formula

$$Z_t - Z_0 = \int_0^t \frac{1}{X_s} dX_s - \frac{1}{2}\int_0^t \frac{1}{X_s^2} d\langle X\rangle_s \quad (1)$$

where

$$dX_s = B(s)X(s)ds + B(s)X(s)dB(s)$$
$$d\langle X\rangle_s = B(s)^2 X(s)^2 ds \qquad (2)$$

Combining (1) and (2), we obtain

$$Z_t - Z_0 = \int_0^t B(s)dB_s + \int_0^t \left(B(s) - \frac{1}{2}B_s^2 \right) ds.$$

By definition, $Z_t = \log X_t$, and therefore we find an explicit expression for X_t. Itô's formula shows that $(X_t)_t$ defines indeed a solution of the given SDE.

Solution II Consider the corresponding ordinary differential equation

$$dx(t) = B(t)x(t)dt \qquad x(0) = 1.$$

Its unique solution is given by

$$x(t) = \exp\left(\int_0^t B(s)ds \right)$$

or, equivalently,

$$x_t \cdot \exp\left(-\int_0^t B(s)ds\right) = 1$$

This indicates that we may try the following Ansatz:

$$Y_t := X_t \cdot \exp\left(-\int_0^t B(s)ds\right).$$

Applying Itô's formula to $f(x,y) := x \cdot e^{-y}$ and the (two-dimensional) Itô process $(X_t, \int_0^t B(s)ds)$ gives

$$Y_t - Y_0 = \int_0^t \exp\left(-\int_0^s B(r)dr\right)dX_s - \int_0^t Y_s B(s)ds = \int_0^t B(s)Y_s\, dB_s$$ Th

is SDE can be solved easily and therefore we obtain an expression for X_t. This approach looks more complicated and lengthy, but actually it works out in a much more general setting.

Solving StochasticDifferential Equation $dY_t = aY_t dt + bdX_t \qquad Y(0) = c$

Please help me in solving this Stochastic Differential Equation for Y_t

$$dY_t = aY_t dt + bdX_t \qquad Y(0) = c$$

where a and b are constants.

Also find the $[Y_t]$ and $\mathrm{var}[Y_t]$.

I have tried to write the Ito Lemma for general Y_t function in the two variables t and X_t

and then tried to compare the coeficients term of dt and dX_t but not getting any proper result.

Thanks,

Answer

Use the integrating-factor approach, where $\mu_t = e^{-at}$.

Rewrite the original SDE as:

$$d\left(\mu_t Y_t\right) = \mu_t b\, dX_t.$$

Y_t can be obtained as

$$Y_t = \frac{b}{\mu_t}\int_0^t \mu_s\, dX_s + \frac{\mu_0 Y_0}{\mu_t} = b e^{at}\int_0^t e^{-as}\, dX_s + c e^{at}.$$

Using Ito's formula, write down a stochastic diferential equation satiesfied by $Y_t := X_t^2$, given both Y_t and X_t

I am trying to solve this exercise and I am stuck in the third part of it. I checked the solution and it makes no sense to me, so I would really

appreciate it if someone could explain to me how Ito's formula is applies in this case (maybe I don't know some rules). Thanks a lot! :)

So We have the following stochastic differential equation

$$dX_t = \alpha X_t dt + 2dW_t, \quad X_0 = 0,$$

where W_t is a standard Brownian motion and $\alpha \in \mathbb{R}$ is a constant.

From this I have obtained, using $Z_t = e^{-\alpha t} X_t$ that

$$\$ X_t = e^{\alpha t} X_0 + 2e^{\alpha t} \int_0^t e^{-\alpha s} dW_s$$

$$**Question: what rules does my professor use to give the solution below?**$$

dY_t=d(X_t^2)=2X_tdX_t+2\frac{1}{2}2^2dt=2\alpha X_t^2 dt+4X_tdW_t+4dt$

I understand the last step - it is simple substitution, but how does he get the previous one?

Answer1

Ito's formula:

$$Y = X^2$$

1.) find dY using the chain rule to order $(dX)^2$ with a taylor expansion and substitute for dX

$$dY = \frac{\partial Y}{\partial t} dt + \frac{\partial Y}{\partial X} dX + \frac{1}{2} \frac{\partial^2 Y}{\partial X^2} (dX)^2$$

$$= 2X(a(t)Xdt + 2dW) + \frac{1}{2} * 2(a(t)Xdt + 2dW)^2 =$$

$$2X(a(t)Xdt + 2dW) + (a(t)Xdt + 2dW)^2$$

2.) use Ito's "box" rule to compute $(dX)^2$ and collect like terms

$$(dW)^2 = dt$$
$$dt.dW = 0$$
$$(dt)^2 = 0$$

$dY = 2X^2 a(t)dt + 4XdW + 4dt$ as desired

Answer2

Ito's formula says that given

$$dX_t = \mu_t dt + \sigma_t dW_t$$

and a C^2-function $f : \mathbf{R}^2 \to \mathbf{R}$, we have for $Y_t = f(t, X_t)$ that

$$dY_t = \left(\frac{\partial f}{\partial t}(t, X_t) + \mu_t \frac{\partial f}{\partial x}(t, X_t) + \frac{\sigma_t^2}{2} \frac{\partial^2 f}{\partial x^2}(t, X_t) \right) dt + \sigma_t \frac{\partial f}{\partial x} dW_t$$

In your case, $\mu_t = aX_t$, $\sigma_t = 2$, $f(t,x) = x^2$, hence $\partial_t f(t,x) = 0$, $\partial_x (t,x) = 2x$, $\partial_x^2 f(t,x) = 2$, giving

$$dY_t = \left(aX_t \cdot 2X_t + \frac{2^2}{2} \cdot 2 \right) dt + 2 \cdot 2X_t dW_t = 2aX_t^2 dt + 4dt + 4X_t dW_t$$

Almost sure convergence of stochastic process

Suppose that we have a (almost surely) continuous stochastic process

$\{X_t\}_{t\geq 0}$ on $[0,1] \, with non-stochastic initial value$ X_{0} = x_{0} \in [0,1]$ and exponentially decreasing expectation

$E(X_t) = x_0 e^{-\gamma t}$, where $\gamma > 0$.

For the corresponding discrete-time process $\{X_n\}_{n=0}^{\infty}$, an application of the Markov inequality and the

Borel–Cantelli lemma shows that $\lim_{n\to\infty} X_n = 0$ almost surely. Is the same true for the continuous-time process

$\{X_t\}$, i.e. do we have $\lim_{t\to\infty} X_t = 0$

almost surely?

Originally, the process $\{X_t\}$ is the (almost surely) unique,

continuous, and Markovian solution of the Ito stochastic differential

equation

\begin{equation*}

dX_{t} = -\gamma X_{t} dt + k \sqrt{\gamma} \sqrt{X_{t}^{3}(1-X_{t})}dW_{t}, \ X_{0} = x_{0} \in [0,1]

\end{equation*}

where $k > 0$ and W_t is a Brownian motion. Does this ensure the almost sure convergence towards 0 ?

answer

We want to prove that if $\{X_t\}_{t\geq 0}$ is a continuous, supermartingale with asymptotically vanishing expectation

$\lim_{t\to\infty} E(X_t)=0$, then $\lim_{t\to\infty} X_t=0$ almost surely. by Doob's first martingale theorem, $\{X_t\}_{t\geq 0}$ has a limit $X_\infty \in [0,\infty[$ almost surely. In particular, we have almost sure

convergence in discrete time, i.e. $\lim_{n\to\infty} X_n = X_\infty$ almost surely, and thus, by Fatou's lemma applied to the non-negative sequence $\{X_n\}$, we get

$$E(X_\infty)=E(\lim_{n\to\infty} X_n)=$$
$$E(\liminf_{n\to\infty} X_n)\leq \liminf_{n\to\infty} E(X_n)$$
$$=\lim_{n\to\infty} E(X_n)=0$$

But by the Markov inequality, $P(X_\infty \geq \ )\leq \frac{1}{\ } E(X_\infty)=0$ for all

>0, and $X_\infty=0$ almost surely.

To see how to apply the above result to the Ito solution $\{X_t\}$, we need to show that it is a supermartingale. However, put $Y_t=e^{\gamma t}X_t$ and notice that $dY_t=k\sqrt{\gamma}\sqrt{e^{-3\gamma t}Y_t^3(1-e^{-\gamma t}Y_t)}dW_t$ by

Ito's lemma, and $\{Y_t\}$ is a non-negative local martingale and

thus a supermartingale. Consequently, $\{X_t\}$ is a

supermartingale.

However, I wonder if some of the assumptions might be weakened or removed,

in particular the supermartingale assumption.

Clarification on Stochastic Exponential

Consider a d-dimensional Brownian motion $B=(B_1,\dots,B_d)$ whose components are independent and

let A be a $d\times d$ squared matrix such that $\sum_{i=1}^{d} A_{ii}^2 = 1$. Define the

$W = AB$ the Brownian motion whose correlation matrix is $\rho = AA^{\dagger}$, where $\dagger$ indicate the transpose.

Consider a d-dimensional process $\lambda_t \in {}^{2}_{loc}$ and a process η_t that solves the stochastic differential equation

$$d\eta_t = -\eta_t \lambda_t \cdot dW_t, \quad \eta_0 = 1,$$

where $\cdot$ indicates Euclidean product. So η_t is a one-dimensional stochastic process. In the [book by A. Pascucci [١٩] it is written that the (unique) solution is

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Authors: Pascucci, Andrea

$\eta_t = \exp\left(-\int_0^t \lambda_s \cdot dW_s - \frac{1}{2}\int_0^t \langle \rho\lambda_s, \lambda_s \rangle ds\right)$.I do not understand how exactly the co-variation $\langle \rho\lambda_s, \lambda_s \rangle$ is defined. It should, in principle, returns a scalar value since η_t is scalar.

At some point in the book the co-variation between two $\mathbb{R}^d$ -valued processes X and Y is defined as

$$\langle X, Y \rangle_t = \left(\left\langle X^i, Y^j \right\rangle\right)_{ij=1,\ldots,d}$$

so it seems that $\langle \rho\lambda_s, \lambda_s \rangle$ is a matrix. Does somebody have some clue?

answer

This is only a matter of notation, by noting $cov_s = \langle \rho\lambda_s, \lambda_s \rangle$ this can be re-ritten written in matrix notation with the d-dimensional vector λ_s and the $d \times d$ matrix ρ in the following way which might seem more amenable to you :

$$cov_s = \lambda_s^t . \rho . \lambda_s$$

which is a scalar.

Best regards

Use stochastic calculus (Ito's lemma) to compute the expectation

Calculate $E[\cos(X)e^X]$, where $X \sim N(0,\sigma^2)$. Use stochastic calculus instead of integrating w.r.t the normal density.

During the discussion with friends, we believe that we should use Brownian motion B_t to represent X and then somehow use Ito's lemma to establish a differential equation. However, we don't understand Ito's lemma well. So we could not do it. Thank you!

answer

Ok, so your idea was right - you should consider

$\mathrm{E}\left[\cos B_t e^{B_t}\right]$ at $t=\sigma^2$ since $B_t \sim \quad (0,t)$. What is Ito lemma about?

Given a function $f \in C^2$ you know that

$$f(B_t)-f(B_0)=\frac{1}{2}\int_0^t f''(B_s)\mathrm{d}s+\int_0^t f'(B_s)\mathrm{d}B_s,$$

so applying expectation to both sides you obtain

$$\mathrm{E}[f(B_t)]-f(0)=\frac{1}{2}\int_0^t \mathrm{E}[f''(B_s)]\mathrm{d}s \quad (\ )$$

which is a simple application of Dynkin's formula. It holds since $\mathrm{E}\int_0^t f'(B_s)\mathrm{d}B_s=0$ since the process under the expectation sign is Ito integral which is always a martingale starting from zero.

Let us focus now on $f(x)=\cos x\cdot e^{x}$, so $f(0)=1$ and $f''(x)=\sin x\cdot e^{x}$. If you denote $m(t)=\mathsf{E}f(B_t)$ then from ( ) we have

$m(t)=1-\int_0^t \mathsf{E}[\sin B_s\cdot e^{B_s}]ds$. Denote $n(s)=E[\sin B_s\cdot e^{B_s}]$, then $m(0)=1$ and

$m'(t)=-n(t)$.

Applying Ito formula to the function $\sin xe^{x}$ we obtain another equation: $n(0)=0$ and

$n'(t)=m(t)$.

We can solve it by substitution: $m''=-n'=-m$, so $m''+m=0$ (do you know how to solve it?). The solution is $m(t)=\alpha\sin t+\beta\cos t$. Based on the initial condition, we find: $\beta=1$ and $\alpha=0$ so

$m(t)=\cos t$ and

$\mathsf{E}[\cos Xe^{X}]=\cos\sigma^{2}$.

I think it's also worth to say that here we have a textbook example in which just two steps were sufficient to calculate the expectation. If the function f is arbitrary then you have to solve a PDE

$m_t=\frac{1}{2}m_{xx}$ with $m(0,x)=f(x)$, see [20]. However, the only way to give a solution to this PDE can be

[20] In statistical mechanics, the **Fokker–Planck equation** is a partial differential equation that describes the time evolution of the probability density function of the velocity of a particle under the influence of drag forces and random forces, as

$m(t,x)=\int_{\mathbb{R}} f(y)\frac{1}{\sqrt{2\pi t}}e^{-y^2/2t^2}dy$ which is exactly just a usual formula for the expectation of the function of a Gaussian random variable.

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in Brownian motion. The equation can be generalized to other observables as well. It is named after Adriaan Fokker and Max Planck[and is also known as the **Kolmogorov forward equation** (diffusion), named after Andrey Kolmogorov, who first introduced it in a 1931 paper. When applied to particle position distributions, it is better known as the **Smoluchowski equation** (after Marian Smoluchowski). The case with zero diffusion is known in statistical mechanics as the Liouville equation.

The first consistent microscopic derivation of the Fokker–Planck equation in the single scheme of classical and quantum mechanics was performed by Nikolay Bogoliubov and Nikolay Krylov.The Smoluchowski equation is the Fokker–Planck equation for the probability density function of the particle positions of Brownian particles.

In one spatial dimension *x*, for an Itō process driven by the standard Wiener process W_t and described by the stochastic differential equation (SDE) $dX_t=\mu(X_t,t)dt+\sigma(X_t,t)dW_t$

with drift $\mu(X_t,t)$ and diffusion coefficient $D(X_t,t)=\sigma^{۲}(X_t,t)/۲$, the Fokker–Planck equation for the probability density $p(x,t)$ of the random variable X_t is $\frac{\partial}{\partial t}p(x,t)=-\frac{\partial}{\partial x}[\mu(x,t)p(x,t)]+\frac{\partial^{۲}}{\partial x^{۲}}[D(x,t)p(x,t)]$.

Brownian bridge

Let $W=(W_t;F_t)$, $t\le 0$ be a standard Wiener process, and let $(X_t)_{0\le t\le 1}$ satisfy the stochastic differential equation

$$dX_t=-\frac{X_t}{1-t}dt+dW_t,\quad 0\le t\le 1,\quad X_0=0$$

Find $(X_t)_{0\le t\le 1}$ and verify that

$$L^2_\lim_{t\to 1}X_t=0$$

X is called *Brownian Bridge*.

answer

I will give you an answer on the general brownian bridge case.

Consider the SDE

$$dX_t=\frac{b-X_t}{1-t}dt+dW_t,\ X_0=a$$

for $t\in[0,1]$ with $a,b\in\mathbb{R}$.

An approach to solve this SDE can be obtained by the constant variation method. Indeed, consider the following ODE

$$x'(t)=\frac{b-x(t)}{(1-t)}+f(t),\ x(0)=a$$

for $t\in[0,1]$. We can easily obtain by constant variation method the solution of this equation which is given by

$$x(t) = a(1-t) + bt + (1-t)\int_0^t \frac{f(s)}{1-s} ds$$

At this point, we would like to simply apply this result with $f(t) = dW_t / ds$, which is formally not possible. But let allow us to consider this formula anyway and then prove that

$$X_t = a(1-t) + bt + (1-t)\int_0^t \frac{1}{1-s} dW_s \qquad (1)$$

is a solution to our SDE. By effect, a stochastic integration by parts give us that

$$X_t = a(1-t) + bt + W_t - (1-t)\int_0^t \frac{W_s}{(1-s)^2} ds.$$

Then, if we consider $Y_t = X_t - W_t$, we have

$$dY_t = \frac{b - Y_t}{1-t} dt - \frac{W_t}{1-t} dt$$

so we can easily conclude the wanted result.

Now, knowing that the brownian bridge have the form given by (1) we are able to calculate it's L^2-limit. Note that X defined by (1) is a gaussian process with mean $a(1-t)+bt$. Using Ito's isometry, we can straight forward calculate

$$\{X_t^2\} =$$

$$[a(1-t)+bt]^2 + (1-t)^2 \int_0^t \frac{1}{(1-s)^2} ds =$$

$$[a(1-t)+bt]^2 + t(1-t) \underset{t \to 1}{\to} b^2$$

which show us that $L^2 - \lim_{t \to 1} X_t = b$ (with $b = 0$ in your case).

Actually this limit holds even $-$ almost ever .

Why do people simulate with Brownian motion instead of "Intuitive Brownian Motion"?

I have just recently begun studying Brownian motion and stochastic calculus at the level of an undergraduate or beginning graduate student of applied mathematics. (Textbooks I've looked at are by Mikosch, Gardiner, Kloeden and Platen.)

Back when I first started thinking about stochastic dynamics, I was a cognitive science student who was simulating neural decision making circuits. We had a differential equation to model our system, and my advisor suggested that I try "adding in some noise."

Intuitively, I would have written a discretized equation to model the evolution of some state variable X like this:

$$\Delta X = [a(t, X) + R]\Delta t$$

where a is a coefficient that depends upon the state variable x and time t, and where R denotes a random variable (probably taken to be normal and centered at zero with variance σ^2)

In other words, at some level, I would have expected an "intuitive stochastic differential equation" to be expressed in the general form

$$\Delta X = [a(t, X, R)]\Delta t$$

where a is now any function of a random variable.

However, crucially for the theory of stochastic differential equations and the models I see (at least the ones that come up in modeling neural decisions -- e.g. Ornstein-Uhlenbeck), the discretized equation has the stochastic part scaled with $\sqrt{\Delta t}$, rather than just Δt. For instance, Brownian motion is discretized as

$$\Delta X = R\sqrt{\Delta t}$$

where R denotes a standard normal.

Finally my question: *Why is it so desirable, for modeling stochastic dynamics generally speaking, to have the stochastic part of a stochastic difference equation scale with the **square root of the change in time** rather than simply linearly with the change in time?*

answer

The fundamental reason is that variances are additive, while standard deviations are not. If you didn't make the variance proportional to the time step, your discretization would not behave consistently when you changed the time step.

To be explicit, let's take $X_{i+1} = X_i + R_{i+1}\sqrt{\Delta t}$, where the R_i are independent and standard normal. After two steps, we have

$X_2 = X_0 + (R_1 + R_2)\sqrt{\Delta t}$. But $R_1 + R_2$ is normal with variance 2, so we can write it as $\sqrt{2}R_{12}$, where R_{12} is again standard normal. Then we get $X_2 = X_0 + \sqrt{2}R_{12}\sqrt{\Delta t} = X_0 + R_{12}\sqrt{2\Delta t}$, so it's exactly the same as if we took a single time step of twice the length. This doesn't happen if you take the stochastic component to be $R_i \Delta t$ instead (try it).

What's really going on is that you're adding a constant variance, say σ^2, to the stochastic part at each time step. So after n time steps you will have accumulated a variance of $n\sigma^2$. If you were to take a single time step of length $n\Delta t$ instead, you'd want your stochastic part to have the same variance $n\sigma^2$, and that corresponds to scaling by $\sqrt{n}$, not by n.

In other words: $\text{variance} \propto \text{time step}$; $\text{scale} \propto \sqrt{\text{variance}}$; so $\text{scale} \propto \sqrt{\text{time step}}$.

On the confinement to [0;1] of the solution of $dX_t = (1 - X_t)X_t dB_t$

One considers the stochastic differential equation $dX_t = (1 - X_t)X_t dB_t$, with B Brownian motion, and one assumes that $0 \le X_0 \le 1$.

One wants to show that $\mathbf{P}(X_t \in [0;1]) = 1$, and that, if $X_s = 0$ then $X_t = 0$ for every $t \ge s$.

Does anybody know how to prove this? Some help would be appreciated

Answer

Assume that X_0 is in $(0,1)$, let $T = \inf\{t>0 \mid X_t = 0 \text{ or } X_t = 1\}$, thus X_t is in $(0,1)$ for every $t<T$. Consider, for every $t<T$,

$Y_t = \log\left(\frac{X_t}{1-X_t}\right)$. Then T is also $T = \inf\{t>0 \mid Y_t = \pm\infty\}$ and Itô's formula shows that, for every $t<T$,

$Y_t = Y_0 + W_t + \frac{1}{2}\int_0^t Z_s \mathrm{d}s, \qquad Z_t = \frac{\mathrm{e}^{Y_t} - 1}{\mathrm{e}^{Y_t} + 1}$. One sees that $|Z_t| < 1$ for every $t<T$, in particular,

$$|Y_t - Y_0 - W_t| \quad \frac{1}{2}t,$$

for every $t<T$. This proves that $|Y_t|$ is finite for every t and that $T = +\infty$ almost surely, that is, X_t is in $(0,1)$ almost surely for every t.

The same technique applies to the strong solution of $\mathrm{d}X_t = C(X_t)\mathrm{d}W_t$,

starting from X_0 in (a,b), for every function $C:[a,b] \to \mathbb{R}$ positive on (a,b), zero at a and at b, and with bounded derivative on $[a,b]$. Then (X_t) stays in (a,b) forever, almost surely.

To show this, one considers the function D defined on (a,b), for some c in (a,b), by

$D(x) = \int_c^x \frac{\mathrm{d}z}{C(z)}$. Then

$D(X_t) = D(X_0) + W_t - \frac{1}{2}\int_0^t C'(X_s)\mathrm{d}s$, thus,

$$|D(X_t)-D(X_0)-W_t| \quad \frac{1}{2}t\|C'\|_\infty,$$

for every t, and, since $\lim_{x\to b^-} D(x)=+\infty$ and $\lim_{x\to a^+} D(x)=-\infty$, the same reasoning applies.

Prove directly from the definition of the Ito's integral $\int_0^t B_s^2 dB_s = \frac{1}{3}B_t^3 - \int_0^t B_s ds$

I am trying to solve the exercises from the book Stochastic differential equations -An Introduction with applications by Bernt Oksendal and I am stuck on 1 question.

Prove directly from the definition of Ito's integral that

$$\int_0^t B_s^2 dB_s = \frac{1}{3}B_t^3 - \int_0^t B_s ds$$

I have tried for a couple of hours but I simply cannot prove it . I would be grateful if somebody could help me out , even a hint would be very helpful

Answer

Note that

$$B_T^3 = \sum_{j=1}^{n} B_{s_j}^3 - B_{s_{j-1}}^3 =$$

$$\sum_{j=1}^{n}(B_{s_{j-1}}+(B_{s_j}-B_{s_{j-1}}))^3-B_{s_{j-1}}^3=$$

$$\underbrace{\sum_{j=1}^{n}(B_{s_j}-B_{s_{j-1}})^3}_{=:I_1}+3\underbrace{\sum_{j=1}^{n}\cdot B_{s_{j-1}}\cdot(B_{s_j}-B_{s_{j-1}})^2}_{=:I_2}$$

$$+3\underbrace{\sum_{j=1}^{n}B_{s_{j-1}}^2\cdot(B_{s_j}-B_{s_{j-1}})}_{=:I_3}\qquad(1)$$

We consider the terms separately and start with I_3:

For

$$g^{\Pi}(s,w):=\sum_{j=1}^{n}\underbrace{B_{s_{j-1}}^2}_{\in L^2}\cdot 1_{[s_{j-1},s_j)}(s)$$

We have

$$\left(\int_0^T|g^{\Pi}(s)-B_s^2|^2ds\right)^{\text{Fub}}=$$

$$\sum_{j=1}^{n}\int_{s_{j-1}}^{s_j}(|B_{s_{j-1}}^2-B_s^2|)ds\le$$

$$\sup_{|s-t|\le|\Pi|}|B_s^2-B_t^2|\sum_{j=1}^{n}\int_{s_{j-1}}^{s_j}1\,ds\to 0\qquad(|\Pi|\to 0)$$

where $|\Pi|:=\max(s_j-s_{j-1})$ denotes the mesh size of the partition $\Pi=\{0=s_0<\ldots<s_n=T\}$. In the last step, we have used that

$[0,T]$ $t \mapsto B_t$ is uniformly continuous (almost surely). From the definition of the Itô integral, it follows that

$$g^{\Pi} \bullet B_T = \sum_{j=1}^{n} B_{s_{j-1}}^2 \cdot (B_{s_j} - B_{s_{j-1}}) \xrightarrow{L^2} \int_0^T B_s^2 \, dB_s$$

Hence, $I_3 \overset{|\Pi| \to 0}{\to} \int_0^T B_s^2 \, dB_s \quad (2)$

Next we consider I_2. To this end, let $X_j := B_{s_{j-1}} \cdot ((B_{s_j} - B_{s_{j-1}})^2 - (s_j - s_{j-1})$.

$$\left\| \left(\sum_{j=1}^{n} X_j \right)^2 \right\|_{L^2(\ )}^2 \overset{*}{=} \sum_{j=1}^{n} (X_j^2) =$$

$$\sum_{j=1}^{n} ((B_{s_{j-1}}^2 \cdot ((B_{s_j} - B_{s_{j-1}})^2 - (s_j - s_{j-1})))^2) =$$

$$\sum_{j=1}^{n} \underbrace{(B_{s_{j-1}}^2)}_{s_{j-1}} \cdot \underbrace{(((B_{s_j} - B_{s_{j-1}})^2 - (s_j - s_{j-1}))^2)}_{\overset{L1.6}{=} (s_j - s_{j-1})^2} =$$

$$\sum_{j=1}^{n} s_{j-1} \cdot (s_j - s_{j-1})^2 \leq |\Pi| \cdot \underbrace{\sum_{j=1}^{n} s_{j-1} \cdot (s_j - s_{j-1})}_{\leq T^2} \to 0 \quad (|\Pi| \to 0)$$

For $(*)$ note that

$$(X_j \cdot X_k) = \quad (\quad (X_j \cdot X_k \mid \quad _{k-1})) =$$

$$(X_j \cdot B_{s_{k-1}} \cdot \underbrace{((B_{s_j} - B_{s_{j-1}})^2) - (s_j - s_{j-1}) \mid \quad _{k-1}))}_{= \ ((B_{s_j} - B_{s_{j-1}})^2 - (s_j - s_{j-1}) = 0} = 0$$

for all $j > k$ where we have used that $B_{s_j} - B_{s_{j-1}}$ is independent from $_{k-1} := \quad _{s_{k-1}}$. Consequently, there exists a subsequence converging almost surely to 0. Since

$$\sum_{j=1}^{n} B_{s_{j-1}} \cdot (s_j - s_{j-1}) \to \int_0^T B_s \, ds \quad \text{f.s.} \quad (|\Pi| \to 0)$$

we conclude $I_2 \to \int_0^T B_s \, ds \quad \text{f.s.} \quad (|\Pi| \to 0) \qquad (3)$

It remains to determine $\lim\limits_{|\Pi| \to 0} I_1$. To this end, we note that

$$\left(\left| \sum_{j=1}^{n} (B_{s_j} - B_{s_{j-1}})^3 \right| \right) \leq$$

$$\sum_{j=1}^{n} \quad (|B_{s_j} - B_{s_{j-1}}| \cdot |(B_{s_j} - B_{s_{j-1}})^2|) \overset{\text{CSI}}{\leq}$$

$$\sum_{j=1}^{n} \underbrace{\| B_{s_j} - B_{s_{j-1}} \|_{L^2(\ )}}_{\sqrt{s_j - s_{j-1}}} \cdot \underbrace{\| (B_{s_j} - B_{s_{j-1}})^2 \|_{L^2(\ )}}_{\sqrt{3(s_j - s_{j-1})^2}} =$$

$$\sqrt{3} \cdot \sum_{j=1}^{n} (s_j - s_{j-1})^{\frac{3}{2}} \leq |\Pi|^{\frac{1}{2}} \cdot \underbrace{\sum_{j=1}^{n} (s_j - s_{j-1})}_{T} \to 0 \quad (|\Pi| \to 0)$$

$I_1(\Pi) \xrightarrow{L^1} 0$ as $|\Pi| \to 0$. Consequently, there exists a subsequence converging to 0 almost surely.

Adding all up, we get

$$B_T^3 \overset{(1)}{=}$$

$$\lim_{|\Pi|\to 0}\Big(\sum_{j=1}^{n}(B_{s_j}-B_{s_{j-1}})^3+3\underbrace{\sum_{j=1}^{n}B_{s_j}\cdot(B_{s_j}-B_{s_{j-1}})^2}_{\overset{(3)}{\to}\int_0^T B_s\,ds}+$$

$$3\cdot\underbrace{\sum_{j=1}^{n}B_{s_{j-1}}^2\cdot(B_{s_j}-B_{s_{j-1}})}_{\overset{(2)}{\to}\int_0^T B_s^2\,dB_s}\Big)=$$

$$0+3\cdot\int_0^T B_s\,ds+3\cdot\int_0^T B_s^2\,dB_s$$

ri

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21 $I_n=\int_0^t B_s^n\,dB_s \Rightarrow I_n=\frac{B_t^{n+1}}{n+1}-\frac{n}{2}\int_0^t B_s^{n-1}\,ds$

A solution's variance and mean

I have a stochastic differential equation of this type :

$$dX(t) = adB(t) + kct^{c-1}\cos(\theta(t))dP(t).$$ with

a, k : constants

$B(t)$: Brownian process

$P(t)$: A Poisson process with parameter λ I tried to calculate the

$\theta(t)$: A uniform distribution on $[0, 2\pi]$

c : a constant $(0 < c < 2)$

variance and the mean of the solution using the Ito general formula but it was too complicated. How can I find these information please ?

Thanks for your help

answer

In other words, assuming that $X_0 = 0$, one asks that

$$X_t = aB_t + k\sum_{n=1}^{+\infty} T_n^c \cdot \cos(\theta_n)\cdot[T_n \quad t],$$ where the sequence $(T_n)_{n \quad 1}$

enumerates the points of a Poisson process with intensity λ and counting process $(N_t)_{t \quad 0}$, and where I guess that one should also assume that $(\theta_n)_{n \quad 1}$ is i.i.d. with the prescribed distribution and that the processes $(B_t)_{t \quad 0}$, $(\theta_n)_{n \quad 1}$ and $(T_n)_{n \quad 1}$ are independent.

(Note that I replaced your ct^{c-1}, which I believe is a mistake, by t^c. If I am wrong, one can adapt the following to ct^{c-1}.)

Since every B_t is centered and every $\cos(\theta_n)$ is centered and independent on T_n,

$$\mathrm{E}(X_t) = 0.$$

Since B_t is independent of everything else and centered, since $\mathrm{E}(B_t^2) = t$ and $\mathrm{E}(\cos(\theta_n)^2) = \frac{1}{2}$, and since the sequence $(\cos(\theta_n))_n$ is independent of everything else and centered,

$$\mathrm{E}(X_t^2) = at + \tfrac{1}{2}k^2 \sum_{n=1}^{+\infty} \mathrm{E}(T_n^{2c} : T_n \quad t).$$ Conditionally on $N_t = k$, the set $\{T_n \mid n \quad k\}$ is distributed like an i.i.d. sample of size n uniformly distributed on $(0,t)$. Let U_t denote a random variable uniformly distributed on $(0,t)$. Hence,

$$\mathrm{E}\left(\sum_{n=1}^{N_t} T_n^{2c} \middle| N_t = k \right) = k \cdot \mathrm{E}(U_t^{2c}) = \frac{kt^{2c}}{2c+1}.$$ Since $\mathrm{E}(N_t) = \lambda t$, this yields

$$\mathrm{E}(X_t^2) = at + \tfrac{1}{2}k^2 \lambda \frac{t^{2c+1}}{2c+1}.$$

If $dX_t = X_t\,dt + dB_t$, why does $e^{-t}dX_t = e^{-t}X_t\,dt + e^{-t}dB_t$?

I'm taking a course in stochastic differential equations, and in order to solve $dX_t = X_t\,dt + dB_t$, the book gives a hint: to multiply both sides of this equation by e^{-t}. (But, as explained below, "multiplication" doesn't really mean usual multiplication because this is an integral equation.)

But the original equation $dX_t = X_t\,dt + dB_t$ really means:

$$X_t - X_0 = \int_0^t X_s\,ds + \int_0^t dB_s.$$

If X_t satisfies the above equation, then according to the hint the following equation also holds.

$$e^{-t}X_t - e^{-t}X_0 = \int_0^t e^{-s}X_s\,ds + \int_0^t e^{-s}\,dB_s$$

But why?

answer

What the book it hinting at is the knowledge that if

$X_t = X_0 + \int_0^t X_s\,ds + B_t$, then X is a continuous semimartingale and

$\int_0^t e^{-s}\,dX_s = \int_0^t e^{-s}X_s\,ds + \int_0^t e^{-s}\,dB_s$. The differential form of an SDE is more than just a way to save ink (or electrons); it can (as here) serve as a guide to deducing one integral equation from another. Finally, the "product rule" for the stochastic integral gives us

$\int_0^t e^{-s}\,dX_s - \int_0^t e^{-s} X_s\,ds = e^{-t}X_t - X_0$, from which we deduce that $e^{-t}X_t = X_0 + \int_0^t e^{-s}\,dB_s$.

Applying the Multivariate Ito Formula

I want to show that the stochastic process

$$S_t^i = S_0^i \exp\left(\int_0^t \left(\mu_s^i - \frac{1}{2}\sum_{j=1}^m (\sigma_s)^{ij}\right)^2 ds + \sum_{j=1}^m \sigma_t^{ij} S_t^i\,dW_t^j\right)$$ solves the stochastic differential equation

$$dS_t^i = \mu_t^i S_t^i\,dt + \sum_{j=1}^m \sigma_t^{ij} S_t^i\,dW_t^j.$$

I just want to plug in this expression and thereby verifying that it solves the SDE, just as in the non-stochastic case. For this I first recognize that the SDE is a shorthand for

$S_t^i = S_0^i + \int_0^t \mu_s^i S_s^i\,ds + \int_0^t \sum_{j=1}^m \sigma_s^{ij} S_s^i\,dW_s^j$. Now first evaluating the integral

$\int_0^t \sum_{j=1}^m \sigma_s^{ij} S_s^i\,dW_s^j$ Here I have no idea how to start, I know I must apply Ito's formula in some clever way, but when I look it up it is always written with some mysterios $dX_i \cdot dX_j$ for the second order terms, and remarks

like "these are evaluated according to the rules $dB_i dB_j = \delta_{ij} dt, dtdB_i = dB_i dt = 0$" which I cannot make sense of, so I have no idea how to apply Ito's formula here to solve the stochastic integral? Any help?

answer

To keep notation simple, we write S_t instead of S_t^i , σ_t^j instead of σ_t^{ij} and so on. That's okay, because i is a fixed number throughout this calculation.

Suppose $(X_t)_{t\geq 0}$ is an Itô process of the form

$$dX_t = b(t)dt + \eta(t)dW_t$$

where $\eta = (\eta_1, \ldots, \eta_m)$ and $(W_t)_{t\geq 0}$ is an m-dimensional Brownian motion. Then Itô's formula states that

$$f(X_t) - f(X_0) = \int_0^t f'(X_s)dX_s + \frac{1}{2}\int_0^t f''(X_s)d\langle X\rangle_s$$

for $f \in C^2$ where

$$\int_0^t f'(X_s)dX_s :=$$

$$\int_0^t f'(X_s)\eta_s dW_s + \int_0^t f'(X_s)b(s)dsd\langle X\rangle_s :=$$

$$\sum_{j=1}^m \eta_j^2(s)ds$$

In your setting, we have

$$b(t)=\left(\mu_t-\frac{1}{2}\sum_{j=1}^{m}\sigma_t^j\right)^2 \qquad \eta(t):=(\sigma_t^1,\ldots,\sigma_t^m).$$

Applying Itô's formula for $f(x):=\exp(x)$ shows that $(S_t)_{t\geq 0}$ is a solution to the given SDE.

Quadratic Variation of Diffusion Process and Geometric Brownian Motion

I'm looking to find out the stochastic differential equation satisfied by the quadratic variation of

1. Geometric Brownian Motion,

2. Diffusion Process.

For example, for a diffusion process that satisfies $dX_t=\mu(t,X_t)dt+\sigma(t,X_t)dW_t$,

I've heard that $d\langle X\rangle_t=\sigma^2 dt$.

But I've tried looking everywhere for a proof or verification of this and not found anything. I attempted it myself but no luck so far; I vaguely think Ito's lemma may be needed.

I already know that the QV of standard Brownian motion is simply the time t .

I'd appreciate if anyone could show me how the equation for QV of Diffusion and GBM comes about, or point out a reference to me that deals with this.

Thanks in advance!

Answer

Assume that X solves

$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t$. In integral form, this means

$X_t = X_0 + \int_0^t \mu(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s$. Now, the quadratic variation of a stochastic integral process $H \cdot Y$, where $(H \cdot Y)_t = \int_0^t H_s dY_s$, is

$[H \cdot Y]_t = \int_0^t H_s^2 d[Y]_s$. You can find this in a special case as Proposition 3.2.17 of Karatzas and Shreve's "Brownian Motion and Stochastic Calculus", or in a general version for continuous semimartingales in Section IV.31 of Rogers and William's "Diffusions, Markov processes and Martingales". Using this, we obtain (with $\mu_t = \mu(t, X_t)$ and $\sigma_t = \sigma(t, X_t)$ as convenient shorthands, and $A_t = t$)

$$[X]_t = [\mu \cdot A]_t + [\sigma \cdot W]_t =$$
$$\int_0^t \mu_s^2 d[A]_s + \int_0^t \sigma_s^2 d[W]_s$$
$$= \int_0^t \sigma(s, X_s)^2 ds,$$

where we have used $[A]_t = 0$, since all continuous processes of finite variation have zero quadratic variation, and $[W]_t = t$. As for the geometric Brownian motion, we have that X is a GBM if it satisfies

$dX_t = \mu X_t dt + \sigma X_t dW_t$, which yields

$[X]_t = \int_0^t \sigma^2 X_s^2 ds$. In particular,

In particular, [X] isn't immediately seen to satisfy any SDE or ODE (as [X] does not figure in the right-hand side of the above). The above is simply a representation of [X] in terms of X, but I doubt that there in general exists and SDE or ODE satisfied by [X].

Numerically solve SDE

I am not really into solving stochastic differential equations, but I was trying to numerically solve an OED given by:

$$\frac{dy}{dt} = f(t, y, p) + N(0, \sigma^2)$$

where normal noise with 0 mean and given variance is added and p is a set of estimated parameters.

I have been reading sime stuff about it and I am solving it using a Euler-Maruyama approach:

$$Y_{n+1} = Y_n + f(t, y, p) \cdot \Delta t + \sigma \cdot \sqrt{\Delta t} \cdot N(0,1)$$

with a given time step. Is this a correct approach?

Kind regards.

answer

Yes, it is a correct scheme for the simulation of the SDE

$\mathrm{d}y_t = f(t, y_t, p)\mathrm{d}t + \sigma \mathrm{d}W_t$ where informally

$\sigma . dW_t \sim \quad (0, \sigma^2 \mathrm{d}t)$

Quadratic covariation of Itô processes

I haven't found any similar question in the forum, so I trust some of you will find this thought-provoking (at the very least). Perhaps you can help me.

Let's consider first the two following stochastic differential equations:

$$dX_t = (6+3X_t)dt + (2+X_t)dB_t, X_0 = 1$$

$$dY_t = 3Y_t dt + Y_t dB_t, Y_0 = 1$$

Of course B_t is a standard Brownian Motion started at zero.

Now the question is: how can I compute the quadratic covariation of these two processes, and then obtain its expectation? That is, how can I compute $E\left(< X, Y >_t\right)$.

Regards and thanks in advance.

answer

Define the quadratic variation of a semimartingale $(X_t)_{t\geq 0}$ by

$$[X,X]_t := -\lim_{n\to\infty} \sum_{j=0}^{n} (X_{t_j} - X_{t_{j-1}})^2$$

where $\Pi_n := \{t_0 < \ldots < t_n < t\}$ is a sequence of partitions such that $|\Pi_n| \to 0$. Moreover, we set $[X,Y]_t := \frac{1}{4}([X+Y]_t - [X-Y]_t)$.

Using this definition it is not difficult to show the following result.

Lemma 1 Let $(M_t)_{t\geq 0}$ a continuous square-integrable martingale and $(A_t)_{t\geq 0}$ a continuous process of bounded variation. Then,

1. $[M]_t$ is the unique previsible process such that $M_t^2 - [M]_t$ is a martingale.

2. $[M,A]_t = [A,A]_t = 0$

Now we are ready to calculate the quadratic covariation. By definition,

$$X_t \pm Y_t = \underbrace{\int_0^t (2 + X_s \pm Y_s)dB_s}_{=:M_t} + \underbrace{c_\pm + \int_0^t (6 + 3X_s \pm 3Y_s)ds}_{=:A_t}.$$

where $c_+ = 2$, $c_- = 0$. Obviously, $(M_t)_{t\geq 0}$ is a (continuous) martingale and $(A_t)_{t\geq 0}$ of bounded variation. Consequently, we obtain by applying Lemma 1

$$[X \pm Y, X \pm Y]_t =$$
$$[M,M]_t + 2[M,A]_t + [A,A]_t =$$
$$\int_0^t (2 + X_s \pm Y_s)^2 ds.$$

Hence,

$$[X,Y]_t = \frac{1}{4}([X+Y]_t - [X-Y]_t) = \int_0^t (2 + X_s)\cdot Y_s\, ds \qquad (1)$$

In order to compute $[X,Y]_t$, we have to find Y_t and $(X_t \cdot Y_t)$. Since stochastic integrals with respect to a Brownian motion are martingales, we find

$$f(t) := (Y_t) = 1 + \underbrace{\left(\int_0^t Y_s \, dB_s\right)}_{0} + 3\int_0^t (Y_s) ds$$

i.e. f satisfies the ODE

$$f'(t) = 3f(t) \qquad f(0) = 1$$

Obviously, the unique solution is given by $Y_t = f(t) = e^{3t}$ (2)

Similarly, we obtain from Itô's formula that

$$g(t) := (X_t \cdot Y_t) =$$

$$1 + \left(\int_0^t (7X_s Y_s + 8Y_s) ds\right) =$$

$$8\int_0^t f(s) ds + 7\int_0^t g(s) ds$$

i.e. $g'(t) = 8f(t) + 7g(t) = 8e^{3t} + 7g(t)$ (3)

This ODE can be solved explicitely; I leave it to you. Combining (1), (2) and (3) allows us to compute $[X,Y]_t$

About an application of Itô's lemma

I need some help with this exercise.

Given the following stochastical differential equation:

$$dX(t) = \frac{-1}{4}(X(t))^3\, dt + \frac{1}{2}(X(t))^2\, dW(t)$$

$$X(0) = \frac{1}{2}$$

I have to obtain $E[X(t)]$ and $E[(X(t))^3]$ using Itô's lemma.

I applied Itô's lemma for $f(t,x) = x$ and $g(t,x) = x^3$, and taking expectations of the resulting equations, I obtained the following pair of differential equations, but they don't seem to get me to the result:

- $E'[X(t)] = \frac{-1}{4}E[(X(t))^3], \quad E[X(0)] = \frac{1}{2}$

and

- $E'[(X(t))^3] = \frac{-3}{4}E[(X(t))^5] + \frac{3}{2}E[(X(t))^3], \quad E[(X(0))^3] = \frac{1}{8}$ An

y ideas?

Thanks a lot for any help!

answer

There is a mistake in the second equation: Instead of

$$'(X_t^3) = -\frac{3}{4} \quad (X_t^5) + \frac{3}{2} \quad (X_t^3)$$

it should read

$$'(X_t^3) = 0$$

Indeed: Applying Itô's formula for $g(x) := x^3$ yields

$$X_t^3 - X_0^3 = 3\int_0^t X_s^2 dX_s + \frac{6}{2}\int_0^t X_s d\langle X\rangle_s \quad (1)$$

By definition, we have

$$dX_s = -\frac{1}{4}X_s^3 ds + \frac{1}{2}X_s^2 dW_s \qquad d\langle X\rangle_s = \frac{1}{4}X_s^4 ds \quad (3)$$

Combining (1) and (2) we find

$$X_t^3 - X_0^3 = \underbrace{\frac{3}{2}\int_0^t X_s^4 dW_s}_{=:M_t} + \underbrace{\int_0^t \left(-\frac{3}{4}X_s^5 + \frac{6}{8}X_s^5\right) ds}_{=0}.$$ As $(M_t)_{t\geq 0}$ is a martingale, hence satisfying $M_t = 0$, we conclude

$$(X_t^3) = \quad (X_0^3) = \frac{1}{8}.$$

Expectation of geometric brownian motion

I was deriving the solution to the stochastic differential equation $dX_t = \mu X_t dt + \sigma X_t dB_t$

where B_t is a brownian motion. After finding $X_t = x_0 \exp((\mu - \frac{\sigma^2}{2})t + \mu B_t)$ I wanted to calculate the expectation of X_t. However I think I'm not quite getting it. I thought that I'd just calculate

$E(x_0 \exp((\mu - \frac{\sigma^2}{2})t + \mu B_t) = x_0 \exp((\mu - \frac{\sigma^2}{2})t) E(\exp(\mu B_t))$ but the book I'm using gives as answer

$E(X_t) = x_0 \exp(\mu t)$. I found this quite surprising as I don't quite see how σ could just disappear. After reading [Wikipedia](http://en.wikipedia.org/wiki/Geometric_Brownian_motion#Properties_of_GBM) I see that the result could be either $E(X_t) = x_0 \exp((\mu + \frac{\sigma^2}{2})t)$ or $E(X_t) = x_0 \exp(\mu t)$, depending on whether you use the Itô interpretation or the Stratanovich interpretation.

Since the book I use only considers the Itô formulation of stochastic integration I am interested in the latter result. But how do I obtain this? Do I just fail in calculating $E(\exp(\mu B_t))$? Thanks in advance.

Answer1

The answer is that $E(X_t) = x_0 e^{\mu t}$. The easiest way to see it is to start from the SDE and to note that $\mathrm{d}E(X_t) = \mu E(X_t)\mathrm{d}t, \quad E(X_0) = x_0$. Hence $a(t) = E(X_t)$ solves $a'(t) = \mu a(t)$ and $a(0) = x_0$, that is, $a(t) = x_0 e^{\mu t}$ as claimed above.

Your solution goes astray when you solve the SDE, the factor of B_t is wrong and, in fact,

$X_t = x_0 e^{(\mu - \sigma^2/2)t + \sigma B_t}$. Hence

$E(X_t) = x_0 e^{(\mu - \sigma^2/2)t} E(e^{\sigma B_t})$.

Since $E(e^{uZ}) = e^{u^2/2}$ for every real number u and every standard normal random variable Z, the identity $E(e^{\sigma B_t}) = e^{\sigma^2 t/2}$ follows from the fact that σB_t is distributed like $\sigma\sqrt{t}Z$. Simplifying, one gets the same expression of $E(X_t)$ than by the direct route, namely, $E(X_t) = E(X_0)e^{\mu t}$.

Answer2

An easy way to calculate an expected value of SDE's solutions is to find an equation on $m(t) = \mathsf{E}[X_t]$. More precisely, from

$dX_t = \mu X_t dt + \sigma X_t dB_t$ taking expectations of both sides (while using that $\mathsf{E}[dB_t] = 0$) we obtain

$dm(t) = \mu m(t)dt$, which gives a desired answer. Indeed, from the latter equation follows that $m(t) = m_0 e^{\mu t}$. @Did has already described such method, but it may be useful for you to apply such method for other SDEs.

You should be aware of taking the expectation in SDE which is quite informal. On the other hand, you can equivalently write

$$X_t = X_0 + \int_0^t \mu X_s ds + \int_0^t \sigma X_s dB_s.$$

Here you can take an expectation since on both sides there are just random variables (at each fixed moment of time). The latter integral has zero expectation since it is Ito integral.

$E[e_t e_s \Delta B_t \Delta B_s]$ for ΔB_t Brownian motion increments and $e_t(\omega)$ a measurable function.

Let $\Delta B_j = B_{t_{j+1}} - B_{t_j}$ where B_t is Brownian motion, and $e_i(\omega)$ measurable with respect to $\sigma(B_{t_i})$. In Oksendal's 'Stochastic Differential Equations' he states:

$$E[e_i e_j \Delta B_i \Delta B_j] = \begin{cases} 0 & i \neq j \\ E[e_j^2] & i = j \end{cases}$$

Justifying this because '$e_i e_j \Delta B_i$ and ΔB_j are independent if $i < j$'. I am trying to understand this.

I understand that Brownian motion is normally distributed with independent increments, so:

$$E[\Delta B_i \Delta B_j] = 0$$

However, how can we justify $e_i e_j \Delta B_i$ being independent to ΔB_j? Even if they are, how can we treat $E[e_i e_j \Delta B_i]$? Basically I believe my confusion stems from how to treat products of measurable functions.

Answer1

For every i, let $F_i = \sigma(B_s, s \quad t_i)$ then each e_i is F_i-measurable and each ΔB_i is centered, independent of F_i, and F_{i+1}-measurable.

Now, fix $i < j$. Then e_i, e_j and ΔB_i are all F_j-measurable and ΔB_j is independent of F_j hence $e_i e_j \Delta B_i$ and ΔB_j are independent, a fact

which implies $E(e_i e_j \Delta B_i \Delta B_j) = E(e_i e_j \Delta B_i) \cdot E(\Delta B_j) = E(e_i e_j \Delta B_i) \cdot 0 = 0.$

Thus, the argument does not require to know the value of $E(e_i e_j \Delta B_i)$.

Likewise, for every i, e_i is F_i-measurable and ΔB_i is independent of F_i hence $E((e_i \Delta B_i)^2) = E(e_i^2) \cdot E((\Delta B_i)^2) = E(e_i^2) \cdot (t_{i+1} - t_i).$

Answer2

You can see it using the "tower property" for instance,

Assume $i < j$, then

$$E[e_i e_j \Delta B_i \Delta B_j] = E\left[E[e_i e_j \Delta B_i \Delta B_j \mid \ _{t_j}]\right].$$

Now observe that since $i < j$ then $_{t_i} \subset \ _{t_j}$ and by hypothesis e_i is $_{t_i}$-measurable and hence also $_{t_j}$-measurable, also $\Delta B_i = B_{t_{i+1}} - B_{t_i}$ is $_{t_j}$-measurable. Hence

$$E[e_i e_j \Delta B_i \Delta B_j] = E\left[e_i e_j \Delta B_i E[\Delta B_j \mid \ _{t_j}]\right].$$

Finally observe that ΔB_j is independent of $_{t_j}$ so $E[\Delta B_j \mid \ _{t_j}] = E[\Delta B_j] = 0$ and hence

$$E[e_i e_j \Delta B_i \Delta B_j] = 0.$$

This was maybe very detailed but useful to understand why.

Growth condition for Ito diffusions

Given a one-dimensional SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dB_t$$

$$X_0 = Z$$

for $t \in [0,T]$ where Z is square integrable: $\mathsf{E}[Z^2] < \infty$ the sufficient conditions for existence and uniqueness are: *the growth condition*

$|b(t,x)| + |\sigma(t,x)| \le C(1+|x|) \quad \forall x \in \mathbb{R}, t \in [0,T]$ and *the Lipschitz condition*

$|b(t,x'') - b(t,x')| + |\sigma(t,x'') - \sigma(t,x')| \le D|x'' - x'| \quad \forall x', x'' \in \mathbb{R}.$

These conditions are stated e.g. in Theorem 5.2.1, "Stochastic Differential Equations" (p. 68)

However, in the definition of an Ito diffusion as a process satisfying

$dX_t = b(X_t)dt + \sigma(X_t)dB_t$ for $t \ge s$, where $X_s = x$, in the same book (p. 114 on the linked webpage) it is written that conditions simplify to Lipschitz condition. Could you help me to understand the reasoning of that?

Answer

Let's consider a simplest case: Let $f:\mathbb{R}\to\mathbb{R}$ satisfy global Lipschitz condition

$$|f(x)-f(y)|\le L|x-y|,\quad x,y\in\mathbb{R}$$

with Lipschitz constant L. Then by triangle inequality, we have

$|f(x)|\le|f(x)-f(0)|+|f(0)|\le L|x|+|f(0)|\le C(|x|+1)$ for large $C\ge\max(L,|f(0)|)$. Same argument applies to this case.

Solution for SDE: $dF_t=\beta_t\left(F_t-\alpha\right)dW_t$

I am trying to derive the solution for the following stochastic differential equation, but I must be doing something wrong in my calculations because I can't arrive to the correct solution.

The SDE in question is the following:

$$dF_t=\beta_t\left(F_t-\alpha\right)dW_t$$

where W is a standard Brownian Motion started at 0, α is a constant and $\beta(t)$ is a deterministic process. I know from the literature that the solution for this SDE for $0<t<T$ can be explicitly written as

$$F_T=\alpha+(F_t-\alpha)e^{-\frac{1}{2}\int_t^T\beta_u^2du+\int_t^T\beta_u dW_u}$$

Surely I am not applying Ito's formula where I should be?

Bonus question: What is the impact of applying a shift to the distribution generated by the above equation? When computing probabilities for

different values, how big will be the differences in the case where $\alpha = 0$ and $\alpha \neq 0$? Intuitively I would like to think that the results would be the same regardless of the shift applied.

Answer

I will first make a few simplifications. Define $G_t := F_t - \alpha$. Then of course $dG_t = dF_t$. So then we are solving for $dG_t = \beta_t G_t dW_t$. Since $t \mapsto \beta_t$ is deterministic, this is still geometric Brownian motion. If $t \mapsto \beta_t$ is Lipschitz, which you would assume to guarantee uniqueness, then G_t has the Markov property. So then, we can look at the solution

$$G_t = G_0 e^{-\frac{1}{2}\int_0^t \beta_s^2 ds + \int_0^t \beta_s dW_s}$$

instead of arbitrary $0 \leq t \leq T$.

Define the function $f_t = \int_0^t \beta_s^2 ds$ and the process $X_t = \int_0^t \beta_s dW_s$.

Hence, $G_t = G_0 e^{-\frac{1}{2} f_t + X_t}$. Note that $dX_t = \beta_t dW_t$ and $d[X,X]_t = \beta_t^2 dt$. Since you wanted a full analytical solution, I will be a bit pedantic and treat G_0 as another stochastic process (constant over time but still random), i.e. $Y_t = G_0$ for all $t \geq 0$. Then of course $dY_t = 0$ and $d[Y,Y]_t = 0$. I hope it is also clear to you that $d[X,Y]_t = 0$ since all the paths of Y are constant.

Now consider the function $r : (t,x,y) \mapsto y e^{-\frac{1}{2} f_t + x}$. You can see that $G_t = r(t, X_t, Y_t)$. This function is infinitely differentiable in all its

arguments so we have more than what we need to apply Ito's lemma. I will use subscripts to denote partial derivatives of r below.

$$dG_t = r_t(t, X_t, Y_t)dt + r_x(t, X_t, Y_t)dX_t + r_y(t, X_t, Y_t)dY_t +$$
$$\frac{1}{2} r_{xx}(t, X_t, Y_t)d[X, X]_t +$$
$$\frac{1}{2} r_{yy}(t, X_t, Y_t)d[Y, Y]_t + r_{xy}(t, X_t, Y_t)d[X, Y]_t$$

The last two terms and the dY term drop due to the considerations above.

$r_t = -\frac{1}{2}\frac{df}{dt} r = -\frac{1}{2}\beta_t^2 r$, $r_x = r$ and $r_{xx} = r$. Putting all these together,

$$dG_t = -\frac{1}{2}\beta_t^2 G_t dt + G_t \beta_t dW_t + \frac{1}{2} G_t \beta_t^2 dt = G_t \beta_t dW_t$$

So then the given solution satisfies the SDE. For the other question we need to look at

$$P\{F_T \le x\} = P\{\alpha + (F_0 - \alpha)e^{-\frac{1}{2}\int_0^t \beta_u^2 du + \int_0^t \beta_u dW_u} \le x\}$$

$$P\{F_T \le x\} = P\{e^{-\frac{1}{2}\int_0^t \beta_u^2 du + \int_0^t \beta_u dW_u} \le \frac{x - \alpha}{F_0 - \alpha}\}$$

$$P\{F_T \le x\} = P\{-\frac{1}{2}\int_0^t \beta_u^2 du + \int_0^t \beta_u dW_u \le \log\left(\frac{x - \alpha}{F_0 - \alpha}\right)\}$$

$$P\{F_T \le x\} = P\{\int_0^t \beta_u dW_u \le \log\left(\frac{x - \alpha}{F_0 - \alpha}\right) + \frac{1}{2}\int_0^t \beta_u^2 du\}$$

Note that $\int_0^t \beta_u \, dW_u \sim \mathrm{N}(0, \int_0^t \beta_u^2 \, du)$. Then we can write

$$P\{F_T \le x\} = \Phi\left(\frac{\log\left(\frac{x-\alpha}{F_0-\alpha}\right) + \frac{1}{2}\int_0^t \beta_u^2 \, du}{\sqrt{\int_0^t \beta_u^2 \, du}} \right)$$

Here Φ is the distribution function of a standard normal random variable. The sensitivity of this expression with respect to α you can judge for yourself by looking at the derivatives or using numerical methods.

One last remark: In the second part I conveniently assumed that F_0 is a constant. F_0 can be a random variable itself. Note that for F_0 to be measurable with respect to $_0$ it would have to be independent of W. Then, all you need to specify is the marginal distribution of F_0 to evaluate the probability $P\{F_T \le x\}$ (rather than the joint distribution of $\int_0^t \beta_u \, dW$ and F_0).

Variance of the Cox Ingersoll Ross model

Consider the Cox-Ingersoll-Ross (CIR) interest rate model: $dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}\,dW_t$ where κ, θ, σ are positive constants and W_t is a standard Brownian motion.

A solution to the CIR stochastic differential equation is given by:

$$r_t = \theta + (r_0 - \theta)e^{-\kappa t} + \sigma e^{-\kappa t}\int_0^t e^{\kappa u}\sqrt{r_u}\,dW_u.$$

Since $dW_u \sim N(0, du)$ it is easy to derive that:

$$\mathrm{E}[r_t] = \theta + (r_0 - \theta)e^{-\kappa t}.$$

However, I am stuck on deriving the variance. I got as far as below. Anyone know how to solve for the variance?

Thanks.

My derivation so far:

$$\mathrm{Var}[r_t] = \mathrm{E}\left[\{r_t - \mathrm{E}(r_t)\}^2\right] = \sigma^2 e^{-2\kappa t}\mathrm{E}\left[\left\{\int_0^t e^{\kappa u}\sqrt{r_u}\,dW_u\right\}^2\right].$$

What to do with the square of that integral? The solution I found states:

$$\sigma^2 e^{-2\kappa t}\mathrm{E}\left[\left\{\int_0^t e^{\kappa u}\sqrt{r_u}\,dW_u\right\}^2\right] = \sigma^2 e^{-2\kappa t}\int_0^t e^{2\kappa u}\mathrm{E}(r_u)\,du$$

and continues onwards from here. But I don't get how you can take the square inside the integral.

Answer

The last equality

$$\sigma^2 e^{-2\kappa t} E\left[\left\{\int_0^t e^{\kappa u}\sqrt{r_u}\,dW_u\right\}^2\right] = \sigma^2 e^{-2\kappa t}\int_0^t e^{2\kappa u} E(r_u)du$$ follows by Ito isometry.

Laplace transform stopping time

Consider a stochastic differential equation:

$$\frac{dX}{dt} = b + \sigma\frac{dW}{dt}, X(0) = x$$

where b, σ are constant, $x \in [0,1]$, and W is a Wiener process. Let $\tau = \inf\{t \geq 0 : X(t) \in \{0,1\}\}$ and $u_c(x) = \ _x(e^{c\tau})$. u_c is clearly finite for any $c \leq 0$, but may be infinite for sufficiently large $c > 0$. The title arises because u_c, as a function of c, is the Laplace transform of the density of τ (if τ has one).

Let $(Lf)(x) = \frac{\sigma^2}{2}f''(x) + bf'(x)$ be the generator of this process. The Feynman-Kac formula implies that we have the following equation for u_c:

$$(Lu_c)(x) + cu_c(x) = 0 \text{ if } x \in (0,1) \backslash\backslash u_c(x) = 1 \text{ if } x \in \{0,1\}.$$

In this simple case, the solution to this problem can be calculated explicitly. We calculate the roots of

$$\frac{\sigma^2}{2}\lambda^2 + b\lambda + c = 0.$$

Then we have our solution $u_c(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$ with

$$\begin{bmatrix} 1 & 1 \\ e^{\lambda_1} & e^{\lambda_2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

This equation has a solution if and only if $e^{\lambda_1} \neq e^{\lambda_2}$ or λ_1 and λ_2 are both integer multiples of $2\pi i$. (We can ignore this latter case by assuming $b \neq 0$.)

My problem is that this only fails at discrete values of c. Yet it should fail for all c larger than some critical c^*, by a straightforward probabilistic argument. So I think I've made an error, but I can't seem to find it. Any thoughts on where I made a mistake?

answer

The set of values of c such that $e^{\lambda_1} = e^{\lambda_2}$ is $\{\gamma_n \mid n \quad 0\}$ where $\gamma_n = (b^2 / 2\sigma^2) + n^2\pi^2(\sigma^2 / 2)$.

When $c = \gamma_0$, $\lambda_1 = \lambda_2$ hence $u_c(x) = (ax+1)e^{\lambda x}$ for some constant a, and the limit conditions yield $u_c(x) = xe^{-\lambda(1-x)} + (1-x)e^{\lambda x}$ for every x in [0,1].

For every c between γ_0 and γ_1, the roots λ_1 and λ_2 have a purely imaginary part and u_c is well defined.

When $c \to \gamma_1^-$, $u_c(x) \to +\infty$ for every x in $(0,1)$ and actually, $E_x(e^{c\tau}) = +\infty$ for every x in $(0,1)$ and every c γ_1.

The solutions u_c when c γ_1 are artefacts with no "physical" reality, actually (if I remember correctly) none of them takes positive values on the whole interval.

Bounding an expected hitting time

Consider a stochastic differential equation:

$$dX_t = dW_t + \sin(X_t)dt, X_0 = x$$

where W_t is a Wiener process. Define

$$\tau_1 = \inf\{t : X_t \in 2\pi\mathbb{Z}\}$$
$$\tau_2 = \inf\left\{t : X_t \in 2\pi\mathbb{Z} \quad \left\{X_{\tau_1}\right\}\right\}.$$

My goal is to show that τ_1 and τ_2 both have finite expectation; this will allow me to solve a larger problem about recurrence using very standard arguments.

Since the drift is 2π periodic in space and homogeneous in time while the diffusion is homogeneous in both space and time, it is enough to consider $x \in (0, 2\pi)$ for the first part and $X_{\tau_1} = 0$ for the second part. In the first case we will be done provided the problem

$$\frac{1}{2}u'' + \sin(x)u' = -1, u(0) = u(2\pi) = 0$$

has a nonnegative solution. In the second case we will be done provided the problem

$$\frac{1}{2}u'' + \sin(x)u' = -1, u(-2\pi) = u(2\pi) = 0$$

has a nonnegative solution. These equations may be solved explicitly by routine ODE techniques, though a closed form solution does not exist. For instance the solution to the first problem may be written as

$$u(x) = -2w(x) + 2\frac{w(2\pi)v(x)}{v(2\pi)}$$

where

$$v(x) = \int_0^x e^{2\cos(y)}dy$$

$$w(x) = \int_0^x \int_0^y e^{2(\cos(y)-\cos(z))}dzdy.$$

If I can prove $u \geq 0$ then I will be done. I'm fairly sure that any proof that this $u \geq 0$ will work on the second one. Any suggestions?

Edit: a quick numerical check shows that $u \geq 0$ should indeed hold.

answer

A simple approach (which avoids the task of checking the nonnegativity of the function in your question) is to compare the hitting times τ_1 and τ_2 with the hitting time θ of $\pm 2\pi$ by the process $\$ dY = \mathrm{d}W - \mathrm{sgn}(Y)\mathrm{d}t, \$$ starting from some $|y| \quad 2\pi$.

Since $|\sin| \quad 1$, τ_1 and τ_2 are dominated stochastically by θ, and θ starting from any $|y| \quad 2\pi$ is stochastically dominated by θ starting from

0. Finally, $s(y) = E_y(\theta)$ solves $\frac{1}{2}s'' - s' = -1$ with $s(\pm 2\pi) = 0$ and $s'(0) = 0$ hence a standard computation shows that $\$8s(x) = e^{4\pi} - e^{2|x|} + 2(|x| - 2\pi)$,

in particular, for every x, $E_x(\tau_1)$ $E_x(\tau_2)$ $E_0(\theta) = \frac{1}{8}(e^{4\pi} - 1 - 4\pi)$,

which is finite.

SDE - removal of the diffusion coefficients

I'm currently looking at stochastic differential equations with irregular coefficients such as $W^{1,p}_{\text{loc}}$. If I have

$dX_t = b(X_t)dt + \sigma dW_t$, where $b \in W^{1,1}_{\text{loc}}$ and σ is a constant, I can write this SDE as a ODE for every Brownian path w by defining $Y_t = X_t - w_t$ and a new vector field $b^W = b(Y,t) = b(Y_t + w_t)$, so the ODE is

$dY = b^W(t,Y)dt$ with initial condition $Y_0 = y$. Since b^W has Sobolev regularity I can then apply DiPerna-Lions theory (1989), which guarantees the existence and uniqueness of the flow of the ODE.

My question is now what happens if σ is not a constant,

$dX_t = b(X_t)dt + \sigma(X_t)dW_t$. Apparently the above argument is NOT correct in general. Is there any ways that the diffusion coefficients $\sigma(X_t)$

can be absorbed into the Brownian motion? or maybe there are some conditions on σ under which the above argument still hold?

Many thanks!!

answer

To answer very partially you can under some (quite stringent) conditions use Lamperti's Transform to transfer the volatility term $\sigma(X_t)$ into the drift term.

The other way around Girsanov's transform allows you to make a change of probability measure that offsets the drift terms still under some conditions and leaves the diffusion term unchanged.

Otherwise there are some other transforms that could be usefull, timechange is one of them but you have to realise that it comes with the cost of a change in the filtration. This transform could allow you to go from $\sigma(X_t)dW_t$ to $dW'_{\phi(t)}$.

I remember a paper about Lie group theory applied to SDE in the same way that it is applied to the ODE case. I'll give you the reference what was nice about it was that (but only in the 1-dim case) it covered almost all transforms that already knew about.

Two last streams of research that could interest you are "quasi sure analysis" and "Rough Path Theory"

Best regards

Edit :

Here is the Title of the paper I mentionned :

Kozlov - The Group Classification of a Scalar Stochastic Differential Equation

Eigenfunctions of an operator using Laguerre Polynomials

I am trying to find the eigenfunctions of the following operator:

$$f = (-\gamma x + \frac{\mu}{x}) f_x + \mu f_{xx}$$

I know that I must somehow use Laguerre polynomials, the solutions to the following ODE. $L_n(x)$ solves

$$xy'' + (1-x)y' + ny = 0.$$

Here is the current ODE I have which should be solved

$$\mu\phi_{xx} + (-\gamma x + \frac{\mu}{x})\phi_x - \lambda\phi = 0$$

That operator is the generator of the following stochastic differential equation

$$dA_t = (-\gamma x + \frac{\mu}{x})dt + \sqrt{2\mu}dW_t$$

where W_t is Brownian motion. I am about 90% sure this is true, but this may be something I'm doing wrong.

In an example in the notes for the section in which I find this problem, the Professor finds the eigenfunctions of

$$f = -\alpha x f_x + \frac{1}{2}\sigma^2 f_{xx}$$

by solving the ODE

$$\phi'' - \frac{2\alpha x}{\sigma^2}\phi' + \frac{2\lambda}{\sigma^2}\phi = 0$$

via substitution with some y to get

$$\phi'' - y\phi' + \frac{\lambda}{\alpha}\phi = 0$$

which is then solved with Hermite polynomials. The reason I include this is that I believe some clever substitution may be necessary in my problem as well, but I cannot figure it out! I greatly appreciate any help.

EDIT: Based on the question, I am almost certain that the eigenvalues are $\lambda = -2\gamma n$ for positive integers n. Maybe this can help.

EDIT2: For clarification, the generator of a process $dX_t = b(X_t,t)dt + \sigma(X_t,t)dW_t$ is found thus:

$f = b(x,t)\cdot\nabla f(x) + \frac{1}{2}a(x,t):\nabla^2 f(x)$ where $a(x,t) = \sigma\sigma^T$. This is how I found the generator, which I'm 90% sure is right.

Answer

Let $x = t^a$ then

Where

$$\frac{dx}{dt} = at^{a-1}$$

$$\frac{d}{dx} = \frac{dt}{dx}\frac{d}{dt} = \frac{1}{a}t^{-a+1}\frac{d}{dt}$$

$$\frac{d^2}{dx^2} = \frac{1}{a}t^{-a+1}\frac{d}{dt}\frac{1}{a}t^{a-1}\frac{d}{dt} =$$

$$\frac{1}{a^2}t^{2(1-a)}\frac{d^2}{dt^2} + \frac{1-a}{a^2}t^{1-2a}\frac{d}{dt}$$

Thus we get

$$\frac{\mu}{a^2}t^{2(1-a)}\frac{d^2y}{dt^2}+\frac{\mu(1-a)}{a^2}t^{1-2a}\frac{dy}{dt}$$
$$-\gamma t^a\frac{1}{a}t^{-a+1}\frac{dy}{dt}+\frac{\mu}{t^a}\frac{1}{a}t^{-a+1}\frac{dy}{dt}-\lambda y=0$$

This leads to

$$\frac{\mu}{a^2}t^{2(1-a)}\frac{d^2y}{dt^2}+\frac{\mu(1-a)}{a^2}t^{1-2a}\frac{dy}{dt}$$ Let $a=1/2$
$$-\gamma t\frac{1}{a}\frac{dy}{dt}+\mu\frac{1}{a}t^{-2a+1}\frac{dy}{dt}-\lambda y=0$$

Then we get

$$\frac{\mu}{a^2}t\frac{d^2y}{dt^2}+\frac{\mu(1-a)}{a^2}\frac{dy}{dt}-\gamma t\frac{1}{a}\frac{dy}{dt}+\mu\frac{1}{a}\frac{dy}{dt}-\lambda y=0$$

Which sort of looks like the Lagurre poly equation, up to normalisation. (But check the calculations).

Subbing in a we find

$t\ddot{y}+\left(1-\frac{\gamma}{2\mu}t\right)\dot{y}-\frac{\lambda}{4\mu}y=0$. Then using the transformation $t\to\beta s$

We find

$s\frac{d^2y}{ds^2}+(1-s)\frac{dy}{ds}-\frac{\lambda}{2\gamma}y=0$ Where $\beta=\frac{2\mu}{\gamma}$ the final statement is fixing

$$n=-\frac{\lambda}{2\gamma}\Rightarrow\lambda=-2\gamma n.$$

Ornstein-Uhlenbeck process and computing $\int_t^{t+\Delta t} e^{\theta s} dWs$.

I am trying to compute a forward equation of an Ornstein-Uhlenbeck process defined in the line below to use in MATLAB to make pretty pictures

$$dX_t = \theta(X_t - \mu)dt + \sigma dW_t$$

by solving the differential equation using variation of parameters on

$$dX_{t+\Delta t} = \theta(X_{t+\Delta t} - \mu)d(t+h) + \sigma dW_{t+h}.$$

After some work, I end up with the following:

$$X_{t+\Delta t} = e^{\theta\Delta t} X_t + \mu(1 - e^{\theta\Delta t}) + \sigma e^{\theta(t+\Delta t)} \int_t^{t+\Delta t} e^{-\theta s} dW_s.$$

From what I've seen online, everything seems fine except the last integral is supposed to look something like

$$\sigma e^{\theta(t+\Delta t)} \int_t^{t+\Delta t} e^{-\theta s} dW_s = \sigma \sqrt{-\frac{1 - e^{-2\theta\Delta t}}{2\theta}} Z_t, \quad Z_t \sim N(0,1)$$

but I don't know how to integrate this with respect to W_s . I do not know a lot about stochastic calculus and this isn't an assignment so I'm really struggling trying to figure out what tool I need to do to finish this calculation.

One observation I made is that the result I am supposed to get looks like the square root of the result of applying the Ito isometry to the $\int \cdot dW_s$ term, but I don't understand where the Z_t would come from or why the Ito isometry is even relevant to my task if I am not trying to compute the expectation or variance.

My other observation is that if this were a non-stochastic integral, at this point I would probably want to approximate the integral with Gaussian Quadrature or something, but I'm not sure if that is the right way to think of it either.

answer

If you know a bit of stochastic calculus theory and gaussian process you might very simply deduce the result you are trying to prove.

First notice that your integrand with respect to the Brownian motion is determistic in such case a stochastic integral is called a Wiener integral and has the very nice property to be a gaussian (centered) process.

This readily means that $Y_t = \int_t^{t+\Delta t} e^{-\theta s} dW_s$ is a gaussian random variable (with mean equal to 0). So there is only one thing left unknown then, to wit the second moment of this random variable. Now using Itô's isometry you get :

$$E[Y_t^2] = E[\int_t^{t+\Delta t} e^{-2.\theta s} ds] = \int_t^{t+\Delta t} e^{-2.\theta s} ds = \ldots$$

The very tractable calculus is left to you.