Values of Rogers-Ramanujan Continued Fraction: Part 3

Evaluation of $R(e^{-2\pi/5})$

In the last post we established the transformation formula

$$\left[\left[\frac{\sqrt{5}+1}{2}\right]^5 + R^5(e^{-2\alpha})\right]\left[\left[\frac{\sqrt{5}+1}{2}\right]^5 + R^5(e^{-2\beta})\right] = 5\sqrt{5}\left[\frac{\sqrt{5}+1}{2}\right]^5 \quad (1)$$

under the condition $\alpha\beta = \pi^2/5$.

If we put $\alpha = \pi$ then $\beta = \pi/5$ and since we already know the value of $R(e^{-2\pi})$ we can use equation (1) to evaluate $R(e^{-2\pi/5})$. But in order to do that we need to calculate $R^5(e^{-2\pi})$ first.

We have from an earlier post

$$R(e^{-2\pi}) = \sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2}$$

Directly raising this to 5^{th} power is bit cumbersome and hence we follow another simpler approach. We note that the value $x = R(e^{-2\pi})$ is a root of the equation

$$x^2 + (\sqrt{5}+1)x - 1 = 0 \quad (2)$$

and it is the greater root of the equation. We propose to find the equation whose roots are 5th powers of the roots of equation (2). Let a, b be roots of equation (2) with $a > b$ so that $a = R(e^{-2\pi})$. Then $a + b = -(\sqrt{5}+1), ab = -1$. It follows that $a^5b^5 = -1$. What we require now is the sum $a^5 + b^5$. To do this let $\Sigma_i = a^i + b^i$. Then we know that $\Sigma_0 = 2, \Sigma_1 = -(\sqrt{5}+1)$.

From equation (2) we can see that $\Sigma_2 + (\sqrt{5}+1)\Sigma_1 - \Sigma_0 = 0$ so that

$$\Sigma_2 = 2 + (\sqrt{5}+1)^2 = 8 + 2\sqrt{5}$$

Again $\Sigma_3 + (\sqrt{5}+1)\Sigma_2 - \Sigma_1 = 0$ and therefore

$$\begin{aligned}\Sigma_3 &= \Sigma_1 - (\sqrt{5}+1)\Sigma_2\\ &= -(\sqrt{5}+1) - (\sqrt{5}+1)(8+2\sqrt{5})\\ &= -(\sqrt{5}+1)(9+2\sqrt{5}) = -(19+11\sqrt{5})\end{aligned}$$

Similarly

$$\begin{aligned}\Sigma_4 &= \Sigma_2 - (\sqrt{5}+1)\Sigma_3\\ &= 8 + 2\sqrt{5} + (\sqrt{5}+1)(19+11\sqrt{5})\\ &= 82 + 32\sqrt{5}\end{aligned}$$

and

$$\begin{aligned}\Sigma_5 &= \Sigma_3 - (\sqrt{5}+1)\Sigma_4\\ &= -(19+11\sqrt{5}) - (\sqrt{5}+1)(82+32\sqrt{5})\\ &= -(19+11\sqrt{5}) - (242+114\sqrt{5})\\ &= -(261+125\sqrt{5})\end{aligned}$$

Hence it follows that a^5, b^5 are roots of the equation

$$x^2 + (261+125\sqrt{5})x - 1 = 0$$

and therefore

$$\begin{aligned}R^5(e^{-2\pi}) = a^5 &= \frac{-(261+125\sqrt{5}) + \sqrt{146250+65250\sqrt{5}}}{2}\\ &= \sqrt{\frac{125(585+261\sqrt{5})}{2}} - \frac{261+125\sqrt{5}}{2}\\ &= 15\sqrt{\frac{5(65+29\sqrt{5})}{2}} - \frac{261+125\sqrt{5}}{2}\end{aligned}$$

and therefore

$$\begin{aligned}R^5(e^{-2\pi}) + \left(\frac{\sqrt{5}+1}{2}\right)^5 &= 15\sqrt{\frac{5(65+29\sqrt{5})}{2}} - \frac{261+125\sqrt{5}}{2} + \frac{11+5\sqrt{5}}{2}\\ &= 15\sqrt{\frac{5(65+29\sqrt{5})}{2}} - 5(25+12\sqrt{5})\\ &= 5\sqrt{5}\left(3\sqrt{\frac{65+29\sqrt{5}}{2}} - (12+5\sqrt{5})\right)\end{aligned}$$

Using the above calculation in equation (1) and setting $\alpha = \pi, \beta = \pi/5$ we get

$$\begin{aligned}\left(\frac{\sqrt{5}+1}{2}\right)^5 + R^5(e^{-2\pi/5}) &= \frac{\left(\frac{\sqrt{5}+1}{2}\right)^5}{3\sqrt{\frac{65+29\sqrt{5}}{2}} - (12+5\sqrt{5})}\\ &= \frac{\left(\frac{\sqrt{5}+1}{2}\right)^5\left(3\sqrt{\frac{65+29\sqrt{5}}{2}} + (12+5\sqrt{5})\right)}{\frac{47+21\sqrt{5}}{2}}\\ &= \left(\frac{\sqrt{5}+1}{2}\right)^5\left(3\sqrt{85-38\sqrt{5}} + \frac{39-17\sqrt{5}}{2}\right)\\ &= 3\sqrt{(85-38\sqrt{5})\cdot\left(\frac{11+5\sqrt{5}}{2}\right)^2}\\ &\quad + \frac{39-17\sqrt{5}}{2}\cdot\frac{11+5\sqrt{5}}{2}\\ &= 3\sqrt{\frac{5+\sqrt{5}}{2}} + 1 + 2\sqrt{5}\end{aligned}$$

so that

$$\begin{aligned}R^5(e^{-2\pi/5}) &= 3\sqrt{\frac{5+\sqrt{5}}{2}}+1+2\sqrt{5}-\frac{11+5\sqrt{5}}{2}\\ &= 3\sqrt{\frac{5+\sqrt{5}}{2}}-\frac{9+\sqrt{5}}{2}\end{aligned}$$

and then

$$R(e^{-2\pi/5}) = \left(3\sqrt{\frac{5+\sqrt{5}}{2}}-\frac{9+\sqrt{5}}{2}\right)^{1/5}$$

In a similar manner using equation (12) of last post we can show that

$$S(e^{-\pi/5}) = \left(3\sqrt{\frac{5-\sqrt{5}}{2}}+\frac{9-\sqrt{5}}{2}\right)^{1/5}$$

Evaluation of $R(e^{-4\pi})$

We next evaluate $R(e^{-4\pi})$ using the values of some class invariants. The first such evaluation for $R(q)$ was done by Ramanathan which was later simplified by Bruce C. Berndt. We follow the approach suggested by Berndt.

Let $x = R(e^{-4\pi})$ so that x satisfies the equation

$$\frac{1}{x}-1-x = e^{4\pi/5}\cdot\frac{f(-e^{-4\pi/5})}{f(-e^{-20\pi})} = A$$

We next turn to the evaluation of constant A defined above. Clearly we know that if $\eta(q) = q^{1/12}f(-q^2)$ then

$$\eta(e^{-\pi/\sqrt{n}}) = n^{1/4}\eta(e^{-\pi\sqrt{n}})$$

Putting $n = 25/4$ we get

$$\begin{aligned}\eta(e^{-2\pi/5}) &= \sqrt{\frac{5}{2}}\eta(e^{-5\pi/2})\\ \Rightarrow e^{-\pi/30}f(-e^{-4\pi/5}) &= \sqrt{\frac{5}{2}}e^{-5\pi/24}f(-e^{-5\pi})\\ \Rightarrow f(-e^{-4\pi/5}) &= \sqrt{\frac{5}{2}}e^{-7\pi/40}f(-e^{-5\pi})\end{aligned}$$

and hence the constant A defined above can be written as

$$\begin{aligned}A &= \sqrt{\frac{5}{2}}e^{5\pi/8}\frac{f(-e^{-5\pi})}{f(-e^{-20\pi})} = \sqrt{\frac{5}{2}}e^{5\pi/8}\frac{f(-q)}{f(-q^4)}\\ &= \sqrt{\frac{5}{2}}e^{5\pi/8}\frac{(q;q)_\infty}{(q^4;q^4)_\infty}\\ &= \sqrt{\frac{5}{2}}e^{5\pi/8}(q;q^2)_\infty(q^2;q^4)_\infty\\ &= \sqrt{\frac{5}{2}}e^{5\pi/8}\cdot 2^{1/4}q^{1/24}g(q)\cdot 2^{1/4}q^{1/12}g(q^2)\\ &= 5^{1/2}e^{5\pi/8}q^{1/8}g(q)g(q^2)\\ &= \sqrt{5}g_{25}g_{100}\end{aligned}$$

where $q = e^{-5\pi}$.

Using equations (13) and (14) of this post and setting $x = \sqrt[4]{5}$ we get

$$\begin{aligned}A &= \sqrt{5}\frac{(1+\sqrt{5})^{3/2}(3+2\sqrt[4]{5})^{1/2}}{2\sqrt{2}}\\ &= \sqrt{5}\frac{(\sqrt{5}+1)(\sqrt{5}-1)(1+\sqrt{5})^{1/2}(3+2\sqrt[4]{5})^{1/2}}{2\sqrt{2}(\sqrt{5}-1)}\\ &= \sqrt{5}\frac{\sqrt{2}}{\sqrt{5}-1}\{(1+\sqrt{5})(3+2\sqrt[4]{5})\}^{1/2}\\ &= \frac{\sqrt{5}}{\sqrt{5}-1}\{2(1+x^2)(3+2x)\}^{1/2}\\ &= \frac{\sqrt{5}}{\sqrt{5}-1}\{6+4x+6x^2+4x^3\}^{1/2}\\ &= \frac{\sqrt{5}}{\sqrt{5}-1}\{1+4x+6x^2+4x^3+x^4\}^{1/2}\\ &= \frac{\sqrt{5}}{x^2-1}\{(1+x)^4\}^{1/2}\\ &= \sqrt{5}\frac{x+1}{x-1}\\ &= \sqrt{5}\frac{\sqrt[4]{5}+1}{\sqrt[4]{5}-1}\end{aligned}$$

It now follows that $x = R(e^{-4\pi})$ is given by

$$x = \sqrt{a^2+1} - a$$

where a is given by

$$2a = 1 + A = 1 + \sqrt{5}\frac{\sqrt[4]{5}+1}{\sqrt[4]{5}-1}$$

General Value of $R(e^{-\pi\sqrt{n}})$

If $q = e^{-\pi\sqrt{n}}$ where n is a positive rational, then we know that the value of $k = k(q) = \theta_2^2(q)/\theta_3^2(q)$ is an algebraic number. Now let $l = k(q^5)$ be the modulus corresponding to q^5 so that l is also an algebraic number. Using notation of Ramanujan we set $\alpha = k^2, \beta = l^2$ so that β is of degree 5 over α. From the transcription formulas in this post we have

$$\begin{aligned}\frac{f^{6}(-q)}{qf^{6}(-q^{5})} &= \frac{2^{-1}z^{3}(q)(1-\alpha)\alpha^{1/4}q^{-1/4}}{q\cdot 2^{-1}z^{3}(q^{5})(1-\beta)\beta^{1/4}q^{-5/4}}\\ &= m^{3}\frac{(1-\alpha)\alpha^{1/4}}{(1-\beta)\beta^{1/4}}\end{aligned}$$

Since the multiplier m itself can be expressed as an algebraic function of α, β it follows from the above that the fraction $f^{6}(-q)/\{qf^{6}(-q^{5})\}$ is an algebraic number provided that $q = e^{-\pi\sqrt{n}}$, n being a positive rational number. And therefore from the identity

$$\frac{1}{R^{5}(q)} - 11 - R^{5}(q) = \frac{f^{6}(-q)}{qf^{6}(-q^{5})}$$

it follows that $R^{5}(q)$ and hence $R(q)$ is also an algebraic number.

Using sophisticated techniques of modular forms and class field theory it can be established that the fraction $f^{6}(-q)/\{qf^{6}(-q^{5})\}$ is an algebraic integer in which case the above identity for $R(q)$ implies that $R(q)$ is a unit whenever $q = e^{-\pi\sqrt{n}}$, n being a positive rational. The same proposition holds for the values of $S(q) = -R(-q)$ and can be established in exactly the same manner as done for $R(q)$.

Following Dr. Bruce C. Berndt we will now establish some general formulas for $R(e^{-2\pi\sqrt{n}})$ in terms of the class invariants. Let us then suppose that $q = e^{-2\pi\sqrt{n}}$ and then we can define a constant A by

$$A = \frac{f(-q)}{q^{1/6}f(-q^{5})} = e^{\pi\sqrt{n}/3}\frac{f(-e^{-2\pi\sqrt{n}})}{f(-e^{-10\pi\sqrt{n}})} \tag{3}$$

so that

$$\frac{1}{R^{5}(q)} - 11 - R^{5}(q) = A^{6}$$

and then

$$R^{5}(e^{-2\pi\sqrt{n}}) = \sqrt{a^{2}+1} - a \tag{4}$$

where $2a = A^{6} + 11$

To calculate the value of A we need to make use of modular equations of degree 5 established in an earlier post. We have from equations (19) and (20) of this post:

$$m = \left(\frac{\beta}{\alpha}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4} \tag{5}$$

$$\frac{5}{m} = \left(\frac{\alpha}{\beta}\right)^{1/4} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/4} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/4} \tag{6}$$

From the above equations we can see that

$$\begin{aligned}\{\alpha(1-\alpha)\}^{1/4}&\left(m+\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4}\right)\\&=\{\beta(1-\beta)\}^{1/4}\left(\frac{5}{m}+\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/4}\right)\end{aligned}\tag{7}$$

Now let q correspond to α so that q^5 corresponds to β and $m=\phi^2(q)/\phi^2(q^5)$. We then have

$$q^{-1/24}\chi(q)=2^{1/6}\{\alpha(1-\alpha)\}^{-1/24}$$

$$q^{-5/24}\chi(q^5)=2^{1/6}\{\beta(1-\beta)\}^{-1/24}$$

and thus the equation (7) above can be transcribed into the following identity for theta functions

$$\frac{2q^{1/4}}{\chi^6(q)}\left(\frac{\phi^2(q)}{\phi^2(q^5)}+\frac{q\chi^6(q)}{\chi^6(q^5)}\right)=\frac{2q^{5/4}}{\chi^6(q^5)}\left(\frac{5\phi^2(q^5)}{\phi^2(q)}+\frac{\chi^6(q^5)}{q\chi^6(q)}\right)$$

which leads to

$$\frac{\phi^2(q)}{\phi^2(q^5)}-\frac{5q\phi^2(q^5)\chi^6(q)}{\phi^2(q)\chi^6(q^5)}=1-\frac{q\chi^6(q)}{\chi^6(q^5)}\tag{8}$$

Noting that

$$\begin{aligned}f(-q^2)&=(q^2;q^2)_\infty\\&=\frac{(-q;q^2)_\infty(-q;q^2)_\infty(q^2;q^2)_\infty}{(-q;q^2)_\infty(-q;q^2)_\infty}\\&=\frac{f(q,q)}{\chi^2(q)}=\frac{\phi(q)}{\chi^2(q)}\end{aligned}$$

we can write $\phi(q)=\chi^2(q)f(-q^2)$ and then the equation (8) is transformed into

$$\frac{\chi^4(q)f^2(-q^2)}{\chi^4(q^5)f^2(-q^{10})}-\frac{5qf^2(-q^{10})\chi^2(q)}{f^2(-q^2)\chi^2(q^5)}=1-\frac{q\chi^6(q)}{\chi^6(q^5)}\tag{9}$$

Next we know that the Ramanujan class invariant function $G(q)$ is given by $G(q)=2^{-1/4}q^{-1/24}\chi(q)$ so that $\chi(q)=2^{1/4}q^{1/24}G(q)$. Using this value of $\chi(q)$ in above equation we get

$$\frac{G^4(q)f^2(-q^2)}{q^{2/3}G^4(q^5)f^2(-q^{10})}-\frac{5q^{2/3}G^2(q)f^2(-q^{10})}{G^2(q^5)f^2(-q^2)}=1-\frac{G^6(q)}{G^6(q^5)}$$

Putting $q=e^{-\pi\sqrt{n}}$ in the above equation and noting that $G(q)=G_n, G(q^5)=G_{25n}$ and also observing that $A=q^{-1/3}f(-q^2)/f(-q^{10})$, we finally get

$$A^2\frac{G_n^4}{G_{25n}^4}-A^{-2}\frac{G_n^2}{G_{25n}^2}=1-\frac{G_n^6}{G_{25n}^6}$$

Setting $U=G_{25n}/G_n$ we can see that the above equation is transformed into

$$\frac{A^2}{U} - 5\frac{U}{A^2} = U^3 - \frac{1}{U^3}$$

If we replace q by $-q$ in the equation (9) and set $V = g_{25n}/g_n$ then we get

$$\frac{A^2}{V} + 5\frac{V}{A^2} = V^3 + \frac{1}{V^3}$$

We can now summarize our formulas as follows:

Let n be a positive rational number and we define

$$U = \frac{G_{25n}}{G_n},\ V = \frac{g_{25n}}{g_n}$$

Then the constant A defined by

$$A = e^{\pi\sqrt{n}/3}\frac{f(-e^{-2\pi\sqrt{n}})}{f(-e^{-10\pi\sqrt{n}})}$$

satisfies the following equations

$$\frac{A^2}{U} - 5\frac{U}{A^2} = U^3 - \frac{1}{U^3}$$
$$\frac{A^2}{V} + 5\frac{V}{A^2} = V^3 + \frac{1}{V^3}$$

and the value of $R^5(e^{-2\pi\sqrt{n}})$ is given by

$$R^5(e^{-2\pi\sqrt{n}}) = \sqrt{a^2+1} - a$$

where a is given by

$$2a = A^6 + 11$$

If we put $n = 1/5$ in the above formula we get $U = 1$ so that $A^2 = \sqrt{5}$ and $a = (11 + 5\sqrt{5})/2$ and the value of $R^5(e^{-2\pi/\sqrt{5}})$ calculated using above formula matches the value calculated earlier.

There are analogous results for $S(q)$ which the reader can formulate and prove himself along the above lines. For more such formulas the reader should consult Dr. Bruce C. Berndt's papers and his *Ramanujan's Notebooks*.

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Labels: Continued Fractions , Mathematical Analysis , Theta Functions