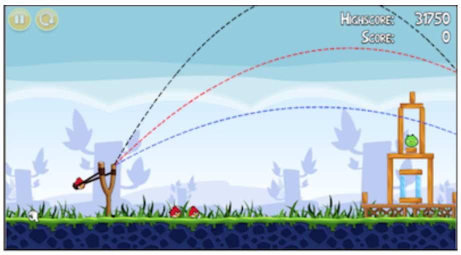
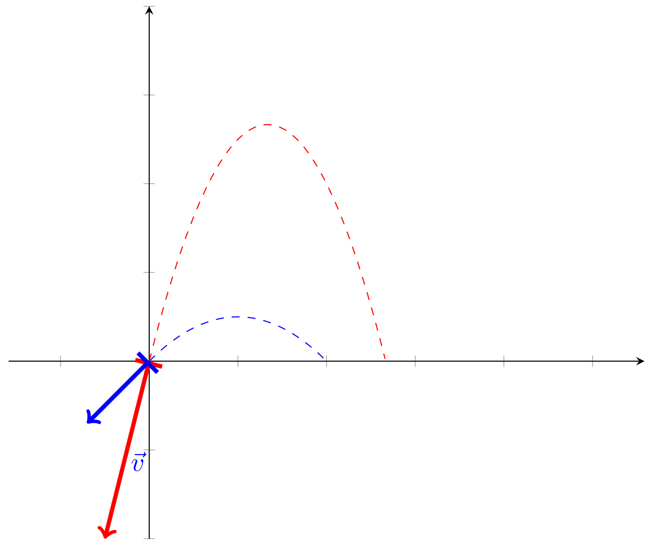
1. You are developing a prototype Angry Birds software to help high school students understand vectors and linear algebra. You would like students to input a vector, $\vec{v} = (v_1, v_2)$, and have the output, $f(\vec{v}) = (a_1, a_2, a_3)$ be the coefficients of a quadratic equation that corresponds to the direction and scale of the vector:

$$y = a_1 x^2 + a_2 x + a_3.$$





(a) Suppose that your vector is based at the origin. What does this mean for the value a_3?

> **Solution:** $a_3 = 0$.

(b) Note that $\vec{v}$ points the **opposite direction** of the flight of the bird. It is required that birds land in front of their launch point at (0,0). What constraints does this impose on v_1 and v_2? Write them down, or explain why there are none.

> **Solution:** $v_1, v_2 < 0$.

(c) Write the equation that corresponds to Span$\{\vec{v}\}$.

> **Solution:**
>
> $$y = \frac{v_2}{v_1} x.$$

(d) You want the slope of the tangent line to the quadratic and the line from part (c) to agree at 0. Write the corresponding linear equation.

> **Solution:** The quadratic has derivative $y' = 2 * a_1 x + a_2$. At $x = 0$ this is equal to a_2. The slope in part (c) is v_2/v_1. So we need
>
> $$a_2 = v_2/v_1$$

(e) Suppose you decide to set $a_1 = v_1 + v_2$. Explain why this is a reasonable choice.

> **Solution:** $v_1 + v_2$ is negative, so the parabola arcs the correct way. Also, as v_1 and v_2 become more negative this will make the arc of the parabola higher.

(f) Write a matrix corresponding to the system of equations for a_1, a_2, a_3.

Solution:
$$\begin{bmatrix} 1 & 0 & 0 & v_1 + v_2 \\ 0 & 1 & 0 & v_2/v_1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

(g) A student asks whether you can now find $\vec{v}$ so that the bird passes through two specified points $(a, b), (c, d)$. Write the system of equations that $\vec{v}$ must satisfy for this to happen.

Solution: We know that $(0, 0)$ is also on the parabola. So we need a parabola that goes through the points $(0, 0), (a, b), (c, d)$. This means that

$$\begin{aligned} a_3 &= 0 \\ a_1 a^2 + a_2 a + a_3 &= b \\ a_1 c^2 + a_2 c + a_3 &= d \end{aligned}$$

(h) Is it possible if instead three arbtrary points are specified? Why or why not?

Solution: No. A parabola is determined by 3 points, and $(0, 0)$ is already implicitly one of them. So we would have 4 points.

(i) You input your answer from part (g) into a computer and obtain the system in echelon form:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \frac{b}{a^2} + \frac{\frac{bc^2}{a^2} - d}{a\left(c - \frac{c^2}{a}\right)} \\ 0 & 1 & 0 & -\frac{\frac{bc^2}{a^2} - d}{c - \frac{c^2}{a}} \\ 0 & 0 & 1 & 0 \end{array}\right)$$

What values of (a, b, c) are not acceptable. Explain why. Don't just say because the matrix equations don't make sense.

Solution: $a = 0$ is not acceptable since this would give a redundant point $(0, 0)$ which is not enough information to uniquely determine the parabola. Also, $c = a$ is not acceptable. This corresponds to (a, b) and (c, d) lying on the same vertical line, which is unacceptable because a parabola is a function, so it cannot go through two points on the same vertical line.

(j) Let

$$g(a, b) = \frac{b}{a^2} + \frac{\frac{bc^2}{a^2} - d}{a\left(c - \frac{c^2}{a}\right)} \quad \text{and} \quad h(a, b) = -\frac{\frac{bc^2}{a^2} - d}{c - \frac{c^2}{a}}.$$

Given $(a, b), (c, d)$ write a **linear** system of equations for v_1 and v_2 in terms of $h(a, b)$ and $g(a, b)$.

Solution: $v_1 + v_2 = g(a, b)$ and $v_2/v_1 = h(a, b)$. So we have a linear system:

$$\begin{aligned} v_1 + v_2 &= g(a, b) \\ -h(a, b)v_1 + v_2 &= 0. \end{aligned}$$

(k) What is the solution to this equation?

Solution: We can add $h(a,b)$ times the first equation to the second. This gives $(h(a,b)+1)v_2 = h(a,b)g(a,b)$. So

$$v_2 = \frac{h(a,b)g(a,b)}{h(a,b)+1}.$$

This means that $v_1 = g(a,b) - \frac{h(a,b)g(a,b)}{h(a,b)+1}$.

2. A video game designer is rectifying the dynamics of two different rooms in a video game. She needs to know if $\left\{ \begin{bmatrix}1\\2\\2\end{bmatrix}, \begin{bmatrix}2\\3\\4\end{bmatrix} \right\}$ and $\left\{ \begin{bmatrix}4\\2\\6\end{bmatrix}, \begin{bmatrix}1\\3\\2\end{bmatrix} \right\}$ have the same span.

   (a) Call (x, y, z) a point in the span of a set of vectors. Write down two augmented matrices that correspond to the span of the two sets of vectors above.

   Solution:
   $$\begin{bmatrix}1 & 2 & x\\2 & 3 & y\\2 & 4 & z\end{bmatrix} \qquad \begin{bmatrix}4 & 1 & x\\2 & 3 & y\\6 & 2 & z\end{bmatrix}$$

   (b) Say you row reduce these matrices. What must happen for them to have the same span?

   Solution: When row reduced they will look like

   $$\begin{bmatrix}1 & 0 & f_1(x,y,z)\\0 & 1 & f_2(x,y,z)\\0 & 0 & f_3(x,y,z)\end{bmatrix} \qquad \begin{bmatrix}1 & 0 & g_1(x,y,z)\\0 & 1 & g_2(x,y,z)\\0 & 0 & g(x,y,z)\end{bmatrix},$$

   with f_i and g_i linear equations. The span is described by the set of points where $f_3(x,y,z) = 0$, and the span for the second set is where $g_3(x,y,z) = 0$. To be the same we need $f_3(x,y,z) = Cg_3(x,y,z)$ for some constant C.

3. Byeol is a programmer at Google and she is given a piece of code which she is told starts with two vectors $\{\vec{u}, \vec{v}\}$ (that she does not know) and ultimately outputs $\{\vec{u}+\vec{v}, \vec{u}-\vec{v}\}$.

   (a) Is it possible to determine the values of $\vec{u}$ and $\vec{v}$ from the output? Explain.

   Solution: Yes. Set $\vec{u}+\vec{v} = \vec{x}$ and $\vec{u}-\vec{v} = \vec{y}$. Adding the two equations we have $2\vec{u} = \vec{x}+\vec{y}$ and thus $\vec{u} = \frac{1}{2}(\vec{x}+\vec{y})$. Similarly, $\vec{v} = \frac{1}{2}(\vec{x}-\vec{y})$.

   (b) Byeol is told that $\vec{u}$ and $\vec{v}$ are linearly independent. She needs to know that the output also is. Show that $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$ are also linearly independent

   Solution: Because $\vec{u}$ and $\vec{v}$ are linearly independent we know that $\vec{u} \neq c\vec{v}$ for all constants c. However, if $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$ were *not* linearly independent, then we could write $\vec{u}+\vec{v} = b(\vec{u}-\vec{v})$ for some constant b. Note that when $b = -1$ we would have

   $$\vec{u}+\vec{v} = -\vec{u}+\vec{v},$$

   which says that $2\vec{u} = \vec{0}$. This is impossible if $\vec{u}$ and $\vec{v}$ are linearly independent, because any collection of vectors containing $\vec{0}$ is linearly dependent.

   Now suppose $b \neq -1$. Rearranging $\vec{u}+\vec{v} = b(\vec{u}-\vec{v})$ we have

   $$\vec{u}+b\vec{u} = \vec{v}-b\vec{v}.$$

This is equivalent to $\vec{u} = \frac{1-b}{1+b}\vec{v}$. Since $b \neq -1$, we can set $c = \frac{1-b}{1+b}$, this contradicts the fact that $\vec{u} \neq c\vec{v}$ for all c. So, there *cannot* exist such a b. This is equivalent to linear independence of the vectors in question.