NOTES

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1. NOTES ON *RAPPORT SUR LA FORMULE DES TRACES*

$p, q, ℓ, \mathbf{F}_q, \mathbf{F}$: $p$ is a prime number, $q = p^\ell$ is a power of $p$ and $\mathbf{F}$ an algebraic closure of the field $\mathbf{F}_q$; $\ell$ is a prime number $\neq p$.

$X_0, X$: $X_0$ is a scheme on $\mathbf{F}_q$, $X = X_0 \times_{\mathbf{F}_q} \mathbf{F}$. If $\mathcal{F}_0$ is an (étale) sheaf on $X_0$, $\mathcal{F}$ denotes its inverse image on $X$.

1.1. **Mise en garde.** The exposition beginning with 1.3 is based on SGA 5, Exposé XIV by C. Houzel. This approach is explicit at the expense being somewhat anti-conceptual and overloaded with cumbersome notation, which makes it hard to remember. On the other hand, Deligne’s approach to Frobenius is conceptually beautiful and easy to remember, but the first time I read it I couldn’t understand it. Now that I do, I write these notes. Deligne uses $\wp$ to denote what he calls the ‘Frobenius substitution’; it is

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the well-known topological generator of $\text{Gal}(\mathbb{F}/\mathbb{F}_q)$. He uses $F$ to denote the ‘Frobenius endomorphism’, notated $\text{fr}_{X_0}$ below, which is the endomorphism of a scheme $X_0/\mathbb{F}_q$ obtained by $x \mapsto x^q$ on the structure sheaf $\mathcal{O}_{X_0}$. Finally, he uses $F^*$ to denote the action of Frobenius on sheaves on a scheme over $\mathbb{F}_q$ (and base extensions of such), and their cohomology.

1.2. Frobenius following Deligne.

1.2.1. Representing the Frobenius correspondence. Let $Q$ be any scheme and $\mathcal{G}$ an (étale) sheaf on $Q$. Let $Q_{\text{pet}}$ denote the category $(\text{Et}/Q)$ of algebraic spaces étale over $Q$, equipped with the étale topology. We ‘recall’ the following

**Proposition.** $\mathcal{G}$ is represented over $Q_{\text{pet}}$ by l’espace étalé $[\mathcal{G}]$ of $\mathcal{G}$: an algebraic space, locally separated and étale over $Q$. If moreover $\mathcal{G}$ is

\begin{align*}
(\alpha) & \text{ locally constant constructible} \\
(\beta) & \text{ locally constant,} \\
(\gamma) & \text{ constructible,}
\end{align*}

then $[\mathcal{G}]$ may be taken to be

\begin{align*}
(\alpha) & \text{ a scheme finite étale over } Q \text{ (SGAA Exp. IX 2.2).} \\
(\beta) & \text{ a scheme étale over } Q \text{ (SGAA Exp. IX 2.2).} \\
(\gamma) & \text{ étale and finitely presented as an algebraic space over } Q \text{ (SGAA Exp. IX 2.7).}
\end{align*}

In other words, $\mathcal{G}$ is the sheaf of local sections of the space $[\mathcal{G}]$: for each $V \in \text{Et}/Q$,

$$\mathcal{G}(V) = \text{Hom}_Q(V, [\mathcal{G}]).$$

M. Artin constructs the espace étalé $[\mathcal{G}]$ associated to an arbitrary étale sheaf $\mathcal{G}$ on a scheme $Q$ in *Théorèmes de représentabilité pour les espaces algébriques* VII §1. A sketch: put

$$\left( U \to \mathcal{G} \right) := \bigsqcup_{U, \xi \in \mathcal{G}(U)} (U \to \mathcal{G}),$$
the sum executed over affine schemes $U$ étale over $Q$ and the $\xi \in \mathcal{G}(U)$. The canonical morphism $\mathcal{U} \times_{\mathcal{G}} \mathcal{U} \to \mathcal{U} \times_{Q} \mathcal{U}$ induced by $\xi \in \mathcal{G}(U)$ and $\eta \in \mathcal{G}(V)$ ($U, V$ affine étale schemes over $Q$) is an open immersion and defines an étale equivalence relation on $\mathcal{U}$. Let $[\mathcal{G}]$ be the quotient of $\mathcal{U}$ by this equivalence relation; it is an algebraic space over $Q$, in general not separated, but only locally separated; in other words, $\mathcal{U} \times_{\mathcal{G}} \mathcal{U} \to \mathcal{U} \times_{Q} \mathcal{U}$ is not necessarily a closed immersion.

Now let $X_0$ be a scheme over $\mathbb{F}_q$ and $\mathcal{F}_0$ a sheaf on $X_0$. The formation of the espace étalé $[\mathcal{F}_0]$ of $\mathcal{F}_0$ yields the commutative square

$$
\begin{array}{ccc}
[\mathcal{F}_0] & \xrightarrow{F} & [\mathcal{F}_0] \\
\downarrow & & \downarrow \\
X_0 & \xrightarrow{F} & X_0.
\end{array}
$$

and hence a morphism $[\mathcal{F}_0] \to F^*[\mathcal{F}_0]$. Replacing $S$ with $X_0$, and $X_0$ with $[\mathcal{F}_0]$, 1.3.1 below finds that this morphism is relative Frobenius $Fr_{[\mathcal{F}_0]/X_0}$; as $[\mathcal{F}_0]$ is étale over $X_0$, 1.3.3 tells us that this morphism is an isomorphism $[\mathcal{F}_0] \xrightarrow{\sim} F^*[\mathcal{F}_0]$ with inverse Deligne’s Frobenius correspondence (1.2.1)

$$
F^*: F^*\mathcal{F}_0 \xrightarrow{\sim} \mathcal{F}_0.
$$

This morphism is the same as the one constructed with different notation in 1.3.4. The point is that by putting additional hypotheses on $\mathcal{F}_0$, one may assume that $[\mathcal{F}_0]$ is in fact a scheme étale over $X_0$.

1.2.2. The Frobenius correspondence and the Frobenius endomorphism. Let $\mathcal{F}_0$ be an abelian sheaf on $X_0$. We wish to elucidate Deligne’s approach in 1.8 of Rapport to show that

$$
F^{* -1} = \varphi \quad \text{(on $H^i_c(X, \mathcal{F})$)}.
$$
Letting \( Y_0 = \text{Spec} \mathbf{F}_q \), as written \( H^*_f(X, \mathcal{F}) = [R^f_!! \mathcal{F}_0](\mathcal{F}) \), and noting \( F = \text{id} \) on \( Y_0 \) we have the (stupid) diagram

\[
\begin{array}{c}
\text{[R}^f_! \mathcal{F}_0] \\
\text{id} \downarrow \\
Y_0 \quad \text{id} \downarrow \\
\end{array}
\]

which just serves to connect this discussion to that of the previous section and show that the morphism defining the inverse of the Frobenius correspondence \( F^* \) on \( [R^f_!! \mathcal{F}_0] \) is indeed \( F : [R^f_!! \mathcal{F}_0] \rightarrow [R^f_!! \mathcal{F}_0] \); as \( F \) acts on geometric points by \( q \), we see that

\[
F^{*-1} : [R^f_!! \mathcal{F}_0](\mathcal{F}) \rightarrow [R^f_!! \mathcal{F}_0](\mathcal{F})
\]

coincides with \( q \).

1.3. **Frobenius following Houzel.**

**Definition.** We denote by \( \text{fr}_{X_0} \) the morphism of schemes \( X_0 \rightarrow X_0 \) which is the identity on the underlying topological space \( |X_0| \) and acts on the structure sheaf \( \mathcal{O}_{X_0} \) by \( g \mapsto g^q \).

This morphism is called **absolute Frobenius**.

1.3.1. If the structure morphism \( X_0 \rightarrow \mathbf{F}_q \) factors through some scheme \( S \), then we denote by \( X_0^{(q/S)} := X_0 \times_{S, \text{fr}_S} S \) the fiber product of \( g : X_0 \rightarrow S \) by the morphism \( \text{fr}_S : S \rightarrow S \) with projection \( \pi_{X_0/S} : X_0^{(q/S)} \rightarrow X_0 \). The absolute Frobenius \( \text{fr}_{X_0} \) then factors through the morphism \( \pi_{X_0/S} \). We can form the diagram

\[
\begin{array}{c}
X_0 \\
\text{fr}_{X_0/S} \downarrow \\
X_0^{(q/S)} \quad \text{fr}_{X_0} \downarrow \\
\pi_{X_0/S} \downarrow \\
S \quad \text{fr}_S \downarrow \\
S
\end{array}
\]
Definition. The morphism $\text{Fr}_{X_0/S}$ is called relative Frobenius.

1.3.2. Frobenius acts on geometric points. Consider the set of geometric points $X(F) = X_0(F)$. Frobenius acts on this set by $\varphi \in \text{Gal}(F/F_q)$, $\varphi(x) = x^q$. In particular, as $F_q$ is perfect, $X^F = X_0(F_q)$, where $X^F$ denotes the geometric points fixed by Frobenius. In slightly more words, consider a geometric point $\overline{x} \to X_0$ centered on $x$. The fiber $X \times_{X_0} x$ is isomorphic to the spectrum of $A = F \otimes_{F_q} k(x)$. As $k(x)/F_q$ is separable, $A \sim \prod_{[k(x):F_q]} F$, and $[k(x):F_q]$ is also equal to the number of $F_q$-embeddings $k(x) \to F$. Such an embedding is fixed by $\varphi$ iff $k(x) = F_q$, and by $\varphi^f$ iff $k(x) \subset F_{q^f}$. So for every point $x \in |X_0|$ with $[k(x):F_q] = f$, there are $f$ geometric points centered on $x$; $F$ acts transitively by $\varphi$ on this set, and $F^f$ fixes each of these geometric points.

1.3.3. Behavior of relative Frobenius. The relative Frobenius $\text{Fr}_{X_0/S}$ is integral, surjective, and radicial, hence a universal homeomorphism. This is clear when $S = F_q$; i.e. for $\text{Fr}_{X_0} = \pi_{X_0/S} \circ \text{Fr}_{X_0/S}$; it follows that $\text{Fr}_{X_0/S}$ is radicial [EGA, I 3.5.6 (ii)]. Moreover, $\pi_{X_0/S}$ is separated and radicial, therefore $\text{Fr}_{X_0/S}$ is integral [EGA, II 6.1.5 (v)] and surjective.

Suppose moreover that $g : X_0 \to S$ is étale. The same is true of $g^{(q)} : X_0^{(q)} \to S$, and therefore $\text{Fr}_{X_0/S} : X_0 \to X_0^{(q)}$ is étale. As $\text{Fr}_{X_0/S}$ is also radicial and surjective, it is an isomorphism.

1.3.4. Frobenius correspondence. Let $X_0$ be a scheme over $F_q$ and $\mathcal{F}_0$ a sheaf of sets on $(X_0)_{\text{ét}}$ (the small étale site of $X_0$ whose underlying category is the category of schemes étale over $X_0$). We have for all $U \to X$ étale $(\text{fr}_{X_0})_* \mathcal{F}_0(U) = \mathcal{F}_0(U^{q/X})$. The isomorphism $\mathcal{F}_0(\text{Fr}_{U/X_0}) : (\text{fr}_{X_0})_* \mathcal{F}_0(U) \to \mathcal{F}_0(U)$ is natural in $U$ and induces an isomorphism of sheaves

$$\mathcal{F}_0(\text{Fr}_{X_0}) : (\text{fr}_{X_0})_* \mathcal{F}_0 \simeq \mathcal{F}_0;$$

by adjunction applied to $\mathcal{F}_0(\text{Fr}_{X_0})^{-1}$ we obtain a morphism

$$F^* : \text{fr}_{X_0}^* \mathcal{F}_0 \to \mathcal{F}_0.$$
As \( \text{fr}_{X_0} \) is integral, surjective, and radicial, \( \text{fr}^*_{X_0} : (X_0)_{\text{et}} \to (X_0)_{\text{et}} \) is an equivalence of sites, and the functors
\[
(\text{fr}_{X_0})^*, \text{fr}^*_{X_0} : (\overline{X_0})_{\text{et}} \to (\overline{X_0})_{\text{et}}
\]
are autoequivalences and quasi-inverses, where \( (\overline{X_0})_{\text{et}} \) denotes the étale topos on \( X_0 \) [SGAA, Exp. VIII, 1.1]. Therefore \( F^* \) is also an isomorphism.

**Definition.** The isomorphism \( F^* : \text{fr}^*_{X_0} \mathcal{F}_0 \to \mathcal{F}_0 \) is called the Frobenius correspondence.

1.3.5. Frobenius acts on cohomology. Consider \( \mathcal{F}_0 \) a sheaf of \( \Lambda \)-modules, for some commutative ring \( \Lambda \). The canonical morphism \( \alpha : \mathcal{F}_0 \to \text{fr}_{X_0} \mathcal{F}_0 \mathcal{F}_0 \) gives rise to
\[
\Gamma(X_0, \mathcal{F}_0) \xrightarrow{\alpha} \Gamma(X_0, \text{fr}^*_{X_0} \mathcal{F}_0) \xrightarrow{F^*} \Gamma(X_0, \mathcal{F}_0);
\]
we also denote the composition of these maps by \( F^* \). When \( \mathcal{F}_0 = \Lambda \), this composition is easily seen to coincide with \( \text{id}_{\Gamma(X_0, \Lambda)} \), and as every section \( s \in \Gamma(X_0, \mathcal{F}_0) \) corresponds to a morphism \( s : \Lambda \to \mathcal{F}_0 \) and \( F^* \) is evidently functorial in \( \mathcal{F}_0 \), we find \( F^* \circ \Gamma(X_0, s) = \Gamma(X_0, s) \circ F^* = \Gamma(X_0, s) \), ergo \( F^* s = s \), so \( F^* \) induces the identity on \( \Gamma(X_0, \mathcal{F}_0) \). Recalling the definition of the Frobenius correspondence via adjunction, this action of Frobenius on \( \Gamma(X_0, \mathcal{F}_0) \) coincides with the composition
\[
(\ast) \quad \Gamma(X_0, \mathcal{F}_0) \xrightarrow{\mathcal{F}_0(\text{fr}_{X_0})^{-1}} \Gamma(X_0, (\text{fr}_{X_0})^* \mathcal{F}_0) = \Gamma(X_0, \mathcal{F}_0).
\]
Considering \( \mathcal{F}_0 \) now as an object of \( D^+(X_0, \Lambda) \), we have \( \text{fr}_{X_0}^* \) preserves injective objects; hence the composition
\[
F^* : \text{R}\Gamma(X_0, \mathcal{F}_0) \xrightarrow{\mathcal{F}_0(\text{fr}_{X_0})^{-1}} \text{R}\Gamma(X_0, \text{fr}_{X_0}^* \mathcal{F}_0) \to \text{R}\Gamma(X_0, \mathcal{F}_0)
\]
can be computed by applying (\ast) term-by-term to an injective resolution of \( \mathcal{F}_0 \), whence we see that \( F^* \) acts by identity on \( \text{R}\Gamma(\mathcal{F}_0) \), and hence
\[
F^* : \mathcal{H}^i(X_0, \mathcal{F}_0) \to \mathcal{H}^i(X_0, \mathcal{F}_0)
\]
is the identity for all \( i \).
Suppose that $X_0$ is separated and of finite type over $\mathbf{F}_q$. Then we can replace $\Gamma$ in the above discussion by $\Gamma_c$ to find that Frobenius acts on compactly supported cohomology

$$F^*: H^i_c(X_0, \mathcal{F}_0) \rightarrow H^i_c(X_0, \mathcal{F}_0).$$

2.1. **The trace formula for** $\text{Spec} \mathbf{F}_q^n \rightarrow \text{Spec} \mathbf{F}_q$. Let $X_0 = \text{Spec} \mathbf{F}_q^n$, $q = p^f$, and $\mathcal{F}_0$ be a constructible $\mathbf{Q}_\ell$ sheaf on $X_0$, $\mathcal{F}$ its inverse image on $X$. In this case, the cohomological description of the $L$-function $L(X_0, \mathcal{F}_0)$ reads very simply

$$(*) \quad \text{det}(1 - F^* t^{d(x)}, \mathcal{F})^{-1} = \text{det}(1 - F^* t^f, H^0_c(X, \mathcal{F}))^{-1},$$

where $d(x) = [k(x) : \mathbf{F}_p]$. We first need to make precise how Frobenius is acting on the left and right sides.

On the left side, we fix a geometric point $\bar{x} \rightarrow x = X$ and construct the action of Frobenius on the fiber $\mathcal{F}_{\bar{x}}$ by picking the smallest power of $F(q)$ which actually fixes the geometric point $\bar{x}$, namely $F(q^n) = F(q)$. The notation $F^*$ denotes the endomorphism of $\mathcal{F}_{\bar{x}}$ induced by $F^*$. Up to isomorphism, $(F^*, \mathcal{F})$ do not depend on the choice of geometric point $\bar{x} \rightarrow x$, and the trace, determinant of this action are denoted by $\text{Tr}(F^*, \mathcal{F})$, etc.

Now, on the right side, we have the Frobenius correspondance on cohomology. We will make use of the identity

$$F^{* -1} = q,$$

where $q$ is the Frobenius considered as the topological generator of $\text{Gal}(\mathbf{F}/\mathbf{F}_q)$ (c.f. remark below). The data of a $\mathbf{Q}_\ell$-sheaf on $X$ is equivalent to the data of a finite-dimensional $\mathbf{Q}_\ell$-vector space $V = \mathcal{F}_{\bar{x}}$ on which $\text{Gal}(\bar{k}(x)/k(x)) = \pi_x$ acts continuously. There is a canonical isomorphism

$$\pi_x = \text{Gal}(\bar{k}(x), k(x)) \cong \hat{\mathbf{Z}}$$

furnished by the Frobenius element

$$q_x \in \text{Gal}(\bar{k}(x), k(x)) = \text{Gal}(\mathbf{F}/\mathbf{F}_q^n), \quad q_x(\lambda) = \lambda^{q^n},$$
so that the action of \( \pi_x \) on \( V = \mathcal{F}_\pi \) is known once one knows the automorphism \( (q_x)_V \) (under the one condition that \((q_x)_V \to \text{id}_V \) as \( \nu \to 0 \) multiplicatively). If

\[
pr_x : X = \text{Spec}(F_{q^n}) \to \text{Spec} F_q = e
\]
is the canonical morphism, \( \pi_x \) is identified via \( pr_x \) with a subgroup of the analogous Galois group \( \pi_e \) for \( e = \text{Spec}(F_q) \), itself topologically generated by \( q_e \), and via this identification we have the identity

\[
q_x = q_e^n.
\]

The sheaf \( \text{pr}_{x*}(\mathcal{F}) \) is defined by the induced module

\[
\text{pr}_{x*}(\mathcal{F})_\pi \simeq \mathcal{F}_\pi \otimes_{\pi_x} \pi_e,
\]

from which one deduces that, letting \( f = q^{-1}_x \), \( q^{-1}_e \) acts on \( \text{pr}_{x*}(\mathcal{F})_\pi \) by

\[
f^{(n)} : (x_1, \ldots, x_n) \mapsto (f(x_n), x_1, \ldots, x_{n-1}),
\]

where here we have written \( \text{pr}_{x*}(\mathcal{F})_\pi \) with respect to a basis as a free \( \pi_x \)-module of rank \( n \). Now the formula (\#) is a matter of verifying the formula

\[
\det(1 - ft^n) = \det(1 - f^{(n)} t)
\]

for \( f \) acting on a free module of rank \( n \). This is Deligne’s corollary 3.4.

**Remark.** Perhaps one way to think about the identification \( F^* = q^{-1} \) is by setting up the usual diagram

\[
\begin{array}{ccc}
X & \xrightarrow{F_{X/X_0}} & X \\
\downarrow & \searrow & \downarrow g \\
X(q/X_0) & \xrightarrow{\pi_{X/X_0}} & X \\
\downarrow & & \downarrow g \\
X_0 & \xrightarrow{F_{X_0}} & X_0
\end{array}
\]

with \( g \) the base extension of the map \( \text{Spec} F \to \text{Spec} F_q \) that arises from fixing an algebraic closure of \( F_q \), and then observing that \( F_{X/X_0} = q \times_{F_q} \text{id}_{X_0} \). Recalling that \( F^* \) on \( \mathcal{O}_0/X_0 \) is induced by \( F_{U/X_0}^{-1} \) for \( U \to X_0 \) étale, and by functoriality \( F^* \) on \( \mathcal{O}/X \) is
induced by pulling back the same, hence by Fr_{U/X_0}^{-1} for U \to X \text{ \acute{e}tale}, in particular we have that F^* on H^3_c(X, \mathcal{F}) is induced by \mathcal{F}(Fr_{X/X_0})^{-1} = \mathcal{F}(q^{-1} \times_{F_q} id_{X_0}).

3.1. \textbf{Le sorite de la notation.} It is very important to note that in Deligne’s notation, Tr(F_x^*, \mathcal{F}) and Tr(F^*, \mathcal{F}_X) are traces of possibly different operators on the fiber \mathcal{F}_x. Namely, if \mathcal{F} is a \mathbb{Q}_\ell\text{-sheaf on } X_0 \text{ a scheme separated and of finite type over } F_q, \text{ then } \mathcal{F}_x \text{ is a } \mathbb{Q}_\ell \text{ vector space, for a choice of geometric point } x \text{ centered on a closed point } x \text{ of } X_0. \text{ Then } F_x^* \text{ denotes the Frobenius } F_{q^n}^* = F_{q^n} \text{ raised to the power of the residue field extension } n = [\text{deg } k(x) : F_q]. \text{ This power of Frobenius is the least that fixes each geometric point centered on } x, \text{ and the notation Tr}(F_x^*, \mathcal{F}) \text{ means Tr}(F_x^*, \mathcal{F}_x).

On the other hand, if, say, } x \in X_0^{F^n} \text{ is a geometric point centered on a point of } X_0 \text{ defined over } F_{q^n}, \text{ Tr}(F^*, \mathcal{F}_x) \text{ denotes (absolute) } q^n \text{-power Frobenius acting on the fiber}. \text{ So, unless } x \in X^F, \text{ Tr}(F^*, \mathcal{F}_x) \text{ and Tr}(F_x^*, \mathcal{F}) \text{ are traces of different operators on the same vector space, the latter a power of the other.

3.2. \textbf{Le sorite des faisceaux localement constants.}

The case of locally constant sheaves of sets. Let } X \text{ be a scheme and } \mathcal{L} \text{ a locally constant sheaf of sets with finite fibers on } X. \text{ (With additional assumptions on } X, \text{ the case of a locally constant sheaf of sets with infinite fibers is reduced to the finite case in the course of the discussion of Weil II 1.7.8.) We know that } \mathcal{L} = h_U \text{ for some } U \to X \text{ rev\-\text{\c{e}t}\-\text{\e}t\-\text{\a}le. We know that every rev\-\text{\c{e}t}\-\text{\e}t\-\text{\a}le of } X \text{ is \acute{e}tale\-locally on } X \text{ trivial; namely for some } V \to X \text{ \acute{e}tale, } U_V \sim \bigsqcup V. \text{ We wish to show that we may take } V \to X \text{ to be a rev\-\text{\c{e}t}\-\text{\e}t\-\text{\a}le \text{ (with no further work, we could then take it to be a galoisian rev\-\text{\c{e}t}\-\text{\e}t\-\text{\a}le, i.e. a connected torsor for the automorphism group of the fiber, as principal Galois objects in a Galois category form a cofinal system).}

First note that if } f : X \to Y \text{ is any morphism of schemes and } V \to Y \text{ is \acute{e}tale, then } f^*h_V \sim h_{V \times_Y X}. \text{ To see this, observe}

\[ \text{Hom}(f^*h_V, \mathcal{G}) = \text{Hom}(h_V, f_*\mathcal{G}) = f_*\mathcal{G}(V) = \mathcal{G}(V \times_Y X) = \text{Hom}(h_{V \times_Y X}, \mathcal{G}). \]
Evidently, this argument holds true for any morphism of sites.

So, it will suffice to show that any revêtement étale can be trivialized after base extension by a revêtement étale. To see this, assume $X$ connected and let $U \to X$ be a revêtement étale of constant degree $d$ and proceed by recurrence on $d$, the case $d = 1$ being trivial. (Of course, in the special case that $U \to X$ is galoisian with Galois group $G$, $U \times_X U \sim U \times G$ is a trivial $G$-torsor, and we are done.)

As $U \to X$ is étale, hence net, and finite, the diagonal morphism $U \to U \times_X U$ is simultaneously an open and closed immersion, hence an isomorphism onto a connected component of $U \times_X U$, allowing us to write $U \times_X U = U \bigsqcup Z$ with $Z \to U$ of constant degree $d - 1$. By hypothesis, there exists a revêtement étale $V \to U$ such that $Z \times_U V \sim \bigsqcup_{d-1} V$. Our desired revêtement is then simply the composition $V \to U \to X$:

$$V \times_X U = V \times_U U \times_X U = V \times_U (U \bigsqcup Z) = V \bigsqcup (V \times_U Z) = \bigsqcup_{d} V.$$

**The case of locally constant constructible sheaves.** Let $\Lambda$ be a commutative, noetherian torsion ring. We adapt the above discussion to locally constant constructible (l.c.c.) sheaves of $\Lambda$-modules. Let $\mathcal{F}$ be a l.c.c. sheaf of $\Lambda$-modules on a connected scheme $X$. Then the fibers of $\mathcal{F}$ are finite sets and the above discussion yields a revêtement étale $f : V \to X$ with $V$ a (connected) galoisian cover with Galois group $H$ (i.e. $V$ is a $H$-torsor) such that $f^* \mathcal{F}$ is a constant sheaf. Its constant value $H^0(V, f^* \mathcal{F})$ is a $\Lambda[H]$-module.

The sheaf $f_* \Lambda$ on $X$, together with the natural action of $H$, is a rank 1 l.c.c. sheaf of $\Lambda[H]$-modules. Relative to the natural action of $H$ on $f_* f^* \mathcal{F}$, the trace morphism $f_* f^* \mathcal{F} \to \mathcal{F}$ factors by an isomorphism

$$(f_* f^* \mathcal{F})_H \to \mathcal{F},$$

and we have $f_* f^* \mathcal{F} = f_* M = f_* \Lambda \otimes_{\Lambda} M$ with the diagonal action of $H$.

The above discussion shows that a l.c.c. sheaf of $\Lambda$-modules on a connected scheme $X$ is determined by its restriction to the small étale site of $X$. Sheaves on the small étale site $\mathcal{U}$ are in turn determined by Grothendieck’s Galois theory: fixing a geometric
point $\overline{x}$ of $X$ and putting $G := \pi_1(X, \overline{x})$, the functor

$$\text{Sh}(\mathcal{U}) \to \text{finite } G\text{-sets}$$

$$\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$$

admits the inverse

$$\text{finite } G\text{-sets} \to \text{Sh}(\mathcal{U})$$

$$\mathcal{F}_{\overline{x}} \mapsto [V \in \mathcal{U} \mapsto \text{Hom}_G(V_{\overline{x}}, \mathcal{F}_{\overline{x}})].$$

To verify this, as the torsors are cofinal in a covering of any $V \in \mathcal{U}$, we may cover $V$ by a torsor $W$ with Galois group $H$ and combine the equalizer description of $\mathcal{F}(V)$

$$\mathcal{F}(V) \to \mathcal{F}(W) \xrightarrow{\epsilon} \mathcal{F}(W \times_V W)$$

with the description of such in the case of a Galois torsor; c.f. [SGA 4 1/2, I §5].

The discussion in the previous section can be rephrased using the monodromy representation of a l.c.c. sheaf. Namely, let $\mathcal{F}$ be a l.c.c. sheaf of $\Lambda$-modules on a connected scheme $X$ pointed by a geometric point $\overline{x}$ as above; $\mathcal{F}$ corresponds to a representation $\pi_1(X, \overline{x}) \to \text{GL}(\mathcal{F}_{\overline{x}})$. As the latter is a finite group, the kernel of this representation is of finite index, and as the Galois coverings are cofinal, we can find a Galois cover of $X$ corresponding to a open subgroup contained in the kernel. The sheaf $\mathcal{F}$ becomes constant when restricted to this cover.

**The case of lisse sheaves.** Let $E \subset \overline{\mathbf{Q}}_\ell$ be an finite extension of $\mathbf{Q}_\ell$ with valuation ring $R$, integral closure of $\mathbf{Z}_\ell$ in $E$, $m$ the maximal ideal of $R$. Every $\overline{\mathbf{Q}}_\ell$-sheaf $\mathcal{F}$ is obtained as $\mathcal{F} \otimes_E \overline{\mathbf{Q}}_\ell$ for some $E$ and some torsion-free (i.e. flat) constructible $E$-sheaf $\mathcal{F}$. This means that $\mathcal{F} = \text{``lim proj''} \mathcal{F}_n$, the latter a flat $R$-sheaf. A lisse $R$-sheaf has all the $\mathcal{F}_n$ locally constant sheaves of $R/m^n$-modules, and for each $n$, the above discussion shows that the functor ‘fiber at $\overline{x}$’ gives an equivalence of categories between the category of lisse $R/m^n$-sheaves and the category of $R/m^n$-modules of finite type together with a continuous action of $\pi_1(X, \overline{x})$. Since the $\mathcal{F}_n$ have $\mathcal{F}_n \otimes_{R/m^{n+1}} R/m^n \simto \mathcal{F}_{n-1}$, by passing to the limit we get an equivalence between the category of lisse $R$-sheaves and the category of finite $R$-modules with continuous action of $\pi_1(X, \overline{x})$. 
REFERENCES

[EGA] Eléments de géométrie algébrique par A. Grothendieck.
[SGA 4\frac{1}{2}] SGA 4\frac{1}{2}, Rapport sur la formule des traces par P. Deligne.
2. Notes on quasi-unipotent monodromy

Some notes about Grothendieck’s theorem on quasi-unipotent monodromy. We study the arithmetic proof. It uses a proposition proved by Grothendieck in the appendix of Serre and Tate’s article *Good Reduction of Abelian Varieties*.

We may assume that $K$ is complete since, following Serre, *Corps Locaux*, II§3 Cor. 4, completing $K$ leaves the decomposition unchanged. Now, we may assume that any matrix in the image of $\rho$ has coefficients in $\mathbb{Z}_l$ and is congruent to $1 \mod l^2$ as these are both open conditions, $\rho$ is continuous, and we are free to pass to an open subgroup of $I(\overline{\nu})$ by making a finite extension of $K$.

Note also that $\text{im} \rho$ is a pro-$l$ group since, while $\text{GL}(n, \mathbb{Z}_l)$ is not a pro-$l$-group, its first congruence subgroup of matrices congruent to $1 \mod l$ is a pro-$l$-group (c.f., e.g., §5.1 of *Analytic Pro-$p$ Groups* by Dixon, du Sautoy, Mann & Segal). We see therefore that the prime-to-$l$ part of the order of $\text{GL}(n, \mathbb{Z}_l)$ is finite. As the image of a pro-$p$ group under a continuous homomorphism is pro-$p$, the continuous image of a pro-$p$ group in $\text{GL}(n, \mathbb{Z}_l)$ is finite. As $\text{im} \rho$ is by construction pro-$l$, the image of a pro-$p$ group in $\text{im} \rho$ is $\{1\}$.

Now, if $L$ is a finite extension of $K_f$, we wish to show that the polynomial $f(T) = T^l - a$ splits in $L$ for any $a \in L$. If it does not, then as $f$ is separable and $L$ contains all $l^{\text{th}}$ roots of unity, $L(\sqrt[l]{a})$ is the splitting field of $f$ and is Galois. The automorphism of $L(\sqrt[l]{a})/L$ sending $\sqrt[l]{a} \mapsto \zeta_l \sqrt[l]{a}$, where $\zeta_l$ is a primitive $l^{\text{th}}$ root of unity, acts transitively on the roots of $f$, hence $f$ is irreducible. But $K_f$ is the $l$-part of the maximal tamely ramified extension of $K_{nr}$, hence $l$ cannot divide $[L : K_f]$. (Recall that $K_f$ is the maximal tamely ramified extension of $K$, and we have

$$\text{Gal}(K_f/K_{nr}) \simeq \prod_{q \neq p} \mathbb{Z}_q \quad \text{Gal}(K_{l}/K_f) \simeq \prod_{q \neq p, l} \mathbb{Z}_q \quad \text{Gal}(K_{s}/K_f) \text{ a pro-$p$ group}$$

as $q$ runs over primes, so $\text{Gal}(K_s/K_f)$ is an extension of a group isomorphic to $\prod_{q \neq p, l} \mathbb{Z}_q$ by a pro-$p$ group, and therefore has no finite quotient of order divisible by $l$.) This
allows one to conclude that $l$ does not divide the order of $\text{Gal}(K_t/K_I)$. The order of $\text{im}\,\rho$ is a power of $l$ as it is a pro-$l$ group.

An alternative way to see that $l$ does not divide the order of $\text{Gal}(K_s/K_I)$ that is more faithful to the original proof proceeds by showing directly that for a finite extension $L/K_I$, every element of $L$ is an $l^{th}$ power. To do this, let $L = K_t[t]/a(t)$ for $a(t)$ an irreducible separable polynomial $a(t) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$, and suppose $t$ is not an $l^{th}$ power in $L$. This implies that the polynomial $a_l(t) = a_n x^{ln} + a_{n-1} x^{(n-1)l} + \ldots + a_0$ is irreducible and separable over $K_I$. Let $K'/K$ be a finite Galois extension containing all $l^{th}$ roots of unity and the $a_i$, and contained in $K_t$. The extension $K'[t]/a_l(t)$ is finite and separable and is contained in a finite Galois extension $K''$ of $K'$ with $l$ dividing $[K'' : K']$, so $l$ divides the ramification index or the residual degree. If the latter, making a finite unramified extension of $K'$ produces a contradiction on the irreducibility of $a_l(t)$ over $K_I$. If the former, replacing $K''$ by $K'[\pi^{1/l}] \subset K_I$ similarly yields a contradiction. Now to see that every finite Galois extension $L$ of $K_I$ cannot have $l$ dividing its degree, note that $\text{Gal}(L/K_I)$ contains a cyclic subgroup $H$ of order $l$, and we claim $L = L^H(a^{1/l})$ for some element $a \in L^H$. Let $\sigma$ generate $H$, and let $b \in L - L^H$. Then the element

$$c = \sum_{m=1}^{l} \zeta_l^m \sigma^m(b)$$

satisfies $c = \zeta_l \sigma(c)$, where $\zeta_l$ is a primitive $l^{th}$ root of unity. So

$$c^l = \prod_{m=1}^{l} \sigma^m(c) \in L^H,$$

and letting $a = c^l$ we find that $L^H(a^{1/l})$ is a nontrivial subextension of $L^H$, hence must actually coincide with $L$.

**References**

3. Notes on Weil II

0.5. Let $X$ be a scheme of finite type over a field $k$. If $X$ is connected, the structure morphism $X \to \text{Spec}(k)$ admits a unique factorization $X \to \text{Spec}(k') \to \text{Spec}(k)$ with $k'/k$ finite separable and $X \to \text{Spec}(k')$ geometrically connected. $\sim \to$ Stacks, tag $04PZ$. Proof uses notion of ‘weakly étale $k$-algebra.’

1.1.2. First of all, to see that if $K \in D^b_c(X, R)$, then $K \otimes^L R/m^n \in D^b_{ctf}(X, R/m^n)$, see Stacks, tag $0942$. Note that $K \otimes^L R/m^n$ can be represented by a bounded complex of flat constructible sheaves by Rapport, 4.7. Also recall that the locally constant sheaves form a weak Serre subcategory of the constructible sheaves on a site ($093U$).

Claim a). On the subject of the category $D^b_c(X, R)$, claim a) is that for each $i$, the projective system of cohomology sheaves $\mathcal{H}^i(K) := \text{“lim proj” } \mathcal{H}^i(K \otimes^L R/m^n)$ of a complex $K$ in $D^b_c(X, R)$ is an $R$-constructible sheaf. First a trivial statement: of course the reduction modulo $m^n$ of a complex of flat sheaves representing $K_{n+1}$ induces a map on cohomology, but a priori it need not induce an isomorphism $\mathcal{H}^i(K_{n+1}) \otimes R/m^n \to \mathcal{H}^i(K_n)$. For example, in

$$
\begin{array}{c}
\begin{array}{c}
0 \\ \downarrow \\
0
\end{array} & \begin{array}{c}
\longrightarrow \\
\downarrow \\
\longrightarrow
\end{array} & \begin{array}{c}
\mathbb{Z}/\ell^n \\ \downarrow \\
\mathbb{Z}/\ell^{n-1}
\end{array} & \begin{array}{c}
\ell^{n-1} \\ \downarrow \\
0
\end{array} & \begin{array}{c}
\longrightarrow \\
\downarrow \\
\longrightarrow
\end{array} & \begin{array}{c}
\mathbb{Z}/\ell^n \\ \downarrow \\
\mathbb{Z}/\ell^{n-1}
\end{array} & \begin{array}{c}
\longrightarrow \\
\downarrow \\
0
\end{array}
\end{array}
$$

the first nonzero cohomology in the top row is $\ell\mathbb{Z}/\ell^n \simeq \mathbb{Z}/\ell^{n-1}$, which gets mapped by the down arrow to $\ell\mathbb{Z}/\ell^{n-1} \simeq \mathbb{Z}/\ell^{n-2}$, even though the first nonzero cohomology in the bottom row is all of $\mathbb{Z}/\ell^{n-1}$. Moreover, $\mathbb{Z}/\ell^{n-1} \sim \mathbb{Z}/\ell^{n-1} \otimes_{\mathbb{Z}/\ell^n} \mathbb{Z}/\ell^{n-1}$, so the map on cohomology after reduction mod $\ell^{n-1}$ is neither injective nor surjective.

It is important to note that in (1.1.1), the term ‘$R$-faisceau constructible’ is used to describe all pro-sheaves in the essential image of the functor $\mathcal{F} \mapsto \text{“lim proj” } \mathcal{F} \otimes R/m^n$. In Exposé V of SGA 5, Jouanolou studies $J$-adic projective systems, where $J$ is an ideal in a commutative ring $A$. All references in this paragraph will be to this exposé unless
indicated otherwise. There is a conflict of indexing, in that Deligne in Weil II has $K_n$ annihilated by $m^n$, while Grothendieck, Jouanolou and Deligne in SGA 4\frac{1}{2} have $K_n$ annihilated by $m^{n+1}$. As a predictable but no less unfortunate result, both conventions are effectively in force in different parts of these notes. Jouanolou begins with an abelian category $C$ and forms $P = \frac{\text{Hom}(\mathbb{N}^\circ, C)}{\text{Hom}(\mathbb{N}^\circ, C)}$, the category of projective systems indeed by the ordered set $\mathbb{N}$ of positive integers with values in $C$. Given an object $X = (X_n, u_n)_{n \geq 0}$ of $P$ and an integer $r \geq 0$, $X[r]$ denotes the projective system $(X_{n+r}, u_{n+r})_{n \geq 0}$. If $r, s$ are integers satisfying $s \geq r \geq 0$, then iterated application of the transition morphisms $u$ define a morphism $w_{sr} : X[s] \to X[r]$. For each integer $r \geq 0$ and $X$ in $P$, the morphism $w_{rt} : X[r] \to X$ is also denoted $V_{rX}$. With this notation, $X$ is said to satisfy the condition Mittag-Leffler-Artin-Rees (MLAR) if there exists an integer $r$ such that for each integer $s \geq r$,

$$\text{im}(X[s] \xrightarrow{w_{s0}} X) = \text{im}(X[r] \xrightarrow{w_{r0}} X).$$

In particular, if there exists an $r \geq 0$ such that the canonical morphism $X[r] \to X$ is null, $X$ is said to be AR-null. The full subcategory $P_0$ of $P$ whose objects are the AR-null projective systems is thick. The quotient category is called the category of projective systems in $C$ up to translation and notated $\frac{\text{Hom}(\mathbb{N}^\circ, C)}{\text{Hom}(\mathbb{N}^\circ, C)}$ or $P_{AR}$. It is abelian and the quotient functor $p_{AR} : P \to P_{AR}$ is exact (2.4.4). Equivalently, $P_{AR}$ is obtained from $P$ by a (left or right) calculus of fractions with respect to the set of morphisms $\{V_{rX}\}$, as $X$ runs over the set of objects in $P$ and $r$ over the set of integers $\geq 0$. Denote this set by AR. Now suppose $p$ equips $C$ with the structure of an $A$-category (1.1) and either that $J$ is of finite type or that $C$ possesses infinite direct sums.

**Definition** (3.1.1). An object $X$ of $P$ is called J-adic if the following two conditions are verified.

(i) For every integer $n \geq 0$, $J^{n+1}X_n = 0$.

(ii) For every couple $(m, n)$ of integers with $m \geq n \geq 0$, the morphism $A/J^{n+1} \otimes_A X_m \to X_n$ deduced from the transition morphism $X_m \to X_n$ is an isomorphism.

If moreover the components of $X$ are noetherian objects of $C$, then $X$ is called noetherian J-adic.
The full subcategory of P (resp. of $\mathcal{P}_{AR}$) generated by the J-adic projective systems (resp. the images of the noetherian J-adic systems) is notated $\text{J–ad}(C)$ (resp. $\text{J–adn}(C)$). Two objects $X$ and $Y$ of the category $P$ are said to be AR-isomorphic if $p_{AR}(X)$ and $p_{AR}(Y)$ are isomorphic in the category $\mathcal{P}_{AR}$.

**Definition (3.2.1-3.2.2).** An object $X$ of $P$ is called AR-J-adic if it satisfies the following conditions.

(i) $J^{n+1}X_n = 0$ for all $n \geq 0$.
(ii) There exists a J-adic projective system $Y$ isomorphic to $X$ in the category $\mathcal{P}_{AR}$.

If moreover the components of $X$ are noetherian objects of $C$, then $X$ is called noetherian AR-J-adic.

The full subcategory of $P$ (resp. of $\mathcal{P}_{AR}$) generated by the noetherian AR-J-adic projective systems (resp. by their images) is denoted $\mathcal{E}_C$ (resp. AR-J-adn($C$)).

**Proposition (3.2.3).** Let $X$ be in $P$. Suppose $J^{n+1}X_n = 0$ for all $n \geq 0$. In order for $X$ to be AR-J-adic, it is necessary and sufficient that it verify the property (MLAR) and that, denoting by $X'$ its projective system of universal images, there exist an integer $r \geq 0$ such that, for each pair $(m,n)$ of integers with $m \geq n + r$, the 'transition morphism' below be an isomorphism:

$$X'_m/J^{n+1}X'_m \longrightarrow X'_{n+r}/J^{n+1}X'_{n+r}.$$

Note that if $X$ verifies (MLAR), it is AR-isomorphic to its projective system of universal images. The hypothesis made on $X'$ implies that the projective system $(X'_{n+r}/J^{n+1}X'_{n+r})_{n \geq 0}$ is J-adic. This projective system is AR-isomorphic to $X'$. This proves sufficiency.

The restriction of the functor $p_{AR}$ to J–ad or J–adn induces an equivalence

$$p^n_{AR} : \text{J–adn}(C) \longrightarrow \text{AR–J–adn}(C).$$

Suppose $X, Y$ are noetherian J-adic projective systems. Then a morphism $X \rightarrow Y$ is represented for a certain integer $r$ by a morphism $X[r] \rightarrow Y$. As $J^{n+1}Y_n = 0$ for all
\( n \geq 0 \) and \( X \) is \( J \)-adic, this morphism is the composition of a morphism \( X \to Y \) with \( X[r] \to X \). Hence \( p^n_{AR} \) is full. Moreover, a given morphism \( X \to X \) goes to zero under \( p^n_{AR} \) if it goes to zero in the inductive limit
\[
\lim_{\to r} \text{Hom}(X[r], Y);
\]
i.e. if precomposition by \( V_{rX} : X[r] \to X \) is null for some \( r \). Such is not the case when \( X \) is \( J \)-adic. Thus \( p^n_{AR} \) is faithful, and, as \( AR-J\text{-adn}(C) \) is evidently the essential image, an equivalence.

**Proposition** (5.2.1). The category \( \mathcal{E}_C \) is stable by kernels and cokernels in \( P \). In other words, \( \mathcal{E}_C \) is an abelian category and the inclusion functor \( \mathcal{E}_C \hookrightarrow P \) is exact.

**Theorem** (5.2.3). The categories \( J\text{-adn}(C) \) and \( AR-J\text{-adn}(C) \) are abelian and noetherian.

We have enough to prove the first statement. (5.2.1) implies on the spot that the category \( AR-J\text{-adn}(C) \) is abelian: given an arrow \( A \to B \) in \( AR-J\text{-adn}(C) \), up to isomorphism of \( A \) and \( B \) this arrow comes from an arrow in \( P \) with kernel and cokernel in \( \mathcal{E}_C \); as the functors \( \mathcal{E}_C \to P \to P_{AR} \) are exact, the kernel and cokernel lie in \( AR-J\text{-adn}(C) \). Therefore \( J\text{-adn}(C) \) must also be abelian as the two categories are equivalent.

**Remark.** If \( A \) is a noetherian (commutative) ring complete and separated with respect to an ideal \( J \) such that \( A/J \) is artinian, then the following is true and provides a kind of ‘spiritual underpinning’ for the category \( J\text{-adn} \).

**Proposition.** The functor \( \lim \) induces an equivalence between the categories \( AR-J\text{-adn}(A\text{-mod}) \) and the category of finite \( A \)-modules.

Specializing to the category \( \text{Ab}(X) \) of abelian sheaves on \( X_{\text{et}} \), \( X \) a scheme, when \( X \) is noetherian, the abelian noetherian sheaves are the abelian constructible sheaves [SGAA, IX 2.9]. Let \( \text{Abc}(X) \) denote the category of abelian constructible sheaves and \( \ell\text{-adc}(X) \) the full subcategory of \( \text{Hom}(N^\infty, \text{Abc}(X)) \) generated by the constructible \( \ell\)-adic sheaves. Then we have shown that, when \( X \) is noetherian,

\[
\ell\text{-adc}(X) = (\ell\mathbb{Z})\text{-ad}(\text{Abc}(X)) = (\ell\mathbb{Z})\text{-adn}(\text{Ab}(X)),
\]
the first equality holding with no assumptions on \( X \). In \( P \), given an exact sequence
\[ 0 \to X \to Y \to Z \to 0 \]
with \( X \) strict and \( Y \)-adic, \( Z \) is \( J \)-adic. Let \( u : \mathcal{F} \to \mathcal{G} \) be a morphism of \( \ell \)-adic sheaves on a scheme \( X \). We apply the above formalism in the case
\( A = R, J = m, C = \text{Ad}(X), P = \text{Hom}(N^o, \text{Ad}(X)) \). Let \( \mathcal{F}' := \text{im} \ u \) and \( \mathcal{G}' := \text{coker} \ u \) computed in the category \( P \). Then the following diagram has exact rows and commutes.
\[
\begin{array}{ccccccccc}
R/m^{n+1} \otimes_R \mathcal{F}_m & \xrightarrow{id \otimes u_m} & R/m^{n+1} \otimes_R \mathcal{G}_m & \to & R/m^{n+1} \otimes_R \mathcal{G}_m' & \to & 0 \\
\downarrow f_{nm} & & \downarrow g_{nm} & & \downarrow h_{nm} & & \\
0 & \to & \mathcal{F}_n' & \to & \mathcal{G}_n & \to & \mathcal{G}_n' & \to & 0
\end{array}
\]

As \( f_{nm} \) is an epimorphism and \( g_{nm} \) an isomorphism, the snake lemma implies that \( h_{nm} \) is an isomorphism; therefore \( \mathcal{G}' \) is a cokernel of \( u \) in \( m \)-adc(\( X \)). On a noetherian \( X \), we have the following simple description of \( \ker \ u \). Let \( \mathcal{H} := \ker \ u \) in \( P \); \( \mathcal{H} \) is AR-\( m \)-adic.

Denoting by \( \mathcal{H}' \) the system of universal images of \( \mathcal{H} \), grâce à (3.2.3) there exists an integer \( r \) such that the projective system
\[
m_r(\mathcal{H}') := \left( \mathcal{H}'_{n+r}/\ell^{n+1} \mathcal{H}'_{n+r} \right)_{n \in \mathbb{N}}
\]
is \( m \)-adic constructible. The composition below is a kernel of \( u \) in \( m \)-adc(\( X \)):
\[
m_r(\mathcal{H}') \xrightarrow{} \mathcal{H}' \xrightarrow{} \mathcal{H}.
\]
Returning to the setting of Deligne’s article, note that the AR-null sheaves become null objects in the category of pro-sheaves on \( X \), as morphisms in that category between two objects \( \underline{X} := (X_i)_{i \in I}, \underline{Y} := (Y_j)_{j \in J} \) indexed by sets \( I, J \) both equipped with filtered preorders are given by
\[
\text{Pro Hom}(\underline{X}, \underline{Y}) := \varprojlim \lim_{i} \text{Hom}(X_i, Y_j),
\]
and if \( \underline{X} \) is an AR-null object of \( P \), it is clear that the identity morphism goes to zero in \( \text{Pro Hom}(\underline{X}, \underline{X}) \), so by the universal property of \( P_{AR} \), the AR-\( m \)-adn(\( \text{Ab}(X) \)) sheaves are in the essential image of the \( m \)-adic constructible sheaves in the category of pro-sheaves.

Returning to claim a), the reduction to the punctual case requires a few words. The equivalence between the categories we have seen above specializes to an equivalence
$\mathcal{Z}_t \to \text{fc} \to (\text{AR}, \mathcal{Z}_t) \to \text{fc}$, where the latter is really the category AR-$m$-ad($\text{Abc}(X)) = \text{AR-}
line m$-adn($\text{Ab}(X)$) [SGA5, VI, 1.5.5], so given the projective system $(\mathcal{K}_n)$ where $\mathcal{K}_n = H^i(K \otimes^L R/m^n)$, letting $j : U \to X$ be an open over which $\mathcal{K}_0$ is locally constant, $i : X - U \to X$ the inclusion of the complement, by the stability of the property AR-$m$-adic in short exact sequences [SGA5, V, 3.2.4], in order to conclude that $\mathcal{K}_n$ is an (AR,R)-constructible sheaf suffices to show that the projective systems $(j^*j_! \mathcal{K}_n)$ and $(i^*i_! \mathcal{K}_n)$ are (AR,R)-constructible. This allows us to reduce to the situation where $(\mathcal{K}_n)$ is a projective system of locally constant constructible sheaves, since the functors $i_+, j_!$ send $m$-adic sheaves to $m$-adic sheaves and AR-null sheaves to AR-null sheaves (this can be checked pointwise), now use [SGA5, V, 2.4.5]. By ‘gluing’ (c.f. proof of [SGA5, VI, 1.5.5]), we can reduce to proving over an open cover trivializing $\mathcal{K}_0$. More concretely, we can cover our space by finitely many opens over which $\mathcal{K}_0$ is constant, and if we can show that $\mathcal{K}_0$ is AR-$m$-adic over each, then the construction (3.2.3) above allows us to find an integer for each open in our cover; taking the maximum $r$ of these, replacing $(\mathcal{K}_n)$ by its system of universal images $(\mathcal{K}_n')$, and forming $(\mathcal{K}_n'/m^{n+1} \mathcal{K}_n')_{n \in \mathbb{N}}$, we have produced an R-constructible sheaf which is AR-isomorphic to the system $(\mathcal{K}_n)$.

In the punctual case, the results of [SGA5, XV p.473] allow us to suppose that we have a projective system $(K_r)_{r \in \mathbb{N}}$, where $K_r$ is a complex of free $R/m^r$-modules of finite type, null outside of an interval $[a, b]$ independent of $r$, with transition morphisms $K_{r+1} \to K_r$ isomorphic (as morphisms of complexes) to $K_{r+1} \to K_{r+1}/m^{r+1}K_{r+1}$, for all $r \in \mathbb{N}$. We will show that the projective system of cohomology “$\lim \text{proj}$” $H^i(K_r)$ is AR-$m$-adic, which will imply that it is an R-constructible sheaf in Deligne’s sense; i.e. that it is isomorphic, as a pro-object, to a bona fide R-constructible sheaf. Abusively, put $K := \lim \limits_{r \in \mathbb{N}} K_r$ (before, K refers to the stalk of an object in $D_b^c(X, R)$). A key ingredient is EGA 0III 13.2.3, which says that if a projective system of complexes such as $(K_r)$ satisfies the Mittag-Leffler condition, and if the projective system $(H^{i-1}(K_r))$ does too, then the canonical map $H^i(K) \to \lim \limits_{r \in \mathbb{N}} H^i(K_r)$ is bijective. As in our situation, the $K_r$ are complexes of finite groups and their cohomology modules $H^i(K_r)$ are also finite groups, the Mittag-Leffler condition is automatic. Our hypotheses on the complexes $K_r$ imply the existence of isomorphisms $K_{r+s} \otimes R/m^{r+s} \to K_s$ and the exactness of the
sequence

$$0 \rightarrow K_{r+s} \otimes_{R/m^r+s} R/m^r \rightarrow K_{r+s} \otimes_{R/m^r+s} R/m^r \rightarrow 0$$

for $1 \leq r, s$. The projective limit as $s \rightarrow \infty$ of the associated cohomology sequence is still exact by the fact that everything is still finite (ML). This projective limit can be broken up into short exact sequences

$$(*) \quad 0 \rightarrow H^i(K)/m^r \rightarrow H^i(K_r) \rightarrow H^{i+1}(K)[m^r] \rightarrow 0.$$ 

Note that $\text{Tor}_1^R(M, R/m^r R) \cong M[m^r]$ for $M$ an $R$-module. Since the modules $H^{i+1}(K)[m^r]$ stabilize as $r \gg 0$ and the transition morphisms on $(\text{Tor}_1^R(H^{i+1}(K), R/m^r))_{r \in \mathbb{N}}$ are multiplication by $m$, this projective system is evidently AR-null. As the projective system $(H^i(K)/m^r)_{r \in \mathbb{N}}$ is evidently $m$-adic, $(H^i(K_r))_{r \in \mathbb{N}}$ is AR-$m$-adic, and hence its image in the category of pro-sheaves “$\lim proj$” $H^i(K_r)$ is isomorphic to the image of the $m$-adic system “$\lim proj$” $H^i(K)/m^r$, which shows the former is $m$-adic in Deligne’s sense.

Remark. To see that each transition morphism on the projective system

$$(\text{Tor}_1^R(H^{i+1}(K), R/m^r))_{r \in \mathbb{N}}$$

is multiplication by $m$, note that by the equivalence $K^-(\mathcal{P}) \rightarrow D^-(R\text{-mod})$, where $K^-(\mathcal{P})$ is the full triangulated subcategory of $K^-(R\text{-mod})$ generated by the complexes with projective objects in all degrees, there is, up to homotopy, a unique map of projective resolutions of $R/m^r$ and $R/m^{r-1}$ inducing the desired map $R/m^r \rightarrow R/m^{r-1}$, namely

$$
\begin{align*}
0 & \rightarrow R \xrightarrow{m^r} R \xrightarrow{id} 0 \\
0 & \rightarrow R \xrightarrow{m^{r-1}} R \xrightarrow{id} 0.
\end{align*}
$$

Tensoring by $H^{i+1}(K)$ we find that the map on $\text{Tor}_1^R$ is indeed multiplication by $m$.

Claim c). Let $f : Y \rightarrow X$ be an arrow between schemes of finite type over $S$ with $S$ regular of dimension $\leq 1$. Claim c) is that about the categories $D^b_{ctf}(X, R/m^n)$ being stable by the six functors, and that these functors also commute with reduction modulo $m^n$. This can be broken into 3 claims about the six functors: (1) that they preserve constructibility, (2) that they preserve finite tor-dimension, and (3) that they commute
with reduction modulo $m^n$. The discussion of Th. finitude 1.1, 1.5–1.7 covers claims 1 & 2 for the four functors $Rf^*, f^*, Rf_1, f^1$, as well as $R\text{Hom}$. Claim 3 for $R\text{Hom}$ is discussed below. The case of $\otimes^L$ is more or less trivial. Claim 3 for the four functors is the presence of isomorphisms like $R/m^n \otimes^L Rf_i(K \otimes^L R/m^{n+1}) \xrightarrow{\sim} Rf_i(K \otimes^L R/m^n)$ in $D^b_{ctf}(X, R/m^n)$. Because of the finitude hypotheses (c.f. note to Th. finitude 1.3), we can apply the recipe of Rapport 4.12 to construct the arrow in $D^b_{ctf}(X, R/m^n)$ in the cases of $Rf_*, Rf_1$. To see the arrow is an isomorphism, we can then copy the reasoning of Rapport 4.9.1. The case of $f^*$ is trivial, since after replacing $K \otimes^L R/m^{n+1}$ by a bounded complex of flat sheaves $M$, we have isomorphisms

$$f^*(K \otimes^L R/m^{n+1}) \otimes^L R/m^n \xrightarrow{\sim} f^*(M) \otimes R/m^n \xrightarrow{\sim} f^*(M \otimes R/m^n) \xrightarrow{\sim} f^*(K \otimes^L R/m^n),$$

as tensor product commutes with inductive limits. As for $Rf^1$, to simplify the notation, let $K_{n+1}$ denote $K \otimes^L R/m^{n+1}$. We obtain an arrow

$$R/m^n \otimes^L Rf^1(K_{n+1}) \rightarrow Rf^1(K_{n+1} \otimes^L R/m^n)$$

in $D^+(Y, R/m^n)$ from the adjunction

$$\text{Hom}_{D^+(X, R/m^n)}(Rf_1 L, M) \simeq \text{Hom}_{D^+(Y, R/m^n)}(L, Rf^1 M)$$

with $L = R/m^n \otimes^L Rf^1(K_{n+1})$ and $M = K_{n+1} \otimes^L R/m^n$ in the following way:

$$R/m^n \otimes^L Rf^1 K_{n+1} \rightarrow Rf^1 Rf_i(R/m^n \otimes^L Rf^1 K_{n+1}) \xrightarrow{\sim} Rf^1(R/m^n \otimes^L Rf Rf^1 K_{n+1}) \rightarrow Rf^1(R/m^n \otimes^L K_{n+1}).$$

As claims 1 & 2 have been verified, this actually yields an arrow in $D^b_{ctf}(Y, R/m^n)$. We can localize this morphism with respect to $u : U \rightarrow Y$ étale, as $R\text{u}^* = u^*$, which we know commutes with reduction modulo $m^n$, and in this way, replacing $Y$ by $U$, assume $f$ factors as $U \xrightarrow{i} Z \xrightarrow{h} X$ with $i$ a closed immersion and $h$ smooth of relative dimension $d$. As the above morphisms are natural, and the composition of the unit and counit $Rf^1 \rightarrow Rf^1 Rf Rf^1 \rightarrow Rf^1$ is the identity transformation of $Rf^1$, and $Rf^1 = Rh^1 Ri^1$ (since the left adjoint $i_*$ of $i^!$ is exact), it suffices to show that $Rh^1$ and $Ri^1$ commute with reduction modulo $m^n$. The case of $Rh^1$ is trivial since $Rh^1 K_{n+1} = K_{n+1}(d)[2d]$. Turning to $Ri^1$, let $j : Z - U \hookrightarrow X$ denote the open immersion
of the complement of \( \mathbb{U} \); we have a commutative diagram with distinguished triangles for rows

\[
R/m^n \otimes^L i_* \mathcal{R}^i K_{n+1} \longrightarrow R/m^n \otimes^L K_{n+1} \longrightarrow R/m^n \otimes^L Rj_* j^* K_{n+1} \overset{d}{\longrightarrow}
\]

\[
i_* \mathcal{R}^i (K_{n+1} \otimes^L R/m^n) \longrightarrow R/m^n \otimes^L K_{n+1} \longrightarrow Rj_* j^*(K_{n+1} \otimes^L R/m^n) \overset{d}{\longrightarrow}.
\]

We obtain an isomorphism \( R/m^n \otimes^L i_* \mathcal{R}^i K_{n+1} \cong i_* \mathcal{R}^i (K_{n+1} \otimes^L R/m^n) \) from (TR3).

Returning now to \( \text{RHom} \), here is a useful lemma which adapts Th. finitude 4.6.

**Lemma.** Let \( X \) be a noetherian scheme, \( \Lambda \) a left noetherian ring, and \( K \in D^-(X, \Lambda) \). Then the following are equivalent.

1. \( K \) is of finite Tor-dimension and the sheaves \( \mathcal{H}^i(K) \) are locally constant constructible.
2. \( K \) is locally isomorphic to a bounded complex of locally constant, flat \( \Lambda \)-modules; i.e. there is a finite étale covering \( \{ U_i \to X \} \) such that \( K|_{U_i} \) is isomorphic (in \( D^-(U_i, \Lambda) \)) to a bounded complex of constant sheaves of projective \( \Lambda \)-modules of finite type.

The argument follows Th. finitude 4.5, except now the \( \mathcal{H}^i(K) \) are moreover locally constant constructible. The sheaves \( A \) and \( B \) are defined by the cartesian diagram

\[
\begin{array}{ccc}
K^n & \longrightarrow & K^n/\text{im }d \\
\uparrow & & \uparrow \\
A & \overset{u}{\longrightarrow} & B \\
\end{array}
\]

and \( B \) is locally free as it sits in the middle of the exact sequence

\[
0 \to \mathcal{H}^n(K) \overset{(\text{id},0)}{\longrightarrow} B \to \ker d \to \mathcal{H}^{n+1}(K)
\]

where \( \ker d \) here denotes the kernel of the differential on the complex \( K' \) being constructed and is hence locally constant constructible.
As \( u \) is surjective, localizing, we may assume \( B \) constant constructible and that \( u \) surjects on global sections, defining a morphism \( v : \Lambda^d \to A \) with the property that \( vu \) is an epimorphism, and we define \( K^m = \Lambda^d \).

Suppose \( K \) is of Tor-dimension \( \leq r \). Propagating the above procedure to the left as far as degree \( -r - 1 \), we produce an étale morphism \( U \to X \), constant constructible sheaves

\[
K^{-r-1} \to K^{-r} \to K^{-r+1} \to \ldots
\]

with the property that \( K^{-r}/\text{im} \, d \) is constant constructible and flat. Then, over \( U \), the complex

\[
\ldots \to 0 \to K^{-r}/\text{im} \, d \to K^{-r+1} \to K^{-r+2} \to \ldots
\]

is quasi-isomorphic to \( K \), and has the desired properties.

**Corollary (Th. finitude 1.7).** If \( \Lambda \) is moreover commutative and of torsion, and in the situation of Th. finitude, then

\[
\mathcal{R} \text{Hom} : D^b_{ctf}(X, \Lambda) \times D^b_{tf}(X, \Lambda) \to D^b_{tf}(X, \Lambda).
\]

Following Th. finitude 1.7, as the finite Tor-dimension is stable by \( \mathcal{R} f_* \), \( \mathcal{R} f^* \), \( \mathcal{R} f^! \), devissant the first variable (say, \( \mathcal{F} \)) relative to a partition of \( X \), and using the adjunction

\[
\mathcal{R} \text{Hom}(j_! \mathcal{F}, \mathcal{G}) \leftarrow \mathcal{R} j_* \mathcal{R} \text{Hom}(\mathcal{F}, \mathcal{R} j^! \mathcal{G})
\]

for \( j : Y \to X \) the inclusion of a locally closed subscheme (c.f. SGA 4, IX 2.5 & XVIII §3.1), one reduces to the situation where the \( \mathcal{H}^i(\mathcal{F}) \) are locally constant. Localizing, we can replace \( \mathcal{F} \) by a bounded complex of constant sheaves of projective \( \Lambda \)-modules of finite type. As in this case \( \text{Hom} \) computes pointwise, we can compute \( \mathcal{R} \text{Hom} \) with respect to such a complex. Finally, if \( N \) and \( M \) are \( \Lambda \)-modules, \( N \) projective and \( M \) of Tor-dimension \( \leq r \), then \( \text{Hom}(N, M) \) is of Tor-dimension \( \leq r \), which can be seen after replacing \( M \) by a complex of flat modules 0 to the left of \( -r \) and writing \( N \) as a direct summand of a free module.
NOTES

Returning to the setting of the paper, with $R$ the ring of integers of a finite extension of $\mathbb{Q}_p$, $m$ its maximal ideal, let’s assume that reduction mod $m^n$ commutes with the four operations $R f_*, f^*, R f$, and $R f^1$, and show that for $\mathcal{F}, \mathcal{G} \in D^b_c(X, R)$,

\[(\dagger) \quad R \text{Hom}(\mathcal{F} \otimes^L R/m^{n+1}, \mathcal{G} \otimes^L R/m^{n+1}) \otimes^L_{R/m^n} R/m^n = R \text{Hom}(\mathcal{F} \otimes^L R/m^n, \mathcal{G} \otimes^L R/m^n) \quad \text{in } D^b_{ctf}(X, R/m^n).\]

Devissant $\mathcal{F} \otimes^L R/m$ with respect to a partition of $X$, we may assume that its cohomology sheaves are locally constant, and therefore the same is true of $\mathcal{F} \otimes^L R/m^{n+1}$ by considering the $m$-adic filtration on a finite complex of flat sheaves representing it. Localizing, we may replace $\mathcal{F} \otimes^L R/m^{n+1}$ with a bounded complex $N^*$ of free $R/m^{n+1}$-modules of finite type and compute $R \text{Hom}$ with respect to $N^*$, since for $\mathcal{F}$ locally free,

$\text{Hom}(\mathcal{F}, \mathcal{G})_x = \text{Hom}(\mathcal{F}_x, \mathcal{G}_x)$. Now the equality $(\dagger)$ is clear.

Consider the example: $X$ finite type over $S$, $\mathcal{F}$ and $\mathcal{G}$ two constructible torsion-free $R$-sheaves. The claim is that the projective system

\[\text{Ext}^i(\mathcal{F}, \mathcal{G}) := H^i R \text{Hom}(\mathcal{F}, \mathcal{G}) = \lim \text{proj } \text{Ext}^i_{R/m^n}(\mathcal{F} \otimes R/m^n, \mathcal{G} \otimes R/m^n)\]

forms a constructible $R$-sheaf. By part (a) of 1.1.2, it suffices to show that $R \text{Hom}(\mathcal{F}, \mathcal{G}) \in D^b_c(X, R)$. By Th. finitude 1.6 and the previous corollary, $R \text{Hom}$ sends $D^b_{ctf}(X, R/m^n) \times D^b_{ctf}(X, R/m^n)$ into $D^b_{ctf}(X, R/m^n)$, so $R \text{Hom}(\mathcal{F} \otimes R/m^n, \mathcal{G} \otimes R/m^n) \in D^b_{ctf}(X, R/m^n)$. Finally, by $(\dagger)$,

\[\lim \text{proj } R \text{Hom}(\mathcal{F} \otimes R/m^n, \mathcal{G} \otimes R/m^n) \in D^-(X, R).\]

Truncation & Tor-dimension. In part (e), Deligne addresses the truncation operators $\tau_{\leq n}$. The issue is that, while a submodule of a flat $R$-module is flat, a submodule of a flat $R/m^n$-module need not be. To address this deficiency, Deligne introduces the modified truncation operators $\tau'_{\leq n}$, which preserve the finite Tor-dimension. As these properties are of a pointwise nature, we may consider the situation in the category of $R$-modules, and the categories $D_{\text{part}}(R), D_{\text{part}}(R/m^k)$. Applying Houzel’s argument at the end of SGA 5, Exp. XV to the stalk of $K \in D^b_c(X, R)$, we may represent $K_k = K \otimes^L R/m^k$ by a bounded complex of free $R/m^k$-modules and the isomorphisms...
$K_{k+1} \otimes_{R/m^{k+1}} R/m^k \xrightarrow{\sim} K_k$ by isomorphisms of complexes $K_{k+1} \otimes R/m^k \xrightarrow{\sim} K_k$. Taking the projective limit of these complexes, we obtain a bounded complex of free $R$ modules which we will again, as in the notes to claim a), (abusively) note $K$. (To see freeness, note that if $r$ equals the rank of $K^i = K^i \otimes R/m$, there exists by Nakayama an exact sequence

$$N \to R' \to K^i \to 0$$

which after tensoring by $R/m^k$ induces an isomorphism $(R/m^k)^r \xrightarrow{\sim} K^i_k$, showing $N \subset m^k R$ for all $k$ and hence $N = 0$. As before, by the Mittag-Leffler condition, $H^i(K) = \varprojlim H^i(K_k)$. Therefore, the submodule of ker $d$ consisting of cycles whose image in $H^i(K_k)$ are in $\text{im}(H^i K \to H^i(K_k))$ is the reduction modulo $m^k$ of a flat $R$ module, as these cycles coincide with the reduction mod $m^k$ of cycles of $K^i$, which form a free submodule of $K^i$. More precisely, in view of the commutative diagram

$$\begin{array}{ccc}
K^{n-1} & \xrightarrow{d^{n-1}} & K^n \\
\downarrow & & \downarrow \\
K^{n-1} \otimes R/m^n & \xrightarrow{d^{n-1} \otimes \text{id}} & K^n \otimes R/m^k \\
\downarrow & & \downarrow \\
K^{n-1} & \xrightarrow{d^n} & K^n \\
\downarrow & & \downarrow \\
K^n \otimes R/m^k & \xrightarrow{d^n \otimes \text{id}} & K^{n+1} \otimes R/m^k
\end{array}$$

the submodule of ker $(d^i \otimes \text{id})$ consisting of cycles whose image in $H^i$ lie in $\text{im}(H^i(K) \to H^i(K \otimes R/m^k))$ is the submodule $(\ker d^n) \otimes R/m^k + \text{im}(d^{n-1} \otimes \text{id}) \subset K^n \otimes R/m^k$. As $\text{im}(d^{n-1} \otimes \text{id}) \subset (\ker d^n) \otimes R/m^k$, confirming the above description and recognizing $\tau_{\leq n} K_k$ as a bounded complex of free $R/m^k$ modules. It may be that $\tau_{\leq n} K_k$ is no longer quasi-isomorphic to $K_k$, but it is clear by construction that the pro-sheaf $H^0(K_k)$ is not affected by the operator $\tau_{\leq n}$; more precisely, by the exact sequence (\*) of claim a), $H^0(\tau_{\leq n} K_k) \simeq H^0(K) \otimes R/m^k$ and hence both are AR-isomorphic to $H^0(K_k)$.

1.1.3. Extension of scalars from $R$ to $E$ for a constructible $R$-sheaf $\mathcal{F}$ is more or less straightforward: letting $E = Q_\ell$, $R = \mathbb{Z}_\ell$, the idea is that if $\mathbb{Z}_\ell$-constructible, the torsion subsheaves $\text{Tor}_1^{\mathbb{Z}_\ell}(\mathcal{F}, \mathbb{Z}/\ell^n) = \ker \ell^n$ stabilize for some $n$, and then multiplication by $\ell$ on $\mathcal{F}$ means multiplying $\mathcal{F} \otimes \mathbb{Z}/\ell^m$ by $\ell^n$ for each $m$ to produce an AR-$\ell$-adic sheaf, and then forming the associated $\ell$-adic sheaf, to produce the exact sequence of $\ell$-adic
sheaves

\[ 0 \rightarrow \text{Tor}^1_{\mathbb{Z}}(\mathcal{F}, \mathbb{Z}/\ell^n) \rightarrow \mathcal{F} \xrightarrow{\ell^n} \mathcal{F}/\ker \ell^n \rightarrow 0. \]

The story in \( D^b_c(X, \mathbb{Z}_\ell) \) works more or less the same way. For an object \( K \) in this category, we represent \( K_m = K \otimes \mathbb{Z}/\ell^n \) by a complex of \( \mathbb{Z}/\ell^n \)-flat sheaves (i.e. with free stalks); multiplication by \( \ell^n \) means multiplying \( K_m \) by \( \ell^n \). This induces multiplication by \( \ell \) on the cohomology, and we’re back in the previous situation, since the cohomology sheaves form AR-\( \ell \)-adic systems. More generally, if \( a \) is in \( \mathbb{Z}_\ell \), letting \( a_m \) denote the image of \( a \) in \( \mathbb{Z}/\ell^n \), the commutativity of the diagram

\[
\begin{array}{ccc}
K_{m+1} & \xrightarrow{a_{m+1}} & K_{m+1} \\
\downarrow \otimes \mathbb{Z}/\ell^n & & \downarrow \otimes \mathbb{Z}/\ell^n \\
K_m & \xrightarrow{a_m} & K_m
\end{array}
\]

shows that \( a \) induces an endomorphism of \( D^b_c(X, \mathbb{Z}_\ell) \).

1.1.7. A representation \( \mathbb{Z} \rightarrow \text{GL}(V) \), sending \( n \) to \( F^n \), where \( V \) is a \( Q_\ell \)-vector space of dimension \( n \), is continuous if and only if the eigenvalues of \( F \) are \( \ell \)-adic units.

To see sufficiency, note that we can choose a basis for \( V \) so that the morphism \( \mathbb{Z} \rightarrow \text{GL}(V) \) factors through \( \text{GL}(\mathbb{Z}_\ell^n) = \varprojlim \text{GL}(\mathbb{Z}/\ell^n)^n \), a profinite group, and we may extend \( \mathbb{Z} \rightarrow G \) to a morphism \( \hat{\mathbb{Z}} \rightarrow G \) for any profinite group \( G \) by the universal property of profinite completion, which we state and prove now.

The profinite completion of a group \( H \) (with respect to normal subgroups of finite index in \( H \)) is denoted \( \hat{H} \), so that \( H \rightarrow \hat{H} \) has dense image. The profinite completion \( \hat{H} \) enjoys the universal property that for every profinite group \( G \) and continuous homomorphism \( H \rightarrow G \), there is a unique homomorphism \( \hat{H} \rightarrow G \) making the diagram

\[
\begin{array}{ccc}
H & \rightarrow & \hat{H} \\
\downarrow & & \downarrow \\
G & \leftarrow &
\end{array}
\]

commute.
To see this, simply use the description of $G$ as $\varprojlim G/N$ as $N$ ranges over open normal subgroups of $G$. The preimage $M$ of $N$ in $H$ is an open normal subgroup of finite index, as $G/N$ is finite. Therefore $H \to G/N$ factors through $H/M$, and to give a continuous morphism from $\hat{H}$ to $G$ it suffices to give compatible continuous maps $\hat{H} \to G/N$. Continuity is assured by the above remark; compatibility is assured by the map $H \to G$, which determines the maps $\hat{H} \to G/N$.

Returning to (1.1.7), to see necessity, we assume we have found a continuous extension $\rho$.

$$
\begin{array}{c}
Z \\
\downarrow \\
\hat{Z}
\end{array} \longrightarrow \begin{array}{c}
\text{GL}(V)
\end{array}
\quad \rho
$$

The image $\rho(\hat{Z})$ is compact, so the set $\rho(\hat{Z})Z^m_\ell$ is compact, for $Z^m_\ell$ a $Z_\ell$-lattice in $V$, so $\rho(\hat{Z})Z^m_\ell \subset \frac{1}{m}Z^m_\ell$ for some $m$. Letting $L$ denote the $Z_\ell$-span of $\rho(\hat{Z})Z^m_\ell$, $L$ is a $Z_\ell$-submodule of $\frac{1}{m}Z^m_\ell$, hence free (of rank $n$). This recognizes $F \in \rho(\hat{Z})$ as an element of $\text{Aut}(L)$, so the eigenvalues of $F$ are $\ell$-adic units indeed.

**1.2.6.** ‘On notera que, pour $k$ un corps fini, une representation $V$ de $W(k,k)$ est automatiquement $\i$-mixte.’ ~ This follows from the existence of Jordan normal form.

**1.3.9.** (When reading the corollary, recall that a semisimple algebraic group is connected by definition.) We wish to understand why $G^{00}$ is reductive. Note first that the sum of the simple $\pi_1(X,\bar{x})$-modules is $W$-stable since if $w \in W$ and $V$ is a $\pi_1(X,\bar{x})$-module, then $wV$ is again a $\pi_1(X,\bar{x})$-module since $\pi_1(X,\bar{x})$ is a normal subgroup of $W$; applying this argument with $w^{-1}$ shows that $wV$ is simple iff $V$ is. Next observe that if $\rho : W(X_0,\bar{x}) \to GL(\mathcal{F}_\pi)$ is the representation defining $\mathcal{F}_0$, then $\rho(\pi_1(X,\bar{x}))$ and its Zariski closure $G^0$ have the same invariant subspaces (to see this, form a basis for $\mathcal{F}_\pi$ beginning with a basis for an invariant subspace). Therefore we see that $G^0$ acts semisimply since $W(X_0,\bar{x})$ does by assumption.
Now recall that $R(G^0)$, the radical of $G^0$, is a connected and solvable normal subgroup of $G^0$. By the argument above, any normal subgroup of $G^0$ acts semisimply; combining this with the Lie-Kolchin theorem, we see that $\mathfrak{F}_G$ decomposes as a direct sum of one-dimensional irreducible $R(G^0)$-modules. The unipotent part of $R(G^0)$, which is the unipotent radical $R_u(G^0)$, must therefore act by the identity, and we see that $R_u(G^0) = R_u(G^{00}) = \{1\}$; i.e. that $G^{00}$ is reductive. Note we have proved the following

**Lemma.** If $V$ is a finite-dimensional vector space over an algebraically closed field and $G$ is a closed subgroup of $GL(V)$, then $G$ is reductive.

This result appears in [Milne, 21.60] as

**Proposition.** Let $G$ be a connected group variety over a perfect field $k$. The following conditions on $G$ are equivalent.

1. $G$ is reductive;
2. The radical $R(G)$ of $G$ is a torus;
3. $G$ is an almost-direct product of a torus and a semisimple group;
4. $G$ admits a semisimple representation with finite kernel.

More is true. In fact, for $G$ a connected reductive group, say, over an algebraically closed field $k$, the maximal central $k$-torus $Z$ coincides with $(\mathfrak{z}G)\circ$, the connected component of the identity of the center of $G$, and the multiplication homomorphism $Z \times \mathfrak{D}(G) \to G$ is a central isogeny, i.e. an isogeny with central kernel, where $\mathfrak{D}(G) = (G,G)$ is the derived subgroup. This implies that $Z \to G/\mathfrak{D}(G)$ is a central isogeny. Here, our $G$ is Deligne’s $G^{00}$, our $Z$ is Deligne’s $T_1$, and our $G/\mathfrak{D}(G)$ is Deligne’s $T$, as a connected, smooth, reductive, and commutative group is a torus [Milne, 19.12], and a quotient of a reductive group over a field of characteristic 0 is reductive.

The set $F$ of characters by which $T_1$ acts on $\mathfrak{F}_\pi$ generates $X(T_1)$ since the representation of $T_1$ is faithful, and, as $T_1$ is a torus, diagonalizable. Therefore, with the right choice of basis, the representation of $T_1$ looks like $\text{diag}(\chi_i)$ for characters $\chi_i \in F$. As the representation is faithful, these characters generate the character group $X(T_1)$. 
(The character group of diag($\chi_i$), which is isomorphic to the character group of $T_1$, is generated by the $\chi_i$.)

The group $W(X_0, \bar{\tau})$ acts on $G^0$ by conjugation. Recall that the neutral component of an algebraic group is a characteristic subgroup, and so is the center. Therefore $T_1$, which can be described as the neutral component of $Z(G^{00})$, is acted upon by $W(X_0, \bar{\tau})$. Recall that the functor $X$ which takes an algebraic group to its character group induces a contravariant equivalence from the category of diagonalizable algebraic groups with the finitely generated commutative groups, and as we have seen, $W(X_0, \bar{\tau})$ acts on $X(T_1)$ by permuting factors, hence through a finite quotient.

We would like to know why the group of outer automorphisms of $G^{00}$ restricting to the identity on $T_1$ is finite. The group $G^{00}$ admits a maximal split torus $T_2$ so that $(G^{00}, T_2)$ is a split reductive group. The radical $R(G^{00}) = T_1$ is the largest subgroup of the multiplicative group $Z(G^{00})$, so the quotient $Z(G^{00})/R(G^{00})$ is finite [Milne, 19.10]. Recall the definition of isomorphism of root data [Milne, 23.2]. An isomorphism $\varphi$ of split reductive groups defines an isomorphism $f$ of root data, and every isomorphism of root data $f$ arises from a $\varphi$, unique up to an inner automorphism [Milne, 23.26]. Moreover for a split reductive group $(G, T)$ we have a canonical isomorphism $\text{Out}(G) \cong \text{Aut}(X, \Phi, \Delta)$, where the latter is automorphisms of based root data [Milne, 23.46]. Given such a $\varphi : (G, T) \rightarrow (G', T')$, the map $f$ is defined by the formula $f(\chi') = \chi' \circ \varphi|_T$ for $\chi' \in X(T')$ [Milne, 23.5]. Suppose $\varphi$ is now an automorphism of $(G^{00}, T_2)$ and restricts to the identity on the radical $T_1$. The isomorphism $f$ is a fortiori a central isogeny and its action on $Z\Phi$ (the $Z$-submodule of $X^*(T_2)$ generated by the roots $\Phi$) preserves the base $\Delta$, hence its action on $Z\Phi$ amounts to permuting a finite set. On the other hand, the quotient $T_2/Z(G^{00})$ has character group the subgroup $Z\Phi$ of $X^*(T_2)$ [Milne, 21.9], hence the the root lattice $Z\Phi$ has finite index in $X^*(T_2/T_1)$. As $Z(G^{00})/T_1$ is finite, this is enough to conclude that subgroup of $\text{Aut}(X, \Phi, \Delta)$ corresponding to automorphisms of $G^{00}$ which restrict to the identity on $T_1$ is finite, hence that the subgroup of $\text{Out}(G^{00})$ consisting of those automorphisms fixing $T_1$ is also finite.

Now, if $w$ is an element of $W(X_0, \bar{\tau})$ of degree 1, and $\bar{w}$ the image of $w$ in $\text{GL}(\bar{F}_2)$, $G$ is the semi-direct product of $Z$ by $G^0 = G^{00}$ relative to the action $\text{int}(\bar{w})$ of $Z$ on $G^0$. As
this action is given by an interior automorphism of $G^0$, by multiplying $w$ by an element of $\pi(X, \bar{x})$, we make the action of $\text{int}(\bar{w})$ trivial, and recognize $G = G^{00} \times Z$. 

The proof of (1.3.9) follows easily from (1.3.8). Note that once one has reduced to $\mathcal{F}_0$ semisimple, it is easy to see that the radical of $G^{00}$ in this case is trivial, as it is by definition the largest connected solvable normal subgroup variety of $G^{00}$, hence contained in the connected component of the identity of $G^0$, so if $G^0$ is an extension of a finite, hence discrete, group, $R(G^{00})$ lies in the kernel of this extension, namely in the semisimple subgroup, so in fact $R(G^{00}) = \{e\}$, as connected normal subgroup varieties of a semisimple group are semisimple [Milne, 21.52].

1.3.10. Note that (iv) should read 'Le centre de $G$ s’envoie sur un sous-groupe d’indice fini de $Z$.' The crux of the direction (iv) $\Rightarrow$ (i) is that, while $G$ is not a priori a linear algebraic group, $G/Z$, as an extension of a finite group by a linear algebraic group, is.

1.3.12. The central element $g$ acts by a scalar by Schur’s lemma.

1.3.13. (i) The claim rests on the following

**Lemma.** Let $X_0, X'_0$ be normal connected schemes of finite type over a field with generic points $\xi, \xi'$ and function fields $K = k(\xi)$ and $K' = k(\xi')$. Let $\Omega, \Omega'$ be algebraically closed extensions of $K, K'$, defining geometric points $a, a'$ of $X_0, X'_0$ centered on $\xi, \xi'$, respectively. If $f : X'_0 \to X_0$ is a dominant morphism, then the image of the induced map $\pi_1(X'_0, a') \to \pi_1(X_0, a)$ is an open subgroup of finite index.

Observing that $\pi_1(X'_0, a')$ acts on $(f^*\mathcal{F})_{a'}$ via the map on $\pi_1$ in the lemma induced by $f$, we see that there is a central element $g \in G'$ of positive degree and a morphism $G' \to G$ sending $g$ to a central element of $G$ of positive degree, and the action of $g$ on $\mathcal{F}_{0,a}$ via this map is the same as the action of $g$ on $(f^*\mathcal{F})_{a'}$.

**Proof of lemma.** The extensions $\Omega, \Omega'$ define geometric points $a_1, a'_1$ of $S = \text{Spec}(K)$ and $S' = \text{Spec}(K')$, respectively. The dominant morphism $f : X'_0 \to X_0$ induces an
extension of fields $K \subset K'$. Then $\pi_1(S, a_1) \to \pi_1(X_0, a)$ is surjective [SGA1, V 8.2], and after identifying $\pi_1(S, a_1)$ with $\text{Gal}(K^{\text{sep}}, K)$, the kernel is identified with those automorphisms which fix all finite extensions of $K$ in $\Omega$ which are unramified over $X_0$, and likewise for $\pi_1(S', a_1') \to \pi_1(X'_0, a')$. If $L$ is an extension of $K$ unramified over $X_0$, then $L \otimes_K K'$ is an extension of $K'$ unramified over $X'_0$ [SGA1, I 10.4(iii)]. The operation on étale covers of $X_0$ consisting of taking inverse image along $f$ followed by fiber at $a'$ is a fiber functor for $X_0$, hence induces a continuous homomorphism of groups $\pi_1(X'_0, a') \to \pi_1(X_0, a)$ [SGA1, V 6.2]. The action of $\pi_1(X'_0, a')$ on $(f^* \mathcal{F}_0)_{a'}$ is by restriction with respect to this homomorphism. This homomorphism, in turn, is induced by restriction of $\pi_1(S, a'_1) \to \pi_1(S, a)$, since if $L$ is as above, an automorphism $\sigma \in \ker(\pi_1(S', a'_1) \to \pi_1(X'_0, a')$ acts on $L \otimes_K K'$ by the identity as $L \otimes_K K'$ is unramified. As $K'/K$ is finitely generated, $K' \cap K^{\text{sep}}$ is a finite extension of $K$, so the image of $\pi_1(S', a'_1) \to \pi_1(X_0, a)$ is an open subgroup of finite index isomorphic to the image of $\text{Gal}(K^{\text{sep}}/K' \cap K^{\text{sep}})$ in $\pi_1(X_0, a)$.

(ii) Choose a basis for representations corresponding to $\mathcal{F}_0$ and $\mathcal{G}_0$ so that Frobenius is upper-triangular in both, and then recall the form of the Kronecker (tensor) product of matrices, which has the property that the Kronecker product of upper-triangular matrices is upper-triangular.

(iii) The claim rests on two observations. The first is that if $\mathcal{F}_0$ is defined by a representation $V$ of $G$, the eigenvalues of any $g \in G$ coincide with the eigenvalues of $g$ acting on the semi-simplification of $V$ with respect to any Jordan-Hölder series. To see this, choose a basis for each graded piece so that $g$ is upper-triangular, and then order a lift of these bases according to the filtration, beginning with the smallest piece. The second observation is that if we begin with an ordered basis $(a_i)$ for $V$ with respect to which $g$ is upper triangular, then a basis $B$ for $\bigwedge^d V$ consisting of $a$-forms in the $a_i$ can be found. If the function $w$ takes an $a$-form in the $a_i$ and outputs the sum of the subscripts which appear (so $w(a_1 \wedge a_3 \wedge a_4) = 8$), then $g$ is upper-triangular with respect to any ordering of $B$ which respects the total order $w$. The claim follows.
1.3.14. It suffices to show that the image of $W(X_0, x)$ in $GL(r, E)$ is bounded by the argument of (1.1.7), which we repeat now. We lose nothing by supposing $E = Q_\ell$, in which case the image $W$ of $W(X_0, x)$ in $GL(r, E)$ is bounded if it is contained in $1/\ell GL(r, Z_\ell)$. Applying $W$ to $Z^\prime_\ell$ and taking the $Z_\ell$ span, we get a free $Z_\ell$-submodule of $1/\ell Z^\prime_\ell$ of rank $r$, on which $W$ acts by automorphisms. This recognizes $W$ as isomorphic to a subgroup of $GL(r, Z_\ell)$, a profinite group to which it is easy to extend a map $W(X_0, x) \to GL(r, Z_\ell)$ to a map from the completion $\pi_1(X_0, x) \to GL(r, Z_\ell)$.

To see that $\rho(W^0_1) \subset G^{00}$ is compact and Zariski dense, observe that $W^0_1$ is a closed subgroup of $\pi_1(X, \bar{x})$, hence a profinite group, and $G^0$ is by definition the Zariski closure of the image of $\pi_1(X, \bar{x})$. In particular, the inverse image of $G^{00}$ is Zariski dense in $G^{00}$.

1.3.15. Relevant sources are Bourbaki, *Lie Groups and Lie Algebras* II, §7, Demazure and Gabriel, *Groupes Algébriques*, II, §6, [Milne, 10, 14d]. Bourbaki explains how to extend the logarithm to the union of all compacta. You need to know that for all compact $G \subset H(E)$, $x \in G$, and neighborhood about $e$, there is a strictly increasing sequence of integers $(n_i)$ such that $x^{n_i} \in V$, which allows one to extend the logarithm by Deligne’s formula. Bourbaki also explains that there is an open subgroup $V$ of $e$ in $H(E)$ such that log is an analytic isomorphism of $V$ onto an open subgroup of Lie $H$, with inverse exp. It follows that $L^1$, the E-linear span of log $K$, coincides with the E-linear span of log($K \cap V$). We have for $X \in \log H$ that $\exp(nX) = \exp(X)^n$, and $\log(g^n) = n \log(g)$ for any $g$ where log is defined.

For $g \in H(E)$, $X \in \text{Lie } H$,

\[ g \cdot \exp(X) \cdot g^{-1} = \exp(\text{Ad}(g)(X)), \]

whenever these expressions converge. Taking $X \in \log(K \cap V)$ and $g \in V \cap K$, we find that the left side is in $K \cap V$. Moreover there is an $n \in \mathbb{Z}$ such that both $n \text{ Ad}(g)(X)$ and $nX$ lie in $\log V$. We find that $\text{Ad}(g)(nX) \in K \cap V$, therefore that for $g \in V \cap K$, $\text{Ad}(g)$ preserves $L^1$, therefore $L^1$ is also preserved by $V$, which is the Zariski closure of $V \cap K$ in $K$. 
Therefore the adjoint representation

$$\text{Ad} : H \to \text{GL} (\text{Lie} H)$$

factors through the algebraic subgroup $F \subset \text{GL} (\text{Lie} H)$ which fixes the $L^1$. Applying the functor Lie, one finds that

$$\text{ad} : \text{Lie} H \to \mathfrak{gl} (\text{Lie} H)$$

factors through $\text{Lie} F$. As $\text{ad}$ induces the bracket on $\text{Lie} H$, it follows that $L^1$ is an ideal in $\text{Lie} H$. As $H$ is semisimple over a field of characteristic 0, Lie subalgebras of $\text{Lie} H$ are in bijection with connected algebraic subgroups of $H$ (c.f. e.g. Demazure and Gabriel, *Groupes Algébriques*, II §6 2.4 & 2.7). As the exponential is functorial (c.f. *ibid*, 3.4), the algebraic subgroup corresponding to $\mathfrak{t}$ contains $K$, hence must equal $H$ by density, hence $L^1 = \text{Lie} H$.

Let $N$ denote the normalizer of $K$ and $g \in N$. Let $X \in K$. Then there is an integer $n$ such that both $X^n \in V$ (hence $n \log X \in \log V$) and $n \text{Ad}(g) (\log X) \in \log V$. Applying $(\dagger)$ to $\log(X^n) = n \log X$ we find that $\exp(\text{Ad}(g)(n \log X)) = g \cdot X^n \cdot g^{-1} \in V \cap K$ so that $\log(g \cdot X \cdot g^{-1}) = \text{Ad}(g)(\log X)$ and we find $\text{Ad}(g)$ preserves the set $\log K$ and *a fortiori* $L^0$.

The morphism $\text{Ad}$ factors as a quotient $H \twoheadrightarrow H/Z(H)$ followed by a closed immersion. As $H$ is semisimple, $Z(H)$ is finite, hence the quotient $H \twoheadrightarrow H/Z(H)$ is finite (*a fortiori* proper) [Milne, 21.7, 7.15, 5.39].

The subgroup $K \subset H(E)$ is a compact subset of a complete metric space, hence closed for the topology induced by the non-archimedean metric on $E$. Hence $K$ is a closed Lie subgroup of $H(E)$ for that metric, and hence $N$ is a closed subgroup of $H(E)$ with respect to the metric induced by the one on $E$. As $L^0$ is compact and isomorphic to an $\mathcal{O}_E$-lattice in $\text{Lie} H$, its automorphism group $\text{Aut} L^0$ is compact; as $\text{Ad}$ is proper, $\text{Ad}^{-1} (\text{Aut} L^1) \subset H(E)$ is compact, and $N \subset \text{Ad}^{-1} (\text{Aut} L^0)$ is a closed subgroup, hence also compact.

1.4.1. (b) See Weil I, (2.9).
1.4.2. Let $\overline{x}$ be a geometric point of $X$; as $X_0$ is absolutely irreducible, $X$ is connected. The pullback of lisse sheaves along the morphism $X \rightarrow X_0$ identifies with the restriction of representations along the continuous homomorphism $\pi_1(X, \overline{x}) \rightarrow W(X_0, \overline{x})$, and likewise the pullback of lisse sheaves along the structural morphism $X_0 \rightarrow \text{Spec}(\mathbf{F}_q)$ with restriction along $W(X_0, \overline{x}) \rightarrow \mathbf{Z}$. Given a lisse sheaf $\mathcal{F}$ on $X_0$ with monodromy representation $V$, the largest subsheaf (resp. quotient sheaf) becoming constant on $X$ is obtained by taking invariants (resp. coinvariants) of $V$ with respect to $\pi_1(X, \overline{x})$. Both $V^{\pi_1(X, \overline{x})}$ and $V_{\pi_1(X, \overline{x})}$ carry natural actions of $\mathbf{Z}$ which induces lisse sheaves $F', F''$ on $\text{Spec}(\mathbf{F}_q)$ with inverse images $V^{\pi_1(X, \overline{x})}$ and $V_{\pi_1(X, \overline{x})}$, respectively. (The exact sequence $0 \rightarrow \pi_1(X, \overline{x}) \rightarrow W(X_0, \overline{x}) \rightarrow \mathbf{Z}$ identifies those lisse sheaves invariant under geometric monodromy with the inverse image of sheaves on $\text{Spec}(\mathbf{F}_q)$.)

1.4.3. The point is that on the one hand, the constituents of the sheaves $F', F''$ are among the constituents of $\mathcal{P}_0$, on the other hand as representations of $W(X_0, \overline{x})$, $F', F''$ are invariant for geometric monodromy, so they have one-dimensional constituents which are determined once Frobenius is put in Jordan normal form. Therefore the eigenvalues of Frobenius on $F'$ and $F''$ appear among the determinental weights for $\mathcal{P}_0$, and, in consideration of (1.4.2), up to a twist the same is true of eigenvalues of Frobenius on $H^0(X, \mathcal{F}), H^1_c(X, \mathcal{F})$, and $H^2_c(X, \mathcal{F})$.

1.4.6. See Ahlfors, Complex Analysis, Ch. 5 §2.2 for a characteristically elegant review of the convergence properties of infinite products, which elucidates the equivalence of the absolute convergence of Deligne’s Euler product with that of of his geometric series.

1.5.1. Perhaps the only thing to remark is that if a lisse sheaf $\mathcal{F}$ on $X$ is $t$-real, then all of its exterior powers are, too: choosing a basis for $\mathcal{F}_x$ with respect to which $F_x$ is upper-triangular, the resulting canonical basis for $\wedge \mathcal{F}$ can be ordered so that $F_x$ remains upper triangular (1.3.13 iii), which makes it easy to see that $t \det(1 - F_x t, \wedge \mathcal{F})$ has coefficients which are symmetric polynomials in the eigenvalues of $F_x$. As the coefficients of $t \det(1 - F_x t, \mathcal{F})$ are the elementary symmetric polynomials in these eigenvalues, and are real, the coefficients of $t \det(1 - F_x t, \wedge \mathcal{F})$ are real too.
1.6.11. To see the Clebsch-Gordon decomposition (1.6.11.2), let
\[
H = du \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]
in the notation of (1.6.8), and let
\[
(\dagger) \quad \chi_d(\lambda) = \text{Tr}(e^{\lambda H}) = \sum_{j=-d}^{d} \chi_{d-2j}(\lambda)
\]
as a function of \( \lambda \in k \). These characters transform additively under direct sum and multiplicatively under tensor product, so finding the decomposition of \( S_d \otimes S_{d'} \) is the same as finding the additive decomposition of
\[
(\lambda^{-d} + \lambda^{-d+2} + \cdots + \lambda^{d})(\lambda^{-d'} + \lambda^{-d'+2} + \cdots + \lambda^{d'})
\]
into sums of the form \((\dagger)\). In this case the decomposition is into \( d' + 1 \) sums of the form \( \chi_j(\lambda) \) for \( j \in P(d, d') \).

1.6.13. It is asserted that the inclusion \( \subset \) of assertion 2) results from the fact that the image of \( N^i M_i \) in \( \text{Gr}_0^W(V) \) is \( N^i M_i(\text{Gr}_0^W V) = M_i(\text{Gr}_0^W V) \). To see this, recall that in the construction (1.6.1), \( N^{d-1} \) sends \( \ker N^d / \text{im} N^d \) onto \( M_{-d+1} / \text{im} N^d \), hence sends \( M_{d-1} \) onto \( M_{-d+1} \), and proceed inductively.

The inequality \( k - 2i - 2 \geq k \geq 2j - k \) should read \( k - 2i - 2 \geq -k \geq 2j - k \) at the end of the discussion of 3). Note \( N^{k-j} : \text{Gr}_k^M G \to \text{Gr}_{2j-k}^M G \).

1.6.14. It seems as though (1.6.14.3) should read
\[
\text{Gr}_i^M V \simeq \bigoplus_{j \in [i]} \bigoplus_{j \in [i]} \left( \frac{-i-j}{2} \right).
\]
For, when \( i \leq 0 \), \( N \) induces an isomorphism of \( \text{Gr}_i^M V / P_i \) onto \( \text{Gr}_{i-2}^M V \). Scaling \( N \mapsto \lambda N \) also multiplies this isomorphism by \( \lambda \), so we need to twist by \( \otimes N^{-1} \). Similarly, the isomorphism \( N^i : \text{Gr}_i^M V \to \text{Gr}_{-i}^M V \) scales by \( \lambda^i \) so we need to twist by \( -i \).
With this modification, the isomorphism

$$P_{-j} \cong \bigoplus_{j \in P(j', j'')} P'_{-j} \otimes P''_{-j} \left( \frac{j - j' - j''}{2} \right)$$

is justifiable by passing through a suitable graded piece

$$P_{-j} \left( \frac{-i - j}{2} \right) \rightarrow \text{Gr}^M_i V \leftarrow \bigoplus_{j \in P(j', j'')} P'_{-j} \otimes P''_{-j} \left( \frac{-i - j' - j''}{2} \right).$$

As for the isomorphism

$$P_{-j}(V^*) \cong P_{-j}(V^*)^{*}(j),$$

consider $V = S_j$ identified with the representation of $\text{SL}(2)$ on homogeneous polynomials in variables $x$ and $y$ of degree $j$. Suppose $P_{-j}(V)$ is generated by the vector $y^j$; then $P_{-j}(V)^*$ is generated by the covector on $y^j$ and $P_{-j}(V^*)$ is the covector on $x^j$. The map $P_{-j}(V^*) \rightarrow P_{-j}(V^*)^*$ is obtained by precomposing by $N^j$. Upon scaling $N \mapsto \lambda N$, this map is scaled by $\lambda^j$. Therefore the canonical isomorphism is obtained by twisting by $N^{-j}$:

$$P_{-j}(V)^* \rightarrow P_{-j}(V^*)^*(-j).$$

1.7.3. ‘Il commute à l’action de $W(\overline{K}, K)$’—this statement is somewhat opaque. Which action(s)? When you let $W(\overline{K}, K)$ act on $V$ via $\rho$ and on $\overline{Q}_l(n)$ via $\text{Gal}(\overline{k}, k)$, the statement is that if $\tau \in W(\overline{K}, K)$,

$$\tau N \tau^{-1} = N : V(1) \rightarrow V.$$ 

Let $\sigma \in \overline{Q}_l(1)$. As $\tau^{-1} \lambda = (\deg \tau)^{-1} \lambda \tau^{-1}$, the statement is equivalent to the statement

$$\tau N \lambda \tau^{-1} = (\deg \tau) N \lambda.$$

$\text{Gal}(\overline{K}, K)$ acts on the inertia character $\iota$ by the formula

$$\iota (\tau \sigma \tau^{-1}) = \tau \iota \sigma,
where on the right $\tau$ acts through $\text{Gal}(\bar{k}, k)$, so if we assume $I_1$ is a normal subgroup of $I$ and take $\sigma \in I_1$,

$$\exp(\rho(\tau) N t_{\ell}(\sigma) \rho(\tau^{-1})) = \rho(\tau \sigma \tau^{-1}) = \exp(N t_{\ell}(\tau \sigma \tau^{-1})) = \exp(N (\deg \tau) t_{\ell}(\sigma)).$$

1.7.4. A normal unipotent subgroup of an algebraic group acts trivially on all semisimple representations, hence the factorization of the action of $I$ by a finite quotient [Milne, 19.16].

The semisimple representation of $W(\bar{k}, K)$ therefore factors through the quotient by $I_1$. The conjugation action of $W(\bar{k}, K)$ on $I/I_1$ is has a kernel $U$ of finite index, as a finite group has only finitely many automorphisms. So for some $m_1$, $F^{m_1}, F^{m_1} \in U$. $F^{m_1} = a F^{m_1}$ for some $a \in I$, but now $F^{m_1}$ commutes with $a$, so if $a^{m_2} = 1$, then $F^{m_1 m_2} = a^{m_2} F^{m_1 m_2} = F^{m_1 m_2}$, so we may take $n = m_1 m_2$.

1.7.5. In the proof it is asserted ‘On a NM’ $\subset M'_{i-2}$.’ This is because of the commutativity of (1.7.3). Namely, $F' N = N F'$ and if $\lambda \in \overline{Q}_i(1)$, $F' N \lambda = \lambda^{-1} N \lambda F'$. Therefore $N \lambda$ sends $M'_{i}$ into $M'_{i-1}$ and $N : M'_{i}(1) \subset M'_{i-2}$.

While proving that $M'$ is independent of $F'$, Deligne introduces $\exp(\lambda N)$ for $\lambda \in \overline{Q}_i(-1)$. Let me instead deduce from the fact $F^n = F'^n \mod I_1$ that for an appropriate choice of $\lambda \in \overline{Q}_i(1)$,

$$F'^n = \exp(N \lambda) F'^n.$$

Leaving the definition of $\mu$ unchanged, we have

$$\exp(N \lambda) F'^n = \exp(N \mu) F'^n \exp(N \mu)^{-1},$$

since if we expand $F'^n \exp(-N \mu)$ as a series

$$F'^n \left(1 - N \mu + \frac{(-N \mu)^2}{2!} + \cdots \right),$$

we have that $N$ commutes with $F'^n$ and that $F'^n \mu^i = q^{n i} \mu^i$, so that

$$\exp(N \mu) F'^n \exp(N \mu)^{-1} = \exp(N \mu) \exp(-q^n N \mu) F'^n = \exp(N \lambda) F'^n.$$
This identity shows that \( \exp(N\mu) \) sends \( M' \) into \( M'' \). As \( (N\mu)M'_i \subset M'_{i-2} \),

\[
\exp(N\mu) = 1 + N\mu + \frac{(N\mu)^2}{2!} + \cdots
\]

also sends \( M'_i \) into \( M'_{i-2} \), so \( M''_i \subset M'_i \).

1.7.7. Let \( F' \) be as in (1.7.5), let \( \alpha \in \overline{Q}_\ell^* \), and let \( V^{\alpha} \) equal the sum of the generalized eigenspaces of \( F' \) acting on \( V \) with eigenvalue in the class of \( \alpha \mod \) roots of 1. Then \( V^{\alpha} \) is independent of lift and \( N\lambda : V^{\alpha} \to V^{\alpha/q} \), and

\[
V^{\alpha} = \sum_{i \in \mathbb{Z}} V^{\alpha q^i},
\]

is stable under \( \exp(N\lambda) \) for \( \lambda \in \overline{Q}_\ell^*(1) \) and hence under the action of \( I_1 \). By the argument of (1.7.5), \( V^{\alpha} \) is therefore stable under \( W(K, K) \).

1.7.8. First, the matter of what it means for the locally constant sheaf of sets \( \mathcal{F} \) on \( X - D \) to be tamely ramified along the divisor \( D \). In Grothendieck-Murre [GM], the notion of a tamely ramified covering is discussed. If \( \mathcal{F} \) is a locally constant sheaf of finite sets on \( X - D \), it is represented by an étale covering of \( X - D \). What about if \( \mathcal{F} \) is a locally constant sheaf of sets on a scheme \( S \) with infinite fibers?

Let \( \{S_i\}_{i \in I} \) be a covering trivializing \( \mathcal{F} \). If \( S \) is quasi-compact we can take this covering to be finite quasi-compact. Over each \( S_i \), \( \mathcal{F} \) is represented by an étale morphism \( X_i \to S_i \), namely the disjoint union \( \bigsqcup_{i \in \Lambda} S_i \) with the set \( \Lambda \) in bijection with a(ny) fiber of \( \mathcal{F} \). This yields a separated, locally of finite presentation, and locally quasi-finite descent datum \( X_i \) relative to the covering \( S_i \) (c.f., e.g. Stacks, tag \texttt{02W4}). By [SGAD, Exp. X, 5.4], this descent datum is effective and yields an \( S \)-scheme \( X \). In the case that \( S \) is quasi-compact, then as the faithfully flat covering \( \bigsqcup_i S_i \) was taken to be quasi-compact, \( X \) is separated [SGA1, Exp. VIII, 4.8 or Exp. IX, 2.4], but in any case it is evidently étale as its restriction to each \( S_i \) is. This elucidates the proof of Lemma 2.2 in [SGAA, Exp. IX], with one refinement:
Lemma. Let \( S \) be a scheme and \( \mathcal{F} \) a locally constant sheaf of sets on \( S \). Then \( \mathcal{F} \) is represented by a \( X/S \) étale. If \( S \) is quasi-compact, then \( X \) is separated. If the fibers of \( \mathcal{F} \) are finite (resp. and non-empty), then \( X \) is an étale covering (resp. and surjective).

Now suppose \( S \) is normal, integral, and quasi-compact with generic point \( \eta \). By EGA IV, 18.10.7 and 18.10.8, the étale \( X \to S \) is isomorphic to a disjoint union of integral normal schemes \( X_\alpha \) such that \( f^{-1}_\alpha(\eta) \) is a finite separable extension of \( k(\eta) \) and moreover if \( S_\alpha \to S \) denotes the normalization of \( S \) in \( f^{-1}_\alpha(\eta) \), then \( S_\alpha \) is a revêtement étale, and \( f_\alpha \) factorizes as \( X_\alpha \to S_\alpha \to S \) with \( f_\alpha' \) an open immersion.

Hence \( \mathcal{F} \) decomposes as a disjoint union of sheaves \( \mathcal{F}_\alpha = h_{X_\alpha} \), with \( h_{X_\alpha} \) locally constant constructible. The property of the \( h_{X_\alpha} \) being locally constant implies that the open immersions \( f_\alpha' \) are in fact isomorphisms, and we sum up in the following

Lemma. Let \( S \) be a normal, integral, and quasi-compact scheme and let \( \mathcal{F} \) be a locally constant sheaf of sets on \( S \). Then \( \mathcal{F} \simeq \bigsqcup_\alpha h_{X_\alpha} \); i.e. \( \mathcal{F} \) decomposes as a disjoint union of locally constant constructible sheaves represented by revêtements étales \( X_\alpha \to S \).

Now we recall the definition of divisors with normal crossings [GM, 1.8]. Let \( S \) be a locally noetherian scheme and \( (D_i)_{i \in I} = D \) a finite set of divisors on \( S \). For simplicity we often denote the inverse image of the \( D_i \) in Spec \( \mathcal{O}_{S,S} \to S \) by the same letter \( D_i \).

Definition. a) We say that the \( (D_i)_{i \in I} \) have strictly normal crossings if for every \( s \in \bigcup_{i \in I} \text{supp } D_i \) we have:
   i) \( \mathcal{O}_{S,s} \) is a regular local ring,
   ii) if \( I_s = \{ i : s \in \text{supp}(D_i) \} \), then for \( i \in I_s \) we have

   \[
   D_i = \sum_\lambda \text{div}(x_{i,\lambda})
   \]

   with \( x_{i,\lambda} \in \mathcal{O}_{S,s} \) and \( (x_{i,\lambda})_{i,\lambda} \) part of a regular system of parameters at \( s \).

b) We say that the set \( (D_i)_{i \in I} \) has normal crossings if for every \( s \in \bigcup_{i \in I} \text{supp } D_i \) there exists an étale neighborhood \( S' \to S \) of \( s \) in \( S \) such that the family of inverse images of the \( (D_i)_{i \in I} \) on \( S' \) has strictly normal crossings.
**Remark.** The concept of (strictly) normal crossings is stable by étale base change, and one can check whether a set of divisors has normal crossings étale-locally.

In the setting of $X$ a regular scheme and $D$ a divisor on $X$ with normal crossings, the definition of a tamely ramified covering $f : V \to X$ relative to $D$ given in Grothendieck-Murre is equivalent to the property that $V$ be étale over $X - D$, and that the inertia at each $d \in D$ of codimension one act trivially on $V$ (see also [SGA1, Exp. XIII, 2.1]). More precisely, let $D$ be a union of lisse divisors $D_i$, $d$, $X(d)$, $\overline{\eta}$ be as in (1.7.8). Then the corresponding inertia group should act trivially on $V_{\overline{\eta}}$. Evidently this definition extends to a locally constant sheaf of sets $\mathcal{F}$. The ramified Kummer covering $\pi : X_n \to X$ is a homeomorphism, and if $\mathcal{F}$ is a locally constant sheaf of sets on $X - D$, $L$-ramified along $D$, for $L$ a set of primes invertible on $X$, then there is an $n$ invertible on $X$ such that the action of inertia with respect to every point of $X_n$ on $\pi^* \mathcal{F}$ is trivial.

More explicitly, let $U = X - D, U_n = \pi^{-1}(U)$. The Kummer covering $\pi$ is a revêtement étale when restricted to $U_n$, which corresponds to the homomorphism $\varphi : \pi_1(U_n, \overline{\eta}) \to \pi_1(U, \overline{\eta})$ of topological groups which is the inclusion of the former group as an open subgroup of the latter corresponding to the connected, pointed étale cover $U_n \to U$. Let $\sigma \in \text{Gal}(\overline{\eta}/\eta)$ be any element of the inertia corresponding to the monodromy around any of the $D_i$ which acts nontrivially on $\mathcal{F}_{\overline{\eta}}$. Then the image of $\sigma$ in $\pi_1(U, \overline{\eta})$ is nonzero in $\pi_1(U, \overline{\eta})/\varphi \pi_1(U_n, \overline{\eta})$; in particular it does not lie in $\text{Gal}(\overline{\eta}/\eta_n)$ where $\eta_n$ is the generic point of $U_n$. Therefore the representation of $\pi_1(U_n, \overline{\eta})$ obtained by restricting $\pi^* \mathcal{F}$ to $U_n$ coincides with the restriction of a continuous representation of $\pi_1(X_n, \overline{\eta})$ by the continuous map $\pi_1(U_n, \overline{\eta}) \to \pi_1(X_n, \overline{\eta})$; i.e. $\pi^* \mathcal{F}$ extends to a locally constant sheaf $\overline{\mathcal{F}}$ on $X_n$.

**1.7.11.** As this paragraph reinterprets the construction of (1.7.8), it is implicit that $X$ is a henselian *trait*. A remark on the equality $\text{Gal}(K_1/K) = \text{Gal}(k_1/k)$: this is effectively saying that there is an equivalence of Galois categories of finite étale covers of the points Spec $K$ and Spec $k$. The functor of restriction to the special fiber does induce an equivalence of the categories of finite étale covers of $X$ with that of Spec $k$; this is [EGA,
IV 18.5.11]. For the equality of Galois groups in the setting of \( R \subset R^{sh}, K \subset K_1 \), \( R \) local henselian normal, \( R^{sh} \) its strict henselization, we use the following facts.

For any finite separable subextension \( k \subset k_2 \subset k_1 \), there exists a unique (up to unique isomorphism) finite étale local ring extension \( R \subset R_2 \) with specified residue field extension. Since the functor of restriction to the special fiber induces an equivalence of categories, the inductive system of finite separable subextensions \( k \subset k_2 \subset k_1 \) specifies an inductive system of local ring homomorphisms \( R \subset R_2 \subset R^{sh} \), of which \( R^{sh} \) is the colimit (Stacks, tag \( \emptyset \)BSL).

The following is a variation on the theme of the aforementioned equivalence. It is Lemma 7, §2.3 of Néron Models by Bosch, Lütkebohmert, and Raynaud

**Lemma.** Let \( R \) be a local ring, \( S' \) an étale \( R \)-scheme, and \( s' \) a point of \( S' \) above the closed point \( s \) of \( S = \text{Spec} R \). Let \( R' \) be the local ring \( \mathcal{O}_{S', s'} \) of \( S' \) at \( s' \) and let \( k' \) be the residue field of \( R' \). Furthermore, let \( A \) be a local \( R \)-algebra with residue field \( k_A \). Then all \( R \)-algebra morphisms from \( R' \) to \( A \) are local. So there is a canonical map

\[
\text{Hom}_R(R', A) \to \text{Hom}_k(k', k_A).
\]

*This map is always injective; it is bijective if \( A \) is henselian.*

The group \( \text{Gal}(K_1, K) \) acts on \( R^{sh} \) with fixed subring \( R \). Let \( k \subset k_2 \subset k_1 \) and \( R \subset R_2 \) be as above, then \( R_2 \) is normal as \( R \) is, and we have an isomorphism

\[
\text{Hom}_K(K_2, K_1) = \text{Hom}_R(R_2, R^{sh}) \cong \text{Hom}_k(k_2, k_1).
\]

As \( R \subset R_2 \) is étale, \( K_2 \) is separable over \( K_1 \), in the inductive limit we find that the induced map \( \text{Gal}(K_1, K) \to \text{Gal}(k_1, k) \) is an isomorphism.

1.8.1. The inclusion \( \tilde{\mathcal{O}} j_* \mathcal{F}_0 \subset j_* \tilde{\mathcal{O}} \mathcal{F}_0 \) comes about by considering that étale locally about a point \( s \in S_0 \) neither sheaf may be locally free, as restricting étale neighborhoods of \( s \) to \( U_0 \) may not be enough to trivialize either sheaf; in other words, a trivialization may ramify when extended to \( X_0 \). Let \( V \to X_0 \) be an étale neighborhood of \( s \). Then the inclusion above simply reflects the fact that sections of \( \tilde{\mathcal{O}} \mathcal{F}_0 \) over \( V |_{U_0} := V \times_{X_0} U_0 \)
include those coming from the tensor product of \( k \) sections in \( \mathcal{F}_0(V_{|U_0}) \), but might include more besides.

Here is an example: let \( X = \text{Spec} \, R \) and let \( \mathcal{F} \) be the locally free sheaf on \( X \) represented by \( \text{Spec} \, R[x]/(x^3 - 1) \); this is the sheaf \( \mu_3 \) of third roots of unity and it is a locally free sheaf of \( \mathbb{Z}/3 \)-modules of rank 1. Then \( \mathcal{F}(R) = \{1\} \) and if \( \zeta \) is a primitive 3rd root of unity, \( \mathcal{F}(\mathbb{C}) = \{1, \zeta, \overline{\zeta}\} \). \( \text{Gal}(\mathbb{C}/\mathbb{R}) \cong \mathbb{Z}/2 \) and \( \mathcal{F} \) corresponds to the representation \( V \) of \( \mathbb{Z}/2 \) given by complex conjugation on the \( \mathbb{Z}/3 \)-module \( \{1, \zeta, \overline{\zeta}\} \). Then \( \mathcal{F} \otimes_{\mathbb{Z}/3} \mathcal{F} \) corresponds to the tensor representation \( V \otimes_{\mathbb{Z}/3} V \). Its sections over \( R \) are its \( \mathbb{Z}/2 \)-invariant sections of \( V \otimes V \). These are \( \{1 \otimes 1, \zeta \otimes \overline{\zeta}\} \).

1.8.4. The fiber of \( j_* \mathcal{F}_0 \) at \( \overline{s} \) can be computed by taking first the inverse image of \( j_* \mathcal{F} \) to \( \text{Spec} \, \mathcal{O}_{X_0,s} \), the local ring of \( X_0 \) at \( s \), and then taking the colimit along all étale ring maps \( \mathcal{O}_{X_0,s} \to U \), these being equivalent to finite separable extensions of \( k(\eta) \) which are non-ramified over \( s \). So in the end we are computing the colimit of sections of the inverse image of \( \mathcal{F}_0 \) along \( \eta \to X_0 \) over finite separable field extensions of \( k(\eta) \) fixed by \( I \); this is nothing other than \( \mathcal{F}^1_{\overline{0I}} \).

The last line of the proof references (1.6.14.3), which has been corrected in the note (1.6.14) above.

1.8.5. The \( t \)-weights of \( \mathcal{F}_\eta \) are integers, as guaranteed by (1.8.4); therefore, we can apply (1.7.5). The nilpotent endomorphism \( N \) respects the filtration \( W \) on \( \mathcal{F}_\eta \), since all of \( W(\eta, \eta) \) respects the filtration, and hence the inertia does, so that the logarithm of the unipotent part of the local monodromy does too. For the local monodromy filtration on \( \mathcal{F}_\eta \), rel. \( W \) to exist, it remains only to check that \( N^k \) induces isomorphisms \( \text{Gr}_{i+k}^M \text{Gr}_i^W(\mathcal{F}_\eta) \to \text{Gr}_{i-k}^M \text{Gr}_i^W(\mathcal{F}_\eta) \); i.e. that the weight filtration \( M_1 \) on \( \text{Gr}_i^W(\mathcal{F}_\eta) \) coincides with the local monodromy monodromy filtration \( M_2 \) on \( \text{Gr}_i^W(\mathcal{F}_\eta) \), shifted by \( i \); i.e. that

\[(\dagger) \quad \text{Gr}_j^{M_1} \text{Gr}_i^W(\mathcal{F}_\eta) = \text{Gr}_j^{M_2} \text{Gr}_i^W(\mathcal{F}_\eta).\]
But (1.7.5) shows that the weight filtration $M_1$ is the unique finite increasing filtration on $Gr^W_i(\mathcal{F}_{\eta})$ which is stable under $W(K, K)$ such that $Gr^M_j Gr^W_i(\mathcal{F}_{\eta})$ is $t$-pure of weight $j$. On the other hand, (1.8.4) shows that the local monodromy filtration $M_2$ on $Gr^W_i(\mathcal{F}_{\eta})$ is a finite increasing filtration which is stable under $W(\eta, \eta)$ such that $Gr^M_j Gr^W_i(\mathcal{F}_{\eta})$ is $t$-pure of weight $i + j$. This shows that the filtration $M_2$, shifted by $i$, coincides with $M_1$; i.e. we have verified (5), and hence the existence of the local monodromy filtration on $\mathcal{F}_{\eta}$, rel. W.

1.8.8. With regards to remark 2), twist the sheaf $\mathcal{F}_0 \sim \mathcal{F}_0^{(b)}$ so that it has weight 0 (following (1.2.7), $b = p^{-b}$). Then apply (1.8.7) with the trivial filtration W to see that $Gr^M_j(j_*\mathcal{F}_0^{(b)}|D_0)$ is punctually $t$-pure of weight $i$. Twist back to conclude that $Gr^M_j(j_*\mathcal{F}_0|D_0)$ is punctually $t$-pure of weight $b + i$.

1.8.9. In c), $j_*\mathcal{F}_0 \hookrightarrow \epsilon_*j_*\epsilon^*\mathcal{F}_0$ follows, after writing $\epsilon_*j_*\epsilon^*\mathcal{F}_0 = j_*\epsilon_*\epsilon^*\mathcal{F}_0$, from the observation that $\mathcal{F}_0 \hookrightarrow \epsilon_*\epsilon^*\mathcal{F}_0$ injects since on stalks, a finite extension of a henselian ring splits as a product of henselian rings; i.e. the adjunction morphism corresponds to the inclusion along the diagonal

$$\mathcal{F}_x \hookrightarrow \prod_{\epsilon^{-1} x} \mathcal{F}_x.$$ 

In d), reduce to a constant sheaf, where it is obvious.

In the explanation for e), the strict henselization of $X_0$ at $x$ is irreducible hence a fortiori the inverse image of any open set is connected. Then, since the fiber product of any étale cover of $x$ with $U_0$ is an étale neighborhood of $z$, there is a map $(i^*j_*\mathcal{F}_0)_x \rightarrow (\mathcal{F}_0)_z$. As the former can be computed as sections over the inverse image of $U_0$ in the strict henselization of $X_0$ at $x$; since this is a connected scheme, and $\mathcal{F}_0$ is lisse, the arrow is injective. The factorization of this arrow as

$$(i^*j_*\mathcal{F}_0)_x \rightarrow (k_*k^*j_*\mathcal{F}_0)_x \rightarrow (j_*\mathcal{F}_0)_y \rightarrow (\mathcal{F}_0)_x$$

can be explained as follows. After rewriting $k_*k^*j_*\mathcal{F}_0$ as $k_*k^*i^*j_*\mathcal{F}_0$, the first arrow is just adjunction for $k$, The middle arrow can be rewritten $(k_*k^*i^*j_*\mathcal{F}_0)_x \rightarrow (i^*j_*\mathcal{F}_0)_y$.
so that it is a statement about sheaves on $F_0$. Take an étale neighborhood $W_0$ of $x$ in $F_0$; then $W_0 \times_{F_0} V_0$ is an étale neighborhood of $y$. The projective system of étale $W'_0 \rightarrow F_0$ s.t. $W_0 \times_{F_0} V_0$ admits an arrow to $W_0 \times_{F_0} V_0$ has the property that the projective system $W'_0$ is a subcategory of the projective system of étale neighborhoods of $y$ in $F_0$. Therefore there is an arrow from the colimit of $i^* j_* \mathcal{F}_0$ applied to the former system to the colimit of $i^* j_* \mathcal{F}_0$ applied to the latter, which is $(i^* j_* \mathcal{F}_0)_y$. This gives an arrow $(k_* k^* i^* j_* F_0)(W_0) \rightarrow (i^* j_* \mathcal{F}_0)_y$, functorial in $W_0$, and hence an arrow $(k_* k^* i^* j_* F_0)_x \rightarrow (i^* j_* \mathcal{F}_0)_y$.

The last arrow is in effect the observation that the fiber product of $U_0$ with an étale neighborhood of $y$ in $X_0$ is an étale neighborhood of $z$.

In the proof of (1.8.9), the reductions are all clear except perhaps the reduction to $\mathcal{F}_0$ tamely ramified at the generic points of $F_0$. Suppose all the other dévissages have been made except that one; then as there are finitely many generic points and $\mathcal{F}_0$ is a lisse $\mathcal{O}_T$ sheaf corresponding to a representation $V$ of $\pi_1(U_0, \overline{x})$ for some geometric point $\overline{x}$ of $U_0$, the image of the wild inertia at the finitely many generic points of $F_0$ in $\text{Aut} V$ is finite. In particular, the preimage of the congruence subgroup $\Gamma_1$ of $\text{Aut} V$ in $\pi_1(U_0, \overline{x})$ is open and of finite index, corresponding to an étale cover of $V_0 \rightarrow U_0$ which extends to a finite surjective morphism to $X_0$. Applying c), we reduce to the desired situation.

To complete the proof, we are almost there, except $F_0$ is just a Weil divisor, and need not satisfy the smoothness assumption of (1.8.6). The idea is to use e) and recurrence on $\dim U_0$ to shrink $X_0$ and throw away the bad points of $F_0$. If we replace $X_0$ by an open set containing $U_0$ whose intersection with $F_0$ is a lisse divisor, then (1.8.8) 2) shows that (1.8.9) is true there. We can find finitely many such open sets with inclusions $j_i$, the union of which, $X'_0$, intersects $F_0$ in a dense set $V_0$. If $j' : U_0 \hookrightarrow X'_0$, then $j'_* \mathcal{F}_0 \hookrightarrow \prod_i j_i* \mathcal{F}_0$ so (1.8.9) is proved for $j'$. By recurrence on dimension we may assume that (1.8.9) holds for $k_*$. In light of this, applying e) to the lisse sheaf $\mathcal{F}_0$, yields that $i^* j_* \mathcal{F}_0$ satisfies the conclusions of (1.8.9); in effect, $k^* j_* = k^* j'_*$. Finally, $j_* \mathcal{F}_0 \hookrightarrow j'_* \mathcal{F}_0 \times i_* i^* j_* \mathcal{F}_0$ allows us to conclude that $j_*$ satisfies (1.8.9).
1.8.11. A Jordan-Hölder series for $\mathcal{F}_0$ allows us to reduce to $\mathcal{F}_0$ irreducible. The restriction of an irreducible lisse sheaf to a nonempty open $U_0$ of a normal connected scheme $X_0$ is still irreducible because if $\eta$ denotes the generic point of $X_0$, and $\overline{\eta}$ a geometric point centered on $\eta$, we have by [SGA1, Exp. V, 8.2]
\[
\text{Gal}(\overline{\eta}, \eta) \rightarrow \pi_1(U_0, \overline{\eta}) \rightarrow \pi_1(X_0, \overline{\eta}).
\]
Now, $\mathcal{F}_0$ is $t$-mixed, so admits an $t$-pure subsheaf $\mathcal{G}_0$ which is lisse when restricted to some $U_0$. Therefore $\mathcal{F}_0|U_0$, irreducible yet containing $\mathcal{G}_0|U_0$, must equal $\mathcal{G}_0|U_0$.

1.8.12. Let $f : X'_0 \rightarrow X_0$ be the normalization morphism. It induces a bijection on irreducible components, and $X'_0$ is a disjoint of normal integral schemes. We make use of the fact, true for any morphism, that if $x_0 \in X_0$, the weights of $\mathcal{F}_0$ at $x_0$ coincide with the weights of $f^*\mathcal{F}_0$ at every point of the fiber $f^{-1}(x_0)$. If $\mathcal{F}_0$ is $t$-pure of weight $\beta$ at a point $x_0$, then all the points of $X'_0$ in the fiber over $x_0$ are also $t$-pure of the same weight; therefore (1.8.11) implies that all the points of the irreducible components of $X'_0$ meeting the fiber of $x_0$ are $t$-pure of weight $\beta$, which in turn implies that all the points in the irreducible components of $X_0$ meeting $x_0$ are $t$-pure of weight $\beta$, which shows that the locus of points where $\mathcal{F}_0$ is $t$-pure of weight $\beta$ is closed. This locus is also open because, taking any $x_0$ in it, there is an open neighborhood of $x_0$ which meets only the irreducible components of $X_0$ on which $x_0$ lies. Then the above construction shows that all the points in this neighborhood are also $t$-pure of weight $\beta$.

References

[WeilIII] Deligne, Weil II.
[GM] The Tame Fundamental Group of a Formal Neighbourhood of a Divisor with Normal Crossings on a Scheme par A. Grothendieck et J. Murre.
[SGA5] SGA 5, dirigé par Grothendieck
[SGA 4\(\frac{1}{2}\)]  SGA 4\(\frac{1}{2}\), *Rapport sur la formule des traces* par P. Deligne.
4. Notes on Faisceaux Pervers

1.11. To verify the anti-commutativity of the 9th square, as the morphism of triangles $(X'', Y'', Z'') \rightarrow (X'[1], Y'[1], Z'[1])$ factors as the composition of two morphisms of triangles $(X'', Y'', Z'') \rightarrow (A, Y'', Z'[1]) \rightarrow (X'[1], Y'[1], Z'[1])$, where the second arrow is the rotation of $(Z', A, Y'') \rightarrow (Z', X'[1], Y'[1])$, it suffices to verify that the triangle $(Z', X'[1], Y'[1])$ which appears in this last morphism of triangles has all arrows induced by the arrows of $(X', Y', Z')$ or translates of them (with the same parity). This is not hard to check from the diagram (1). (The stated explanation appears to be an un-explanation.)

1.3.3. Though it is not stated explicitly, it is immediate from the definition $D^{\leq n} := D^{\leq 0}[-n], D^{\geq n} := D^{\geq 0}[-n]$ that $(\tau_{\leq n} X)[m] = \tau_{\leq n-m}(X[m])$ and $(\tau_{\geq n} X)[m] = \tau_{\geq n-m}(X[m])$. Namely, for $X$ in $D$ and $T$ in $D^{\leq n}, T = T'[-m]$ for some $T'$ in $D^{\leq n-m}$, so

$$\text{Hom}(T, (\tau_{\leq n-m}(X[m]))[-m]) = \text{Hom}(T'[-m], (\tau_{\leq n-m}(X[m]))[-m])$$

$$= \text{Hom}(T'[-m], \tau_{\leq n-m}(X[m]))[-m] = \text{Hom}(T', X[m])[-m] = \text{Hom}(T'[-m], X)$$

$$= \text{Hom}(T, X).$$

1.4.2.1. The argument for why the derived functors continue to satisfy the stated adjunctions is as follows (this argument is also found in SGA 4 Exp. XVIII 3.1.4.11). Given $F, G$ an adjoint pair of functors on abelian categories

$$\mathcal{A} \xrightarrow{F} \mathcal{B} \xleftarrow{G}$$

where both categories have enough injectives and $L$ is exact. The functors $F$ and $G$ extend to functors $D^+(\mathcal{A}) \rightleftarrows D^+(\mathcal{B})$. Given $K' \in D^+(\mathcal{A}), L' \in D^+(\mathcal{B})$, we may assume $L'$ is a complex of injective objects; we have an isomorphism of triple complexes

$$\text{Hom}'(F(K'), L) \xrightarrow{\sim} \text{Hom}'(K', G(L')).$$

As $G$ preserves injectives, taking $H^0$ of the associated simple complex (calculated with products) finds the desired

$$\text{Hom}_{K(\mathcal{B})}(F(K'), L') \xlongleftarrow{\sim} \text{Hom}_{K(\mathcal{A})}(K', G(L')).$$
where both sides are also Hom in the respective derived categories, since $L'$ and $G(L')$ are complexes of injectives.

1.4.4. The question is, why are the adjoints to the Verdier quotients fully faithful? Let’s consider the quotient $Q : \mathcal{T} \to \mathcal{T}/\mathcal{U}$, where $\mathcal{U}$ is the strictly full coreflective triangulated subcategory of $\mathcal{T}$; $(\mathcal{U}, \mathcal{V})$ form a t-structure on $\mathcal{T}$; $\mathcal{U} = ^\perp \mathcal{V}$, and $\mathcal{V} = \mathcal{U}^\perp$. Since the embedding $u : \mathcal{U} \to \mathcal{T}$ admits a right adjoint $u$, it follows that $Q$ admits a right adjoint $Q$, [CD, I 6-5]. There is a natural isomorphism of functors $Q \circ Q \sim v \circ v^\prime$, where $v^\prime$ is the left adjoint to the inclusion $v : \mathcal{V} \to \mathcal{T}$ [CD, I 6-6]. The functor $v^\prime$ is nothing other than $\tau_{\geq 0}$ for the t-structure $(\mathcal{U}, \mathcal{V})$, and therefore, restricted to $\mathcal{V}$, $v \circ v^\prime = \text{id}|_{\mathcal{V}}$. On the other hand, the functor $Q$ when restricted to $\mathcal{V}$ is fully faithful [CD, I 5-3]. Therefore $Q$, restricted to the essential image of $\mathcal{V}$ under $Q$ is fully faithful. This essential image is all of $\mathcal{T}/\mathcal{U}$, since every object $X$ in $\mathcal{T}$ belongs to an exact triangle $(U, X, V)$ with $U$, $V$ objects in $\mathcal{U}$, $\mathcal{V}$, respectively. The assertion that $v^\prime$ yields an equivalence $v^\prime : \mathcal{T}/\mathcal{U} \sim \mathcal{V}$ (functor obtained by applying the universal property of $\mathcal{T}/\mathcal{U}$ to $\mathcal{V}$) is easy since $Q \circ v : \mathcal{V} \to \mathcal{T}/\mathcal{U}$ is an equivalence, and $v^\prime \circ Q \circ v = v^\prime \circ v = \text{id}$. The corresponding statement for $\mathcal{T} \to \mathcal{T}/\mathcal{V}$ follows identically.

1.4.6. In part b), there is the following consideration. Given triangulated categories $\mathcal{T}, \mathcal{T}'$, a thick subcategory $\mathcal{U} \subset \mathcal{T}$ with Verdier quotient $Q : \mathcal{T} \to \mathcal{T}/\mathcal{U}$, exact functors $F, G : \mathcal{T}/\mathcal{U} \to \mathcal{T}'$ and a natural transformation $\overline{q} : F \circ Q \to G \circ Q$, there is an obvious candidate for a natural transformation $q : F \to G$, since $\text{Ob}(\mathcal{T}/\mathcal{U}) = \text{Ob}(\mathcal{T})$. But is it still a natural transformation? Let $f : X \to Y$ in $\mathcal{T}/\mathcal{U}$ be represented by $X \overset{s}{\leftarrow} Z \overset{a}{\rightarrow} Y$, where $s$ is in the saturated multiplicative system of morphisms corresponding to $\mathcal{U}$. The commutative diamond

```
+-----------------+  +-----------------+
| F(X)            |  | G(X)            |
|                 |  |                 |
| F(s)            |  | G(s)            |
|                 |  |                 |
+-----------------+  +-----------------+
| id              |  |                 |
|                 |  |                 |
+-----------------+  +-----------------+
| F(Z)            |  | G(Z)            |
|                 |  |                 |
| q(Z)            |  |                 |
|                 |  |                 |
+-----------------+  +-----------------+
| q(X)            |  | G(\alpha)       |
|                 |  |                 |
```

where both sides are also Hom in the respective derived categories, since $L'$ and $G(L')$ are complexes of injectives.
shows that $\varphi(X) \circ G(f)$ coincides with $F(X) \xrightarrow{F(s)} F(Z) \xrightarrow{G(a) \circ \varphi(Z)} G(Y)$, but as

\[
\begin{array}{ccc}
F(Z) & \xrightarrow{F(s)} & F(Y) \\
\downarrow q(Z) & & \downarrow q(Y) \\
G(Z) & \xrightarrow{G(a)} & G(Y)
\end{array}
\]

commutes, this morphism is just $F(X) \xleftarrow{F(s)} F(Z) \xrightarrow{q(Y) \circ F(a)} G(Y) = F(f) \circ \varphi(Y)$, which shows that $\varphi$ is indeed a natural transformation.

1.4.7. In c), by 1.1.9, the morphism $B \to C$ is the unique such that completes the morphism of triangles $(B, j_! j^* X[1], j_* j^* X[1]) \to (C, X[1], j_* j^* X[1])$.

1.4.13. The distinguished triangle $(\tau^F_{\leq p} X, X, i_* \tau_{> p} i^* X)$ is the distinguished triangle $(A, Y, i_* \tau_{> p} i^* Y)$ of 1.4.10, since as remarked, $X = Y$ since $\tau_{> 0} j^* X = 0$. To check that $A = \tau^F_{\leq p} X$, note that $A$ belongs to $\mathcal{D}_{= p}$, as $i^* A \simeq \tau_{\leq p} i^* X$, and that if $T$ belongs to $\mathcal{D}_{= p}$, then by applying $\text{Hom}(T, -)$ to the above distinguished triangle and observing that as $i_*$ is t-exact, it commutes with truncation, so $i_* \tau_{> p} i^* X[-1]$ lies in $\mathcal{D}_{> p+1}$, $\text{Hom}(T, i_* \tau_{> p} i^* X) = 0 = \text{Hom}^{-1}(T, i_* \tau_{> p} i^* X)$, and $\text{Hom}(T, X) \simeq \text{Hom}(T, A)$.

The distinguished triangle $(\tau^F_{\leq p-1} X, X, i_* \tau_{> p-1} i^* X)$ and the fact that $i_*$ commutes with truncation establishes $i_* \tau_{> p-1} i^* X$ as $\tau_{> p-1} X$ for the t-structure on $\mathcal{D}$; applying $\tau_{\leq p}$ and passing it through the $i_*$ gives the statement about cohomology.

1.4.14. To find the dual statement at the end of the proof, reverse arrows and exchange $j_! \leftrightarrow j_*$, $i^* \leftrightarrow i^!$ to obtain the distinguished triangle $(i_* i^* X[-1], j_! Y, X)$, then use (b'), the isomorphism $j_*/j_! \simeq i^! j_! [1]$ of 1.4.6.4, (and the note to 1.3.3) to write

$$i_* i^* X[-1] = i_* (\tau_{\leq p-1} (j_*/j_! Y)[-1] = i_* \tau_{\leq p} ((j_*/j_!) Y[-1]) = i_* \tau_{\leq p} i^! j_! Y,$$

establishing $X$ as $\tau^F_{> p+1} j_! Y$.

1.4.17.1. A little note: $p^! i^* X$ is the largest quotient of $X$ belonging to $C_F$. First we check that it is a quotient from 1.4.17 (ii). Then, suppose $A$ belongs to $C_F$ and $X \to A$; then
\[ p^*_i X \to p^*_i A \sim A, \text{ as } p^*_i \text{ is fully faithful, so the adjunction morphism } p^*_i p_! \to \text{id is an isomorphism, and } p^*_i X \text{ is indeed the largest quotient of } X \text{ in } C_F. \text{ Dually for } p^*_i. \]

1.4.18. A little note about \( \mathcal{T} \) faithful: as \( p^*_j \) is an exact functor, if \( p^*_j f_i = 0 \), this means that \( p^*_j \text{ im}(f_i) = \text{ im}(p^*_j f_i) = 0 \), which is to say that \( \text{ im } f_i \) belongs to \( \bar{C}_F \).

1.4.23. In the distinguished triangle \( (i_* H^0 j_! B, \tau^{F}_{\geq 0} j_! B, \tau^{F}_{\geq 1} j_! B), \) as \( j_! B = \tau^{U}_{\geq 0} j_! B, \tau^{p}_{\geq p} = \tau^{F}_{\geq p} \tau^{U}_{\geq p}, \) and \( j_! \) is right \( t \)-exact, \( \tau^{F}_{\geq 1} j_! B \) sits in \( C \). Likewise, \( i_* \) is \( t \)-exact, so \( i_* H^0 j_! B \) also sits in \( C \), and from the long exact sequence of \( H^i \) one finds that \( \tau^{F}_{\geq 1} j_! B \) is in \( D[1,0] \).

2.1.2. In the discussion ‘\( Si les foncteurs \circ i^*_S \) sont de dimension cohomologique finie...’ it is claimed that there is a neighborhood of \( S \) in which \( H^j \tau_{<a} K \) is supported on \( S \). To find such a neighborhood, simply discard \( \bar{S} - S \) and the closure of any stratum which doesn’t meet \( S \). The assumption that the closure of each stratum is a union of strata implies that the induced stratification of the resulting neighborhood of \( S \) has the property that every stratum contains \( S \) in its closure, and therefore \( H^j \tau_{<a} K \) vanishes on every stratum distinct from \( S \). By construction, \( S \) is a closed set in this neighborhood.

As for the isomorphism \( H^j(\tau_{<a} K) \sim H^j(i^*_S \tau_{<a} K) \) for \( i < a \), let us replace \( X \) by the neighborhood above, in which case the adjunction morphism \( \tau_{>a} K \to i_S i^*_S \tau_{<a} K \) is an isomorphism as \( \circ i^*_S \) and \( \circ i^*_S \) are exact and the induced morphism on cohomology \( H^j(\tau_{>a} K) \to i_S i^*_S H^j(\tau_{<a} K) \) is an isomorphism for all \( j \). Therefore by 1.4.1.2, \( i^*_S \tau_{<a} K \sim i^*_S i^*_S \tau_{<a} K \sim i^*_S i^*_S \tau_{<a} K \).

2.1.13. To get the desired conclusion from the spectral sequence \( R^p j_* H^q K \Rightarrow H^{p+q} R j_* K \), recall that the locally constant constructible sheaves form a weak Serre subcategory of the category of constructible sheaves of \( O \)-modules. For \( j^i \) when \( j \) is a closed immersion, just use that \( j^i = j_* \) is exact in the long exact sequence of cohomology for the distinguished triangle \( (j^i j^! K, K, k^* K) \).

One can deduce that the truncation operators \( \tau_{\leq p} \) and \( \tau_{\geq p} \) respect \( D_c(X,R) \) from the proof of 1.4.10 by induction on the number of strata à la 2.1.3.
2.1.14. For the isomorphism $i^!L \cong i^*L \otimes_{\mathbb{Z}} \text{or}[-d]$, combine Proposition 4.3.6 from Dimca, *Sheaves in Topology* with the description of the relative dualizing complex in, e.g., Remark 3.3.5 in Kashiwara-Schapira, *Sheaves on Manifolds*, taking $S = \{\ast\}$ there and noting that $f^!A_X$ is the relative dualizing complex $\omega_{Y/X}$ (Definition 3.1.16), and the orientation sheaves are self-dual (so that Deligne’s or coincides with $\text{or}_T \otimes i^* \text{or}_S$). The ‘non-characteristic’ hypothesis in Dimca’s Proposition is trivial in light of the result that precedes it and the fact that the $\text{H}^n j^!K$ are locally constant.

Note that by the description of $i^!$ in terms of local cohomology, this gives a statement like, $\text{H}^n_T(S, \mathcal{F})$ vanishes in degrees less than the codimension of $T$, for $\mathcal{F}$ a locally free sheaf (with obvious extension to $\mathcal{F}$ a bounded below complex with locally free $\text{H}^0$).

2.1.16. In the remarks about Verdier duality, the fact that the functor $D$ exchanges $j_! \leftrightarrow j_*$, $j^! \leftrightarrow j^*$ follows by taking $L$ to be the dualizing complex in the local formulas of adjunction

\[
j_!DK = j_!\text{RHom}(K, j^!L) \sim \text{RHom}(j_!K, L) = Dj_*K
\]

\[
Dj^*K = \text{RHom}(j^*K, j^!L) \sim j^!\text{RHom}(K, L) = j^!DK
\]

(the latter may be found, e.g. in SGA 4 Exp. XVIII 3.1.12.2, or Dimca, 3.3.7, but see especially SGA 5 Exp. I 1.12). These formulæ hold for more general morphisms than the inclusion of a locally closed subscheme; the condition is compactifiability.

It is essential that the cohomology sheaves be locally constant of finite rank when restricted to each stratum and that $R$ have the proscribed properties so that $\text{RHom}$ may be computed locally and $D$ be indeed involutive. A brief sketch of the latter, which goes under the name ‘biduality.’ The situation in general appears to be complicated (c.f. SGA 5 Exp. I), but in this special case subject to the conditions 2.1.13, dévissage brings us to the consideration of $j_!K$ a locally constant sheaf (concentrated in degree 0) and $j : S \to X$ the inclusion of a stratum. The morphism $j_!K \to DDj_!K$ is an isomorphism if $\text{R}_{j_!K} \to j^!\text{DDK}$ is, and on $S$ it is seen in light of 2.1.13 c) that $K \sim DDK$ by the given formula involving the orientation sheaf.
About the formula $H^iDK = (H^{-d-i}K)^\vee \otimes \text{or}$, let’s do it instead in the case $R = \mathbb{Z}/\ell$, in which case the dualizing complex is $\mathbb{Z}/\ell(d)[2d]$ and the formula is

\[(\dagger) \quad H^iDK = (H^{-2d-i}K)^\vee(d).\]

In light of the note 2.1 in §7, this is the immediate consequence of the weakly convergent spectral sequence which collapses at the $E_2$ page

\[E_2^{pq} = \text{Ext}^p(H^{-q}(K), \mathbb{Z}/\ell(d)) \Rightarrow \text{Ext}^{p+q}(K, \mathbb{Z}/\ell(d)).\]

The business about exchanging $H^i$ and $H^{-i}$ is seen to be true for $H^0$, then use $H^iK = H^0(K[i])$ and $D(K[n]) = \text{RHom}(K[n], -) = \text{RHom}(K, -)[-n] = (DK)[-n]$

2.2.2. A word about the ‘trivial’ implication (ii)$\Rightarrow$(iii). As each $S$ in $\mathcal{T}$ is lisse equidimensional, applying (ii) to each irreducible component we find there exists a Zariski dense open $i : U \hookrightarrow S$ such that $i^*H^iS^\vee K = H^iS^\vee K$ (resp. $i^*H^iS^\vee K = H^iS^\vee K$) vanish in degrees $i > p(S)$ (resp. $i < p(S)$); as $H^iS^\vee$ and $H^iS^\vee$ are locally constant, this implies they also vanish when their restrictions to $U$ do.

In the second paragraph, ‘il reste à montrer que chacune implique que $H^iK = 0$ pour $i > b$ (resp. $i < a$).’ Without securing this, we would not have that $\tau_{\le b}K$ (resp. $\tau_{\ge a}K$) belongs to $D^b_c$ and the proof in the first paragraph wouldn’t apply. In both cases, the verification for $i > b$ is easy as “$i^*$ is exact. The verification for (iii) $i < a$ follows the proof of (2.1.2.1) exactly, after $\mathcal{T}$ has been replaced by a finer stratification. The verification for (ii) follows immediately by the noetherian property from the stated claim that for each irreducible subvariety $S'$ and each $i < a$, there exists a dense open $S$ of $S'$ such that $H^iK$ vanishes on $S$. To prove this claim we proceed by descending induction on dim $S'$, the maximal nonvacuous case being easy since such an $S'$ is an irreducible component of $X$, and $H^iK$ vanishes on the dense open $S$ obtained from (ii) since $S$ is open in $X$ and $i'_S = i_S^\vee$ is exact. The case of general $S'$ will follow from the argument of the proof of (2.1.2.1) (see note for 2.1.2) if we can find a neighborhood $U$ of the $S$ obtained from (ii) so that $\text{supp}(H^i\tau_{<a}K)|_U \subset S$. Begin with the irreducible component containing $S'$; the inductive assumption gives an open set which has either empty or nonempty intersection with $S'$. If nonempty, then this intersection is the desired open
of $S'$ of the claim. If empty, $S'$ belongs to the complement. Throw away all irreducible components of the complement that do not contain $S'$ and apply the same process to the irreducible component of the complement that contains $S'$. After finitely many steps we are left only with $S'$, and we throw away $S' - S$. The open neighborhood $U$ is the set we are left with after throwing away finitely many closed sets of $X$, and $S$ is closed in $U$.

2.2.3. In the discussion of the intermediate extension, the triangle $(\tau_{<l}i^*j_*A, i^*j_*A, \tau_{\geq l}i^*j_*A)$ is distinguished, not the one written, and if $\tau_{\geq l}i^*j_*A$ is in $pD_{\mathcal{C}}^{\geq 0}$, then indeed $p\tau_{<l}i^*j_*A \sim \tau_{<l}i^*j_*A$ and $\tau_{\geq l}i^*j_*A \sim p\tau_{<l}i^*j_*A$; the latter isomorphism establishes an isomorphism $p\tau_{<l}A \sim \tau_{<l}A$, which differs by one character from what is written.

2.2.8. The discussion of $\text{RHom}$ differs from what is proved in 2.1.20 in that there is no longer a fixed stratification. Fortunately, if $p \leq a, b \geq q$, there are only finitely many solutions to $i = m - n$ for a fixed $i$ and for $n \leq a, m \geq b$. Therefore for each $i$, we can apply the reasoning of 2.1.20 to a common refinement of only finitely many stratifications.

2.2.10. In order to have a good (involutive) Verdier duality on each stratum as in 2.1.16, it is necessary to secure that the complex of sheaves has l.c.c. cohomology, and that the stratum is smooth. The program outlined in this paragraph achieves both. Here is an explication of the smoothness condition (a).

**Lemma** (EGA 0IV 22.5.8 & IV 6.7.6, 6.7.8, Stacks 07EL & 038X). Let $X$ be a scheme locally of finite type over a field $k$ and $x \in X$. Then the following are equivalent:

(i) $X \to \text{Spec } k$ is smooth at $x$.

(ii) $X$ is geometrically regular at $x$, i.e. for every finite extension $k'$ of $k$, the semi-local ring $(\mathcal{O}_X)_x \otimes_k k'$ is regular.

(iii) $X \times_k \overline{k}$ is regular at any point lying over $x$. 
The smoothness condition on strata is that over \( \overline{k} \), each stratum \( S \), with the reduced subscheme structure, is smooth. The claim is then that on \( S \) equidimensional of dimension \( d \), the dualizing complex is given by \( \mathbf{Z}/\ell(d)[2d] \). After replacing \( k \) by its perfect closure, anodyne operation with respect to the étale topology, we may assume \( S \) is of finite type over a perfect field \( k \). The fact that \( S \times_k \overline{k} \) is smooth implies (in light of the lemma and Stacks tag \( \text{030U} \)) that \( S \), with its reduced scheme structure, is smooth. In this case, the fact about the dualizing complex is standard (SGAA Exp. XVIII 3.2.5).

2.2.14. A brief review of Galois cohomology of a finite field \( k \), to recall why the groups \( H^i(\text{Gal}(\overline{k}/k'), \mathbf{Z}/\ell) \) are finite for every finite extension \( k' \) of \( k \). We have

\[
H^0 = H^1 = \mathbf{Z}/\ell \\
H^i = 0 \quad i > 1.
\]

The case of \( H^0 \) is obvious as it corresponds to taking \( G = \text{Gal}(\overline{k}/k') \)-invariants of a trivial \( G \)-module. The \( H^1 \) is the corollary of a formula given in Serre, \textit{Corps Locaux}, Ch. XIII Prop. 1 (p. 197 in the 1968 édition Hermann). The vanishing in degrees > 1 is because a finite field is \( C_1 \) and hence has finite cohomological dimension; now see (1.6) in \textit{Arcata}, SGA 4\( 1 \frac{1}{2} \). For more details see Serre, \textit{Cohomologie Galoisienne}, Ch. II, §3.

Let \( D_n = D^b_{\text{ctf}}(X, \mathbf{Z}/\ell^n) \), \( K, L \) objects of \( D_n \), \( G = \text{Gal}(\overline{k}, k) \), and \( f : X \to \text{Spec } k \) the structure morphism. Why does the above imply that \( \text{Hom}_{D_n}(K, L) \) is finite? Deligne’s finiteness theorems show that the sheaves \( \underline{\text{RHom}}(K, L) \) belong to \( D_n \). The Hochschild-Serre spectral sequence gives

\[
E^2_{ij} = H^i(G, R^j\Gamma(X \times_k \overline{k}, \underline{\text{RHom}}(K, L))) \Rightarrow R^{i+j}\Gamma(X, \underline{\text{RHom}}(K, L)).
\]

As \( R\Gamma(X \times_k \overline{k}, \underline{\text{RHom}}(K, L)) \) coincides with the stalk of the constructible sheaf \( Rf_*\underline{\text{RHom}}(K, L) \) at any geometric point of \( \text{Spec } k \), it belongs to \( D^+_\ell(\mathbf{Z}/\ell^n) \) (even \( D^+_{\text{ctf}} \); c.f. Th. finitude 1.7).

Fix \( j \) and let \( A := R^j\Gamma(X \times_k \overline{k}, \underline{\text{RHom}}(K, L)) \); it is a finite \( G \)-module. Let \( U \) denote the kernel of \( G \to \text{Aut } A \); it is an open normal subgroup of finite index corresponding to a finite extension \( k' \) of \( k \). The Galois group \( U = \text{Gal}(\overline{k}, k') \) acts trivially on \( A \) and a simple dévissage reducing to the case \( \mathbf{Z}/\ell \) shows that the \( H^i(U, A) \) are finite. As \( G/U \)
is a finite group, the spectral sequence (c.f. *Cohomologie Galoisienne* §2.6b)

\[ H^p(G/U, H^q(U, A)) \Rightarrow H^{p+q}(G, A) \]

shows that the groups \( H^i(G, A) \) are also finite, and therefore that the objects on the \( E_2 \) page of (†) are finite \( \mathbb{Z}/\ell^n \)-modules so that \( R^0\Gamma(X, R\text{Hom}(K, L)) = \text{Hom}_{\mathcal{D}_h}(K, L) \) is finite.

**2.2.16.** See note to Weil II (1.1.2). In that paper, the functor \( H^i(K) \) is defined as the pro-sheaf which is the projective system defined by the \( H^i(K_n) \); the corresponding projective system is AR-isomorphic to an \((\ell\mathbb{Z})\)-adic sheaf in the naïve sense. This allows us to upgrade the pointwise exact sequence (*) of those notes to the corresponding sequence of sheaves (2.2.16.1). There, he uses the notation \( K \otimes^L \mathbb{Z}/\ell^n \) for \( K_n \), whereas here \( \otimes \) is used instead of \( \otimes^L \). In the interest of consistency, I will continue with the notation \( K \otimes^L \mathbb{Z}/\ell^n \) for \( K_n \). So in this paragraph, we are implicitly in the AR category or in the category of pro-sheaves. (In Weil II, Deligne uses the definition of \( \mathbb{Z}_\ell \)-sheaf as any pro-sheaf in the essential image of the \( \mathbb{Z}_\ell \)-sheaves; i.e. the \((\ell\mathbb{Z})\)-adic objects of the category of abelian constructible sheaves.) For the business about \( H^0 \) inducing an equivalence between \( \mathcal{D}^b_c(X, \mathbb{Z}_\ell)^{\leq 0} \cap \mathcal{D}^b_c(X, \mathbb{Z}_\ell)^{> 0} \) and \( \mathbb{Z}_\ell \)-constructible sheaves, this is simple, since for such a \( K \) we can represent \( K \otimes^L \mathbb{Z}/\ell \) by a sheaf concentrated in degree 0. As every complex of flat sheaves representing the \( K_n \) admits an \( \ell \)-adic filtration with successive quotients quasi-isomorphic to \( K \otimes^L \mathbb{Z}/\ell \), we can represent \( K \) by a projective system of flat sheaves concentrated in degree 0. In this case, \( K \) is a bona fide \( \ell \)-adic sheaf.

As for checking whether \( K \) belongs to \( \mathcal{D}^b_c(X, \mathbb{Z}_\ell)^{\leq 0} \), the statement is punctual and we may consider the problem in \( \mathcal{D}_{\text{parf}} \). If \( H^i(K \otimes^L \mathbb{Z}/\ell^n) \) is null for one \( n \), then \( H^i(K) \otimes \mathbb{Z}/\ell^n \) is null by (2.2.16.1), so \( H^i(K) \otimes \mathbb{Z}/\ell \) is null. This implies by the exact sequence

\[ 0 \to \ell H^i(K)/\ell^n H^i(K) \to H^i(K)/\ell^n H^i(K) \to H^i(K)/\ell H^i(K) \to 0 \]

and the fact that \( H^i(K)/\ell^{n-1} \to H^i(K)/\ell^n \) surjects that \( H^i(K) \otimes \mathbb{Z}/\ell^n \) is null for all \( n \) and so \( K \) is in \( \mathcal{D}^b_c(X, \mathbb{Z}_\ell)^{\leq 0} \). On the other hand, if \( K \) is in \( \mathcal{D}^b_c(X, \mathbb{Z}_\ell)^{\leq 0} \), the exact sequence (2.2.16.1) for \( n = 1 \) tells us that \( H^i(K \otimes^L \mathbb{Z}/\ell) = 0 \) for \( i > 0 \). The \( \ell \)-adic filtration on
flat complexes representing $K \otimes^L \mathbb{Z}/\ell^n$ has successive quotients quasi-isomorphic to
$K \otimes^L \mathbb{Z}/\ell$ and the sequence of cohomology then establishes that $H^i(K \otimes^L \mathbb{Z}/\ell^n)$ is null.

2.2.17. On the equivalent conditions: suppose $K \otimes^L_{\mathbb{Z}/\ell} \mathbb{Z}/\ell^n$ is in $D^b_{S,L}(X, \mathbb{Z}/\ell^n)$ for
one $n$. In the spirit of Weil II (1.1.2) claim a), observe that as the projective system
$H^i(K \otimes^L \mathbb{Z}/\ell^k)$ is noetherian AR-(\ell\mathbb{Z})-adic, the projective system $H^i(K \otimes^L \mathbb{Z}/\ell^{kn})$ is
noetherian AR-(\ell^n\mathbb{Z})-adic and AR-isomorphic by (3.2.3) of that section (i.e. [SGA5,
V, 3.2.3]) to the (\ell^n\mathbb{Z})-adic system $(\mathcal{H}_k^{i}/\ell^{kn}, \mathcal{H}_k^{i})_{k \in \mathbb{N}}$ for some integer $r \geq 0$, where
$\mathcal{H}_k^{i}$ denotes the projective system of universal images of the system $H^i(K \otimes^L \mathbb{Z}/\ell^{kn})$.
As $K \otimes^L \mathbb{Z}/\ell^n$ is in $D^b_{S,L}(X, \mathbb{Z}/\ell^n)$, the $H^i(K \otimes^L \mathbb{Z}/\ell^n)$ are locally constant on the strata
in $S$. Taking successive quotients on the $\ell^n$-adic filtration on bounded complexes of
flat sheaves representing the $K \otimes^L \mathbb{Z}/\ell^k$, we find that the $H^i(K \otimes^L \mathbb{Z}/\ell^{kn})$ are also
locally constant on the strata. Therefore the sheaves $\mathcal{H}_k^{i}$ are, as well as the sheaves
$\mathcal{H}_k^{i}/\ell^{kn} \mathcal{H}_k^{i}$. For $k = 0$, the latter sheaf is isomorphic to $H^i(K) \otimes \mathbb{Z}/\ell^n$, so we
have shown that $H^i(K) \otimes \mathbb{Z}/\ell^n$ is locally constant on the strata $S$.

(An equivalent way to argue is again to use the description of [SGA5, V, 3.2.3] and
just note that when computing the universal image subsheaves of $H^i(K \otimes^L \mathbb{Z}/\ell^{kn})$, one
can restrict to looking at the images of the sheaves $H^i(K \otimes^L \mathbb{Z}/\ell^{kn})$, and when finding
an $r$, if $r$ works, then $s$ works for any $s \geq r$, so $r$ can be taken to be a multiple of $kn$.)

As $H^i(K) \otimes \mathbb{Z}/\ell^n$ is locally constant and includes into $H^i(K \otimes^L \mathbb{Z}/\ell^n)$, on each stratum
$S \in S$ consider a Jordan-Hölder series for both. The constituents of the former are
a subset of the constituents of the latter and therefore also belong to $L(S)$. We have
shown that $H^i(K) \otimes \mathbb{Z}/\ell^n$, and therefore $H^i(K) \otimes \mathbb{Z}/\ell$, is $(S, L)$-constructible.

On the other hand, suppose the $H^i(K) \otimes \mathbb{Z}/\ell$ are $(S, L)$-constructible, and without
loss of generality let $K$ be in $D^b_{\ell}(X, \mathbb{Z}/\ell)$. Proceed by recurrence on $-j$; in the case
$j = 0$, $H^1(K) = 0$ and $H^0(K) \otimes \mathbb{Z}/\ell \to H^0(K \otimes^L \mathbb{Z}/\ell)$. The $\ell$-adic filtration on
any bounded complex of flat sheaves representing $K \otimes^L \mathbb{Z}/\ell^n$ has successive quotients
quasi-isomorphic to $K \otimes^L \mathbb{Z}/\ell$; taking the long exact sequence of cohomology finds that
$H^0(K \otimes^L \mathbb{Z}/\ell^n) \to H^0(K) \otimes \mathbb{Z}/\ell^n$ are $(S, L)$-constructible for all $n$. In particular,
this implies that each $Tor_1^{\mathbb{Z}/\ell}(H^0(K), \mathbb{Z}/\ell^n)$ is $(S, L)$-constructible. To see why, note that the
increasing sequence of ℓ-adic subsheaves ker ℓ^n ⊂ H^0(K) must stabilize, say at a = N as H^0(K) is noetherian. Let H denote ker ℓ^N; it is an ℓ-adic subsheaf of H^0(K) and is (S, ℒ)-constructible since H^0(K) ⊗ ℤ/ℓ^N is. Then

\( \text{Tor}^1_1(\text{H}^0(K), \text{ℤ}/ℓ^n) \simeq \text{Tor}^1_1(H, \text{ℤ}/ℓ^n) \simeq H[ℓ^n] \)

This last sheaf is easily seen to be (S, ℒ)-constructible, as it is locally constant wherever \( H \) is, and as a subsheaf, its constituents on a stratum are a subset of the constituents of \( H \) on that stratum. Since \( \text{Tor}^1_1(H^0(K), \text{ℤ}/ℓ^n) \) and \( H^{-1}(K) \otimes ℤ/ℓ \) are (S, ℒ)-constructible, (2.2.16.1) shows that \( H^{-1}(K \otimes ℤ/ℓ) \) is (S, ℒ)-constructible, hence that \( H^{-1}(K \otimes ℤ/ℓ^n) \) is for all \( n \). The argument above then shows that \( H^{-1}(K) \otimes ℤ/ℓ^n \) are (S, ℒ)-constructible for all \( n \), and hence that \( \text{Tor}^1_1(H^{-1}(K), \text{ℤ}/ℓ^n) \) are for all \( n \), etc.

In order to proceed to define the t-structure in imitation of 2.2.10, one needs to extend the six functors to \( D^b_ξ(X, ℤ_ℓ) \). This is trivial because they commute with reduction modulo \( ℓ^n \); see note to Weil II 1.1.2c. Then the claim about \( K \) belonging to \( p D^b_ξ(X, ℤ_ℓ) \) iff its reduction modulo \( ℓ \) belongs to \( p D^b_ξ(X, ℤ/ℓ) \) is also trivial.

2.2.18. See note to Weil II 1.1.3. Multiplication by \( ℓ \) on complexes of flat sheaves representing \( K \) in \( D^b_ξ(X, ℤ_ℓ) \) induces a multiplication by \( ℓ \) on their cohomology. To see that the image \( D^b_ξ(X, ℤ_ℓ) \otimes ℚ_ℓ \) consists of those \( K \) such that each \( H^iK \) is the \( ℚ_ℓ \) of an (S, ℒ)-constructible \( ℤ_ℓ \)-sheaf, well, certainly it is contained in it. On the other hand, given \( K \) in \( D^b_ξ(X, ℤ_ℓ) \) with each \( H^iK \) the \( ℚ_ℓ \) of an (S, ℒ)-constructible \( ℤ_ℓ \)-sheaf \( ℱ \), then for each of finitely many nonzero \( i \), there exist nonzero \( a_i, b_i \in ℤ_ℓ \) such that \( a_i H^iK = b_i ℱ \); \( b_i ℱ \) is a subsheaf of \( ℱ \) and is also (S, ℒ)-constructible. Then \( (\prod_i a_i)K \) has cohomology sheaves which are (S, ℒ)-constructible, showing the reverse containment.

Turning now to the claim that the forgetful functor \( ω \) induces an equivalence

\( D^b_ξ(X, E_κ) \rightarrow \{ \text{category of objects } K \text{ of } D^b_ξ(X, ℚ_ℓ) \text{ equipped with a morphism of } ℚ_ℓ\text{-algebras } E_κ \rightarrow \text{End}(K) \} \),
the essential surjectivity of $\omega$ follows from the fact that if $K$ is in $D^b_\ell(X, Q_\ell)$ and equipped with an action $\phi : E_\lambda \to \text{End}(K)$, and $K \otimes_{Q_\ell} E_\lambda$ has $E_\lambda$ acting on itself, then there is an $E_\lambda$-equivariant embedding $K \to K \otimes E_\lambda$ with retraction $r$. Let $\alpha$ be a primitive element for the extension $E_\lambda/Q_\ell$ so that $E_\lambda \simeq Q(\alpha)$, and let $d$ denote the degree $[E_\lambda : Q_\ell]$. The maps are given by

$$i : K \to K \otimes E_\lambda \quad r : K \otimes E_\lambda \to K$$

$$K \mapsto \frac{1}{d} \sum_{i=0}^{d-1} \phi(\alpha)^i K \otimes \alpha^{-i} \quad K \otimes a \mapsto \phi(a)K.$$  

This implies that $K$ is indeed a direct factor of $K \otimes E_\lambda$; see Neeman, *Triangulated Categories* 1.2.10. This in turn gives an idempotent in $\text{End}(K \otimes E_\lambda)$, and if the image of this idempotent is represented in $D^b_\ell(X, E_\lambda)$, then this implies a splitting of $K \otimes E_\lambda$ in $D^b_\ell(X, E_\lambda)$ which is sent by $\omega$ to the direct factor $K$. A category in which every idempotent splits is called alternatively Cauchy complete, idempotent complete, or Karoubi complete (see SGA 4, I 8.7.8), so we are done if we show that $D^b_\ell(X, E_\lambda)$ is Karoubi complete. The splitting of an idempotent $e$ in the endomorphism ring of an object in some category is equivalent to the existence of the equalizer $i = \ker(e, \text{id})$ or the coequalizer $r = \text{coker}(e, \text{id})$, and this (co)equalizer, if it exists, is an absolute (co)limit; i.e. it is preserved by every functor (see Proposition 1 of Borceaux and Dejean, *Cauchy Completion in Category Theory*). As $D^b_\ell(X, E_\lambda)$ is a projective limit of categories $D^b_{ctf}(X, R/m^n)$ for $R$ the ring of integers in $E_\lambda$, it is easily seen that if the categories $D_n := D^b_{ctf}(X, R/m^n)$ are Karoubi complete, if $e$ is an idempotent in $\text{End}(K \otimes E)$, its reductions in $D_n$ are idempotents which split, and as these splittings are absolute (co)limits, they automatically give an object in $D^b_\ell(X, E_\lambda)$ splitting $e$. It will suffice to show that $D_n$ is Karoubi complete. Let’s say that a triangulated category has direct sums if it has (arbitrary) categorical direct sums and if the (arbitrary) direct sum of distinguished triangles is distinguished. Bökstedt and Neeman show in *Homotopy limits in triangulated categories* 3.2 that if a triangulated category has direct sums, it is Karoubi complete. The category $D(X, R/m^n)$ is therefore Karoubi complete. An object $C$ of an additive category with arbitrary direct sums is said to be compact if $\text{Hom}(C, -)$ commutes with arbitrary direct sums. Any direct summand of a compact object $C$ is compact, since a finite colimit of compact objects is compact (maps out of
a finite colimit is a finite limit, and finite limits commute with filtered colimits in Set). Therefore as $D(X, R/m^n)$ is Karoubi complete, so is the full subcategory generated by compact objects. That $D^{(b)}_{ctf}(X, R/m^n)$ coincides with the subcategory generated by compact objects of $D(X, R/m^n)$ is 6.4.8 of Bhatt-Scholze, *The pro-étale topology for schemes*, after you recall that the objects of $D^{(b)}_{ctf}(X, R/m^n)$ can be represented by bounded complexes of $R/m^n$-flat constructible sheaves (*Rapport 4.6*).

**2.2.19.** The fact that $i^! f_* = f_* i^!$ is SGAA XVIII 3.1.12.3. The ‘argument habituel d’homotopie’ referenced in the last sentence of the proof is referring to, e.g. Th. 5.7.1 in Godement, *Théorie des faisceaux*, but a more accessible reference is Stacks 09UY. If you have two open coverings $\mathcal{U} = (U_i)_{i \in I}$ and $\mathcal{V} = (V_j)_{j \in J}$ of a space $X$, and $\mathcal{U}$ is a refinement of $\mathcal{V}$, so that we can choose a map $\phi : I \to J$ such that $U_i \subset V_{\phi(i)}$ for all $i \in I$. This induces a map of Čech complexes $\phi^* : \check{C}(\mathcal{V}, \mathcal{F}) \to \check{C}(\mathcal{U}, \mathcal{F})$ for any sheaf $\mathcal{F}$ on $X$. The result then says that if you have a second $\phi' : I \to J$ such that $U_i \subset V_{\phi'(i)}$, the maps $\phi^*, \phi'^*$ are homotopic. This instantly implies that if $\mathcal{U}, \mathcal{V}$ are mutual refinements, they have the same Čech cohomology, since in this case we can choose $\phi : I \to J$ and $\psi : J \to I$ satisfying $U_i \subset V_{\phi(i)}$ and $V_j \subset U_{\psi(j)}$; the maps $\phi \circ \psi$ and $\psi \circ \phi$ must then induce the identity on Čech cohomology as they are homotopic to the identity on the chain level. Our case is formally equivalent to this statement for $\check{H}^0$.

In more words: $A$ is defined as an equalizer of two maps

$$\prod_i p_j i_* A_{U_i} \Rightarrow \prod_{i \times I} p_j i_* A_{U_{ij}}$$

where one map $d_0^i$ is the product over $(i, j) \in I \times I$ of the maps

$$\prod_i p_j i_* A_{U_i} \xrightarrow{pr_j} p_j i_* A_{U_{ij}} \to p_j j_* A_{U_{ij}},$$

(here the second map is $j_i*$ of the unit of the adjunction on $U_i$ if $\cdot = i$ and 0 if $\cdot \neq i$), and the other map $d_1^i$ is the product over $(i, j) \in I \times I$ of the maps

$$\prod_i p_j i_* A_{U_i} \xrightarrow{pr_j} p_j j_* A_{U_{ij}} \to p_j i_* A_{U_{ij}}.$$
(the second map is \( j_{j*} \) of the unit of the adjunction on \( U_j \) if \( j = j \) and 0 if \( j \neq j \)). This recognizes \( A \) as \( \ker(d_0^0 - d_1^0) \). Let \( f \) denote maps of the type

\[
U_{i_0} \times \cdots \times U_{i_a} \times \cdots U_{i_p} \to U_{i_0} \times \cdots \times U_{i_a} \times \cdots U_{i_p}.
\]

In higher degrees, the differential is determined by the formula

\[
\text{pr}_{i_0 \ldots i_{p+1}} \circ d = \sum_{a=0}^{p+1} (-1)^a p_{j_{i_0 \ldots i_a \ldots i_{p+1}}}(\eta(A_{i_0 \ldots i_p})) \circ \text{pr}_{i_0 \ldots i_a \ldots i_{p+1}},
\]

where \( \eta \) is the unit of the adjunction \( \text{id} \to p f_\ast p f^\ast \). Let \( \mathcal{U} \) be the cover of \( X \) as above and \( s : I \to J \) satisfying \( s_i : U_i \to V_{s(i)} \) as above. The chain map induced by \( s \) has the explicit description

\[
\text{pr}_{i_\Lambda} \circ s^\ast = p_{s(i_\Lambda)}(\eta(A_{i_\Lambda})) \circ \text{pr}_{s(i_\Lambda)},
\]

where \( i_\Lambda \) is a multi-index \( i_0 i_1 \ldots i_p \), \( s(i_\Lambda) = s(i_0) \ldots s(i_p) \), and \( \eta \) is the unit of the adjunction \( \text{id} \to p s_i^\ast p s_i^\ast \) on \( V_{s(i_\Lambda)} \), where \( s_i : U_i \times \cdots \times U_{i_p} \to V_{s(i_0)} \times \cdots \times V_{s(i_p)} \) is deduced by taking the product of the maps \( s_i \).

Now given another map \( t : I \to J \) satisfying \( t_i : U_i \to V_{t(i)} \), let \( 0 \leq a \leq p \) and let \( f \) now denote maps of the sort

\[
U_{i_0} \times \cdots \times U_{i_p} \to V_{s(i_0)} \times \cdots \times V_{s(i_a)} \times V_{t(i_a)} \times \cdots \times V_{t(i_p)},
\]

where here we use the map \( U_{i_a} \to V_{s(i_a)} \times V_{t(i_a)} \). We set up a homotopy \( h \) by the formula

\[
\text{pr}_{i_0 \ldots i_p} \circ h = \sum_{a=0}^{p} (-1)^a p_{j_{i_0 \ldots s(i_a) \ldots t(i_a) \ldots t(i_p)}}(\eta(A_{s(i_0) \ldots s(i_a) \ldots t(i_a) \ldots t(i_p)}) \circ \text{pr}_{s(i_0) \ldots s(i_a) \ldots t(i_a) \ldots t(i_p)}
\]

where here \( \eta \) is the unit of the adjunction \( \text{id} \to p f_\ast p f^\ast \).

Now we can follow the argument of Stacks 01FP.

4.1.7. The Čech spectral sequence is also called the Cartan-Leray spectral sequence and its existence in an arbitrary topos is established in SGAA, Exp. V 3.3.
4.1.8. The only thing worth mentioning is that the entire second paragraph is implicitly local to $U_i$. After all, on $U_i$ we have that $\tau_{\leq -1}K$ is in $\mathcal{D}^{\leq 0}$, and to show that $K|U_i$ is in $\mathcal{D}^{\leq 0}$, it suffices to show that $H^q(\tau_{\geq 1}K)|U_i$ is 0 for all $q$ (i.e., for all $q > -1$). Proceeding by descending induction on $q > -1$, the induction step consists of showing that $H^q(V \cap W_i, \mathcal{L}) = 0$ for all affine open $V$ implies that $\mathcal{L} = 0$.

4.2.5. As $f^*$ and $\mathcal{R}{\mathcal{H}}om$ commute with reduction modulo $\ell''$, it is enough to prove the statement in the category $\mathcal{D}^{b}_{ct}(X, \mathbb{Z}/\ell'')$, and the equality $\Gamma\mathcal{H}^{0}\mathcal{R}{\mathcal{H}}om = \mathcal{H}^{0}\Gamma\mathcal{R}{\mathcal{H}}om$ holds in $\mathcal{D}(X, \mathbb{Z}/\ell'')$ because, as $\mathcal{R}{\mathcal{H}}om$ is in $\mathcal{D}^{\leq 0}_{c}$, $H^0 = \tau_{\leq 0} \ker d_0$, and $\Gamma$ commutes with the formation of kernels. For (4.2.5.3), the retraction $\mathcal{H} \to \circ f_*f^*\mathcal{H}$ is $\circ f_*(\eta(f^*\mathcal{H})$ where $\eta$ is the unit id $\to \circ e_*e^*$:

$$\mathcal{H} \to \circ f_*f^*\mathcal{H} \to \circ f_*\circ e_*e^*f^*\mathcal{H} = \mathcal{H},$$

and it remains to check that the first arrow is surjective. The references in the rest of this paragraph are to SGA A Exposé XV. By (1.1), it suffices to show that $H^0(Y', \mathcal{H}) \to H^0(X', f'^*\mathcal{H})$ is bijective for each $Y' \to Y$ étale. Replacing $Y$ by $Y'$, we know by (1.5) that $f$ is $(1)$-acyclic; i.e., that $\alpha : H^0(Y, \mathcal{H}) \to H^0(X, f^*\mathcal{H})$ is injective, as $f$ is surjective. Moreover, $f$ is locally acyclic as it is smooth. Then, (1.16) shows that $\alpha$ is surjective iff for every geometric point $\overline{y}$ of $Y$ algebraic over a point $y$ of $Y$, $\overline{\alpha} : H^0(\overline{y}, \mathcal{H}_{\overline{y}}) \to H^0(X_{\overline{y}}, f^*\mathcal{H}|X_{\overline{y}})$ is, so we may assume $Y$ is the spectrum of an algebraically closed field, in which case $X$ is connected, as the fibers of $f$ were assumed geometrically connected. Then $\alpha$ is seen to be bijective by another application of (1.1), which reduces the matter to the corresponding question for a constant sheaf. Note that the existence of a retraction discussed above is irrelevant to this argument.

4.2.6. This paragraph, as written, is nonsense. The correct (equivalent) statements are

(a) $u^*$ identifies $\mathcal{A}$ with a subcategory of $\mathcal{B}$ closed under subquotients.
(b) the unit of adjunction $\eta_1 : id_{\mathcal{B}} \to u^*u_!$ is a natural epimorphism;
(b') the counit of adjunction $\varepsilon_\circ : u^*u_\circ \to id_{\mathcal{B}}$ is a natural monomorphism.
Note that statement (a) is different from \( \mathcal{A} \) being épaisse, as an épaisse subcategory is also closed under extensions. A general remark: an adjunction \( F : C \rightleftarrows D : G \) with unit \( \eta \) and counit \( \epsilon \) is called idempotent if it can be factored as a reflection and a coreflection, in which case many things are true; see the nLab page ‘idempotent adjunction.’ In particular, \( \epsilon F \) and \( \eta G \) are natural isomorphisms. In our situation, both \( u ! \dashv u ^* \) and \( u ^* \dashv u_* \) are idempotent adjunctions as they are reflective and coreflective, respectively, so we get that \( \epsilon_* u ^* \) and \( \eta_* u ^* \) are natural isomorphisms. Of course, \( u ! \) is right exact and \( u_* \) is left exact. We prove \( (a) \iff (b') \); dual arguments give \( (a) \iff (b) \).

To prove \( (a) \Rightarrow (b') \), let \( B \) be an object of \( \mathcal{B} \). In the commutative diagram with exact rows,

\[
\begin{array}{ccc}
0 & \longrightarrow & \ker \epsilon_*(B) \longrightarrow u^*u_*B \longrightarrow B \\
\uparrow \alpha & & \uparrow \beta \\
0 & \longrightarrow & u^*u_* \ker \epsilon_*(B) \longrightarrow u^*u_*u^*u_*B \longrightarrow u^*u_*B \\
\end{array}
\]

\( \ker \epsilon_*(B) \) is in the essential image of \( \mathcal{A} \) as \( \mathcal{A} \) is closed under subobjects. Therefore \( \alpha \), \( \beta \), and \( \gamma \) are isomorphisms, which shows \( u^*u_* \ker \epsilon_*(B) = 0 \) and therefore \( \ker \epsilon_*(B) = 0 \).

Now let’s prove \( (b') \Rightarrow (a) \). In the commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & A \longrightarrow u^*B \longrightarrow C \longrightarrow 0 \\
\uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & u^*u_*A \longrightarrow u^*u_*u^*B \longrightarrow u^*u_*C \\
\end{array}
\]

with exact rows, the middle arrow is an isomorphism and the outer arrows are monomorphisms. By the four-lemma, the first arrow is also an epimorphism, hence an isomorphism, identifying \( A \) in the essential image of \( \mathcal{A} \). Therefore this essential image is closed under subobjects and hence also under subquotients, as \( u^* \) is exact.

Identifying \( \mathcal{A} \) with its essential image, a full subcategory of \( \mathcal{B} \), every object \( B \) of \( \mathcal{B} \) has a largest subobject in \( \mathcal{A} \), viz. \( u^*u_*B \), and a largest quotient in \( \mathcal{A} \), viz. \( u^*u_*B \). To see this, simply observe that both candidates are indeed in \( \mathcal{A} \), and if \( A \) is in \( \mathcal{A} \) and a subobject of \( \mathcal{B} \), then \( A \cong u^*u_*A \iff u^*u_*B \iff B \), and dually.
The example adjunction is backwards: \( u_l(X \rightarrow Y) = (Y \sim \rightarrow Y) \) is the left adjoint, \( u_r(X \rightarrow Y) = (X \sim \rightarrow X) \) is the right adjoint, since the diagrams

\[
\begin{array}{c}
A \rightarrow B \\
\downarrow \\
C \sim \rightarrow D
\end{array} \quad \quad \begin{array}{c}
A \sim \rightarrow B \\
\downarrow \\
C \rightarrow D
\end{array}
\]

have unique completions to commutative squares. Then \( \varepsilon : u^*u_r \rightarrow \text{id}_{\mathcal{B}} \) needn’t be a natural monomorphism, as a commutative diagram of the sort

\[
\begin{array}{c}
A \rightarrow B \\
\downarrow \\
C \sim \rightarrow C
\end{array} \quad \quad \begin{array}{c}
A \rightarrow C \\
\downarrow \quad \downarrow \\
C \rightarrow D
\end{array}
\]

determines the morphism \( A \rightarrow C \) but isn’t enough to determine the morphism \( B \rightarrow D \).

4.2.6.1. The Jordan-Hölder theorem holds in any abelian category, and since \( u^* \) is exact and preserves simple objects, identifying \( \mathcal{A} \) with its essential image, it follows that the components of any \( A \) in \( \mathcal{A} \) also belong to \( \mathcal{A} \). To show that \( \mathcal{A} \) is closed under subobjects, and therefore subquotients, it will suffice to show that if 0 \( \rightarrow B \rightarrow A \rightarrow S \rightarrow 0 \) with \( A \) in \( \mathcal{A} \), \( B \) in \( \mathcal{B} \), and \( S \) simple, then \( B \) is in \( \mathcal{A} \). Since \( S \) is simple, it is a component of \( A \) and therefore in \( \mathcal{A} \). The five-lemma, applied to the diagram with exact rows obtained by applying \( \varepsilon \),

\[
\begin{array}{c}
0 \rightarrow B \rightarrow A \rightarrow S \\
\uparrow \\
0 \rightarrow u^*u_rB \rightarrow u^*u_rA \rightarrow u^*u_rS
\end{array}
\]

shows that indeed \( u^*u_rB \sim \rightarrow B \), and \( B \) is in \( \mathcal{A} \).

4.2.6.2. Notes on the proof are below; the proof depends on the middle perversity insofar as the commutativity of intermediate extension with \( f^*[d] \) relies on the relative
dimension coinciding with the change in perversity between an irreducible component and its inverse image by $f$. The proof doesn’t work for $\mathcal{Z}_\ell$-sheaves as it relies on 4.3.1 which fails for $\mathcal{Z}_\ell$-cohomology.

**Commutation of $f^*[d]$ with intermediate extension.** We wish to show that $f^*[d] j_*! = j_+ f^*[d]$. The transitivity of $j_+$ for $j$ the inclusion of a locally closed subset allows us to factorize the inclusion of $V$ as the open immersion $V \hookrightarrow \overline{V}$ followed by the closed immersion $\overline{V} \hookrightarrow Y$; the latter posing no problem as in this case lower shriek, lower star, and intermediate extension all coincide and commute with $f^*[d]$ by smooth base change. Reduced to the case of open immersion $Y = \overline{V}$, let $m$ be minimal such that $F = Y - V$ is of dimension $\leq m$, and put $t := p(2m)$. We can find a stratification of $F$ into strata satisfying 2.2.10 (a) so that for $A$ in $D^b_\ell(V, \mathbb{R}/m^n)$, the $H^i j_* A$ are locally constant on each stratum for $i \geq t$. Let $U_n$ (resp. $F_n$) denote the union of strata $S$ in this stratification of $F$ satisfying $p(S) \leq n$ (resp. $p(S) \geq n$). For such an $S$, $\overline{S} - S$ is a union of strata of dimension strictly less than $\dim S$. Therefore each $S$ in the stratification of $U_n - U_{n-1}$ is closed in $U_n$. By transitivity of the intermediate extension, it will suffice to extend $A$ from $V$ to $V \cup U_t$, as the complement of the latter in $Y$ is $F_{i+1}$, closed; $V \cup U_t - V = U_t$ is also closed in $V \cup U_t$, and nonempty. $U_t$ is a disjoint union of equidimensional strata smooth over $\overline{k}$ (hence by the note to 2.2.10, the étale topology sees them as smooth), so that the $H^i j_* A$ are locally constant on $U_t$ for $i \geq t$. We have reduced to the setting of 2.2.4: $Y = \overline{V}$, $F = Y - V$ smooth of dimension $m$, $H^i j_* A$ are locally constant on $F$ for $i \geq t := p(2m)$, and $j_+ A = \tau_{t-1} F \leq j_* A$. As $f$ is smooth of relative dimension $d$, $f^{-1} F$ is also equidimensional, now of dimension $m + d$. As pullbacks of locally constant sheaves, the $H^i j_* f^* A$ are still locally constant on $f^{-1} F$. If $t' = p(2(d + m)) = -d - m$, then shifting by $d$ and applying $\tau_{t'}$ is the same as applying $\tau_{t'}$ and then shifting by $d$. It only remains to verify that $f^*$ commutes with $\tau_{t'}^F$, but this is easy: letting $i : F \hookrightarrow Y$ be the closed immersion, $(f^* \tau_{\leq 0} A, f^* A, f^* i_\tau_{> 0} i^* A)$ defines $f^* \tau_{\leq 0} A$, but $f^* i_\tau_{> 0} i^* A = i_\tau_{> 0} i^* A$ (notating $i$ as usual for the base extension as well), hence also defines $\tau_{\leq 0} f^* A$.

**Irreducibility of the inverse image of an irreducible local system on an irreducible scheme by a smooth morphism with geometrically connected fibers.** SGA 1, Exp. IX, 5.6 shows
that if \( f : S' \to S \) is universally submersive (e.g. faithfully flat and quasi-compact) with geometrically connected fibers, and \( S \) is connected, then \( S' \) is connected, and, choosing a geometric point \( s' \) of \( S' \) and letting \( s \) be the image of \( s' \) in \( S \), the homomorphism \( \pi_1(S', s') \to \pi_1(S, s) \) is surjective. This immediately implies that if \( \mathcal{L} \) is an irreducible local system on \( S \), \( f^* \mathcal{L} \) is an irreducible local system on \( S' \). Of course, smooth with geometrically connected fibers means smooth with geometrically irreducible fibers as these fibers are themselves smooth (SGA 1, Exp II, 2.1) and therefore regular. It is a topological fact (Stacks 004Z) that if \( Y \) is irreducible and \( f : X \to Y \) is open with irreducible fibers, then \( X \) is irreducible. This verifies that the if \( V \) is irreducible and \( f \) is smooth with connected fibers, the inverse image on \( f^{-1} V \) of an irreducible local system on \( V \) is again irreducible.

**4.2.7.** Let \( X', X, Y \) be of finite type over a field \( k \) and \( f : X' \to X \). We wish to show that \( \Box \) commutes with direct image; i.e. that the below arrow is an isomorphism.

\[
f_* (K \boxtimes L) \to (f \times \text{id})_* (K \boxtimes L)
\]

We will compute locally about a geometric point \( \xi := (x, y) \to X \times Y \) (\( x, y \) geometric points of \( X, Y \) respectively), so that all the objects in \( X_x \leftarrow (X \times Y)_\xi \to Y_y \) are spectra of strictly henselian local rings. Let \( t \to X_t \) be a geometric point centered on the generic point of an irreducible component of \( X_x \). Th. finitude 2.16 gives that \( Y \to \text{Spec} \, k \) is universally locally acyclic, so for any \( M \) in \( D^+(X \times Y, \mathbb{Z}/\ell) \),

\[
\Gamma((X \times Y)_\xi, M) = \Gamma((X \times Y)_{\xi, t}, M),
\]

where \((X \times Y)_{\xi, t}\) denotes the geometric fiber in \( t \) of \((X \times Y)_\xi \to Y_y\). To pass from \( X_x \) to \((X_x)_t = t\) we can first pass to the limit of Zariski neighborhoods of \( t \), which is the spectrum of an artinian local ring, then kill nilpotents and extend scalars. As lower star commutes with smooth base change and the étale topology doesn’t see nilpotents, we may therefore assume \( X = t \), the spectrum of a separably closed extension \( k(t) \) of \( k \), and \( Y = Y_y \). As \( X_x \to \text{Spec} \, \overline{k} \leftarrow Y_y \), \( \text{Spec} \, \overline{k} \times_k Y_y \) is the disjoint union of copies of \( Y_y \), and \( \xi \) picks one of them; i.e. \((\text{Spec} \, \overline{k} \times_k Y_y)_{\xi} = Y_y \). So we may assume \( Y_y \to \overline{k}, \)
and write
\[ \Gamma((X \times Y)_{\mathbb{Z}_r}, M) = \Gamma(Y_y \times_{\overline{k}} k', M) = \Gamma(Y_y, M), \]
where the second equality is Arcata V.3.3. We are reduced to \( k = \overline{k}, X = \overline{k}, Y = Y_y, \)
in which case the formula is
\[ \Gamma(X', K) \otimes \Gamma(Y, L) \to \Gamma(X' \times Y, K \boxtimes L), \]
which is obtained from the Künneth formula of *Th. finitude* 1.11 by smooth base change and passage to the limit along \( Y_y \to Y. \)

4.2.8. It would appear that \( \boxtimes \) is only right t-exact in \( \mathbb{Z}_\ell \)-cohomology, due to the possible appearance of Tor.

4.3.1. This is a theorem in \( \mathbb{Q}_\ell \)-cohomology and not in \( \mathbb{Z}_\ell \)-cohomology because the latter has few irreducible objects; in particular, the category of lisse \( \mathbb{Z}_\ell \)-sheaves is not artinian; the irreducible lisse \( \mathbb{Z}_\ell \) sheaves are torsion . . .

4.3.3. ‘la monodromie de \( L \) ne change pas par restriction à \( U \)’ \( \rightsquigarrow \) SGA 1 Exp. 1 10.3.

4.3.4. The dévissage of the perverse sheaf \( \mathcal{F} \) should occur over an irreducible affine smooth open so that we can apply the results of 4.1.10–4.1.12. The sequence 4.1.10.1 has outer terms supported on \( X - U \) and so reduces the problem for \( \mathcal{F} \) to that for \( j_! j^! \mathcal{F} = j_! \mathcal{L} [\text{dim } U]; 4.1.12.3 \) then reduces the problem for \( j_! \mathcal{L} [\text{dim } U] \) to that for \( j_! j^! \mathcal{L} [\text{dim } U] \). This allows us to proceed by induction on the length of \( \mathcal{L} \). Let \( \mathcal{L}' \subset \mathcal{L} \) be a simple lisse subsheaf. It suffices to remark that the kernel of \( j_* j^* \mathcal{L}' [\text{dim } U] \to j_* \mathcal{L} [\text{dim } U] \) is supported on \( X - U \), and the restriction of the cokernel to \( U \) has strictly lesser length than that of \( \mathcal{L}' \).

### References

[BBD] *Faisceaux Pervers* par Beilinson, Bernstein, Deligne (& Gabber!).

[CD] *Catégories Dérivées (Etat 0)* par Verdier, dans SGA 4\frac{1}{2}.
[SGA5] SGA 5, dirigé par Grothendieck
5. Notes on *Applications de la formule des traces aux sommes trigonométriques*

1.2. Suppose $X$ is locally noetherian and $X'$ is a connected component of $X$ pointed by a geometric point $a \rightarrow X'$. Via Grothendieck’s Galois theory, over $X'$, the $A$-torsor $T$ can be identified (after a choice $e$ of identity for the stalk of $T$ at $a$) with the set $A$ with (continuous) left action of $\pi_1(X', a)$ and right action of $A$.

Given a homomorphism $\tau : A \rightarrow B$, we wish to produce a map of sets $\tau : A \rightarrow B$ so that $\tau(ta) = \tau(t)\tau(a)$ and give the set $B$ with right action of $B$ a left action of $\pi_1(X', a)$ with respect to which $\tau$ is equivariant. The given homomorphism of groups sets up a map of sets compatible with the actions of $A$ and $B$, and the formula

$$g\tau(t) := \tau(gt)$$

defines a left $\pi_1$ action on $\tau(A)$. For $s \in B$ not necessarily in $\tau(A)$,

$$gs := g\tau(e)s = \tau(ge)s$$

extends the left action of $\pi_1$ to all of $B$. The $B$-torsor $\tau(T)$ is then identified with the set $B$ with right action of $B$ and the given left action of $\pi_1$, with respect to which $\tau : T \rightarrow \tau(T)$ is equivariant by construction. This discussion depends on the choice of identity for the torsor $T$ (the identity for $\tau(T)$ is obtained from the choice for $T$ and the homomorphism $\tau$) and therefore only defines the torsor $\tau(T)$ up to isomorphism.

Let $R$ (resp. $E_{\lambda}$) be a finite $\mathbb{Z}_\ell$-algebra (resp. finite $\mathbb{Q}_\ell$-algebra). Recall (SGA 5 Exp. VI (α) 1.2.4, (β) 1.4.1, (γ) 1.4.2, (δ) 1.4.3) that the category of

(α) lisse $\mathbb{Z}_\ell$-sheaves
(β) lisse $\mathbb{R}$-sheaves
(γ) lisse $\mathbb{Q}_\ell$-sheaves
(δ) lisse $E_{\lambda}$-sheaves

on a connected locally noetherian scheme $X$, pointed by a geometric point $a \rightarrow X$ is equivalent to the category of
(α) $\mathbb{Z}_\ell$-modules of finite type on which $\pi_1(X) := \pi_1(X, a)$ acts continuously for the
$\ell$-adic topology

(β) R-modules of finite type equipped with a continuous and R-linear action of
$\pi_1(X)$ on the underlying $\mathbb{Z}_\ell$-module of finite type

(γ) $\mathbb{Q}_\ell$-vector spaces of finite dimension equipped with a continuous action of $\pi_1(X)$

(δ) $E_k$-vector spaces of finite dimension equipped with a continuous and $E_k$-linear
action of $\pi_1(X)$ on the underlying $\mathbb{Q}_\ell$-vector space.

Given an $A$-torsor $T$ on $X$ and a finite-dimensional $E_k$-vector space $V$, the data of an $E_k$-
sheaf $\mathcal{F}$, lisse of rank dim $V$, together with a morphism of sheaves $\rho : T \to \text{Isom}(V, \mathcal{F})$
satisfying $\rho(ta) = \rho(t)\rho(a)$ is the same (after a choice of identity $e$ for the stalk of $T$ at $a$)
as the data of a continuous and $E_k$-linear action of $\pi_1(X)$ on the underlying $\mathbb{Q}_\ell$-vector
space of $V$ and a $\pi_1$-equivariant group homomorphism $\rho : A \to \text{GL}(V)$, where the
action of $\pi_1$ on $\text{GL}(V)$ is induced by the action of $\pi_1$ on the second factor of $\text{Hom}(V, V)$.
Given $\rho : A \to \text{GL}(V)$ a linear representation of $A$, there is a unique continuous and
$E_k$-linear action of $\pi_1$ on the $\mathbb{Q}_\ell$-vector space underneath $V$ that makes $\rho$ $\pi_1$-equivariant:

(†) $\pi_1(X) \times V \to V$

$(g, v) \mapsto \rho(ge)v \rightsquigarrow \rho(gea) = \rho(ge)\rho(a) = g\rho(a)$.

1.3. Torsors can be discussed in the language of schemes or sheaves; the distinction
comes down to whether the torsor is representable as a sheaf, and this distinction
motivates the introduction of algebraic spaces. In SGA 1 Exp. V §2 and SGAD Exp.
III §0 & Exp. IV §5 the notion of principal homogeneous space is developed; these
are the representing objects for certain (sheaf) torsors. There, an $A$-torsor is called
a principal homogeneous space under $A$. More precisely, given a category $\mathcal{C}$, let $\widehat{\mathcal{C}}$
denote its category of set-valued presheaves $\text{Hom}(\mathcal{C}^\circ, (\text{Set}))$. If $A$ is a $\widehat{\mathcal{C}}$-group acting
on $X$ an object of $\widehat{\mathcal{C}}$, $X$ is formally principal homogeneous under $A$ (i.e. an $A$-pseudo
torsor) if the equivalent conditions below are satisfied:

(i) for each object $S$ of $\mathcal{C}$, the set $X(S)$ is empty or principal homogeneous under $A(S)$;
(ii) the morphism of functors $A \times X \to X \times X$ defined setwise by $(a, x) \mapsto (ax, x)$ is an isomorphism.

It amounts to the same to say that the canonical morphism of functors

$$X \times A \to X \times X$$

is an isomorphism. If $\mathcal{C}$ is equipped with a topology, then one says that the $S$-object $X$ with $S$-group of operators $A$ is fibered principal homogeneous under $A$ (i.e. is an $A$-torsor) if it is locally trivial; i.e. there exists a covering family $\{S_i \to S\}$ such that for each $i$, the $S_i$-functor $X \times_S S_i$ with $S_i$-functor-group of operators $A \times_S S_i$ is trivial.

The category $\mathcal{C}$ has a final object $\mathbf{e}$ which sends an object of $\mathcal{C}$ to $\{0\}$, the set with one element. This functor is representable if $\mathcal{C}$ admits a final object. The ‘sections’ functor $\Gamma$ is defined on $\mathcal{C}$ as $\text{Hom}(\_ , \mathbf{e})$ and on $\mathcal{C}$ via $X \mapsto h_X$; if $\mathcal{C}$ admits a final object, this latter functor is isomorphic to $\text{Hom}(\mathbf{e} , \_ )$. SGAD Exp. IV 5.1.2, 5.1.3 observes that $X$ is formally principal homogeneous under $A$, there is an isomorphism

$$\Gamma(X) \xrightarrow{\sim} \text{Isom}_{A-\text{obj}}(A, X)$$

of principal homogeneous sets under $\Gamma(A)$; therefore an isomorphism of $A$-objects

$$X \xrightarrow{\sim} \text{Isom}_{A-\text{obj}}(A, X).$$

The proof is simply that to each section $x$ of $X$ one associates the morphism $A \to X$ defined setwise by $a \mapsto xa$. This implies that an object with group of operators is trivial iff it is formally principal homogeneous and possesses a section.

The algebraic group schemes in the given extension can also be considered as sheaves for the fpqc topology on $S$. **En effet**, the surjectivity of $\pi$ implies that $\pi$ is faithfully flat, and therefore an fpqc covering.

**Proposition** (SGAD Exp. IV 5.1.7.1). Let $\mathcal{C}$ denote a category possessing a final object, stable by fiber products, and equipped with a subcanonical topology $\mathcal{T}$ (such as $\text{Sch}$ equipped with fppf, fpqc, étale). Suppose $\pi : G' \to G$ is a morphism of $\mathcal{C}$-groups which is covering for the topology $\mathcal{T}$, and $A = \ker \pi$. Then $G$ represents the quotient sheaf $G'/A$, and $\pi$ is an $A_G$-torsor; i.e. $G'$ is an $A$-torsor on $G$. 
Therefore it makes sense to say ‘$G$’ is an $A$-torsor on $G$’ or that ‘the sheaf $T$ of local sections of $\pi$, $T = \text{Hom}_G(-, G)$, is an $A$-torsor on $G$’; the former represents the latter. The $A$-torsor $G'$ on $G$ is indeed locally trivial for fpqc: pulling back along the faithfully flat covering $\pi$, we find

$$G' \times_G A \to G' \times_G G'$$

$$(g', a) \mapsto (g', g'a)$$

is indeed an isomorphism, as can be checked setwise.

In order to understand how to add torsors, it is instructive to first recall how to add extensions of abelian groups. Suppose

$$0 \to A \to G' \to G \to 0$$

$$0 \to A \to G'' \to G \to 0$$

are exact sequences of abelian groups. $G'$ and $G''$ are $A$-torsors in $\text{Set}$. The Baer sum $G' + G''$ is constructed from the direct sum of extensions by pushout along addition for $A$ and pullback along the diagonal for $G$.

$$
\begin{array}{ccccccccc}
0 & \to & A \oplus A & \to & G' \oplus G & \to & G \oplus G & \to & 0 \\
\downarrow & & \downarrow^\gamma & & \downarrow & & \downarrow & & \\
0 & \to & A & \to & +_A(G' \oplus G'') & \to & G \oplus G & \to & 0 \\
\uparrow & & \uparrow^\Delta & & \uparrow & & \Delta & & \\
0 & \to & A & \to & \Delta^* +_A(G' \oplus G'') & \to & G & \to & 0
\end{array}
$$

Here, the square marked with $\uparrow$ is cartesian while the one marked with $\downarrow$ is cocartesian. The Baer sum of $G'$ and $G''$ is $\Delta^* +_A(G' \oplus G'')$. Now consider the case of 1.3 where we are given an extension of commutative algebraic group schemes over $S$, and $G'$ is an $A$-torsor over $G$. Pulling back $G'$ along addition for $G$ yields the fiber product

$$+_G G' = G' \times_G G \times_S G.$$ 

If $X$ is a scheme over $S$, to give a morphism over $S$ to $+_G G'$ amounts to giving two objects $g_1, g_2 \in G(X)$ and an object in $g \in G'(X)$ such that $\pi g = g_1 + g_2$. This data is equivalent to the data of two objects $g'_1, g'_2$ in $G'(X)$ mapping to $g_1, g_2$, respectively, modulo the relation which considers two such pairs equivalent.
if their sum is the same. This generalizes pushing out by addition on A to the case of torsors. The pullback of torsors is given on representing objects by fiber product, so that given an S-morphism \( f : X \to G \), \( f^*T = \text{Hom}_X(-, X \times_{f, \pi} G') \). This torsor is trivial if there is a section \( X \to X \times_{f, \pi} G' \) over \( X \); i.e. a commutative diagram

\[
\begin{array}{ccc}
X & \longrightarrow & G' \\
\downarrow f & & \downarrow \pi \\
G & \longleftarrow & X
\end{array}
\]

In other words, the torsor \( f^*T \) is defined up to isomorphism by the image of \( f \) in \( \text{Hom}_S(X, G) / \pi \text{Hom}_S(X, G') \).

If \( g : X \to G \) is another S-morphism, then \( g^*T \) is represented by \( X \times_{g, \pi} G' \). The fiber products \( X \times_{f, \pi} G' \) and \( X \times_{g, \pi} G' \) are A-torsors over \( X \). In analogy with Baer sum, a sum of torsors is given by pushing out by the addition on A followed by pulling back by the diagonal \( \Delta_X : X \to X \times_S X \). Since \( G' \) is defined by an extension and both torsors come from pulling back \( G' \) (i.e. \( T \)), this is the same as pulling back \( G' \) along the composition of morphisms

\[
X \xrightarrow{(f, g)} G \times_S G \xrightarrow{+} G.
\]

More explicitly, the morphism \( X \to G \times_S G \) factors as

\[
X \xrightarrow{\Delta} X \times_S X \xrightarrow{(f \circ \text{pr}_1, g \circ \text{pr}_2)} G \times_S G
\]

and therefore pulling back along this morphism corresponds to pulling back \((f^*T, g^*T)\) along the diagonal \( \Delta_X \). Of course, the fiber product \( G' \times_{G'} (G' \times_S G') \approx G' \times_S G' \) corresponds to the \( A \times A \)-torsor \((T, T)\). As we have already seen, pulling back \( G' \) along \( G \times_S G \xrightarrow{+} G \) is the quotient of \((T, T)\) by the relation the action of \((a_1, a_2)\) and \((a_1', a_2')\) if \( a_1 + a_2 = a_1' + a_2' \). Therefore, \((f + g)^*T\) is the quotient of \((f^*T, g^*T)\) by this same relation, which verifies the formula

\[
(f + g)^*T = f^*T + g^*T.
\]
The Lang torsor. For the development of the Lang isogeny and its properties, see Borel, *Algebraic Groups* §16. There, he proves that $\mathcal{L}$ is surjective when $G^0$ is connected. If the group $G^0$ is smooth (i.e. geometrically reduced), then $\mathcal{L}$ is seen to be étale (as a morphism of schemes) by computing its differential and Milne, *Algebraic Groups*, 1.63. That the functor $G^0 = G_0(\mathbb{F}_q)$ is a consequence of the description of $F$ on the set $G_0(\mathbb{F})$; see Rapport 1.1.

1.6. If $G_0$ is a commutative algebraic group defines over $\mathbb{F}_q$, then the norm map

$$N : G_0(\mathbb{F}_q^n) \to G_0(\mathbb{F}_q) \subset G_0(\mathbb{F}_q^n)$$

is defined, as $F$ is a group homomorphism. In the case $G_0 = G_\alpha$, then $G_\alpha(\mathbb{F}_q^n) = \mathbb{F}_q^n$ and $N$ coincides with the field trace $N(x) = x + x^q + \cdots + x^{q^{n-1}}$, as the group operation is addition. If $G_0 = G_m$, on the other hand, then $G_m(\mathbb{F}_q^n) = \mathbb{F}_q^\times$, and $N$ coincides with the field norm $N(x) = x \cdot x^q \cdots x^{q^{n-1}}$, as the group operation is multiplication.

1.7. The formulæ $\mathcal{F}(\chi, f_0) = f_0^* \mathcal{F}(\chi, \text{id}_{G_0})$, etc. result from the fact that the pullback of lisse sheaves corresponds to the homomorphism of $\pi_1$ induced by $f_0$; i.e. if $x \to X_0$ is a geometric point and $g = f_0(x)$, then both sides of these identities result from the composition of maps

$$\pi_1(X_0, x) \to \pi_1(G_0, g) \xrightarrow{L_0} G_0(\mathbb{F}_q) \xrightarrow{N} E^*_x,$$

where the $L_0$ indicates the homomorphism of groups defining the Lang torsor $L_0(G_0)$.

The formula that motivates this whole business

$$F_x^* = \chi f_0(x)$$

follows from the hint that the fiber in $x$ of the morphism 1.2.2 commutes with $F_x^*$. This is true for the following reason: the geometric point $x \in X^F$ and $f_0$ give homomorphisms

$$\text{Gal}(\mathbb{F}, \mathbb{F}_q) \to \pi_1(X_0, x) \to \pi_1(G_0, x)$$
and $F_x^{r-1}$ coincides with the image of the Frobenius substitution $q \in \text{Gal}(F,F_q)$. Commutation of $F_x^r$ with $f_0^*$ is clear and it remains to show that $\rho_x$ of 1.2.2 is $\pi_1(X,x)$-equivariant. The formula (†) of the note to 1.2 shows this explicitly.

1.8. (ii) The inverse image by $f$ of the the $E_x$-sheaf obtained from $\chi^{-1}(L_0(G_0))$ by extension of scalars from $F_q$ to $k$ is probably better written $f^*F(\chi)_1$ than $F(\chi f)$.

1.10. The subtlety in this argument is building a bridge between $C$ and $F_q$. Of course, the isomorphisms on singular cohomology with $Q$ coefficients imply isomorphisms on singular cohomology with $Q_\ell$ coefficients for all $\ell$. The quadric hypersurface $X'_0$ and hyperplane $Y'_0$ in $P^N_0$ over $F_q$ are each defined by the vanishing of a homogeneous polynomial of degree two and one, respectively, in the ring $F_q[x_0,x_1,\ldots,x_{2N}]$; let $f_0$ denote the polynomial defining $X'_0$. As $X'_0$ is a nonsingular variety over a perfect field, it is smooth over $F_q$. Let $m(x)$ be the minimal polynomial for a primitive element of $F_q$ over $F_q$; lifting the coefficients of $m$ to $Z$, $m$ remains irreducible, and defines a finite extension of domains $Z \to Z[x]/(m(x)) = A$ so that $A/(p) = F_q$. The coefficients of $f_0$ now admit lifts to $A$ and define a projective quadric $X'_S$ in $P^N_S$ which is, in particular, flat over $S := \text{Spec } A$, and such that the fiber over $(p)$ is $X'_0$. Recalling EGA IV 12.2.4, the set of points $s \in S$ such that $(X'_S)_s$ is smooth over $k(s)$ is open. In particular, the fiber over the generic point $\xi \in S$ is smooth over $k(\xi)$, which is a finite extension of $Q$. The strict henselization $\tilde{A}$ of $A$ at $(p)$ is a regular local ring with spectrum $\tilde{S}$; $X'_S \to \tilde{S}$ is proper and smooth, and $k(\xi)$ is an algebraic extension of $k(\xi)$. Let $\text{Spec } C = \tilde{\eta} \to \tilde{S}$ be a geometric point centered on $\xi$, and put $X := (X'_S)_\tilde{\eta}$. The specialization morphism

$$H^*(X',Q_\ell) \to H^*(X,Q_\ell)$$

is an isomorphism (Arcata V 3.1). As $X$ is a projective nonsingular quadric in $P^N_C$, the comparison theorem between ordinary cohomology and étale cohomology (Arcata V 3.5.1) allow us to apply the transcendental argument to conclude.

1.13. To compute $R^if_!Q_\ell$, we may assume $Y$ is the spectrum of a separably closed field and $X = A^1$. By Poincaré duality and Artin’s theorem, $H^0_c(A^1,Q_\ell) = 0$. As for $H^1_c$, the
short exact sequence of sheaves on $\mathbf{P}^1$

$$0 \to (\mathbf{Q}_\ell)_A \to \mathbf{Q}_\ell \to (\mathbf{Q}_\ell)_\infty \to 0$$
gives rise to a long exact sequence of cohomology

$$0 \to \mathbf{Q}_\ell \to \mathbf{Q}_\ell \to H^1_c(\mathbf{A}^1, \mathbf{Q}_\ell) \to H^1(\mathbf{P}^1, \mathbf{Q}_\ell) = 0$$
as $\text{Pic}^0(\mathbf{P}^1) = 0$,

verifying $H^1_c(\mathbf{A}^1, \mathbf{Q}_\ell) = 0$. Evidently $H^0(\mathbf{A}^1, \mathbf{Q}_\ell) = \mathbf{Q}_\ell$; by Poincaré we conclude

$$\begin{align*}
R^i f_! \mathbf{Q}_\ell &= 0 & i \neq 2 \\
R^2 f_! \mathbf{Q}_\ell &= \mathbf{Q}_\ell(-1).
\end{align*}$$

2.3*. Leray spectral sequence for cohomology with proper support is a particular case of spectral sequence of composed functors, but the proof that for all composition of morphisms $f = gh : X \to h : Y \to Z$ we have ‘well-behaved’ transitivity isomorphisms between $Rf_!$ and $Rg_!Rh_!$ is not straight forward; it is SGAA Exp. XVII 5.1.8, which also proves that the functors $Rf_!$ are ‘way out’ and triangulated. The proof of transitivity is formal, relying on §3 of the same exposé, which reduces the problem to the analogous ones for proper morphisms and open immersions, provided one ‘compatibility’ isomorphism, which is 5.1.6. With transitivity in hand, the spectral sequence

$$E_2^{pq} = R^p g_! R^q h_!(K) \Rightarrow R^{p+q} f_!(K),$$

valid for $K$ in $D(X, \mathcal{A}_X)$ ($\mathcal{A}$ a sheaf of rings), is a consequence of a general spectral sequence written down by Verdier: for all $L$ in $D(Y, \mathcal{A}_Y)$

$$E_2^{pq} = R^p g_!(\mathcal{H}^q(L)) \Rightarrow R^{p+q} g_!(L);$$

see *Des catégories dérivées des catégories abéliennes* 4.4.6.

2.4*. Küneth formula in cohomology with proper support is SGA 4 Exp. XVII 5.4.

2.5*. The spectral sequence (2.5.2)* is the spectral sequence associated to a filtration of chain complex. Namely, the complements of the closed subsets in the filtration on $X$ gives rise to a filtration on $X$ by open subsets $U_p \subseteq U_{p+1}$. To calculate cohomology
with proper support, fix a compactification \( j : X \hookrightarrow \bar{X} \). In what follows, we write \( \mathcal{F} \) for the sheaf and its various inverse images. Let \( j_p : U_p \hookrightarrow \bar{X} \) and \( i_{p+1} : U_{p+1} \setminus U_p = X_p \setminus X_{p+1} \hookrightarrow \bar{X} \) be the immersions. The filtration \( j_p! \mathcal{F} \subset j_{p+1}! \mathcal{F} \) on \( j_! \mathcal{F} \) has successive quotients isomorphic to \( i_{p+1}! \mathcal{F} \). In light of the spectral sequence (2.3.1)*, the spectral sequence associated to a filtered injective resolution of \( j_! \mathcal{F} \) (Stacks 05TT) gives rise (Stacks 015W) to the spectral sequence

\[
E_1^{pq} = H^{p+q}(\bar{X}, i_{p+1!} \mathcal{F}) = H_c^{p+q}(X_p \setminus X_{p+1}, \mathcal{F}) \Rightarrow H^{p+q}(\bar{X}, j_! \mathcal{F}) = H_c^{p+q}(X, \mathcal{F}).
\]

Remark. Let \( C \) be a curve of finite type over an algebraically closed field \( k \), \( P \) the set of rational points of the curve, \( \mathcal{F} \) the constant sheaf with value \( A \) on \( C \), and \( x \in C(k) \). The fiber of \( P_* P^* \mathcal{F} \) is not \( \mathcal{F}_x \); rather, it can be identified with functions from \( P \) to \( A \) continuous for the discrete topology on \( P \), modulo the relation that \( f \sim g \) if \( f(x) = g(x) \) and \( f \) and \( g \) disagree on only finitely many points.

2.6*. SGAA Exp. V §3 introduces the Čech complex and the Cartan-Leray spectral sequence associated to a covering, which is simply the spectral sequence of the composition of functors, where the first is inclusion of sheaves into presheaves and the second is \( \check{H}^0 \) (it is shown that the \( \check{H}^i \) associated to the Čech complex are indeed right derived functors of \( \check{H}^0 \) on presheaves). The condition on the morphisms in the covering is simply that fibered products are representable, which is true in the category of schemes. However, this is not done for cohomology with support. In the introduction to SGAA Exp. Vbis, the Leray spectral sequence is discussed for an open covering and for a locally finite closed covering, and this guides SGAA Exp. XVII 6.2.8–6.2.10, which discusses in full detail the construction of the so-called ‘extraordinary’ spectral sequence (2.6.2)*. One can also obtain the more ‘ordinary’ spectral sequence of (2.6.1)* from the argument there by replacing the trace morphism \( u! u^* \rightarrow \text{id} \) in the case of \( u : Y \rightarrow X \) separated, étale, surjective, and finite type with the unit of adjunction \( \text{id} \rightarrow u_* u^* \) when \( u \) arises from a finite covering by closed subschemes. Then instead of getting a left resolution of the sheaf, one gets a right resolution, as in the usual Čech resolution.
Given $(X_i)_{i \in I}$ a finite covering by closed subschemes, let $u : Y = \bigsqcup_i X_i := Y \to X$,

$$Y_n = Y \times_X \cdots \times_X Y,$$

and $u_n : Y_n \to X$. The sheaves $\mathcal{F}_n = u_{n*} u_n^* \mathcal{F}$ form, via the units of adjunction $\mathcal{F} \to u_{n*} u_n^* \mathcal{F}$, a simplicial sheaf coaugmented by $\mathcal{F}$. As discussed in SGAA Exp. Vbis, if $\mathcal{C}^*(\mathcal{F})$ denotes an injective resolution of $\mathcal{F}$, the double complex

$$\mathcal{F} \to (u_{p*} \mathcal{C}^q(u_p^*(\mathcal{F})))_{p,q}$$

defines a resolution of $\mathcal{F}$ by injective sheaves.

\[ \begin{array}{ccc}
Y & \xrightarrow{u} & X \\
\downarrow{f} & \downarrow{g} & \downarrow{Spec \ k} \\
\Spec k & \xrightarrow{u_n} & X
\end{array} \]

If $f_n$ is the projection of $Y_n$ to $Spec \ k$, then $Rf_n! = Rg_! u_n = Rg_! u_{n*}$. Applying the functor $g_!$ to the double complex and filtering by semi-simplicial degree gives rise to the Čech spectral sequence

$$E_1^{p,q} = R^q f_p!(u^*_p \mathcal{F}) \Rightarrow R^{p+q} g_! \mathcal{F};$$

this is (2.6.1)* for cohomology with support.

The problem when taking the $X_i$ to be open is that $u_{n*} \neq u_n!$ and there is no longer a coaugmentation map $id \to u_n! u_{n*}$. The trace morphism, however, provides an augmentation map $u_n! u_{n*} \to id$ and one proceeds by resolving to the left to produce the extraordinary Čech resolution and spectral sequence

$$E_1^{-p,q} = R^q f_p!(u^*_p \mathcal{F}) \Rightarrow R^{-p+q} g_! \mathcal{F};$$

this is (2.6.2)*. For more on the extraordinary Čech resolution, see SGAA Exp. XVII 6.2.8–6.2.10.
REFERENCES

[Sommes trig.] Application de la formule des traces aux sommes trigonométriques dans SGA 4\textsuperscript{1\!/\!2}.
As a warm-up, let’s recall some Galois theory from SGA 1 in connection with the beginning of Sommes trig. All references in this paragraph are to SGA 1 Exposé V. The Lang isogeny for $G_{a,F_q}$ is written

$$0 \rightarrow F_q \rightarrow G_a \xrightarrow{x^q-x} G_a \rightarrow 0$$

and is a revêtement étale called the Artin-Schreier revêtement. The Galois group of the Artin-Schreier revêtement is therefore $F_q$, as a connected torsor under a finite group $G$ has Galois (= automorphism) group $G$. Just as in the theory of fields, the choice of words ‘Galois group’ in place of ‘automorphism group’ is reserved for Galois objects in the Galois category. Remark 5.11 characterizes the Galois objects in a Galois category $\mathcal{C}$ equipped with fiber functor $F$ as the connected torsors $X$ under a finite group $G$ (torsor = principal homogeneous space). The implications go as follows: an object $X$ is galoisian if it is connected, not isomorphic to $\emptyset_{\mathcal{C}}$, the initial object of $\mathcal{C}$ ($\Leftrightarrow F(X) \neq \emptyset$), and $\text{Aut} X$ is transitive ($\Leftrightarrow$ simply transitive) ($N^\circ 4$, f) & 5.4). $X$ is a torsor under the group opposite $\text{Aut} X$ iff $F(X)$ is a torsor under the group opposite $\text{Aut} X$; i.e. $\text{Aut} X$ acts simply transitively. Therefore $X$ is a connected torsor under the group opposite $\text{Aut} X$. On the other hand, suppose $X$ is a torsor under $G$, i.e. $G$ acts on $X$ on the right and on $F(X)$ simply transitively, yielding a natural injection from $G$ into the group opposite $\text{Aut} X$. As $G$ acts simply transitively, $\text{Aut} X$ acts transitively. If moreover $X$ is connected, then $N^\circ 4$ f) gives that $\text{Aut} X$ acts transitively iff it acts simply transitively, showing that the injection above is actually an isomorphism between $G$ and the group opposite $\text{Aut} X$. This is justification for the fact ‘In a Galois category, a connected torsor under a finite group $G$ is a Galois object with Galois group $G$.’
1.1.1.5. En vue de l’additivité de $t$ (0.9), l’énoncé $t_{R,f,\overline{Q}_{k,N}}(\infty) = 1 - q$ résulte de la
pureté relative pour $\mathbb{A}^1 \subset \mathbb{P}^1 \leftrightarrow \infty$ (Arcata V 3.4) en ce que
\[
\begin{align*}
&j_\ast \mathbb{Q}_\ell = \mathbb{Q}_\ell \\
&R_1^1 j_\ast \mathbb{Q}_\ell = \mathbb{Q}_\ell(-1)_\infty \\
&R_q^q j_\ast \mathbb{Q}_\ell = 0 \text{ pour } q \geq 2.
\end{align*}
\]

1.1.3. The action of Frobenius on the fiber is clearer in Sommes trig. 1.5, 1.6 since
Deligne’s notation for Frobenius is clear and consistent.

The rigidification (1.1.3.1) is a consequence of the fact that there is a distinguished
element of $L^{-1}(1)$, namely 1.

The trivialization (1.1.3.2) depends on the construction of $\mathcal{L}_x$ from a torsor defined
by an extension. In the setting of Sommes trig. 1.3, the $A$-torsor $T$ is defined as the sheaf
of local sections of an extension $\pi$. As $H$ represents $m^*T$ in the commutative diagram
with exact rows ($\amalg$ = cartesian and $m$ denotes the group law)
\[
\begin{array}{cccccccc}
0 & \longrightarrow & A \times A & \longrightarrow & G' \times G' & \stackrel{\pi \times \pi}{\longrightarrow} & G \times G & \longrightarrow & 0 \\
& & \downarrow m & & \downarrow & & \| & & \\
0 & \longrightarrow & A & \longrightarrow & H & \longrightarrow & G \times G & \longrightarrow & 0 \\
& & \| & & \| & & \| & & \\
0 & \longrightarrow & A & \longrightarrow & G' & \stackrel{\pi}{\longrightarrow} & G & \longrightarrow & 0,
\end{array}
\]
in the language of torsors there is a canonical isomorphism
\[
m(T \times T) = m^*T.
\]
The structure of group on $G$ provides commutative diagrams

\[
\begin{array}{c}
G \\
\downarrow \text{id} \times 1 \\
\downarrow m \\
\downarrow 1 \times \text{id} \\
G \\
\end{array}
\quad \quad
\begin{array}{c}
\pi_1(G, 1) \\
\downarrow \text{pr}_1 \\
\downarrow \pi_1(G, 1) \\
\downarrow \text{pr}_2 \\
\pi_1(G, 1) \\
\end{array}
\quad \quad
\begin{array}{c}
\pi_1(G, 1) \\
\downarrow \text{pr}_1 \\
\downarrow \pi_1(G, 1) \\
\downarrow \text{pr}_2 \\
\pi_1(G, 1) \\
\end{array}
\]

The diagram

\[
\pi_1(G \times G, (1, 1)) \xrightarrow{\text{pr}_1 \times \text{pr}_2} \pi_1(G, 1) \times \pi_1(G, 1) \xrightarrow{m} \pi_1(G, 1)
\]

gives the action of $\pi_1(G \times G, (1, 1)$ on the torsors $T \times T$ and $m(T \times T) = m^*T$. Pushing by $\chi^{-1}$ gives a representation of $\pi_1(G \times G, (1, 1))$ on $\overline{Q}_L$ which is induced by

\[(g_1, g_2) \mapsto \chi^{-1}(g_1) \otimes \chi^{-1}(g_2),\]

demonstrating explicitly an isomorphism

\[m^* \mathcal{L}_\chi \cong \text{pr}_1^* \mathcal{L}_\chi \otimes \text{pr}_2^* \mathcal{L}\]

and hence a trivialization of $\mathcal{D}_2(\mathcal{L}_\chi)$ compatible with the identification of the fibers of $\mathcal{D}_2(\mathcal{L}_\chi)$ and $\overline{Q}_L$ at $(1, 1)$.

(1.1.3.3) = Sommes trig. 1.7.7 (see note to Sommes trig. 1.6)

(1.1.3.4) = Sommes trig. 2.7*.

The remark (1.1.3.7) can be summed up by the morphism of torsors induced by the commutative diagram below, the projective limit of which defines $\lim_{l(k)} \mu_N(k)$:

\[
\begin{array}{ccc}
1 & \longrightarrow & \mu_{NM}(k) \\
\downarrow & & \downarrow \text{[M]} \\
1 & \longrightarrow & \mu_N(k)
\end{array}
\quad \quad
\begin{array}{ccc}
\mathbf{G}_{m,k} & \xrightarrow{[\text{NM}]} & \mathbf{G}_{m,k} \\
\downarrow \text{[M]} & & \downarrow \text{[M]} \\
\mathbf{G}_{m,k} & \longrightarrow & 1
\end{array}
\quad \quad
\begin{array}{ccc}
1 & \longrightarrow & \mu_{NM}(k) \\
\downarrow & & \downarrow \text{[M]} \\
1 & \longrightarrow & \mu_N(k)
\end{array}
\quad \quad
\begin{array}{ccc}
\mathbf{G}_{m,k} & \xrightarrow{[\text{N}]} & \mathbf{G}_{m,k} \\
\downarrow \text{[N]} & & \downarrow \text{[N]} \\
\mathbf{G}_{m,k} & \longrightarrow & 1
\end{array}
\]
As $\mathbb{F}_q^\times \simeq \mathbb{Z}/(q - 1)$, $I(\mathbb{F}_q) = \{n \in \mathbb{N}_{>0} : n|q - 1\}$.

1.2.1. The pairing $\langle , \rangle : E \times_S E' \to G_{a,k}$ should rather land in $G_{a,S}$. See Sommes trig. 1.7 c) to make sense of what this does to $\mathcal{L}_\psi$.

To be maximally pedantic, if $q = p^a$, with the notation of (1.1.3.3) we should write $\psi_n$ where $\psi_q$ appears in the definition of $\hat{\iota}$.

1.2.2. Locally $S = \text{Spec} A$, $E = \text{Spec} A[t_1, \ldots, t_r]$, $E \times_S E = \text{Spec} A[t_1, \ldots, t_r, t'_1, \ldots, t'_r]$, and the addition $E \times_S E \to E$ is given by $t_i \mapsto t_i + t'_i$ while $[-1]$ is given by $t_i \mapsto -t_i$. More to the point, writing $\mathcal{E} \simeq \text{Spec} \text{Sym}(\mathcal{E})$ with $\text{Sym}(\mathcal{E})$ the symmetric algebra on a locally free sheaf $\mathcal{E}$, then addition of sections gives addition on $E$, and vice versa, as the sheaf of local sections of $E$ coincides with $\mathcal{E}^\vee$.

(1.2.2.1) Identifying $E$ with $E''$ via $e \mapsto \langle e, \rangle = -a(e)$, the diagram becomes the base change by $\pi' : E' \to S$ of addition on $E''$,

\[\begin{array}{ccc}
E'' \times_S E'' & \to & E'' \\
\downarrow^{pr} & & \downarrow^{pr} \\
E'' & \to & E'' \\
\downarrow & & \downarrow \\
S & \to & E''
\end{array}\]

and the isomorphism

$\text{pr}_{12}^* \mathcal{L}(\langle , \rangle) \otimes \text{pr}_{23}^* \mathcal{L}(\langle , \rangle) = \alpha^* \mathcal{L}(\langle , \rangle)$
becomes (1.1.3.2) on the nose.

\[
\mathcal{F}' \circ \mathcal{F}(K) = R \text{pr}_!''((\text{pr}'''(R \text{pr}_!(\text{pr}'(K) \otimes \mathcal{L}(\langle , \rangle))) \otimes \mathcal{L}(\langle , \rangle))[2r]
\]

\[
\simeq R(\text{pr}''' \circ \text{pr}_{23})!((\text{pr} \circ \text{pr}_{13})^*(K) \otimes \alpha^* \mathcal{L}(\langle , \rangle) \otimes \text{pr}_{23}^* \mathcal{L}(\langle , \rangle))[2r]
\]

\[
\simeq R(\text{pr}''' \circ \text{pr}_{23})!((\text{pr} \circ \text{pr}_{13})^*(K) \otimes \alpha^* \mathcal{L}(\langle , \rangle))[2r]
\]

\[
\simeq R \text{pr}_!''((\text{pr}' K \otimes R \text{pr}_{13}! \alpha^* \mathcal{L}(\langle , \rangle))[2r]
\]

\[
\simeq R \text{pr}_!''((\text{pr}' K \otimes \beta^* \text{pr}''' \mathcal{L}(\langle , \rangle))[2r].
\]

In (1.2.2.2), the first equation of the proof is missing a \(\pi^*\); it should read

\[
\mathcal{F}(\pi^*\mathcal{L}[r]) = \pi''\mathcal{L} \otimes R \text{pr}_! \mathcal{L}(\langle , \rangle)[2r].
\]

The rest of the proof of (1.2.2.2) is straightforward. It allows us to write

\[
\mathcal{F}' \circ \mathcal{F}(K) \simeq R \text{pr}_!''((\text{pr}' K \otimes \beta^* \mathcal{F}'(\overline{Q}_{E'[r]})
\]

\[
\simeq R \text{pr}_!''((\text{pr}' K \otimes \beta^* \sigma''_s \overline{Q}_{E,-r}))
\]

\[
\simeq a_*K(-r),
\]

as \(\beta^* \sigma''_s\) is supported precisely on the locus where \(e'' = a(e)\), and after base change along \(E \times_{a, \text{id}} E'' \to E \times_S E''\), which commutes with \(\text{pr}_!''\), we find \(R \text{pr}_!'' \text{pr}' K = a_*\).

**Remark.** Note that the inverse image under \(a\) of this identity is \(\mathcal{F}_{\psi^{-1}} \circ \mathcal{F}(K) \simeq K(-r)\), since \(a^* \mathcal{F}' = \mathcal{F}_{\psi^{-1}}\) in light of the fact that \(a^* \psi((e'', e')) = \psi((-e, e')) = \psi^{-1}((e, e'))\).

(1.2.2.4) The adjunction upon which the equality rests is

\[
\langle f(s), \varphi \rangle_2 = \langle s, f'' \varphi := \varphi \circ f \rangle_1,
\]
as pullback by $f'$ on a section $\varphi$ of $E_2'$ gives, by definition, the element of $E_1'$ which, to a section $s$ of $E_1$, applies $f$ then $\varphi$. We then write

$$\mathcal{F}_2(\mathcal{R}f_1 K_1) = \mathcal{R} \text{pr}_2^*(\text{pr}_2^*(\mathcal{R}f_1 K_1) \otimes \mathcal{L}(\langle \ , \ \rangle))[r_2]$$

$$\simeq \mathcal{R} \text{pr}_2^*(\mathcal{R}(f \times 1)_!(\text{pr}_1^* K_1) \otimes \mathcal{L}(\langle \ , \ \rangle))[r_2]$$

$$\simeq \mathcal{R}(\text{pr}_2^* \circ (f \times 1)_!(\text{pr}_1^* K_1) \otimes (f \times 1)^* \mathcal{L}(\langle \ , \ \rangle))[r_2]$$

$$\simeq \mathcal{R}(\text{pr}_2^* \circ (f \times 1)_!(\text{pr}_1^* K_1) \otimes (1 \times f')^* \mathcal{L}(\langle \ , \ \rangle))[r_2]$$

$$\simeq \mathcal{R} \text{pr}_2^*(1 \times f')^* \mathcal{R}(f \times 1)_!(\text{pr}_1^* K_1 \otimes \mathcal{L}(\langle \ , \ \rangle))[r_2]$$

$$\simeq f'^* \mathcal{F}_1(K_1)[r_2 - r_1].$$

(1.2.2.5) If $(\pi')'$ is the morphism of dual bundles $S \to E''$ induced by $\pi'$, then $(\pi')'^* a_* \simeq \sigma^*$. To make sense of the morphism $(\pi')' : S \to E''$, you have to unpack what it means to consider $S$ as a vector bundle of rank 0 over $S$. The only section of $S$ is the zero section, as at all points of $S$, as a rank 0 vector bundle returns the vector space $\{0\}$ at every point of $S$. In other words, we dispose of a tautological isomorphism $S \simeq S \times_k * = S \times \{0\}$. Thinking about $\pi' : E' \to S$ as sending $(s, e') \mapsto (s, 0)$, the map $(\pi')' : S \to E''$ turns the 0 section of $S$ into a section of $E''$, i.e. the one which to a section $s'$ of $E'$ returns the 0 section of $S$. Therefore $(\pi')'$ is the embedding by zero section $0 : S \hookrightarrow E''$; as $-0 = 0$, the zero section of $E''$ corresponds under the isomorphism $a$ to the zero section of $E$, which sees $(\pi')'^* a_* \simeq \sigma^*$.

(1.2.2.7) For the isomorphism

$$\mathcal{F}(K_1 \boxtimes_S K_2) \simeq \mathcal{F}(K_1) \boxtimes_S \mathcal{F}(K_2),$$
perhaps it helps to write the commutative diagram

\[
\begin{array}{c}
E \leftarrow \quad E \times_S E \quad \rightarrow \quad E \\
\uparrow \quad \quad \quad \quad \uparrow \quad \quad \quad \quad \uparrow \\
E \times_S E' \leftarrow (E \times_S E) \times_S (E' \times_S E') \quad \rightarrow \quad E \times_S E \\
\downarrow \quad \quad \quad \quad \downarrow \\
E' \leftarrow \quad \quad E' \times_S E' \quad \rightarrow \quad E'.
\end{array}
\]

If one imagines this diagram drawn on the xy-plane in 3D with the center at \((0,0,0)\), places \(S\) at \((0,0,1)\), and connects every node of this diagram to this \(S\), every row and column of the \(3 \times 3\) above will then form a cartesian square with this \(S\). The equality of Fourier transform then follows from the Küneth formula in light of the isomorphism

\[
\text{pr}_1^* \mathcal{L}(\langle \ , \ \rangle_{E_1}) \otimes \text{pr}_2^* \mathcal{L}(\langle \ , \ \rangle_{E_2}) \cong \mathcal{L}(\langle \ , \ \rangle_{E_1 \oplus E_2}),
\]

which is an obvious consequence of the fact that given vector bundles \(E_1, E_2\),

\[
\langle \ , \ \rangle_{E_1 \oplus E_2} = \langle \ , \ \rangle_{E_1} + \langle \ , \ \rangle_{E_2}.
\]

The character \(\psi\) carries this additive identity to a multiplicative one, hence the \(\otimes\).

To go from here to the stated isomorphism

\[
\mathcal{F}(K_1 \ast K_2) \cong \mathcal{F}(K_1) \otimes \mathcal{F}(K_2)[-r],
\]

after applying (1.2.2.4) in the given way, one is faced with the problem of justifying

\[
\sigma^*(\mathcal{F}(K_1) \boxtimes_S \mathcal{F}(K_2)) \cong \mathcal{F}(K_1) \otimes \mathcal{F}(K_2).
\]

It is useful to reason by adjunction, writing the following adjoint pair of diagrams.
The point is that given $e, e'$ sections of $E, E'$ respectively,
\[ \langle e, (\text{pr}_1 \circ s')(e') \rangle = \langle s_1(e), s'(e') \rangle = \langle (s \circ s_1)(e), e' \rangle = \langle e, e' \rangle. \]

Ergo,
\[ s'^*(\mathcal{F}(K_1) \boxtimes_S \mathcal{F}(K_2)) \simeq (\text{pr}_1 \circ s')^* \mathcal{F}(K_1) \otimes_S (\text{pr}_2 \circ s')^* \mathcal{F}(K_2) \simeq \mathcal{F}(K_1) \otimes \mathcal{F}(K_2). \]

1.2.3. Quelques détails supplémentaires pour les exemples donnés suivent.

(1.2.3.1) With $\sigma^*_E : S \to F'$,
\[ \mathcal{F}(i_* \overline{Q}_{L,F}[s]) \simeq i'^* \mathcal{F}(\overline{Q}_{L,F}[s])[r - s] \simeq i'^* \sigma^*_F(\overline{Q}_{L,S}(-s))[r - s]; \]

now use proper base change for the cartesian square
\[
\begin{array}{ccc}
E' & \xleftarrow{i'^*} & F' \\
\downarrow{\pi'} & & \downarrow{\pi'} \\
F' & \xleftarrow{\sigma^*_F} & S.
\end{array}
\]

(1.2.3.2) A few useless words: $e_\ast \overline{Q}_{L,S}$ is the sheaf on $E$ that is supported precisely on the section $e$ of $E$. Restricted to the closed subscheme which is the image of $e$, $e_\ast \overline{Q}_{L,S}$ is constant with value $\overline{Q}_L$. Therefore $(e_\ast \overline{Q}_{L,S}) \boxtimes_S K$ has support contained in $e \times_S E$, and restricted to this closed subscheme, $s$ is an isomorphism with inverse $E \overset{\sim}{\to} (-e) \times_S E \overset{s}{\to} E \overset{\sim}{\to} e \times_S E$. En effet, $\tau_e$ factors as $E \overset{\sim}{\to} e \times_S E \overset{s}{\to} E$.

After applying (1.2.2.7), one uses (1.2.3.1) to compute
\[ \mathcal{F}(e_\ast \overline{Q}_{L,S}) \simeq e_\ast^* \overline{Q}_{L,E}[r] \simeq \mathcal{L}'(\langle e, \rangle)[r]. \]

(1.2.3.3) The isomorphism $\alpha$ gives rise to a nondegenerate bilinear form $B : E \times_S E \to G_{\alpha,S}$ via $B(e_1, e_2) = \langle e_1, \alpha(e_2) \rangle$, and $\alpha$ is symmetric if $B$ is; equivalently,
\[ \langle \alpha^{-1}(e'), \alpha(e) \rangle = \langle e, e' \rangle. \]
This allows the easy verification of the identity

\((\dagger)\quad q(e) + \langle e, 2e' \rangle = q(e + \alpha^{-1}(e')) - q'(e').\)

The cartesian square

\[
\begin{array}{ccc}
E \times E' & \xrightarrow{\text{id} \times [2]} & E \times E' \\
\downarrow & & \downarrow \\
E' & \xrightarrow{[2]} & E'
\end{array}
\]

and proper base change gives

\([2]^* \mathcal{F}(\mathcal{L}(q)) \simeq \text{R pr}_!{\text{pr}^*_\mathcal{L}(q) \otimes \mathcal{L}(\langle, 2 \rangle)},\)

where here we write \(\mathcal{L}(\langle, 2 \rangle)\) for \((\text{id} \times [2])^* \mathcal{L}(\langle, \rangle).\) Let \(f\) denote the composition of maps in the diagram

\[
\begin{array}{ccc}
E \times E' & \xrightarrow{\text{id} \times \alpha^{-1}} & E \times S E \\
\downarrow & & \downarrow s \\
E & \xrightarrow{s} & E.
\end{array}
\]

We have the following correspondences between functions and sheaves on \(E \times_S E'.\)

\[
q(e + \alpha^{-1}(e')) \leftrightarrow f^* \mathcal{L}(q)
\]

\[
-q'(e') \leftrightarrow \mathcal{L}(-q')
\]

\[
q(e) \leftrightarrow \text{pr}^*_\mathcal{L}(q)
\]

\[
\langle e, 2e' \rangle \leftrightarrow (\text{id} \times [2])^* \mathcal{L}(\langle, \rangle) =: \mathcal{L}(\langle, 2 \rangle).
\]

Disposing of this dictionary, the identity \((\dagger)\) gives an isomorphism of sheaves on \(E \times_S E'.\)

\[
\text{pr}^*_\mathcal{L}(q) \otimes \mathcal{L}(\langle, 2 \rangle) \simeq \text{pr}^*_f \mathcal{L}(-q') \otimes f^* \mathcal{L}(q).
\]

Therefore the stated isomorphism rests on showing that

\[
\text{R pr}_!(\text{pr}^*_f \mathcal{L}(-q) \otimes f^* \mathcal{L}(q)) \simeq \mathcal{L}(-q') \otimes \pi^* \text{R}_{\pi,1} \mathcal{L}(q).
\]
The following diagram is commutative with cartesian squares.

\[
\begin{array}{ccc}
E \times_S E' & \xrightarrow{\text{id} \times \alpha^{-1}} & E \times_S E \\
\downarrow{pr'} & & \downarrow{pr_2} \\
E' & \xrightarrow{\alpha^{-1}} & E \\
\end{array}
\xrightarrow{\beta} 
\begin{array}{ccc}
E \times_S E & \xrightarrow{pr_1} & E \\
\downarrow{pr_2} & & \downarrow{\pi} \\
E & \xrightarrow{\pi} & S
\end{array}
\]

To see that the right square is cartesian, note that the isomorphism

\[
\beta : E \times_S E \longrightarrow E \times_S E
\]

\[
(e_1, e_2) \mapsto (e_1 - e_2, e_2)
\]

induces an isomorphism of cartesian squares

\[
\begin{array}{ccc}
E \times_S E & \xrightarrow{pr_1} & E \\
\downarrow{pr_2} & & \downarrow{\pi} \\
E & \xrightarrow{\pi} & S
\end{array}
\xrightarrow{\beta} 
\begin{array}{ccc}
E \times_S E & \xrightarrow{\text{id}} & E \\
\downarrow{pr_2} & & \downarrow{\pi} \\
E & \xrightarrow{\pi} & S
\end{array}
\]

By the projection formula and proper base change, we conclude

\[
R \text{pr}_!(\text{pr}_*^* \mathcal{L}(-q) \otimes f^* \mathcal{L}(q)) \simeq \mathcal{L}(-q') \otimes R \text{pr}_! f^* \mathcal{L}(q) \simeq \mathcal{L}(-q') \pi'_* R \pi_! \mathcal{L}(q).
\]

(1.2.3.4) Let \( p : G \rightarrow S \) denote the structure morphism and its various base extensions along \( \pi, \pi' \). Let \( f = (\text{pr}_G, m)^{-1} \), with transpose \( f' = (\text{pr}_G, m') \). The diagram

\[
\begin{array}{ccc}
G \times_S E' & \xrightarrow{f'} & G \times_S E' \\
\downarrow{m'} & & \downarrow{p} \\
& & E'
\end{array}
\]

commutes, so that

\[
m'^* \mathcal{F}(K) = f'^* p^* \mathcal{F}(K) \simeq f'^* \mathcal{F}(p^* K) \simeq \mathcal{F}(f; p^* K).
\]
where in the first isomorphism we have used (1.2.2.9) and in the second (1.2.2.4). The commutative diagram

\[
\begin{array}{cccc}
G \times_S E & \xrightarrow{f} & G \times_S E & \xrightarrow{(pr_G, m)} \xrightarrow{p} G \times_S E & \xrightarrow{id} \xrightarrow{m} G \times_S E & \xrightarrow{id} E \\
& \xleftarrow{\pi} & \xleftarrow{\pi'} & \xleftarrow{\pi} & \xleftarrow{p} & \xleftarrow{p} & \xleftarrow{p} \\
& \xleftarrow{pr} & \xleftarrow{pr'} & \xleftarrow{pr} & \xleftarrow{pr} & \xleftarrow{pr} & \xleftarrow{pr} \\
E & \xrightarrow{\pi} & G & \xrightarrow{\pi'} & \xrightarrow{\pi} & \xrightarrow{p} & \xrightarrow{p} \\
& \xleftarrow{p} & \xleftarrow{p} & \xleftarrow{p} & \xleftarrow{p} & \xleftarrow{p} & \xleftarrow{p} \\
\end{array}
\]

together with the isomorphism \( L \sim f_* f^* L = f'_* f'^* L \) for all \( L \) in \( D_c^b(G \times_S E, \overline{Q}_\ell) \) and the hypothesis \( m^* K \simeq M \otimes_S L \) let us write

\[ \mathcal{F}(f_* p^* K) \simeq \mathcal{F}(m^* K) \simeq \mathcal{F}(M \otimes_S L). \]

Letting \( pr, pr' \) denote \( E \xleftarrow{pr} E \times_S E' \xrightarrow{pr'} E' \) and their base extensions by \( p \), the commutative diagrams below have cartesian diamonds marked.
The projection formula and proper base change find

\[ \mathcal{F}(M \boxtimes_S L) \simeq R\text{pr}_1^!(pr^* (M \boxtimes_S L) \otimes \mathcal{L}(\langle \ , \rangle)) \]
\[ \simeq R\text{pr}_1^!(pr^* \pi^* M \otimes pr^* p^* L \otimes \mathcal{L}(\langle \ , \rangle)) \]
\[ \simeq R\text{pr}_1^!(pr^* \pi'^* M \otimes (pr^* p^* L \otimes \mathcal{L}(\langle \ , \rangle))) \]
\[ \simeq \pi'^* M \otimes R\text{pr}_1^!(p^*(pr^* L \otimes \mathcal{L}(\langle \ , \rangle))) \]
\[ \simeq \pi'^* M \otimes p^* \mathcal{F}(L) = M \boxtimes_S \mathcal{F}(L). \]

1.2.3.5. As \( E_1 \times_S E' \simeq E_1 \times_{S_1} E'_1 \simeq E \times_S E'_1 \), the diagram below has cartesian diamonds.

\[ \begin{tikzcd}
S_1 \ar[rd] \ar[dd] & \\
& E_1 \times_{S_1} E'_1 \ar[ld] \ar[d] \ar[rd] \\
E_1 \ar[rd] \ar[r]_{\text{pr}_1} & E \times_S E' \ar[r]_{\text{pr}_1'} & E'_1 \ar[ld] \ar[l] \\
& E \ar[d] \ar[l]_{f_E} \ar[r]_{f_E'} \\
& S
\end{tikzcd} \]

In light of the fact that \( \mathcal{L}(\langle \ , \rangle) \) on \( E_1 \times_{S_1} E'_1 \) coincides with the inverse image under \( f \) of \( \mathcal{L}(\langle \ , \rangle) \) on \( E \times_S E' \), proper base change and the projection formula for \( f \) give

\[ \mathcal{F}(Rf_{E!}K_1) \simeq R\text{pr}_1^!(Rf_! pr^*_1 K_1 \otimes \mathcal{L}(\langle \ , \rangle)) \simeq R(pr' \circ f)_!(pr^*_1 K_1 \otimes f^* \mathcal{L}(\langle \ , \rangle)) \]
\[ \simeq R(f_{E'} \circ pr'_1)_!(pr^*_1 K_1 \otimes \mathcal{L}(\langle \ , \rangle)) \simeq Rf_{E'!} \mathcal{F}_1(K_1). \]

1.3.1–1.3.2.

\( t \)-exactness of \( \mathcal{F} \). Laumon deduces the \( t \)-exactness of \( \mathcal{F} \) from the fact that the ‘forget supports’ map is an isomorphism. However, the following direct argument (lifted from the appendix to the reprinted Astérisque 100) is immediate. As in the remark to the
note to (1.2.2.1), $\mathcal{F}_g^{-1} \circ \mathcal{F}_g(K) = K(-r)$. As $\mathcal{F}_g$ is the composition of exact functors $\text{pr}^*[n]$, $\mathcal{L}(\langle , \rangle)$ and the left t-exact functor $\text{pr}_i$ (BBD 4.1.2), it is left t-exact. But it is also left adjoint to its inverse, and this inverse is also left t-exact since up to a Tate twist it coincides with $\mathcal{F}_g^{-1}$. Therefore $\mathcal{F}_g$ is also right t-exact (BBD 1.3.17 (iii)).

‘Forget supports’. The theorem (1.3.1.1) in Laumon’s paper states that for all $K$ in $\mathcal{D}^b_c(E, \overline{Q}_\ell)$, the ‘forget supports’ map

$$R \text{pr}_i^*(\text{pr}^* K \otimes \mathcal{L}(\langle , \rangle)) \to R \text{pr}_i^*(\text{pr}^* K \otimes \mathcal{L}(\langle , \rangle))$$

is an isomorphism. Laumon refers the reader to his paper with Katz, where they give an involved geometric proof that ends up yielding more. But Verdier gave the first proof of this isomorphism, and his proof is very short and completely formal. It can be found in Katz’s 1988 Séminaire Bourbaki talk ‘Travaux de Laumon.’

References


[Sommes trig.] *Application de la formule des traces aux sommes trigonométriques* dans SGA 4½.
7. Notes on *Théorèmes de finitude en cohomologie ℓ-adique*

1.3. Under the hypothesis of 1.1, \( Rf_* \) is of finite cohomological dimension. In light of the description of the fiber of \( f_* \) (Arcata 3.3), we may assume we are studying the cohomology of a scheme \( X \) of finite type over \( S \) the spectrum of a separably closed field (which has cohomological dimension 0) or a strictly henselian local DVR. SGAA X 4.2 proves this, but we have to satisfy two conditions. The condition (i) can be deduced from SGAA X 2.1, 2.3, which give bounds on the cohomological dimension of residue fields of points of \( X \) depending on which points of \( S \) they sit over; the condition (ii) follows from SGAA X 3.2, which gives a bound on the cohomological dimension of the function field of strict localizations of points of \( X \).

1.6. Given a morphism of schemes \( f : X \to Y \) and sheaves \( \mathcal{F}, \mathcal{G} \) on \( Y \), there is always a morphism

\[
 f^* \text{Hom}(\mathcal{F}, \mathcal{G}) \to \text{Hom}(f^*\mathcal{F}, f^*\mathcal{G}).
\]

When \( \mathcal{F} \) is locally constant constructible, this map is a bijection.

If \( I \) is an injective abelian sheaf on \( X \) and \( j : U \to X \) belongs to the topology on \( X \), then \( J(U) \) is an injective object, since if \( \underline{A} \) is the constant sheaf on \( U \) with value \( A \),

\[
 \text{Hom}(\underline{A}, J(U)) = \text{Hom}(j_!\underline{A}, I),
\]

and \( j_! \) is left exact. The restriction \( I_\mathcal{U} \) is also an injective sheaf, since for \( \mathcal{F} \) on \( U \)

\[
 \text{Hom}(\mathcal{F}, I_\mathcal{U}) = \text{Hom}(j_!\mathcal{F}, I).
\]

Likewise, for any (geometric) point \( x \in X \), \( I_\mathcal{x} \) is an injective object, by forming constant sheaves on \( X \) and taking stalks.

All this to say that \( \text{RHom}(\mathcal{F}, \mathcal{G})|_U = \text{RHom}(\mathcal{F}|_U, \mathcal{G}|_U) \), and when \( \mathcal{F} \) is l.c.c.,

\[
 \text{RHom}(\mathcal{F}, \mathcal{G})|_x = \text{RHom}(\mathcal{F}|_x, \mathcal{G}|_x), \quad \text{and R Hom}(\mathcal{F}, \mathcal{G}) \text{ is constructible (resp. l.c.c.) when } \mathcal{G} \text{ is. This also shows that for } \mathcal{F} \text{ constant, one can compute R Hom by a projective resolution of the constant value of } \mathcal{F}. \]
1.11. $Rpr_2, pr_1^* K = b^* R\Gamma(X, K)$ is base change for $Ra_s$ along $b$. In the string of equalities, the first one is just Leray, and the second and last are again base change morphisms, discussed at length in SGAA XVII §4. Given

\[
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow{f'} & & \downarrow{f} \\
S' & \xrightarrow{g} & S,
\end{array}
\]

at the level of complexes, the map $L \otimes g^* f_* K \to f'_*(f^* L \otimes g^* K)$ factors as

\[
L \otimes g^* f_* K \to L \otimes f'_* g'^* K \to f'_* f^* f'' L \otimes f'_* g'^* K \to f'_*(f^* L \otimes g^* K),
\]

where the last arrow comes from the fact that $\mathcal{F}(U) \otimes \mathcal{G}(U)$ are among the sections of $\mathcal{F} \otimes \mathcal{G}$ over $U$.

2.1. It is claimed that $R\text{Hom}(\mathcal{F}, \mathbb{Z}/m) \xrightarrow{\sim} \text{Hom}(\mathcal{F}, \mathbb{Z}/m)$ when $\mathcal{F}$ is a l.c.c. sheaf (c.f. SGAA XVIII 3.2.6). There is evidently an arrow, and by the note to 1.6 (and since $\mathbb{Z}/m$ is an injective $\mathbb{Z}/m$-module), it is an isomorphism. It holds more generally with $\mathbb{Z}/m$ replaced by any l.c.c. sheaf locally isomorphic to $\mathbb{Z}/m$, such as one obtained by twisting.

References

[Th. finitude] *Théorèmes de finitude en cohomologie $l$-adique* par Deligne, dans SGA 4½.