NOTES

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1. NOTES ON RAPPORT SUR LA FORMULE DES TRACES

\[ p, q, \ell, \mathbf{F}_q, \mathbf{F} \]: \( p \) is a prime number, \( q = p^\ell \) is a power of \( p \) and \( \mathbf{F} \) an algebraic closure of the field \( \mathbf{F}_q \); \( \ell \) is a prime number \( \neq p \).

\( X_0, X \): \( X_0 \) is a scheme on \( \mathbf{F}_q \), \( X = X_0 \times_{\mathbf{F}_q} \mathbf{F} \). If \( \mathcal{F}_0 \) is an (étale) sheaf on \( X_0 \), \( \mathcal{F} \) denotes its inverse image on \( X \).

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Definition. We denote by \( \text{fr}_{X_0} \) the morphism of schemes \( X_0 \to X_0 \) which is the identity on the underlying topological space \(|X_0|\) and acts on the structure sheaf \( \mathcal{O}_{X_0} \) by \( g \mapsto g^q \).

This morphism is called absolute Frobenius.

1.1. If the structure morphism \( X_0 \to \mathbf{F}_q \) factors through some scheme \( S \), then we denote by \( X_0^{(q/S)} := X_0 \times_{S, \text{fr}_S} S \) the fiber product of \( g : X_0 \to S \) by the morphism \( \text{fr}_S : S \to S \) with projection \( \pi_{X_0/S} : X_0^{(q/S)} \to X_0 \). The absolute Frobenius \( \text{fr}_{X_0} \) then factors through the morphism \( \pi_{X_0/S} \). We can form the diagram

\[
\begin{array}{ccc}
X_0 & \xrightarrow{\text{fr}_{X_0}} & X_0^{(q/S)} \\
\downarrow & & \downarrow \\
S & \xrightarrow{\text{fr}_S} & S
\end{array}
\]

Definition. The morphism \( \text{Fr}_{X_0/S} \) is called relative Frobenius.

1.1.1. Frobenius acts on geometric points. Consider the set of geometric points \( X(\mathbf{F}) = X_0(\mathbf{F}) \). Frobenius acts on this set by \( \varphi \in \text{Gal}(\mathbf{F}/\mathbf{F}_q) \), \( \varphi(x) = x^q \). In particular, as \( \mathbf{F}_q \) is perfect, \( X^F = X_0(F_q) \), where \( X^F \) denotes the geometric points fixed by Frobenius.

In slightly more words, consider a geometric point \( \bar{x} \to X_0 \) centered on \( x \). The fiber \( X \times_{X_0} x \) is isomorphic to the spectrum of \( A = \mathbf{F} \otimes_{\mathbf{F}_q} k(x) \). As \( k(x)/\mathbf{F}_q \) is separable, \( A \sim \prod_{[k(x) : \mathbf{F}_q]} \mathbf{F} \), and \( [k(x) : \mathbf{F}_q] \) is also equal to the number of \( \mathbf{F}_q \)-embeddings \( k(x) \to \mathbf{F} \). Such an embedding is fixed by \( \varphi \) iff \( k(x) = \mathbf{F}_q \), and by \( \varphi^f \) iff \( k(x) \subset \mathbf{F}_q^f \).

So for every point \( x \in |X_0| \) with \( [k(x) : \mathbf{F}_q] = f \), there are \( f \) geometric points centered on \( x \); \( F \) acts transitively by \( \varphi \) on this set, and \( F^f \) fixes each of these geometric points.

1.2. Behavior of relative Frobenius. The relative Frobenius \( \text{Fr}_{X_0/S} \) is integral, surjective, and radical, hence a universal homeomorphism. This is clear when \( S = \mathbf{F}_q \); i.e. for \( \text{fr}_{X_0} = \pi_{X_0/S} \circ \text{Fr}_{X_0/S} \) it follows that \( \text{Fr}_{X_0/S} \) is radical [EGA, I 3.5.6 (ii)]. Moreover, \( \pi_{X_0/S} \) is separated and radical, therefore \( \text{Fr}_{X_0/S} \) is integral [EGA, II 6.1.5 (v)] and surjective.
Suppose moreover that $g : X_0 \to S$ is étale. The same is true of $g^{(q)} : X_0^{(q)} \to S$, and therefore $\text{Fr}_{X_0/S} : X_0 \to X_0^{(q)}$ is étale. As $\text{Fr}_{X_0/S}$ is also radicial and surjective, it is an isomorphism.

1.3. Frobenius correspondence. Let $X_0$ be a scheme over $F_q$ and $\mathscr{F}_0$ a sheaf of sets on $(X_0)_{\text{et}}$. We have for all $U \to X$ étale $(\text{fr}_{X_0})_* \mathscr{F}_0(U) = \mathscr{F}_0(U^{(q/X)})$. The isomorphism $\mathscr{F}_0(\text{Fr}_{U/X_0}) : (\text{fr}_{X_0})_* \mathscr{F}_0(U) \to \mathscr{F}_0(U)$ is natural in $U$ and induces an isomorphism of sheaves

$$\mathscr{F}_0(\text{Fr}_{X_0}) : (\text{fr}_{X_0})_* \mathscr{F}_0 \cong \mathscr{F}_0;$$

by adjunction applied to $\mathscr{F}_0(\text{Fr}_{X_0})^{-1}$ we obtain a morphism

$$F^* : \text{fr}_{X_0}^* \mathscr{F}_0 \to \mathscr{F}_0.$$ 

As $\text{fr}_{X_0}$ is integral, surjective, and radicial, $\text{fr}_{X_0}^* : (X_0)_{\text{et}} \to (X_0)_{\text{et}}$ is an equivalence of sites, and the functors

$$(\text{fr}_{X_0})_* \circ \text{fr}_{X_0}^* : (X_0)_{\text{et}} \to (X_0)_{\text{et}}$$

are autoequivalences and quasi-inverses, where $(X_0)_{\text{et}}$ denotes the étale topos on $X_0$ [SGAA, Exp. VIII, 1.1]. Therefore $F^*$ is also an isomorphism.

**Definition.** The isomorphism $F^* : \text{fr}_{X_0}^* \mathscr{F}_0 \to \mathscr{F}_0$ is called the *Frobenius correspondence*.

1.4. Frobenius acts on cohomology. Consider $\mathscr{F}_0$ a sheaf of $\Lambda$-modules, for some commutative ring $\Lambda$. The canonical morphism $\alpha : \mathscr{F}_0 \to \text{fr}_{X_0}^* \text{fr}_{X_0}^* \mathscr{F}_0$ gives rise to

$$\Gamma(X_0, \mathscr{F}_0) \xrightarrow{\alpha} \Gamma(X_0, \text{fr}_{X_0}^* \mathscr{F}_0) \xrightarrow{F^*} \Gamma(X_0, \mathscr{F}_0);$$

we also denote the composition of these maps by $F^*$. When $\mathscr{F}_0 = \Lambda$, this composition is easily seen to coincide with $\text{id}_{\Gamma(X_0, \Lambda)}$, and as every section $s \in \Gamma(X_0, \mathscr{F}_0)$ corresponds to a morphism $s : \Lambda \to \mathscr{F}_0$ and $F^*$ is evidently functorial in $\mathscr{F}_0$, we find $F^* \circ \Gamma(X_0, s) = \Gamma(X_0, s) \circ F^* = \Gamma(X_0, s)$, ergo $F^* s = s$, so $F^*$ induces the identity on $\Gamma(X_0, \mathscr{F}_0)$. Recalling
the definition of the Frobenius correspondence via adjunction, this action of Frobenius on \(\Gamma(X_0, \mathcal{F}_0)\) coincides with the composition

\[
(*) \quad \Gamma(X_0, \mathcal{F}_0) \xrightarrow{\mathcal{F}_0(Fr_0)^{-1}} \Gamma(X_0, (fr_{X_0})_*\mathcal{F}_0) = \Gamma(X_0, \mathcal{F}_0).
\]

Considering \(\mathcal{F}_0\) now as an object of \(D^+(X_0, \Lambda)\), we have \(fr_{X_0*}\) preserves injective objects; hence the composition

\[
F^* : R\Gamma(X_0, \mathcal{F}_0) \xrightarrow{\mathcal{F}_0(Fr_0)^{-1}} R\Gamma(X_0, fr_{X_0*}\mathcal{F}_0) \longrightarrow R\Gamma(X_0, \mathcal{F}_0)
\]

can be computed by applying \((*)\) term-by-term to an injective resolution of \(\mathcal{F}_0\), whence we see that \(F^*\) acts by identity on \(R\Gamma(\mathcal{F}_0)\), and hence

\[
F^* : H^i(X_0, \mathcal{F}_0) \longrightarrow H^i(X_0, \mathcal{F}_0)
\]

is the identity for all \(i\).

Suppose that \(X_0\) is separated and of finite type over \(\mathbf{F}_q\). Then we can replace \(\Gamma\) in the above discussion by \(\Gamma_c\) to find that Frobenius acts on compactly supported cohomology

\[
F^* : H^i_c(X_0, \mathcal{F}_0) \longrightarrow H^i_c(X_0, \mathcal{F}_0).
\]

### 2.1. The trace formula for \(\text{Spec} \mathbf{F}_q^n \to \text{Spec} \mathbf{F}_q\)

Let \(X_0 = \text{Spec} \mathbf{F}_q^n\), \(q = p^f\), and \(\mathcal{F}_0\) be a constructible \(\mathbf{Q}_\ell\) sheaf on \(X_0\), \(\mathcal{F}\) its inverse image on \(X\). In this case, the cohomological description of the L-function \(L(X_0, \mathcal{F}_0)\) reads very simply

\[
(*) \quad \det(1 - F^*_x t^{d(x)}, \mathcal{F})^{-1} = \det(1 - F^* t^f, H^0_c(X, \mathcal{F}))^{-1},
\]

where \(d(x) = [k(x) : \mathbf{F}_p]\). We first need to make precise how Frobenius is acting on the left and right sides.

On the left side, we fix a geometric point \(\overline{x} \to x = X\) and construct the action of Frobenius on the fiber \(\mathcal{F}_x\) by picking the smallest power of \(F_{(q)}\) which actually fixes the geometric point \(\overline{x}\), namely \(F_{(q^n)} = F_{(q)}\). The notation \(F^*_x\) denotes the endomorphism of \(\mathcal{F}_x\) induced by \(F^*_{(q^n)}\). Up to isomorphism, \((F_x, \mathcal{F}_x)\) do not depend on the choice of geometric point \(\overline{x} \to x\), and the trace, determinant of this action are denoted by \(\text{Tr}(F^*_x, \mathcal{F})\), etc.
Now, on the right side, we have the Frobenius correspondance on cohomology. We will make use of the identity
\[ F^{* \kappa - 1} = \varphi, \]
where \( \varphi \) is the Frobenius considered as the topological generator of \( \text{Gal}(\mathbf{F}, \mathbf{F}_q) \) (c.f. remark below). The data of a \( \mathbb{Q}_\ell \)-sheaf on \( X \) is equivalent to the data of a finite-dimensional \( \mathbb{Q}_\ell \)-vector space \( V = \mathcal{F} \pi \) on which \( \text{Gal}(\overline{k(x)} / k(x)) = \pi_x \) acts continuously. There is a canonical isomorphism
\[ \pi_x = \text{Gal}(\overline{k(x)}, k(x)) \cong \mathcal{Z} \]
furnished by the Frobenius element
\[ \varphi_x \in \text{Gal}(\overline{k(x)}, k(x)) = \text{Gal}(\mathbf{F} / \mathbf{F}_q^n), \quad \varphi_x(\lambda) = \lambda^{q^n}, \]
so that the action of \( \pi_x \) on \( V = \mathcal{F} \pi \) is known once one knows the automorphism \( (\varphi_x)_V \) (under the one condition that \( (\varphi_x)_V \to \text{id}_V \) as \( v \to 0 \) multiplicatively). If
\[ \text{pr}_x : X = \text{Spec}(\mathbf{F}_q) \to \text{Spec} \mathbf{F}_q = e \]
is the canonical morphism, \( \pi_x \) is identified via \( \text{pr}_{x*} \) with a subgroup of the analogous Galois group \( \pi_e \) for \( e = \text{Spec}(\mathbf{F}_q) \), itself topologically generated by \( \varphi \), and via this identification we have the identity
\[ \varphi_x = \varphi^n. \]
The sheaf \( \text{pr}_{x*}(\mathcal{F}) \) is defined by the induced module
\[ \text{pr}_{x*}(\mathcal{F})_\pi \simeq \mathcal{F} \pi \otimes_{\pi_x} \pi_e, \]
from which one deduces that, letting \( f = \varphi_x^{-1}, \varphi^{-1} \) acts on \( \text{pr}_{x*}(\mathcal{F})_\pi \) by
\[ f^{(n)} : (x_1, \ldots, x_n) \mapsto (f(x_n), x_1, \ldots, x_{n-1}), \]
where here we have written \( \text{pr}_{x*}(\mathcal{F})_\pi \) with respect to a basis as a free \( \pi_x \)-module of rank \( n \). Now the formula \( (*) \) is a matter of verifying the formula
\[ \det(1 - ft^n) = \det(1 - f^{(n)}t) \]
for \( f \) acting on a free module of rank \( n \). This is Deligne’s corollary 3.4.
Remark. Perhaps one way to think about the identification $F^* = q^{-1}$ is by setting up the usual diagram

![Diagram](image)

with $g$ the base extension of the only $F_q$-linear map $\text{Spec} F \to \text{Spec} F_q$, and then observing that $\text{Fr}_{X/X_0} = q \times_{F_q} \text{id}_{X_0}$. Recalling that $F^*$ on $\mathcal{F}/X_0$ is induced by $\text{Fr}_{U/X_0}^{-1}$ for $U \to X_0$ étale, and by functoriality $F^*$ on $\mathcal{F}/X$ is induced by pulling back the same, hence by $\text{Fr}_{U/X_0}^{-1}$ for $U \to X$ étale, in particular we have that $F^*$ on $H^1_c(X, \mathcal{F})$ is induced by $\mathcal{F}(\text{Fr}_{X/X_0})^{-1} = \mathcal{F}(q^{-1} \times_{F_q} \text{id}_{X_0})$.

3.1. Le sorite de la notation. It is very important to note that in Deligne’s notation, $\text{Tr}(F^*_x, \mathcal{F})$ and $\text{Tr}(F^*, \mathcal{F}_x)$ are traces of possibly different operators on the fiber $\mathcal{F}_x$. Namely, if $\mathcal{F}$ is a $\mathbb{Q}_\ell$-sheaf on $X_0$ a scheme separated and of finite type over $F_q$, then $\mathcal{F}_x$ is a $\mathbb{Q}_\ell$ vector space, for a choice of geometric point $\overline{x}$ centered on a closed point $x$ of $X_0$. Then $F^*_x$ denotes the Frobenius $F^*_q^n = F^*_q$ raised to the power of the residue field extension $n = [\deg k(x) : F_q]$. This power of Frobenius is the least that fixes each geometric point centered on $x$, and the notation $\text{Tr}(F^*_x, \mathcal{F})$ means $\text{Tr}(F^*_x, \mathcal{F}_x)$.

On the other hand, if, say, $x \in X^{F^n}$ is a geometric point centered on a point of $X_0$ defined over $F_q^n$, $\text{Tr}(F^*, \mathcal{F}_x)$ denotes (absolute) $q$-power Frobenius acting on the fiber. So, unless $x \in X^F$, $\text{Tr}(F^*, \mathcal{F}_x)$ and $\text{Tr}(F^*_x, \mathcal{F})$ are traces of different operators on the same vector space, the latter a power of the other.

3.2. Le sorite des faisceaux localement constantes.

The case of locally constant sheaves of sets. Let $X$ be a scheme and $\mathcal{L}$ a locally constant sheaf of sets with finite fibers on $X$. (With additional assumptions on $X$, the case of a locally constant sheaf of sets with infinite fibers is reduced to the finite case in the
course of the discussion of Weil II 1.7.8.) We know that \( \mathcal{L} = h_U \) for some \( U \to X \) revêtement étale. We know that every revêtement étale of \( X \) is étale-locally on \( X \) trivial; namely for some \( V \to X \) étale, \( U_V \sim \bigsqcup V \). We wish to show that we may take \( V \to X \) to be a revêtement étale (with no further work, we could then take it to be a \textit{galoisian} revêtement, i.e. a connected torsor for the automorphism group of the fiber, as principal Galois objects in a Galois category form a cofinal system).

First note that if \( f : X \to Y \) is any morphism of schemes and \( V \to Y \) is étale, then \( f^*h_V \sim h_{V \times_Y X} \). To see this, observe

\[
\text{Hom}(f^*h_V, \mathcal{G}) = \text{Hom}(h_V, f_*\mathcal{G}) = f_*\mathcal{G}(V) = \mathcal{G}(V \times_Y X) = \text{Hom}(h_{V \times_Y X}, \mathcal{G}).
\]

Evidently, this argument holds true for any morphism of sites.

So, it will suffice to show that any revêtement étale can be trivialized after base extension by a revêtement étale. To see this, assume \( X \) connected and let \( U \to X \) be a revêtement étale of constant degree \( d \) and proceed by recurrence on \( d \), the case \( d = 1 \) being trivial. (Of course, in the special case that \( U \to X \) is galoisian with Galois group \( G \), \( U \times_X U \sim U \times G \) is a trivial \( G \)-torsor, and we are done.)

As \( U \to X \) is étale, hence net, and finite, the diagonal morphism \( U \to U \times_X U \) is simultaneously an open and closed immersion, hence an isomorphism onto a connected component of \( U \times_X U \), allowing us to write \( U \times_X U = U \bigsqcup Z \) with \( Z \to U \) of constant degree \( d - 1 \). By hypothesis, there exists a revêtement étale \( V \to U \) such that \( Z \times_U V \sim \bigsqcup_{d-1} V \). Our desired revêtement is then simply the composition \( V \to U \to X \):

\[
V \times_X U = V \times_U U \times_X U = V \times_U (U \bigsqcup Z) = V \bigsqcup (V \times_U Z) = \bigsqcup_d V.
\]

\textit{The case of locally constant constructible sheaves.} Let \( \Lambda \) be a commutative, noetherian torsion ring. We adapt the above discussion to locally constant constructible (l.c.c.) sheaves of \( \Lambda \)-modules. Let \( \mathcal{F} \) be a l.c.c. sheaf of \( \Lambda \)-modules on a connected scheme \( X \). Then the fibers of \( \mathcal{F} \) are finite sets and the above discussion yields a revêtement étale \( f : V \to X \) with \( V \) a (connected) galoisian cover with Galois group \( H \) (i.e. \( V \) is a \( H \)-torsor) such that \( f^*\mathcal{F} \) is a constant sheaf. Its constant value \( H^0(V, f^*\mathcal{F}) \) is a \( \Lambda[H] \)-module.
The sheaf \( f_*\Lambda \) on \( X \), together with the natural action of \( H \), is a rank 1 l.c.c. sheaf of \( \Lambda[H] \)-modules. Relative to the natural action of \( H \) on \( f_*f^*\mathcal{F} \), the trace morphism 
\[ (f_*f^*\mathcal{F})_H \to \mathcal{F}, \]
and we have \( f_*f^*\mathcal{F} = f_*\Lambda = f_*\Lambda \otimes \Lambda M \) with the diagonal action of \( H \).

The above discussion shows that a l.c.c. sheaf of \( \Lambda \)-modules on a connected scheme \( X \) is determined by its restriction to the small étale site of \( X \). Sheaves on the small étale site \( \mathcal{U} \) are in turn determined by Grothendieck’s Galois theory: fixing a geometric point \( \overline{x} \) of \( X \) and putting \( G := \pi_1(X, \overline{x}) \), the functor
\[ \text{Sh}(\mathcal{U}) \to \text{finite } G\text{-sets} \]
\[ \mathcal{F} \mapsto \mathcal{F}_{\overline{x}} \]
admits the inverse
\[ \text{finite } G\text{-sets} \to \text{Sh}(\mathcal{U}) \]
\[ \mathcal{F}_{\overline{x}} \mapsto [V \in \mathcal{U} \mapsto \text{Hom}_G(V_{\overline{x}}, \mathcal{F}_{\overline{x}})]. \]

To verify this, as the torsors are cofinal in a covering of any \( V \in \mathcal{U} \), we may cover \( V \) by a torsor \( W \) with Galois group \( H \) and combine the equalizer description of \( \mathcal{F}(V) \)
\[ \mathcal{F}(V) \to \mathcal{F}(W) \rightrightarrows \mathcal{F}(W \times_V W) \]
with the description of such in the case of a Galois torsor; c.f. [SGA 4\( \frac{1}{2} \), I 85].

The discussion in the previous section can be rephrased using the monodromy representation of a l.c.c. sheaf. Namely, let \( \mathcal{F} \) be a l.c.c. sheaf of \( \Lambda \)-modules on a connected scheme \( X \) pointed by a geometric point \( \overline{x} \) as above; \( \mathcal{F} \) corresponds to a representation \( \pi_1(X, \overline{x}) \to \text{GL}(\mathcal{F}_{\overline{x}}) \). As the latter is a finite group, the kernel of this representation is of finite index, and as the Galois coverings are cofinal, we can find a Galois cover of \( X \) corresponding to a open subgroup contained in the kernel. The sheaf \( \mathcal{F} \) becomes constant when restricted to this cover.
The case of lisse sheaves. Let $E \subset \overline{Q}_\ell$ be an finite extension of $Q_\ell$ with valuation ring $R$, integral closure of $Z_\ell$ in $E$, $m$ the maximal ideal of $R$. Every $\overline{Q}_\ell$-sheaf $\mathcal{G}$ is obtained as $\mathcal{F} \otimes_E \overline{Q}_\ell$ for some $E$ and some torsion-free (i.e. flat) constructible $E$-sheaf $\mathcal{F}$. This means that $\mathcal{F} = \lim \text{proj} \mathcal{F}_n$, the latter a flat $R$-sheaf. A lisse $R$-sheaf has all the $\mathcal{F}_n$ locally constant sheaves of $R/m^n$-modules, and for each $n$, the above discussion shows that the functor ‘fiber at $\overline{x}$’ gives an equivalence of categories between the category of lisse $R/m^n$-sheaves and the category of $R/m^n$-modules of finite type together with a continuous action of $\pi_1(X, \overline{x})$. Since the $\mathcal{F}_n$ have $\mathcal{F}_n \otimes_{R/m^{n+1}} R/m^n \sim \mathcal{F}_{n-1}$, by passing to the limit we get an equivalence between the category of lisse $R$-sheaves and the category of finite $R$-modules with continuous action of $\pi_1(X, \overline{x})$.

References

[EGA] Eléments de géométrie algébrique par A. Grothendieck.
[SGA 4\frac{1}{2}] SGA 4\frac{1}{2}, Rapport sur la formule des traces par P. Deligne.
2. Notes on quasi-unipotent monodromy

Some notes about Grothendieck’s theorem on quasi-unipotent monodromy. We study the arithmetic proof. It uses a proposition proved by Grothendieck in the appendix of Serre and Tate’s article *Good Reduction of Abelian Varieties*.

We may assume that $K$ is complete since, following Serre, *Corps Locaux*, II§3 Cor. 4, completing $K$ leaves the decomposition unchanged. Now, we may assume that any matrix in the image of $\rho$ has coefficients in $\mathbb{Z}_l$ and is congruent to $1 \mod l^2$ as these are both open conditions, $\rho$ is continuous, and we are free to pass to an open subgroup of $I(\overline{\nu})$ by making a finite extension of $K$.

Note also that $\text{im} \, \rho$ is a pro-$l$ group since, while $GL_n(\mathbb{Z}_l)$ is not a pro-$l$-group, its first congruence subgroup of matrices congruent to $1 \mod l$ is a pro-$l$-group (c.f., e.g., §5.1 of *Analytic Pro-$p$ Groups* by Dixon, du Sautoy, Mann & Segal). We see therefore that the prime-to-$l$ part of the order of $GL_n(\mathbb{Z}_l)$ is finite. As the image of a pro-$p$ group under a continuous homomorphism is pro-$p$, the continuous image of a pro-$p$ group in $GL_n(\mathbb{Z}_l)$ is finite. As $\text{im} \, \rho$ is by construction pro-$l$, the image of a pro-$p$ group in $\text{im} \, \rho$ is $\{1\}$.

Now, if $L$ is a finite extension of $K_l$, we wish to show that the polynomial $f(T) = T^l - a$ splits in $L$ for any $a \in L$. If it does not, then as $f$ is separable and $L$ contains all $l^{th}$ roots of unity, $L(\sqrt[l]{a})$ is the splitting field of $f$ and is Galois. The automorphism of $L(\sqrt[l]{a})/L$ sending $\sqrt[l]{a} \mapsto \zeta_l \sqrt[l]{a}$, where $\zeta_l$ is a primitive $l^{th}$ root of unity, acts transitively on the roots of $f$, hence $f$ is irreducible. But $K_l$ is the $l$-part of the maximal tamely ramified extension of $K_{nr}$, hence $l$ cannot divide $[L : K_l]$. (Recall that $K_l$ is the maximal tamely ramified extension of $K$, and we have

$$\text{Gal}(K_l/K_{nr}) \cong \prod_{q \neq p} \mathbb{Z}_q \quad \text{Gal}(K_l/K_l) \cong \prod_{q \neq p, l} \mathbb{Z}_q \quad \text{Gal}(K_s/K_l) \text{ a pro-$p$ group}$$

as $q$ runs over primes, so $\text{Gal}(K_s/K_l)$ is an extension of a group isomorphic to $\prod_{q \neq p, l} \mathbb{Z}_q$ by a pro-$p$ group, and therefore has no finite quotient of order divisible by $l$.) This
allows one to conclude that \( l \) does not divide the order of \( \text{Gal}(K_f/K_f) \). The order of \( \text{im} \rho \) is a power of \( l \) as it is a pro-\( l \) group.

An alternative way to see that \( l \) does not divide the order of \( \text{Gal}(K_f/K_f) \) that is more faithful to the original proof proceeds by showing directly that for a finite extension \( L/K_f \), every element of \( L \) is an \( l^{th} \) power. To do this, let \( L = K_f[t]/a(t) \) for \( a(t) \) an irreducible separable polynomial \( a(t) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0 \), and suppose \( t \) is not an \( l^{th} \) power in \( L \). This implies that the polynomial \( a_l(t) = a_n x^{ln} + a_{n-1} x^{l(n-1)} + \ldots + a_0 \) is irreducible and separable over \( K_f \). Let \( K'/K \) be a finite Galois extension containing all \( l^{th} \) roots of unity and the \( a_l \), and contained in \( K_f \). The extension \( K'[t]/a_l(t) \) is finite and separable and is contained in a finite Galois extension \( K'' \) of \( K' \) with \( l \) dividing \( [K'' : K'] \), so \( l \) divides the ramification index or the residual degree. If the latter, making a finite unramified extension of \( K' \) produces a contradiction on the irreducibility of \( a_l(t) \) over \( K_f \). If the former, replacing \( K'' \) by \( K''[\pi^{1/l}] \subset K_f \) similarly yields a contradiction.

Now to see that every finite Galois extension \( L \) of \( K_f \) cannot have \( l \) dividing its degree, note that \( \text{Gal}(L/K_f) \) contains a cyclic subgroup \( H \) of order \( l \), and we claim \( L = L^H(a^{1/l}) \) for some element \( a \in L^H \). Let \( \sigma \) generate \( H \), and let \( b \in L - L^H \). Then the element

\[
c = \sum_{m=1}^{l} \zeta_l^m \sigma^m(b)
\]

satisfies \( c = \zeta_l \sigma(c) \), where \( \zeta_l \) is a primitive \( l^{th} \) root of unity. So

\[
c^l = \prod_{m=1}^{l} \sigma^m(c) \in L^H,
\]

and letting \( a = c^l \) we find that \( L^H(a^{1/l}) \) is a nontrivial subextension of \( L^H \), hence must actually coincide with \( L \).

**References**

3. Notes on Weil II

0.5. Let X be a scheme of finite type over a field k. If X is connected, the structure morphism $X \to \text{Spec}(k)$ admits a unique factorization $X \to \text{Spec}(k') \to \text{Spec}(k)$ with $k'/k$ finite separable and $X \to \text{Spec}(k')$ geometrically connected. $\leadsto$ Stacks, tag 04PZ. Proof uses notion of ‘weakly étale $k$-algebra.’

1.1.2. First of all, to see that if $K \in D^b(X, R)$, then $K \otimes^L R/m^n \in D^b_{ctf}(X, R/m^n)$, see Stacks, tag 0942. Note that $K \otimes^L R/m^n$ can be represented by a bounded complex of flat constructible sheaves by Rapport, 4.7. Also recall that the locally constant sheaves form a weak Serre subcategory of the constructible sheaves on a site (093U). Here is a useful lemma which adapts Th. finitude 4.6.

**Lemma.** Let $X$ be a noetherian scheme, $\Lambda$ a left noetherian ring, and $K \in D^{-}(X, \Lambda)$. Then the following are equivalent.

1. $K$ is of finite Tor-dimension and the sheaves $\mathcal{H}^i(K)$ are locally constant constructible.

2. $K$ is locally isomorphic to a bounded complex of locally constant, flat $\Lambda$-modules; i.e. there is a finite étale covering $\{U_i \to X\}$ such that $K|_{U_i}$ is isomorphic (in $D^{-}(U_i, \Lambda)$) to a bounded complex of constant sheaves of projective $\Lambda$-modules of finite type.

The argument follows Th. finitude 4.5, except now the $\mathcal{H}^i(K)$ are moreover locally constant constructible. The sheaves A and B are defined by the cartesian diagram

\[
\begin{array}{ccc}
K^n & \longrightarrow & K^n/\text{im } d \\
\uparrow & & \uparrow \\
A & \longrightarrow & \ker d \\
\end{array}
\]

and B is locally free as it sits in the middle of the exact sequence

\[
0 \to \mathcal{H}^n(K) \xrightarrow{(\text{id}, 0)} B \to \ker d \to \mathcal{H}^{n+1}(K)
\]
where ker \(d\) here denotes the kernel of the differential on the complex \(K'\) being constructed and is hence locally constant constructible.

As \(u\) is surjective, localizing, we may assume \(B\) constant constructible and that \(u\) surjects on global sections, defining a morphism \(v : \Lambda^{\oplus d} \to A\) with the property that \(vu\) is an epimorphism, and we define \(K''' = \Lambda^{\oplus d}\).

Suppose \(K\) is of Tor-dimension \(\leq r\). Propagating the above procedure to the left as far as degree \(-r - 1\), we produce an étale morphism \(U \to X\), constant constructible sheaves

\[
K'^{-r-1} \to K'^{-r} \to K'^{-r+1} \to \ldots
\]

with the property that \(K'^{-r}/\text{im} \, d\) is constant constructible and flat. Then, over \(U\), the complex

\[
\ldots \to 0 \to K'^{-r}/\text{im} \, d \to K'^{-r+1} \to K'^{-r+2} \to \ldots
\]

is quasi-isomorphic to \(K\), and has the desired properties.

**Corollary (Th. finitude 1.7).** If \(\Lambda\) is moreover commutative and of torsion, and in the situation of Th. finitude, then

\[
\operatorname{RHom} : D^b_{\text{ctf}}(X, \Lambda) \times D^b_{\text{lf}}(X, \Lambda) \to D^b_{\text{lf}}(X, \Lambda).
\]

Following Th. finitude 1.7, as the finite Tor-dimension is stable by \(\text{R}f_*\), \(\text{R}f^!\), \(f^*\), \(\text{R}f^!\), devissant the first variable (say, \(\mathcal{F}\)) relative to a partition of \(X\), and using the adjunction

\[
\operatorname{RHom}(j_!\mathcal{F}, \mathcal{G}) \leftrightarrow \text{R}j_* \operatorname{RHom}(\mathcal{F}, \text{R}j^!\mathcal{G})
\]

for \(j : Y \to X\) the inclusion of a locally closed subscheme (c.f. SGA 4, IX 2.5 & XVIII §3.1), one reduces to the situation where the \(\mathcal{H}^i(\mathcal{F})\) are locally constant. Localizing, we can replace \(\mathcal{F}\) by a bounded complex of constant sheaves of projective \(\Lambda\)-modules of finite type. As in this case \(\operatorname{Hom}\) computes pointwise, we can compute \(\text{R} \operatorname{Hom}\) with respect to such a complex. Finally, if \(N\) and \(M\) are \(\Lambda\)-modules, \(N\) projective and \(M\) of Tor-dimension \(\leq r\), then \(\operatorname{Hom}(N, M)\) is of Tor-dimension \(\leq r\), which can be seen after replacing \(M\) by a complex of flat modules 0 to the left of \(-r\) and writing \(N\) as a direct summand of a free module.
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Returning to the setting of the paper, with $R$ the ring of integers of a finite extension of $Q_p$, $m$ its maximal ideal, let’s assume that reduction mod $m^n$ commutes with the four operations $R f_1, f^*$, $R f_1$, and $R f^1$, and show that for $\mathcal{F}, \mathcal{G} \in D^b_c(X, R)$,

$$
(\dagger) \quad R \mathcal{H}om(\mathcal{F} \otimes^L R/m^{n+1}, \mathcal{G} \otimes^L R/m^{n+1}) \otimes^L_{R/m^{n+1}} R/m^n
= R \mathcal{H}om(\mathcal{F} \otimes^L R/m^n, \mathcal{G} \otimes^L R/m^n) \quad \text{in } D^b_{ctf}(X, R/m^n).
$$

Devissant $\mathcal{F} \otimes^L R/m$ with respect to a partition of $X$, we may assume that its cohomology sheaves are locally constant, and therefore the same is true of $\mathcal{F} \otimes^L R/m^{n+1}$ by considering the $m$-adic filtration on a finite complex of flat sheaves representing it. Localizing, we may replace $\mathcal{F} \otimes^L R/m^{n+1}$ with a bounded complex $N^*$ of free $R/m^{n+1}$ modules of finite type and compute $R \mathcal{H}om$ with respect to $N^*$, since for $\mathcal{F}$ locally free, $\mathcal{H}om(\mathcal{F}, \mathcal{G})_x = \mathcal{H}om(\mathcal{F}_x, \mathcal{G}_x)$. Now the equality $(\dagger)$ is clear.

Consider the example: $X$ finite type over $S$, $\mathcal{F}$ and $\mathcal{G}$ two constructible torsion-free $R$-sheaves. The claim is that the projective system

$$
\mathcal{E}xt^i(\mathcal{F}, \mathcal{G}) := H^i R \mathcal{H}om(\mathcal{F}, \mathcal{G}) = \text{"lim proj" } \mathcal{E}xt^i_{R/m^n}(\mathcal{F} \otimes R/m^n, \mathcal{G} \otimes R/m^n)
$$

forms a constructible $R$-sheaf. By part (a) of 1.1.2, it suffices to show that $R \mathcal{H}om(\mathcal{F}, \mathcal{G}) \in D^b_c(X, R)$. By Th. finitude 1.6 and the previous corollary, $R \mathcal{H}om$ sends $D^b_{ctf}(X, R/m^n) \otimes D^b_{ctf}(X, R/m^n)$ into $D^b_{ctf}(X, R/m^n)$, so $R \mathcal{H}om(\mathcal{F} \otimes R/m^n, \mathcal{G} \otimes R/m^n) \in D^b_{ctf}(X, R/m^n)$. Finally, by $(\dagger)$,

$$
\text{"lim proj" } R \mathcal{H}om(\mathcal{F} \otimes R/m^n, \mathcal{G} \otimes R/m^n) \in D^-(X, R).
$$

**Truncation & Tor-dimension.** In part (e), Deligne addresses the truncation operators $\tau_{\leq n}$. The issue is that, while a submodule of a flat $R$-module is flat, a submodule of a flat $R/m^n$-module need not be. To address this deficiency, Deligne introduces the modified truncation operators $\tau'_{\leq n}$, which preserve the finite Tor-dimension. As these properties are of a pointwise nature, we may consider the situation in the category of $R$-modules, and the categories $D_{par}$, $D_{par}$.

Applying Houzel’s argument at the end of SGA 5, Exp. XV to the stalk of $K \in D^b_c(X, R)$, we may represent $K$ by a bounded complex of free $R/m$-modules and the isomorphisms

$$
K_{k+1} \otimes_{R/m^{k+1}} R/m^k \sim K_k \otimes R/m \sim K_k.
$$

Taking
the projective limit of these complexes, we obtain a bounded complex $K$ of free $R$ modules. (To see freeness, note that if $r$ equals the rank of $K'_1 = K^i \otimes R/m$, there exists by Nakayama an exact sequence

$$N \to R' \to K^i \to 0$$

which after tensoring by $R/m^k$ induces an isomorphism $(R/m^k)^r \cong K^i_k$, showing $N \subset m^kR$ for all $k$ and hence $N = 0$.) By the Mittag-Leffler condition, $H^k(K) = \lim_{\leftarrow} H^k(K_k)$. Therefore, the submodule of $\ker d$ consisting of cycles whose image in $H^0(K_k)$ are in $\text{im}(H^nK \to H^0(K_k))$ is the reduction modulo $m^k$ of a flat $R$-module, which recognizes $\tau'_{\leq n}K_k$ as a bounded complex of free $R/m^k$ modules. It may be that $\tau'_{\leq n}K_n$ is no longer quasi-isomorphic to $K_k$, but it is clear by construction that the projective system of $H^n(K_k)$ is not affected by the operator $\tau'_{\leq n}$.

1.1.7. A representation $Z \to \text{GL}(V)$, sending $n$ to $F^n$, where $V$ is a $\mathbb{Q}_\ell$-vector space of dimension $n$, is continuous if and only if the eigenvalues of $F$ are $\ell$-adic units.

To see sufficiency, note that we can choose a basis for $V$ so that the morphism $Z \to \text{GL}(V)$ factors through $\text{GL}(Z_\ell^n) = \lim_{\leftarrow} \text{GL}((Z/\ell^m)^n)$, a profinite group, and we may extend $Z \to \text{G}$ to a morphism $\hat{Z} \to \text{G}$ for any profinite group $\text{G}$ by the universal property of profinite completion, which we state and prove now.

The profinite completion of a group $H$ (with respect to normal subgroups of finite index in $H$) is denoted $\hat{H}$, so that $H \to \hat{H}$ has dense image. The profinite completion $\hat{H}$ enjoys the universal property that for every profinite group $\text{G}$ and continuous homomorphism $H \to \text{G}$, there is a unique homomorphism $\hat{H} \to \text{G}$ making the diagram

$$\begin{array}{ccc}
H & \longrightarrow & \hat{H} \\
\downarrow & & \downarrow \\
\text{G} & \swarrow & \\
\end{array}$$

commute.

To see this, simply use the description of $\text{G}$ as $\lim_{\leftarrow} \text{G}/N$ as $N$ ranges over open normal subgroups of $\text{G}$. The preimage $M$ of $N$ in $H$ is an open normal subgroup of finite index,
as $G/N$ is finite. Therefore $H \to G/N$ factors through $H/M$, and to give a continuous morphism from $\hat{H}$ to $G$ it suffices to give compatible continuous maps $\hat{H} \to G/N$. Continuity is assured by the above remark; compatibility is assured by the map $H \to G$, which determines the maps $\hat{H} \to G/N$.

Returning to (1.1.7), to see necessity, we assume we have found a continuous extension $\rho$.

\[
\begin{array}{ccc}
\mathbb{Z} & \overset{\rho}{\longrightarrow} & \text{GL}(V) \\
\downarrow & & \downarrow \\
\hat{\mathbb{Z}} & \overset{\rho}{\longrightarrow} & \hat{\text{GL}(V)}
\end{array}
\]

The image $\rho(\hat{\mathbb{Z}})$ is compact, so the set $\rho(\hat{\mathbb{Z}})\mathbb{Z}^n_\ell$ is compact, for $\mathbb{Z}^n_\ell$ a $\mathbb{Z}_\ell$-lattice in $V$, so $\rho(\hat{\mathbb{Z}})\mathbb{Z}^n_\ell \subset \frac{1}{m}\mathbb{Z}^n_\ell$ for some $m$. Letting $L$ denote the $\mathbb{Z}_\ell$-span of $\rho(\hat{\mathbb{Z}})\mathbb{Z}^n_\ell$, $L$ is a $\mathbb{Z}_\ell$-submodule of $\frac{1}{m}\mathbb{Z}^n_\ell$, hence free (of rank $n$). This recognizes $F \in \rho(\hat{\mathbb{Z}})$ as an element of $\text{Aut}(L)$, so the eigenvalues of $F$ are $\ell$-adic units indeed.

1.2.6. ‘On notera que, pour $k$ un corps fini, une representation $V$ de $W(\overline{k}, k)$ est automatiquement $\ell$-mixte.’ $\rightsquigarrow$ This follows from the existence of Jordan normal form.

1.3.9. (When reading the corollary, recall that a semisimple algebraic group is connected by definition.) We wish to understand why $G^{00}$ is reductive. Note first that the sum of the simple $\pi_1(X, \overline{x})$-modules is $W$-stable since if $w \in W$ and $V$ is a $\pi_1(X, \overline{x})$-module, then $wV$ is again a $\pi_1(X, \overline{x})$-module since $\pi_1(X, \overline{x})$ is a normal subgroup of $W$; applying this argument with $w^{-1}$ shows that $wV$ is simple iff $V$ is. Next observe that if $\rho : W(X_0, \overline{x}) \to \text{GL}(\mathcal{F}_\overline{x})$ is the representation defining $\mathcal{F}_0$, then $\rho(\pi_1(X, \overline{x}))$ and its Zariski closure $G^0$ have the same invariant subspaces (to see this, form a basis for $\mathcal{F}_\overline{x}$ beginning with a basis for an invariant subspace). Therefore we see that $G^0$ acts semisimply since $W(X_0, \overline{x})$ does by assumption.

Now recall that $R(G^0)$, the radical of $G^0$, is a connected and solvable normal subgroup of $G^0$. By the argument above, any normal subgroup of $G^0$ acts semisimply; combining this with the Lie-Kolchin theorem, we see that $\mathcal{F}_\overline{x}$ decomposes as a direct sum of
one-dimensional irreducible $R(G^0)$-modules. The unipotent part of $R(G^0)$, which is the unipotent radical $R_u(G^0)$, must therefore act by the identity, and we see that $R_u(G^0) = R_u(G^{(0)}) = \{1\}$; i.e. that $G^{(0)}$ is reductive. Note we have proved the following

**Lemma.** If $V$ is a finite-dimensional vector space over an algebraically closed field and $G$ is a closed subgroup of $GL(V)$, then $G$ is reductive.

This result appears in [Milne, 21.60] as

**Proposition.** Let $G$ be a connected group variety over a perfect field $k$. The following conditions on $G$ are equivalent.

(1) $G$ is reductive;
(2) The radical $R(G)$ of $G$ is a torus;
(3) $G$ is an almost-direct product of a torus and a semisimple group;
(4) $G$ admits a semisimple representation with finite kernel.

More is true. In fact, for $G$ a connected reductive group, say, over an algebraically closed field $k$, the maximal central $k$-torus $Z$ coincides with $(\mathcal{Z}(G) \gg e)^0$, the connected component of the identity of the center of $G$, and the multiplication homomorphism $Z \times \mathcal{D}(G) \to G$ is a central isogeny, i.e. an isogeny with central kernel, where $\mathcal{D}(G) = (G,G)$ is the derived subgroup. This implies that $Z \to G/\mathcal{D}(G)$ is a central isogeny. Here, our $G$ is Deligne’s $G^{(0)}$, our $Z$ is Deligne’s $T_1$, and our $G/\mathcal{D}(G)$ is Deligne’s $T$, as a connected, smooth, reductive, and commutative group is a torus [Milne, 19.12], and a quotient of a reductive group over a field of characteristic 0 is reductive.

The set $F$ of characters by which $T_1$ acts on $\mathcal{X}$ generates $X(T_1)$ since the representation of $T_1$ is faithful, and, as $T_1$ is a torus, diagonalizable. Therefore, with the right choice of basis, the representation of $T_1$ looks like $\text{diag}(\gamma_i)$ for characters $\gamma_i \in F$. As the representation is faithful, these characters generate the character group $X(T_1)$. (The character group of $\text{diag}(\gamma_i)$, which is isomorphic to the character group of $T_1$, is generated by the $\gamma_i$.)
The group $\mathcal{W}(X_0, \bar{x})$ acts on $G^0$ by conjugation. Recall that the neutral component of an algebraic group is a characteristic subgroup, and so is the center. Therefore $T_1$, which can be described as the neutral component of $Z(G^{00})$, is acted upon by $\mathcal{W}(X_0, \bar{x})$. Recall that the functor $X$ which takes an algebraic group to its character group induces a contravariant equivalence from the category of diagonalizable algebraic groups with the finitely generated commutative groups, and as we have seen, $\mathcal{W}(X_0, \bar{x})$ acts on $X(T_1)$ by permuting factors, hence through a finite quotient.

We would like to know why the group of outer automorphisms of $G^{00}$ restricting to the identity on $T_1$ is finite. The group $G^{00}$ admits a maximal split torus $T_2$ so that $(G^{00}, T_2)$ is a split reductive group. The radical $R(G^{00}) = T_1$ is the largest subgroup of the multiplicative group $Z(G^{00})$, so the quotient $Z(G^{00})/R(G^{00})$ is finite [Milne, 19.10]. Recall the definition of isomorphism of root data [Milne, 23.2]. An isomorphism $\varphi$ of split reductive groups defines an isomorphism $f$ of root data, and every isomorphism of root data $f$ arises from a $\varphi$, unique up to an inner automorphism [Milne, 23.26]. Moreover for a split reductive group $(G, T)$ we have a canonical isomorphism $\text{Out}(G) \simeq \text{Aut}(X, \Phi, \Delta)$, where the latter is automorphisms of based root data [Milne, 23.46]. Given such a $\varphi : (G, T) \to (G', T')$, the map $f$ is defined by the formula $f(\chi') = \chi' \circ \varphi|_T$ for $\chi' \in X(T')$ [Milne, 23.5]. Suppose $\varphi$ is now an automorphism of $(G^{00}, T_2)$ and restricts to the identity on the radical $T_1$. The isomorphism $f$ is a fortiori a central isogeny and its action on $Z\Phi$ (the $Z$-submodule of $X^*(T_2)$ generated by the roots $\Phi$) preserves the base $\Delta$, hence its action on $Z\Phi$ amounts to permuting a finite set. On the other hand, the quotient $T_2/Z(G^{00})$ has character group the subgroup $Z\Phi$ of $X^*(T_2)$ [Milne, 21.9], hence the the root lattice $Z\Phi$ has finite index in $X^*(T_2/T_1)$. As $Z(G^{00})/T_1$ is finite, this is enough to conclude that subgroup of $\text{Aut}(X, \Phi, \Delta)$ corresponding to automorphisms of $G^{00}$ which restrict to the identity on $T_1$ is finite, hence that the subgroup of $\text{Out}(G^{00})$ consisting of those automorphisms fixing $T_1$ is also finite.

Now, if $w$ is an element of $\mathcal{W}(X_0, \bar{x})$ of degree 1, and $\bar{w}$ the image of $w$ in $\text{GL}(\bar{\mathbb{F}}_p)$, $G$ is the semi-direct product of $Z$ by $G^0 = G^{00}$ relative to the action $\text{int}(\bar{w})$ of $Z$ on $G^0$. As this action is given by an interior automorphism of $G^0$, by multiplying $w$ by an element of $\pi(X, \bar{x})$, we make the action of $\text{int}(\bar{w})$ trivial, and recognize $G \simeq G^{00} \times Z$. 

The proof of (1.3.9) follows easily from (1.3.8). Note that once one has reduced to $\mathscr{F}_0$ semisimple, it is easy to see that the radical of $G^{00}$ in this case is trivial, as it is by definition the largest connected solvable normal subgroup variety of $G^{00}$, hence contained in the connected component of the identity of $G^0$, so if $G^0$ is an extension of a finite, hence discrete, group, $R(G^{00})$ lies in the kernel of this extension, namely in the semisimple subgroup, so in fact $R(G^{00}) = \{e\}$, as connected normal subgroup varieties of a semisimple group are semisimple [Milne, 21.52].

1.3.10. Note that (iv) should read ‘Le centre de G s’envoie sur un sous-groupe d’indice fini de Z.’ The crux of the direction $\text{iv} \Rightarrow \text{i}$ is that, while G is not a priori a linear algebraic group, $G/Z$, as an extension of a finite group by a linear algebraic group, is.

1.3.12. The central element $g$ acts by a scalar by Schur’s lemma.

1.3.13. (i) The claim rests on the following

**Lemma.** Let $X_0, X'_0$ be normal connected schemes of finite type over a field with generic points $\xi, \xi'$ and function fields $K = k(\xi)$ and $K' = k(\xi')$. Let $\Omega, \Omega'$ be algebraically closed extensions of $K, K'$, defining geometric points $a, a'$ of $X_0, X'_0$ centered on $\xi, \xi'$, respectively. If $f : X'_0 \to X_0$ is a dominant morphism, then the image of the induced map $\pi_1(X'_0, a') \to \pi_1(X_0, a)$ is an open subgroup of finite index.

Observing that $\pi_1(X'_0, a')$ acts on $(f^*\mathcal{F})_{a'}$ via the map on $\pi_1$ in the lemma induced by $f$, we see that there is a central element $g \in G'$ of positive degree and a morphism $G' \to G$ sending $g$ to a central element of $G$ of positive degree, and the action of $g$ on $\mathcal{F}_{0,a}$ via this map is the same as the action of $g$ on $(f^*\mathcal{F}_0)_{a'}$.

**Proof of lemma.** The extensions $\Omega, \Omega'$ define geometric points $a_1, a'_1$ of $S = \text{Spec}(K)$ and $S' = \text{Spec}(K')$, respectively. The dominant morphism $f : X'_0 \to X_0$ induces an extension of fields $K \subset K'$. Then $\pi_1(S, a_1) \to \pi_1(X_0, a)$ is surjective [SGA1, V 8.2], and after identifying $\pi_1(S, a_1)$ with $\text{Gal}(K^\text{sep}, K)$, the kernel is identified with those automorphisms which fix all finite extensions of $K$ in $\Omega$ which are unramified over $X_0$. 

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and likewise for $\pi_1(S', a'_1) \to \pi_1(X'_0, a')$. If $L$ is an extension of $K$ unramified over $X_0$, then $L \otimes_K K'$ is an extension of $K'$ unramified over $X'_0$ [SGA1, I 10.4(iii)]. The operation on étale covers of $X_0$ consisting of taking inverse image along $f$ followed by fiber at $a'$ is a fiber functor for $X_0$, hence induces a continuous homomorphism of groups $\pi_1(X'_0, a') \to \pi_1(X_0, a)$ [SGA1, V 6.2]. The action of $\pi_1(X'_0, a')$ on $(f^*\mathcal{F}_0)_{a'}$ is by restriction with respect to this homomorphism. This homomorphism, in turn, is induced by restriction of $\pi_1(S', a'_1) \to \pi_1(S, a)$, since if $L$ is as above, an automorphism $\sigma \in \ker(\pi_1(S', a'_1) \to \pi_1(X'_0, a')$ acts on $L \otimes_K K'$ by the identity as $L \otimes_K K'$ is unramified. As $K'/K$ is finitely generated, $K' \cap K^{\text{sep}}$ is a finite extension of $K$, so the image of $\pi_1(S', a'_1) \to \pi_1(X_0, a)$ is an open subgroup of finite index isomorphic to the image of $\text{Gal}(K^{\text{sep}}/K' \cap K^{\text{sep}})$ in $\pi_1(X_0, a)$. □

(ii) Choose a basis for representations corresponding to $\mathcal{F}_0$ and $\mathcal{G}_0$ so that Frobenius is upper-triangular in both, and then recall the form of the Kronecker (tensor) product of matrices, which has the property that the Kronecker product of upper-triangular matrices is upper-triangular.

(iii) The claim rests on two observations. The first is that if $\mathcal{F}_0$ is defined by a representation $V$ of $G$, the eigenvalues of any $g \in G$ coincide with the eigenvalues of $g$ acting on the semi-simplification of $V$ with respect to any Jordan-Hölder series. To see this, choose a basis for each graded piece so that $g$ is upper-triangular, and then order a lift of these bases according to the filtration, beginning with the smallest piece. The second observation is that if we begin with an ordered basis $(a_i)$ for $V$ with respect to which $g$ is upper triangular, then a basis $B$ for $\wedge^a V$ consisting of $a$-forms in the $a_i$ can be found. If the function $w$ takes an $a$-form in the $a_i$ and outputs the sum of the subscripts which appear (so $w(a_1 \wedge a_3 \wedge a_4) = 8$), then $g$ is upper-triangular with respect to any ordering of $B$ which respects the total order $w$. The claim follows.

1.3.14. It suffices to show that the image of $W(X_0, x)$ in $\text{GL}(r, E)$ is bounded by the argument of (1.1.7), which we repeat now. We lose nothing by supposing $E = \mathbb{Q}_\ell$, in which case the image $W$ of $W(X_0, x)$ in $\text{GL}(r, E)$ is bounded if it is contained in $\frac{1}{\ell} \text{GL}(r, \mathbb{Z}_\ell)$. Applying $W$ to $\mathbb{Z}_\ell$ and taking the $\mathbb{Z}_\ell$ span, we get a free $\mathbb{Z}_\ell$-submodule of
\(1/2 \mathbb{Z}_\ell^r\) of rank \(r\), on which \(W\) acts by automorphisms. This recognizes \(W\) as isomorphic to a subgroup of \(\text{GL}(r, \mathbb{Z}_\ell)\), a profinite group to which it is easy to extend a map \(W(X_0, x) \to \text{GL}(r, \mathbb{Z}_\ell)\) to a map from the completion \(\pi_1(X_0, x) \to \text{GL}(r, \mathbb{Z}_\ell)\).

To see that \(\rho(W_1^0) \subset G^0\) is compact and Zariski dense, observe that \(W_1^0\) is a closed subgroup of \(\pi_1(X, \overline{x})\), hence a profinite group, and \(G^0\) is by definition the Zariski closure of the image of \(\pi_1(X, \overline{x})\). In particular, the inverse image of \(G^0\) is Zariski dense in \(G^0\).

1.3.15. Relevant sources are Bourbaki, *Lie Groups and Lie Algebras* II, §7, Demazure and Gabriel, *Groupes Algébriques*, II, §6, [Milne, 10, 14d]. Bourbaki explains how to extend the logarithm to the union of all compacta. You need to know that for all compact \(G \subset H(E)\), \(x \in G\), and neighborhood about \(e\), there is a strictly increasing sequence of integers \((n_i)\) such that \(x^{n_i} \in V\), which allows one to extend the logarithm by Deligne’s formula. Bourbaki also explains that there is an open subgroup \(V\) of \(e\) in \(H(E)\) such that log is an analytic isomorphism of \(V\) onto an open subgroup of \(\text{Lie} H\), with inverse \(\exp\). It follows that \(L^1\), the \(E\)-linear span of \(\log K\), coincides with the \(E\)-linear span of \(\log(K \cap V)\). We have for \(X \in \log H\) that \(\exp(nX) = \exp(X)^n\), and \(\log(g^n) = n \log(g)\) for any \(g\) where \(\log\) is defined.

For \(g \in H(E)\), \(X \in \text{Lie} H\),

\[
(\dagger) \quad g \cdot \exp(X) \cdot g^{-1} = \exp(\text{Ad}(g)(X)),
\]

whenever these expressions converge. Taking \(X \in \log(K \cap V)\) and \(g \in V \cap K\), we find that the left side is in \(K \cap V\). Moreover there is an \(n \in \mathbb{Z}\) such that both \(n \text{Ad}(g)(X)\) and \(nX\) lie in \(\log V\). We find that \(\text{Ad}(g)(nX) \in K \cap V\), therefore that for \(g \in V \cap K\), \(\text{Ad}(g)\) preserves \(L^1\), therefore \(L^1\) is also preserved by \(V\), which is the Zariski closure of \(V \cap K\) in \(K\).

Therefore the adjoint representation

\[\text{Ad} : H \to \text{GL}(\text{Lie} H)\]
factors through the algebraic subgroup \( F \subset \text{GL}(\text{Lie} \, H) \) which fixes the \( L^1 \). Applying
the functor \text{Lie}, one finds that
\[
\text{ad} : \text{Lie} \, H \rightarrow \mathfrak{gl}(\text{Lie} \, H)
\]
factors through \text{Lie} \, F. As \text{ad} induces the bracket on \text{Lie} \, H, it follows that \( L^1 \) is an ideal in 
\text{Lie} \, H. As \( H \) is semisimple over a field of characteristic 0, \text{Lie} subalgebras of \text{Lie} \, H are 
in bijection with connected algebraic subgroups of \( H \) \cite{DemazureGabriel, II §6 2.4 & 2.7}. As the exponential is functorial (c.f. \textit{ibid}, 3.4),
the algebraic subgroup corresponding to \( \mathfrak{t} \) contains \( K \), hence must equal \( H \) by density,
hence \( L^1 = \text{Lie} \, H \).

Let \( N \) denote the normalizer of \( K \) and \( g \in N \). Let \( X \in K \). Then there is an integer \( n \)
such that both \( X^n \in V \) (hence \( n \log X \in \log V \)) and \( n \text{Ad}(g)(\log X) \in \log V \). Applying
\((\dagger)\) to \( n \log X \) we find that \( \exp(\text{Ad}(g)(n \log X)) = g \cdot X^n \cdot g^{-1} \in V \cap K \) so that
\( \log(g \cdot X \cdot g^{-1}) = \text{Ad}(g)(\log X) \) and we find \( \text{Ad}(g) \) preserves the set \( \log K \) and \textit{a fortiori}
\( L^0 \).

The morphism \( \text{Ad} \) factors as a quotient \( H \rightarrow H/Z(H) \) followed by a closed immersion.
As \( H \) is semisimple, \( Z(H) \) is finite, hence the quotient \( H \rightarrow H/Z(H) \) is finite \textit{(a fortiori)
proper} \cite[21.7, 7.15, 5.39]{Milne}.

The subgroup \( K \subset H(E) \) is a compact subset of a complete metric space, hence
closed for the topology induced by the non-archimedean metric on \( E \). Hence \( K \) is a
closed Lie subgroup of \( H(E) \) for that metric, and hence \( N \) is a closed subgroup of \( H(E) \)
with respect to the metric induced by the one on \( E \). As \( L^0 \) is compact and isomorphic
to an \( \theta_E \)-lattice in \( \text{Lie} \, H \), its automorphism group \( \text{Aut} \, L^0 \) is compact; as \( \text{Ad} \) is proper,
\( \text{Ad}^{-1}(\text{Aut} \, L^1) \subset H(E) \) is compact, and \( N \subset \text{Ad}^{-1}(\text{Aut} \, L^0) \) is a closed subgroup, hence
also compact.

\textbf{1.4.1.} \( (b) \) See Weil I, (2.9).

\textbf{1.4.2.} Let \( \overline{x} \) be a geometric point of \( X \); as \( X_0 \) is absolutely irreducible, \( X \) is connected.
The pullback of lisse sheaves along the morphism \( X \rightarrow X_0 \) identifies with the restriction
of representations along the continuous homomorphism $\pi_1(X, \overline{x}) \to W(X_0, \overline{x})$, and
likewise the pullback of lisse sheaves along the structural morphism $X_0 \to \text{Spec}(\mathbb{F}_q)$
with restriction along $W(X_0, \overline{x}) \to \mathbb{Z}$. Given a lisse sheaf $\mathcal{F}$ on $X_0$ with monodromy
representation $V$, the largest subsheaf (resp. quotient sheaf) becoming constant on $X$ is
obtained by taking invariants (resp. coinvariants) of $V$ with respect to $\pi_1(X, \overline{x})$. Both
$V^{\pi_1(X, \overline{x})}$ and $V_{\pi_1(X, \overline{x})}$ carry natural actions of $\mathbb{Z}$ which induces lisse sheaves $F'_0, F''_0$ on
$\text{Spec}(\mathbb{F}_q)$ with inverse images $V^{\pi_1(X, \overline{x})}$ and $V_{\pi_1(X, \overline{x})}$, respectively. (The exact sequence
$0 \to \pi_1(X, \overline{x}) \to W(X_0, \overline{x}) \to \mathbb{Z}$ identifies those lisse sheaves invariant under geometric
monodromy with the inverse image of sheaves on $\text{Spec}(\mathbb{F}_q)$.)

1.4.3. The point is that on the one hand, the constituents of the sheaves $F', F''$ are
among the constituents of $\mathcal{F}_0$, on the other hand as representations of $W(X_0, \overline{x})$, $F', F''$
are invariant for geometric monodromy, so they have one-dimensional constituents
which are determined once Frobenius is put in Jordan normal form. Therefore the
eigenvalues of Frobenius on $F'$ and $F''$ appear among the determinantal weights for $\mathcal{F}_0$,
and, in consideration of (1.4.2), up to a twist the same is true of eigenvalues of Frobenius
on $H^0(X, \mathcal{F}), H^0_c(X, \mathcal{F})$, and $H^2_c(X, \mathcal{F})$.

1.4.6. See Ahlfors, Complex Analysis, Ch. 5 §2.2 for a characteristically elegant review
of the convergence properties of infinite products, which elucidates the equivalence of
the absolute convergence of Deligne’s Euler product with that of of his geometric series.

1.5.1. Perhaps the only thing to remark is that if a lisse sheaf $\mathcal{F}$ on $X$ is $t$-real, then
tall of its exterior powers are, too: choosing a basis for $\mathcal{F}_x$ with respect to which $F_x$
is upper-triangular, the resulting canonical basis for $\wedge \mathcal{F}$ can be ordered so that $F_x$
remains upper triangular (1.3.13 iii), which makes it easy to see that $t \det(1 - F_x t, \wedge \mathcal{F})$
has coefficients which are symmetric polynomials in the eigenvalues of $F_x$. As the
coefficients of $t \det(1 - F_x t, \mathcal{F})$ are the elementary symmetric polynomials in these
eigenvalues, and are real, the coefficients of $t \det(1 - F_x t, \wedge \mathcal{F})$ are real too.
1.6.11. To see the Clebsch-Gordon decomposition (1.6.11.2), let
\[ H = du \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \]
in the notation of (1.6.8), and let
\[ (\dagger) \quad \chi_d(\lambda) = \text{Tr}(e^{\lambda H}) = \sum_{j=-d}^{d} \lambda^j \]
as a function of $\lambda \in k$. These characters transform additively under direct sum and multiplicatively under tensor product, so finding the decomposition of $S_d \otimes S_{d'}$ is the same as finding the additive decomposition of
\[ (\lambda^{-d} + \lambda^{-d+2} + \ldots + \lambda^d)(\lambda^{-d'} + \lambda^{-d'+2} + \ldots + \lambda^{d'}) \]
into sums of the form $(\dagger)$. In this case the decomposition is into $d' + 1$ sums of the form $\chi_j(\lambda)$ for $j \in P(d, d')$.

1.6.13. It is asserted that the inclusion $\subset$ of assertion 2) results from the fact that the image of $N^i M_i$ in $\text{Gr}^W_0(V)$ is $N^i M_i(\text{Gr}^W_0 V) = M_i(\text{Gr}^W_0 V)$. To see this, recall that in the construction (1.6.1), $N^{d-1}$ sends $\ker N^d / \text{im} N^d$ onto $M_{-d+1} / \text{im} N^d$, hence sends $M_{d-1}$ onto $M_{-d+1}$, and proceed inductively.

The inequality $k - 2i - 2 \geq k \geq 2j - k$ should read $k - 2i - 2 \geq -k \geq 2j - k$ at the end of the discussion of 3). Note $N^{k-j} : \text{Gr}^M_k G \to \text{Gr}^M_{2j-k} G$.

1.6.14. It seems as though (1.6.14.3) should read
\[ \text{Gr}^M_i V \cong \bigoplus_{j \equiv i \pmod{2}} P_{-j} \left( \frac{-i - j}{2} \right) . \]
For, when $i \leq 0$, $N$ induces an isomorphism of $\text{Gr}^M_i V/P_i$ onto $\text{Gr}^M_{i-2} V$. Scaling $N \mapsto \lambda N$ also multiplies this isomorphism by $\lambda$, so we need to twist by $\otimes N^{-1}$. Similarly, the isomorphism $N^i : \text{Gr}^M_i V \cong \text{Gr}^M_{-i} V$ scales by $\lambda^i$ so we need to twist by $-i$. 
With this modification, the isomorphism
\[ P_{-j} \simeq \bigoplus_{j \in P(j',j'')} P'_j \otimes P''_j \left( \frac{j - j' - j''}{2} \right) \]
is justifiable by passing through a suitable graded piece
\[ P_{-j} \left( \frac{-i - j}{2} \right) \rightarrow \text{Gr}^M_i V \leftarrow \bigoplus_{j \in P(j',j'')} P'_j \otimes P''_j \left( \frac{-i - j' - j''}{2} \right). \]

As for the isomorphism
\[ P_{-j}(V^*) \simeq P_{-j}(V)^*(j), \]
consider \( V = S_j \) identified with the representation of \( \text{SL}(2) \) on homogeneous polynomials in variables \( x \) and \( y \) of degree \( j \). Suppose \( P_{-j}(V) \) is generated by the vector \( y^j \); then \( P_{-j}(V)^* \) is generated by the covector on \( y^j \) and \( P_{-j}(V^*) \) is the covector on \( x^j \). The map \( P_{-j}(V)^* \rightarrow P_{-j}(V^*) \) is obtained by precomposing by \( N^j \). Upon scaling \( N \mapsto \lambda N \), this map is scaled by \( \lambda^j \). Therefore the canonical isomorphism is obtained by twisting by \( N^{-j} \):
\[ P_{-j}(V)^* \rightarrow P_{-j}(V^*) (-j). \]

1.7.3. ‘Il commute à l’action de \( W(\overline{K}, K) \)’—this statement is somewhat opaque. Which action(s)? When you let \( W(\overline{K}, K) \) act on \( V \) via \( \rho \) and on \( \overline{Q}_l(n) \) via \( \text{Gal}(\overline{k}, k) \), the statement is that if \( \tau \in W(\overline{K}, K) \),
\[ \tau N \tau^{-1} = N : V(1) \rightarrow V. \]
Let \( \sigma \in \overline{Q}_l(1) \). As \( \tau^{-1} \lambda = (\deg \tau)^{-1} \lambda \tau^{-1} \), the statement is equivalent to the statement
\[ \tau N \lambda \tau^{-1} = (\deg \tau) N \lambda. \]
\( \text{Gal}(\overline{K}, K) \) acts on the inertia character \( t \) by the formula
\[ t(\sigma \tau^{-1}) = t \tau \sigma, \]
where on the right \( \tau \) acts through \( \text{Gal}(\overline{k}, k) \), so if we assume \( I_1 \) is a normal subgroup of \( I \) and take \( \sigma \in I_1 \),
\[ \exp(\rho(\tau) N t_t(\sigma) \rho(\tau^{-1})) = \rho(\tau \sigma \tau^{-1}) = \exp(N t_t(\tau \sigma \tau^{-1})) = \exp(N (\deg \tau) t_t(\sigma)). \]
1.7.4. A normal unipotent subgroup of an algebraic group acts trivially on all semisimple representations, hence the factorization of the action of I by a finite quotient [Milne, 19.16).

The semisimple representation of \( \text{W}(\bar{K}, K) \) therefore factors through the quotient by \( I_1 \). The conjugation action of \( \text{W}(\bar{K}, K) \) on \( I/I_1 \) is has a kernel \( U \) of finite index, as a finite group has only finitely many automorphisms. So for some \( m_1, F^{m_1}, F^{m_1} \in U \). \( F^{m_1} = aF^{m_1} \) for some \( a \in I \), but now \( F^{m_1} \) commutes with \( a \), so if \( a^{m_2} = 1 \), then \( F^{m_1m_2} = a^{m_2}F^{m_1m_2} = F^{m_1m_2} \), so we may take \( n = m_1m_2 \).

1.7.5. In the proof it is asserted ‘On a \( \text{NM}_i' \subset M'_{i-2} \).’ This is because of the commutativity of (1.7.3). Namely, \( F'N = NF' \) and if \( \lambda \in \overline{Q}_l(1) \), \( F'N\lambda = q^{-1}N\lambda F' \). Therefore \( N\lambda \) sends \( M'_i \) into \( M'_{i-1} \) and \( N : M'_i(1) \subset M'_{i-2} \).

While proving that \( M' \) is independent of \( F' \), Deligne introduces \( \exp(\lambda N) \) for \( \lambda \in \overline{Q}_l(-1) \). Let me instead deduce from the fact \( F'' = F'' \mod I_1 \) that for an appropriate choice of \( \lambda \in \overline{Q}_l(1) \),

\[
F'' = \exp(N\lambda)F''.
\]

Leaving the definition of \( \mu \) unchanged, we have

\[
\exp(N\lambda)F'' = \exp(N\mu)F'' \exp(N\mu)^{-1},
\]

since if we expand \( F'' \exp(-N\mu) \) as a series

\[
F'' \left( 1 - N\mu + \frac{(-N\mu)^2}{2!} + \cdots \right),
\]

we have that \( N \) commutes with \( F'' \) and that \( F''\mu^i = q^n\mu^i \), so that

\[
\exp(N\mu)F'' \exp(N\mu)^{-1} = \exp(N\mu) \exp(-q^nN\mu)F'' = \exp(N\lambda)F''.
\]

This identity shows that \( \exp(N\mu) \) sends \( M' \) into \( M'' \). As \( (N\mu)M'_i \subset M''_{i-2} \),

\[
\exp(N\mu) = 1 + N\mu + \frac{(N\mu)^2}{2!} + \cdots
\]

also sends \( M'_i \) into \( M''_{i-2} \), so \( M''_i \subset M'_i \).
1.7.7. Let $F'$ be as in (1.7.5), let $\alpha \in \overline{Q}_l^+$, and let $V'^\alpha$ equal the sum of the generalized eigenspaces of $F'$ acting on $V$ with eigenvalue in the class of $\alpha \mod$ roots of 1. Then $V'^\alpha$ is independent of lift and $N_\lambda : V'^\alpha \to V'^\alpha/q$, and

$$V'^\alpha = \sum_{i \in \mathbb{Z}} V'^{\alpha + i}.$$ 

is stable under $\exp(N_\lambda)$ for $\lambda \in \overline{Q}_l(1)$ and hence under the action of $I_1$. By the argument of (1.7.5), $V'^\alpha$ is therefore stable under $W(\overline{K}, K)$.

1.7.8. First, the matter of what it means for the locally constant sheaf of sets $\mathcal{F}$ on $X - D$ to be tamely ramified along the divisor $D$. In Grothendieck-Murre [GM], the notion of a tamely ramified covering is discussed. If $\mathcal{F}$ is a locally constant sheaf of finite sets on $X - D$, it is represented by an étale covering of $X - D$. What about if $\mathcal{F}$ is a locally constant sheaf of sets on a scheme $S$ with infinite fibers?

Let $\{S_i\}_{i \in I}$ be a covering trivializing $\mathcal{F}$. If $S$ is quasi-compact we can take this covering to be finite quasi-compact. Over each $S_i$, $\mathcal{F}$ is represented by an étale morphism $X_i \to S_i$, namely the disjoint union $\bigsqcup_{j \in \Lambda} S_i$ with the set $\Lambda$ in bijection with any fiber of $\mathcal{F}$. This yields a separated, locally of finite presentation, and locally quasi-finite descent datum $X_i$ relative to the covering $S_i$ (c.f., e.g. Stacks, tag 02W4). By [SGAD, Exp. X, 5.4], this descent datum is effective and yields an $S$-scheme $X$. In the case that $S$ is quasi-compact, then as the faithfully flat covering $\bigsqcup S_i$ was taken to be quasi-compact, $X$ is separated [SGA1, Exp. VIII, 4.8 or Exp. IX, 2.4], but in any case it is evidently étale as its restriction to each $S_i$ is. This elucidates the proof of Lemma 2.2 in [SGAA, Exp. IX], with one refinement:

**Lemma.** Let $S$ be a scheme and $\mathcal{F}$ a locally constant sheaf of sets on $S$. Then $\mathcal{F}$ is represented by a $X/S$ étale. If $S$ is quasi-compact, then $X$ is separated. If the fibers of $\mathcal{F}$ are finite (resp. and non-empty), then $X$ is an étale covering (resp. and surjective).

Now suppose $S$ is normal, integral, and quasi-compact with generic point $\eta$. By EGA IV$_4$ 18.10.7 and 18.10.8, the étale $X \to S$ is isomorphic to a disjoint union of integral normal schemes $X_\alpha$ such that $f^{-1}_\alpha(\eta)$ is a finite separable extension of $k(\eta)$ and
moreover if $S'_\alpha \xrightarrow{\cal{g}_\alpha} S$ denotes the normalization of $S$ in $f_\alpha^{-1}(\eta)$, then $S'_\alpha$ is a revêtement étale, and $f_\alpha$ factorizes as $X_\alpha \xrightarrow{f_\alpha} S'_\alpha \xrightarrow{\cal{g}_\alpha} S$ with $f'_\alpha$ an open immersion.

Hence $\mathcal{F}$ decomposes as a disjoint union of sheaves $\mathcal{F}_\alpha = h_{X_\alpha}$, with $h_{X_\alpha}$ locally constant constructible. The property of the $h_{X_\alpha}$ being locally constant implies that the open immersions $f'_\alpha$ are in fact isomorphisms, and we sum up in the following

**Lemma.** Let $S$ be a normal, integral, and quasi-compact scheme and let $\mathcal{F}$ be a locally constant sheaf of sets on $S$. Then $\mathcal{F} \cong \bigsqcup_{\alpha} h_{X_\alpha}$, i.e. $\mathcal{F}$ decomposes as a disjoint union of locally constant constructible sheaves represented by revêtements étales $X_\alpha \to S$.

Now we recall the definition of divisors with normal crossings [GM, 1.8]. Let $S$ be a locally noetherian scheme and $(D_i)_{i \in I} = D$ a finite set of divisors on $S$. For simplicity we often denote the inverse image of the $D_i$ in Spec $\mathcal{O}_{S,s} \to S$ by the same letter $D_i$.

**Definition.**

a) We say that the $(D_i)_{i \in I}$ have **strictly normal crossings** if for every $s \in \bigcup_{i \in I} \text{supp } D_i$ we have:

i) $\mathcal{O}_{S,s}$ is a regular local ring,

ii) if $I_s = \{i : s \in \text{supp}(D_i)\}$, then for $i \in I_s$ we have

\[ D_i = \sum_{k} \text{div}(x_{i,k}) \]

with $x_{i,k} \in \mathcal{O}_{S,s}$ and $(x_{i,k})_{i,k}$ part of a regular system of parameters at $s$.

b) We say that the set $(D_i)_{i \in I}$ has **normal crossings** if for every $s \in \bigcup_{i \in I} \text{supp } D_i$ there exists an étale neighborhood $S' \to S$ of $s$ in $S$ such that the family of inverse images of the $(D_i)_{i \in I}$ on $S'$ has strictly normal crossings.

**Remark.** The concept of (strictly) normal crossings is stable by étale base change, and one can check whether a set of divisors has normal crossings étale-locally.

In the setting of $X$ a regular scheme and $D$ a divisor on $X$ with normal crossings, the definition of a tamely ramified covering $f : V \to X$ relative to $D$ given in Grothendieck-Murre is equivalent to the property that $V$ be étale over $X - D$, and that the inertia at each $d \in D$ of codimension one act trivially on $V$ (see also [SGA1, Exp. XIII, 2.1]).
More precisely, let $D$ be a union of lisse divisors $D_i$, $d$, $X_{(d)}$, $\bar{n}$ be as in (1.7.8). Then the corresponding inertia group should act trivially on $V_{\bar{n}}$. Evidently this definition extends to a locally constant sheaf of sets $\mathcal{F}$. The ramified Kummer covering $\pi : X_n \rightarrow X$ is a homeomorphism, and if $\mathcal{F}$ is a locally constant sheaf of sets on $X - D$, $L$-ramified along $D$, for $L$ a set of primes invertible on $X$, then there is an $n$ invertible on $X$ such that the action of inertia with respect to every point of $X_n$ on $\pi^* \mathcal{F}$ is trivial.

More explicitly, let $U = X - D$, $U_n = \pi^{-1}(U)$. The Kummer covering $\pi$ is a revêtement étale when restricted to $U_n$, which corresponds to the homomorphism $\varphi : \pi_1(U_n, \bar{n}) \rightarrow \pi_1(U, \bar{n})$ of topological groups which is the inclusion of the former group as an open subgroup of the latter corresponding to the connected, pointed étale cover $U_n \rightarrow U$. Let $\sigma \in \text{Gal}(\bar{n}/\eta)$ be any element of the inertia corresponding to the monodromy around any of the $D_i$ which acts nontrivially on $\mathcal{F}_{\bar{n}}$. Then the image of $\sigma$ in $\pi_1(U, \bar{n})$ is nonzero in $\pi_1(U, \bar{n})/\varphi \pi_1(U_n, \bar{n})$; in particular it does not lie in $\text{Gal}(\bar{n}/\eta_n)$ where $\eta_n$ is the generic point of $U_n$. Therefore the representation of $\pi_1(U_n, \bar{n})$ obtained by restricting $\pi^* \mathcal{F}$ to $U_n$ coincides with the restriction of a continuous representation of $\pi_1(X_n, \bar{n})$ by the continuous map $\pi_1(U_n, \bar{n}) \rightarrow \pi_1(X_n, \bar{n})$; i.e. $\pi^* \mathcal{F}$ extends to a locally constant sheaf $\mathcal{F}$ on $X_n$.

1.7.11. As this paragraph reinterprets the construction of (1.7.8), it is implicit that $X$ is a henselian trait. A remark on the equality $\text{Gal}(K_1/K) = \text{Gal}(k_1/k)$: this is effectively saying that there is an equivalence of Galois categories of finite étale covers of the points $\text{Spec }K$ and $\text{Spec }k$. The functor of restriction to the special fiber does induce an equivalence of the categories of finite étale covers of $X$ with that of $\text{Spec }k$; this is [EGA, IV$_4$ 18.5.11]. For the equality of Galois groups in the setting of $R \subset R^h$, $K \subset K_1$, $R$ local henselian normal, $R^h$ its strict henselization, we use the following facts.

For any finite separable subextension $k \subset k_2 \subset k_1$, there exists a unique (up to unique isomorphism) finite étale local ring extension $R \subset R_2$ with specified residue field extension. Since the functor of restriction to the special fiber induces an equivalence of categories, the inductive system of finite separable subextensions $k \subset k_2 \subset k_1$ specifies
an inductive system of local ring homomorphisms $R \subset R_2 \subset R^{sh}$, of which $R^{sh}$ is the colimit (Stacks, tag 0BSL).

The following is a variation on the theme of the aforementioned equivalence. It is Lemma 7, §2.3 of *Néron Models* by Bosch, Lütkebohmert, and Raynaud

**Lemma.** Let $R$ be a local ring, $S'$ an étale $R$-scheme, and $s'$ a point of $S'$ above the closed point $s$ of $S = \text{Spec } R$. Let $R'$ be the local ring $\mathcal{O}_{S', s'}$ of $S'$ at $s'$ and let $k'$ be the residue field of $R'$. Furthermore, let $A$ be a local $R$-algebra with residue field $k_A$. Then all $R$-algebra morphisms from $R'$ to $A$ are local. So there is a canonical map

$$\text{Hom}_R(R', A) \to \text{Hom}_k(k', k_A).$$

*This map is always injective; it is bijective if $A$ is henselian.*

The group $\text{Gal}(K_1, K)$ acts on $R^{sh}$ with fixed subring $R$. Let $k \subset k_2 \subset k_1$ and $R \subset R_2$ be as above, then $R_2$ is normal as $R$ is, and we have an isomorphism

$$\text{Hom}_k(K_2, K_1) = \text{Hom}_R(R_2, R^{sh}) \cong \text{Hom}_k(k_2, k_1).$$

As $R \subset R_2$ is étale, $K_2$ is separable over $K_1$, in the inductive limit we find that the induced map $\text{Gal}(K_1, K) \to \text{Gal}(k_1, k)$ is an isomorphism.

1.8.1. The inclusion $\mathcal{F}_s \otimes_{j_*} \mathcal{F}_0 \subset j_* \mathcal{F}_0$ comes about by considering that étale locally about a point $s \in S_0$ neither sheaf may be locally free, as restricting étale neighborhoods of $s$ to $U_0$ may not be enough to trivialize either sheaf; in other words, a trivialization may ramify when extended to $X_0$. Let $V \to X_0$ be an étale neighborhood of $s$. Then the inclusion above simply reflects the fact that sections of $\mathcal{F}_s \otimes \mathcal{F}_0$ over $V|_{U_0} := V \times_{X_0} U_0$ include those coming from the tensor product of $k$ sections in $\mathcal{F}_0(V|_{U_0})$, but might include more besides.

Here is an example: let $X = \text{Spec } R$ and let $\mathcal{F}$ be the locally free sheaf on $X$ represented by $\text{Spec } R[x]/(x^3 - 1)$; this is the sheaf $\mu_3$ of third roots of unity and it is a locally free sheaf of $\mathbb{Z}/3$-modules of rank 1. Then $\mathcal{F}(R) = \{1\}$ and if $\zeta$ is a primitive 3rd root of unity, $\mathcal{F}(\mathbb{C}) = \{1, \zeta, \zeta^2\}$. $\text{Gal}((R/\mathbb{C}) \cong \mathbb{Z}/2$ and $\mathcal{F}$ corresponds to
the representation $V$ of $\mathbb{Z}/2$ given by complex conjugation on the $\mathbb{Z}/3$-module $\{1, \zeta, \bar{\zeta}\}$. Then $\mathcal{F} \otimes_{\mathbb{Z}/3} \mathcal{F}$ corresponds to the tensor representation $V \otimes_{\mathbb{Z}/3} V$. Its sections over $\mathcal{R}$ are its $\mathbb{Z}/2$-invariant sections of $V \otimes V$. These are $\{1 \otimes 1, \zeta \otimes \bar{\zeta}\}$.

1.8.4. The fiber of $j_* \mathcal{F}_0$ at $\bar{s}$ can be computed by taking first the inverse image of $j_* \mathcal{F}$ to $\text{Spec } \mathcal{O}_{X_0,s}$, the local ring of $X_0$ at $s$, and then taking the colimit along all étale ring maps $\mathcal{O}_{X_0,s} \to U$, these being equivalent to finite separable extensions of $k(\eta)$ which are non-ramified over $s$. So in the end we are computing the colimit of sections of the inverse image of $\mathcal{F}_0$ along $\eta \to X_0$ over finite separable field extensions of $k(\eta)$ fixed by $I$; this is nothing other than $\mathcal{F}_0^{I}$.

The last line of the proof references (1.6.14.3), which has been corrected in the note (1.6.14) above.

1.8.5. The $t$-weights of $\mathcal{F}_\eta$ are integers, as guaranteed by (1.8.4); therefore, we can apply (1.7.5). The nilpotent endomorphism $N$ respects the filtration $W$ on $\mathcal{F}_\eta$, since all of $W(t, \eta)$ respects the filtration, and hence the inertia does, so that the logarithm of the unipotent part of the local monodromy does too. For the local monodromy filtration on $\mathcal{F}_\eta$, rel. $W$ to exist, it remains only to check that $N^k$ induces isomorphisms $\text{Gr}^M_{i-k} \mathcal{F}^W_\eta \simeq \text{Gr}^M_{i-k} \mathcal{F}^W_\eta$; i.e. that the weight filtration $M_1$ on $\text{Gr}^W_i(\mathcal{F}_\eta)$ coincides with the local monodromy monodromy filtration $M_2$ on $\text{Gr}^W_i(\mathcal{F}_\eta)$, shifted by $i$; i.e. that

\[(\dagger) \quad \text{Gr}^M_{i-j} \mathcal{F}^W_j(\mathcal{F}_\eta) = \text{Gr}^M_{i-i} \mathcal{F}^W_j(\mathcal{F}_\eta).\]

But (1.7.5) shows that the weight filtration $M_1$ is the unique finite increasing filtration on $\text{Gr}^W_i(\mathcal{F}_\eta)$ which is stable under $W(\overline{K}, \mathcal{K})$ such that $\text{Gr}^M_{i-j} \mathcal{F}^W_j(\mathcal{F}_\eta)$ is $t$-pure of weight $j$. On the other hand, (1.8.4) shows that the local monodromy filtration $M_2$ on $\text{Gr}^W_i(\mathcal{F}_\eta)$ is a finite increasing filtration which is stable under $W(t, \eta)$ such that $\text{Gr}^M_{i-j} \mathcal{F}^W_j(\mathcal{F}_\eta)$ is $t$-pure of weight $i + j$. This shows that the filtration $M_2$, shifted by $i$, coincides with $M_1$; i.e. we have verified $(\dagger)$, and hence the existence of the local monodromy filtration on $\mathcal{F}_\eta$, rel. $W$. 
1.8.8. With regards to remark 2), twist the sheaf $\mathcal{F}_0 \rightsquigarrow \mathcal{F}_0^{(b)}$ so that it has weight 0 (following (1.2.7), $b = p^{-\beta}$). Then apply (1.8.7) with the trivial filtration $W$ to see that $\mathrm{Gr}_i^M(j_*\mathcal{F}_0^{(b)}|D_0)$ is punctually $i$-pure of weight $i$. Twist back to conclude that $\mathrm{Gr}_i^M(j_*\mathcal{F}_0|D_0)$ is punctually $i$-pure of weight $\beta + i$.

1.8.9. In c), $j_*\mathcal{F}_0 \hookrightarrow \varepsilon_*j_*\varepsilon^*\mathcal{F}_0$ follows, after writing $\varepsilon_*j_*\varepsilon^*\mathcal{F}_0 = j_*\varepsilon_*\varepsilon^*\mathcal{F}_0$, from the observation that $\mathcal{F}_0 \hookrightarrow \varepsilon_*\varepsilon^*\mathcal{F}_0$ injects since on stalks, a finite extension of a henselian ring splits as a product of henselian rings; i.e. the adjunction morphism corresponds to the inclusion along the diagonal

$$\mathcal{F}_x \hookrightarrow \prod_{\varepsilon^{-1}x} \mathcal{F}_x.$$

In d), reduce to a constant sheaf, where it is obvious.

In the explanation for e), the strict henselization of $X_0$ at $x$ is irreducible hence a fortiori the inverse image of any open set is connected. Then, since the fiber product of any étale cover of $x$ with $U_0$ is an étale neighborhood of $z$, there is a map $(i^*j_*\mathcal{F}_0)_x \rightarrow (\mathcal{F}_0)_z$. As the former can be computed as sections over the inverse image of $U_0$ in the strict henselization of $X_0$ at $x$; since this is a connected scheme, and $\mathcal{F}_0$ is lisse, the arrow is injective. The factorization of this arrow as

$$(i^*j_*\mathcal{F}_0)_x \rightarrow (k_*k^*j_*\mathcal{F}_0)_x \rightarrow (j_*\mathcal{F}_0)_y \rightarrow (\mathcal{F}_0)_x$$

can be explained as follows. After rewriting $k_*k^*j_*\mathcal{F}_0$ as $k_*k^*i^*j_*\mathcal{F}_0$, the first arrow is just adjunction for $k$, The middle arrow can be rewritten $(k_*k^*i^*j_*\mathcal{F}_0)_x \rightarrow (i^*j_*\mathcal{F}_0)_y$ so that it is a statement about sheaves on $F_0$. Take an étale neighborhood $W_0$ of $x$ in $F_0$; then $W_0 \times_{F_0} V_0$ is an étale neighborhood of $y$. The projective system of étale $W'_0 \rightarrow F_0$ s.t. $W'_0 \times_{F_0} V_0$ admits an arrow to $W_0 \times_{F_0} V_0$ has the property that the projective system $W'_0$ is a subcategory of the projective system of étale neighborhoods of $y$ in $F_0$. Therefore there is an arrow from the colimit of $i^*j_*\mathcal{F}_0$ applied to the former system to the colimit of $i^*j_*\mathcal{F}_0$ applied to the latter, which is $(i^*j_*\mathcal{F}_0)_y$. This gives an arrow $(k_*k^*i^*j_*F_0)(W_0) \rightarrow (i^*j_*\mathcal{F}_0)_y$, functorial in $W_0$, and hence an arrow $(k_*k^*i^*j_*F_0)_x \rightarrow (i^*j_*\mathcal{F}_0)_y$. 

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The last arrow is in effect the observation that the fiber product of $U_0$ with an étale neighborhood of $y$ in $X_0$ is an étale neighborhood of $z$.

In the proof of (1.8.9), the reductions are all clear except perhaps the reduction to $\mathcal{F}_0$ tamely ramified at the generic points of $F_0$. Suppose all the other dévissages have been made except that one; then as there are finitely many generic points and $\mathcal{F}_0$ is a lisse $\overline{Q}_e$ sheaf corresponding to a representation $V$ of $\pi_1(U_0, \overline{x})$ for some geometric point $\overline{x}$ of $U_0$, the image of the wild inertia at the finitely many generic points of $F_0$ in $\text{Aut} V$ is finite. In particular, the preimage of the congruence subgroup $\Gamma_1$ of $\text{Aut} V$ in $\pi_1(U_0, \overline{x})$ is open and of finite index, corresponding to an étale cover of $V_0 \to U_0$ which extends to a finite surjective morphism to $X_0$. Applying c), we reduce to the desired situation.

To complete the proof, we are almost there, except $F_0$ is just a Weil divisor, and need not satisfy the smoothness assumption of (1.8.6). The idea is to use e) and recurrence on dim $U_0$ to shrink $X_0$ and throw away the bad points of $F_0$. If we replace $X_0$ by an open set containing $U_0$ whose intersection with $F_0$ is a lisse divisor, then (1.8.8) 2) shows that (1.8.9) is true there. We can find finitely many such open sets with inclusions $j_i$, the union of which, $X'_0$, intersects $F_0$ in a dense set $V_0$. If $j' : U_0 \hookrightarrow X'_0$, then $j'_*\mathcal{F}_0 \hookrightarrow \prod_i j_i*\mathcal{F}_0$ so (1.8.9) is proved for $j'$. By recurrence on dimension we may assume that (1.8.9) holds for $k_*$. In light of this, applying e) to the lisse sheaf $\mathcal{F}_0$, yields that $i^*j_*\mathcal{F}_0$ satisfies the conclusions of (1.8.9); in effect, $k^*j_* = k^*j'_*$. Finally, $j_*\mathcal{F}_0 \hookrightarrow j'_*\mathcal{F}_0 \times i_*i^*j_*\mathcal{F}_0$ allows us to conclude that $j_*$ satisfies (1.8.9).

1.8.11. A Jordan-Hölder series for $\mathcal{F}_0$ allows us to reduce to $\mathcal{F}_0$ irreducible. The restriction of an irreducible lisse sheaf to a nonempty open $U_0$ of a normal connected scheme $X_0$ is still irreducible because if $\eta$ denotes the generic point of $X_0$, and $\overline{\eta}$ a geometric point centered on $\eta$, we have by [SGA1, Exp. V, 8.2]

$$\text{Gal}(\overline{\eta}, \eta) \to \pi_1(U_0, \overline{\eta}) \to \pi_1(X_0, \overline{\eta}).$$

Now, $\mathcal{F}_0$ is $\tau$-mixed, so admits an $\iota$-pure subsheaf $\mathcal{G}_0$ which is lisse when restricted to some $U_0$. Therefore $\mathcal{F}_0|U_0$, irreducible yet containing $\mathcal{G}_0|U_0$, must equal $\mathcal{G}_0|U_0$. 
1.8.12. Let \( f : X_0' \to X_0 \) be the normalization morphism. It induces a bijection on irreducible components, and \( X_0' \) is a disjoint of normal integral schemes. We make use of the fact, true for any morphism, that if \( x_0 \in X_0 \), the weights of \( \mathcal{F}_0 \) at \( x_0 \) coincide with the weights of \( f^*\mathcal{F}_0 \) at every point of the fiber \( f^{-1}(x_0) \). If \( \mathcal{F}_0 \) is \( t \)-pure of weight \( \beta \) at a point \( x_0 \), then all the points of \( X'_0 \) in the fiber over \( x_0 \) are also \( t \)-pure of the same weight; therefore (1.8.11) implies that all the points of the irreducible components of \( X'_0 \) meeting the fiber of \( x_0 \) are \( t \)-pure of weight \( \beta \), which in turn implies that all the points in the irreducible components of \( X_0 \) meeting \( x_0 \) are \( t \)-pure of weight \( \beta \), which shows that the locus of points where \( \mathcal{F}_0 \) is \( t \)-pure of weight \( \beta \) is closed. This locus is also open because, taking any \( x_0 \) in it, there is an open neighborhood of \( x_0 \) which meets only the irreducible components of \( X_0 \) on which \( x_0 \) lies. Then the above construction shows that all the points in this neighborhood are also \( t \)-pure of weight \( \beta \).

**References**

[GM] *The Tame Fundamental Group of a Formal Neighbourhood of a Divisor with Normal Crossings on a Scheme* par A. Grothendieck et J. Murre.
[Milne] *Algebraic Groups*.
[SGA5] SGA 5, Exposé XV par C. Houzel.
[SGA 4\( \frac{1}{2} \)] SGA 4\( \frac{1}{2} \), *Rapport sur la formule des traces* par P. Deligne.
1.1.11 To verify the anti-commutativity of the 9th square, as the morphism of triangles $(X'', Y'', Z'') \to (X'[1], Y'[1], Z'[1])$ factors as the composition of two morphisms of triangles $(X'', Y'', Z'') \to (A, Y'', Z'[1]) \to (X'[1], Y'[1], Z'[1])$, where the second arrow is the rotation of $(Z', A, Y'') \to (Z', X'[1], Y'[1])$, it suffices to verify that the triangle $(Z', X'[1], Y'[1])$ which appears in this last morphism of triangles has all arrows induced by the arrows of $(X', Y', Z')$ or translates of them (with the same parity). This is not hard to check from the diagram (1). (The stated explanation appears to be an un-explanation.)

1.3.3 Though it is not stated explicitly, it is immediate from the definition $\mathcal{D}^{\leq n} := \mathcal{D}^{0}\leq -n, \mathcal{D}^{\geq n} := \mathcal{D}^{0}\geq -n$ that $(\tau_{\leq n} X)[m] = \tau_{\leq n-m}(X[m])$ and $(\tau_{\geq n} X)[m] = \tau_{\geq n-m}(X[m])$. Namely, for $X$ in $\mathcal{D}$ and $T$ in $\mathcal{D}^{\leq n}$, $T = T'[-m]$ for some $T'$ in $\mathcal{D}^{\leq n-m}$, so

\[
\text{Hom}(T, (\tau_{\leq n-m}(X[m]))[-m]) = \text{Hom}(T'[-m], (\tau_{\leq n-m}(X[m]))[-m]) = \text{Hom}(T'[-m], X) = \text{Hom}(T, X).
\]

1.4.2.1 The argument for why the derived functors continue to satisfy the stated adjunctions is as follows (this argument is also found in SGA 4 Exp. XVIII 3.1.4.11). Given $F, G$ an adjoint pair of functors on abelian categories

\[
\mathcal{A} \xleftarrow{F} G \xrightarrow{\approx} \mathcal{B}
\]

where both categories have enough injectives and $L$ is exact. The functors $F$ and $G$ extend to functors $D^+(\mathcal{A}) \subseteq D^+(\mathcal{B})$. Given $K' \in D^+(\mathcal{A}), L' \in D^+(\mathcal{B})$, we may assume $L'$ is a complex of injective objects; we have an isomorphism of triple complexes

\[
\text{Hom}'(F(K'), L') \xleftarrow{\approx} \text{Hom}'(K', G(L')).
\]

As $G$ preserves injectives, taking $H^0$ of the associated simple complex (calculated with products) finds the desired

\[
\text{Hom}_{K(\mathcal{B})}(F(K'), L') \xleftarrow{\approx} \text{Hom}_{K(\mathcal{A})}(K', G(L')).
\]
where both sides are also Hom in the respective derived categories, since \( L' \) and \( G(L') \) are complexes of injectives.

1.4.4 The question is, why are the adjoints to the Verdier quotients fully faithful? Let’s consider the quotient \( Q : \mathcal{T} \to \mathcal{T}/\mathcal{U} \), where \( \mathcal{U} \) is the strictly full coreflective triangulated subcategory of \( \mathcal{T} \); \((\mathcal{U}, \mathcal{V})\) form a t-structure on \( \mathcal{T} \); \( \mathcal{U} = ^{\perp}\mathcal{V} \), and \( \mathcal{V} = \mathcal{U}^{\perp} \).

Since the embedding \( u : \mathcal{U} \to \mathcal{T} \) admits a right adjoint \( u \), it follows that \( Q \) admits a right adjoint \( Q \). [CD, I 6-5]. There is a natural isomorphism of functors \( Q \circ Q \sim v \circ v' \), where \( v' \) is the left adjoint to the inclusion \( v : \mathcal{V} \to \mathcal{T} \). [CD, I 6-6]. The functor \( v' \) is nothing other than \( \tau_{\geq 0} \) for the t-structure \((\mathcal{U}, \mathcal{V})\), and therefore, restricted to \( \mathcal{V} \), \( v' \circ v' = \text{id}_{|\mathcal{V}} \). On the other hand, the functor \( Q \) when restricted to \( \mathcal{V} \) is fully faithful [CD, I 5-3]. Therefore \( Q \), restricted to the essential image of \( \mathcal{V} \) under \( Q \) is fully faithful. This essential image is all of \( \mathcal{T}/\mathcal{U} \), since every object \( X \) in \( \mathcal{T} \) belongs to an exact triangle \((U, X, V)\) with \( U, V \) objects in \( \mathcal{U}, \mathcal{V} \), respectively. The assertion that \( v' \) yields an equivalence \( v' : \mathcal{T}/\mathcal{U} \to \mathcal{V} \) (functor obtained by applying the universal property of \( \mathcal{T}/\mathcal{U} \) to \( v' \)) is easy since \( Q \circ v : \mathcal{V} \to \mathcal{T}/\mathcal{U} \) is an equivalence, and \( v' \circ Q \circ v = v' \circ v = \text{id} \). The corresponding statement for \( \mathcal{T} \to \mathcal{T}/\mathcal{V} \) follows identically.

1.4.6 In part b), there is the following consideration. Given triangulated categories \( \mathcal{T}, \mathcal{T}' \), a thick subcategory \( \mathcal{U} \subset \mathcal{T} \) with Verdier quotient \( Q : \mathcal{T} \to \mathcal{T}/\mathcal{U} \), exact functors \( \mathcal{F}, \mathcal{G} : \mathcal{T}/\mathcal{U} \to \mathcal{T}' \) and a natural transformation \( \overline{\varphi} : \mathcal{F} \circ Q \to \mathcal{G} \circ Q \), there is an obvious candidate for a natural transformation \( \varphi : \mathcal{F} \to \mathcal{G} \), since \( \text{Ob}(\mathcal{T}/\mathcal{U}) = \text{Ob}(\mathcal{T}) \).

But is it still a natural transformation? Let \( f : X \to Y \) in \( \mathcal{T}/\mathcal{U} \) be represented by \( X \xrightarrow{s} Z \xrightarrow{a} Y \), where \( s \) is in the saturated multiplicative system of morphisms corresponding to \( \mathcal{U} \). The commutative diamond

\[
\begin{array}{ccc}
F(X) & \xrightarrow{id} & F(X) \\
\downarrow q(X) & & \downarrow q(X) \\
G(X) & \xrightarrow{G(a)} & G(Y) \\
\downarrow G(s) & & \downarrow G(s) \\
F(Z) & \xleftarrow{q(Z)} & G(Z) \\
\end{array}
\]
shows that \( q(X) \circ G(f) \) coincides with \( F(X) \leftarrow F(Z) \xrightarrow{G(a) \circ q(Z)} G(Y) \), but as

\[
\begin{align*}
F(Z) & \xrightarrow{F(a)} F(Y) \\
\downarrow q(Z) & \quad \downarrow q(Y) \\
G(Z) & \xrightarrow{G(a)} G(Y)
\end{align*}
\]

commutes, this morphism is just \( F(X) \leftarrow F(Z) \xrightarrow{q(Y) \circ F(a)} G(Y) = F(f) \circ q(Y) \), which shows that \( q \) is indeed a natural transformation.

**1.4.7** In c), by 1.1.9, the morphism \( B \to C \) is the unique such that completes the morphism of triangles \((B, j_! j^* X[1], j_! j^* X[1]) \to (C, X[1], j_! j^* X[1])\).

**1.4.13** The distinguished triangle \((\tau_{\leq p}^F X, X, i_* \tau_{> p}^F i^* X)\) is the distinguished triangle \((A, Y, i_* \tau_{> p}^F i^* Y)\) of 1.4.10, since as remarked, \( X = Y \) since \( \tau_{> 0}^* X = 0 \). To check that \( A = \tau_{\leq p}^F X \), note that \( A \) belongs to \( \mathcal{D}^{\leq p} \), as \( i^! A \simeq \tau_{\leq p}^F i^* X \), and that if \( T \) belongs to \( \mathcal{D}^{\leq p} \), then by applying \( \text{Hom}(T, -) \) to the above distinguished triangle and observing that as \( i_* \) is t-exact, it commutes with truncation, so \( i_* \tau_{> p}^F i^* X[-1] \) lies in \( \mathcal{D}^{> p+1} \), \( \text{Hom}(T, i_* \tau_{> p}^F i^* X) = 0 = \text{Hom}^{-1}(T, i_* \tau_{> p}^F i^* X) \), and \( \text{Hom}(T, X) \simeq \text{Hom}(T, A) \).

The statement about cohomology follows by applying \( \tau_{\geq p} \tau_{\leq p} \) to the distinguished triangle \((\tau_{\leq p-1}^F X, X, i_* \tau_{\geq p-1}^F i^* X)\) and using the fact that \( i_* \) commutes with truncation.

**1.4.14** To find the dual statement at the end of the proof, reverse arrows and exchange \( j_! \leftrightarrow j_* \) to obtain the distinguished triangle \((i_* i^* X[-1], j_! Y, X)\), then use \((b')\), the isomorphism \( j_! / j_! \simeq i_!^* j_! [i] \) of 1.4.6.4, (and the note to 1.3.3) to write

\[
i_* i^* X[-1] = i_*(\tau_{\leq p-1}(j_! / j_!) Y)[-1] = i_* \tau_{\leq p}((j_! / j_!) Y)[-1] = i_* \tau_{\leq p} i_!^* j_! Y,
\]

establishing \( X \) as \( \tau_{\geq p+1} j_! Y \).

**1.4.17.1** A little note: \( p_! i^* X \) is the largest quotient of \( X \) belonging to \( C_F \). First we check that it is a quotient from 1.4.17 (ii). Then, suppose \( A \) belongs to \( C_F \) and \( X \to A \); then \( p_! i^* X \twoheadrightarrow p_! i^* A \twoheadrightarrow A \), as \( p_! i_! \) is fully faithful, so the adjunction morphism \( p_! i^* p_! i_* \to \text{id} \) is an isomorphism, and \( p_! i^* X \) is indeed the largest quotient of \( X \) in \( C_F \). Dually for \( p_! i^! \).
1.4.18 A little note about $T$ faithful: as $p^*j^*$ is an exact functor, if $p^*j^*f_1 = 0$, this means that $p^*\text{im}(f_1) = \text{im}(p^*j^*f_1) = 0$, which is to say that $f_1$ belongs to $\overline{C}_F$.

1.4.23 In the distinguished triangle $(i_*\mathcal{H}^0j^!$, $\tau^F_{\geq 0}j^!$, $\tau^F_{\geq 1}j^!B)$, as $j^!B = \tau^U_{\geq 0}j^!B$, $\tau^F_{\geq p} = \tau^F_{\geq p}U_{\geq p}$, and $j^!$ is right $t$-exact, $\tau^F_{\geq 0}j^!B$ sits in $C$. Likewise, $i^*$ is $t$-exact, so $i_*\mathcal{H}^0j^!B$ also sits in $C$, and from the long exact sequence of $\mathcal{H}^t$ one finds that $\tau^F_{\geq 1}j^!B$ is in $\mathcal{D}[-1,0]$.

2.1.2 In the discussion ‘Si les foncteurs $\tau^!_S$ sont de dimension cohomologique finie...’ it is claimed that there is a neighborhood of $S$ in which $\mathcal{H}^t\tau^a_SK$ is supported on $S$. To find such a neighborhood, simply discard $\overline{S} - S$ and the closure of any stratum which doesn’t meet $S$. The assumption that the closure of each stratum is a union of strata implies that the induced stratification of the resulting neighborhood of $S$ has the property that every stratum contains $S$ in its closure, and therefore $\mathcal{H}^t\tau^a_SK$ vanishes on every stratum distinct from $S$. By construction, $S$ is a closed set in this neighborhood.

As for the isomorphism $\mathcal{H}^t(i^*_S\tau^a_SK) \sim \mathcal{H}^t(i^*_S\tau^a_SK)$ for $i < a$, let us replace $X$ by the neighborhood above, in which case the adjunction morphism $\tau^a_SK \to i'_S\tau^a_SK$ is an isomorphism as $\tau^a_S$ and $\tau^a_S$ are exact and the induced morphism on cohomology $\mathcal{H}^t(\tau^a_SK) \to i'_S\tau^a_S\mathcal{H}^t(\tau^a_SK)$ is an isomorphism for all $j$. Therefore by 1.4.1.2, $i^*_S\tau^a_SK \sim i^*_S\tau^a_SK \sim i^*_S\tau^a_SK$.

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