Notes
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1. Rapport sur la formule des traces

$p, q, \ell, F_q, F$: $p$ is a prime number, $q = p^f$ is a power of $p$ and $F$ an algebraic closure of the field $F_q$; $\ell$ is a prime number $\neq p$.

$X_0, X$: $X_0$ is a scheme on $F_q$, $X = X_0 \times_{F_q} F$. If $\mathcal{F}_0$ is an (étale) sheaf on $X_0$, $\mathcal{F}$ denotes its inverse image on $X$.

1.1. Mise en garde. The exposition beginning with [1.3] is based on SGA 5, Exposé XIV by C. Houzel. This approach is explicit at the expense being somewhat anti-conceptual and overloaded with cumbersome notation, which makes it hard to remember. On the other hand, Deligne’s approach to Frobenius is conceptually beautiful and easy to remember, but the first time I read it I couldn’t understand it. Now that I do, I write these notes. Deligne uses $\varphi$ to denote what he calls the ‘Frobenius substitution’; it is the well-known topological generator of $\text{Gal}(F/F_q)$. He uses $F$ to denote the ‘Frobenius endomorphism’, notated $\text{fr}_{X_0}$ below, which is the endomorphism of a scheme $X_0/F_q$ obtained by $x \mapsto x^q$ on the structure sheaf $\mathcal{O}_{X_0}$. Finally, he uses $F^*$ to denote the action of Frobenius on sheaves on a scheme over $F_q$ (and base extensions of such), and their cohomology.

1.2. Frobenius following Deligne.

1.2.1. Representing the Frobenius correspondence. Let $Q$ be any scheme and $\mathcal{G}$ an (étale) sheaf on $Q$. Let $Q_{\text{pet}}$ denote the category $(\text{Et}/Q)$ of algebraic spaces étale over $Q$, equipped with the étale topology. We ‘recall’ the following

*Proposition.* — $\mathcal{G}$ is represented over $Q_{\text{pet}}$ by l’espace étalé $[\mathcal{G}]$ of $\mathcal{G}$: an algebraic space, locally separated and étale over $Q$. If moreover $\mathcal{G}$ is

- (α) locally constant constructible
- (β) locally constant,
- (γ) constructible,

then $[\mathcal{G}]$ may be taken to be

- (α) a scheme finite étale over $Q$ (SGAA Exp. IX 2.2).
(β) a scheme étale over \( Q \) (SGAA Exp. IX 2.2).

(γ) étale and finitely presented as an algebraic space over \( Q \) (SGAA Exp. IX 2.7).

In other words, \( \mathcal{G} \) is the sheaf of local sections of the space \( [\mathcal{G}] \): for each \( V \in \text{Et}/Q \),

\[
\mathcal{G}(V) = \text{Hom}_Q(V, [\mathcal{G}]).
\]

M. Artin constructs the espace étalé \([\mathcal{G}]\) associated to an arbitrary étale sheaf \( \mathcal{G} \) on a scheme \( Q \) in \textit{Théorèmes de représentabilité pour les espaces algébriques} VII §1. A sketch: put

\[
(U \to \mathcal{G}) := \bigcup_{U, \xi \in \mathcal{G}(U)} (U \xrightarrow{\xi} \mathcal{G}),
\]

the sum executed over affine schemes \( U \) étale over \( Q \) and the \( \xi \in \mathcal{G}(U) \). The canonical morphism \( U \times_{\mathcal{G}} U \to U \times_Q U \) induced by \( \xi \in \mathcal{G}(U) \) and \( \eta \in \mathcal{G}(V) \) (\( U, V \) affine étale schemes over \( Q \)) is an open immersion and defines an étale equivalence relation on \( U \). Let \([\mathcal{G}]\) be the quotient of \( U \) by this equivalence relation; it is an algebraic space over \( Q \), in general not separated, but only locally separated; in other words, \( U \times_{\mathcal{G}} U \to U \times_Q U \) is not necessarily a closed immersion.

Now let \( X_0 \) be a scheme over \( \mathbb{F}_q \) and \( \mathcal{F}_0 \) a sheaf on \( X_0 \). The formation of the espace étalé \([\mathcal{F}_0]\) of \( \mathcal{F}_0 \) yields the commutative square

\[
\begin{array}{ccc}
[\mathcal{F}_0] & \longrightarrow & [\mathcal{F}_0] \\
\downarrow & & \downarrow \\
X_0 & \longrightarrow & X_0.
\end{array}
\]

and hence a morphism \([\mathcal{F}_0] \to \mathcal{F}^*[\mathcal{F}_0]\). Replacing \( S \) with \( X_0 \), and \( X_0 \) with \([\mathcal{F}_0]\), \textit{[1.3.1]} below finds that this morphism is relative Frobenius \( \text{Fr}_{[\mathcal{F}_0]/X_0} \); as \([\mathcal{F}_0]\) is étale over \( X_0 \), \textit{[1.3.3]} tells us that this morphism is an isomorphism \([\mathcal{F}_0] \sim \mathcal{F}^*[\mathcal{F}_0]\) with inverse Deligne’s Frobenius correspondence (1.2.1)

\[
\mathcal{F}^* : \mathcal{F}^*[\mathcal{F}_0] \sim [\mathcal{F}_0].
\]
This morphism is the same as the one constructed with different notation in [1.3.4]. The point is that by putting additional hypotheses on \( \mathcal{F}_0 \), one may assume that \([\mathcal{F}_0]\) is in fact a scheme étale over \( X_0 \).

1.2.2. The Frobenius correspondence and the Frobenius endomorphism. Let \( \mathcal{F}_0 \) be an abelian sheaf on \( X_0 \). We wish to elucidate Deligne’s approach in 1.8 of Rapport to show that

\[
F^* - 1 = \varphi \quad \text{(on } H^i_c(X, \mathcal{F}))\).
\]

Letting \( Y_0 = \text{Spec} \ F_q \), as written \( H^i_c(X, \mathcal{F}) = [R^i f_! \mathcal{F}_0](\mathcal{F}) \), and, noting \( F = \text{id} \) on \( Y_0 \), we have the (stupid) diagram

\[
\begin{array}{ccc}
[R^i f_! \mathcal{F}_0] & \to & [R^i f_! \mathcal{F}_0]
\\
\downarrow & & \downarrow
\\
Y_0 & \to & Y_0
\end{array}
\]

which just serves to connect this discussion to that of the previous section and show that the morphism defining the inverse of the Frobenius correspondence \( F^* \) on \([R^i f_! \mathcal{F}_0]\) is indeed \( F : [R^i f_! \mathcal{F}_0] \to [R^i f_! \mathcal{F}_0] \); as \( F \) acts on geometric points by \( \varphi \), we see that

\[
F^* - 1 : [R^i f_! \mathcal{F}_0](\mathcal{F}) \to [R^i f_! \mathcal{F}_0](\mathcal{F})
\]

coincides with \( \varphi \).

1.3. Frobenius following Houzel.

Definition. We denote by \( f_{X_0} \) the morphism of schemes \( X_0 \to X_0 \) which is the identity on the underlying topological space \(|X_0|\) and acts on the structure sheaf \( \mathcal{O}_{X_0} \) by \( g \mapsto g^q \).

This morphism is called absolute Frobenius.

1.3.1. If the structure morphism \( X_0 \to \mathbf{F}_q \) factors through some scheme \( S \), then we denote by \( X_0^{(q/S)} := X_0 \times_{S, f_{S}} S \) the fiber product of \( g : X_0 \to S \) by the morphism
fr\(_S : S \to S\) with projection \(\pi_{X_0/S} : X_0^{(q/S)} \to X_0\). The absolute Frobenius \(fr_{X_0}\) then factors through the morphism \(\pi_{X_0/S}\). We can form the diagram

\[
\begin{array}{c}
X_0 \\
\downarrow fr_{X_0} \\
X_0^{(q/S)} \\
\downarrow \pi_{X_0/S} \\
X_0 \\
\downarrow g \\
S \\
\end{array}
\]

\[
\begin{array}{c}
S \\
\downarrow fr_{S} \\
S \\
\end{array}
\]

**Definition.** The morphism \(Fr_{X_0/S}\) is called *relative Frobenius*.

### 1.3.2. Frobenius acts on geometric points

Consider the set of geometric points \(X(F) = X_0(F)\). Frobenius acts on this set by \(q \in \text{Gal}(F/F_q)\), \(q(x) = x^q\). In particular, as \(F_q\) is perfect, \(X^F = X_0(F_q)\), where \(X^F\) denotes the geometric points fixed by Frobenius. In slightly more words, consider a geometric point \(x \to X_0\) centered on \(x\). The fiber \(X \times_{X_0} x\) is isomorphic to the spectrum of \(A = F \otimes_{F_q} k(x)\). As \(k(x)/F_q\) is separable, \(A \sim \prod_{[k(x):F_q]} F\), and \([k(x) : F_q]\) is also equal to the number of \(F_q\)-embeddings \(k(x) \to F\). Such an embedding is fixed by \(q\) iff \(k(x) = F_q\), and by \(q^{f}\) iff \(k(x) \subset F_{q^f}\).

So for every point \(x \in |X_0|\) with \([k(x) : F_q] = f\), there are \(f\) geometric points centered on \(x\); \(F\) acts transitively by \(q\) on this set, and \(F^{f}\) fixes each of these geometric points.

### 1.3.3. Behavior of relative Frobenius

The relative Frobenius \(Fr_{X_0/S}\) is integral, surjective, and radical, hence a universal homeomorphism. This is clear when \(S = F_q\); i.e. for \(fr_{X_0} = \pi_{X_0/S} \circ Fr_{X_0/S}\); it follows that \(Fr_{X_0/S}\) is radical [EGA I 3.5.6 (ii)]. Moreover, \(\pi_{X_0/S}\) is separated and radical, therefore \(Fr_{X_0/S}\) is integral [EGA II 6.1.5 (v)] and surjective.

Suppose moreover that \(g : X_0 \to S\) is étale. The same is true of \(g^{(q)} : X_0^{(q)} \to S\), and therefore \(Fr_{X_0/S} : X_0 \to X_0^{(q)}\) is étale. As \(Fr_{X_0/S}\) is also radical and surjective, it is an isomorphism.

### 1.3.4. Frobenius correspondence

Let \(X_0\) be a scheme over \(F_q\) and \(\mathcal{F}_0\) a sheaf of sets on \((X_0)_{\acute{e}t}\) (the small étale site of \(X_0\) whose underlying category is the category of schemes étale over \(X_0\)). We have for all \(U \to X\) étale \((fr_{X_0})_*\mathcal{F}_0(U) = \mathcal{F}_0(U^{(q/X)})\).
The isomorphism \( \mathcal{F}_0(\text{Fr}_{U/X_0}) : (\text{fr}_{X_0})_* \mathcal{F}_0(U) \to \mathcal{F}_0(U) \) is natural in \( U \) and induces an isomorphism of sheaves
\[
\mathcal{F}_0(\text{Fr}_{U/X_0}) : (\text{fr}_{X_0})_* \mathcal{F}_0 \xrightarrow{\sim} \mathcal{F}_0;
\]
by adjunction applied to \( \mathcal{F}_0(\text{Fr}_{U/X_0})^{-1} \) we obtain a morphism
\[
F^* : \text{fr}^*_{X_0} \mathcal{F}_0 \to \mathcal{F}_0.
\]
As \( \text{fr}_{X_0} \) is integral, surjective, and radicial, \( \text{fr}^*_{X_0} : (X_0)_{\text{ét}} \to (X_0)_{\text{ét}} \) is an equivalence of sites, and the functors
\[
(\text{fr}_{X_0})_* , \text{fr}^*_{X_0} : J(X_0)_{\text{ét}} \to J(X_0)_{\text{ét}}
\]
are autoequivalences and quasi-inverses, where \( J(X_0)_{\text{ét}} \) denotes the étale topos on \( X_0 \) [SGAA, Exp. VIII, 1.1]. Therefore \( F^* \) is also an isomorphism.

Definition. The isomorphism \( F^* : \text{fr}^*_{X_0} \mathcal{F}_0 \to \mathcal{F}_0 \) is called the Frobenius correspondence.

1.3.5. Frobenius acts on cohomology. Consider \( \mathcal{F}_0 \) a sheaf of \( \Lambda \)-modules, for some commutative ring \( \Lambda \). The canonical morphism \( \alpha : \mathcal{F}_0 \to \text{fr}_{X_0}^* \mathcal{F}_0 \) gives rise to
\[
\Gamma(X_0, \mathcal{F}_0) \xrightarrow{\alpha} \Gamma(X_0, \text{fr}_{X_0}^* \mathcal{F}_0) \xrightarrow{F^*} \Gamma(X_0, \mathcal{F}_0);
\]
we also denote the composition of these maps by \( F^* \). When \( \mathcal{F}_0 = \Lambda \), this composition is easily seen to coincide with \( \text{id}_{\Gamma(X_0, \Lambda)} \), and as every section \( s \in \Gamma(X_0, \mathcal{F}_0) \) corresponds to a morphism \( s : \Lambda \to \mathcal{F}_0 \) and \( F^* \) is evidently functorial in \( \mathcal{F}_0 \), we find \( F^* \circ \Gamma(X_0, s) = \Gamma(X_0, s) \circ F^* = \Gamma(X_0, s) \), ergo \( F^* s = s \), so \( F^* \) induces the identity on \( \Gamma(X_0, \mathcal{F}_0) \).

Recalling the definition of the Frobenius correspondence via adjunction, this action of Frobenius on \( \Gamma(X_0, \mathcal{F}_0) \) coincides with the composition
\[
\Gamma(X_0, \mathcal{F}_0) \xrightarrow{\mathcal{F}_0(\text{Fr}_{X_0})^{-1}} \Gamma(X_0, (\text{fr}_{X_0})_* \mathcal{F}_0) = \Gamma(X_0, \mathcal{F}_0).
\]

(\star)

Considering \( \mathcal{F}_0 \) now as an object of \( D^+(X_0, \Lambda) \), we have \( \text{fr}_{X_0}^* \) preserves injective objects; hence the composition
\[
F^* : R\Gamma(X_0, \mathcal{F}_0) \xrightarrow{\mathcal{F}_0(\text{Fr}_{X_0})^{-1}} R\Gamma(X_0, \text{fr}_{X_0}^* \mathcal{F}_0) \to R\Gamma(X_0, \mathcal{F}_0)
\]
can be computed by applying (⋆) term-by-term to an injective resolution of $\mathcal{F}_0$, whence we see that $F^*$ acts by identity on $R\Gamma(\mathcal{F}_0)$, and hence

$$F^* : H^i(X_0, \mathcal{F}_0) \rightarrow H^i(X_0, \mathcal{F}_0)$$

is the identity for all $i$.

Suppose that $X_0$ is separated and of finite type over $F_q$. Then we can replace $\Gamma$ in the above discussion by $\Gamma_c$ to find that Frobenius acts on compactly supported cohomology

$$F^* : H^i_c(X_0, \mathcal{F}_0) \rightarrow H^i_c(X_0, \mathcal{F}_0).$$

2.1. The trace formula for $\text{Spec } F_{q^n} \rightarrow \text{Spec } F_q$. Let $X_0 = \text{Spec } F_{q^n}$, $q = p^f$, and $\mathcal{F}_0$ be a constructible $Q_\ell$ sheaf on $X_0$, $\mathcal{F}$ its inverse image on $X$. In this case, the cohomological description of the L-function $L(X_0, \mathcal{F}_0)$ reads very simply

$$\det(1 - F^*_{x}t^{d(x)}, \mathcal{F})^{-1} = \det(1 - F^*_{t_{x}}t^{f}, H^0_c(X, \mathcal{F}))^{-1},$$

where $d(x) = [k(x) : F_p]$. We first need to make precise how Frobenius is acting on the left and right sides.

On the left side, we fix a geometric point $\bar{x} \rightarrow x = X$ and construct the action of Frobenius on the fiber $\mathcal{F}_\bar{x}$ by picking the smallest power of $F(q)$ which actually fixes the geometric point $\bar{x}$, namely $F_{q^n} = F_{q^n}(q)$. The notation $F^*_x$ denotes the endomorphism of $\mathcal{F}_\bar{x}$ induced by $F_{q^n}$. Up to isomorphism, $(F_x, \mathcal{F}_x)$ do not depend on the choice of geometric point $\bar{x} \rightarrow x$, and the trace, determinant of this action are denoted by $\text{Tr}(F^*_x, \mathcal{F})$, etc.

Now, on the right side, we have the Frobenius correspondance on cohomology. We will make use of the identity

$$F^{*-1} = \varphi,$$

where $\varphi$ is the Frobenius considered as the topological generator of $\text{Gal}(F,F_q)$ (c.f. remark below). The data of a $Q_\ell$-sheaf on $X$ is equivalent to the data of a finite-dimensional $Q_\ell$-vector space $V = \mathcal{F}_{\bar{x}}$ on which $\text{Gal}(\bar{k}(x)/k(x)) = \pi_x$ acts continuously.
There is a canonical isomorphism
\[ \pi_x = \text{Gal}(\overline{k(x)}, k(x)) \cong \hat{\mathbb{Z}} \]
furnished by the Frobenius element
\[ \varphi_x \in \text{Gal}(\overline{k(x)}, k(x)) = \text{Gal}(F/F_{q^n}), \quad \varphi_x(\lambda) = \lambda q^n, \]
so that the action of \( \pi_x \) on \( V = \mathcal{F}_x \) is known once one knows the automorphism \((\varphi_x)_V\) (under the one condition that \((\varphi_x)_V \to \text{id}_V\) as \( v \to 0 \) multiplicatively). If
\[ \text{pr}_x : X = \text{Spec}(F_{q^n}) \to \text{Spec } F_q = e \]
is the canonical morphism, \( \pi_x \) is identified via \( \text{pr}_x^* \) with a subgroup of the analogous Galois group \( \pi_e \) for \( e = \text{Spec}(F_q) \), itself topologically generated by \( \varphi \), and via this identification we have the identity
\[ \varphi_x = \varphi^n. \]

The sheaf \( \text{pr}_{x*}(\mathcal{F}) \) is defined by the induced module
\[ \text{pr}_{x*}(\mathcal{F})_\pi \cong \mathcal{F}_x \otimes_{\pi_x} \pi_e, \]
from which one deduces that, letting \( f = \varphi^{-1}_x \), \( \varphi^{-1} \) acts on \( \text{pr}_{x*}(\mathcal{F})_\pi \) by
\[ f^{(n)} : (x_1, \ldots, x_n) \mapsto (f(x_n), x_1, \ldots, x_{n-1}), \]
where here we have written \( \text{pr}_{x*}(\mathcal{F})_\pi \) with respect to a basis as a free \( \pi_x \)-module of rank \( n \). Now the formula \((*)\) is a matter of verifying the formula
\[ \det(1 - ft^n) = \det(1 - f^{(n)}t) \]
for \( f \) acting on a free module of rank \( n \). This is Deligne’s corollary 3.4.
Remark. Perhaps one way to think about the identification $F^* = q^{-1}$ is by setting up the usual diagram

$$
\begin{array}{c}
\text{X} \\
\downarrow \Fr_{X_0} \\
X_{(q/X_0)} & \xrightarrow{\Fr_{(q/X_0)}} & X \\
\downarrow \g^{(q)} \quad \text{and} \quad \downarrow \g \\
X_0 & \xrightarrow{\Fr_{X_0}} & X_0
\end{array}
$$

with $g$ the base extension of the map $\Spec F \to \Spec F_q$ that arises from fixing an algebraic closure of $F_q$, and then observing that $\Fr_{X_0} = q_{X_0} \times F_q \id_{X_0}$. Recalling that $F^*$ on $F_{X_0}$ is induced by $\Fr_{X_0}^{-1}$ for $U \to X_0$ étale, and by functoriality $F^*$ on $\mathcal{F}/X$ is induced by pulling back the same, hence by $\Fr_{U/X_0}$ for $U \to X$ étale, in particular we have that $F^*$ on $H^0_c(X, \mathcal{F})$ is induced by $\Fr(F_{X_0})^{-1} = \mathcal{F}(q^{-1} \times F_q \id_{X_0})$.

3.1. Le sorite de la notation. It is very important to note that in Deligne’s notation, $\Tr(F^*_x, \mathcal{F})$ and $\Tr(F^*, \mathcal{F}_x)$ are traces of possibly different operators on the fiber $\mathcal{F}_x$. Namely, if $\mathcal{F}$ is a $\mathbb{Q}_\ell$-sheaf on $X_0$ a scheme separated and of finite type over $F_q$, then $\mathcal{F}_x$ is a $\mathbb{Q}_\ell$ vector space, for a choice of geometric point $\bar{x}$ centered on a closed point $x$ of $X_0$. Then $F^*_x$ denotes the Frobenius $F_{q^n}^* = F_q^n$ raised to the power of the residue field extension $n = [\deg k(x) : F_q]$. This power of Frobenius is the least that fixes each geometric point centered on $x$, and the notation $\Tr(F^*_x, \mathcal{F})$ means $\Tr(F^*_x, \mathcal{F}_x)$.

On the other hand, if, say, $x \in X^{F^n}$ is a geometric point centered on a point of $X_0$ defined over $F_q^n$, $\Tr(F^*, \mathcal{F}_x)$ denotes (absolute) $q$-power Frobenius acting on the fiber. So, unless $x \in X^F$, $\Tr(F^*, \mathcal{F}_x)$ and $\Tr(F^*_x, \mathcal{F})$ are traces of different operators on the same vector space, the latter a power of the other.

3.2. Le sorite des faisceaux localement constants.

The case of locally constant sheaves of sets. Let $X$ be a scheme and $\mathcal{L}$ a locally constant sheaf of sets with finite fibers on $X$. (With additional assumptions on $X$, the case of a locally constant sheaf of sets with infinite fibers is reduced to the finite case in the course of the discussion of Weil II [1.7.8]) We know that $\mathcal{L} = h_U$ for some $U \to X$
revêtement étale. We know that every revêtement étale of \( X \) is étale-locally on \( X \) trivial; namely for some \( V \to X \) étale, \( U_V \sim \bigsqcup V \). We wish to show that we may take \( V \to X \) to be a revêtement étale (with no further work, we could then take it to be a galoisian revêtement, i.e. a connected torsor for the automorphism group of the fiber, as principal Galois objects in a Galois category form a cofinal system).

First note that if \( f : X \to Y \) is any morphism of schemes and \( V \to Y \) is étale, then \( f^* h_V \sim h_{V \times_Y X} \). To see this, observe

\[
\text{Hom}(f^* h_V, \mathcal{G}) = \text{Hom}(h_V, f_* \mathcal{G}) = f_* \mathcal{G}(V) = \mathcal{G}(V \times_Y X) = \text{Hom}(h_{V \times_Y X}, \mathcal{G}).
\]

Evidently, this argument holds true for any morphism of sites.

So, it will suffice to show that any revêtement étale can be trivialized after base extension by a revêtement étale. To see this, assume \( X \) connected and let \( U \to X \) be a revêtement étale of constant degree \( d \) and proceed by recurrence on \( d \), the case \( d = 1 \) being trivial. (Of course, in the special case that \( U \to X \) is galoisian with Galois group \( G \), \( U \times_X U \sim U \times G \) is a trivial \( G \)-torsor, and we are done.)

As \( U \to X \) is étale, hence net, and finite, the diagonal morphism \( U \to U \times_X U \) is simultaneously an open and closed immersion, hence an isomorphism onto a connected component of \( U \times_X U \), allowing us to write \( U \times_X U = U \bigsqcup Z \) with \( Z \to U \) of constant degree \( d - 1 \). By hypothesis, there exists a revêtement étale \( V \to U \) such that \( Z \times_U V \sim \bigsqcup_{d-1} V \). Our desired revêtement is then simply the composition \( V \to U \to X \):

\[
V \times_X U = V \times_U U \times_X U = V \times_U (U \bigsqcup Z) = V \bigsqcup (V \times_U Z) = \bigsqcup_d V.
\]

The case of locally constant constructible sheaves. Let \( \Lambda \) be a commutative, noetherian torsion ring. We adapt the above discussion to locally constant constructible (l.c.c.) sheaves of \( \Lambda \)-modules. Let \( \mathcal{F} \) be a l.c.c. sheaf of \( \Lambda \)-modules on a connected scheme \( X \). Then the fibers of \( \mathcal{F} \) are finite sets and the above discussion yields a revêtement étale \( f : V \to X \) with \( V \) a (connected) galoisian cover with Galois group \( H \) (i.e. \( V \) is a \( H \)-torsor) such that \( f^* \mathcal{F} \) is a constant sheaf. Its constant value \( H^0(V, f^* \mathcal{F}) \) is a \( \Lambda[H] \)-module.
The sheaf $f_!\Lambda$ on $X$, together with the natural action of $H$, is a rank 1 l.c.c. sheaf of $\Lambda[H]$-modules. Relative to the natural action of $H$ on $f_!f^!\mathcal{F}$, the trace morphism $f_!f^!\mathcal{F} \to \mathcal{F}$ factors by an isomorphism

$$(f_!f^!\mathcal{F})_H \to \mathcal{F},$$

and we have $f_!f^!\mathcal{F} = f_!M = f_!\Lambda \otimes_\Lambda M$ with the diagonal action of $H$.

The above discussion shows that a l.c.c. sheaf of $\Lambda$-modules on a connected scheme $X$ is determined by its restriction to the small étale site of $X$. Sheaves on the small étale site $\mathcal{U}$ are in turn determined by Grothendieck’s Galois theory: fixing a geometric point $\overline{x}$ of $X$ and putting $G := \pi_1(X,\overline{x})$, the functor

$$\text{Sh}(\mathcal{U}) \to \text{finite G-sets}$$

$$\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$$

admits the inverse

$$\text{finite G-sets} \to \text{Sh}(\mathcal{U})$$

$$\mathcal{F}_{\overline{x}} \mapsto [V \in \mathcal{U} \mapsto \text{Hom}_G(V_{\overline{x}}, \mathcal{F}_{\overline{x}})].$$

To verify this, as the torsors are cofinal in a covering of any $V \in \mathcal{U}$, we may cover $V$ by a torsor $W$ with Galois group $H$ and combine the equalizer description of $\mathcal{F}(V)$

$$\mathcal{F}(V) \to \mathcal{F}(W) \rightrightarrows \mathcal{F}(W \times_V W)$$

with the description of such in the case of a Galois torsor; c.f. \cite{SGA 4\frac{1}{2}} I §5.

The discussion in the previous section can be rephrased using the monodromy representation of a l.c.c. sheaf. Namely, let $\mathcal{F}$ be a l.c.c. sheaf of $\Lambda$-modules on a connected scheme $X$ pointed by a geometric point $\overline{x}$ as above; $\mathcal{F}$ corresponds to a representation $\pi_1(X,\overline{x}) \to \text{GL}(\mathcal{F}_{\overline{x}})$. As the latter is a finite group, the kernel of this representation is of finite index, and as the Galois coverings are cofinal, we can find a Galois cover of $X$ corresponding to an open subgroup contained in the kernel. The sheaf $\mathcal{F}$ becomes constant when restricted to this cover.

The case of lisse sheaves. Let $E \subset \overline{Q}_\ell$ be an finite extension of $Q_\ell$ with valuation ring $R$, integral closure of $Z_\ell$ in $E$, $m$ the maximal ideal of $R$. Every $\overline{Q}_\ell$-sheaf $\mathcal{G}$ is
obtained as \( \mathcal{F} \otimes_{E} \overline{\mathbb{Q}}_{\ell} \) for some \( E \) and some torsion-free (i.e. flat) constructible \( E \)-sheaf \( \mathcal{F} \). This means that \( \mathcal{F} = \text{"lim proj"} \mathcal{F}_{n} \), the latter a flat \( R \)-sheaf. A lisse \( R \)-sheaf has all the \( \mathcal{F}_{n} \) locally constant sheaves of \( R/m^{n} \)-modules, and for each \( n \), the above discussion shows that the functor ‘fiber at \( \overline{x} \)’ gives an equivalence of categories between the category of lisse \( R/m^{n} \)-sheaves and the category of \( R/m^{n} \)-modules of finite type together with a continuous action of \( \pi_{1}(X, \overline{x}) \). Since the \( \mathcal{F}_{n} \) have \( \mathcal{F}_{n} \otimes_{R/m^{n+1}} R/m^{n} \to \mathcal{F}_{n-1} \), by passing to the limit we get an equivalence between the category of lisse \( R \)-sheaves and the category of finite \( R \)-modules with continuous action of \( \pi_{1}(X, \overline{x}) \).
Bibliography

[EGA]  *Eléments de géométrie algébrique* par A. Grothendieck.


[SGA5] SGA 5, Exposé XV par C. Houzel.

2. Quasi-unipotent monodromy

Some notes about Grothendieck’s theorem on quasi-unipotent monodromy. We study the arithmetic proof. It uses a proposition proved by Grothendieck in the appendix of Serre and Tate’s article *Good Reduction of Abelian Varieties*.

We may assume that $K$ is complete since, following Serre, *Corps Locaux*, II§3 Cor. 4, completing $K$ leaves the decomposition unchanged. Now, we may assume that any matrix in the image of $\rho$ has coefficients in $\mathbb{Z}_l$ and is congruent to 1 mod $l^2$ as these are both open conditions, $\rho$ is continuous, and we are free to pass to an open subgroup of $I(\overline{\nu})$ by making a finite extension of $K$.

Note also that $\text{im} \rho$ is a pro-$l$ group since, while $\text{GL}(n, \mathbb{Z}_l)$ is not a pro-$l$-group, its first congruence subgroup of matrices congruent to 1 mod $l$ is a pro-$l$-group (c.f., e.g., §5.1 of *Analytic Pro-$p$ Groups* by Dixon, du Sautoy, Mann & Segal). We see therefore that the prime-to-$l$ part of the order of $\text{GL}(n, \mathbb{Z}_l)$ is finite. As the image of a pro-$p$ group under a continuous homomorphism is pro-$p$, the continuous image of a pro-$p$ group in $\text{GL}(n, \mathbb{Z}_l)$ is finite. As $\text{im} \rho$ is by construction pro-$l$, the image of a pro-$p$ group in $\text{im} \rho$ is $\{1\}$.

Now, if $L$ is a finite extension of $K_v$, we wish to show that the polynomial $f(T) = T^l - a$ splits in $L$ for any $a \in L$. If it does not, then as $f$ is separable and $L$ contains all $l^{th}$ roots of unity, $L(\sqrt[l]{a})$ is the splitting field of $f$ and is Galois. The automorphism of $L(\sqrt[l]{a})/L$ sending $\sqrt[l]{a} \mapsto \zeta_l \sqrt[l]{a}$, where $\zeta_l$ is a primitive $l^{th}$ root of unity, acts transitively on the roots of $f$, hence $f$ is irreducible. But $K_v$ is the $l$-part of the maximal tamely ramified extension of $K_{nr}$, hence $l$ cannot divide $[L : K_v]$. (Recall that $K_v$ is the maximal tamely ramified extension of $K$, and we have

$$\text{Gal}(K_v/K_{nr}) \simeq \prod_{q \neq p} \mathbb{Z}_q$$

$$\text{Gal}(K_v/K_v) \simeq \prod_{q \neq p, l} \mathbb{Z}_q$$

$$\text{Gal}(K_{s}/K_v) \text{ a pro-$p$ group}$$

as $q$ runs over primes, so $\text{Gal}(K_s/K_v)$ is an extension of a group isomorphic to $\prod_{q \neq p, l} \mathbb{Z}_q$ by a pro-$p$ group, and therefore has no finite quotient of order divisible by $l$.) This allows one to conclude that $l$ does not divide the order of $\text{Gal}(K_s/K_v)$. The order of $\text{im} \rho$ is a power of $l$ as it is a pro-$l$ group.
An alternative way to see that \( l \) does not divide the order of \( \text{Gal}(K_s/K_l) \) that is more faithful to the original proof proceeds by showing directly that for a finite extension \( L/K_l \), every element of \( L \) is an \( l \)th power. To do this, let \( L = K_l[t]/a(t) \) for \( a(t) \) an irreducible separable polynomial \( a(t) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0 \), and suppose \( t \) is not an \( l \)th power in \( L \). This implies that the polynomial \( a_l(t) = a_n x^{ln} + a_{n-1} x^{l(n-1)} + \ldots + a_0 \) is irreducible and separable over \( K_l \). Let \( K'/K \) be a finite Galois extension containing all \( l \)th roots of unity and the \( a_i \), and contained in \( K_l \). The extension \( K'[t]/a_l(t) \) is finite and separable and is contained in a finite Galois extension \( K'' \) of \( K' \) with \( l \) dividing \( [K'':K'] \), so \( l \) divides the ramification index or the residual degree. If the latter, making a finite unramified extension of \( K' \) produces a contradiction on the irreducibility of \( a_l(t) \) over \( K_l \). If the former, replacing \( K'' \) by \( K'[\pi^{1/l}] \subset K_l \) similarly yields a contradiction. Now to see that every finite Galois extension \( L \) of \( K_l \) cannot have \( l \) dividing its degree, note that \( \text{Gal}(L/K_l) \) contains a cyclic subgroup \( H \) of order \( l \), and we claim \( L = L^H(a^{1/l}) \) for some element \( a \in L^H \). Let \( \sigma \) generate \( H \), and let \( b \in L - L^H \). Then the element

\[
c = \sum_{m=1}^{l} \zeta_l^m \sigma^m(b)
\]

satisfies \( c = \zeta_l \sigma(c) \), where \( \zeta_l \) is a primitive \( l \)th root of unity. So

\[
c^l = \prod_{m=1}^{l} \sigma^m(c) \in L^H,
\]

and letting \( a = c^l \) we find that \( L^H(a^{1/l}) \) is a nontrivial subextension of \( L^H \), hence must actually coincide with \( L \).
Bibliography

3. WEIL II

0.5. Let $X$ be a scheme of finite type over a field $k$. If $X$ is connected, the structure
morphism $X \to \text{Spec}(k)$ admits a unique factorization $X \to \text{Spec}(k') \to \text{Spec}(k)$ with
$k'/k$ finite separable and $X \to \text{Spec}(k')$ geometrically connected. \(\sim\) \textit{Stacks, tag 04PZ.}
Proof uses notion of ‘weakly étale $k$-algebra.’

1.1.2. First of all, to see that if $K \in D^b_c(X, R)$, then $K \otimes^L R/m^n \in D^b_{cft}(X, R/m^n)$, see \textit{Stacks, tag 0942}. Note that $K \otimes^L R/m^n$ can be represented by a bounded complex of
flat constructible sheaves by \textit{Rapport}, 4.7. Also recall that the locally constant sheaves
form a weak Serre subcategory of the constructible sheaves on a site \((093U)).

Claim a). On the subject of the category $D^b_c(X, R)$, claim a) is that for each $i$, the
projective system of cohomology sheaves $H^i(K) := \text{“lim proj”} H^i(K \otimes^L R/m^n)$ of a
complex $K$ in $D^b_c(X, R)$ is an $R$-constructible sheaf. First a trivial statement: of course
the reduction modulo $m^n$ of a complex of flat sheaves representing $K_{n+1}$ induces a map
on cohomology, but a priori it need not induce an isomorphism $H^i(K_{n+1}) \otimes R/m^n \to
H^i(K_n)$. For example, in

\[
\begin{array}{c}
0 \longrightarrow \mathbb{Z}/\ell^n \longrightarrow \mathbb{Z}/\ell^{n-1} \longrightarrow \mathbb{Z}/\ell^n \longrightarrow 0 \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
0 \longrightarrow \mathbb{Z}/\ell^{n-1} \longrightarrow 0 \quad \longrightarrow \mathbb{Z}/\ell^{n-1} \longrightarrow \mathbb{Z}/\ell^n \longrightarrow 0
\end{array}
\]

the first nonzero cohomology in the top row is $\ell\mathbb{Z}/\ell^n \simeq \mathbb{Z}/\ell^{n-1}$, which gets mapped by
the down arrow to $\ell\mathbb{Z}/\ell^{n-1} \simeq \mathbb{Z}/\ell^{n-2}$, even though the first nonzero cohomology in the
bottom row is all of $\mathbb{Z}/\ell^{n-1}$. Moreover, $\mathbb{Z}/\ell^{n-1} \simeq \mathbb{Z}/\ell^{n-1} \otimes_{\ell^{n-1}} \mathbb{Z}/\ell^n$, so the map on
cohomology after reduction mod $\ell^{n-1}$ is neither injective nor surjective.

It is important to note that in (1.1.1), the term ‘$R$-faisceau constructible’ is used to
describe all pro-sheaves in the essential image of the functor $\mathcal{F} \mapsto \text{“lim proj”} \mathcal{F} \otimes R/m^n$.
In Exposé V of SGA 5, Jouanolou studies $J$-adic projective systems, where $J$ is an ideal
in a commutative ring $A$. All references in this paragraph will be to this exposé unless
indicated otherwise. There is a conflict of indexing, in that Deligne in Weil II has $K_n$
annihilated by $m^n$, while Grothendieck, Jouanolou and Deligne in SGA 4 1/2 have $K_n$
annihilated by \( n^{n+1} \). As a predictable but no less unfortunate result, both conventions are effectively in force in different parts of these notes. Jouanolou begins with an abelian category \( \mathcal{C} \) and forms \( P = \text{Hom}(\mathbb{N}^\circ, \mathcal{C}) \), the category of projective systems indeed by the ordered set \( \mathbb{N} \) of positive integers with values in \( \mathcal{C} \). Given an object \( X = (X_n, u_n)_{n \geq 0} \) of \( P \) and an integer \( r \geq 0 \), \( X[r] \) denotes the projective system \( (X_{n+r}, u_{n+r})_{n \geq 0} \). If \( r, s \) are integers satisfying \( s \geq r \geq 0 \), then iterated application of the transition morphisms \( u \) define a morphism \( w_{sr} : X[s] \to X[r] \). For each integer \( r \geq 0 \) and \( X \) in \( P \), the morphism \( w_{r0} : X[r] \to X \) is also denoted \( V_rX \). With this notation, \( X \) is said to satisfy the condition Mittag-Leffler-Artin-Rees (MLAR) if there exists an integer \( r \) such that for each integer \( s \geq r \),

\[
\text{im}(X[s] \xrightarrow{w_{sr}} X) = \text{im}(X[r] \xrightarrow{w_{r0}} X).
\]

In particular, if there exists an \( r \geq 0 \) such that the canonical morphism \( X[r] \to X \) is null, \( X \) is said to be AR-null. The full subcategory \( P_0 \) of \( P \) whose objects are the AR-null projective systems is thick. The quotient category is called the category of projective systems in \( \mathcal{C} \) up to translation and notated \( \text{Hom}_{\text{AR}}(\mathbb{N}^\circ, \mathcal{C}) \) or \( P_{\text{AR}} \). It is abelian and the quotient functor \( p_{\text{AR}} : P \to P_{\text{AR}} \) is exact (2.4.4). Equivalently, \( P_{\text{AR}} \) is obtained from \( P \) by a (left or right) calculus of fractions with respect to the set of morphisms \( \{V_rX\} \), as \( X \) runs over the set of objects in \( P \) and \( r \) over the set of integers \( \geq 0 \). Denote this set by \( \text{AR} \). Now suppose \( \rho \) equips \( \mathcal{C} \) with the structure of \( A \)-category (1.1) and either that \( J \) is of finite type or that \( \mathcal{C} \) possesses infinite direct sums.

**Definition (3.1.1).** An object \( X \) of \( P \) is called \( J \)-adic if the following two conditions are verified.

(i) For every integer \( n \geq 0 \), \( J^{n+1}X_n = 0 \).

(ii) For every couple \( (m,n) \) of integers with \( m \geq n \geq 0 \), the morphism \( A/J^{n+1} \otimes_A X_m \to X_n \) deduced from the transition morphism \( X_m \to X_n \) is an isomorphism.

If moreover the components of \( X \) are noetherian objects of \( \mathcal{C} \), then \( X \) is called noetherian \( J \)-adic.

The full subcategory of \( P \) (resp. of \( P_{\text{AR}} \)) generated by the \( J \)-adic projective systems (resp. the images of the noetherian \( J \)-adic systems) is notated \( J_{\text{ad}}(\mathcal{C}) \) (resp. \( J_{\text{adn}}(\mathcal{C}) \)).
Two objects \(X\) and \(Y\) of the category \(P\) are said to be AR-isomorphic if \(p_{\text{AR}}(X)\) and \(p_{\text{AR}}(Y)\) are isomorphic in the category \(P_{\text{AR}}\).

**Definition (3.2.1-3.2.2).** An object \(X\) of \(P\) is called AR-J-adic if it satisfies the following conditions.

(i) \(J^{n+1}X_n = 0\) for all \(n \geq 0\).

(ii) There exists a J-adic projective system \(Y\) isomorphic to \(X\) in the category \(P_{\text{AR}}\).

If moreover the components of \(X\) are noetherian objects of \(\mathcal{C}\), then \(X\) is called noetherian AR-J-adic.

The full subcategory of \(P\) (resp. of \(P_{\text{AR}}\)) generated by the noetherian AR-J-adic projective systems (resp. by their images) is denoted \(E_{\mathcal{C}}\) (resp. AR-J-adn(\(\mathcal{C}\))).

**Proposition (3.2.3).** — Let \(X\) be in \(P\). Suppose \(J^{n+1}X_n = 0\) for all \(n \geq 0\). In order for \(X\) to be AR-J-adic, it is necessary and sufficient that it verify the property (MLAR) and that, denoting by \(X'\) its projective system of universal images, there exist an integer \(r \geq 0\) such that, for each pair \((m, n)\) of integers with \(m \geq n + r\), the ‘transition morphism’ below be an isomorphism:

\[
X'_m / J^{n+1}X'_m \to X'_{n+r} / J^{n+1}X'_{n+r}.
\]

Note that if \(X\) verifies (MLAR), it is AR-isomorphic to its projective system of universal images. The hypothesis made on \(X'\) implies that the projective system \((X'_{n+r} / J^{n+1}X'_{n+r})_{n \geq 0}\) is J-adic. This projective system is AR-isomorphic to \(X'\). This proves sufficiency.

The restriction of the functor \(p_{\text{AR}}\) to J–ad or J–adn induces an equivalence

\[
p^n_{\text{AR}} : \text{J–adn}(\mathcal{C}) \to \text{AR–J–adn}(\mathcal{C}).
\]

Suppose \(X, Y\) are noetherian J-adic projective systems. Then a morphism \(X \to Y\) is represented for a certain integer \(r\) by a morphism \(X[r] \to Y\). As \(J^{n+1}Y_n = 0\) for all \(n \geq 0\) and \(X\) is J-adic, this morphism is the composition of a morphism \(X \to Y\) with \(X[r] \to X\). Hence \(p^n_{\text{AR}}\) is full. Moreover, a given morphism \(X \to X\) goes to zero under
$p^n_{AR}$ if it goes to zero in the inductive limit

$$\lim_{r \to} \text{Hom}(X[r], Y);$$

i.e. if precomposition by $V_{r, X} : X[r] \to X$ is null for some $r$. Such is not the case when $X$ is J-adic. Thus $p^n_{AR}$ is faithful, and, as AR-J-adn($\mathcal{C}$) is evidently the essential image, an equivalence.

**Proposition (5.2.1).** — The category $\mathcal{E}_\mathcal{C}$ is stable by kernels and cokernels in $\mathcal{P}$. In other words, $\mathcal{E}_\mathcal{C}$ is an abelian category and the inclusion functor $\mathcal{E}_\mathcal{C} \to \mathcal{P}$ is exact.

**Theorem (5.2.3).** — The categories $J$-adn($\mathcal{C}$) and AR-J-adn($\mathcal{C}$) are abelian and noetherian.

We have enough to prove the first statement. (5.2.1) implies on the spot that the category AR-J-adn($\mathcal{C}$) is abelian: given an arrow $A \to B$ in AR-J-adn($\mathcal{C}$), up to isomorphism of $A$ and $B$ this arrow comes from an arrow in $\mathcal{P}$ with kernel and cokernel in $\mathcal{E}_\mathcal{C}$; as the functors $\mathcal{E}_\mathcal{C} \to \mathcal{P} \to \mathcal{P}_{AR}$ are exact, the kernel and cokernel lie in AR-J-adn($\mathcal{C}$). Therefore J-adn($\mathcal{C}$) must also be abelian as the two categories are equivalent.

**Remark.** If $A$ is a noetherian (commutative) ring complete and separated with respect to an ideal $J$ such that $A/J$ is artinian, then the following is true and provides a kind of ‘spiritual underpinning’ for the category J-adn.

**Proposition.** — The functor $\lim_{\leftarrow}$ induces an equivalence between the categories AR-J-adn($A$-mod) and the category of finite $A$-modules.

Specializing to the category $\text{Ab}(X)$ of abelian sheaves on $X_{\text{et}}$, $X$ a scheme, when $X$ is noetherian, the abelian noetherian sheaves are the abelian constructible sheaves [SGA4, IX 2.9]. Let $\text{Abc}(X)$ denote the category of abelian constructible sheaves and $\ell$–ad($\mathcal{C}$) the full subcategory of $\text{Hom}(N^\circ, \text{Abc}(X))$ generated by the constructible $\ell$-adic sheaves. Then we have shown that, when $X$ is noetherian,

$$\ell$–ad($X$) = ($\ell Z$–ad($\text{Abc}(X)$) = ($\ell Z$–adn($\text{Ab}(X)$),

the first equality holding with no assumptions on $X$. In $\mathcal{P}$, given an exact sequence $0 \to X \to Y \to Z \to 0$ with $X$ strict and $Y$ J-adic, $Z$ is J-adic. Let $u : \mathcal{T} \to \mathcal{G}$ be a
morphism of $\ell$-adic sheaves on a scheme $X$. We apply the above formalism in the case $A = R$, $J = m$, $C = \text{Ad}(X)$, $P = \text{Hom}(\mathcal{N}^\circ, \text{Ad}(X))$. Let $\mathcal{F}' := \text{im} u$ and $\mathcal{G}' := \text{coker} u$ computed in the category $P$. Then the following diagram has exact rows and commutes.

As $f_{nm}$ is an epimorphism and $g_{nm}$ an isomorphism, the snake lemma implies that $h_{nm}$ is an isomorphism; therefore $\mathcal{G}'$ is a cokernel of $u$ in $\text{m-adc}(X)$. On a noetherian $X$, we have the following simple description of $\ker u$. Let $\mathcal{K} := \ker u$ in $P$; $\mathcal{K}$ is $\text{AR}-m$-adic. Denoting by $\mathcal{K}'$ the system of universal images of $\mathcal{K}$, grâce à (3.2.3) there exists an integer $r$ such that the projective system

$$m_r(\mathcal{K}') := (\mathcal{K}'_{n+r}/\ell^{n+1} \mathcal{K}'_{n+r})_{n \in \mathbb{N}}$$

is $m$-adic constructible. The composition below is a kernel of $u$ in $\text{m-adc}(X)$:

$$m_r(\mathcal{K}') \longrightarrow \mathcal{K}' \longrightarrow \mathcal{K}.$$
i : X → U → X the inclusion of the complement, by the stability of the property AR-
m-adic in short exact sequences \[ \text{SGA}5, \text{V}, 3.2.4 \], in order to conclude that \( \mathcal{H}_n \) is an
(AR,R)-constructible sheaf suffices to show that the projective systems \( (j^*j_!, \mathcal{H}_n) \) and
\( (i^*i_!, \mathcal{H}_n) \) are (AR,R)-constructible. This allows us to reduce to the situation where \( \mathcal{H}_n \)
is a projective system of locally constant constructible sheaves, since the functors \( i_*, j_! \); send m-adic sheaves to m-adic sheaves and AR-null sheaves to AR-null sheaves (this can be checked pointwise), now use \[ \text{SGA}5, \text{V}, 2.4.5 \]. By ‘gluing’ (c.f. proof of \[ \text{SGA}5, \text{VI}, 1.5.5 \]), we can reduce to proving over an open cover trivializing \( \mathcal{H}_0 \). More concretely,
we can cover our space by finitely many opens over which \( \mathcal{H}_0 \) is constant, and if we can
show that \( \mathcal{H}_0 \) is AR-m-adic over each, then the construction (3.2.3) above allows us to
find an integer for each open in our cover; taking the maximum \( r \) of these, replacing
\( \mathcal{H}_n \) by its system of universal images \( \mathcal{H}'_n \), and forming \( \mathcal{H}'_{n+r}/m^{n+1}\mathcal{H}'_{n+r} \), we have produced an R-constructible sheaf which is AR-isomorphic to the system \( \mathcal{H}_n \).

In the punctual case, the results of \[ \text{SGA}5 \text{ XV p. 473} \] allow us to suppose that
we have a projective system \( (K_r)_{r \in \mathbb{N}} \), where \( K_r \) is a complex of free \( R/m^r \)-modules
of finite type, null outside of an interval \([a, b]\) independent of \( r \), with transition morphisms
\( K_{r+1} \to K_r \) isomorphic (as morphisms of complexes) to \( K_{r+1} \to K_{r+1}/m^{r+1}K_{r+1} \), for
all \( r \in \mathbb{N} \). We will show that the projective system of cohomology \( \text{lim proj} H^i(K_r) \) is
AR-m-adic, which will imply that it is an R-constructible sheaf in Deligne’s sense; i.e.
that it is isomorphic, as a pro-object, to a bona fide R-constructible sheaf. Abusively, put
\( K := \lim \leftarrow K_r \) (before, \( K \) refers to the stalk of an object in \( D^\beta(X, R) \)). A key ingredient
is EGA 0 \text{III} 13.2.3, which says that if a projective system of complexes such as \( (K_r) \)
satisfies the Mittag-Leffler condition, and if the projective system \( (H^{i-1}(K_r)) \) does too,
then the canonical map \( H^i(K) \to \lim \leftarrow H^i(K_r) \) is bijective. As in our situation, the \( K_r \)
are complexes of finite groups and their cohomology modules \( H^i(K_r) \) are also finite
groups, the Mittag-Leffler condition is automatic. Our hypotheses on the complexes \( K_r \)
imply the existence of isomorphisms \( K_{r+s} \otimes_{R/m^r} R/m^s \to K_s \), and the exactness of the sequence
\[ 0 \to K_{r+s} \otimes_{R/m^r} R/m^s \to K_{r+s} \to K_{r+s} \otimes_{R/m^r} R/m^r \to 0 \]
for $1 \leq r,s$. The projective limit as $s \to \infty$ of the associated cohomology sequence is still exact by the fact that everything is still finite (ML). This projective limit can be broken up into short exact sequences

$$0 \to H^i(K)/m^r \to H^i(K_r) \to H^{i+1}(K)[m^r] \to 0.$$  \hspace{1cm} (*)

Note that $\operatorname{Tor}^R_1(M,R/m^r R) \simeq M[m^r]$ for $M$ an $R$-module. Since the modules $H^{i+1}(K)[m^r]$ stabilize as $r \gg 0$ and the transition morphisms on $(\operatorname{Tor}^R_1(H^{i+1}(K),R/m^r))_{r \in \mathbb{N}}$ are multiplication by $m$, this projective system is evidently AR-null. As the projective system $(H^i(K)/m^r)_{r \in \mathbb{N}}$ is evidently $m$-adic, $(H^i(K_r))_{r \in \mathbb{N}}$ is AR-$m$-adic, and hence its image in the category of pro-sheaves “$\limproj$” $H^i(K)$ is isomorphic to the image of the $m$-adic system “$\limproj$” $H^i(K)/m^r$, which shows the former is $m$-adic in Deligne’s sense.

**Remark.** To see that each transition morphism on the projective system

$$(\operatorname{Tor}^R_1(H^{i+1}(K),R/m^r))_{r \in \mathbb{N}}$$

is multiplication by $m$, note that by the equivalence $\mathcal{K}^{-}(\mathscr{P}) \to \mathcal{D}^{-}(\mathcal{R}-\text{mod})$, where $\mathcal{K}^{-}(\mathscr{P})$ is the full triangulated subcategory of $\mathcal{K}^{-}(\mathcal{R}-\text{mod})$ generated by the complexes with projective objects in all degrees, there is, up to homotopy, a unique map of projective resolutions of $R/m^r$ and $R/m^{r-1}$ inducing the desired map $R/m^r \to R/m^{r-1}$, namely

$$0 \longrightarrow R \overset{m^r}{\longrightarrow} R \overset{\text{id}}{\longrightarrow} 0$$

Tensoring by $H^{i+1}(K)$ we find that the map on $\operatorname{Tor}^R_1$ is indeed multiplication by $m$.

**Claim c.** Let $f : Y \to X$ be an arrow between schemes of finite type over $S$ with $S$ regular of dimension $\leq 1$. Claim c) is that about the categories $\mathcal{D}^b_{\text{ctf}}(X,R/m^n)$ being stable by the six functors, and that these functors also commute with reduction modulo $m^n$. This can be broken into 3 claims about the six functors: (1) that they preserve constructibility, (2) that they preserve finite tor-dimension, and (3) that they commute with reduction modulo $m^n$. The discussion of Th. finitude 1.1, 1.5–1.7 covers claims 1 & 2 for the four functors $Rf_*, f^*, Rf_!, Rf^!$, as well as $\mathcal{R}\text{Hom}$. Claim 3 for $\mathcal{R}\text{Hom}$ is discussed below. The case of $\otimes^L$ is more or less trivial. Claim 3 for the four functors is
We can localize this morphism with respect to \( u : U \to Y \) étale, as \( R^i u^* = u^* \), which we know commutes with reduction modulo \( m^n \), and in this way, replacing \( Y \) by \( U \), assume \( f \) factors as \( U \hookrightarrow Z \xrightarrow{h} X \) with \( i \) a closed immersion and \( h \) smooth of relative dimension \( d \). As the above morphisms are natural, and the composition of the unit and counit \( R f^! \to R f^! R f_! f^! \to R f^! \) is the identity transformation of \( R f^! \), and \( R f^! = R h^! i^! \) (since the left adjoint \( i_* \) of \( i^! \) is exact), it suffices to show that \( R h^! \) and \( i^! \) commute with reduction modulo \( m^n \). The case of \( R h^! K_{n+1} = K_{n+1}(d[2d]) \). Turning to \( R i^! \), let \( j : Z - U \hookrightarrow X \) denote the open immersion of the complement of \( U \); we have a commutative diagram with distinguished triangles.
for rows

\[ \frac{R/m^n \otimes^{L} i_* R^j K_{n+1}}{\sim} \xrightarrow{d} \frac{R/m^n \otimes^{L} R^j j^* K_{n+1}}{\sim} \frac{R/m^n \otimes^{L} R^j j^* (K_{n+1} \otimes^{L} R/m^n)}{\sim} . \]

We obtain an isomorphism \( R/m^n \otimes^{L} i_* R^j K_{n+1} \sim \to i_* R^j (K_{n+1} \otimes^{L} R/m^n) \) from (TR3).

Returning now to \( \text{RHom} \), here is a useful lemma which adapts Th. finitude 4.6.

**Lemma.** — Let \( X \) be a noetherian scheme, \( \Lambda \) a left noetherian ring, and \( K \in D^{-}(X, \Lambda) \). Then the following are equivalent.

1. \( K \) is of finite Tor-dimension and the sheaves \( \mathcal{H}^i(K) \) are locally constant constructible.

2. \( K \) is locally isomorphic to a bounded complex of locally constant, flat \( \Lambda \)-modules; i.e. there is a finite étale covering \( \{U_i \to X\} \) such that \( K|_{U_i} \) is isomorphic (in \( D^{-}(U_i, \Lambda) \)) to a bounded complex of constant sheaves of projective \( \Lambda \)-modules of finite type.

The argument follows Th. finitude 4.5, except now the \( \mathcal{H}^i(K) \) are moreover locally constant constructible. The sheaves \( A \) and \( B \) are defined by the cartesian diagram

\[
\begin{array}{ccc}
K^n & \rightarrow & K^n/\text{im } d \\
\uparrow & & \uparrow \\
A & \xrightarrow{u} & B \\
& & \uparrow \\
& & \text{ker } d
\end{array}
\]

and \( B \) is locally free as it sits in the middle of the exact sequence

\[ 0 \to \mathcal{H}^n(K) \xrightarrow{(id,0)} B \to \text{ker } d \to \mathcal{H}^{n+1}(K) \]

where \( \text{ker } d \) here denotes the kernel of the differential on the complex \( K' \) being constructed and is hence locally constant constructible.

As \( u \) is surjective, localizing, we may assume \( B \) constant constructible and that \( u \) surjects on global sections, defining a morphism \( v : \Lambda^{@d} \to A \) with the property that \( vu \) is an epimorphism, and we define \( K^{m} = \Lambda^{@d} \).
Suppose $K$ is of Tor-dimension $\leq r$. Propagating the above procedure to the left as far as degree $-r - 1$, we produce an étale morphism $U \to X$, constant constructible sheaves

$$K'_{-r-1} \to K'_{-r} \to K'_{-r+1} \to \cdots$$

with the property that $K'_{-r}/\text{im } d$ is constant constructible and flat. Then, over $U$, the complex

$$\cdots \to 0 \to K'_{-r}/\text{im } d \to K'_{-r+1} \to K'_{-r+2} \to \cdots$$

is quasi-isomorphic to $K$, and has the desired properties.

**Corollary (Th. finitude 1.7).** — If $\Lambda$ is moreover commutative and of torsion, and in the situation of Th. finitude, then

$$R \text{Hom} : D^b_{ctf}(X, \Lambda) \times D^b_{tf}(X, \Lambda) \to D^b_{tf}(X, \Lambda).$$

Following Th. finitude 1.7, as the finite Tor-dimension is stable by $R f_*, R f^!, f^*, R f^!$, devissant the first variable (say, $\mathcal{F}$) relative to a partition of $X$, and using the adjunction

$$R \text{Hom}(j_! \mathcal{F}, \mathcal{G}) \leftrightarrow R j_* R \text{Hom}(\mathcal{F}, R j^! \mathcal{G})$$

for $j : Y \to X$ the inclusion of a locally closed subscheme (c.f. SGA 4, IX 2.5 & XVIII §3.1), one reduces to the situation where the $\mathcal{H}^i(\mathcal{F})$ are locally constant. Localizing, we can replace $\mathcal{F}$ by a bounded complex of constant sheaves of projective $\Lambda$-modules of finite type. As in this case $\text{Hom}$ computes pointwise, we can compute $R \text{Hom}$ with respect to such a complex. Finally, if $N$ and $M$ are $\Lambda$-modules, $N$ projective and $M$ of Tor-dimension $\leq r$, then $\text{Hom}(N, M)$ is of Tor-dimension $\leq r$, which can be seen after replacing $M$ by a complex of flat modules 0 to the left of $-r$ and writing $N$ as a direct summand of a free module.

Returning to the setting of the paper, with $R$ the ring of integers of a finite extension of $\mathbb{Q}_p$, $m$ its maximal ideal, let’s assume that reduction mod $m^n$ commutes with the four
operations $R f_*, f^*, R f_!$, and $R f_!^\natural$, and show that for $\mathcal{F}, \mathcal{G} \in D^b_c(X, R)$,

$$\begin{align*}
R \text{Hom}(\mathcal{F} \otimes^L R/m^{n+1}, \mathcal{G} \otimes^L R/m^{n+1}) \otimes_{R/m^{n+1}} R/m^n &= R \text{Hom}(\mathcal{F} \otimes^L R/m^n, \mathcal{G} \otimes^L R/m^n) \quad \text{in } D^b_{ctf}(X, R/m^n). \quad (\dagger)
\end{align*}$$

Devissant $\mathcal{F} \otimes^L R/m$ with respect to a partition of $X$, we may assume that its cohomology sheaves are locally constant, and therefore the same is true of $\mathcal{F} \otimes^L R/m^{n+1}$ by considering the $m$-adic filtration on a finite complex of flat sheaves representing it. Localizing, we may replace $\mathcal{F} \otimes^L R/m^{n+1}$ with a bounded complex $N^*$ of free $R/m^{n+1}$-modules of finite type and compute $R \text{Hom}$ with respect to $N^*$, since for $\mathcal{F}$ locally free, $\text{Hom}(\mathcal{F}, \mathcal{G})_x = \text{Hom}(\mathcal{F}_x, \mathcal{G}_x)$. Now the equality $(\dagger)$ is clear.

Consider the example: $X$ finite type over $S$, $\mathcal{F}$ and $\mathcal{G}$ two constructible torsion-free $R$-sheaves. The claim is that the projective system

$$\text{Ext}^i(\mathcal{F}, \mathcal{G}) := H^i R \text{Hom}(\mathcal{F}, \mathcal{G}) = \lim \text{proj } \text{Ext}^i_{R/m^n}(\mathcal{F} \otimes R/m^n, \mathcal{G} \otimes R/m^n)$$

forms a constructible $R$-sheaf. By part (a) of 1.1.2, it suffices to show that $R \text{Hom}(\mathcal{F}, \mathcal{G}) \in D^b_c(X, R)$. By Th. finitude 1.6 and the previous corollary, $R \text{Hom}$ sends $D^b_{ctf}(X, R/m^n) \times D^b_{ctf}(X, R/m^n)$ into $D^b_{ctf}(X, R/m^n)$, so $R \text{Hom}(\mathcal{F} \otimes R/m^n, \mathcal{G} \otimes R/m^n) \in D^b_{ctf}(X, R/m^n)$. Finally, by $(\dagger)$,

$$\lim \text{proj } R \text{Hom}(\mathcal{F} \otimes R/m^n, \mathcal{G} \otimes R/m^n) \in D^-(X, R).$$

**Truncation & Tor-dimension.** In part (e), Deligne addresses the truncation operators $\tau_{\leq n}$. The issue is that, while a submodule of a flat $R$-module is flat, a submodule of a flat $R/m^n$-module need not be. To address this deficiency, Deligne introduces the modified truncation operators $\tau'_{\leq n}$, which preserve the finite Tor-dimension. As these properties are of a pointwise nature, we may consider the situation in the category of $R$-modules, and the categories $D_{\text{parf}}(R), D_{\text{parf}}(R/m^k)$. Applying Houzel’s argument at the end of SGA 5, Exp. XV to the stalk of $K \in D^b_c(X, R)$, we may represent $K_k = K \otimes^L R/m^k$ by a bounded complex of free $R/m^k$-modules and the isomorphisms $K_{k+1} \otimes_{R/m^{k+1}} R/m^k \to K_k$ by isomorphisms of complexes $K_{k+1} \otimes R/m^k \to K_k$. Taking the projective limit of these complexes, we obtain a bounded complex of free $R$ modules which we will again, as in the notes to claim a), (abusively) notate $K$. (To see freeness, note that if $r$ equals the
rank of $K_i^j = K^j \otimes R/m$, there exists by Nakayama an exact sequence

$$N \to R^r \to K^j \to 0$$

which after tensoring by $R/m^k$ induces an isomorphism $(R/m^k)^r \sim K^j_k$, showing $N \subseteq m^kR$ for all $k$ and hence $N = 0$.) As before, by the Mittag-Leffler condition, $H^i(K) = \lim_{\leftarrow} H^i(K_k)$. Therefore, the submodule of $\ker d$ consisting of cycles whose image in $H^i(K_k)$ are in $\text{im}(H^r K \to H^r(K_k))$ is the reduction modulo $m^k$ of a flat $R$-module, as these cycles coincide with the reduction mod $m^k$ of cycles of $K^j$, which form a free submodule of $K^j$. More precisely, in view of the commutative diagram

$$
\begin{array}{c}
K^{n-1} \xrightarrow{d^{n-1}} K^n \xrightarrow{d^n} K^{n+1} \\
\downarrow \quad \downarrow \quad \downarrow \\
K^{n-1} \otimes R/m^n \xrightarrow{d^{n-1} \otimes \text{id}} K^n \otimes R/m^k \xrightarrow{d^n \otimes \text{id}} K^{n+1} \otimes R/m^k
\end{array}
$$

the submodule of $\ker (d^i \otimes \text{id})$ consisting of cycles whose image in $H^i$ lie in $\text{im}(H^r(K) \to H^r(K_k))$ is the submodule $(\ker d^n) \otimes R/m^k + \text{im}(d^{n-1} \otimes \text{id}) \subset K^n \otimes R/m^k$. As $\text{im}(d^{n-1} \otimes \text{id}) \subset (\ker d^n) \otimes R/m^k$, confirming the above description and recognizing $\tau'_{\leq n} K_k$ as a bounded complex of free $R/m^k$ modules. It may be that $\tau'_{\leq n} K_k$ is no longer quasi-isomorphic to $K_k$, but it is clear by construction that the pro-sheaf $H^n(K_k)$ is not affected by the operator $\tau'_{\leq n}$; more precisely, by the exact sequence (*) of claim a), $H^n(\tau'_{\leq n} K_k) \simeq H^n(K) \otimes_R R/m^k$ and hence both are AR-isomorphic to $H^n(K_k)$.

1.1.3. Extension of scalars from $R$ to $E$ for a constructible $R$-sheaf $\mathcal{F}$ is more or less straightforward: letting $E = Q_\ell, R = Z_\ell$, the idea is that if is $Z_\ell$-constructible, the torsion subsheaves $\text{Tor}_1^Z(\mathcal{F}, Z/\ell^n) = \ker \ell^n$ stabilize for some $n$, and then multiplication by $\ell$ on $\mathcal{F}$ means multiplying $\mathcal{F} \otimes Z/\ell^m$ by $\ell^n$ for each $m$ to produce an AR-$\ell$-adic sheaf, and then forming the associated $\ell$-adic sheaf, to produce the exact sequence of $\ell$-adic sheaves

$$0 \to \text{Tor}_1^Z(\mathcal{F}, Z/\ell^n) \to \mathcal{F} \xrightarrow{\ell^n} \mathcal{F}/\ker \ell^n \to 0.$$ 

The story in $D^b_c(X, Z_\ell)$ works more or less the same way. For an object $K$ in this category, we represent $K_m = K \otimes^L Z/\ell^m$ by a complex of $Z/\ell^m$-flat sheaves (i.e. with free stalks);
multiplication by $\ell^n$ means multiplying $K_m$ by $\ell^n$. This induces multiplication by $\ell$ on the cohomology, and we’re back in the previous situation, since the cohomology sheaves form AR-$\ell$-adic systems. More generally, if $a$ is in $\mathbb{Z}_\ell$, letting $a_m$ denote the image of $a$ in $\mathbb{Z}/\ell^m$, the commutativity of the diagram

\[
\begin{array}{ccc}
K_{m+1} & \overset{a_{m+1}}{\longrightarrow} & K_{m+1} \\
\downarrow & & \downarrow \\
\mathbb{Z}/\ell^m & \overset{a_m}{\longrightarrow} & \mathbb{Z}/\ell^m \\
K_m & \overset{a_m}{\longrightarrow} & K_m
\end{array}
\]

shows that $a$ induces an endomorphism of $D^b_c(X, \mathbb{Z}_\ell)$.

1.1.7. A representation $\mathbb{Z} \to \text{GL}(V)$, sending $n$ to $F^n$, where $V$ is a $\mathbb{Q}_\ell$-vector space of dimension $n$, is continuous if and only if the eigenvalues of $F$ are $\ell$-adic units.

To see sufficiency, note that we can choose a basis for $V$ so that the morphism $\mathbb{Z} \to \text{GL}(V)$ factors through $\text{GL}(\mathbb{Z}_\ell^n) = \lim_{\leftarrow} \text{GL}((\mathbb{Z}/\ell^m)^n)$, a profinite group, and we may extend $\mathbb{Z} \to G$ to a morphism $\hat{\mathbb{Z}} \to G$ for any profinite group $G$ by the universal property of profinite completion, which we state and prove now.

The profinite completion of a group $H$ (with respect to normal subgroups of finite index in $H$) is denoted $\hat{H}$, so that $H \to \hat{H}$ has dense image. The profinite completion $\hat{H}$ enjoys the universal property that for every profinite group $G$ and continuous homomorphism $H \to G$, there is a unique homomorphism $\hat{H} \to G$ making the diagram

\[
\begin{array}{ccc}
H & \longrightarrow & \hat{H} \\
\downarrow & & \downarrow \\
G & \longrightarrow & 
\end{array}
\]

commute.

To see this, simply use the description of $G$ as $\lim G/N$ as $N$ ranges over open normal subgroups of $G$. The preimage $M$ of $N$ in $H$ is an open normal subgroup of finite index, as $G/N$ is finite. Therefore $H \to G/N$ factors through $H/M$, and to give a continuous morphism from $\hat{H}$ to $G$ it suffices to give compatible continuous maps $\hat{H} \to G/N$. Continuity is assured by the above remark; compatibility is assured by the map $H \to G$, which determines the maps $\hat{H} \to G/N$. 
Returning to (1.1.7), to see necessity, we assume we have found a continuous extension $\rho$.

$$
\begin{array}{ccc}
\mathbb{Z} & \longrightarrow & \text{GL}(V) \\
\downarrow \rho & & \nearrow \\
\hat{\mathbb{Z}} & & \\
\end{array}
$$

The image $\rho(\hat{\mathbb{Z}})$ is compact, so the set $\rho(\hat{\mathbb{Z}})^n$ is compact, for $\mathbb{Z}_\ell^n$ a $\mathbb{Z}_\ell$-lattice in $V$, so $\rho(\hat{\mathbb{Z}})\mathbb{Z}_\ell^n \subset \frac{1}{\ell^m}\mathbb{Z}_\ell^n$ for some $m$. Letting $L$ denote the $\mathbb{Z}_\ell$-span of $\rho(\hat{\mathbb{Z}})^n$, $L$ is a $\mathbb{Z}_\ell$-submodule of $\frac{1}{\ell^m}\mathbb{Z}_\ell^n$, hence free (of rank $n$). This recognizes $F \in \rho(\hat{\mathbb{Z}})$ as an element of $\text{Aut}(L)$, so the eigenvalues of $F$ are $\ell$-adic units indeed.

1.2.6. ‘On notera que, pour $k$ un corps fini, une representation $V$ de $W(\overline{k}, k)$ est automatiquement $\iota$-mixte.’ This follows from the existence of Jordan normal form.

1.3.9. (When reading the corollary, recall that a semisimple algebraic group is connected by definition.) We wish to understand why $G^{00}$ is reductive. Note first that the sum of the simple $\pi_1(X, \overline{x})$-modules is $W$-stable since if $w \in W$ and $V$ is a $\pi_1(X, \overline{x})$-module, then $wV$ is again a $\pi_1(X, \overline{x})$-module since $\pi_1(X, \overline{x})$ is a normal subgroup of $W$; applying this argument with $w^{-1}$ shows that $wV$ is simple iff $V$ is. Next observe that if $\rho : W(X_0, \overline{x}) \to \text{GL}(\mathcal{F}_\pi)$ is the representation defining $\mathcal{F}_0$, then $\rho(\pi_1(X, \overline{x}))$ and its Zariski closure $G^0$ have the same invariant subspaces (to see this, form a basis for $\mathcal{F}_\pi$ beginning with a basis for an invariant subspace). Therefore we see that $G^0$ acts semisimply since $W(X_0, \overline{x})$ does by assumption.

Now recall that $R(G^0)$, the radical of $G^0$, is a connected and solvable normal subgroup of $G^0$. By the argument above, any normal subgroup of $G^0$ acts semisimply; combining this with the Lie-Kolchin theorem, we see that $\mathcal{F}_\pi$ decomposes as a direct sum of one-dimensional irreducible $R(G^0)$-modules. The unipotent part of $R(G^0)$, which is the unipotent radical $R_u(G^0)$, must therefore act by the identity, and we see that $R_u(G^0) = R_u(G^{00}) = \{1\}$; i.e. that $G^{00}$ is reductive. Note we have proved the following

**Lemma.** — *If $V$ is a finite-dimensional vector space over an algebraically closed field and $G$ is a closed subgroup of $\text{GL}(V)$, then $G$ is reductive.*
This result appears in [Milne] 21.60 as

**Proposition.** — Let $G$ be a connected group variety over a perfect field $k$. The following conditions on $G$ are equivalent.

1. $G$ is reductive;
2. The radical $R(G)$ of $G$ is a torus;
3. $G$ is an almost-direct product of a torus and a semisimple group;
4. $G$ admits a semisimple representation with finite kernel.

More is true. In fact, for $G$ a connected reductive group, say, over an algebraically closed field $k$, the maximal central $k$-torus $Z$ coincides with $(G^0)^\circ$, the connected component of the identity of the center of $G$, and the multiplication homomorphism $Z \times \mathcal{D}(G) \to G$ is a central isogeny, i.e. an isogeny with central kernel, where $\mathcal{D}(G) = (G,G)$ is the derived subgroup. This implies that $Z \to G/\mathcal{D}(G)$ is a central isogeny. Here, our $G$ is Deligne’s $G^{00}$, our $Z$ is Deligne’s $T_1$, and our $G/\mathcal{D}(G)$ is Deligne’s $T$, as a connected, smooth, reductive, and commutative group is a torus [Milne] 19.12, and a quotient of a reductive group over a field of characteristic 0 is reductive.

The set $F$ of characters by which $T_1$ acts on $\mathcal{F}_\overline{\kappa}$ generates $X(T_1)$ since the representation of $T_1$ is faithful, and, as $T_1$ is a torus, diagonalizable. Therefore, with the right choice of basis, the representation of $T_1$ looks like diag($\chi_i$) for characters $\chi_i \in F$. As the representation is faithful, these characters generate the character group $X(T_1)$. (The character group of diag($\chi_i$), which is isomorphic to the character group of $T_1$, is generated by the $\chi_i$.)

The group $W(X_0,\overline{\kappa})$ acts on $G^0$ by conjugation. Recall that the neutral component of an algebraic group is a characteristic subgroup, and so is the center. Therefore $T_1$, which can be described as the neutral component of $Z(G^{00})$, is acted upon by $W(X_0,\overline{\kappa})$. Recall that the functor $X$ which takes an algebraic group to its character group induces a contravariant equivalence from the category of diagonalizable algebraic groups with the finitely generated commutative groups, and as we have seen, $W(X_0,\overline{\kappa})$ acts on $X(T_1)$ by permuting factors, hence through a finite quotient.
We would like to know why the group of outer automorphisms of $G^{00}$ restricting to the identity on $T_1$ is finite. The group $G^{00}$ admits a maximal split torus $T_2$ so that $(G^{00}, T_2)$ is a split reductive group. The radical $R(G^{00}) = T_1$ is the largest subgroup of the multiplicative group $Z(G^{00})$, so the quotient $Z(G^{00})/R(G^{00})$ is finite \[Milne\] 19.10. Recall the definition of isomorphism of root data \[Milne\] 23.2. An isomorphism $\varphi$ of split reductive groups defines an isomorphism $f$ of root data, and every isomorphism of root data $f$ arises from a $\varphi$, unique up to an inner automorphism \[Milne\] 23.26. Moreover for a split reductive group $(G, T)$ we have a canonical isomorphism $\text{Out}(G) \cong \text{Aut}(X, \Phi, \Delta)$, where the latter is automorphisms of based root data \[Milne\] 23.46. Given such a $\varphi : (G, T) \to (G', T')$, the map $f$ is defined by the formula $f(\chi') = \chi' \circ \varphi|_T$ for $\chi' \in X(T')$ \[Milne\] 23.5. Suppose $\varphi$ is now an automorphism of $(G^{00}, T_2)$ and restricts to the identity on the radical $T_1$. The isomorphism $f$ is a fortiori a central isogeny and its action on $Z\Phi$ (the $Z$-submodule of $X^*(T_2)$ generated by the roots $\Phi$) preserves the base $\Delta$, hence its action on $Z\Phi$ amounts to permuting a finite set. On the other hand, the quotient $T_2/Z(G^{00})$ has character group the subgroup $Z\Phi$ of $X^*(T_2)$ \[Milne\] 21.9, hence the the root lattice $Z\Phi$ has finite index in $X^*(T_2/T_1)$. As $Z(G^{00})/T_1$ is finite, this is enough to conclude that subgroup of $\text{Aut}(X, \Phi, \Delta)$ corresponding to automorphisms of $G^{00}$ which restrict to the identity on $T_1$ is finite, hence that the subgroup of $\text{Out}(G^{00})$ consisting of those automorphisms fixing $T_1$ is also finite.

Now, if $w$ is an element of $W(X_0, \overline{x})$ of degree 1, and $\overline{w}$ the image of $w$ in $\text{GL}(\overline{F}_T)$, $G$ is the semi-direct product of $Z$ by $G^0 = G^{00}$ relative to the action $\text{int}(\overline{w})$ of $Z$ on $G^0$. As this action is given by an interior automorphism of $G^0$, by multiplying $w$ by an element of $\pi(X, \overline{x})$, we make the action of $\text{int}(\overline{w})$ trivial, and recognize $G \cong G^0 \times Z$.

The proof of (1.3.9) follows easily from (1.3.8). Note that once one has reduced to $\mathcal{R}_0$ semisimple, it is easy to see that the radical of $G^{00}$ in this case is trivial, as it is by definition the largest connected solvable normal subgroup variety of $G^{00}$, hence contained in the connected component of the identity of $G^0$, so if $G^0$ is an extension of a finite, hence discrete, group, $R(G^{00})$ lies in the kernel of this extension, namely in the semisimple subgroup, so in fact $R(G^{00}) = \{e\}$, as connected normal subgroup varieties of a semisimple group are semisimple \[Milne\] 21.52.
1.3.10. Note that (iv) should read ‘Le centre de $G$ s’envoie sur un sous-groupe d’indice fini de $\mathbb{Z}$.’ The crux of the direction (iv) $\Rightarrow$ (i) is that, while $G$ is not a priori a linear algebraic group, $G/\mathbb{Z}$, as an extension of a finite group by a linear algebraic group, is.

1.3.12. The central element $g$ acts by a scalar by Schur’s lemma.

1.3.13. (i) The claim rests on the following

Lemma. — Let $X_0, X'_0$ be normal connected schemes of finite type over a field with generic points $\xi, \xi'$ and function fields $K = k(\xi)$ and $K' = k(\xi')$. Let $\Omega, \Omega'$ be algebraically closed extensions of $K, K'$, defining geometric points $a, a'$ of $X_0, X'_0$ centered on $\xi, \xi'$, respectively. If $f : X'_0 \to X_0$ is a dominant morphism, then the image of the induced map $\pi_1(X'_0, a') \to \pi_1(X_0, a)$ is an open subgroup of finite index.

Observing that $\pi_1(X'_0, a')$ acts on $(f^*\mathcal{F})_{a'}$ via the map on $\pi_1$ in the lemma induced by $f$, we see that there is a central element $g \in G'$ of positive degree and a morphism $G' \to G$ sending $g$ to a central element of $G$ of positive degree, and the action of $g$ on $\mathcal{F}_{0,a}$ via this map is the same as the action of $g$ on $(f^*\mathcal{F}_{0,a})_{a'}$.

Proof of lemma. — The extensions $\Omega, \Omega'$ define geometric points $a_1, a'_1$ of $S = \text{Spec}(K)$ and $S' = \text{Spec}(K')$, respectively. The dominant morphism $f : X'_0 \to X_0$ induces an extension of fields $K \subset K'$. Then $\pi_1(S, a_1) \to \pi_1(X_0, a)$ is surjective [SGA 1 V 8.2], and after identifying $\pi_1(S, a_1)$ with $\text{Gal}(K^{\text{sep}}, K)$, the kernel is identified with those automorphisms which fix all finite extensions of $K$ in $\Omega$ which are unramified over $X_0$, and likewise for $\pi_1(S', a'_1) \to \pi_1(X'_0, a')$. If $L$ is an extension of $K$ unramified over $X_0$, then $L \otimes_K K'$ is an extension of $K'$ unramified over $X'_0$ [SGA 1 I 10.4(iii)]. The operation on étale covers of $X_0$ consisting of taking inverse image along $f$ followed by fiber at $a'$ is a fiber functor for $X_0$, hence induces a continuous homomorphism of groups $\pi_1(X'_0, a') \to \pi_1(X_0, a)$ [SGA 1 V 6.2]. The action of $\pi_1(X'_0, a')$ on $(f^*\mathcal{F}_{0,a'})$ is by restriction with respect to this homomorphism. This homomorphism, in turn, is induced by restriction of $\pi_1(S', a'_1) \to \pi_1(S, a)$, since if $L$ is as above, an automorphism $\sigma \in \ker(\pi_1(S', a'_1) \to \pi_1(X'_0, a')$ acts on $L \otimes_K K'$ by the identity as $L \otimes_K K'$ is unramified. As $K'/K$ is finitely generated, $K' \cap K^{\text{sep}}$ is a finite extension of $K$, so the image of
\[ \pi_1(S', a') \rightarrow \pi_1(X_0, a) \] is an open subgroup of finite index isomorphic to the image of \( \text{Gal}(K^\text{sep}/K' \cap K^\text{sep}) \) in \( \pi_1(X_0, a) \). \( \square \)

(ii) Choose a basis for representations corresponding to \( R_0 \) and \( G_0 \) so that Frobenius is upper-triangular in both, and then recall the form of the Kronecker (tensor) product of matrices, which has the property that the Kronecker product of upper-triangular matrices is upper-triangular.

(iii) The claim rests on two observations. The first is that if \( R_0 \) is defined by a representation \( V \) of \( G \), the eigenvalues of any \( g \in G \) coincide with the eigenvalues of \( g \) acting on the semi-simplification of \( V \) with respect to any Jordan-Hölder series. To see this, choose a basis for each graded piece so that \( g \) is upper-triangular, and then order a lift of these bases according to the filtration, beginning with the smallest piece. The second observation is that if we begin with an ordered basis \( (a_i) \) for \( V \) with respect to which \( g \) is upper triangular, then a basis \( B \) for \( \bigwedge a V \) consisting of \( a \)-forms in the \( a_i \) can be found. If the function \( w \) takes an \( a \)-form in the \( a_i \) and outputs the sum of the subscripts which appear (so \( w(a_1 \wedge a_3 \wedge a_4) = 8 \)), then \( g \) is upper-triangular with respect to any ordering of \( B \) which respects the total order \( w \). The claim follows.

1.3.14. It suffices to show that the image of \( W(X_0, x) \) in \( \text{GL}(r, E) \) is bounded by the argument of (1.1.7), which we repeat now. We lose nothing by supposing \( E = Q_{\ell} \), in which case the image \( W \) of \( W(X_0, x) \) in \( \text{GL}(r, E) \) is bounded if it is contained in \( \frac{1}{\ell} \text{GL}(r, Z_{\ell}) \). Applying \( W \) to \( Z'_{\ell} \) and taking the \( Z_{\ell} \) span, we get a free \( Z_{\ell} \)-submodule of \( \frac{1}{\ell} Z'_{\ell} \) of rank \( r \), on which \( W \) acts by automorphisms. This recognizes \( W \) as isomorphic to a subgroup of \( \text{GL}(r, Z_{\ell}) \), a profinite group to which it is easy to extend a map \( W(X_0, x) \rightarrow \text{GL}(r, Z_{\ell}) \) to a map from the completion \( \pi_1(X_0, x) \rightarrow \text{GL}(r, Z_{\ell}) \).

To see that \( \rho(W'_1) \subset G^{00} \) is compact and Zariski dense, observe that \( W'_1 \) is a closed subgroup of \( \pi_1(X, \overline{x}) \), hence a profinite group, and \( G^0 \) is by definition the Zariski closure of the image of \( \pi_1(X, \overline{x}) \). In particular, the inverse image of \( G^{00} \) is Zariski dense in \( G^{00} \).

1.3.15. Relevant sources are Bourbaki, *Lie Groups and Lie Algebras* II, §7, Demazure and Gabriel, *Groupes Algébriques* II, §6, [Milne] 10, 14d. Bourbaki explains how to extend the logarithm to the union of all compacta. You need to know that for
all compact $G \subset H(E)$, $x \in G$, and neighborhood about $e$, there is a strictly increasing sequence of integers $(n_i)$ such that $x^{n_i} \in V$, which allows one to extend the logarithm by Deligne’s formula. Bourbaki also explains that there is an open subgroup $V$ of $e$ in $H(E)$ such that log is an analytic isomorphism of V onto an open subgroup of Lie H, with inverse exp. It follows that $L^1$, the E-linear span of log $K$, coincides with the E-linear span of log$(K \cap V)$. We have for $X \in \text{log} H$ that $\exp(nX) = \exp(X)^n$, and $\log(g^n) = n \log(g)$ for any $g$ where log is defined.

For $g \in H(E)$, $X \in \text{Lie} H$,

$$g \cdot \exp(X) \cdot g^{-1} = \exp(\text{Ad}(g)(X)),$$

(†)

whenever these expressions converge. Taking $X \in \log(K \cap V)$ and $g \in V \cap K$, we find that the left side is in $K \cap V$. Moreover there is an $n \in \mathbb{Z}$ such that both $n \text{Ad}(g)(X)$ and $nX$ lie in $\log V$. We find that $\text{Ad}(g)(nX) \in K \cap V$, therefore that for $g \in V \cap K$, $\text{Ad}(g)$ preserves $L^1$, therefore $L^1$ is also preserved by $V$, which is the Zariski closure of $V \cap K$ in $K$.

Therefore the adjoint representation

$$\text{Ad} : H \to \text{GL} \left( \text{Lie} H \right)$$

factors through the algebraic subgroup $F \subset \text{GL} \left( \text{Lie} H \right)$ which fixes the $L^1$. Applying the functor Lie, one finds that

$$\text{ad} : \text{Lie} H \to \text{gl} \left( \text{Lie} H \right)$$

factors through Lie F. As $\text{ad}$ induces the bracket on Lie H, it follows that $L^1$ is an ideal in Lie H. As H is semisimple over a field of characteristic 0, Lie subalgebras of Lie H are in bijection with connected algebraic subgroups of H (c.f. e.g. Demazure and Gabriel, *Groupes Algébriques*, II §6 2.4 & 2.7). As the exponential is functorial (c.f. *ibid*, 3.4), the algebraic subgroup corresponding to $\mathfrak{t}$ contains $K$, hence must equal $H$ by density, hence $L^1 = \text{Lie} H$.

Let N denote the normalizer of $K$ and $g \in N$. Let $X \in K$. Then there is an integer $n$ such that both $X^n \in V$ (hence $n \log X \in \log V$) and $n \text{Ad}(g)(\log X) \in \log V$. Applying (†) to $\log(X^n) = n \log X$ we find that $\exp(\text{Ad}(g)(n \log X)) = g \cdot X^n \cdot g^{-1} \in V \cap K$ so
that \( \log(g \cdot X \cdot g^{-1}) = \text{Ad}(g)(\log X) \) and we find \( \text{Ad}(g) \) preserves the set \( \log K \) and \textit{a fortiori} \( L^0 \).

The morphism \( \text{Ad} \) factors as a quotient \( H \to H/Z(H) \) followed by a closed immersion. As \( H \) is semisimple, \( Z(H) \) is finite, hence the quotient \( H \to H/Z(H) \) is finite (\textit{a fortiori} proper) [Milne 21.7, 7.15, 5.39].

The subgroup \( K \subset H(E) \) is a compact subset of a complete metric space, hence closed for the topology induced by the non-archimedean metric on \( E \). Hence \( K \) is a closed Lie subgroup of \( H(E) \) for that metric, and hence \( N \) is a closed subgroup of \( H(E) \) with respect to the metric induced by the one on \( E \). As \( L^0 \) is compact and isomorphic to an \( O_E \)-lattice in \( \text{Lie} H \), its automorphism group \( \text{Aut} L^0 \) is compact; as \( \text{Ad} \) is proper, \( \text{Ad}^{-1}(\text{Aut} L^1) \subset H(E) \) is compact, and \( N \subset \text{Ad}^{-1}(\text{Aut} L^0) \) is a closed subgroup, hence also compact.

\[ \text{1.4.1. (b) See Weil I, (2.9).} \]

\[ \text{1.4.2. Let } \bar{x} \text{ be a geometric point of } X; \text{ as } X_0 \text{ is absolutely irreducible, } X \text{ is connected. The pullback of lisse sheaves along the morphism } X \to X_0 \text{ identifies with the restriction of representations along the continuous homomorphism } \pi_1(X,\bar{x}) \to W(X_0,\bar{x}), \text{ and likewise the pullback of lisse sheaves along the structural morphism } X_0 \to \text{Spec}(F_q) \text{ with restriction along } W(X_0,\bar{x}) \to \mathbb{Z}. \text{ Given a lisse sheaf } \mathcal{F} \text{ on } X_0 \text{ with monodromy representation } V, \text{ the largest subsheaf (resp. quotient sheaf) becoming constant on } X \text{ is obtained by taking invariants (resp. coinvariants) of } V \text{ with respect to } \pi_1(X,\bar{x}). \text{ Both } \mathcal{V}_{\pi_1(X,\bar{x})} \text{ and } V_{\pi_1(X,\bar{x})} \text{ carry natural actions of } \mathbb{Z} \text{ which induces lisse sheaves } F'_0,F''_0 \text{ on } \text{Spec}(F_q) \text{ with inverse images } \mathcal{V}_{\pi_1(X,\bar{x})} \text{ and } V_{\pi_1(X,\bar{x})}, \text{ respectively. (The exact sequence } 0 \to \pi_1(X,\bar{x}) \to W(X_0,\bar{x}) \to \mathbb{Z} \text{ identifies those lisse sheaves invariant under geometric monodromy with the inverse image of sheaves on Spec}(F_q)). \]

\[ \text{1.4.3. The point is that on the one hand, the constituents of the sheaves } F',F'' \text{ are among the constituents of } \mathcal{F}_0, \text{ on the other hand as representations of } W(X_0,\bar{x}), F',F'' \text{ are invariant for geometric monodromy, so they have one-dimensional constituents which are determined once Frobenius is put in Jordan normal form. Therefore the eigenvalues of Frobenius on } F' \text{ and } F'' \text{ appear among the determinental weights for } \mathcal{F}_0, \]
and, in consideration of (1.4.2), up to a twist the same is true of eigenvalues of Frobenius on $H^0(X, \mathcal{F}), H^0_c(X, \mathcal{F})$, and $H^2_c(X, \mathcal{F})$.

1.4.6. See Ahlfors, *Complex Analysis*, Ch. 5 §2.2 for a characteristically elegant review of the convergence properties of infinite products, which elucidates the equivalence of the absolute convergence of Deligne’s Euler product with that of his geometric series.

1.5.1. Perhaps the only thing to remark is that if a lisse sheaf $\mathcal{F}$ on $X$ is $t$-real, then all of its exterior powers are, too: choosing a basis for $\mathcal{F}_x$ with respect to which $F_x$ is upper-triangular, the resulting canonical basis for $\wedge \mathcal{F}$ can be ordered so that $F_x$ remains upper triangular (1.3.13 iii), which makes it easy to see that $t \det(1 - F_x t, \wedge \mathcal{F})$ has coefficients which are symmetric polynomials in the eigenvalues of $F_x$. As the coefficients of $t \det(1 - F_x t, \mathcal{F})$ are the elementary symmetric polynomials in these eigenvalues, and are real, the coefficients of $t \det(1 - F_x t, \mathcal{F})$ are real too.

1.6.11. To see the Clebsch-Gordon decomposition (1.6.11.2), let

$$H = du \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

in the notation of (1.6.8), and let

$$\chi_d(\lambda) = \text{Tr}(e^{\lambda H}) = \sum_{j=-d}^{d} \lambda^j$$

as a function of $\lambda \in k$. These characters transform additively under direct sum and multiplicatively under tensor product, so finding the decomposition of $S_d \otimes S_{d'}$ is the same as finding the additive decomposition of

$$(\lambda^{-d} + \lambda^{-d+2} + \cdots + \lambda^d)(\lambda^{-d'} + \lambda^{-d'+2} + \cdots + \lambda^{d'})$$

into sums of the form ($\dagger$). In this case the decomposition is into $d' + 1$ sums of the form $\chi_j(\lambda)$ for $j \in P(d, d')$. 
It is asserted that the inclusion $\subset$ of assertion 2) results from the fact that the image of $N_iM_i$ in $\text{Gr}^W_0(V)$ is $N_iM_i(\text{Gr}^W_0V) = M_i(\text{Gr}^W_0V)$. To see this, recall that in the construction (1.6.1), $N^{d-1}_i$ sends $\text{ker}N^{d}/\text{im}N^d$ onto $M_{-d+1}/\text{im}N^d$, hence sends $M_{d-1}$ onto $M_{-d+1}$, and proceed inductively.

The inequality $k - 2i - 2 \geq k \geq 2j - k$ should read $k - 2i - 2 \geq -k \geq 2j - k$ at the end of the discussion of 3). Note $N^{k-j}_i : \text{Gr}^M_kG \rightarrow \text{Gr}^M_{2j-k}G$.

It seems as though (1.6.14.3) should read

$$\text{Gr}^M_i V \simeq \bigoplus_{j \geq |i|} P_{-j} \left( \frac{-i - j}{2} \right).$$

For, when $i \leq 0$, $N$ induces an isomorphism of $\text{Gr}^M_i V/P_i$ onto $\text{Gr}^M_{i-2} V$. Scaling $N \mapsto \lambda N$ also multiplies this isomorphism by $\lambda$, so we need to twist by $\otimes N^{-1}$. Similarly, the isomorphism $N^i : \text{Gr}^M_i V \sim \text{Gr}^M_{-i} V$ scales by $\lambda^i$ so we need to twist by $-i$.

With this modification, the isomorphism

$$P_{-j} \simeq \bigoplus_{j \in P(j',j'')} P'_{-j} \otimes P''_{-j} \left( \frac{j - j' - j''}{2} \right)$$

is justifiable by passing through a suitable graded piece

$$P_{-j} \left( \frac{-i - j}{2} \right) \rightarrow \text{Gr}^M_i V \leftarrow \bigoplus_{j \in P(j',j'')} P'_{-j} \otimes P''_{-j} \left( \frac{-i - j' - j''}{2} \right).$$

As for the isomorphism

$$P_{-j}(V^*) \simeq P_{-j}(V)^*(j),$$

consider $V = S_j$ identified with the representation of $\text{SL}(2)$ on homogeneous polynomials in variables $x$ and $y$ of degree $j$. Suppose $P_{-j}(V)$ is generated by the vector $y^j$; then $P_{-j}(V)^*$ is generated by the covector on $y^j$ and $P_{-j}(V^*)$ is the covector on $x^j$. The map $P_{-j}(V)^* \rightarrow P_{-j}(V^*)$ is obtained by precomposing by $N^j$. Upon scaling $N \mapsto \lambda N$, this map is scaled by $\lambda^j$. Therefore the canonical isomorphism is obtained by twisting by $N^{-j}$:

$$P_{-j}(V)^* \rightarrow P_{-j}(V^*)(-j).$$
1.7.2. Note that in this case $V$ is a $\overline{Q}_\ell$-vector space. The claim about the existence of $\overline{\rho}$ satisfying

$$\rho(\sigma) = \exp(\overline{\rho}_t(t(\sigma))) \quad \text{for} \quad \sigma \in I_1 \subset I$$

in particular implies that $P \cap I_1$ acts trivially on $V$. Let’s recall why this is the case. As $p \neq \ell$, $\rho(P)$ is finite. As $\overline{Q}_\ell$ has characteristic zero, by Maschke’s theorem, the action of $P$ on $V$ is semisimple. As $\overline{Q}_\ell$ is algebraically closed and $V$ is finite-dimensional, this implies that each element in $V$ is diagonalizable. A diagonalizable element of $\GL(V)$ is unipotent iff it is the identity. This proves our claim. (Recall also the Jordan-Chevalley decomposition.)

1.7.3. ‘Il commute à l’action de $W(\overline{K}, K)$’ this statement is somewhat opaque. Which action(s)? When you let $W(\overline{K}, K)$ act on $V$ via $\rho$ and on $\overline{Q}_\ell(n)$ via $\Gal(\kappa, k)$, the statement is that if $\tau \in W(\overline{K}, K)$,

$$\tau N \tau^{-1} = N : V(1) \to V.$$

Let $\sigma \in \overline{Q}_\ell(1)$. As $\tau^{-1}\lambda = (\deg \tau)^{-1}\lambda \tau^{-1}$, the statement is equivalent to the statement

$$\tau N \lambda \tau^{-1} = (\deg \tau) N \lambda.$$

$\Gal(\overline{K}, K)$ acts on the inertia character $t$ by the formula

$$t(\tau \sigma \tau^{-1}) = \tau t \sigma,$$

where on the right $\tau$ acts through $\Gal(\kappa, k)$, so if we assume $I_1$ is a normal subgroup of $I$ and take $\sigma \in I_1$,

$$\exp(\rho(\tau) N t_\ell(\sigma) \rho(\tau^{-1})) = \rho(\tau \sigma \tau^{-1})$$

$$= \exp(N t_\ell(\tau \sigma \tau^{-1})) = \exp(N (\deg \tau) t_\ell(\sigma)).$$

1.7.4. A normal unipotent subgroup of an algebraic group acts trivially on all semisimple representations, hence the factorization of the action of $I$ by a finite quotient $[\text{Milne}]$ 19.16.
The semisimple representation of $W(\overline{K}, K)$ therefore factors through the quotient by $I_1$. The conjugation action of $W(\overline{K}, K)$ on $I/I_1$ is has a kernel $U$ of finite index, as a finite group has only finitely many automorphisms. So for some $m_1, F^{m_1}, F^{m_1} \in U$. $F^{m_1} = aF^{m_1}$ for some $a \in I$, but now $F^{m_1}$ commutes with $a$, so if $a^{m_2} = 1$, then $F^{m_1m_2} = a^{m_2}F^{m_1m_2} = F^{m_1m_2}$, so we may take $n = m_1m_2$.

1.7.5. In the proof it is asserted ‘On a $\text{NM}'_i \subset \text{M}'_{i-2}$’. This is because of the commutativity of (1.7.3). Namely, $F'N = NF'$ and if $\lambda \in \overline{Q}_\ell(1)$, $F'N\lambda = q^{-1}N\lambda F'$. Therefore $N\lambda$ sends $M'_i$ into $M'_{i-1}$ and $N : M'_i(1) \subset M'_{i-2}$.

While proving that $M'$ is independent of $F'$, Deligne introduces $\exp(\lambda N)$ for $\lambda \in \overline{Q}_\ell(-1)$. Let me instead deduce from the fact $F'^n = F'^n \mod I_1$ that for an appropriate choice of $\lambda \in \overline{Q}_\ell(1)$,

$$F'^n = \exp(N\lambda)F'^n.$$  

Leaving the definition of $\mu$ unchanged, we have

$$\exp(N\lambda)F'^n = \exp(N\mu)F'^m \exp(N\mu)^{-1},$$

since if we expand $F'^m \exp(-N\mu)$ as a series

$$F'^m \left(1 - N\mu + \frac{(-N\mu)^2}{2!} + \cdots \right),$$

we have that $N$ commutes with $F'^m$ and that $F'^m\mu^i = q^m\mu^i$, so that

$$\exp(N\mu)F'^m \exp(N\mu)^{-1} = \exp(N\mu) \exp(-q^nN\mu)F'^m = \exp(N\lambda)F'^m.$$  

This identity shows that $\exp(N\mu)$ sends $M'$ into $M''$. As $(N\mu)M'_i \subset M'_{i-2}$,

$$\exp(N\mu) = 1 + N\mu + \frac{(N\mu)^2}{2!} + \cdots$$

also sends $M'_i$ into $M'_{i-2}$, so $M''_i \subset M'_i$.

1.7.7. Let $F'$ be as in (1.7.5), let $\alpha \in \overline{Q}_\ell^*$, and let $V^{\alpha}$ equal the sum of the generalized eigenspaces of $F'$ acting on $V$ with eigenvalue in the class of $\alpha \mod$ roots of 1. Then
$V^{\alpha}$ is independent of lift and $N\lambda : V^{\alpha} \to V^{\alpha/q}$, and
$$V^\alpha = \sum_{i \in \mathbb{Z}} V^{\alpha q^i},$$
is stable under $\exp(N\lambda)$ for $\lambda \in \overline{\mathbb{Q}}_\ell(1)$ and hence under the action of $I_1$. By the argument of (1.7.5), $V^\alpha$ is therefore stable under $W(\overline{K}, K)$.

1.7.8. First, the matter of what it means for the locally constant sheaf of sets $\mathcal{F}$ on $X - D$ to be tamely ramified along the divisor $D$. In Grothendieck-Murre [GM], the notion of a tamely ramified covering is discussed. If $\mathcal{F}$ is a locally constant sheaf of finite sets on $X - D$, it is represented by an étale covering of $X - D$. What about if $\mathcal{F}$ is a locally constant sheaf of sets on a scheme $S$ with infinite fibers?

Let $\{S_i\}_{i \in I}$ be a covering trivializing $\mathcal{F}$. If $S$ is quasi-compact we can take this covering to be finite quasi-compact. Over each $S_i$, $\mathcal{F}$ is represented by an étale morphism $X_i \to S_i$, namely the disjoint union $\bigsqcup_{\lambda \in \Lambda} S_i$ with the set $\Lambda$ in bijection with a(ny) fiber of $\mathcal{F}$. This yields a separated, locally of finite presentation, and locally quasi-finite descent datum $X_i$ relative to the covering $S_i$ (c.f., e.g. Stacks, tag 02W4). By [SGAD], Exp. X, 5.4, this descent datum is effective and yields an $S$-scheme $X$. In the case that $S$ is quasi-compact, then as the faithfully flat covering $\bigsqcup_i S_i$ was taken to be quasi-compact, $X$ is separated [SGA 1], Exp. VIII, 4.8 or Exp. IX, 2.4], but in any case it is evidently étale as its restriction to each $S_i$ is. This elucidates the proof of Lemma 2.2 in [SGAA], Exp. IX], with one refinement:

**Lemma.** — Let $S$ be a scheme and $\mathcal{F}$ a locally constant sheaf of sets on $S$. Then $\mathcal{F}$ is represented by a $X/S$ étale. If $S$ is quasi-compact, then $X$ is separated. If the fibers of $\mathcal{F}$ are finite (resp. and non-empty), then $X$ is an étale covering (resp. and surjective).

Now suppose $S$ is normal, integral, and quasi-compact with generic point $\eta$. By EGA IV$_1$ 18.10.7 and 18.10.8, the étale $X \to S$ is isomorphic to a disjoint union of integral normal schemes $X_\alpha$ such that $f_\alpha^{-1}(\eta)$ is a finite separable extension of $k(\eta)$ and moreover if $S'_\alpha \xrightarrow{g_\alpha} S$ denotes the normalization of $S$ in $f_\alpha^{-1}(\eta)$, then $S'_\alpha$ is a revêtement étale, and $f_\alpha$ factorizes as $X_\alpha \xrightarrow{f'_\alpha} S'_\alpha \xrightarrow{g_\alpha} S$ with $f'_\alpha$ an open immersion.
Hence $\mathcal{F}$ decomposes as a disjoint union of sheaves $\mathcal{F}_\alpha = h_{X_\alpha}$, with $h_{X_\alpha}$ locally constant constructible. The property of the $h_{X_\alpha}$ being locally constant implies that the open immersions $f'_\alpha$ are in fact isomorphisms, and we sum up in the following

**Lemma.** — Let $S$ be a normal, integral, and quasi-compact scheme and let $\mathcal{F}$ be a locally constant sheaf of sets on $S$. Then $\mathcal{F} = \bigsqcup_\alpha h_{X_\alpha}$; i.e. $\mathcal{F}$ decomposes as a disjoint union of locally constant constructible sheaves represented by revêtements étales $X_\alpha \to S$.

Now we recall the definition of divisors with normal crossings [GM 1.8]. Let $S$ be a locally noetherian scheme and $(D_i)_{i \in I} = D$ a finite set of divisors on $S$. For simplicity we often denote the inverse image of the $D_i$ in $\text{Spec} \mathcal{O}_{S, s} \to S$ by the same letter $D_i$.

**Definition.**

a) We say that the $(D_i)_{i \in I}$ have strictly normal crossings if for every $s \in \bigcup_{i \in I} \text{supp } D_i$ we have:

i) $\mathcal{O}_{S, s}$ is a regular local ring,

ii) if $I_s = \{i : s \in \text{supp}(D_i)\}$, then for $i \in I_s$ we have

$$D_i = \sum_k \text{div}(x_{i,\lambda})$$

with $x_{i,\lambda} \in \mathcal{O}_{S, s}$ and $(x_{i,\lambda})_{i,\lambda}$ part of a regular system of parameters at $s$.

b) We say that the set $(D_i)_{i \in I}$ has normal crossings if for every $s \in \bigcup_{i \in I} \text{supp } D_i$ there exists an étale neighborhood $S' \to S$ of $s$ in $S$ such that the family of inverse images of the $(D_i)_{i \in I}$ on $S'$ has strictly normal crossings.

**Remark.** The concept of (strictly) normal crossings is stable by étale base change, and one can check whether a set of divisors has normal crossings étale-locally.

In the setting of $X$ a regular scheme and $D$ a divisor on $X$ with normal crossings, the definition of a tamely ramified covering $f : V \to X$ relative to $D$ given in Grothendieck-Murre is equivalent to the property that $V$ be étale over $X - D$, and that the inertia at each $d \in D$ of codimension one act trivially on $V$ (see also [SGA I, Exp. XIII, 2.1]). More precisely, let $D$ be a union of lisse divisors $D_i, d, X_{(d)}, \bar{\eta}$ be as in (1.7.8). Then the corresponding inertia group should act trivially on $V_{\bar{\eta}}$. Evidently this definition extends to a locally constant sheaf of sets $\mathcal{F}$. The ramified Kummer covering $\pi : X_n \to X$ is
a homeomorphism, and if $\mathcal{F}$ is a locally constant sheaf of sets on $X - D$, $L$-ramified along $D$, for $L$ a set of primes invertible on $X$, then there is an $n$ invertible on $X$ such that the action of inertia with respect to every point of $X_n$ on $\pi^* \mathcal{F}$ is trivial.

More explicitly, let $U = X - D, U_n = \pi^{-1}(U)$. The Kummer covering $\pi$ is a revêtement étalement when restricted to $U_n$, which corresponds to the homomorphism $\varphi: \pi_1(U_n, \eta) \to \pi_1(U, \eta)$ of topological groups which is the inclusion of the former group as an open subgroup of the latter corresponding to the connected, pointed étale cover $U_n \to U$. Let $\sigma \in \text{Gal}(\eta/\eta_n)$ be any element of the inertia corresponding to the monodromy around any of the $D_i$ which acts nontrivially on $\mathcal{F}_{\eta}$. Then the image of $\sigma$ in $\pi_1(U_n, \eta)$ is nonzero in $\pi_1(U, \eta)/\varphi \pi_1(U_n, \eta)$; in particular it does not lie in $\text{Gal}(\eta/\eta_n)$ where $\eta_n$ is the generic point of $U_n$. Therefore the representation of $\pi_1(U_n, \eta)$ obtained by restricting $\pi^* \mathcal{F}$ to $U_n$ coincides with the restriction of a continuous representation of $\pi_1(X_n, \eta)$ by the continuous map $\pi_1(U_n, \eta) \to \pi_1(X_n, \eta)$; i.e. $\pi^* \mathcal{F}$ extends to a locally constant sheaf $\mathcal{F}$ on $X_n$.

1.7.11. As this paragraph reinterprets the construction of (1.7.8), it is implicit that $X$ is a henselian trait. A remark on the equality $\text{Gal}(K_1/K) = \text{Gal}(k_1/k)$: this is effectively saying that there is an equivalence of Galois categories of finite étale covers of the points $\text{Spec } K$ and $\text{Spec } k$. The functor of restriction to the special fiber does induce an equivalence of the categories of finite étale covers of $X$ with that of $\text{Spec } k$; this is [EGA IV4 18.5.11]. For the equality of Galois groups in the setting of $R \subset R^\text{sh}, K \subset K_1$, $R$ local henselian normal, $R^\text{sh}$ its strict henselization, we use the following facts.

For any finite separable subextension $k \subset k_2 \subset k_1$, there exists a unique (up to unique isomorphism) finite étale local ring extension $R \subset R_2$ with specified residue field extension. Since the functor of restriction to the special fiber induces an equivalence of categories, the inductive system of finite separable subextensions $k \subset k_2 \subset k_1$ specifies an inductive system of local ring homomorphisms $R \subset R_2 \subset R^\text{sh}$, of which $R^\text{sh}$ is the colimit (Stacks, tag [0BSL]).

The following is a variation on the theme of the aforementioned equivalence. It is Lemma 7, §2.3 of *Néron Models* by Bosch, Lütkebohmert, and Raynaud (compare *Corps Locaux* III §5 Th. 3).
Lemma. — Let $R$ be a local ring, $S'$ an étale $R$-scheme, and $s'$ a point of $S'$ above the closed point $s$ of $S = \text{Spec } R$. Let $R'$ be the local ring $\mathcal{O}_{S', s'}$ of $S'$ at $s'$ and let $k'$ be the residue field of $R'$. Furthermore, let $A$ be a local $R$-algebra with residue field $k_A$. Then all $R$-algebra morphisms from $R'$ to $A$ are local. So there is a canonical map

$$\text{Hom}_R(R', A) \rightarrow \text{Hom}_k(k', k_A).$$

This map is always injective; it is bijective if $A$ is henselian.

The group $\text{Gal}(K_1, K)$ acts on $R_\text{sh}$ with fixed subring $R$. Let $k \subset k_2 \subset k_1$ and $R \subset R_2$ be as above, then $R_2$ is normal as $R$ is, and we have an isomorphism

$$\text{Hom}_K(K_2, K_1) = \text{Hom}_R(R_2, R_\text{sh}) \rightarrow \text{Hom}_k(k_2, k_1).$$

As $R \subset R_2$ is étale, $K_2$ is separable over $K$, in the inductive limit we find that the induced map $\text{Gal}(K_1, K) \rightarrow \text{Gal}(k_1, k)$ is an isomorphism.

1.8.1. The inclusion $\hat{k} \otimes_{j_* \mathcal{F}_0} \subset j_* \hat{k} \otimes_{j_* \mathcal{F}_0}$ comes about by considering that étale locally about a point $s \in S_0$ neither sheaf may be locally free, as restricting étale neighborhoods of $s$ to $U_0$ may not be enough to trivialize either sheaf; in other words, a trivialization may ramify when extended to $X_0$. Let $V \rightarrow X_0$ be an étale neighborhood of $s$. Then the inclusion above simply reflects the fact that sections of $\hat{k} \otimes_{j_* \mathcal{F}_0}$ over $V|_{U_0} := V \times_{X_0} U_0$ include those coming from the tensor product of $k$ sections in $\mathcal{F}_0(V|_{U_0})$, but might include more besides.

Here is an example: let $X = \text{Spec } R$ and let $\mathcal{F}$ be the locally free sheaf on $X$ represented by $\text{Spec } R[x]/(x^3 - 1)$; this is the sheaf $\mu_3$ of third roots of unity and it is a locally free sheaf of $\mathbb{Z}/3$-modules of rank 1. Then $\mathcal{F}(R) = \{1\}$ and if $\zeta$ is a primitive 3rd root of unity, $\mathcal{F}(\mathbb{C}) = \{1, \zeta, \zeta^2\}$. $\text{Gal}(\mathbb{C}/R) \cong \mathbb{Z}/2$ and $\mathcal{F}$ corresponds to the representation $V$ of $\mathbb{Z}/3$-module $\{1, \zeta, \zeta^2\}$. Then $\mathcal{F} \otimes_{\mathbb{Z}/3} \mathcal{F}$ corresponds to the tensor representation $V \otimes_{\mathbb{Z}/3} V$. Its sections over $R$ are its $\mathbb{Z}/2$-invariant sections of $V \otimes V$. These are $\{1 \otimes 1, \zeta \otimes \zeta, \zeta^2 \otimes \zeta^2\}$.

1.8.4. The fiber of $j_* \mathcal{F}_0$ at $s$ can be computed by taking first the inverse image of $j_* \mathcal{F}$ to $\text{Spec } \mathcal{O}_{X_0, s}$, the local ring of $X_0$ at $s$, and then taking the colimit along all étale ring maps $\mathcal{O}_{X_0, s} \rightarrow U$, these being equivalent to finite separable extensions of $k(\eta)$.
which are non-ramified over \( s \). So in the end we are computing the colimit of sections
of the inverse image of \( \mathcal{F}_0 \) along \( \eta \to X_0 \) over finite separable field extensions of \( k(\eta) \)
fixed by \( I \); this is nothing other than \( \mathcal{F}^I_{0\eta} \).

The last line of the proof references (1.6.14.3), which has been corrected in the note
(1.6.14) above.

**1.8.5.** The \( \iota \)-weights of \( \mathcal{F}_{\bar{\eta}} \) are integers, as guaranteed by (1.8.4); therefore, we can
apply (1.7.5). The nilpotent endomorphism \( N \) respects the filtration \( W \) on \( \mathcal{F}_{\bar{\eta}} \), since
all of \( W(\bar{\eta}, \eta) \) respects the filtration, and hence the inertia does, so that the logarithm
of the unipotent part of the local monodromy does too. For the local monodromy
filtration on \( \mathcal{F}_{\bar{\eta}} \), rel. \( W \) to exist, it remains only to check that \( N^k \) induces isomorphisms
\( \text{Gr}^{M_i}_{i+k} \mathcal{F}^W_{\bar{\eta}} \rightarrow \text{Gr}^{M_i}_{i-k} \mathcal{F}^W_{\bar{\eta}} \); i.e. that the weight filtration \( M_1 \) on \( \text{Gr}^W_i(\mathcal{F}_{\bar{\eta}}) \)
coincides with the local monodromy monodromy filtration \( M_2 \) on \( \text{Gr}^W_i(\mathcal{F}_{\bar{\eta}}) \), shifted by
\( i \); i.e. that
\[
\text{Gr}^{M_i}_j \mathcal{F}^W_i(\mathcal{F}_{\bar{\eta}}) = \text{Gr}^{M_{i+j}}_j \mathcal{F}^W_{i+j}(\mathcal{F}_{\bar{\eta}}). \tag{\dagger}
\]

But (1.7.5) shows that the weight filtration \( M_1 \) is the unique finite increasing filtration on
\( \text{Gr}^W_i(\mathcal{F}_{\bar{\eta}}) \) which is stable under \( W(\bar{K}, \bar{K}) \) such that \( \text{Gr}^{M_i}_j \mathcal{F}^W_i(\mathcal{F}_{\bar{\eta}}) \) is \( \iota \)-pure of weight \( j \).

On the other hand, (1.8.4) shows that the local monodromy filtration \( M_2 \) on \( \text{Gr}^W_i(\mathcal{F}_{\bar{\eta}}) \)
is a finite increasing filtration which is stable under \( W(\bar{\eta}, \eta) \) such that \( \text{Gr}^{M_2}_j \mathcal{F}^W_{i+j}(\mathcal{F}_{\bar{\eta}}) \) is
\( \iota \)-pure of weight \( i + j \). This shows that the filtration \( M_2 \), shifted by \( i \), coincides with \( M_1 \); i.e. we have verified (\dagger), and hence the existence of the local monodromy filtration on
\( \mathcal{F}_{\bar{\eta}} \), rel. \( W \).

**Remark.** Both the local monodromy filtration and the weight filtration involve a
choice of point \( s \in S \), but end up defining a filtration on the fiber \( \mathcal{F}_{\bar{\eta}} \).

**1.8.8.** With regards to remark 2), twist the sheaf \( \mathcal{F}_0 \rightarrow \mathcal{F}_0^{(b)} \) so that it has weight 0
(following (1.2.7), \( b = p^{-\beta} \)). Then apply (1.8.7) with the trivial filtration \( W \) to see
that \( \text{Gr}^{M_i}_j(\mathcal{F}_0^{(b)}|D_0) \) is punctually \( \iota \)-pure of weight \( i \). Twist back to conclude that
\( \text{Gr}^{M}_i(j_* \mathcal{F}_0|D_0) \) is punctually \( \iota \)-pure of weight \( \beta + i \).
In c), \( j_* \mathcal{F}_0 \hookrightarrow \epsilon_* j_* \epsilon^* \mathcal{F}_0 \) follows, after writing \( \epsilon_* j_* \epsilon^* \mathcal{F}_0 = j_* \epsilon_* \epsilon^* \mathcal{F}_0 \), from the observation that \( \mathcal{F}_0 \hookrightarrow \epsilon^* \mathcal{F}_0 \) injects since on stalks, a finite extension of a henselian ring splits as a product of henselian rings; i.e. the adjunction morphism corresponds to the inclusion along the diagonal

\[
\mathcal{F}_x \hookrightarrow \prod_{\epsilon^{-1} x} \mathcal{F}_{x'}.
\]

In d), reduce to a constant sheaf, where it is obvious.

In the explanation for e), the strict henselization of \( X_0 \) at \( x \) is irreducible hence a fortiori the inverse image of any open set is connected. Then, since the fiber product of any étale cover of \( x \) with \( U_0 \) is an étale neighborhood of \( z \), there is a map \( (i^* j_* \mathcal{F}_0)_x \rightarrow (\mathcal{F}_0)_z \). As the former can be computed as sections over the inverse image of \( U_0 \) in the strict henselization of \( X_0 \) at \( x \); since this is a connected scheme, and \( \mathcal{F}_0 \) is lisse, the arrow is injective. The factorization of this arrow as

\[
(i^* j_* \mathcal{F}_0)_x \rightarrow (k_* k^* i^* j_* \mathcal{F}_0)_x \rightarrow (j_* \mathcal{F}_0)_y \rightarrow (\mathcal{F}_0)_x
\]

can be explained as follows. After rewriting \( k_* k^* i^* j_* \mathcal{F}_0 \) as \( k_* k^* i^* j_* \mathcal{F}_0 \), the first arrow is just adjunction for \( k \), The middle arrow can be rewritten \( (k_* k^* i^* j_* \mathcal{F}_0)_x \rightarrow (i^* j_* \mathcal{F}_0)_y \) so that it is a statement about sheaves on \( \mathcal{F}_0 \). Take an étale neighborhood \( W_0 \) of \( x \) in \( F_0 \); then \( W_0 \times_{F_0} V_0 \) is an étale neighborhood of \( y \). The projective system of étale \( W'_0 \rightarrow F_0 \) s.t. \( W'_0 \times_{F_0} V_0 \) admits an arrow to \( W_0 \times_{F_0} V_0 \) has the property that the projective system \( W'_0 \) is a subcategory of the projective system of étale neighborhoods of \( y \) in \( F_0 \). Therefore there is an arrow from the colimit of \( i^* j_* \mathcal{F}_0 \) applied to the former system to the colimit of \( i^* j_* \mathcal{F}_0 \) applied to the latter, which is \( (i^* j_* \mathcal{F}_0)_y \). This gives an arrow \( (k_* k^* i^* j_* \mathcal{F}_0)(W_0) \rightarrow (i^* j_* \mathcal{F}_0)_y \), functorial in \( W_0 \), and hence an arrow \( (k_* k^* i^* j_* \mathcal{F}_0)_x \rightarrow (i^* j_* \mathcal{F}_0)_y \).

The last arrow is in effect the observation that the fiber product of \( U_0 \) with an étale neighborhood of \( y \) in \( X_0 \) is an étale neighborhood of \( z \).

In the proof of (1.8.9), the reductions are all clear except perhaps the reduction to \( \mathcal{F}_0 \) tamely ramified at the generic points of \( F_0 \). Suppose all the other dévissages have been made except that one; then as there are finitely many generic points and \( \mathcal{F}_0 \) is a lisse \( \overline{\mathbb{Q}}_\ell \).
sheaf corresponding to a representation \( V \) of \( \pi_1(U_0, \overline{x}) \) for some geometric point \( \overline{x} \) of \( U_0 \), the image of the wild inertia at the finitely many generic points of \( F_0 \) in \( \text{Aut} V \) is finite. In particular, the preimage of the congruence subgroup \( \Gamma_1 \) of \( \text{Aut} V \) in \( \pi_1(U_0, \overline{x}) \) is open and of finite index, corresponding to an étale cover of \( V_0 \to U_0 \) which extends to a finite surjective morphism to \( X_0 \). Applying c), we reduce to the desired situation.

To complete the proof, we are almost there, except \( F_0 \) is just a Weil divisor, and need not satisfy the smoothness assumption of (1.8.6). The idea is to use e) and recurrence on \( \dim U_0 \) to shrink \( X_0 \) and throw away the bad points of \( F_0 \). If we replace \( X_0 \) by an open set containing \( U_0 \) whose intersection with \( F_0 \) is a lisse divisor, then (1.8.8) 2) shows that (1.8.9) is true there. We can find finitely many such open sets with inclusions \( j_i \), the union of which, \( X'_0 \), intersects \( F_0 \) in a dense set \( V_0 \). If \( j'_i : U_0 \hookrightarrow X'_0 \), then \( j'_i \ast F_0 \hookrightarrow \prod j_i \ast F_0 \) so (1.8.9) is proved for \( j'_i \). By recurrence on dimension we may assume that (1.8.9) holds for \( k \ast \). In light of this, applying e) to the lisse sheaf \( F_0 \), yields that \( i^* j_* F_0 \) satisfies the conclusions of (1.8.9); in effect, \( k^* j_* = k^* j'_* \). Finally, \( j_* F_0 \leftarrow j'_* F_0 \times i_* i^* j_* F_0 \) allows us to conclude that \( j_* \) satisfies (1.8.9).

### 1.8.11. A Jordan-Hölder series for \( F_0 \) allows us to reduce to \( F_0 \) irreducible. The restriction of an irreducible lisse sheaf to a nonempty open \( U_0 \) of a normal connected scheme \( X_0 \) is still irreducible because if \( \eta \) denotes the generic point of \( X_0 \), and \( \overline{\eta} \) a geometric point centered on \( \eta \), we have by [SGA 1, Exp. V, 8.2]

\[
\text{Gal}(\overline{\eta}, \eta) \to \pi_1(U_0, \overline{\eta}) \to \pi_1(X_0, \overline{\eta}).
\]

Now, \( F_0 \) is \( \iota \)-mixed, so admits an \( \iota \)-pure subsheaf \( \mathcal{G}_0 \) which is lisse when restricted to some \( U_0 \). Therefore \( F_0|U_0 \), irreducible yet containing \( \mathcal{G}_0|U_0 \), must equal \( \mathcal{G}_0|U_0 \).

### 1.8.12. Let \( f : X'_0 \to X_0 \) be the normalization morphism. It induces a bijection on irreducible components, and \( X'_0 \) is a disjoint of normal integral schemes. We make use of the fact, true for any morphism, that if \( x_0 \in X_0 \), the weights of \( F_0 \) at \( x_0 \) coincide with the weights of \( f^* F_0 \) at every point of the fiber \( f^{-1}(x_0) \). If \( F_0 \) is \( \iota \)-pure of weight \( \beta \) at a point \( x_0 \), then all the points of \( X'_0 \) in the fiber over \( x_0 \) are also \( \iota \)-pure of the same weight; therefore (1.8.11) implies that all the points of the irreducible components of \( X'_0 \) meeting the fiber of \( x_0 \) are \( \iota \)-pure of weight \( \beta \), which in turn implies that all the points in
the irreducible components of $X_0$ meeting $x_0$ are $\iota$-pure of weight $\beta$, which shows that the locus of points where $\mathcal{F}_0$ is $\iota$-pure of weight $\beta$ is closed. This locus is also open because, taking any $x_0$ in it, there is an open neighborhood of $x_0$ which meets only the irreducible components of $X_0$ on which $x_0$ lies. Then the above construction shows that all the points in this neighborhood are also $\iota$-pure of weight $\beta$. 
Bibliography

[WeilII] Deligne, Weil II.
[GM] The Tame Fundamental Group of a Formal Neighbourhood of a Divisor with Normal Crossings on a Scheme par A. Grothendieck et J. Murre.
[SGA5] SGA 5, dirigé par Grothendieck
[SGA 4 1/2] SGA 4 1/2, Rapport sur la formule des traces par P. Deligne.
4. **Faisceaux Pervers**

1.1.11. To verify the anti-commutativity of the 9th square, as the morphism of triangles \((X'',Y'',Z'') \to (X'[1],Y'[1],Z'[1])\) factors as the composition of two morphisms of triangles \((X'',Y'',Z'') \to (A,Y'',Z'[1]) \to (X'[1],Y'[1],Z'[1])\), where the second arrow is the rotation of \((Z',A,Y'') \to (Z',X'[1],Y'[1])\), it suffices to verify that the triangle \((Z',X'[1],Y'[1])\) which appears in this last morphism of triangles has all arrows induced by the arrows of \((X',Y',Z')\) or translates of them (with the same parity). This is not hard to check from the diagram (1). (The stated explanation appears to be an un-explanation.)

1.3.3. Though it is not stated explicitly, it is immediate from the definition \(\mathcal{D}^{\leq n} := \mathcal{D}^{\leq 0}[-n], \mathcal{D}^{\geq n} := \mathcal{D}^{\geq 0}[-n]\) that \((\tau_{\leq n}X)[m] = \tau_{\leq n-m}(X[m])\) and \((\tau_{\geq n}X)[m] = \tau_{\geq n-m}(X[m])\). Namely, for \(X\) in \(\mathcal{D}\) and \(T\) in \(\mathcal{D}^{\leq n}\), \(T = T'[-m]\) for some \(T'\) in \(\mathcal{D}^{\leq n-m}\), so
\[
\text{Hom}(T, (\tau_{\leq n-m}(X[m]))[-m]) = \text{Hom}(T'[-m], (\tau_{\leq n-m}(X[m]))[-m])
\]
\[
= \text{Hom}(T'[-m], \tau_{\leq n-m}(X[m]))[-m] = \text{Hom}(T', X[m])[-m] = \text{Hom}(T'[-m], X)
\]
\[
= \text{Hom}(T, X).
\]

1.4.2.1. The argument for why the derived functors continue to satisfy the stated adjunctions is as follows (this argument is also found in SGA 4 Exp. XVIII 3.1.4.11). Given \(F,G\) an adjoint pair of functors on abelian categories
\[
\mathcal{A} \xrightarrow{F} \mathcal{B} \xleftarrow{G}
\]
where both categories have enough injectives and \(L\) is exact. The functors \(F\) and \(G\) extend to functors \(D^+(\mathcal{A}) \xrightarrow{\sim} D^+(\mathcal{B})\). Given \(K' \in D^+(\mathcal{A}), L' \in D^+(\mathcal{B})\), we may assume \(L'\) is a complex of injective objects; we have an isomorphism of triple complexes
\[
\text{Hom}'(F(K'), L) \xleftarrow{\sim} \text{Hom}'(K', G(L')).
\]
As \(G\) preserves injectives, taking \(H^0\) of the associated simple complex (calculated with products) finds the desired
\[
\text{Hom}_{K(\mathcal{B})}(F(K'), L') \xleftarrow{\sim} \text{Hom}_{K(\mathcal{A})}(K', G(L')).
\]
where both sides are also Hom in the respective derived categories, since \( L' \) and \( G(L') \) are complexes of injectives.

**1.4.4.** The question is, why are the adjoints to the Verdier quotients fully faithful? Let’s consider the quotient \( Q : \mathcal{I} \to \mathcal{I}/\mathcal{U} \), where \( \mathcal{U} \) is the strictly full coreflective triangulated subcategory of \( \mathcal{I} \); \((\mathcal{U}, \mathcal{V})\) form a t-structure on \( \mathcal{I} \); \( \mathcal{U} = \perp \mathcal{V} \), and \( \mathcal{V} = \mathcal{U}^\perp \). Since the embedding \( u : \mathcal{U} \to \mathcal{I} \) admits a right adjoint \( u^\cdot \) it follows that \( Q \) admits a right adjoint \( Q^\cdot \). There is a natural isomorphism of functors \( Q^\cdot \circ Q \cong \sim v \circ v^\cdot \), where \( v^\cdot \) is the left adjoint to the inclusion \( v : \mathcal{V} \to \mathcal{I} \). The functor \( v^\cdot \) is nothing other than \( \tau_{\geq 0} \) for the t-structure \((\mathcal{U}, \mathcal{V})\), and therefore, restricted to \( \mathcal{V} \), \( v \circ v^\cdot = id \mid \mathcal{V} \). On the other hand, the functor \( Q \) when restricted to \( \mathcal{V} \) is fully faithful \([CD] \ I \ 5-3\]. Therefore \( Q^\cdot \) restricted to the essential image of \( \mathcal{V} \) under \( Q \) is fully faithful. This essential image is all of \( \mathcal{I}/\mathcal{U} \), since every object \( X \) in \( \mathcal{I} \) belongs to an exact triangle \((U, X, V)\) with \( U, V \) objects in \( \mathcal{U}, \mathcal{V} \), respectively. The assertion that \( v^\cdot \) yields an equivalence \( v^\cdot : \mathcal{I}/\mathcal{U} \to \mathcal{V} \) (functor obtained by applying the universal property of \( \mathcal{I}/\mathcal{U} \) to \( v^\cdot \)) is easy since \( Q \circ v : \mathcal{V} \to \mathcal{I}/\mathcal{U} \) is an equivalence, and \( v^\cdot Q \circ v = v^\cdot v = id \). The corresponding statement for \( \mathcal{I} \to \mathcal{I}/\mathcal{V} \) follows identically.

**1.4.6.** In part b), there is the following consideration. Given triangulated categories \( \mathcal{I}, \mathcal{I}' \), a thick subcategory \( \mathcal{U} \subset \mathcal{I} \) with Verdier quotient \( Q : \mathcal{I} \to \mathcal{I}/\mathcal{U} \), exact functors \( F, G : \mathcal{I}/\mathcal{U} \to \mathcal{I}' \) and a natural transformation \( \phi : F \circ Q \to G \circ Q \), there is an obvious candidate for a natural transformation \( \varphi : F \to G \), since \( \text{Ob}(\mathcal{I}/\mathcal{U}) = \text{Ob}(\mathcal{I}) \). But is it still a natural transformation? Let \( f : X \to Y \) in \( \mathcal{I}/\mathcal{U} \) be represented by \( X \leftarrow Z \rightarrow Y \), where \( s \) is in the saturated multiplicative system of morphisms corresponding to \( \mathcal{U} \). The commutative diamond

\[
\begin{array}{ccc}
F(X) & \xrightarrow{q(X)} & G(X) \\
\downarrow{id} & & \downarrow{G(s)} \\
F(X) & & G(Y)
\end{array}
\]

\[
\begin{array}{ccc}
F(Z) & \xrightarrow{q(Z)} & G(Z) \\
\downarrow{F(s)} & & \downarrow{G(a)} \\
F(X) & & G(Y)
\end{array}
\]
shows that $\varphi(X) \circ G(f)$ coincides with $F(X)$ $\xleftarrow{F(f)} F(Z) \xrightarrow{G(a) \circ \varphi(Z)} G(Y)$, but as 

$$
\begin{align*}
F(Z) \xrightarrow{F(a)} & F(Y) \\
\downarrow q(Z) & \downarrow q(Y) \\
G(Z) \xrightarrow{G(a)} & G(Y)
\end{align*}
$$

commutes, this morphism is just $F(X) \xleftarrow{F(f)} F(Z) \xrightarrow{q(Y) \circ F(a)} G(Y) = F(f) \circ \varphi(Y)$, which shows that $\varphi$ is indeed a natural transformation.

**1.4.7.** In c), by 1.1.9, the morphism $B \to C$ is the unique such that completes the morphism of triangles $(B, j_! j^* X[1], j_* j^* X[1]) \to (C, X[1], j_* j^* X[1])$.

**1.4.13.** The distinguished triangle $(\tau^F_{\leq p} X, X, i_* \tau_{> p} i^* X)$ is the distinguished triangle $(A, Y, i_* \tau_{> p} i^* Y)$ of 1.4.10, since as remarked, $X = Y$ since $\tau_{> 0} j^* X = 0$. To check that $A = \tau^F_{\leq p} X$, note that $A$ belongs to $D_{\leq p}$, as $i^* A \simeq \tau_{\leq p} i^* X$, and that if $T$ belongs to $D_{\leq p}$, then by applying $\text{Hom}(T, -)$ to the above distinguished triangle and observing that as $i_*$ is t-exact, it commutes with truncation, so $i_* \tau_{> p} i^* X[-1]$ lies in $D_{> p+1}$, $\text{Hom}(T, i_* \tau_{> p} i^* X) = 0 = \text{Hom}^{-1}(T, i_* \tau_{> p} i^* X)$, and $\text{Hom}(T, X) \simeq \text{Hom}(T, A)$.

The distinguished triangle $(\tau^F_{\leq -p+1} X, X, i_* \tau_{> -p-1} i^* X)$ and the fact that $i_*$ commutes with truncation establishes $i_* \tau_{> -p-1} i^* X$ as $\tau_{> -p-1} X$ for the t-structure on $D$; applying $\tau_{\leq p}$ and passing it through the $i_*$ gives the statement about cohomology.

**1.4.14.** To find the dual statement at the end of the proof, reverse arrows and exchange $j! \leftrightarrow j_*$, $i^* \leftrightarrow i!$ to obtain the distinguished triangle $(i_* i^* X[-1], j_! Y, X)$, then use (b’), the isomorphism $j_* / j! \simeq i! j_! [1]$ of 1.4.6.4, (and the note to 1.3.3) to write

$$
i_* i^* X[-1] = i_* (\tau_{\leq -p-1} (j_* / j!) Y)[-1] = i_* \tau_{\leq -p} ((j_* / j!) Y[-1]) = i_* \tau_{\leq -p+1} j_! Y,
$$

establishing $X$ as $\tau^F_{\geq -p+1} j_! Y$.

**1.4.17.** A little note: $p^i i^* X$ is the largest quotient of $X$ belonging to $C_F$. First we check that it is a quotient from 1.4.17 (ii). Then, suppose $A$ belongs to $C_F$ and $X \to A$; then $p^i i^* X \to p^i i^* A \to A$, as $p^i$, is fully faithful, so the adjunction morphism $p^i i^* p_{i*} \to \text{id}$ is an isomorphism, and $p^i i^* X$ is indeed the largest quotient of $X$ in $C_F$. Dually for $p^i i_!$. 

1.4.18. A little note about $T$ faithful: as $p^* j^*$ is an exact functor, if \( p^* j^* f_1 = 0 \), this means that $p^* j^* \text{im}(f_1) = \text{im}(p^* j^* f_1) = 0$, which is to say that $\text{im} f_1$ belongs to $\overline{C}_F$.

1.4.23. In the distinguished triangle \((i^* H^0 i^! B, \tau^F_{\geq 0} j^! B, \tau^F_{\geq 1} j^! B)\), as $j^! B = \tau^U_{\geq 0} j^! B$, \( \tau^F_{\geq p} \tau^U_{\geq p} \), and $j^!$ is right t-exact, $\iota_* H^0 i^! j^! B$ sits in $\mathcal{C}$. Likewise, $\iota_*$ is t-exact, so $\iota_* H^0 i^! j^! B$ also sits in $\mathcal{C}$, and from the long exact sequence of $H^j$ one finds that $\tau^F_{\geq 1} j^! B$ is in $D([-1, 0])$.

2.1.2. In the discussion ‘Si les foncteurs $\circ i^!$ sont de dimension cohomologique finie...’ it is claimed that there is a neighborhood of $S$ in which $H^j \tau_{<a} K$ is supported on $S$. To find such a neighborhood, simply discard $\overline{S} - S$ and the closure of any stratum which doesn’t meet $S$. The assumption that the closure of each stratum is a union of strata implies that the induced stratification of the resulting neighborhood of $S$ has the property that every stratum contains $S$ in its closure, and therefore $H^j \tau_{<a} K$ vanishes on every stratum distinct from $S$. By construction, $S$ is a closed set in this neighborhood.

As for the isomorphism $H^j (i^! S \tau_{<a} K) \sim H^j (i^! S \tau_{<a} K)$ for $i < a$, let us replace $X$ by the neighborhood above, in which case the adjunction morphism $\tau_{>a} K \to i^! S i^* \tau_{<a} K$ is an isomorphism as $\iota_* i^!$ and $\iota^* i^!$ are exact and the induced morphism on cohomology $H^j (\tau_{>a} K) \to i^! S i^* H^j (\tau_{<a} K)$ is an isomorphism for all $j$. Therefore by 1.4.1.2, $i^! S \tau_{<a} K \sim i^! S i^* \tau_{<a} K \sim i^* \tau_{<a} K$.

2.1.13. To get the desired conclusion from the spectral sequence $R^p j^* H^q K \Rightarrow H^{p+q} R_j K$, recall that the locally constant constructible sheaves form a weak Serre subcategory of the category of constructible sheaves of $\mathcal{O}$-modules. For $j^!$ when $j$ is a closed immersion, just use that $j^! = j_*$ is exact in the long exact sequence of cohomology for the distinguished triangle $(j^! j^! K, K, k_* k^* K)$.

One can deduce that the truncation operators $\tau_{\leq p}$ and $\tau_{\geq p}$ respect $D_c (X, R)$ from the proof of 1.4.10 by induction on the number of strata à la 2.1.3.

2.1.14. For the isomorphism $i^! L \sim i^* L \otimes_Z \mathcal{O} [-d]$, combine Proposition 4.3.6 from Dimca, *Sheaves in Topology* with the description of the relative dualizing complex in, e.g., Remark 3.3.5 in Kashiwara-Schapira, *Sheaves on Manifolds*, taking $S = \{ * \}$ there and noting that $f^! A_X$ is the relative dualizing complex $\omega_{Y/X}$ (Definition 3.1.16), and the
orientation sheaves are self-dual (so that Deligne’s or coincides with \( \text{or}_T \otimes \text{or}_S \)). The ‘non-characteristic’ hypothesis in Dimca’s Proposition is trivial in light of the result that precedes it and the fact that the \( H^n \) of \( j^! K \) are locally constant.

Note that by the description of \( i^! \) in terms of local cohomology, this gives a statement like, \( H^n_T(S, F) \) vanishes in degrees less than the codimension of \( T \), for \( F \) a locally free sheaf (with obvious extension to \( F \) a bounded below complex with locally free \( H^n \)).

**2.1.16.** In the remarks about Verdier duality, the fact that the functor \( D \) exchanges \( j! \leftrightarrow j_*, \ j^! \leftrightarrow j^* \) follows by taking \( L \) to be the dualizing complex in the local formulas of adjunction

\[
\begin{align*}
j_* DK &= j_* \text{RHom}(K, j^! L) \sim \text{RHom}(j_* K, L) = D j^* K \\
D j^* K &= \text{RHom}(j^* K, j^! L) \sim j^! \text{RHom}(K, L) = j^! DK
\end{align*}
\]

(the latter may be found, e.g. in SGA 4 Exp. XVIII 3.1.12.2, or Dimca, 3.3.7, but see especially SGA 5 Exp. I 1.12). These formulæ hold for more general morphisms than the inclusion of a locally closed subscheme; the condition is compactifiability.

In Th. finitude §4 ‘Bidualité locale,’ Deligne puts Verdier duality for étale cohomology on firm footing in the case of \( a : X \to S \) a scheme of finite type over \( S \) a regular scheme of dimension 0 or 1. If \( A = \mathbb{Z}/n \), \( K_S \) constant sheaf on \( S \) with value \( A \), and \( K \) in \( D^b_{ctf}(X, A) \), put \( K_X := R a^! K_S \) and \( DK := \text{RHom}(K, K_X) \). Then \( K_X \) is dualizing; i.e.

**Theorem (Deligne).** — \( K \sim DDK \).

This involutivity establishes the stated duality in the formalism, since we may write

\[
D j_* K = D j_* DDK = DDD j^* K = j^! DK.
\]

It is essential that the cohomology sheaves be locally constant of finite rank when restricted to each stratum and that \( R \) have the proscribed properties so that Poincaré duality holds on each stratum. In fact, the definition of perverse sheaf is engineered expressly so that on each stratum we have Poincaré duality, and this data determines the Verdier dual of the sheaf on the stratified space.
About the formula $\mathcal{H}^i \mathcal{D}(K) = (H^{-d-i} \mathcal{K})^\vee \otimes \mathcal{O}$ or, let’s do it instead in the case $R = \mathbb{Z}/\ell$, in which case the dualizing complex is $\mathbb{Z}/\ell(d)[2d]$ and the formula is

$$\mathcal{H}^i \mathcal{D}(K) = (H^{-2d-i} \mathcal{K})^\vee(d).$$

(†)

This is simply Poincaré duality (SGA A XVIII 3.2.5). (In light of the note to 2.1 in Th. finitude, this is the immediate consequence of the weakly convergent spectral sequence

$$E_2^{p,q} = \text{Ext}_R^{p}(H^{-q}(K), \mathbb{Z}/\ell(d)) \Rightarrow \text{Ext}_R^{p+q}(K, \mathbb{Z}/\ell(d))$$

which collapses at the $E_2$ page.)

The business about exchanging $p^* \mathcal{H}^i$ and $p_! \mathcal{H}^{-i}$ is seen to be true for $\mathcal{H}^0$, then use

$$\mathcal{H}^i K = \mathbb{H}^0(K[i])$$

and $\mathcal{D}(K[n]) = R\text{Hom}(K[n], -) = R\text{Hom}(K, -)[-n] = (\mathcal{D}K)[-n]$

2.2.2. A word about the ‘trivial’ implication (ii)$\Rightarrow$(iii). As each $S$ in $\mathcal{S}$ is lisse equidimensional, applying (ii) to each irreducible component we find there exists a Zariski dense open $i : U \rightarrow S$ such that $i^* \mathcal{H}^i i_S^* \mathcal{K} = \mathcal{H}^i i_S^* \mathcal{K}$ (resp. $i^* \mathcal{H}^i i_S^! \mathcal{K} = \mathcal{H}^i i_S^! \mathcal{K}$) vanish in degrees $i > \rho(S)$ (resp. $i < \rho(S)$); as $\mathcal{H}^i i_S^* \mathcal{K}$ and $\mathcal{H}^i i_S^! \mathcal{K}$ are locally constant, this implies they also vanish when their restrictions to $U$ do.

In the second paragraph, ‘il reste à montrer que chacune implique que $\mathcal{H}^i K = 0$ pour $i > b$ (resp. $i < a$).’ Without securing this, we would not have that $\tau_{\leq a} K$ (resp. $\tau_{\geq b} K$) belongs to $\mathcal{D}^b$ and the proof in the first paragraph wouldn’t apply. In both cases, the verification for $i > b$ is easy as $i^*$ is exact. The verification for (iii) $i < a$ follows the proof of (2.1.2.1) exactly, after $\mathcal{S}$ has been replaced by a finer stratification. The verification for (ii) follows immediately by the noetherian property from the stated claim that for each irreducible subvariety $S'$ and each $i < a$, there exists a dense open $S$ of $S'$ such that $\mathcal{H}^i K$ vanishes on $S$. To prove this claim we proceed by descending induction on dim $S'$, the maximal nonvacuous case being easy since such an $S'$ is an irreducible component of $X$, and $\mathcal{H}^i K$ vanishes on the dense open $S$ obtained from (ii) since $S$ is open in $X$ and $i^!_S = i_S^*$ is exact. The case of general $S'$ will follow from the argument of the proof of (2.1.2.1) (see note for 2.1.2) if we can find a neighborhood $U$ of the $S$ obtained from (ii) so that $\text{supp}((\mathcal{H}^i \tau_{\leq a} K)|_U) \subset S$. Begin with the irreducible component containing $S'$; the inductive assumption gives an open set which has either empty or nonempty intersection with $S'$. If nonempty, then this intersection is the desired open
2.2.3. In the discussion of the intermediate extension, the triangle \((\tau_{\leq i}^* j_! A, \tau_{\geq i}^* j_! A, \tau_{< i}^* j_! A)\) is distinguished, not the one written, and if \(\tau_{\leq i}^* j_! A\) is in \(pD_{c}^{\geq 0}\), then indeed \(\tau_{\leq i}^* j_! A \sim \tau_{< i}^* j_! A\) and \(\tau_{\geq i}^* j_! A \simeq p\tau_{\leq 0} j_! A\); the latter isomorphism establishes an isomorphism \(p\tau_{\geq 0} j_! A \sim \tau_{< i}^* j_! A\), which differs by one character from what is written.

2.2.8. The discussion of \(\text{RHom}\) differs from what is proved in 2.1.20 in that there is no longer a fixed stratification. Fortunately, if \(p \leq a, b \geq q\), there are only finitely many solutions to \(i = m - n\) for a fixed \(i\) and for \(n \leq a, m \geq b\). Therefore for each \(i\), we can apply the reasoning of 2.1.20 to a common refinement of only finitely many stratifications.

2.2.10. As remarked in the note to 2.1.16, we will an involutive Verdier duality regardless of whether each stratum is smooth or the sheaf on it is lisse (or has lisse cohomology). The reason we restrict to smooth strata and complexes of sheaves \(K\) with lisse cohomology on each stratum is so that we can determine precisely the degrees in which the Verdier dual \(DK\) is concentrated due to the simple description of the dualizing complex and the simple computation of \(\text{RHom}\) (c.f. the note to Th. finitude 1.6). This is necessary to make Verdier duality a compatible operation vis à vis the t-structures attached to a given perversity function and its dual.

The program outlined in this paragraph achieves both the smoothness of the strata and the lisseté of the cohomology sheaves on each stratum so that the latter property is moreover respected by the six functors (and hence also by Verdier duality). Here is an explication of the smoothness condition (a).

Lemma (EGA 0IV 22.5.8 & IV 6.7.6, 6.7.8, Stacks tags 07EL & 038X). — Let \(X\) be a scheme locally of finite type over a field \(k\) and \(x \in X\). Then the following are equivalent:
(i) $X \to \text{Spec} \, k$ is smooth at $x$.
(ii) $X$ is geometrically regular at $x$, i.e. for every finite extension $k'$ of $k$, the semi-local ring $(\mathcal{O}_X)_x \otimes_k k'$ is regular.
(iii) $X \times_k \overline{k}$ is regular at every point lying over $x$.

The smoothness condition on strata is that over $\overline{k}$, each stratum $S$, with the reduced subscheme structure, is smooth. The claim is then that on $S$ equidimensional of dimension $d$, the dualizing complex is given by $\mathbb{Z}/\ell(d)[2d]$. After replacing $k$ by its perfect closure, anodyne operation with respect to the étale topology, we may assume $S$ is of finite type over a perfect field $k$. The fact that $S \times_k \overline{k}$ is smooth implies (in light of the lemma and Stacks tag 030U) that $S$, with its reduced scheme structure, is smooth. In this case, the fact about the dualizing complex is standard (SGAA Exp. XVIII 3.2.5).

2.2.14. A brief review of Galois cohomology of a finite field $k$, to recall why the groups $H^i(\text{Gal}(\overline{k}/k'), \mathbb{Z}/\ell)$ are finite for every finite extension $k'$ of $k$. We have

$$
H^0 = H^1 = \mathbb{Z}/\ell
$$

$$
H^i = 0 \quad i > 1.
$$

The case of $H^0$ is obvious as it corresponds to taking $G = \text{Gal}(\overline{k}/k')$-invariants of a trivial $G$-module. The $H^1$ is the corollary of a formula given in Serre, Corps Locaux, Ch. XIII Prop. 1 (p. 197 in the 1968 édition Hermann). The vanishing in degrees $> 1$ is because a finite field is $C_1$ and hence has finite cohomological dimension; now see (1.6) in Arcata, SGA 4$\frac{1}{2}$. For more details see Serre, Cohomologie Galoisienne, Ch. II, §3.

Let $D_n = D_{ctf}^b(X, \mathbb{Z}/\ell^n)$, $K, L$ objects of $D_n$, $G = \text{Gal}(\overline{k}, k)$, and $f : X \to \text{Spec} \, k$ the structure morphism. Why does the above imply that $\text{Hom}_{D_n}(K, L)$ is finite? Deligne’s finiteness theorems show that the sheaves $\text{RHom}(K, L)$ belong to $D_n$. The Hochschild-Serre spectral sequence gives

$$
E_2^{i,j} = H^i(G, R^j\Gamma(X \times_k \overline{k}, \text{RHom}(K, L))) \Rightarrow R^{i+j}\Gamma(X, \text{RHom}(K, L)). \quad (\dagger)
$$

As $R\Gamma(X \times_k \overline{k}, \text{RHom}(K, L))$ coincides with the stalk of the constructible sheaf $Rf_*\text{RHom}(K, L)$ at any geometric point of Spec $k$, it belongs to $D_{ctf}^+(\mathbb{Z}/\ell^n)$ (even $D_{ctf}^+$; c.f. Th. finitude 1.7).

Fix $j$ and let $A := R^j\Gamma(X \times_k \overline{k}, \text{RHom}(K, L))$; it is a finite $G$-module. Let $U$ denote the
kernel of $G \to \text{Aut} A$; it is an open normal subgroup of finite index corresponding to a finite extension $k'$ of $k$. The Galois group $U = \text{Gal}(\overline{k}, k')$ acts trivially on $A$ and a simple dévissage reducing to the case $\mathbb{Z}/\ell$ shows that the $H^i(U, A)$ are finite. As $G/U$ is a finite group, the spectral sequence (c.f. *Cohomologie Galoisienne* §2.6b)

$$H^p(G/U, H^q(U, A)) \Rightarrow H^{p+q}(G, A)$$

shows that the groups $H^i(G, A)$ are also finite, and therefore that the objects on the $E^2$ page of $(\dagger)$ are finite $\mathbb{Z}/\ell^n$-modules so that $R^0\Gamma(X, R\text{Hom}(K, L)) = \text{Hom}_{\mathcal{D}_n}(K, L)$ is finite.

### 2.2.16. See note to Weil II (1.1.2)

In that paper, the functor $H^i(K)$ is defined as the pro-sheaf which is the projective system defined by the $H^i(K_n)$; the corresponding projective system is AR-isomorphic to an $(\ell\mathbb{Z})$-adic sheaf in the naïve sense. This allows us to upgrade the pointwise exact sequence (*) of those notes to the corresponding sequence of sheaves (2.2.16.1). There, he uses the notation $K \otimes^{L} \mathbb{Z}/\ell^n$ for $K_n$, whereas here $\otimes$ is used instead of $\otimes^L$. In the interest of consistency, I will continue with the notation $K \otimes^{L} \mathbb{Z}/\ell^n$ for $K_n$. So in this paragraph, we are implicitly in the AR category or in the category of pro-sheaves. (In Weil II, Deligne uses the definition of $\mathbb{Z}_\ell$-sheaf as any pro-sheaf in the essential image of the $\mathbb{Z}_\ell$-sheaves; i.e. the $(\ell\mathbb{Z})$-adic objects of the category of abelian constructible sheaves.) For the business about $H^0$ inducing an equivalence between $D^b_c(X, \mathbb{Z}_\ell)^{\leq 0} \cap D^b_c(X, \mathbb{Z}_\ell)^{\geq 0}$ and $\mathbb{Z}_\ell$-constructible sheaves, this is simple, since for such a $K$ we can represent $K \otimes^{L} \mathbb{Z}/\ell$ by a sheaf concentrated in degree 0. As every complex of flat sheaves representing the $K_n$ admits an $\ell$-adic filtration with successive quotients quasi-isomorphic to $K \otimes^{L} \mathbb{Z}/\ell$, we can represent $K$ by a projective system of flat sheaves concentrated in degree 0. In this case, $K$ is a bona fide $\ell$-adic sheaf.

As for checking whether $K$ belongs to $D^b_c(X, \mathbb{Z}_\ell)^{\leq 0}$, the statement is punctual and we may consider the problem in $D_{\text{parf}}$. If $H^i(K \otimes^{L} \mathbb{Z}/\ell^n)$ is null for one $n$, then $H^i(K) \otimes \mathbb{Z}/\ell^n$ is null by (2.2.16.1), so $H^i(K) \otimes \mathbb{Z}/\ell$ is null. This implies by the exact sequence

$$0 \rightarrow \ell H^i(K)/\ell^n H^i(K) \rightarrow H^i(K)/\ell^n H^i(K) \rightarrow H^i(K)/\ell H^i(K) \rightarrow 0$$
and the fact that $H^i(K)/\ell^{n-1} \to H^i(K)/\ell^n$ surjects that $H^i(K) \otimes \mathbb{Z}/\ell^n$ is null for all $n$ and so $K$ is in $D^b_c(X, \mathbb{Z}_\ell)^{\leq 0}$. On the other hand, if $K$ is in $D^b_c(X, \mathbb{Z}_\ell)^{\leq 0}$, the exact sequence (2.2.16.1) for $n = 1$ tells us that $H^i(K \otimes^L \mathbb{Z}/\ell) = 0$ for $i > 0$. The $\ell$-adic filtration on flat complexes representing $K \otimes^L \mathbb{Z}/\ell$ has successive quotients quasi-isomorphic to $K \otimes^L \mathbb{Z}/\ell$ and the sequence of cohomology then establishes that $H^i(K \otimes^L \mathbb{Z}/\ell^n)$ is null.

**2.2.17.** On the equivalent conditions: suppose $K \otimes^L \mathbb{Z}/\ell^n$ is in $D^b_{S, \mathcal{L}}(X, \mathbb{Z}/\ell^n)$ for one $n$. In the spirit of Weil II (1.1.2) claim a) observe that as the projective system $H^i(K \otimes^L \mathbb{Z}/\ell^k)$ is noetherian AR-$(\ell \mathbb{Z})$-adic, the projective system $H^i(K \otimes^L \mathbb{Z}/\ell^{kn})$ is noetherian AR-$(\ell^n \mathbb{Z})$-adic and AR-isomorphic by (3.2.3) of that section (i.e. [SGA5 V, 3.2.3]) to the $(\ell^n \mathbb{Z})$-adic system $(\mathcal{H}^i_{k+r}/\ell^{kn} \mathcal{H}^i_{k+r})_{k \in \mathbb{N}}$ for some integer $r \geq 0$, where $\mathcal{H}^i_{k}$ denotes the projective system of universal images of the system $H^i(K \otimes^L \mathbb{Z}/\ell^{kn})$. As $K \otimes^L \mathbb{Z}/\ell^n$ is in $D^b_{S, \mathcal{L}}(X, \mathbb{Z}/\ell^n)$, the $H^i(K \otimes^L \mathbb{Z}/\ell^n)$ are locally constant on the strata in $S$. Taking successive quotients on the $\ell^n$-adic filtration on bounded complexes of flat sheaves representing the $K \otimes^L \mathbb{Z}/\ell^{kn}$, we find that the $H^i(K \otimes^L \mathbb{Z}/\ell^{kn})$ are also locally constant on the strata. Therefore the sheaves $\mathcal{H}^i_{k}$ are, as well as the sheaves $\mathcal{H}^i_{k+r}/\ell^{(k+1)n} \mathcal{H}^i_{k+r}$. For $k = 0$, the latter sheaf is isomorphic to $H^i(K) \otimes \mathbb{Z}/\ell^n$, so we have shown that $H^i(K) \otimes \mathbb{Z}/\ell^n$ is locally constant on the strata $S$.

(An equivalent way to argue is again to use the description of [SGA5 V, 3.2.3] and just note that when computing the universal image subsheaves of $H^i(K \otimes^L \mathbb{Z}/\ell^{kn})$, one can restrict to looking at the images of the sheaves $H^i(K \otimes^L \mathbb{Z}/\ell^{kn})$, and when finding an $r$, if $r$ works, then $s$ works for any $s \geq r$, so $r$ can be taken to be a multiple of $kn$.)

As $H^i(K) \otimes \mathbb{Z}/\ell^n$ is locally constant and includes into $H^i(K \otimes^L \mathbb{Z}/\ell^n)$, on each stratum $S \in \mathcal{S}$ consider a Jordan-Hölder series for both. The constituents of the former are a subset of the constituents of the latter and therefore also belong to $\mathcal{L}(S)$. We have shown that $H^i(K) \otimes \mathbb{Z}/\ell^n$, and therefore $H^i(K) \otimes \mathbb{Z}/\ell$, is $(\mathcal{S}, \mathcal{L})$-constructible.

On the other hand, suppose the $H^i(K) \otimes \mathbb{Z}/\ell$ are $(\mathcal{S}, \mathcal{L})$-constructible, and without loss of generality let $K$ be in $D^b_c(X, \mathbb{Z}_\ell)^{\leq 0}$. Proceed by recurrence on $-j$; in the case $j = 0$, $H^1(K) = 0$ and $H^0(K) \otimes \mathbb{Z}/\ell \sim H^0(K \otimes^L \mathbb{Z}/\ell)$. The $\ell$-adic filtration on any bounded complex of flat sheaves representing $K \otimes^L \mathbb{Z}/\ell^n$ has successive quotients quasi-isomorphic to $K \otimes^L \mathbb{Z}/\ell$; taking the long exact sequence of cohomology finds that
This last sheaf is easily seen to be \((\mathcal{S}, \mathcal{L})\)-constructible, as it is locally constant wherever \(\mathcal{K}\) is, and as a subsheaf, its constituents on a stratum are a subset of the constituents of \(\mathcal{K}\) on that stratum. Since \(\text{Tor}_1^Z(H^0(K), \mathbb{Z}/\ell^n)\) and \(H^{-1}(K) \otimes \mathbb{Z}/\ell\) are \((\mathcal{S}, \mathcal{L})\)-constructible, (2.2.16.1) shows that \(H^{-1}(K \otimes^L \mathbb{Z}/\ell^n)\) is for all \(n\). The argument above then shows that \(H^{-1}(K) \otimes \mathbb{Z}/\ell^n\) are \((\mathcal{S}, \mathcal{L})\)-constructible for all \(n\), and hence that \(\text{Tor}_1^Z(H^{-1}(K), \mathbb{Z}/\ell^n)\) are for all \(n\), etc.

In order to proceed to define the t-structure in imitation of 2.2.10, one needs to extend the six functors to \(D^b_c(X, \mathbb{Z}_\ell)\). This is trivial because they commute with reduction modulo \(\ell^n\); see note to Weil II 1.1.2c. Then the claim about \(K\) belonging to \(pD^b_c(X, \mathbb{Z}_\ell)\) iff its reduction modulo \(\ell\) belongs to \(pD^b_c(X, \mathbb{Z}_\ell)\) is also trivial.

\[\text{2.2.18. See note to Weil II 1.1.3} \] Multiplication by \(\ell\) on complexes of flat sheaves representing \(K\) in \(D^b_c(X, \mathbb{Z}_\ell)\) induces a multiplication by \(\ell\) on their cohomology. To see that the image \(D^b_{c, \mathcal{S}, \mathcal{L}}(X, \mathbb{Z}_\ell) \otimes \mathbb{Q}_\ell\) consists of those \(K\) such that each \(H^iK\) is the \(\mathbb{Q}_\ell\otimes\) of an \((\mathcal{S}, \mathcal{L})\)-constructible \(\mathbb{Z}_\ell\)-sheaf, well, certainly it is contained in it. On the other hand, given \(K\) in \(D^b_c(X, \mathbb{Z}_\ell)\) with each \(H^iK\) the \(\mathbb{Q}_\ell\otimes\) of an \((\mathcal{S}, \mathcal{L})\)-constructible \(\mathbb{Z}_\ell\)-sheaf \(\mathcal{F}_i\), then for each of finitely many nonzero \(i\), there exist nonzero \(a_i, b_i \in \mathbb{Z}_\ell\) such that \(a_iH^iK = b_i\mathcal{F}_i\); \(b_i\mathcal{F}_i\) is a subsheaf of \(\mathcal{F}_i\) and is also \((\mathcal{S}, \mathcal{L})\)-constructible. Then \((\prod_i a_i)K\) has cohomology sheaves which are \((\mathcal{S}, \mathcal{L})\)-constructible, showing the reverse containment.

Turning now to the claim that the forgetful functor \(\omega\) induces an equivalence

\[D^b_c(X, E_h) \to \{\text{category of objects } K \text{ of } D^b_c(X, \mathbb{Q}_\ell) \text{ equipped with a morphism of } \mathbb{Q}_\ell\text{-algebras } E_h \to \text{End}(K)\},\]
the essential surjectivity of \( \omega \) follows from the fact that if \( K \) is in \( \text{D}^b_c(X, \mathbb{Q}_\ell) \) and equipped with an action \( \phi: E_\lambda \to \text{End}(K) \), and \( K \otimes_{\mathbb{Q}_\ell} E_\lambda \) has \( E_\lambda \) acting on itself, then there is an \( E_\lambda \)-equivariant imbedding \( K \to K \otimes E_\lambda \) with retraction \( r \). Let \( \alpha \) be a primitive element for the extension \( E_\lambda/\mathbb{Q}_\ell \) so that \( E_\lambda \cong \mathbb{Q}(\alpha) \), and let \( d \) denote the degree \([E_\lambda : \mathbb{Q}_\ell]\). The maps are given by 

\[
i: K \to K \otimes E_\lambda \quad \quad r: K \otimes E_\lambda \to K
\]

\[
K \mapsto \frac{1}{d} \sum_{i=0}^{d-1} \phi(\alpha)^i K \otimes \alpha^{-i} \quad \quad K \otimes a \mapsto \phi(a) K.
\]

This implies that \( K \) is indeed a direct factor of \( K \otimes E_\lambda \); see Neeman, *Triangulated Categories* 1.2.10. This in turn gives an idempotent in \( \text{End}(K \otimes E_\lambda) \), and if the image of this idempotent is represented in \( \text{D}^b_c(X, E_\lambda) \), then this implies a splitting of \( K \otimes E_\lambda \) in \( \text{D}^b_c(X, E_\lambda) \) which is sent by \( \omega \) to the direct factor \( K \). A category in which every idempotent splits is called alternatively Cauchy complete, idempotent complete, or Karoubi complete (see SGA 4, I 8.7.8), so we are done if we show that \( \text{D}^b_c(X, E_\lambda) \) is Karoubi complete. The splitting of an idempotent \( e \) in the endomorphism ring of an object in some category is equivalent to the existence of the equalizer \( i = \ker(e, \text{id}) \) or the coequalizer \( r = \text{coker}(e, \text{id}) \), and this (co)equalizer, if it exists, is an absolute (co)limit; i.e. it is preserved by every functor (see Proposition 1 of Borceaux and Dejean, *Cauchy Completion in Category Theory*). As \( \text{D}^b_c(X, E_\lambda) \) is a projective limit of categories \( \text{D}^b_{ctf}(X, \mathbb{R}/m^n) \) for \( \mathbb{R} \) the ring of integers in \( E_\lambda \), it is easily seen that if the categories \( \mathbb{D}_n := \text{D}^b_{ctf}(X, \mathbb{R}/m^n) \) are Karoubi complete, if \( e \) is an idempotent in \( \text{End}(K \otimes E) \), its reductions in \( \mathbb{D}_n \) are idempotents which split, and as these splittings are absolute (co)limits, they automatically give an object in \( \text{D}^b_c(X, E_\lambda) \) splitting \( e \). It will suffice to show that \( \mathbb{D}_n \) is Karoubi complete. Let’s say that a triangulated category has direct sums if it has (arbitrary) categorical direct sums and if the (arbitrary) direct sum of distinguished triangles is distinguished. Bökstedt and Neeman show in *Homotopy limits in triangulated categories* 3.2 that if a triangulated category has direct sums, it is Karoubi complete. (Actually, all that is needed is that countable coproducts of objects in the triangulated category exist; a simple exposition is Neeman, *Triangulated Categories* §1.6, in particular (1.6.8).) The category \( \text{D}(X, \mathbb{R}/m^n) \) is therefore Karoubi complete. An object \( C \) of an additive category with arbitrary direct sums is said to be
compact if Hom(C, −) commutes with arbitrary direct sums. Any direct summand of a compact object C is compact, since a finite colimit of compact objects is compact*. Therefore as D(X, R/m^n) is Karoubi complete, so is the full subcategory generated by compact objects. That D_{ctf}^b(X, R/m^n) coincides with the subcategory generated by compact objects of D(X, R/m^n) is 6.4.8 of Bhatt-Scholze, *The pro-étale topology for schemes*, after you recall that the objects of D_{ctf}^b(X, R/m^n) can be represented by bounded complexes of R/m^n-flat constructible sheaves (Rapport 4.6).

(*) To recognize A, a direct summand of C, as a finite colimit, note that for the retraction A → C → A, A is both the equalizer and coequalizer of id_C and ip (by the maps i and p, respectively). As coequalizer, A is a finite colimit; we can therefore write
\[
\text{Hom} \left( \text{colim}_i C, \bigoplus_{i \in I} U_i \right) = \text{lim} \text{Hom} \left( C, \bigoplus_{i \in I} U_i \right) = \text{lim} \bigoplus_{i \in I} \text{Hom}(C, U_i),
\]
where the last equality is because C is compact; now use that finite limits commute with filtered colimits in Set to write
\[
\text{lim} \bigoplus_{i \in I} \text{Hom}(C, U_i) = \bigoplus_{i \in I} \text{lim} \text{Hom}(C, U_i) = \bigoplus_{i \in I} \text{Hom}(\text{colim}_i C, U_i).
\]
As A = colim_i C, this proves A compact.

2.2.19. The fact that i^!f_* = f_*i^! is SGAA XVIII 3.1.12.3. The ‘argument habituel d’homotopie’ referenced in the last sentence of the proof is referring to, e.g. Th. 5.7.1 in Godement, *Théorie des faisceaux*, but a more accessible reference is [Stacks 09UY](https://stacks.math.columbia.edu/tag/09UY). If you have two open coverings U = (U_i)_{i \in I} and V = (V_j)_{j \in J} of a space X, and U is a refinement of V, so that we can choose a map \( \phi : I \to J \) such that \( U_i \subset V_{\phi(i)} \) for all \( i \in I \). This induces a map of Čech complexes \( \phi^* : \check{\mathcal{C}}(\mathcal{V}, \mathcal{F}) \to \check{\mathcal{C}}(\mathcal{U}, \mathcal{F}) \) for any sheaf \( \mathcal{F} \) on X. The result then says that if you have a second \( \phi' : I \to J \) such that \( U_i \subset V_{\phi'(i)} \), the maps \( \phi^*, \phi'^* \) are homotopic. This instantly implies that if \( \mathcal{U}, \mathcal{V} \) are mutual refinements, they have the same Čech cohomology, since in this case we can choose \( \phi : I \to J \) and \( \psi : J \to I \) satisfying \( U_i \subset V_{\phi(i)} \) and \( V_j \subset U_{\psi(j)} \); the maps \( \phi \circ \psi \) and \( \psi \circ \phi \) must then induce the identity on Čech cohomology as they are homotopic to the identity on the chain level. Our case is formally equivalent to this statement for \( \check{H}^0 \).
In more words: A is defined as an equalizer of two maps

\[
\prod_I p_{j_i^*A_{U_i}} \rightrightarrows \prod_{I \times I} p_{j_{ij}^*A_{U_{ij}}}
\]

where one map \(d_0^0\) is the product over \((i, j) \in I \times I\) of the maps

\[
\prod_I p_{j_i^*A_{U_i}} \xrightarrow{pr_i} p_{j_i^*A_{U_i}} \rightarrow p_{j_i^*A_{U_{ij}}},
\]

(here the second map is \(j_i^*\) of the unit of the adjunction on \(U_i\) if \(\bullet = i\) and 0 if \(\bullet \neq i\)), and the other map \(d_1^0\) is the product over \((i, j) \in I \times I\) of the maps

\[
\prod_I p_{j_i^*A_{U_i}} \xrightarrow{pr_j} p_{j_j^*A_{U_j}} \rightarrow p_{j_i^*A_{U_{ij}}},
\]

(the second map is \(j_j^*\) of the unit of the adjunction on \(U_j\) if \(\bullet = j\) and 0 if \(\bullet \neq j\)). This recognizes A as \(\text{ker}(d_0^0 - d_1^0)\). Let \(f\) denote maps of the type

\[
U_{i_0} \times \cdots \times U_{i_a} \times \cdots U_{i_p} \rightarrow U_{i_0} \times \cdots \times \hat{U}_{i_a} \times \cdots \times U_{i_p}.
\]

In higher degrees, the differential is determined by the formula

\[
pr_{i_0 \ldots i_+1} \circ d = \sum_{a=0}^{p+1} (-1)^a p_{j_{i_0 \ldots i_a \ldots i_p}^*(\eta(A_{i_0 \ldots i_p})) \circ pr_{i_0 \ldots i_a \ldots i_{p+1}},
\]

where \(\eta\) is the unit of the adjunction \(id \rightarrow p_{f^*}p_f^*\). Let \(U\) be the cover of \(X\) as above and \(s : I \rightarrow J\) satisfying \(s_i : U_i \rightarrow V_{s(i)}\) as above. The chain map induced by \(s\) has the explicit description

\[
pr_{i_A} \circ s^* = p_{j_{s(i_A)}^*(\eta(A_{i_A}))} \circ pr_{s(i_A)},
\]

where \(i_A\) is a multi-index \(i_0i_1 \ldots i_p, s(i_A) = s(i_0) \ldots s(i_p)\), and \(\eta\) is the unit of the adjunction \(id \rightarrow p_{s^*}p_{s^*}\) on \(V_{s(i_A)}\), where \(s_{i_A} : U_{i_0} \times \cdots \times U_{i_p} \rightarrow V_{s(i_0)} \times \cdots \times V_{s(i_p)}\) is deduced by taking the product of the maps \(s_i\).

Now given another map \(t : I \rightarrow J\) satisfying \(t_i : U_i \rightarrow V_{t(i)}\), let \(0 \leq a \leq p\) and let \(f\) now denote maps of the sort

\[
U_{i_0} \times \cdots \times U_{i_p} \rightarrow V_{s(i_0)} \times \cdots \times V_{s(i_a)} \times V_{t(i_a)} \times \cdots \times V_{t(i_p)},
\]
where here we use the map $U_{i_a} \to V_{s(i_a)} \times V_{t(i_a)}$. We set up a homotopy $h$ by the formula

$$pr_{i_0 \ldots i_p} \circ h = \sum_{a=0}^{p} (-1)^{a} p_{j_s(i_0) \ldots s(i_a)t(i_a) \ldots t(i_p) *} (\eta(A_{s(i_0) \ldots s(i_a)t(i_a) \ldots t(i_p)}) \circ pr_{s(i_0) \ldots s(i_a)t(i_a) \ldots t(i_p)}$$

where here $\eta$ is the unit of the adjunction $\text{id} \to pf_* pf^*$. Now we can follow the argument of \textbf{Stacks 01FP}.

**3.1.2.** (3.1.2.7) The morphism $K_{n+1} \to K_n$ in $\text{D}A$ can be represented by $K_{n+1} \xrightarrow{\alpha} K_{n+1}' \xrightarrow{\beta} K_n$ with $\alpha$ a quasi-isomorphism and $\beta$ a homotopy class of morphisms. Replacing $K_n$ by $K_n \oplus \text{cone}(K_{n+1}')$ allows us to write a commutative square

$$\begin{array}{ccc}
K_{n+1}' & \xrightarrow{(\beta, \iota)} & K_n \oplus \text{cone}(K_{n+1}') \\
\downarrow \alpha & & \uparrow \gamma \\
K_{n+1} & \xrightarrow{} & K_n
\end{array}$$

where $\iota$ denotes the canonical map $K_{n+1}' \to \text{cone} K_{n+1}'$. Now $(\beta, \iota)$ is injective and $\gamma$ is a homotopy equivalence so that it can be inverted. Doing this finitely many times allows us to produce a filtered complex of objects $(K, F)$ of $\mathcal{A}$ with the sequence of $F^iK$ isomorphic in $\text{D}A$ to the given sequence.

(3.1.2.8) The intersection of two subobjects $A, B$ of an object $C$ in an abelian category $\mathcal{A}$ is defined as the pullback

$$\begin{array}{ccc}
A \cap B & \xrightarrow{} & A \\
\downarrow & & \downarrow \\
B & \xrightarrow{} & C
\end{array}$$

or equivalently by the exact sequence

$$0 \to A \cap B \to A \oplus B \to C.$$
The sum $A + B$ is defined as the image of $A \oplus B$ under the same map $A \oplus B \to C$ defined by the monomorphisms $A \to C$ and $B \to C$. The inclusion
\[
\sum_{i+j=p} F^i \cap G^j \subset \bigcap_{i+j=p+1} F^i + G^j
\]
is evident from the fact that if $a + b = p$, every pair $(i,j)$ satisfying $i + j = p + 1$ also satisfies $i \leq a$ or $j \leq b$. For the reverse inclusion, find some $a$ in the intersection; then we have expressions $a = f_i + g_j$ for $f_i \in F^i, g_j \in G^j, i + j = p + 1$; as the filtration is finite & decreasing there exists an $n$ such that $a = f_n$ and $g_n = 0$. Then $a - f_{n+1} = g_{p-1-n(i+1)}$ is in $F^i \cap G^{p-n}$. Likewise $f_{n+i} - f_{n+i+1} = g_{p+1-(n+i+1)} - g_{p+1-(n+i)}$ is in $F^{n+i} \cap G^{p-(n+i)}$ for $i \geq 0$. By finiteness, $f_{n+i} = 0$ for $i \gg 0$, so that
\[
a = \sum_{i=0}^{\infty} f_{n+i} - f_{n+i+1} \quad (f_{n+i} - f_{n+i+1} \in F^{n+i} \cap G^{p-(n+i)})
\]
has only finitely many nonzero terms.

3.1.3. This paragraph should be read along with Illusie Complexes cotangent et déformations I Ch. V §1, cited [12]. Equations (3.1.3.4) and (3.1.3.5) are written with the notation $\text{Hom}^{p+q}$ and $\text{Hom}^{p+q}_{DA}$ and $\text{Hom}^{p+q}_{DF_A}$ instead of $\text{Ext}^{p+q}$ and $\text{Ext}^{p+q}_{DA}$ and $\text{Ext}^{p+q}_{DF_A}$. This notation, which continues into (3.1.4) and beyond, is standard but creates something of a conflict, as it should not be conflated with the $\text{Hom}^n$ defined in the first part of (3.1.3), as the $\text{Hom}^n$ there specifies a component of a familiar chain complex. For objects $K, L$ of $DA$, $\text{Ext}^n(K, L) = \text{Hom}_{DA}(K, L[n])$; in these notes we will tend to write $\text{Ext}^n$ instead of $\text{Hom}^n$ even when Deligne writes $\text{Hom}^n$.

3.1.4. The statement of the proposition of course has grammatical grouping ‘lorsque $\{n = 0 \text{ ou } -1\}$ et $i > j$’ and ‘lorsque $\{n = 0 \text{ ou } -1\}$ et $n + i - j < 0’$. The proof of (i) is obtained from the distinguished triangle given by taking the sequence of cohomology and using the vanishing of $H^0$ and $H^{-1}$ of $F^{-\infty}/F^0$. In the proof of (ii), the differential at $(0,0)$ on the $E_r$ page for goes $(0,0) \to (r, 1-r)$; as $E_1^{pq} = 0$ for $p < 0$, $E_0^{00} = \cap r \ker d_r$. The assumptions give that $E_r^{pq} = 0$ whenever $q < 0$ and either $p + q = 0$ or $p + q = 1$, so that $\ker d_r = E_2^{00}$ for $r \geq 2$ and therefore $\text{E}_2^{00} = E_\infty^{00}$. The assumption that $E_1^{pq} = 0$
if $n = 0$ and $q < 0$ implies that the only nonzero graded piece in the filtration on $\text{Hom}_{D_{\text{DF}}}^1(K,L) = \text{Ext}_{D_{\text{DF}}}^0(K,L)$ is the $0^{th}$ and it coincides with $E_{2}^{00}$.

Note: $\text{Ext}^n(\text{Gr}^i_F K, \text{Gr}^j_F L) = \text{Ext}^{n+i-j}(\text{Gr}^i_F K[i], \text{Gr}^j_F L[j])$ so for example the condition $\text{Ext}^n(\text{Gr}^j_F K[-i], \text{Gr}^j_F L[-j]) = 0$ for $n < 0$ equates to $\text{Ext}^{n-j+i}(\text{Gr}^i_F K, \text{Gr}^j_F L) = 0$ for $n < 0$, which implies $\text{Ext}^n(\text{Gr}^i_F K, \text{Gr}^j_F L) = 0$ whenever $i > j$ and $n < 1$.

Remark (vague). As discussed in the note to (3.1.7) below, considering the $\text{Gr}^i_F$ as filtered objects in $D_{\text{DF}}$, $\text{Hom}_{D_{\text{DF}}}^n(\text{Gr}^i_F K, \text{Gr}^j_F K) = 0$ when $i > j$ (any $n$). Of course, forgetting the filtration, $\text{Hom}_{D_{\text{DF}}}^n(\omega \text{Gr}^j_F K, \omega \text{Gr}^i_F K)$ need not vanish. The result of (3.1.4) (i) can be seen as saying that this is precisely the difference between computing $\text{Hom}$ in $D_{\text{DF}}$ versus in $D$, and that moreover it only need be checked for $n = 0$ and $-1$.

3.1.6. The filtration $T_i$ is now taken to be an increasing filtration, corresponding to the decreasing filtration $T^{-i}$. Under this correspondence $(\text{Gr}^i_T K)[i]$ becomes $(\text{Gr}^{-i}_T K)[i]$. $\text{Hom}^n(A,B)$ should be written $\text{Ext}^n(A,B) = \text{Hom}(A,B[n])$, null for $n < 0$ when $A, B$ are in $C$ by (1.3.1) (i).

3.1.7. A priori there is the matter of the uniqueness of the arrow of degree 1 $\text{Gr}^i_F K \rightarrow \text{Gr}^{i+1}_F K$ ‘defined’ by the distinguished triangle $(\text{Gr}^{i+1}_F K, \text{F}^i/\text{F}^{i+2}, \text{Gr}^i_F K)$. Recall [BBD, 1.1.10] which says that this arrow is unique (and therefore actually defined by the property that $(\text{Gr}^{i+1}_F K, \text{F}^i/\text{F}^{i+2}, \text{Gr}^i_F K)$ is distinguished) if $\text{Hom}_{D_{\text{DF}}}^{-1}(\text{Gr}^{i+1}_F K, \text{Gr}^j_F K) = 0$.

\[
\text{Hom}_{D_{\text{DF}}}^{-1}(\text{Gr}^{i+1}_F K, \text{Gr}^j_F K) = \text{Hom}(\text{Gr}^{i+1}_F K, (\text{Gr}^j_F K)[−1]) = \text{Hom}(\text{Gr}^{i+1}_F K, \text{Gr}^j_F (K[−1])) = 0
\]

(all Hom are in $D_{\text{DF}}$) as of course $F^i K \in D_{\text{DF}}$, $\text{Gr}^i_F K = 0$. 

For \( d^{i+1} \circ d^i = 0 \) the point is that in the commutative diagram of distinguished triangles below, the two morphisms \( F^{i+1}/F^{i+3} \rightarrow Gr^{i+1}_F \) coincide.

\[
\begin{array}{cccccc}
Gr^i_F[-1] & \rightarrow & F^{i+1}/F^{i+3} & \rightarrow & F^i/F^{i+3} \\
\downarrow & & \downarrow & & \downarrow \\
Gr^{i+1}_F & \rightarrow & F^i/F^{i+2} & \rightarrow & Gr^i_F \\
& & \downarrow & & \downarrow \\
F^{i+1}/F^{i+3} & \rightarrow & Gr^{i+1}_F & \rightarrow & Gr^{i+2}_F[1]
\end{array}
\]

3.1.8. On the subject of the differential \( d_1(f) = df - fd \), we have to compute it. Illusie discusses the differential in [I2 V 1.4.10]. The spectral sequence (3.1.3.4) is isomorphic to the spectral sequence

\[
E_1^{pq} = H^{p+q} Gr^p_F R\text{Hom}(L, M) \Rightarrow \text{Ext}^{p+q}(L, M).
\]

We replace \( M \) by a filtered injective resolution of \( M \), which is therefore (non-canonically) termwise-split as a direct sum of its graded pieces, which are also injective (05TP), and we compute in the abelian category. General considerations (012N) dictate that the differential \( d_1 \) is equal to the coboundary map associated to the short exact sequence of complexes

\[
0 \rightarrow Gr^{p+1}(\text{Hom}(L, M)) \rightarrow F^p/F^{p+2} \text{Hom}(L, M) \rightarrow Gr^p(\text{Hom}(L, M))) \rightarrow 0,
\]

where the differential on \( \text{Hom}(L, M) \) is defined by

\[
\text{Hom}^n(L, M) = \prod_i \text{Hom}(L^i, M^{i+n}), \quad d^n_p = d_L + (-1)^{n+1}d_M.
\]

We need only consider \( n = 0 \) and for simplicity we look at a single \( i \) at a time. Basically we need to translate this differential via the isomorphism

\[
E_1^{pq} \cong \oplus_k \text{Ext}^{p+q}(Gr^k L, Gr^{k+p} M), \quad \text{or, more properly, through}
\]

\[
Gr^p(\text{Hom}(L^i, M^i)) \cong \oplus_k \text{Hom}(Gr^k L^i, Gr^{k+p} M^i).
\]
The diagrams on the next page, combined with the fact that \( d^0 = d_L - d_M \), show that \( d_1 \), the coboundary map induced by \( d^0 \) and the relevant exact sequence, coincides with

\[
\prod_i \text{Hom}(\text{Gr}^k L^i, \text{Gr}^{k+p} M^i) \ni (f_i) \mapsto (f_i d''_L - d'_M f_{i-1}),
\]

for each \( k \), where \( d'_M \) and \( d''_M \) are induced on \( \text{Hom} \) by coboundaries associated to the exact sequences

\[
0 \to \text{Gr}^{k+p+1} M \to F^{k+p} M/F^{k+p+2} M \to \text{Gr}^{k+p} M \to 0 \quad \text{and} \quad 0 \to \text{Gr}^k L \to F^{k+1} L/F^{k+1} L \to \text{Gr}^{k-1} L \to 0,
\]

respectively.

So \((f_i)\) belongs to \( \ker d_1 \) if \( f_i d''_L - d'_M f_{i-1} = 0 \) for all \( i \), and all that is left to observe is that \( d'_M \) and \( d''_M \) correspond with Deligne’s differential \( d \).
Let $a, b \in \mathbb{Z}$ (we care about $(a, b) = (i - 1, i), (i, i)$, or $(i, i + 1)$) and fix retractions to the monomorphisms $F^p/F^{p+k} M^b := F^p M^b / F^{p+k} M^b \hookrightarrow F^{p+1}/F^{p+k} M^b$ so that we can identify $M^b$ with the direct sum of its graded pieces, which are almost all zero.

$$\begin{align*}
0 &\longrightarrow \text{Gr}^{p+1}(\text{Hom}(L^i, M^i)) \\
&\quad \downarrow \\
&\quad \text{Gr}^p(\text{Hom}(L^i, M^i)) \\
&\quad \downarrow \\
0 &\longrightarrow \text{Gr}^{p+1}(\text{Hom}(L^{i-1}, M^i) \oplus \text{Hom}(L^i, M^{i+1})) \\
&\quad \overset{d_p = d_{l-d}}{\longrightarrow} \\
&\quad \text{Gr}^p(\text{Hom}(L^{i-1}, M^i) \oplus \text{Hom}(L^i, M^{i+1})) \\
&\quad \downarrow \\
0 &\longrightarrow \text{Gr}^{p+1}(\text{Hom}(L^{i-1}, M^i) \oplus \text{Hom}(L^i, M^{i+1})) \\
&\quad \longrightarrow \text{Gr}^p(\text{Hom}(L^{i-1}, M^i) \oplus \text{Hom}(L^i, M^{i+1})) \\
&\quad \longrightarrow 0
\end{align*}$$

The above diagram commutes. The bottom row and diagonal row are induced by the exact sequences

$$\begin{align*}
&0 \longrightarrow \oplus_k \text{Hom}(G^{k-p-1} L^a, Gr^k M^b) \\
&\quad \longrightarrow \oplus_k \text{Hom}(G^{k-p} L^a, Gr^k M^b) \\
&\quad \longrightarrow \oplus_k \text{Hom}(G^{k-p} L^a, Gr^k M^b) \\
&\quad \longrightarrow 0
\end{align*}$$

The above diagram commutes. The bottom row and diagonal row are induced by the exact sequences

$$\begin{align*}
0 \longrightarrow \text{Gr}^{k+p+1} M^b &\longrightarrow F^{k+p} / F^{k+p+2} M^b \longrightarrow \text{Gr}^{k+p} M^b \longrightarrow 0 \\
&\text{and} \\
0 \longrightarrow \text{Gr}^k L^a &\longrightarrow F^{k-1} / F^{k+1} L^a \longrightarrow \text{Gr}^{k-1} L^a \longrightarrow 0,
\end{align*}$$

respectively.
Returning to the proof of (3.1.8), let’s examine why the filtered complex $X$ in the first solution is taken by $G$ to a bounded complex of objects of $\mathcal{C}$ isomorphic to the given one. In the (unconventionally indexed) double complex

$$
\begin{array}{cccc}
\uparrow & \rightarrow & K_{i+1,j-1} & \rightarrow \\
& \uparrow & \uparrow & d'' \\
\rightarrow & K_{i,j-1} & \rightarrow & K_{i,j} \\
& \uparrow & \uparrow & d' \\
\rightarrow & K_{i-1,j-1} & \rightarrow & K_{i-1,j} \\
& \uparrow & \uparrow & \\
\end{array}
$$

the terms $X^n$ correspond to the diagonals with slope $-1$ capturing all terms with biindex summing to $n$.

$$
\text{Gr}^i X \cong (K^i[-i], (-1)^i d'') = K^i[-i].
$$

It remains only to show that the morphism of degree one in the distinguished triangle associated to the exact sequence

$$
0 \rightarrow \text{Gr}^{i+1} X \rightarrow F^i/F^{i+2} X \rightarrow \text{Gr}^i X \rightarrow 0
$$

coincides with $d''$. Actually we will show that it coincides with $-d''$. This is OK, as any complex $L = (L, d)$ is isomorphic to the complex $L' = (L, -d)$ by the morphisms $(-1)^i \text{id} : L^i \rightarrow L^i$ which induce an isomorphism of chain complexes $L \sim L'$. Therefore we will still have shown that $0 \rightarrow K^a \rightarrow \cdots \rightarrow K^b \rightarrow 0$ is in the essential image of $G$.

We have an isomorphism of distinguished triangles

$$
\begin{array}{cccc}
\text{Gr}^{i+1} X & \xrightarrow{f} & F^i/F^{i+2} X & \xrightarrow{g} \text{Gr}^i X \\
\| & & \uparrow \gamma & \| \\
\text{Gr}^{i+1} X & \xrightarrow{\text{Cyl}(f)} & C(f) & \xrightarrow{\delta} \text{Gr}^{i+1} X[1]
\end{array}
$$
where \( \delta : C(f) = \text{Gr}^{i+1} X[1] \oplus F'/F^{i+2}X \rightarrow \text{Gr}^{i+1} X[1] \) corresponds componentwise to projection to the first factor and \( \gamma : C(f) \rightarrow \text{Gr}^1 X \) is given by the composition \( C(f) \xrightarrow{\text{pr}_2} F'/F^{i+2}X \xrightarrow{g} \text{Gr}^1 X \), where here \( \text{pr}_2 \) denotes abusively the morphism of complexes induced by componentwise projection to the second factor.

\[
C(f) = K^{i+1}[i] \oplus K'[\neg i] \oplus K^{i+1}[-i - 1] \quad \text{with differential}
\]
\[
d^n(k^{i+1,n-i}, k^{i,n-i}, k^{i+1,n-i-1}) = ((-1)^i d'', (-1)^i d''', f(k^{i+1,n-i}) + d'k^{i,n-i} + (-1)^{i+1} d'' k^{i+1,n-i-1}).
\]

It suffices to show that \(-d' \circ \gamma \) is homotopic to \( \delta \). Consider the homotopy given by

\[
h^n : C(f)^n = K^{i+1,n-i} \oplus K^{i,n-i} \oplus K^{i+1,n-i-1} \xrightarrow{\text{pr}_3} K^{i+1,n-i-1} = (K^{i+1}[-i])^{n-1}
\]
\[
dh(k^{i+1,n-i}, k^{i,n-i}, k^{i+1,n-i-1}) = (-1)^i d'' k^{i+1,n-i-1}
\]
\[
hd(k^{i+1,n-i}, k^{i,n-i}, k^{i+1,n-i-1}) = d'(k^{i,n-i}) + (-1)^{i+1} d'' k^{i+1,n-i-1} + k^{i+1,n-i}
\]
so that
\[
dh + hd = k^{i+1,n-i} + d'(k^{i,n-i}).
\]

Therefore \( h \) defines a homotopy between the two morphisms

\[
C(f) \xrightarrow{\delta} \xrightarrow{-d' \circ \gamma} K^{i+1}[-i].
\]

As for the construction of the differential of the complex \( X \), let’s check that

\[
d_2 = H_0 + H_1 + H_2 = (-1)^j d'' + d' + \sum_i (-1)^{i+1} H^i
\]

satisfies (*) ; i.e. that \( d_2 \circ d_2 \) is of filtration \( \geq 3 \). To do this, it suffices to look at the terms of \( d_2 \circ d_2 \) which are of filtration \( < 3 \). We denote by \((\geq n)\) terms of filtration \( \geq n \). It suffices to compute what \( d_2 \circ d_2 \) does to \( K^{i,j} \).

\[
d_2 \circ d_2 = d_1 \circ d_1 + d_1 \circ H_2 + H_2 \circ D_1 + H_2 \circ H_2
\]
\[
= d^{i+1} \circ d'' + (-1)^{i+2} d'' \circ H_2 + H_2 \circ (-1)^i d''
\]
\[
= d'' \circ H_2 + H_2 \circ d'' + (-1)^{2i+3} d'' \circ H_2 + (-1)^{2i+1} H_2 \circ d'' + (\geq 3)
\]
\[
= 0 + (\geq 3)
\]

as \( H_2 \circ H_2 \) is of filtration 4 and \( d' \circ H_2 \) and \( H_2 \circ d' \) are of filtration 3.
In the case \( p \geq 2 \), \( q^i : K^i \to K^{i+p+1}[1-p] \) is a morphism of complexes. This is because the component of degree \( p+1 \) of \( d_p \circ d_p \) is bihomogenous of degree \((p+1, 1-p)\): indeed, it is a sum of morphisms obtained by composing two morphisms bihomogenous of degree \((a, -a+1)\) and \((b, -b+1)\), respectively, so that \( a + b = p + 1 \). Let’s verify that with

\[
H_{p+1} = \sum_i (-1)^{i+p} H_{p+1}^i, \quad d_{p+1} \circ d_{p+1} \text{ is of filtration } \geq p + 2.
\]

We check what \( d_{p+1} \circ d_{p+1} \) does to \( K^{i,j} \).

\[
d_{p+1} \circ d_{p+1} = (d_p + \sum_i (-1)^{i+p} H_{p+1}^i) \circ (d_p + \sum_i (-1)^{i+p} H_{p+1}^i)
\]

\[
= q + H_0 \circ H_{p+1} + H_{p+1} \circ H_0 + (\geq p + 2)
\]

\[
= q^i + (-1)^{i+p+1} d'' \circ (-1)^{i+p} H_{p+1}^i + (-1)^{i+p} H_{p+1}^i \circ (-1)^i d'' + (\geq p + 2)
\]

\[
= q^i - d'' \circ H_{p+1}^i + (-1)^p H_{p+1}^i \circ d'' + (\geq p + 2)
\]

\[
= 0 + (\geq p + 2),
\]

in light of the identity that shows \( q^i \) homotopic to 0 (as a morphism of complexes \( K^i \to K^{i+p+1}[1-p] \)) with homotopy \( H_{p+1}^i \).

2ème solution. Let’s check that the object \((C, F)\) of \( \mathcal{D}^bF_{bête} \) defined up to isomorphism by the distinguished triangle

\[
\text{id}_A \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ Quad
diagram with distinguished rows and columns.

\[
\begin{array}{c}
\text{Gr}_F^{i+2} A \rightarrow \text{Gr}_F^{i+1} B \rightarrow \text{Gr}_F^{i+1} C \\
\downarrow \quad \downarrow \quad \downarrow \\
\text{F}^{i+1}/\text{F}^{i+3} A \rightarrow \text{F}^{i}/\text{F}^{i+2} B \rightarrow \text{F}^{i}/\text{F}^{i+2} C \\
\downarrow \quad \downarrow \quad \downarrow \\
\text{Gr}_F^{i+1} A \rightarrow \text{Gr}_F^{i} B \rightarrow \text{Gr}_F^{i} C
\end{array}
\]

When \( i > p \), \( \text{Gr}_F^i A = \text{Gr}_F^{i+1} A = 0 \), the left column vanishes, and we find an isomorphism of bête-truncated complexes \( \sigma_{>p} G(B) \sim \sigma_{>p} G(C) \):

\[
\begin{array}{c}
(\text{Gr}_F^{p+1} C)[p+1] \xrightarrow{d_{C}^{p+1}} \cdots \xrightarrow{\sim} (\text{Gr}_F^{b} C)[b] \xrightarrow{\sim} (\text{Gr}_F^{b+1} C)[b+1] \xrightarrow{\sim} \cdots \\
\xrightarrow{\sim} \xrightarrow{\sim} \xrightarrow{\sim} \\
(\text{Gr}_F^{p+1} B)[p+1] \xrightarrow{\sim} \cdots \xrightarrow{\sim} (\text{Gr}_F^{b} B)[b] \xrightarrow{\sim} (\text{Gr}_F^{b+1} B)[b+1] \xrightarrow{\sim} \cdots \\
\xrightarrow{\sim} \xrightarrow{\sim} \xrightarrow{\sim} \\
K^{p+1} \xrightarrow{d_{A}^{p+1}} \cdots \xrightarrow{\sim} K^{b} \xrightarrow{\sim} 0 \xrightarrow{\sim} \cdots
\end{array}
\]

When \( i \leq p \), \( \text{Gr}_F^i B = 0 \) and \( \text{Gr}_F^i C \xrightarrow{\sim} (\text{Gr}_F^{i+1} A)[1] \). When \( i < p \), moreover the entire middle column vanishes, finding an isomorphism of bête truncated complexes \( \sigma_{<p} G(C) \xrightarrow{\sim} \sigma_{<p}(G(A)[1]) \):

\[
\begin{array}{c}
\cdots \xrightarrow{\sim} (\text{Gr}_F^{a-1} C)[a-1] \xrightarrow{\sim} (\text{Gr}_F^{a} C)[a] \xrightarrow{d_{C}^{a}} \cdots \xrightarrow{\sim} (\text{Gr}_F^{p} C)[p] \\
\xrightarrow{\sim} \xrightarrow{\sim} \xrightarrow{\sim} \\
\cdots \xrightarrow{\sim} (\text{Gr}_F^{a-1} A)[a] \xrightarrow{\sim} (\text{Gr}_F^{a} A)[a+1] \xrightarrow{\sim} \cdots \xrightarrow{\sim} (\text{Gr}_F^{p} A)[p+1] \\
\xrightarrow{\sim} \xrightarrow{\sim} \xrightarrow{\sim} \\
\cdots \xrightarrow{\sim} 0 \xrightarrow{\sim} K^{a} \xrightarrow{d_{A}^{a}} \cdots \xrightarrow{\sim} K^{p}
\end{array}
\]
We will be done if we can join these diagrams with a commutative diagram

\[
\begin{array}{c}
(\text{Gr}_F^{p} C)[p] \xrightarrow{d_{C}^p} (\text{Gr}_F^{p+1} C)[p+1] \\
\downarrow \sim \hspace{1cm} \sim \uparrow
\end{array}
\]

\[
(\text{Gr}_F^{p} A)[p+1] \rightarrow (\text{Gr}_F^{p+1} B)[p+1]
\]

\[
\begin{array}{c}
\downarrow \sim \hspace{1cm} \sim \downarrow
\end{array}
\]

\[
K^{p} \xrightarrow{d^{p}} K^{p+1}
\]

When \(i = p\), part of our earlier commutative diagram looks like

\[
\begin{array}{c}
\text{Gr}_F^{p+1} B \xrightarrow{\sim} \text{Gr}_F^{p+1} C \\
\beta \downarrow \sim \hspace{1cm} \downarrow
\end{array}
\]

\[
\begin{array}{c}
\text{F}^{p+1}/\text{F}^{p+3} A \xrightarrow{\alpha} \text{F}_p/\text{F}^{p+2} B \rightarrow \text{F}_p/\text{F}^{p+2} C
\end{array}
\]

\[
\begin{array}{c}
\gamma \downarrow \sim \hspace{1cm} \downarrow
\end{array}
\]

\[
\begin{array}{c}
\text{Gr}_F^{p+1} A \hspace{1cm} \text{Gr}_F^{p} C
\end{array}
\]

\[
\begin{array}{c}
\downarrow d^{p}_{[1-p]}
\end{array}
\]

with distinguished row and column, inducing an isomorphism of distinguished triangles

\[
\begin{array}{c}
\text{Gr}_F^{p+1} B \rightarrow \text{F}_p/\text{F}^{p+2} C \rightarrow (\text{Gr}_F^{p+1} A)[1] \xrightarrow{-(\beta^{-1} \alpha \alpha^{-1} \gamma^{-1})[1]} \rightarrow
\end{array}
\]

\[
\begin{array}{c}
\downarrow \sim \hspace{1cm} \sim \uparrow
\end{array}
\]

\[
\begin{array}{c}
\text{Gr}_F^{p+1} C \rightarrow \text{F}_p/\text{F}^{p+2} C \rightarrow \text{Gr}_F^{p} C \xrightarrow{d^{p}_{[1-p]}}
\end{array}
\].
The commutative diagram

\[
\begin{array}{c}
\text{Gr}_{F[1]}^p A & \xrightarrow{\sim} & \text{Gr}_F^{p+1} A & \xrightarrow{\delta} & \text{Gr}_F^{p+1} B \\
\uparrow & & \downarrow & & \downarrow \\
F^{p+1}/F^{p+3} A & = (F[1])^p/(F[1])^{p+2} A & \xrightarrow{\text{id}_A} & F^p/F^{p+2} A & \xrightarrow{\sim} & F^p/F^{p+2} B
\end{array}
\]

shows that \(\delta = \beta^{-1} \circ \alpha \circ \gamma^{-1} : \text{Gr}_F^{p+1} A \to \text{Gr}_F^{p+1} B\) has the property that \(-\delta[p+1]\) makes a commutative diagram

\[
\begin{array}{c}
(\text{Gr}_F^p C)[p] & \xrightarrow{d_c^p} & (\text{Gr}_F^{p+1} C)[p+1] \\
\downarrow & & \uparrow \\
(\text{Gr}_F^p A)[p+1] & \xrightarrow{-\delta[p+1]} & (\text{Gr}_F^{p+1} B)[p+1] \\
\downarrow & & \downarrow \\
K^p & \xrightarrow{-f^{p+1}} & K^{p+1}
\end{array}
\]

This is why we should take \(f^{p+1} = -d^p\).

**Remark.** There is a related construction involving décalage as defined by Deligne in *Théorie de Hodge II* (1.3.3). (The choice of name, décalage, refers not to a shift of the filtration, but rather to a shift in the page numbering of the spectral sequences associated to the filtration.) The décalage functor \((X, F) \mapsto (X, \text{Dec}(F))\) is defined componentwise by

\[
(\text{Dec} F)^a(X^i) := \ker(d : F^{i+a}(X^i) \to F^{i+a}(X^{i+1})/F^{i+a+1}(X^{i+1})).
\]

By construction, the differential on \((\text{Dec} F)^a\) coincides with the coboundary map on cohomology so that if we define (for each \(a\)) the complex \(H^a(X, F)\) with \(H^a(X, F)^i := H^i \text{Gr}_F^{i+a} X\) with differential coming from the coboundary, the obvious map

\[
\text{Gr}_{\text{Dec} F}^a X \to H^a(X, F)
\]
is a quasi-isomorphism. There is a t-structure on the filtered derived category $DF$ with

$$(DF)^{\leq 0} = \{(X, F) \text{ s.t. } Gr^i X \text{ is acyclic in degrees } > i\}$$

$$(DF)^{\geq 0} = \{(X, F) \text{ s.t. } Gr^i X \text{ is acyclic in degrees } < i\}$$

whose heart coincides with the category of bête-filtered complexes. The truncation $\tau_{\leq 0}$ for this t-structure is the functor

$$(X, F) \mapsto ((\operatorname{Dec} F)^0 X, F \cap (\operatorname{Dec} F)^0)$$

which lands in $(DF)^{\leq 0}$ since $(\operatorname{Dec} F)^0$ lies in $F^{i+1}$ in degrees $> i$. Under the equivalence of categories $G$, the corresponding cohomological functor is the functor $H^\bullet$. This is (3.11) in Beilinson’s *Notes on Absolute Hodge Cohomology*.

3.1.9. The isomorphisms (3.1.9.2) render anticommutative the given square because the isomorphisms $Gr^i_F(K[1]) = (Gr^i_F K)[1]$ together with their translations render anticommutative the square

$\begin{array}{c}
\begin{bmatrix}
Gr^i_F(K[1]) & \xrightarrow{d_1[-i]} & Gr^{i+1}_F(K[1])[1] \\
\downarrow & & \downarrow \\
(Gr^i_F K)[1] & \xrightarrow{d_0[1-i]} & (Gr^{i+1}_F K)[2]
\end{bmatrix}
\end{array}$

where we have added subscripts for clarity. As discussed in the note to (3.1.7) above, $d_0[1-i] : (Gr^i_F K)[1] \rightarrow (Gr^{i+1}_F K)[2]$ is the translation of the morphism $d_0[-i]$ defined by the distinguished triangle

$$\begin{array}{c}
Gr^{i+1}_F K \xrightarrow{\nu} F^i/F^{i+2} K \xrightarrow{\nu[1]} Gr^i_F K \xrightarrow{d_0[-i]}
\end{array}$$

On the other hand, $d_1[-i] : Gr^i_F(K[1]) \rightarrow Gr^{i+1}_F(K[1])$ is the morphism defined by the triangle in the second row

$$\begin{array}{c}
\begin{bmatrix}
(Gr^{i+1}_F K)[1] & \xrightarrow{u[1]} & (F^i/F^{i+2} K)[1] & \xrightarrow{v[1]} & (Gr^i_F K)[1] \\
\downarrow & & \downarrow & & \downarrow \\
Gr^{i+1}_F(K[1]) & \rightarrow & F^i/F^{i+2}(K[1]) & \rightarrow & Gr^i_F(K[1]) \xrightarrow{d_1[-i]}
\end{bmatrix}
\end{array}$$
once we show that this triangle is distinguished. If we can show there is a morphism of degree 1 \(d_1[-i]\) making this triangle distinguished, this morphism of degree 1 will be unique by the considerations of the note to (3.1.7). The diagram

\[
\begin{array}{cccccc}
(\text{Gr}_{i+1} F)[1] & \xrightarrow{-u[1]} & (F^i/F^{i+2} K)[1] & \xrightarrow{-v[1]} & (\text{Gr}_F K)[1] & \xrightarrow{-d_0[1-i]} \\
\downarrow{\text{id}} & & \downarrow{-\text{id}} & & \downarrow{\text{id}} & \\
(\text{Gr}_{i+1} F)[1] & \xrightarrow{u[1]} & (F^i/F^{i+2} K)[1] & \xrightarrow{v[1]} & (\text{Gr}_F K)[1] & \xrightarrow{-d_0[1-i]} \\
\text{Gr}_{i+1} (K[1]) & \xrightarrow{\text{id}} & F^i/F^{i+2} (K[1]) & \xrightarrow{\text{id}} & \text{Gr}_F (K[1]) & \xrightarrow{d_1[-i]} \\
\end{array}
\]

is an isomorphism of triangles with the top row distinguished. We conclude \(d_1[-i] = -d_0[1-i]\) after the canonical isomorphisms \(\text{Gr}_F (K[1])[j] = (\text{Gr}_F K)[1 + j]\).

3.1.11. The point is that we already computed the differential on the \(E_1\) page to be \(d_1(f) = df - fd\) in the note to 3.1.8 and the image of \(d_1 : E_{1,-1}^{0,0} \to E_{1,0}^{0,0}\) consists of precisely the homotopies. As computed in 3.1.8, the kernel of \(d_1 : E_{1,0}^{0,0} \to E_{1,1}^{1,0}\) consists of precisely the morphisms of complexes in \(C^b(\mathcal{C})\).

3.1.12. We can choose a representative \(f\) for a homotopy class of morphisms of complexes in \(K^b(\mathcal{C})\). Typos & clarifications: ‘\(G(C(f), \tilde{F}) = C(G(f))\). En particulier, si \((K, F)\) et \((L, F)\) sont bêtes, \((C(f), \tilde{F})\) l’est aussi, et \(C(\omega f) = \omega(C(f), \tilde{F})\) s’identifie à \(\text{real}(C(f))\).’ The isomorphism \(G(C(f), \tilde{F}) = C(G(f))\) relies on the earlier calculation \(G(K[1], F[1]) = (GK)[1]\). Interestingly,

\[G(K, L, C(f)) = \omega((K, F), (L, F), (C(f), \tilde{F})) = (K, L, C(f))\]

where the middle triangle is not distinguished, but after forgetting the filtration, it is.

3.1.13. We will compute the differentials of \(G(\Sigma)\) to verify that \(s\Sigma^* = G(\Sigma)\), but before we do, a few words on the matter of sign that was also an issue in the discussion of both solutions to (3.1.8) in the note to 3.1.8. Given a map \(f : A^* \to B^*\) of complexes (considered as a double complex with first degree concentrated in 0 and 1) and the filtration by first degree on the associated simple complex \(K\), it is easy to compute that
the coboundary $\partial$ on cohomology associated to the exact sequence

$$
\begin{array}{cccccc}
0 & \rightarrow & \text{Gr}^1 K & \rightarrow & F^0/F^2 K & \rightarrow & \text{Gr}^0 K & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & B[-1] & \rightarrow & B[-1] \oplus A & \rightarrow & A & \rightarrow & 0
\end{array}
$$

coincides with $f$, as the differential on $F^0/F^2 K \cong B[-1] \oplus A$ is given by

$$
\begin{pmatrix}
d_{B[-1]} & f \\
0 & d_A
\end{pmatrix}.
$$

The issue appears to be that when one defines the differential on the cone of a morphism of complexes $u : L \rightarrow M$ in the usual way as

$$
\begin{pmatrix}
d_{L[1]} & 0 \\
u & d_M
\end{pmatrix},
$$

then given the exact sequence of complexes with coboundary $\partial$

$$
0 \rightarrow L \stackrel{u}{\rightarrow} M \rightarrow \text{coker } u \rightarrow 0,
$$

the morphism of degree one $\delta$ in the distinguished triangle $(L, \text{Cyl}(u), C(u))$ coincides on cohomology with $-\partial$ after the isomorphism with $(L, M, \text{coker } u)$. (This is consistent with Weibel’s book but Gelfand-Manin say, I believe incorrectly, $\partial$.)

Applying this in the case of $u : \text{Gr}^1 K \rightarrow F^0/F^2 K$, we expect that the map of degree one $\text{Gr}^0 K \rightarrow (\text{Gr}^1 K)[1]$ will coincide with $-f$. As before, this isn’t really a problem in view of the isomorphism

$$
\begin{array}{cccccc}
0 & \rightarrow & A & \stackrel{f}{\rightarrow} & B & \stackrel{g}{\rightarrow} & C & \rightarrow & 0 \\
\downarrow^{\text{id}} & & \downarrow^{-\text{id}} & & \downarrow^{\text{id}} & & \downarrow & & \\
0 & \rightarrow & A & \stackrel{-f}{\rightarrow} & B & \stackrel{-g}{\rightarrow} & C & \rightarrow & 0.
\end{array}
$$

It is also possible to define the differential of the cone of the morphism $u$ to be

$$
\begin{pmatrix}
d_{L[1]} & 0 \\
-u & d_M
\end{pmatrix},
$$

and although less standard, this would correct our sign, but we don’t proceed this way.
Returning to the notation in the beginning of this paragraph, we will show that the differentials on $G(s\Sigma^*)$ coincide with the differentials of $\Sigma$ with the sign reversed. In fact, it suffices to do so for the single morphism of complexes $f$. In the double complex

\[
\begin{array}{cc}
A^p & \rightarrow & B^p \\
\Uparrow & & \Uparrow \\
A^{p-1} & \rightarrow & B^{p-1}
\end{array}
\]

we consider first degree concentrated in 0, 1, and we let $K$ denote the associated simple complex. With $v$ as above,

\[
\begin{array}{ccc}
C(v) & \xrightarrow{\delta} & (\text{Gr}^1 K)[1] \\
\downarrow & & \approx \\
\text{Gr}^0 K & \xrightarrow{f} & (\text{Gr}^1 K)[1]
\end{array}
\]

where $\delta$ is projection to the first factor, $\gamma$ to the last, and we wish to show these squares commute up to homotopy. The differential on $C(v) = B \oplus B[-1] \oplus A$ is given by

\[
\begin{pmatrix}
    d_B \\
    \text{id} \\
    d_B[-1] & f \\
    d_A
\end{pmatrix}
\]

and we compute $dh - hd$ for the homotopy $h^p : B^p \oplus B^{p-1} \oplus A^p \rightarrow B^{p-1}$ which is projection to the middle factor:

\[
dh(b^p, b^{p-1}, a^p) = db^{p-1} \\
hd(b^p, b^{p-1}, a^p) = b^p + f(a^p) - db^{p-1}.
\]

This shows that $\delta \sim -f$ so that if $\Sigma^* = A^* \xrightarrow{f} B^* \xrightarrow{g} C^*$, the differentials of $G(s\Sigma^*)$ coincide with $-f$ and $-g$ which, as discussed above, is ok.
3.1.14. We check that the cone $C$ of the inclusion $\iota : \tau_{\leq i} K^* \to \tau'_{\leq i} K^*$ is homotopic to the complex $L$ given by $\ker d \to K^i \to \im d$ in degrees $i - 1, i, i + 1$. The morphisms of complexes $f$ and $g$ are given componentwise by

\[
\begin{array}{ccccccccc}
\cdots & \to & 0 & \to & \ker d & \to & K^i & \to & \im d & \to & 0 & \to & \cdots \\
\downarrow^{f_{i-2}} & & \downarrow^{f_{i-1}} & & \downarrow^{f_i} & & \downarrow^{f_{i+1}} & & \downarrow^{f_{i+2}} & & \\
\cdots & \to & K^{i-1} & \xrightarrow{d^{[1]}} & \ker d & \to & 0 & \to & 0 & \to & 0 & \to & \cdots \\
\oplus & \xrightarrow{\iota_{i-1}} & \oplus & \xrightarrow{\iota_i} & \oplus & \oplus & \oplus & \oplus & \oplus & & \\
\cdots & \to & K^{i-2} & \xrightarrow{d} & K^{i-1} & \xrightarrow{d} & K^i & \xrightarrow{d} & \im d & \to & 0 & \to & \cdots \\
\downarrow^{g_{i-2}} & & \downarrow^{g_{i-1}} & & \downarrow^{g_i} & & \downarrow^{g_{i+1}} & & \downarrow^{g_{i+2}} & & \\
\cdots & \to & 0 & \to & \ker d & \to & K^i & \to & \im d & \to & 0 & \to & \cdots 
\end{array}
\]

where $f_{i-1}$ is inclusion into the first factor and $f_i, f_{i+1}$ are inclusion into the second factor, while $g_i, g_{i+1}$ are projection to the second factor, $g_i = 0$ for $j < i - 1$, and $g_{i-1} = \text{pr}_1 - d \circ \text{pr}_2$. Evidently $g \circ f = \text{id}_L$ and we need to construct a homotopy $h$ so that $f \circ g = \text{id}_C + dh + hd$. Set each $h^p : C^p \to C^{p-1}$ for $p > i - 1$ to zero, while for $p \leq i - 1$, let $h^p$ be the projection to the second factor of $C^p$ followed by inclusion into the first factor of $C^{p-1}$ so that if we only draw arrows for the components of $h$ which are nonzero, $h$ is given by

\[
\begin{array}{cccccccc}
\cdots & \to & K^{i-2} & \oplus & K^{i-1} & \oplus & \ker d & \oplus & 0 & \oplus & 0 & \oplus & 0 & \oplus & \cdots \\
\oplus & \xrightarrow{\iota_{i-1}} & \oplus & \xrightarrow{\iota_i} & \oplus & \oplus & \oplus & \oplus & & & & & & & \\
\cdots & \to & K^{i-3} & \oplus & K^{i-2} & \oplus & K^{i-1} & \oplus & K^i & \oplus & \im d & \oplus & 0 & \oplus & \cdots 
\end{array}
\]

Now let’s examine why an isomorphism between $H^i K^*[-i]$ and a cone on $\text{real}(\tau_{\leq i} K^*)$ is enough to conclude that $\text{real}(\tau_{\leq i} K^*) \simeq \tau_{\leq i}(\text{real} K^*)$ and $H^i \text{real} K^* \simeq H^i K^*$. As $K^*$ is an object of $C^b(\mathcal{C})$, there are integers $a, b$ such that $K^i = 0$ for $i < a$ and $i > b$. Then $\tau_{\leq a-1} K^* = 0$ and we find $\text{real}(\tau_{\leq a} K^*) \simeq (H^a K^*)[-a]$. As $H^i K^*$ is in $\mathcal{C}$, proceeding inductively we find that $H^j(\text{real}(\tau_{\leq i} K^*))$ equals $H^j K^*$ for $j \leq i$ and is null for $j > i$. As $H^{b+1} K^* = 0$, $\text{real}(\tau_{\leq b} K^*) \simeq \text{real}(\tau_{\leq b+1} K^*) = \text{real} K^*$ so that $\text{real}(\tau_{\leq b} K^*) \simeq \text{real} K^* \simeq \tau_{\leq b} \text{real} K^*$ since as we have seen, $H^j(\text{real} K^*) = 0$ for $j > b$. Now supposing $\text{real}(\tau_{\leq i} K^*) \simeq \tau_{\leq i}(\text{real} K^*)$ has been established, the isomorphism of
distinguished triangles
\[
\begin{array}{cccccc}
\text{real}(\tau_{\leq i-1}K^*) & \longrightarrow & \text{real}(\tau_{\leq i}K^*) & \longrightarrow & (H^iK^*)[-i] & \longrightarrow \\
\downarrow & & \downarrow & & \downarrow & \\
\tau_{\leq i-1}(\text{real }K^*) & \longrightarrow & \tau_{\leq i}(\text{real }K^*) & \longrightarrow & \rho H^i(\text{real }K^*)[-i] & \longrightarrow \\
\end{array}
\]
achieves the same for \( i - 1 \).

Now we show that the cone \( C^* \) on the morphism \( \iota : \tau'_{\leq i-1}K^* \rightarrow \tau_{\leq i}K^* \) is homotopic to the complex \( L^* \) reduced to \( \im d^{i-1} \rightarrow \ker d^i \) in degrees \( i - 1 \) and \( i \). The story is the same as what we just did. We define morphisms \( f : L^* \rightarrow C^* \) and \( g : C^* \rightarrow L^* \) which compose to morphisms homotopic to the identity.

\[
\begin{array}{cccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \im d^{i-1} & \longrightarrow & \im d & \longrightarrow & 0 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & K^{i-2} & \longrightarrow & K^{i-1} & \longrightarrow & \im d^{i-1} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
\oplus & \longrightarrow & \oplus & \longrightarrow & \oplus & \longrightarrow & \oplus & \longrightarrow & \oplus & \longrightarrow & \oplus & \longrightarrow & \\
\iota^{i-2} & \longrightarrow & \iota^{i-1} & \longrightarrow & \iota^i & \longrightarrow & \iota^i & \longrightarrow & \iota^i & \longrightarrow & \iota^i & \longrightarrow & \\
\cdots & \longrightarrow & K^{i-3} & \longrightarrow & K^{i-2} & \longrightarrow & K^{i-1} & \longrightarrow & \ker d^i & \longrightarrow & 0 & \longrightarrow & \cdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \im d^{i-1} & \longrightarrow & \im d & \longrightarrow & 0 & \longrightarrow & \cdots \\
\end{array}
\]

Here the morphisms \( f^i, g^i \) is the obvious inclusion and projection, while \( f^{i-1} \) is the inclusion to the first factor and \( g^{i-1} = \text{pr}_1 - d^{i-1} \circ \text{pr}_2 \). Then \( g \circ f = \text{id}_{L^*} \) and \( f \circ g = \text{id}_{C^*} + dh + hd \) where \( h \) is the homotopy with all components nonzero except for the arrows in the below diagram.

\[
\begin{array}{cccccccc}
\cdots & \longrightarrow & K^{i-2} & \oplus & K^{i-1} & \oplus & \im d^{i-1} & \oplus & 0 & \oplus & 0 & \oplus & 0 & \oplus & \cdots \\
\oplus & \longrightarrow & \oplus & \longrightarrow & \oplus & \longrightarrow & \oplus & \longrightarrow & \oplus & \longrightarrow & \oplus & \longrightarrow & \oplus & \longrightarrow & \\
\iota^{i-2} & \longrightarrow & \iota^{i-1} & \longrightarrow & \iota^i & \longrightarrow & \iota^i & \longrightarrow & \iota^i & \longrightarrow & \iota^i & \longrightarrow & \\
\cdots & \longrightarrow & K^{i-3} & \longrightarrow & K^{i-2} & \longrightarrow & K^{i-1} & \longrightarrow & \ker d^i & \longrightarrow & 0 & \longrightarrow & \cdots \\
\end{array}
\]

The complex \( L^* \) is the cone of the morphism
\[
\im(d^{i-1})[-i] \rightarrow \ker(d^i)[-i]
\]
(shifted by \([-i]\) not \([i]\)) and the point is that the realization functor sends an object \( D \) of \( \mathcal{C} \) shifted by \( i \) to the same object shifted by \( i \): the complex \( D[i] \) with one-step
filtration $F^j(D[i]) = D[i]$ for $j \leq -i$, $F^j(D[i]) = 0$ for $j > -i$ is bête-filtered with $\text{Gr}^{-i}(D[i])[-i] = D$ and all other $\text{Gr}^j = 0$, so that $\text{real}(D[i]) = D[i]$. Of course, the exact sequence

$$0 \to \text{im} \ d^{i-1} \to \ker d^i \to H^i K^* \to 0$$

gives rise to a distinguished triangle (where the cone is unique up to unique isomorphism)

$$(\text{im} \ d^{i-1})[-i] \to (\ker d^i)[-i] \to (H^i K^*)[-i] \to .$$

**Remarque.** The functor real is exact and $t$-exact.

### 3.1.16.

The claim about the characterization of the $\text{Hom}_{\text{D}C}(A, B[n])$ is effectively the classical Tôhoku statement, with the twist that it suffices to efface morphisms one at a time, rather than the whole Hom group at once. Swan calls this weaker effaceability property described in 3.1.16 ‘weak effaceability.’ Grothendieck’s statement about a $\delta$-functor $F^n$ being universal if $F^n$ is effaceable for $n > 0$ holds with ‘effaceable’ replaced by ‘weakly effaceable’ if $F$ takes values in a module category. The $\delta$-functor $\text{Hom}_{\text{D}C}(A, [n])$ takes values in abelian groups and is weakly effaceable. See Buchsbaum, *Satellites and Universal Functors* (4.2) & (4.3) and Swan, *Cup products in sheaf cohomology, pure injectives, and a substitute for projective resolutions* (1.1) & (6.1) for the full story.

The bit about every object of $\text{D}^b C$ se dévissant en objets de $\mathcal{C}$ is combined with the exact sequence of $\text{Hom}^i$ of proposition 1-2 of ‘état 0’ of Verdier’s thesis in SGA 4 1/2.

### 3.1.17.

As remark (ii) is a corollary of remark (i), let it suffice to prove (i). There is some trivial ambiguity: where remark (i) writes ‘pour $A, B$ in $\mathcal{C}$’ it means ‘for any $A, B$ in $\mathcal{C}$.’ On the one hand, if $f \in \text{Hom}_{\text{D}}(A, B[N])$ is in the image of real, say $\text{real}(\tilde{f}) = f$, then $\tilde{f}$ can be represented in $\text{Hom}_{\text{D}C}(A, B[N])$ by a Yoneda Ext and either the monomorphism at the left end or the epimorphism at the right will efface $\tilde{f}$, and, after realization, $f$ (see below). On the other hand, let’s suppose $f$ is effaceable in the sense of (3.1.16) and we have an epimorphism $u : A' \to A$ and monomorphism $v'[−N] : B \hookrightarrow B'$ such that $v'fu = 0$. In light of (1.2.3), these morphisms give rise to distinguished triangles $(A'', A', A)$ and $(B, B', B'')$ in $\mathcal{D}$ which are unique up to unique isomorphism, and, in
view of (1.1.9), a morphism of triangles

\[
\begin{array}{cccc}
A' & \longrightarrow & A & \longrightarrow & A''[1] \\
\downarrow g & & \downarrow f & & \downarrow h \\
B''[N - 1] & \longrightarrow & B[N] & \longrightarrow & B'[N]
\end{array}
\]

where now the two dashed arrows lie in a $\text{Hom}^{N-1}_{\mathcal{C}}$. All the data of this morphism of triangles (except $f$) is lifted uniquely to $D\mathcal{C}$, where we can complete it to a morphism of triangles (in possibly several different ways) and then apply real. We would like to show that real establishes a bijection between the morphisms $\tilde{f}$ which complete the lift of the above to a morphism of triangles in $D\mathcal{C}$ and those $f$ which complete the morphism of triangles in $\mathcal{D}$. This will simultaneously show that our original $f$ is in the image of real and prove that real : $\text{Hom}^N_{D\mathcal{C}}(A,B[N]) \to \text{Hom}^N_{\mathcal{D}}(A,B[N])$ is injective.

We have a commutative diagram with exact rows and columns on the next page. From this diagram we see that if $g \in \text{Hom}^{-1}(A',B''[N])$ and $h \in \text{Hom}(A''[1],B'[N])$ such that there is at least one $f \in \text{Hom}(A,B[N])$ such that $fu$ is in the image of $g$ and $v'f$ is in the image of $h$, the set of all $f \in \text{Hom}(A,B[N])$ satisfying this same property is in bijection with

\[X \cap Y,\] where

\[X := \text{coker}(\text{Hom}^{-1}(A',B[N]) \to \text{Hom}(A''[1],B[N]))\] and

\[Y := \text{coker}(\text{Hom}^{-1}(A,B'[N]) \to \text{Hom}^{-1}(A,B''[N])).\]

Both $X$ and $Y$ are considered as subgroups of $\text{Hom}(A,B[N])$. The point is that real sends $X$ and $Y$ (Hom taken in $D\mathcal{C}$) isomorphically to $X$ and $Y$, respectively (Hom taken in $\mathcal{D}$) by the hypothesis that real is an isomorphism on $\text{Hom}^n$ for $n < N$. Therefore real does the same for $X \cap Y$. This proves our claim.
Annihilating Yoneda Ext. Now we verify that given \( f \in \text{Hom}_{\mathcal{D}}(A, B[N]) \) represented by the Yoneda Ext
\[
Q := \cdots \to 0 \to B = K^{-N} \xrightarrow{\alpha} K^{-N+1} \to \cdots \to K^0 \xrightarrow{\beta} A \to 0 \to \cdots
\]
either the monomorphism \( \alpha \) or the epimorphism \( \beta \) will efface \( f \). The postcomposition with \( \alpha[N] \) is represented by the diagram
\[
\begin{array}{cccccccccccc}
\cdots & \to & 0 & \to & 0 & \to & 0 & \to & \cdots & \to & A & \to & 0 & \to & \cdots \\
& & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \beta & & \uparrow & & \uparrow \\
\cdots & \to & 0 & \to & B = K^{-N} & \xrightarrow{\alpha} & K^{-N+1} & \to & \cdots & \to & K^0 & \to & 0 & \to & \cdots \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \to & 0 & \to & B & \to & 0 & \to & \cdots & \to & 0 & \to & 0 & \to & \cdots \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \to & 0 & \to & K^{-N+1} & \to & 0 & \to & \cdots & \to & 0 & \to & 0 & \to & \cdots \\
\end{array}
\]
with \( \beta \) inducing a quasi-isomorphism, and the point is that the morphism of complexes
\[
\begin{array}{cccccccccccc}
\cdots & \to & 0 & \to & B = K^{-N} & \xrightarrow{\alpha} & K^{-N+1} & \to & \cdots & \to & K^0 & \to & 0 & \to & \cdots \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \to & 0 & \to & K^{-N+1} & \to & 0 & \to & \cdots & \to & 0 & \to & 0 & \to & \cdots \\
\end{array}
\]
is homotopic to 0 via the obvious homotopy. On the other hand, the precomposition with \( \beta \) is represented by the diagram
\[
\begin{array}{cccccccccccc}
\cdots & \to & 0 & \to & 0 & \to & 0 & \to & \cdots & \to & K_0 & \to & 0 & \to & \cdots \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \beta \downarrow & & \downarrow & & \downarrow \\
\cdots & \to & 0 & \to & 0 & \to & 0 & \to & \cdots & \to & A & \to & 0 & \to & \cdots \\
& & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \beta \uparrow & & \uparrow & & \uparrow \\
\cdots & \to & 0 & \to & B = K^{-N} & \xrightarrow{\alpha} & K^{-N+1} & \to & \cdots & \to & K^0 & \to & 0 & \to & \cdots \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots & \to & 0 & \to & B & \to & 0 & \to & \cdots & \to & 0 & \to & 0 & \to & \cdots \\
\end{array}
\]
and here the point is that this morphism is in the same equivalence class as the morphism

\[ \cdots \to 0 \to 0 \to 0 \to \cdots \to K_0 \to 0 \to \cdots \]

\[ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \id \quad \id \]

\[ \cdots \to 0 \to B = K^{-N} \to \cdots \to K^{-N+1} \to \cdots \to K^0 \to 0 \to \cdots \]

\[ \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \id \quad \id \]

\[ \cdots \to 0 \to B \to 0 \to \cdots \to 0 \to 0 \to \cdots \]

as the diagram

\[ \begin{array}{ccc}
K^0 & \xrightarrow{\alpha} & K^{-N+1} \\
\id & \downarrow & \id \\
K^0 & \xrightarrow{\beta} & Q \\
\beta & \downarrow & \beta \\
A & \xrightarrow{\beta} & B
\end{array} \]

commutes, where here the arrow decorations refer only to the degree 0 component of the morphism, the other components being obvious. Now it is obvious this composition is null.

4.1.1. The following proof of Artin’s theorem was given to me by Sasha Beilinson. The \( \eta \) notation, along with the construction of a retraction, is discussed after the proof.

Below \( D(X) := D^b_c(X, \mathbf{Z}/\ell^n) \), \( D(X)_{\leq 0} := \{ \mathcal{F} \in \text{D}(X) \colon \dim \text{supp} \mathcal{H}^i \mathcal{F} \leq -i \} \), i.e., this is \( pD(X)_{\leq 0} \) where \( p \) is the middle perverse \( t \)-structure.

**Theorem (Artin). — For an affine map \( f : X \to Y \) of schemes of finite type over a field \( k \) (with \( \text{char } k \) prime to \( \ell \)) the functor \( f_* : \text{D}(X) \to \text{D}(Y) \) is right \( t \)-exact.**

**Proof. —** Pick \( \mathcal{F} \in \text{D}(X)_{\leq 0} \); we want to show that \( f_* \mathcal{F} \in \text{D}(Y)_{\leq 0} \). Let \( d(\mathcal{F}) \) be the dimension of support of \( \mathcal{F} \). We use induction by \( d(\mathcal{F}) \), so we assume that for every \( f, k \) as in the theorem and \( \mathcal{G} \in \text{D}(X)_{\leq 0} \) with \( d(\mathcal{G}) < d(\mathcal{F}) \) one has \( f_* \mathcal{G} \in \text{D}(Y)_{\leq 0} \).

(o) We can assume that \( k \) is algebraically closed (since \( f_* \) commutes with the base change to an algebraic closure of \( k \)).
(i) It is enough to show that for every closed point \( y \in Y \) the complex \( (f_*\mathcal{F})_y \) is connective (i.e., acyclic in degrees \( > 0 \)): We need to check that for a point \( \eta \) of \( Y \) of dimension \( \delta > 0 \) the complex \( (f_*\mathcal{F})_\eta[-\delta] \) is connective. Let \( Z \subset Y \) be the closure of \( \eta \). Replacing \( Y \) by an étale neighborhood of \( \eta \), choose a retraction \( Y \to Z \) (it exists since \( k \) is perfect). Consider \( Y \) as a \( Z \)-scheme and \( f \) as a map of \( Z \)-schemes. Let \( f^0 : X^0 \to Y^0 \) be the map of the generic fibers (over \( \eta = \text{Spec} k_\eta \in Z \)); this is an affine morphism of \( k_\eta \)-schemes. Since \( \mathcal{F}|_{X^0} \in D(X^0)^{\leq -\delta} \) and \( d(\mathcal{F}|_{X^0}) \leq d(\mathcal{F}) - \delta \), one has \( f^0_* (\mathcal{F}|_{X^0})[-\delta] \in D(Y^0)^{\leq 0} \) by the induction assumption applied to \( f^0, k_\eta, \) and \( \mathcal{F}|_{X^0}[-\delta] \). Now \( (f_*\mathcal{F})|_{Y^0} = f^0_* (\mathcal{F}|_{X^0}) \), hence \( (f_*\mathcal{F})_\eta[-\delta] = f^0_* (\mathcal{F}|_{X^0})_\eta[-\delta] \) is connective, q.e.d.

(ii) The case when \( f \) is an open embedding \( j : X \hookrightarrow \overline{X} \) with \( Q := \overline{X} - X \) a Cartier divisor: We can assume that \( y \) as in (i) lies in \( Q \) which is a principal divisor \( h = 0 \). Let \( K \) be the field of fractions of the henselian local ring at \( 0 \in A^1_k, \tilde{K} \) its separable closure, \( G := \text{Gal}(\tilde{K}/K) \), and \( \Psi = \Psi_h : D(X) \to D(Q) \) be the nearby cycles functor. One has \( (j_*\mathcal{F})_y = \Gamma(G, \Psi(\mathcal{F})_y) \), so, since \( G \) has cohomological dimension 1, it is enough to check that \( \Psi(\mathcal{F})_y[-1] \) is connective. By definition, \( \Psi(\mathcal{F})_y \) is inductive limit of complexes \( R\Gamma(U_{\tilde{K}}, \mathcal{F}|_{U_{\tilde{K}}}) \) where \( U/\overline{X} \) runs the category of affine étale neighborhoods of \( y, U_{\tilde{K}} := U \times_{A^1} \text{Spec} \tilde{K}, \mathcal{F}|_{U_{\tilde{K}}} \) is the pullback of \( \mathcal{F} \) by the map \( U_{\tilde{K}} \to X \). Since \( \mathcal{F}|_{U_{\tilde{K}}} \in D(U_{\tilde{K}})^{\leq -1} \) and \( D(U_{\tilde{K}}) < d(\mathcal{F}) \), each complex \( R\Gamma(U_{\tilde{K}}, \mathcal{F}|_{U_{\tilde{K}}})[-1] \) is connective by the induction assumption applied to the affine map \( U_{\tilde{K}} \to \text{Spec} \tilde{K} \), and so \( \Psi(\mathcal{F})_y[-1] \) is connective.

(iii) The case when \( f \) is the projection \( p : X = A^1 \times Y \to Y \): For \( y \) as in (i) consider the complementary embeddings \( i_y : X_y = A^1_y \hookrightarrow X, j_y : X - X_y \hookrightarrow X \). Applying \( p_*(-)_y \) to the exact triangle \( j_y \circ i_y^* \mathcal{F} \to \mathcal{F} \to i_y^* j_y^* \mathcal{F} \) we see that it is enough to show that \( (p_*i_y^*j_y^* \mathcal{F})_y \) and \( (p_*j_y^*i_y^* \mathcal{F})_y \) are connective.

\((p_*i_y^*j_y^* \mathcal{F})_y \) is connective: One has \( (p_*i_y^*j_y^* \mathcal{F})_y = R\Gamma(A^1_{\tilde{K}}, i^*_y \mathcal{F}) \), so it is enough to check that for every successive quotient of the (usual) canonical filtration on \( i^*_y \mathcal{F} \) the complex \( R\Gamma(A^1_{\tilde{K}}, \text{Gr}_n i^*_y \mathcal{F}) \) is connective. Since \( \text{Gr}_{\geq 0} \mathcal{F} = 0 \) and \( \text{Gr}_{0} \mathcal{F} \) is supported at finitely many points, we are reduced to the claim that for a (usual) sheaf \( \mathcal{G} \) on \( A^1_k \) one has \( H^{>1}(A^1_{\tilde{K}}, \mathcal{G}) = 0 \) which is SGA 4 IX 5.7.
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(p \cdot j_y j^*_y \mathcal{F})_y \text{ is connective: } One has \mathcal{G} := j_y j^*_y \mathcal{F} \in D(X)_{\geq 0}. Consider the open embedding \( j : X = (\mathbb{P}^1 - \{\infty\}) \times Y \hookrightarrow \overline{X} := \mathbb{P}^1 \times Y \). Let \( \overline{p} : \overline{X} \to Y \) be the projection, and \( \overline{i}_y : \mathbb{P}^1_k \to \overline{X} \) be the embedding \( \overline{i}_y(a) = (a, y) \). Then \( p = \overline{p} j, (p_* \mathcal{G})_y = (\overline{p}_* \overline{i}_y^* \mathcal{G})_y = R\Gamma(\mathbb{P}^1_k, \overline{i}_y^* \mathcal{G}) \) be proper base change, and so \( (p_* \mathcal{G})_y = (\overline{p}_* \overline{i}_y^* \mathcal{G})_y = 0 \). We are done by (ii) applied to \( j \) and \( \mathcal{G} \).

(iv) The general case: It is enough to write \( f \) as a composition \( f = f_n f_{n-1} \cdots f_0 \) of affine maps \( f_i \) such that our claim is true for each \( f_i \) (indeed, the sheaves \( \mathcal{F}_i := (f_i f_{i-1} \cdots f_0)_* \mathcal{F} \) satisfy \( d(\mathcal{F}_i) \leq d(\mathcal{F}) \), and so \( \mathcal{F}_i = f_i_* \mathcal{F}_{i-1} \in D(\geq 0) \) by induction by \( i \)). Now locally on \( Y \) we can factor \( f \) as composition \( X \hookrightarrow A^n \times Y \to Y \) where \( \hookrightarrow \) is a closed embedding and \( \to \) is the projection. Thus \( f = f_n f_{n-1} \cdots f_0 \) where \( f_0 \) is \( \hookrightarrow \) and \( f_i \) is the projection \( A^{n-i+1} \times Y \to A^{n-i} \times Y \) for \( i > 0 \). Our claim is true for \( f_0 \) since \( f_0 \) is t-exact and for \( f_i, i > 0 \), by (iii), and we are done. \( \square \)

Constructing retractions. In the above, \( \eta \) is used simultaneously for a point of the topological space of the scheme \( Y \) and for a geometric point centered on this (scheme-theoretic) point. Let \( \overline{\eta} \) be a geometric point of \( Y \) centered on a point \( \eta \) of \( Y \) and \( Z = \{ \eta \} \).

As \( k \) is perfect, the smooth locus of \( Z \) is nonempty, so we may assume that \( Z \) factors as \( Z \hookrightarrow A^\delta \to \text{Spec} k \) for \( h \) étale. Suppose \( Y = \text{Spec} A, Z = \text{Spec} B \) and form the pullback

\[
\begin{array}{ccc}
A' & \longrightarrow & k[x_1, \ldots, x_\delta] \\
\downarrow r & & \downarrow \\
A & \longrightarrow & B
\end{array}
\]

of rings; it is easy to find a retract \( k[x_1, \ldots, x_\delta] \to A' \). Let \( Y' = \text{Spec} A' \).

\[
\begin{array}{ccc}
Z & \longleftarrow & Y \\
\downarrow h & & \downarrow \\
A^\delta & \longleftarrow & Y'
\end{array}
\]

The map \( Z \times_{A^\delta} Y \to Y \) is étale and the base change by \( Z \to Y \) is given by \( Z \times_{A^\delta} Z \to Z \), which admits the diagonal as section. As \( h \) is étale, this diagonal is is an isomorphism onto a connected component of \( Z \times_{A^\delta} Z \) (SGA 1 I 9.3) and we identify \( Z \) with this component. Let \( U \) denote \( Z \times_{A^\delta} Y \) minus the closed subscheme \( Z \times_{A^\delta} Z - Z \); \( U \) is an
étagé neighborhood of $\eta$ and is equipped with a retract $U \to Z$, namely the one

$$Z \hookrightarrow U = Z \times_A \bar{Y} - (Z \times_A Z - Z) \to Z$$

induced by the first projection $Z \times_A \bar{Y} \to Z$.

4.1.7. The Čech spectral sequence is also called the Cartan-Leray spectral sequence and its existence in an arbitrary topos is established in SGAA, Exp. V 3.3.

4.1.8. The only thing worth mentioning is that the entire second paragraph is implicitly local to $U_i$. After all, on $U_i$ we have that $\tau \leq -i K$ is in $pD^{\leq 0}$, and to show that $K|_{U_i}$ is in $pD^{\leq 0}$, it suffices to show that $H^q(\tau_{\leq -i} K)|_{U_i} = 0$ for all $q (i.e. for all $q > -i$). Proceeding by descending induction on $q > -i$, the induction step consists of showing that $H^i(V \cap W_i, L) = 0$ for all affine open $V$ implies that $L = 0$.

4.2.5. As $f^*$ and $\text{RHom}$ commute with reduction modulo $\ell^n$, it is enough to prove the statement in the category $D^{b}_{ctf}(X, Z/\ell^n)$, and the equality $\Gamma H^0\text{RHom} = H^0\Gamma \text{RHom}$ holds in $D(X, Z/\ell^n)$ because, as $\text{RHom}$ is in $D^{\geq 0}$, $H^0 = \tau_{\leq 0} \ker d_0$, and $\Gamma$ commutes with the formation of kernels. For (4.2.5.3), the retraction $H \to \circ f_* f^* H$ is $\circ f_* (\eta(f^* H))$ where $\eta$ is the unit $\text{id} \to \circ e_* e^*$:

$$H \to \circ f_* f^* H \to \circ f_* \circ e_* e^* f^* H = H,$$

and it remains to check that the first arrow is surjective. The references in the rest of this paragraph are to SGAA Exposé XV. By, (1.1), it suffices to show that $H^0(Y', \mathcal{H}) \to H^0(X', f'^* \mathcal{H})$ is bijective for each $Y' \to Y$ étale. Replacing $Y$ by $Y'$, we know by (1.5) that $f$ is $(-1)$-acyclic; i.e. that $\alpha : H^0(Y, \mathcal{H}) \to H^0(X, f^* \mathcal{H})$ is injective, as $f$ is surjective. Moreover, $f$ is locally acyclic as it is smooth. Then, (1.16) shows that $\alpha$ is surjective iff for every geometric point $\bar{y}$ of $Y$ algebraic over a point $y$ of $Y$, $\alpha : H^0(\mathcal{H}^\vee, f^* \mathcal{H}) \to H^0(X^\vee, f'^* \mathcal{H}|X^\vee)$ is, so we may assume $Y$ is the spectrum of an algebraically closed field, in which case $X$ is connected, as the fibers of $f$ were assumed geometrically connected. Then $\alpha$ is seen to be bijective by another application of (1.1), which reduces the matter to the corresponding question for a constant sheaf. Note that the existence of a retraction discussed above is irrelevant to this argument.
4.2.6. This paragraph, as written, is nonsense. The correct (equivalent) statements are

(a) \( u^* \) identifies \( \mathcal{A} \) with a subcategory of \( \mathcal{B} \) closed under subquotients.

(b) the unit of adjunction \( \eta_! : \text{id}_\mathcal{B} \to u^* u_! \) is a natural epimorphism;

(b') the counit of adjunction \( \varepsilon_* : u^* u_* \to \text{id}_\mathcal{B} \) is a natural monomorphism.

Note that statement (a) is different from \( \mathcal{A} \) being épaisse, as an épaisse subcategory is also closed under extensions. A general remark: an adjunction \( F : \mathcal{C} \rightleftarrows \mathcal{D} : G \) with unit \( \eta \) and counit \( \varepsilon \) is called idempotent if it can be factored as a reflection and a coreflection, in which case many things are true; see the nLab page ‘idempotent adjunction.’ In particular, \( \varepsilon F \) and \( \eta G \) are natural isomorphisms. In our situation, both \( u_! \dashv u^* \) and \( u^* \dashv u_* \) are idempotent adjunctions as they are reflective and coreflective, respectively, so we get that \( \varepsilon_* u^* \) and \( \eta_! u_* \) are natural isomorphisms. Of course, \( u_! \) is right exact and \( u_* \) is left exact. We prove \((a) \Leftrightarrow (b')\); dual arguments give \((a) \Leftrightarrow (b)\).

To prove \((a) \Rightarrow (b')\), let \( B \) be an object of \( \mathcal{B} \). In the commutative diagram with exact rows,

\[
\begin{array}{ccc}
0 & \longrightarrow & \ker \varepsilon_*(B) \\
& \alpha \downarrow & \downarrow \beta \\
0 & \longrightarrow & u^* u_* B \\
& \downarrow \gamma & \downarrow \\
& B & \ker \varepsilon_*(B) & u^* u_* B & u^* u_* B & u^* u_* B
\end{array}
\]

\( \ker \varepsilon_*(B) \) is in the essential image of \( \mathcal{A} \) as \( \mathcal{A} \) is closed under subobjects. Therefore \( \alpha, \beta, \) and \( \gamma \) are isomorphisms, which shows \( u^* u_* \ker \varepsilon_*(B) = 0 \) and therefore \( \ker \varepsilon_*(B) = 0 \).

Now let’s prove \((b') \Rightarrow (a)\). In the commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & A \\
& \downarrow & \downarrow \\
0 & \longrightarrow & u^* u_* A \\
& \downarrow & \downarrow \\
& u^* u_* B & u^* u_* C
\end{array}
\]

with exact rows, the middle arrow is an isomorphism and the outer arrows are monomorphisms. By the four-lemma, the first arrow is also an epimorphism, hence an isomorphism, identifying \( A \) in the essential image of \( \mathcal{A} \). Therefore this essential image is closed under subobjects and hence also under subquotients, as \( u^* \) is exact.
Identifying \( \mathcal{A} \) with its essential image, a full subcategory of \( \mathcal{B} \), every object \( B \) of \( \mathcal{B} \) has a largest subobject in \( \mathcal{A} \), viz. \( u^*u_*B \), and a largest quotient in \( \mathcal{A} \), viz. \( u^*u!B \). To see this, simply observe that both candidates are indeed in \( \mathcal{A} \), and if \( A \) is in \( \mathcal{A} \) and a subobject of \( B \), then \( A \cong u^*u_*A \leftarrow u^*u_*B \leftrightarrow B \), and dually.

The example adjunction is backwards: \( u_!(X \to Y) = (Y \sim Y) \) is the left adjoint, \( u_*(X \to Y) = (X \sim X) \) is the right adjoint, since the diagrams

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
C & \sim & D
\end{array}
\quad
\begin{array}{ccc}
A & \sim & B \\
\downarrow & & \downarrow \\
C & \rightarrow & D
\end{array}
\]

have unique completions to commutative squares. Then \( \varepsilon_* : u^*u_* \to \text{id}_B \) needn’t be a natural monomorphism, as a commutative diagram of the sort

\[
\begin{array}{ccc}
A & \rightarrow & B \\
\downarrow & & \downarrow \\
C & \sim & C \\
\downarrow & & \downarrow \\
C & \rightarrow & D
\end{array}
\]

\( \sim \text{id} \)

determines the morphism \( A \rightarrow C \) but isn’t enough to determine the morphism \( B \rightarrow D \).

4.2.6.1. The Jordan-Hölder theorem holds in any abelian category, and since \( u^* \) is exact and preserves simple objects, identifying \( \mathcal{A} \) with its essential image, it follows that the components of any \( A \) in \( \mathcal{A} \) also belong to \( \mathcal{A} \). To show that \( \mathcal{A} \) is closed under subobjects, and therefore subquotients, it will suffice to show that if \( 0 \rightarrow B \rightarrow A \rightarrow S \rightarrow 0 \) with \( A \) in \( \mathcal{A} \), \( B \) in \( \mathcal{B} \), and \( S \) simple, then \( B \) is in \( \mathcal{A} \). Since \( S \) is simple, it is a component of \( A \) and therefore in \( \mathcal{A} \). The five-lemma, applied to the diagram with exact rows obtained by applying \( \varepsilon_* \),

\[
\begin{array}{ccc}
0 & \rightarrow & B \rightarrow A \rightarrow S \\
\uparrow & & \sim \uparrow \\
0 & \rightarrow & u^*u_*B \rightarrow u^*u_*A \rightarrow u^*u_*S,
\end{array}
\]

shows that indeed \( u^*u_*B \sim B \), and \( B \) is in \( \mathcal{A} \).
4. FAISCEAUX PERVERS

4.2.6.2. Notes on the proof are below; the proof depends on the middle perversity insofar as the commutativity of intermediate extension with \( f^*[d] \) relies on the relative dimension coinciding with the change in perversity between an irreducible component and its inverse image by \( f \). The proof doesn’t work for \( \mathbb{Z}_\ell \)-sheaves as it relies on 4.3.1 which fails for \( \mathbb{Z}_\ell \)-cohomology.

Commutation of \( f^*[d] \) with intermediate extension. We wish to show that \( f^*[d]j_\ast! = j_\ast!f^*[d] \). The transitivity of \( j_\ast! \) for \( j \) the inclusion of a locally closed subset allows us to factorize the inclusion of \( V \) as the open immersion \( V \hookrightarrow \overline{V} \) followed by the closed immersion \( \overline{V} \hookrightarrow Y \); the latter posing no problem as in this case lower shriek, lower star, and intermediate extension all coincide and commute with \( f^*[d] \) by smooth base change. Reduced to the case of open immersion \( Y = \overline{V} \), let \( m \) be minimal such that \( F = Y - V \) is of dimension \( \leq m \), and put \( t := p(2m) \). We can find a stratification of \( F \) into strata satisfying 2.2.10 (a) so that for \( A \) in \( D^b_c(V, \mathbb{R}/m^n) \), the \( H^i j_\ast A \) are locally constant on each stratum for \( i \geq t \). Let \( U_n \) (resp. \( F_n \)) denote the union of strata \( S \) in this stratification of \( F \) satisfying \( p(S) \leq n \) (resp. \( p(S) \geq n \)). For such an \( S \), \( \overline{S} - S \) is a union of strata of dimension strictly less than \( \dim S \). Therefore each \( S \) in the stratification of \( U_n - U_{n-1} \) is closed in \( U_n \). By transitivity of the intermediate extension, it will suffice to extend \( A \) from \( V \) to \( V \cup U_t \), as the complement of the latter in \( Y \) is \( F_{t+1} \), closed; \( V \cup U_t - V = U_t \) is also closed in \( V \cup U_t \), and nonempty. \( U_t \) is a disjoint union of equidimensional strata smooth over \( \overline{k} \) (hence by the note to 2.2.10, the étale topology sees them as smooth), so that the \( H^i j_\ast A \) are locally constant on \( U_t \) for \( i \geq t \). We have reduced to the setting of 2.2.4: \( Y = \overline{V} \), \( F = Y - V \) smooth of dimension \( m \), \( H^i j_\ast A \) are locally constant on \( F \) for \( i \geq t := p(2m) \), and \( j_\ast A = \tau^F_{\leq t-1} j_\ast A \). As \( f \) is smooth of relative dimension \( d \), \( f^{-1} F \) is also equidimensional, now of dimension \( m + d \). As pullbacks of locally constant sheaves, the \( H^i j_\ast f^* A \) are still locally constant on \( f^{-1} F \). If \( t' = p(2(d + m)) = -d - m \), then shifting by \( d \) and applying \( \tau^F_{<t'} \) is the same as applying \( \tau^F_{<t} \) and then shifting by \( d \). It only remains to verify that \( f^* \) commutes with \( \tau^F_{\leq 0} \), but this is easy: letting \( i : F \hookrightarrow Y \) be the closed immersion, \( (f^* \tau^F_{\leq 0} A, f^* A, f^* i_\ast \tau^F_{>0} i^* A) \) defines \( f^* \tau^F_{\leq 0} A \), but \( f^* i_\ast \tau^F_{>0} i^* A = i_\ast \tau^F_{>0} i^* f^* A \) (notating \( i_\ast \) as usual for the base extension as well), hence also defines \( \tau^F_{\leq 0} f^* A \).
Irreducibility of the inverse image of an irreducible local system on an irreducible scheme by a smooth morphism with geometrically connected fibers. SGA 1, Exp. IX, 5.6 shows that if \( f : S' \to S \) is universally submersive (e.g. faithfully flat and quasi-compact) with geometrically connected fibers, and \( S \) is connected, then \( S' \) is connected, and, choosing a geometric point \( s' \) of \( S' \) and letting \( s \) be the image of \( s' \) in \( S \), the homomorphism \( \pi_1(S', s') \to \pi_1(S, s) \) is surjective. This immediately implies that if \( \mathcal{L} \) is an irreducible local system on \( S \), \( f^* \mathcal{L} \) is an irreducible local system on \( S' \). Of course, smooth with geometrically connected fibers means smooth with geometrically irreducible fibers as these fibers are themselves smooth (SGA 1, Exp II, 2.1) and therefore regular. It is a topological fact (Stacks 004Z) that if \( Y \) is irreducible and \( f : X \to Y \) is open with irreducible fibers, then \( X \) is irreducible. This verifies that the if \( V \) is irreducible and \( f \) is smooth with connected fibers, the inverse image on \( f^{-1} V \) of an irreducible local system on \( V \) is again irreducible.

4.2.7. Let \( X', X, Y \) be of finite type over a field \( k \) and \( f : X' \to X \). We wish to show that \( \boxtimes \) commutes with direct image; i.e. that the below arrow is an isomorphism.

\[
f_* K \boxtimes L \to (f \times \text{id})_* (K \boxtimes L)
\]

We will compute locally about a geometric point \( \xi := (x, y) \to X \times Y \) (\( x, y \) geometric points of \( X, Y \) respectively), so that all the objects in \( X_x \leftarrow (X \times Y)_\xi \to Y_y \) are spectra of strictly henselian local rings. Let \( t \to X_x \) be a geometric point centered on the generic point of an irreducible component of \( X_x \). Th. finitude 2.16 gives that \( Y \to \text{Spec } k \) is universally locally acyclic, so for any \( M \) in \( D^+(X \times Y, \mathbb{Z}/\ell) \),

\[
\Gamma((X \times Y)_\xi, M) = \Gamma((X \times Y)_{\xi, t}, M),
\]

where \( (X \times Y)_{\xi, t} \) denotes the geometric fiber in \( t \) of \( (X \times Y)_\xi \to Y_y \). To pass from \( X_x \) to \( (X_x)_t = t \) we can first pass to the limit of Zariski neighborhoods of \( t \), which is the spectrum of an artinian local ring, then kill nilpotents and extend scalars. As lower star commutes with smooth base change and the étale topology doesn’t see nilpotents, we may therefore assume \( X = t \), the spectrum of a separably closed extension \( k(t) \) of \( k \), and \( Y = Y_y \). As \( X_x \to \text{Spec } \overline{k} \leftarrow Y_y \), Spec \( \overline{k} \times_k Y_y \) is the disjoint union of copies of \( Y_y \), and
ξ picks one of them; i.e. \((\text{Spec } \bar{k} \times_k Y_y)_{\xi} = Y_y\). So we may assume \(Y_y \to \bar{k}\), and write
\[
\Gamma((X \times Y)_{\xi}, M) = \Gamma(Y \times_{\bar{k}} k', M) = \Gamma(Y, M),
\]
where the second equality is Arcata V 3.3. We are reduced to \(k = \bar{k}, X = \bar{k}, Y = Y_y\), in which case the formula is
\[
\Gamma(X', K) \otimes \Gamma(Y, L) \to \Gamma(X' \times Y, K \boxtimes L),
\]
which is obtained from the Künneth formula of Th. finitude 1.11 by smooth base change and passage to the limit along \(Y_y \to Y\).

4.2.8. It would appear that \(\boxtimes\) is only right t-exact in \(\mathbb{Z}_\ell\)-cohomology, due to the possible appearance of Tor.

4.3.1. This is a theorem in \(\mathbb{Q}_\ell\)-cohomology and not in \(\mathbb{Z}_\ell\)-cohomology because the latter has few irreducible objects; in particular, the category of lisse \(\mathbb{Z}_\ell\)-sheaves is not artinian; the irreducible lisse \(\mathbb{Z}_\ell\) sheaves are torsion.

4.3.3. ‘l’a monodromie de \(L\) ne change pas par restriction à \(U’ \leadsto\text{SGA 1 Exp. 1 10.3.}

4.3.4. The dévissage of the perverse sheaf \(\mathcal{F}\) should occur over an irreducible affine smooth open so that we can apply the results of 4.1.10–4.1.12. The sequence 4.1.10.1 has outer terms supported on \(X \setminus U\) and so reduces the problem for \(\mathcal{F}\) to that for \(j_! j^* \mathcal{F} = j_! \mathcal{L}[\dim U]\); 4.1.12.3 then reduces the problem for \(j_! \mathcal{L}[\dim U]\) to that for \(j_! \mathcal{L}[\dim U]\). This allows us to proceed by induction on the length of \(\mathcal{L}\). Let \(\mathcal{L}' \subset \mathcal{L}\) be a simple lisse subsheaf. It suffices to remark that the kernel of \(j_! \mathcal{L}'[\dim U] \to j_! \mathcal{L}[\dim U]\) is supported on \(X \setminus U\), and the restriction of the cokernel to \(U\) has strictly lesser length than that of \(\mathcal{L}'\).

4.4 & Appendix A: t-exactness of nearby & vanishing cycles. To understand the argument in Appendix A, [I] §3.1 is very helpful. In Appendix A, the calculation of \(R\Gamma\) proceeds along identical lines to [D] 7.11.3 & 10.7.
There is a discrepancy between (4.4.2) the proposition of Appendix A; namely (4.4.2) says that $\Psi_\eta := R\Psi_\overline{\eta}$ is right t-exact, while the appendix claims that the same functor has perverse amplitude $-1$; i.e. that $\Psi_\overline{\eta}[-1]$ is t-exact. This discrepancy is due to a difference of t-structures; both are relative to the middle perversity function $p_1/2$, but the dimension function in (4.4.2) is the naïve one, while the dimension function in the appendix is the rectified dimension function introduced by Artin in [SGAA XIV 2.2], which is also the one described in Remark (i) to Appendix A. The point is that if $S$ admits structure morphism to a field $k$ and $x \in X$ maps to the generic point of $S$,

$$\text{tr. deg.}(k(x)/k) = \text{tr. deg.}(k(x)/k(\eta)) + 1;$$

in general if $x$ has image $s$ in $S$, Artin rectifies the naïve dimension function by adding $\text{tr. deg.}(k(x)/k(y))$ so that

$$\delta(x) = \text{dim \{y\}} + \text{tr. deg.}(k(x)/k(y)).$$

This has the effect of shifting the naïve t-structure on $X_\eta$ by 1 to the left, so that if $\Psi_{\overline{\eta}}$ is t-exact with respect to the naïve t-structure (4.4.2), $\Psi_{\overline{\eta}}[-1]$ is t-exact with respect to the rectified t-structure. The point is that we would like to think of $S$ as the henselization of a curve at a regular point; if $S$ were instead the curve instead of its localization and $X \to S$ were still of finite type, any point $x$ of $X$ lying over the generic point of $S$ would have strictly positive dimension (dim \{x\} > 0), and indeed if $S$ were a curve of finite type over a field $k$, dim \{x\} would be given by precisely tr. deg. $(k(x)/k)$.

On the matter of invariants of a $\Lambda[G]$-module when $G$ is a finite group of order invertible in $\Lambda$: the functor ‘invariants under $G$’ $\text{Hom}_{\Lambda[G]}(\Lambda, -)$ is exact if $\Lambda$ is projective as $\Lambda[G]$-module. The canonical surjection $\Lambda[G] \twoheadrightarrow \Lambda$ is split by

$$1 \mapsto \frac{1}{|G|} \sum_{g \in G} g =: \omega,$$

which recognizes $\Lambda$ as projective $\Lambda[G]$-module. Therefore when $Q$ is a profinite group of order prime to $\ell$ acting on a finite $\Lambda$-module ($\Lambda \supseteq \mathbb{Z}/\ell^n$), the functor ‘invariants under $Q$’ is exact. This argument can be applied stalkwise, and gives more. Namely, if $M$ is a $\Lambda[G]$-module, the map $M \twoheadrightarrow M_G$ factors as

$$M \xrightarrow{\omega} M^G \hookrightarrow M_G.$$
This shows that the map \( M^G \hookrightarrow M_G \) is an isomorphism; the inverse is given by the map which to a class \([m] \in M_G\) associates \( \omega m \).

There is the matter of how to define the nearby cycles: in [SGA 7 XIII 2.1] the functor \( R^\Psi \) is defined in a somewhat different way from the functor \( \Psi \) of [SGA 7 I 2.2]. Let \((S, \eta, s)\) be a henselian trait and let \( \tilde{\eta}, \bar{\eta} \) denote the spectra respectively of the maximal unramified extension of \( k(\eta) \) and the separable closure of \( \eta \). Let \( \tilde{S} \) denote the normalization of \( S \) in \( k(\tilde{\eta}) \); and \( \tilde{s} \) its closed point; \( \tilde{s} \) is the spectrum of the separable closure of \( k(s) \) and \( \tilde{S} \) is the strict henselization of \( S \) at \( \tilde{s} \). Let \( \bar{S} \) denote the normalization of \( S \) in \( k(\bar{\eta}) \): \( \bar{S} \) is the spectrum of a valuation ring with value group \( \mathbb{Q} \), generic point \( \bar{\eta} \), and closed point \( \bar{s} \) with residue field a purely inseparable extension of \( k(\bar{s}) \). Let \( \rho : \bar{\eta} \to \eta \) denote the chosen geometric generic point of \( S \). We have a commutative diagram

\[
\begin{array}{ccc}
\tilde{s} & \xrightarrow{i} & \tilde{S} \\
\downarrow & & \downarrow \\
\tilde{s} & \xrightarrow{i} & \tilde{S} \\
\downarrow & & \downarrow \\
s & \xrightarrow{i} & S \\
\end{array}
\]

Let \( f : X \to S \) and \( G := \text{Gal}(\bar{\eta}/\eta) \). Exposé 1 (and Appendix A) suppose \( S \) is strictly henselian to begin with, in which case the tilde is superfluous, and define a functor \( \psi \). On the other hand, in Exposé XIII, Deligne defines for (a not necessarily strictly henselian \( S \))

\[
R^\Psi(\eta)(K) := i^\ast R_j^\ast \bar{K}, \quad K \in D^+(X_\eta, \Lambda).
\]

Let \( \bar{\rho} : \bar{s} \to s \) denote the morphism obtained by base change of \( \bar{S} \to S \) along \( i : s \to S \) (technically the base change of the latter morphism has its source in the spectrum of a field purely inseparable over \( k(\bar{s}) \), but the map to \( s \) from the spectrum of this larger field factors through our \( \bar{\rho} \)), and define

\[
\psi(K) := \bar{\rho}^\ast i^\ast R_j^\ast \rho_* \rho^\ast K, \quad K \in D_\psi^b(X_\eta, \Lambda).
\]
When $S$ is strictly henselian, this coincides with the functor $i^*Rj_!\rho_*\rho^*K$. But now $\psi$ and $R\Psi_\eta$ coincide on an arbitrary henselian trait $S$: the fact that an integral morphism commutes with all base extension [SGA, VIII 5.6] allows one to write

$$\psi(K) = \bar{\rho}^*i^*Rj_!\rho_*\rho^*K = \bar{\rho}^*i^*\rho_*Rj_*K_{\bar{\eta}} = \bar{\rho}^*\bar{\rho}_*\bar{i}^*Rj_*K_{\bar{\eta}} = \bar{i}^*Rj_*K_{\bar{\eta}} = R\Psi(K).$$

Note that in the case $X = S$, $R\Psi(K)$ just gives the sheaf $K_{\bar{\eta}}$ on the topos $s \times_s \eta \simeq \eta$, and the distinguished triangle defining $R\Psi$ can be written as

$$sp^*K_s \to K_{\bar{\eta}} \to R\Psi(K) \to .$$

One obtains the isomorphism

$$K \sim \to R\Gamma(G,\rho_*\rho^*K) \quad K \in D^b_c(X_\eta,\Lambda)$$

of Appendix A from the same argument, namely by passage to the limit from

$$K \sim \to R\Gamma(G_i,\rho_i^*\rho_i^*K) \quad K \in D^b_c(X_\eta,\Lambda).$$

This isomorphism can be proved after a base change $\eta_i \to \eta$ after which it is literally Shapiro’s lemma (compare [SGA 7, XIII §1]). (The passage to the limit is justified in light of the isomorphisms

$$H^q(G,A) \sim \to H^q(G/U,A_U)$$

as $U$ runs over open subgroups of $G$ and $A$ is a discrete $G$-module (c.f. Serre, Cohomologie Galoisienne §2.2).)

Appendix A (i). In light of the above, the identity $i^*Rj_*K = R\Gamma(G,R\Psi(K))$ is definitional [L. 3.1.3], but perhaps not quite – a detailed argument is written in the note to Morel’s article on gluing perverse sheaves. The spectral sequence associated to the perverse canonical filtration is the spectral sequence associated to the filtration of complexes given by the $p^*\tau_\leq$; see [BBD, §3.1.5] for the definition of the filtration and [012N] for the formulation of the spectral sequence arising from a filtration of complexes (see also [015X]). Unfortunately when writing down the spectral sequence in Appendix A there is a collision of notation, as $a, b$ are used as indexes in the spectral sequence while $R\Psi(\mathcal{F})$ still has perverse amplitude $[a, b]$. The perverse amplitude of $i^*j_!\mathcal{F}$ is contained
in $[-1,0]$ by [BBD 4.1.10 (ii)]. The compatibility of $R\Psi$ with change of trait is Th. finitude (3.7).

Appendix A (ii). $Y$ must denote the closed fiber $X_s$. When $\mathcal{G}$ is supported on the closed fiber, $R\Psi_\eta(\mathcal{G}) = 0$ and the claim is clear. Therefore we assume the support of $\mathcal{G}$ has nonempty intersection with $X_\eta$ and $\mathcal{G}$ is a simple perverse sheaf. [BBD 4.3.1 & 4.3.2] gives $\mathcal{G} = j_! j^* \mathcal{G}$, and [BBD 4.1.12] allows us to write the distinguished triangle

$$i^* \mathcal{G} \rightarrow R\Psi(\mathcal{G}_\eta) \rightarrow R\Phi(\mathcal{G}) \rightarrow$$

as

$$(\rho H^{-1} i^* j_* \mathcal{F})[1] \rightarrow R\Psi(\mathcal{F}) \rightarrow R\Phi(\mathcal{G}) \rightarrow$$

This shows that $R\Psi(\mathcal{G})[-1]$ is a perverse sheaf, and using (i) and (*) obtains

$$(R\Psi(\mathcal{G})[-1] = \rho H^{-1} R\Psi(\mathcal{F})/(\rho H^{-1} R\Psi(\mathcal{F}))^G. \]$$
Bibliography

[BBD]  *Faisceaux Pervers* par Beilinson, Bernstein, Deligne (& Gabber!)
[CD]   *Catégories Dérivées (Etat 0)* par Verdier, dans SGA 4½
[D]    *Les constantes des équations fonctionnelles des fonctions* L par Deligne, dans LNM 349
[I]    *Autour du théorème de monodromie locale* par Illusie, dans Astérisque 223
[I2]   *Complexe Cotangent et déformations I* par Illusie, dans LNM 239
[SGA5] SGA 5, dirigé par Grothendieck
5. Applications de la formule des traces aux sommes trigonométriques

1.2. Suppose $X$ is locally noetherian and $X'$ is a connected component of $X$ pointed by a geometric point $a \to X'$. Via Grothendieck’s Galois theory, over $X'$, the $A$-torsor $T$ can be identified (after a choice $e$ of identity for the stalk of $T$ at $a$) with the set $A$ with (continuous) left action of $\pi_1(X', a)$ and right action of $A$.

Given a homomorphism $\tau : A \to B$, we wish to produce a map of sets $\tau : A \to B$ so that $\tau(ta) = \tau(t)\tau(a)$ and give the set $B$ with right action of $B$ a left action of $\pi_1(X', a)$ with respect to which $\tau$ is equivariant. The given homomorphism of groups sets up a map of sets compatible with the actions of $A$ and $B$, and the formula

$$g\tau(t) := \tau(gt)$$

defines a left $\pi_1$ action on $\tau(A)$. For $s \in B$ not necessarily in $\tau(A)$,

$$g s := g\tau(e)s = \tau(g e)s$$

extends the left action of $\pi_1$ to all of $B$. The $B$-torsor $\tau(T)$ is then identified with the set $B$ with right action of $B$ and the given left action of $\pi_1$, with respect to which $\tau : T \to \tau(T)$ is equivariant by construction. This discussion depends on the choice of identity for the torsor $T$ (the identity for $\tau(T)$ is obtained from the choice for $T$ and the homomorphism $\tau$) and therefore only defines the torsor $\tau(T)$ up to isomorphism.

Let $R$ (resp. $E_\lambda$) be a finite $\mathbb{Z}_\ell$-algebra (resp. finite $\mathbb{Q}_\ell$-algebra). Recall (SGA 5 Exp. VI (α) 1.2.4, (β) 1.4.1, (γ) 1.4.2, (δ) 1.4.3) that the category of

- (α) lisse $\mathbb{Z}_\ell$-sheaves
- (β) lisse $R$-sheaves
- (γ) lisse $\mathbb{Q}_\ell$-sheaves
- (δ) lisse $E_\lambda$-sheaves

on a connected locally noetherian scheme $X$, pointed by a geometric point $a \to X$ is equivalent to the category of

- (α) $\mathbb{Z}_\ell$-modules of finite type on which $\pi_1(X) := \pi_1(X, a)$ acts continuously for the $\ell$-adic topology
5. APPLICATIONS DE LA FORMULE DES TRACES AUX SOMMES TRIGONOMÉTRIQUES

(β) R-modules of finite type equipped with a continuous and R-linear action of \( \pi_1(X) \) on the underlying \( \mathbb{Z}_\ell \)-module of finite type

(γ) \( \mathbb{Q}_\ell \)-vector spaces of finite dimension equipped with a continuous action of \( \pi_1(X) \)

(δ) \( \mathbb{E}_\lambda \)-vector spaces of finite dimension equipped with a continuous and \( \mathbb{E}_\lambda \)-linear action of \( \pi_1(X) \) on the underlying \( \mathbb{Q}_\ell \)-vector space.

Given an A-torsor T on X and a finite-dimensional \( \mathbb{E}_\lambda \)-vector space \( V \), the data of an \( \mathbb{E}_\lambda \)-sheaf \( F \), lisse of rank \( \dim V \), together with a morphism of sheaves \( \rho : T \to \text{Isom}(V, \mathcal{F}) \) satisfying \( \rho(ta) = \rho(t)\rho(a) \) is the same (after a choice of identity \( e \) for the stalk of T at \( a \)) as the data of a continuous and \( \mathbb{E}_\lambda \)-linear action of \( \pi_1(X) \) on the underlying \( \mathbb{Q}_\ell \)-vector space of \( V \) and a \( \pi_1 \)-equivariant group homomorphism \( \rho : A \to \text{GL}(V) \), where the action of \( \pi_1 \) on \( \text{GL}(V) \) is induced by the action of \( \pi_1 \) on the second factor of \( \text{Hom}(V, V) \).

Given \( \rho : A \to \text{GL}(V) \) a linear representation of \( A \), there is a unique continuous and \( \mathbb{E}_\lambda \)-linear action of \( \pi_1 \) on the \( \mathbb{Q}_\ell \)-vector space underneath \( V \) that makes \( \rho \pi_1 \)-equivariant:

\[
\pi_1(X) \times V \to V \\
(g, v) \mapsto \rho(ge)v \quad \sim \quad \rho(ga) = \rho(gea) = \rho(ge)\rho(a) = gp(a).
\]

1.3. Torsors can be discussed in the language of schemes or sheaves; the distinction comes down to whether the torsor is representable as a sheaf, and this distinction motivates the introduction of algebraic spaces. In SGA 1 Exp. V §2 and SGAD Exp. III §0 & Exp. IV §5 the notion of principal homogeneous space is developed; these are the representing objects for certain (sheaf) torsors. There, an A-torsor is called a principal homogeneous space under \( A \). More precisely, given a category \( \mathcal{C} \), let \( \mathcal{C} \) denote its category of set-valued presheaves \( \text{Hom}(\mathcal{C}^\circ, (\text{Set})) \). If \( A \) is a \( \mathcal{G} \)-group acting on \( X \) an object of \( \mathcal{C} \), \( X \) is formally principal homogeneous under \( A \) (i.e. an A-pseudo torsor) if the equivalent conditions below are satisfied:

(i) for each object \( S \) of \( \mathcal{C} \), the set \( X(S) \) is empty or principal homogeneous under \( A(S) \);

(ii) the morphism of functors \( A \times X \to X \times X \) defined setwise by \( (a, x) \mapsto (ax, x) \) is an isomorphism.
It amounts to the same to say that the canonical morphism of functors

$$X \times A \to X \times X$$

is an isomorphism. If \( \mathcal{C} \) is equipped with a topology, then one says that the \( S \)-object \( X \) with \( S \)-group of operators \( A \) is fibered principal homogeneous under \( A \) (i.e. is an \( A \)-torsor) if it is locally trivial; i.e. there exists a covering family \( \{ S_i \to S \} \) such that for each \( i \), the \( S_i \)-functor \( X \times_S S_i \) with \( S_i \)-functor-group of operators \( A \times_S S_i \) is trivial.

The category \( \widehat{\mathcal{C}} \) has a final object \( \varepsilon \) which sends an object of \( \mathcal{C} \) to \( \{ \emptyset \} \), the set with one element. This functor is representable iff \( \mathcal{C} \) admits a final object. The ‘sections’ functor \( \Gamma \) is defined on \( \widehat{\mathcal{C}} \) as \( \text{Hom}(\cdot, \varepsilon) \) and on \( \mathcal{C} \) via \( X \mapsto h_X \); if \( \mathcal{C} \) admits a final object, this latter functor is isomorphic to \( \text{Hom}(\varepsilon, \cdot) \). SGAD Exp. IV 5.1.2, 5.1.3 observes that \( X \) is formally principal homogeneous under \( \Gamma \); therefore an isomorphism of \( \mathcal{C} \)-objects

$$\Gamma(X) \xrightarrow{\sim} \text{Isom}_{\mathcal{C} \text{-obj.}}(A, X)$$

of principal homogeneous sets under \( \Gamma(A) \); therefore an isomorphism of \( \mathcal{C} \)-objects

$$X \xrightarrow{\sim} \text{Isom}_{\mathcal{C} \text{-obj.}}(A, X).$$

The proof is simply that to each section \( x \) of \( X \) one associates the morphism \( A \to X \) defined setwise by \( a \mapsto xa \). This implies that an object with group of operators is trivial iff it is formally principal homogeneous and possesses a section.

The algebraic group schemes in the given extension can also be considered as sheaves for the fpqc topology on \( S \). En effet, the surjectivity of \( \pi \) implies that \( \pi \) is faithfully flat, and therefore an fpqc covering.

**Proposition (SGAD Exp. IV 5.1.7.1).** — Let \( \mathcal{C} \) denote a category possessing a final object, stable by fiber products, and equipped with a subcanonical topology \( \mathcal{T} \) (such as \( \text{Sch} \) equipped with fppf, fpqc, étale). Suppose \( \pi : G' \to G \) is a morphism of \( \mathcal{C} \)-groups which is covering for the topology \( \mathcal{T} \), and \( A = \ker \pi \). Then \( G \) represents the quotient sheaf \( G'/A \), and \( \pi \) is an \( A \)-torsor; i.e. \( G' \) is an \( A \)-torsor on \( G \).

Therefore it makes sense to say ‘\( G' \) is an \( A \)-torsor on \( G \)’ or that ‘the sheaf \( T \) of local sections of \( \pi \), \( T = \text{Hom}_G(\cdot, G') \), is an \( A \)-torsor on \( G \)’; the former represents the latter. The \( A \)-torsor \( G' \) on \( G \) is indeed locally trivial for fpqc: pulling back along the faithfully
flat covering \( \pi \), we find

\[
G' \times_G A \to G' \times_G G'
\]

\((g', a) \mapsto (g', g'a)\)

is indeed an isomorphism, as can be checked setwise.

In order to understand how to add torsors, it is instructive to first recall how to add extensions of abelian groups. Suppose

\[
0 \to A \to G' \to G \to 0
\]

\[
0 \to A \to G'' \to G \to 0
\]

are exact sequences of abelian groups. \( G' \) and \( G'' \) are \( A \)-torsors in \( \text{Set} \). The Baer sum \( G' + G'' \) is constructed from the direct sum of extensions by pushout along addition for \( A \) and pullback along the diagonal for \( G \).

\[
\begin{array}{ccc}
0 & \to & A \oplus A \\
\downarrow & & \downarrow \cup \\
0 & \to & G' \oplus G \\
\downarrow & & \downarrow \\
0 & \to & G \oplus G \\
\end{array}
\]

Here, the square marked with \( \cup \) is cartesian while the one marked with \( \Delta \) is cocartesian. The Baer sum of \( G' \) and \( G'' \) is \( \Delta^* +_\pi \): \( G' \oplus G'' \). Now consider the case of 1.3 where we are given an extension of commutative algebraic group schemes over \( S \), and \( G' \) is an \( A \)-torsor over \( G \). Pulling back \( G' \) along addition for \( G \) yields the fiber product \( +_G G' = G' \times_G G \times_S G \). If \( X \) is a scheme over \( S \), to give a morphism over \( S \) to \( +_G G' \) amounts to giving two objects \( g_1, g_2 \in G(X) \) and an object in \( g \in G'(X) \) such that \( \pi g = g_1 + g_2 \). This data is equivalent to the data of two objects \( g'_1, g'_2 \) in \( G'(X) \) mapping to \( g_1, g_2 \), respectively, modulo the relation which considers two such pairs equivalent if their sum is the same. This generalizes pushing out by addition on \( A \) to the case of torsors. The pullback of torsors is given on representing objects by fiber product, so that given an \( S \)-morphism \( f: X \to G \), \( f^*T = \text{Hom}_X(-, X \times_{f, \pi} G') \). This torsor is trivial if
there is a section $X \to X \times_{f,\pi} G'$ over $X$; i.e. a commutative diagram

\[
\begin{array}{ccc}
X & \longrightarrow & G' \\
\downarrow f & & \downarrow \pi \\
G & \longrightarrow & G'.
\end{array}
\]

In other words, the torsor $f^*T$ is defined up to isomorphism by the image of $f$ in $\text{Hom}_S(X,G)/\pi \text{Hom}_S(X,G')$.

If $g : X \to G$ is another $S$-morphism, then $g^*T$ is represented by $X \times_{g,\pi} G'$. The fiber products $X \times_{f,\pi} G'$ and $X \times_{g,\pi} G'$ are $A$-torsors over $X$. In analogy with Baer sum, a sum of torsors is given by pushing out by the addition on $A$ followed by pulling back by the diagonal $\Delta_X : X \to X \times_S X$. Since $G'$ is defined by an extension and both torsors come from pulling back $G'$ (i.e. $T$), this is the same as pulling back $G'$ along the composition of morphisms

\[
X \xrightarrow{(f,g)} G \times_S G \xrightarrow{+} G.
\]

More explicitly, the morphism $X \to G \times_S G$ factors as

\[
X \xrightarrow{\Delta} X \times_S X \xrightarrow{(f \circ \text{pr}_1, g \circ \text{pr}_2)} G \times_S G
\]

and therefore pulling back along this morphism corresponds to pulling back $(f^*T, g^*T)$ along the diagonal $\Delta_X$. Of course, the fiber product $G' \times_{G'} (G' \times_S G') \simeq G' \times_S G'$ corresponds to the $A \times A$-torsor $(T,T)$. As we have already seen, pulling back $G'$ along $G \times_S G \xrightarrow{+} G$ is the quotient of $(T,T)$ by the relation the action of $(a_1, a_2)$ and $(a'_1, a'_2)$ if $a_1 + a_2 = a'_1 + a'_2$. Therefore, $(f + g)^*T$ is the quotient of $(f^*T, g^*T)$ by this same relation, which verifies the formula

\[(f + g)^*T = f^*T + g^*T.\]

The Lang torsor. For the development of the Lang isogeny and its properties, see Borel, Algebraic Groups §16. There, he proves that $\mathcal{L}$ is surjective when $G^0$ is connected. If the group $G^0$ is smooth (i.e. geometrically reduced), then $\mathcal{L}$ is seen to be étale (as a morphism of schemes) by computing its differential and Milne, Algebraic Groups, 1.63.
That the functor $G_0^F = G_0(F_q)$ is a consequence of the description of $F$ on the set $G_0(F)$: see Rapport 1.1.

1.6. If $G_0$ is a commutative algebraic group defines over $F_q$, then the norm map

$$N : G_0(F_{q^n}) \rightarrow G_0(F_q) \subset G_0(F_{q^n})$$

is defined, as $F$ is a group homomorphism. In the case $G_0 = G_a$, then $G_a(F_{q^n}) = F_{q^n}$ and $N$ coincides with the field \textit{trace} $N(x) = x + x^{q} + \cdots + x^{q^{n-1}}$, as the group operation is addition. If $G_0 = G_m$, on the other hand, then $G_m(F_{q^n}) = F_{q^n}$ and $N$ coincides with the field \textit{norm} $N(x) = x \cdot x^{q} \cdots x^{q^{n-1}}$, as the group operation is multiplication.

1.7. The formulæ $\mathcal{F}(\chi, f_0) = f_0^* \mathcal{F}(\chi, \text{id}_{G_0})$, etc. result from the fact that the pullback of lisse sheaves corresponds to the homomorphism of $\pi_1$ induced by $f_0$; i.e. if $x \rightarrow X_0$ is a geometric point and $g = f_0(x)$, then both sides of these identities result from the composition of maps

$$\pi_1(X_0, x) \rightarrow \pi_1(G_0, g) \xrightarrow{L_0} G_0(F_q) \xrightarrow{\chi^{-1}} E_{\chi}^*,$$

where the $L_0$ indicates the homomorphism of groups defining the Lang torsor $L_0(G_0)$.

The formula that motivates this whole business

$$F^*_{\chi} = \chi f_0(x)$$

follows from the hint that the fiber in $x$ of the morphism 1.2.2 commutes with $F_0^*$. This is true for the following reason: the geometric point $x \in X^F$ and $f_0$ give homomorphisms

$$\text{Gal}(F, F_q) \rightarrow \pi_1(X_0, x) \rightarrow \pi_1(G_0, x)$$

and $F_{\chi}^{x-1}$ coincides with the image of the Frobenius substitution $\varphi \in \text{Gal}(F, F_q)$. Commutation of $F^*_{\chi}$ with $f_0^*$ is clear and it remains to show that $\rho_\chi$ of 1.2.2 is $\pi_1(X, x)$-equivariant. The formula (†) of \textit{the note to 1.2} shows this explicitly.

1.8. (ii) The inverse image by $f$ of the the $E_{\chi}$-sheaf obtained from $\chi^{-1}(L_0(G_0))$ by extension of scalars from $F_q$ to $k$ is probably better written $f^* \mathcal{F}(\chi)_1$ than $\mathcal{F}(\chi f)$.
1.10. The subtlety in this argument is building a bridge between $C$ and $F_q$. Of course, the isomorphisms on singular cohomology with $Q$ coefficients imply isomorphisms on singular cohomology with $Q_\ell$ coefficients for all $\ell$. The quadric hypersurface $X_0'$ and hyperplane $Y_0'$ in $P^{2N}_0$ over $F_q$ are each defined by the vanishing of a homogeneous polynomial of degree two and one, respectively, in the ring $F_q[X_0, X_1, \ldots, X_{2N}]$; let $f_0$ denote the polynomial defining $X_0'$. As $X_0'$ is a nonsingular variety over a perfect field, it is smooth over $F_q$. Let $m(x)$ be the minimal polynomial for a primitive element of $F_q$ over $F_p$; lifting the coefficients of $m(x)$ to $Z$, $m(x)$ remains irreducible, and defines a finite extension of domains $Z \to Z[x]/(m(x)) =: A$ so that $A/(p) \cong F_q$. The coefficients of $f_0$ now admit lifts to $A$ and define a projective quadric $X'_S$ in $P^{2N}_S$ which is, in particular, flat over $S := \text{Spec } A$, and such that the fiber over $(p)$ is $X_0'$. Recalling EGA IV 12.2.4, the set of points $s \in S$ such that $(X'_S)_s$ is smooth over $k(s)$ is open. In particular, the fiber over the generic point $\xi \in S$ is smooth over $k(\xi)$, which is a finite extension of $Q$. The strict henselization $\tilde{A}$ of $A$ at $(p)$ is a regular local ring with spectrum $\tilde{S}$; $X'_S \to \tilde{S}$ is proper and smooth, and $k(\tilde{\xi})$ is an algebraic extension of $k(\xi)$. Let $\text{Spec } C = t \to \tilde{S}$ be a geometric point centered on $\tilde{\xi}$, and put $X := (X'_S)_t$. The specialization morphism

$$H^* (X', Q_\ell) \sim H^* (X, Q_\ell)$$

is an isomorphism (Arcata V 3.1). As $X$ is a projective nonsingular quadric in $P^{2N}_C$, the comparison theorem between ordinary cohomology and étale cohomology (Arcata V 3.5.1) allow us to apply the transcendental argument to conclude.

1.13. To compute $R^i f_! Q_\ell$, we may assume $Y$ is the spectrum of a separably closed field and $X = A^1$. By Poincaré duality and Artin’s theorem, $H^0_\ell (A^1, Q_\ell) = 0$. As for $H^1_\ell$, the short exact sequence of sheaves on $P^1$

$$0 \to (Q_\ell)_{A^1} \to Q_\ell \to (Q_\ell)_\infty \to 0$$

gives rise to a long exact sequence of cohomology

$$0 \to Q_\ell \to Q_\ell \to H^1_\ell (A^1, Q_\ell) \to H^1 (P^1, Q_\ell) = 0 \quad \text{(as Pic}^0 (P^1) = 0),$$
verifying \( H^1_c(A^1, \mathbb{Q}_\ell) = 0 \). Evidently \( H^0_c(A^1, \mathbb{Q}_\ell) = \mathbb{Q}_\ell \); by Poincaré we conclude
\[
\begin{align*}
R^i f_! \mathbb{Q}_\ell &= 0 \quad i \neq 2 \\
R^2 f_! \mathbb{Q}_\ell &= \mathbb{Q}_\ell(-1).
\end{align*}
\]

2.3*. Leray spectral sequence for cohomology with proper support is a particular case of spectral sequence of composed functors, but the proof that for all composition of morphisms \( f = gh : X \to Y \to Z \) we have ‘well-behaved’ transitivity isomorphisms between \( Rf_* \) and \( Rg_*Rh_* \) is not straightforward; it is SGAA Exp. XVII 5.1.8, which also proves that the functors \( Rf_* \) are ‘way out’ and triangulated. The proof of transitivity is formal, relying on §3 of the same exposé, which reduces the problem to the analogous ones for proper morphisms and open immersions, provided one ‘compatibility’ isomorphism, which is 5.1.6. With transitivity in hand, the spectral sequence
\[
E^{pq}_2 = R^p g_* R^q h_! (K) \Rightarrow R^{p+q} f_! (K),
\]
valid for \( K \) in \( D(X, \mathcal{A}_X) \) (\( \mathcal{A} \) a sheaf of rings), is a consequence of a general spectral sequence written down by Verdier: for all \( L \) in \( D(Y, \mathcal{A}_Y) \)
\[
E^{pq}_2 = R^p g_! (\mathcal{H}^q(L)) \Rightarrow R^{p+q} g_! (L);
\]
see *Des catégories dérivées des catégories abéliennes* 4.4.6.

2.4*. Küneth formula in cohomology with proper support is SGA 4 Exp. XVII 5.4.

2.5*. The spectral sequence (2.5.2)* is the spectral sequence associated to a filtration of chain complex. Namely, the complements of the closed subsets in the filtration on \( X \) gives rise to a filtration on \( X \) by open subsets \( U_p \subset U_{p+1} \). To calculate cohomology with proper support, fix a compactification \( j : X \hookrightarrow \overline{X} \). In what follows, we write \( \mathcal{F} \) for the sheaf and its various inverse images. Let \( j_p : U_p \hookrightarrow \overline{X} \) and \( i_{p+1} : U_{p+1} - U_p = X_p - X_{p+1} \hookrightarrow \overline{X} \) be the immersions. The filtration \( j_{p!}\mathcal{F} \subset j_{p+1!}\mathcal{F} \) on \( j_!\mathcal{F} \) has successive quotients isomorphic to \( i_{p+1!}\mathcal{F} \). In light of the spectral sequence (2.3.1)*, the spectral sequence associated to a filtered injective resolution of \( j_!\mathcal{F} \) (Stacks 05TT) gives rise
Remark. Let $C$ be a curve of finite type over an algebraically closed field $k$, $P$ the set of rational points of the curve, $\mathcal{F}$ the constant sheaf with value $A$ on $C$, and $x \in C(k)$. The fiber of $P^* P^* F$ is not $F_x$; rather, it can be identified with functions from $P$ to $A$ continuous for the discrete topology on $P$, modulo the relation that $f \sim g$ if $f(x) = g(x)$ and $f$ and $g$ disagree on only finitely many points.

2.6*. SGAA Exp. V §3 introduces the Čech complex and the Cartan-Leray spectral sequence associated to a covering, which is simply the spectral sequence of the composition of functors, where the first is inclusion of sheaves into presheaves and the second is $\check{H}^0$ (it is shown that the $\check{H}^i$ associated to the Čech complex are indeed right derived functors of $\check{H}^0$ on presheaves). The condition on the morphisms in the covering is simply that fibered products are representable, which is true in the category of schemes. However, this is not done for cohomology with support. In the introduction to SGAA Exp. Vbis, the Leray spectral sequence is discussed for an open covering and for a locally finite closed covering, and this guides SGAA Exp. XVII 6.2.8–6.2.10, which discusses in full detail the construction of the so-called ‘extraordinary’ spectral sequence (2.6.2)*. One can also obtain the more ‘ordinary’ spectral sequence of (2.6.1)* from the argument there by replacing the trace morphism $u!u^* \to \text{id}$ in the case of $u : Y \to X$ separated, étale, surjective, and finite type with the unit of adjunction $\text{id} \to u_*u^*$ when $u$ arises from a finite covering by closed subschemes. Then instead of getting a left resolution of the sheaf, one gets a right resolution, as in the usual Čech resolution.

Given $(X_i)_{i \in I}$ a finite covering by closed subschemes, let $u : Y = \bigsqcup_i X_i := Y \to X$, $Y_n = Y \times_X \cdots \times_X Y$, and $u_n : Y_n \to X$. The sheaves $\mathcal{F}_n = u_n^*u_n^* \mathcal{F}$ form, via the units of adjunction $\mathcal{F} \to u_n^*u_n^* \mathcal{F}$, a simplicial sheaf coaugmented by $\mathcal{F}$. As discussed in SGAA Exp. Vbis, if $\mathcal{E}^*(\mathcal{F})$ denotes an injective resolution of $\mathcal{F}$, the double complex

$$\mathcal{F} \to (u_p^* \mathcal{E}^q(u_p^*(\mathcal{F})))_{p,q}$$
defines a resolution of $\mathscr{F}$ by injective sheaves.

If $f_n$ is the projection of $Y_n$ to $\text{Spec } k$, then $Rf_n! = Rg!u_n! = Rg!u_n^*$. Applying the functor $g_!$ to the double complex and filtering by semi-simplicial degree gives rise to the Čech spectral sequence

$$E_1^{p,q} = R^q f_p!(u_p^* \mathscr{F}) \Rightarrow R^p g_! \mathscr{F};$$

this is (2.6.1)* for cohomology with support.

The problem when taking the $X_i$ to be open is that $u_{n^*} \neq u_n!$ and there is no longer a coaugmentation map $\text{id} \to u_n!u_n^*$. The trace morphism, however, provides an augmentation map $u_n!u_n^* \to \text{id}$ and one proceeds by resolving to the left to produce the extraordinary Čech resolution and spectral sequence

$$E_1^{-p,q} = R^q f_p!(u_p^* \mathscr{F}) \Rightarrow R^{-p+q} g_! \mathscr{F};$$

this is (2.6.2)*. For more on the extraordinary Čech resolution, see SGAA Exp. XVII 6.2.8–6.2.10.
Bibliography

[Sommes trig.] *Application de la formule des traces aux sommes trigonométriques* dans SGA 4$\frac{1}{2}$. 
As a warm-up, let’s recall some Galois theory from SGA 1 in connection with the beginning of Sommes trig. All references in this paragraph are to SGA 1 Exposé V. The Lang isogeny for $G_{a,F_q}$ is written

$$0 \to F_q \to G_a \xrightarrow{x^q-x} G_a \to 0$$

and is a revêtatement étale called the Artin-Schreier revêtement. The sheaf of sections defines an $F_q$-torsor. The Galois group of the Artin-Schreier revêtement is therefore $F_q$, as a connected torsor under a finite group $G$ has Galois (= automorphism) group $G$. Just as in the theory of fields, the choice of words ‘Galois group’ in place of ‘automorphism group’ is reserved for Galois objects in the Galois category. Remark 5.11 characterizes the Galois objects in a Galois category $\mathcal{C}$ equipped with fiber functor $F$ as the connected torsors $X$ under a finite group $G$ (torsor = principal homogeneous space). The implications go as follows: an object $X$ is galoisian if it is connected, not isomorphic to $\emptyset_{\mathcal{C}}$, the initial object of $\mathcal{C}$ ($\iff F(X) \neq \emptyset$), and $\text{Aut } X$ is transitive ($\iff$ simply transitive) (N° 4, f) & 5.4). $X$ is a torsor under the group opposite $\Aut X$ iff $F(X)$ is a torsor under the group opposite $\Aut X$; i.e. $\Aut X$ acts simply transitively. Therefore $X$ is a connected torsor under the group opposite $\Aut X$. On the other hand, suppose $X$ is a torsor under $G$, i.e. $G$ acts on $X$ on the right and on $F(X)$ simply transitively, yielding a natural injection from $G$ into the group opposite $\Aut X$. As $G$ acts simply transitively, $\Aut X$ acts transitively. If moreover $X$ is connected, then N° 4 f) gives that $\Aut X$ acts transitively iff it acts simply transitively, showing that the injection above is actually an isomorphism between $G$ and the group opposite $\Aut X$. This is justification for the fact ‘In a Galois category, a connected torsor under a finite group $G$ is a Galois object with Galois group $G.$’
1.1.1.5. En vue de l’additivité de $t(0.9)$, l’énoncé $t_{Rf,Q_{λ,X}}(∞) = 1 - q$ résulte de la pureté relative pour $A^1 \xhookrightarrow{j} P^1 \longleftrightarrow ∞$ (Arcata V 3.4) en ce que

$$\begin{align*}
j_∗Q_λ &= Q_λ \\
R^1j_∗Q_λ &= Q_λ(-1)_∞ \\
R^qj_∗Q_λ &= 0 \text{ pour } q \geq 2.
\end{align*}$$

1.1.3. The action of Frobenius on the fiber is clearer in Sommes trig. 1.5, 1.6 since Deligne’s notation for Frobenius is clear and consistent.

The rigidification (1.1.3.1) is a consequence of the fact that there is a distinguished element of $L^{-1}(1)$, namely 1.

The trivialization (1.1.3.2) depends on the construction of $L^\chi$ from a torsor defined by an extension. In the setting of Sommes trig. 1.3, the $A$-torsor $T$ is defined as the sheaf of local sections of an extension $π$. As $H$ represents $m^∗T$ in the commutative diagram with exact rows ($^\pi = \text{cartesian and } m$ denotes the group law)

$$\begin{array}{cccccc}
0 & \longrightarrow & A \times A & \longrightarrow & G \times G' & \xrightarrow{π \times π} & G \times G & \longrightarrow & 0 \\
& & \downarrow{m} & & \downarrow & & \| & & \\
0 & \longrightarrow & A & \longrightarrow & H & \xrightarrow{r} & G \times G & \longrightarrow & 0 \\
& & \downarrow & & \downarrow{r} & & \downarrow{m} & & \\
0 & \longrightarrow & A & \longrightarrow & G' & \xrightarrow{π} & G & \longrightarrow & 0
\end{array}$$

in the language of torsors there is a canonical isomorphism

$m(T \times T) = m^∗T$.

The structure of group on $G$ provides commutative diagrams
The diagram

$$\pi_1(G \times G, (1,1)) \xrightarrow{\pr_1 \times \pr_2} \pi_1(G,1) \times \pi_1(G,1) \xrightarrow{m} \pi_1(G,1)$$

$$\xrightarrow{\sim} A \times A \quad \xrightarrow{\sim} A$$

gives the action of $\pi_1(G \times G, (1,1))$ on the torsors $T \times T$ and $m(T \times T) = m^*T$. Pushing by $\chi^{-1}$ gives a representation of $\pi_1(G \times G, (1,1))$ on $\overline{Q}_\ell$ which is induced by

$$(g_1, g_2) \mapsto \chi^{-1}(g_1) \otimes \chi^{-1}(g_2),$$

demonstrating explicitly an isomorphism

$$m^*L_\chi \simeq \pr_1^*L_\chi \otimes \pr_2^*L$$

and hence a trivialization of $D_2(L_\chi)$ compatible with the identification of the fibers of $D_2(L_\chi)$ and $\overline{Q}_\ell$ at $(1,1)$.

(1.1.3.3) = Sommes trig. 1.7.7 (see note to Sommes trig. 1.6)

(1.1.3.4) = Sommes trig. 2.7*.

The remark (1.1.3.7) can be summed up by the morphism of torsors induced by the commutative diagram below, the projective limit of which defines $\lim_{\longleftarrow I(k)} \mu_N(k)$:

$$1 \xrightarrow{[M]} \mu_{NM}(k) \xrightarrow{[NM]} G_{m,k} \xrightarrow{[N]} G_{m,k} \xrightarrow{[M]} G_{m,k} \xrightarrow{[N]} G_{m,k} \xrightarrow{[M]} 1.$$  

As $F_q^\times \simeq \mathbb{Z}/(q - 1)$, $I(F_q) = \{ n \in \mathbb{N} : n | q - 1 \}$.

1.2.1. The pairing $\langle \ , \ \rangle : E \times_S E' \rightarrow G_{a,k}$ should rather land in $G_{a,S}$. See Sommes trig. 1.7 c) to make sense of what this does to $L_\psi$.

To be maximally pedantic, if $q = p^n$, with the notation of (1.1.3.3) we should write $\psi_n$ where $\psi_q$ appears in the definition of $\hat{i}$. 

1.2.2. Locally $S = \text{Spec } A$, $E = \text{Spec } A[t_1, \ldots, t_r]$, $E \times_S E = \text{Spec } A[t_1, \ldots, t_r, t'_1, \ldots, t'_r]$, and the addition $E \times_S E \to E$ is given by $t_i \mapsto t_i + t'_i$ while $[-1]$ is given by $t_i \mapsto -t_i$.

More to the point, writing $E = \text{Spec } \text{Sym}(E)$ with $\text{Sym}(E)$ the symmetric algebra on a locally free sheaf $E$, then addition of sections gives addition on $E$, and vice versa, as the sheaf of local sections of $E$ coincides with $E^\vee$.

Identifying $E$ with $E''$ via $e \mapsto \langle e, \rangle = -a(e)$, the diagram becomes the base change by $\pi' : E' \to S$ of addition on $E''$, $E'' \times_S E'' \to E''$

and the isomorphism

$$\text{pr}^*_{12} \mathcal{L}(\langle , \rangle) \otimes \text{pr}^*_{23} \mathcal{L}(\langle , \rangle) = \alpha^* \mathcal{L}(\langle , \rangle)$$

becomes (1.1.3.2) on the nose.

$$\mathcal{F} \circ \mathcal{F}(K) = R \text{pr}''_! (R \text{pr}_1^*(R \text{pr}_1^*(K) \otimes \mathcal{L}(\langle , \rangle)) \otimes \mathcal{L}(\langle , \rangle))[2r]$$

In (1.2.2.2), the first equation of the proof is missing a $\pi^*$; it should read

$$\mathcal{F}(\pi^* L[r]) = \pi''^* L \otimes R \text{pr}_1^* \mathcal{L}(\langle , \rangle)[2r].$$
The rest of the proof of (1.2.2.2) is straightforward. It allows us to write

\[ F' \circ F(K) \simeq R \text{pr}'''(\text{pr}^* K \otimes \beta^* F'(\overline{Q_{E'}}[r])) \]
\[ \simeq R \text{pr}'''(\text{pr}^* K \otimes \beta^* \sigma'' \overline{Q_{E'}}(-r)) \]
\[ \simeq a_* K(-r), \]

as \( \beta^* \sigma'' \) is supported precisely on the locus where \( e'' = a(e) \), and after base change along \( E \times_{a, \text{id}} E'' \to E \times_S E'' \), which commutes with \( \text{pr}''' \), we find \( R \text{pr}''' \text{pr}^* K = a_* \).

**Remark.** Identifying \( E \simeq (E^\vee)^\vee \) in the canonical way, the involutivity can be written

\[ F' \circ F(K) \simeq [-1]_* K(-r), \]

and the inverse image under \([-1]\) of this identity is \( F'_{\psi^{-1}} \circ F(K) \simeq K(-r) \), since \([{-1}]^* F'_{\psi} = F'_{\psi^{-1}} \) in light of the fact that \([{-1}]^* \psi(e'' e') = \psi(-e, e') = \psi^{-1}(e, e') \).

(1.2.2.4) The adjunction upon which the equality rests is

\[ \langle f(s), \varphi \rangle_2 = \langle s, f'' \varphi := \varphi \circ f \rangle_1. \]
as pullback by $f'$ on a section $q$ of $E'_2$ gives, by definition, the element of $E'_1$ which, to a section $s$ of $E_1$, applies $f$ then $q$. We then write

$$\mathcal{F}_2(Rf_!K_1) = R\text{pr}'_2!(\text{pr}'_2^*(Rf_!K_1) \otimes \mathcal{L}(\langle , \rangle))[r_2]$$

$$\simeq R\text{pr}'_2!(R(f \times 1)_!(\text{pr}_1^*K_1) \otimes \mathcal{L}(\langle , \rangle))[r_2]$$

$$\simeq R(\text{pr}_2' \circ (f \times 1)_!(\text{pr}_1^*K_1 \otimes (f \times 1)^*\mathcal{L}(\langle , \rangle))[r_2]$$

$$\simeq R(\text{pr}_2' \circ (f \times 1)_!(\text{pr}_1 \circ (1 \times f')^*K_1 \otimes (1 \times f')^*\mathcal{L}(\langle , \rangle)))[r_2]$$

$$\simeq R\text{pr}'_2!(1 \times f')^*R(f \times 1)_!(\text{pr}_1^*K_1 \otimes \mathcal{L}(\langle , \rangle))[r_2]$$

$$\simeq f''R\text{pr}'_1!(\text{pr}_1^*K_1 \otimes \mathcal{L}(\langle , \rangle))[r_2]$$

$$\simeq f''\mathcal{F}_1(K_1)[r_2 - r_1].$$

(1.2.2.5) If $(\pi')'$ is the morphism of dual bundles $S \to E''$ induced by $\pi'$, then $(\pi')'^*a_* \simeq \sigma^*$. To make sense of the morphism $(\pi')' : S \to E''$, you have to unpack what it means to consider $S$ as a vector bundle of rank 0 over $S$. The only section of $S$ is the zero section, as at all points of $S$, as a rank 0 vector bundle returns the vector space $\{0\}$ at every point of $S$. In other words, we dispose of a tautological isomorphism $S \sim S \times_k 0 = S \times \{0\}$. Thinking about $\pi' : E' \to S$ as sending $(s, e') \mapsto (s, 0)$, the map $(\pi')' : S \to E''$ turns the 0 section of $S$ into a section of $E''$, i.e. the one which to a section $s'$ of $E'$ returns the 0 section of $S$. Therefore $(\pi')'$ is the embedding by zero section $0 : S \hookrightarrow E''$; as $-0 = 0$, the zero section of $E''$ corresponds under the isomorphism $a$ to the zero section of $E$, which sees $(\pi')'^*a_* \simeq \sigma^*$.

(1.2.2.7) For the isomorphism

$$\mathcal{F}(K_1 \otimes_S K_2) \simeq \mathcal{F}(K_1) \otimes_\mathcal{F}(K_2),$$

perhaps it helps to write the commutative diagram

\[
\begin{array}{cccc}
E & \xrightarrow{\text{pr}_1} & E \times_S E & \xrightarrow{\text{pr}_2} & E \\
\uparrow & & \uparrow & & \uparrow \\
E \times_S E' & \text{pr}_1 & (E \times_S E) \times_S (E' \times_S E') & \text{pr}_2 & E \times_S E \\
\downarrow & & \downarrow & & \downarrow \\
E' & \xrightarrow{\text{pr}_1} & E' \times_S E' & \xrightarrow{\text{pr}_2} & E' \\
\end{array}
\]
If one imagines this diagram drawn on the \(xy\)-plane in 3D with the center at \((0,0,0)\), places \(S\) at \((0,0,1)\), and connects every node of this diagram to this \(S\), every row and column of the \(3 \times 3\) above will then form a cartesian square with this \(S\). The equality of Fourier transform then follows from the Künneth formula in light of the isomorphism

\[
\text{pr}_1^* \mathcal{L}(\langle\ ,\ \rangle_{E_1}) \otimes \text{pr}_2^* \mathcal{L}(\langle\ ,\ \rangle_{E_2}) \simeq \mathcal{L}(\langle\ ,\ \rangle_{E_1 \oplus E_2}),
\]

which is an obvious consequence of the fact that given vector bundles \(E_1, E_2\),

\[
\langle\ ,\ \rangle_{E_1 \oplus E_2} = \langle\ ,\ \rangle_{E_1} + \langle\ ,\ \rangle_{E_2}.
\]

The character \(\psi\) carries this additive identity to a multiplicative one, hence the \(\otimes\).

To go from here to the stated isomorphism

\[
\mathcal{F}(K_1 * K_2) \simeq \mathcal{F}(K_1) \otimes \mathcal{F}(K_2)[-r],
\]

after applying (1.2.2.4) in the given way, one is faced with the problem of justifying

\[
s''(\mathcal{F}(K_1) \boxtimes_{S} \mathcal{F}(K_2)) \simeq \mathcal{F}(K_1) \otimes \mathcal{F}(K_2).
\]

It is useful to reason by adjunction, writing the following adjoint pair of diagrams.

![Diagram](image)

The point is that given \(e, e'\) sections of \(E, E'\) respectively,

\[
\langle e, (\text{pr}_i \circ s')(e') \rangle = \langle \sigma_i(e), s'(e') \rangle = \langle (s \circ \sigma_i)(e), e' \rangle = \langle e, e' \rangle.
\]

Ergo,

\[
s''(\mathcal{F}(K_1) \boxtimes_{S} \mathcal{F}(K_2)) \simeq (\text{pr}_1 \circ s')^* \mathcal{F}(K_1) \otimes_{S} (\text{pr}_2 \circ s')^* \mathcal{F}(K_2)) \simeq \mathcal{F}(K_1) \otimes \mathcal{F}(K_2).
\]

1.2.3. Quelques détails supplémentaires pour les exemples donnés suivent.
(1.2.3.1) With $\sigma'_F : S \to F'$,
\[
\mathcal{F}(i_*\overline{Q}_{E'}[s]) \simeq i'^*\mathcal{F}(\overline{Q}_{E}[s])[r-s] \simeq i'^*\sigma'_F(\overline{Q}_{E'}(-s))[r-s];
\]
now use proper base change for the cartesian square

\[
\begin{array}{ccc}
E' & \xleftarrow{i_{11}} & F^1 \\
\downarrow{i'} & \searrow & \downarrow{\pi'} \\
F' & \xleftarrow{\sigma'_F} & S.
\end{array}
\]

(1.2.3.2) A few useless words: $e_*\overline{Q}_{E}$ is the sheaf on $E$ that is supported precisely on the section $e$ of $E$. Restricted to the closed subscheme which is the image of $e$, $e_*\overline{Q}_{E}$ is constant with value $\overline{Q}_E$. Therefore $(e_*\overline{Q}_{E}) \boxtimes_k K$ has support contained in $e \times_S E$, and restricted to this closed subscheme, $s$ is an isomorphism with inverse $E \to (-e) \times_S E \xrightarrow{s} E \to e \times_S E$. En effet, $\tau_e$ factors as $E \to e \times_S E \xrightarrow{s} E$.

After applying (1.2.2.7), one uses (1.2.3.1) to compute
\[
\mathcal{F}(e_*\overline{Q}_{E'}) \simeq e_+\overline{Q}_{E'}[r] \simeq \mathcal{L}(\langle e, \rangle)[r].
\]

(1.2.3.3) The isomorphism $\alpha$ gives rise to a nondegenerate bilinear form $B : E \times_S E \to G_a$ via $B(e_1, e_2) = \langle e_1, \alpha(e_2) \rangle$, and $\alpha$ is symmetric if $B$ is; equivalently,
\[
\langle \alpha^{-1}(e'), \alpha(e) \rangle = \langle e, e' \rangle.
\]
This allows the easy verification of the identity
\[
q(e) + \langle e, 2e' \rangle = q(e + \alpha^{-1}(e')) - q'(e'). \quad (\dagger)
\]

The cartesian square

\[
\begin{array}{ccc}
E \times E' & \xrightarrow{\text{id} \times [2]} & E \times E' \\
\downarrow & & \downarrow \\
E' & \xrightarrow{[2]} & E'
\end{array}
\]

and proper base change gives
\[
[2]^*\mathcal{F}(\mathcal{L}(q)) \simeq R\text{pr}'_!(\text{pr}^*\mathcal{L}(q) \otimes \mathcal{L}(\langle e, 2 \rangle)),
\]
where here we write $\mathcal{L}(\langle ,2 \rangle)$ for $(\text{id} \times [2])^* \mathcal{L}(\langle , \rangle)$. Let $f$ denote the composition of maps in the diagram

$$
E \times E' \xrightarrow{\text{id} \times \alpha^{-1}} E \times_S E \xrightarrow{s} E.
$$

We have the following correspondences between functions and sheaves on $E \times_S E'$.

\[
\begin{align*}
q(e + \alpha^{-1}(e')) & \longleftrightarrow f^* \mathcal{L}(q) \\
-q'(e') & \longleftrightarrow \mathcal{L}(-q') \\
q(e) & \longleftrightarrow \text{pr}^* \mathcal{L}(q) \\
\langle e, 2e' \rangle & \longleftrightarrow (\text{id} \times [2])^* \mathcal{L}(\langle , \rangle) =: \mathcal{L}(\langle ,2 \rangle).
\end{align*}
\]

Disposing of this dictionary, the identity \((\dag)\) gives an isomorphism of sheaves on $E \times_S E'$

$$
\text{pr}^* \mathcal{L}(q) \otimes \mathcal{L}(\langle ,2 \rangle) \simeq \text{pr}^* \mathcal{L}(-q') \otimes f^* \mathcal{L}(q).
$$

Therefore the stated isomorphism rests on showing that

$$
\mathcal{R} \text{pr}_1(\text{pr}^* \mathcal{L}(-q) \otimes f^* \mathcal{L}(q)) \simeq \mathcal{L}(-q') \otimes \pi'^* \mathcal{R} \pi_! \mathcal{L}(q).
$$

The following diagram is commutative with cartesian squares.

To see that the right square is cartesian, note that the isomorphism

$$\beta : E \times_S E \to E \times_S E$$

\[
(e_1, e_2) \mapsto (e_1 - e_2, e_2)
\]
induces an isomorphism of cartesian squares

\[
\begin{array}{ccc}
E \times_S E & \xrightarrow{\beta} & E \\
\downarrow & & \downarrow \\
E \times_S E & \xrightarrow{s} & E \\
\downarrow & & \downarrow \\
E & \xrightarrow{\pi} & S.
\end{array}
\]

By the projection formula and proper base change, we conclude

\[
R \text{pr}'_!(\text{pr}'^* \mathcal{L}(-q) \otimes f^* \mathcal{L}(q)) \cong \mathcal{L}(-q') \otimes R \text{pr}'_! f^* \mathcal{L}(q) \cong \mathcal{L}(-q') \pi'^* R \pi_! \mathcal{L}(q).
\]

(1.2.3.4) Let \( p : G \to S \) denote the structure morphism and its various base extensions along \( \pi, \pi' \). Let \( f = (\text{pr}_G, m)^{-1} \), with transpose \( f' = (\text{pr}_G, m') \). The diagram

\[
\begin{array}{ccc}
G \times_S E' & \xrightarrow{f'} & G \times_S E' \\
\downarrow & \searrow & \downarrow \\
G \times_S E' & \xrightarrow{p} & E'
\end{array}
\]

commutes, so that

\[
m'^* \mathcal{F}(K) = f'^* p^* \mathcal{F}(K) \cong f'^* \mathcal{F}(p^*K) \cong \mathcal{F}(f'_! p^*K).
\]

where in the first isomorphism we have used (1.2.2.9) and in the second (1.2.2.4). The commutative diagram

\[
\begin{array}{ccc}
G \times_S E & \xrightarrow{f} & G \times_S E \\
\downarrow & \searrow \text{id} & \downarrow \\
G \times_S E & \xrightarrow{(\text{pr}_G, m)} & G \times_S E \\
\downarrow & \searrow & \downarrow \\
E & \xrightarrow{m} & E
\end{array}
\]

together with the isomorphism \( \widetilde{f_! f^*} L = f_! f^* L \) for all \( L \) in \( D^b_c(G \times_S E, \overline{Q}_\ell) \) and the hypothesis \( m^* K \cong M \otimes_S L \) let us write

\[
\mathcal{F}(f'_! p^*K) \cong \mathcal{F}(m^* K) \cong \mathcal{F}(M \otimes_S L).
\]
Letting \( pr, pr' \) denote \( E \leftarrow E \times_S E' \xrightarrow{pr'} E' \) and their base extensions by \( p \), the commutative diagrams below have cartesian diamonds marked.

The projection formula and proper base change find

\[
\mathcal{F}(M \boxtimes_S L) \simeq R pr'_!(pr^*(M \boxtimes_S L) \otimes L(\langle , \rangle)) \\
\simeq R pr'_!(pr^* \pi^*M \otimes p^*L \otimes L(\langle , \rangle)) \\
\simeq R pr'_!(pr'' \pi''M \otimes p^*(pr^* L \otimes L(\langle , \rangle))) \\
\simeq \pi''M \otimes p^* \mathcal{F}(L) = M \boxtimes_S \mathcal{F}(L).
\]
1.2.3.5. As $E_1 \times_S E' \simeq E_1 \times_{S_1} E'_1 \simeq E \times_S E'_1$, the diagram below has cartesian diamonds.

\[ \begin{array}{ccc}
S_1 & \xrightarrow{pr_1} & E_1 \times _{S_1} E'_1 \\
\downarrow f & & \downarrow f_{E_1} \\
E & \xrightarrow{pr} & E' \\
\downarrow pr' & & \downarrow f_{E'} \\
S & & \\
\end{array} \]

In light of the fact that $\mathcal{L}(' , )$ on $E_1 \times_{S_1} E'_1$ coincides with the inverse image under $f$ of $\mathcal{L}(' , )$ on $E \times_S E'$, proper base change and the projection formula for $f$ give

\[
\mathcal{F}(Rf_{E_1!}K_1) \simeq R\text{pr}_1^!(Rf_1^!\text{pr}_1^*K_1 \otimes \mathcal{L}(', )) \simeq R(\text{pr}' \circ f)_!(\text{pr}_1^*K_1 \otimes f^*\mathcal{L}(', )) \\
\simeq R(f_{E_1} \circ \text{pr}'_1)_!(\text{pr}_1^*K_1 \otimes \mathcal{L}(', )) \simeq Rf_{E_1'}!\mathcal{F}_1(K_1).
\]

1.3.1–1.3.2. I would like to advocate for something of a shortcut through these sections.

*t-exactness of $\mathcal{F}$. *Laumon deduces the t-exactness of $\mathcal{F}$ from the fact that the ‘forget supports’ map is an isomorphism. However, the following direct argument (lifted from the appendix to the reprinted Astérisque 100) is immediate. As in the remark to the note to (1.2.2.1), $\mathcal{F}_{\psi^{-1}} \circ \mathcal{F}_\psi(K) = K(-r)$. As $\mathcal{F}_\psi$ is the composition of exact functors $\text{pr}^*[n], \mathcal{L}(', )$ and the left t-exact functor $\text{pr}_1$ (BBD 4.1.2), it is left t-exact. But it is also left adjoint to its inverse, and this inverse is also left t-exact since up to a Tate twist it coincides with $\mathcal{F}_{\psi^{-1}}$. Therefore $\mathcal{F}_\psi$ is also right t-exact (BBD 1.3.17 (iii)).

*‘Forget supports’. *The theorem (1.3.1.1) in Laumon’s paper states that for all $K$ in $D^b_c(E, \mathbb{Q}_\ell)$, the ‘forget supports’ map

\[
R \text{pr}_1^!(\text{pr}^*K \otimes \mathcal{L}(', )) \to R \text{pr}'_!(\text{pr}^*K \otimes \mathcal{L}(', ))
\]
is an isomorphism. Laumon refers the reader to his paper with Katz, where they give an involved geometric proof that ends up yielding more. But Verdier gave the first proof of this isomorphism, and his proof is very short and completely formal. It can be found in Katz’s 1988 Séminaire Bourbaki talk ‘Travaux de Laumon.’

As for the proof of (1.3.2.1), surely the stated isomorphism should read
\[ \text{RHom}(F_\psi(K), \pi^1L) \cong \text{R pr}_*(\text{pr}^!(\text{RHom}(K, \pi^1L)) \otimes \mathcal{L}^{-1}(\langle , \rangle)). \]

1.4.1. 
\[
\begin{array}{ccc}
A \times A' & \xrightarrow{\text{pr}'} & A' \\
\downarrow_{\alpha \times \alpha'} & & \downarrow_{\alpha'} \\
D \times D' & \xrightarrow{\text{pr}'} & D'
\end{array}
\]

\[
\alpha'_! \mathcal{F}(K) = \text{R pr}'_*(\alpha \times \alpha')!(\text{pr}^* K \otimes \mathcal{L}(xx'))[1]
= \text{R pr}'_*(\alpha \times \alpha')!(\text{pr}^* K \otimes \mathcal{L}(xx'))[1]
= \text{R pr}'_*(\text{pr}^* \alpha K \otimes \mathcal{L}(xx'))[1].
\]

1.4.2. On a curve \( X \) with \( j : U \hookrightarrow X \) a dense smooth open with complement \( F \) and \( A \) a lisse sheaf on \( U \), \( A[1] \) is perverse on \( U \) and Verdier’s formula [BBD, 2.2.4] gives
\[
\]

The simple perverse sheaves on a curve come as (a) \( i_* \) of an irreducible \( \overline{Q}_\ell \)-sheaf on a closed point or (b) from \( j_! [1] \) of an irreducible sheaf on a dense open. (T_1) takes care of (a), but we must verify that a \( K \) of type (b) is either (T_2) or (T_3). Given a \( K \) of type (T_2),
\[
K|\overline{A} = (\text{pr}_{\overline{A}!}(\mathcal{L}(x.s') \otimes \text{pr}_{s'}^* F'))[1]
\]
where \( s' \) denotes the geometric fiber of \( s' \), a discrete set on which \( F' \) is a constant sheaf, so that \( K|\overline{A}[-1] \) has all of its constituents (as a lisse sheaf [Weil II, 1.1.6]) isomorphic to \( \mathcal{L}(x.a') \) for some \( a' \in \overline{k} \).

**Lemma.** — Suppose \( K = j_! F[1] = j_* F[1] \) for \( F \) irreducible lisse on dense \( U \subset A \), and \( K = \mathcal{F}'(K) \), where \( \mathcal{F}' = j'_* F'[1] \), \( F' \) irreducible lisse on dense \( U' \subset A' \). Then \( K \) is (T_3).
Proof. — Let \( \varepsilon \) denote \( \text{Spec} \, \overline{k} \to \text{Spec} \, k \) and its various extensions. The constituents of \( \varepsilon^* K' \) coincide with \( j'_* [1] \) of the constituents of the lisse sheaf \( \varepsilon^* F' \) (exact sequences of lisse sheaves give rise to distinguished triangles concentrated in degree 0, apply triangulated functor \( j'_*[1] \) followed by \( p^H \)); in particular they are all of type (b). As \( \mathcal{F} \) induces an equivalence of perverse sheaves on \( A \) and \( A' \), \( \mathcal{F} \) is a fortiori exact (likewise for \( \mathcal{F}' \)), so that the constituents of \( \varepsilon^* K' \) coincide with those of \( \mathcal{F} \circ \mathcal{F}'(\varepsilon^* K') \); in particular, they are still of type (b), and this implies that none of the constituents of \( \mathcal{F}'(\varepsilon^* K') \) are of type (T\(_2\)). As the formation of Fourier transform commutes with any base change (1.2.2.9), this implies the same for the constituents of \( \varepsilon^* \mathcal{F}'(K') = \varepsilon^* K \); i.e. that none are isomorphic to \( L(x.a')[1] \) for some \( a' \in \overline{k} \). In light of [BBD] 4.3.2 or the lemma in the note to 4.3.2 below, which says that if \( F \) is a lisse sheaf on a normal connected curve, the unit of adjunction \( F \to j_! j^* F \) is an isomorphism, this implies that \( K \) is of type (T\(_3\)): if \( \varepsilon^* F \) had a constituent isomorphic to \( L(x.a')[1] \), by the above \( \varepsilon^* K \) would have a constituent isomorphic to \( j_!(L(x.a')[\overline{U}])[1] \approx L(x.a')[1] \), where \( j = \varepsilon^* j_* : \overline{U} \to \overline{A} \). \( \square \)

Corollary. — \( \mathcal{F} \) exchanges (T\(_3\)) and (T\(_3'\)).

Proof. — Given a simple perverse sheaf \( K' \) of type (T\(_3'\)), \( \mathcal{F}'(K') \) is simple and is not (T\(_1\)) by (1.4.2.1 (i)), therefore must be of type (b); i.e. \( \mathcal{F}'(K') \) satisfies the hypotheses of the lemma, so is (T\(_3\)). \( \square \)

Corollary (Dichotomy). — An irreducible lisse \( \mathbb{Q}_\ell \)-sheaf \( \mathcal{F} \) on a dense open \( U \hookrightarrow A \)

(a) has every constituent of \( \varepsilon^* \mathcal{F} \) isomorphic to \( L(x.a')[\overline{U}] \) for various \( a' \in \overline{k} \), or

(b) has no constituent of \( \varepsilon^* \mathcal{F} \) isomorphic to \( L(x.a')[\overline{U}] \), for any \( a' \in \overline{k} \).

Remark. In analogy with the Fourier transform on function spaces on \( \mathbb{R} \),

constant functions \( \leftrightarrow \) (T\(_2\))

\( L^2(\mathbb{R}) \leftrightarrow \) (T\(_3\))

point masses \( \leftrightarrow \) (T\(_1\)).

Remark. It is tempting to observe that if we were in the abelian category of constructible sheaves (perverse of perversity \( p = 0 \)) shifted by 1, of course \( j_! A \leftrightarrow j_* A \),
although in the category of perverse sheaves for the middle perversity, $j_* A[1]$ is simple. In this category,

$$\ker(j_! A[1] \to j_* A[1]) = i_* \mathcal{H}^0 i^! R j_* A = i_* R^1 i^! j_* A$$

placed in degree 0,

where $i$ denotes the immersion of the complement \cite[4.1.2]{BBD}. The point is that although $i^! j_* A = 0$, $R^1 i^! j_* A$ vanishes if $j_* A$ extends to a lisse sheaf on $X$. Assuming it doesn’t, $j_* A[1]$ is not simple, as it admits a nontrivial subobject $i_* p^! \mathcal{H}^0 R i^! j_* A[1] =: i_* p^! i^! j_* A[1]$; this is nothing other than $i_* R^1 i^! j_* A$ placed in degree 0, and coincides with the largest sub-object of $j_* A[1]$ in the essential image (via $p^!$) of the category of perverse sheaves on $S$ (for the middle perversity – this is simply the category of constructible sheaves on the finite set $S$) \cite[1.4.25]{BBD}.

### 2.1.1. A first curiosity: are the conventions (0.3) en rigeur? Evidently $T$ isn’t essentially of finite type over $k$, but perhaps it is implicit that it is the spectrum of a ring ind-étale over a $k$-algebra of finite type. Does the inclusion $k \{ \pi \} \subset R \subset k[[\pi]]$ require that $k \subset R$ by assumption, or as in the complete case, does a coefficient field exist automatically for $R$? The answer is that we must assume that $k \subset R$, as the following stupid example shows.

Let $k_1 = \mathbb{F}_q(t)$, $A_1 := k_1[\pi]_{(\pi)}$, the local ring at 0 of $A_{k_1}^1$, and put

$$A_n := A_1[x_n]/(x_n^{p^n} - (\pi + t))$$

$$A := \lim_{\longrightarrow} A_n.$$ 

For each $n$, $A_n$ is a d.v.r. with uniformizer $\pi$. For $m < n$ the map goes $x_m \mapsto x_n^{p^{n-m}}$. Let $A^h$ denote the henselization of $A$; both $A$ and $A^h$ have the perfect closure $k_1^{p^{n-\infty}}$ of $k_1$ as residue field, but neither contain $k_1^{p^{\infty}}$.

**Remark.** As a partial converse, if $A$ is an excellent henselian d.v.r. with perfect residue field, then its completion $\hat{A}$ contains a canonical coefficient field, and Artin’s approximation theorem gives that $A$ contains a coefficient field.

As $R$ is equicharacteristic, the inertia admits this simple description; c.f. \cite[Weil II, 1.7.11]{note to}. 

---

One must always repair to *Corps Locaux* Ch. IV for the ultra-mystical ‘upper numbering’ filtration on $I$. In Proposition 3 of §1, it is claimed that $s(f) - f$ has all its coefficients divisible by $s(y) - y$. If we let $p_K'$ denote the maximal ideal of $A_{K'}$, the definition of $i_{G/H}$ means that $s(y) - y$ is of order $i_{G/H}(s)$ in $A_{K'}$; i.e. $(s(y) - y) = p_{K'}^{i_{G/H}(s)}$. Lemma 1 then shows that all the coefficients of $s(f) - f$ have order $\geq i_{G/H}(s)$, hence are divisible by $s(y) - y$.

In light of *Corps Locaux* IV §3 Prop. 14 & Rmk. 1, if $L/K$ is an infinite Galois extension with Galois group $G$, one defines $G^v := \lim\leftarrow G(L'/K)^v$ as $L'$ runs over the set of finite Galois subextensions of $L$. This description shows that $G^v$ is a compact subgroup of $G$, hence closed in $G$, hence also in the compact open subgroup $I = G_0 = G^0$, (provided of course $v \geq 0$). It also shows that $G^v$ is normal, as it is a projective limit of the normal groups $G(L'/K)^v$ (*Corps Locaux* IV Prop. 1). Left continuity

$$G^v = \bigcap_{w < v} G^w$$

amounts to the statement that if $s \in G$ is not in $G^v$, then $s \not\in G^w$ for some $w < v$. An element $s \in G$ belongs to $G^v$ if for every finite Galois subextension $L \supset L' \supset K$ with $\text{Gal}(L'/K) = H$, $i_{G/H}(s) \geq \psi(v)$, so it will suffice to show that if there is some $L'$ as above with $i_{G/H}(s) < \psi(v)$, then there is some $w < v$ such that $i_{G/H}(s) < \psi(w)$. As $\psi$ is continuous and increasing, this is trivial.

Laumon considers the induced filtration on $I = G_0 = G^0$. The filtration is separated,

$$\bigcap_{i \geq 0} I^{(i)} = \{1\},$$

as the same is true for $G/H$ for every normal open subgroup $H$ of $G$ (*Corps Locaux* IV Prop. 1).

It’s clear that $I^{(i)} \subset I^{(i)}$ as $I^{(i)}$ is closed. But why $I^{(0 +)} = P$? On the level of a finite Galois extension $L'$ of $K$, *Corps Locaux* IV §2 explains that over a perfect field of characteristic $p$, $G_1$ is a $p$-group and the quotient $G_0/G_1$ is sent isomorphically by the inertia character to a subgroup of the group of roots of unity of the residue field of $\overline{L'}$; this is the tame inertia, and $G_1$ is the wild inertia. It is necessary to switch to the upper numbering filtration in order for the filtration to play well with quotients, and
as the index \((G_0 : G_1)\) increases (corresponding to more tame inertia in the extension \(L')\), \(q(1)\) approaches 0 from the right. This means that (provided the maximal tamely ramified extension of \(K\) is not finite over \(K_r\), the maximal unramified extension of \(K\)) for every \(\varepsilon > 0\) and \(s \in I^{(0+\varepsilon)}\) there exists an extension \(L'\) of \(K\) with \(1/(G_0 : G_1) < \varepsilon\), so that \(s \notin G_1\), \(G := \text{Gal}(L'/K)\). So \(P\) coincides with the completion of \(\cup I^{(0+\varepsilon)}\), which coincides with the closure of \(\cup I^{(0+\varepsilon)}\) in \(I\).

2.1.2. (2.1.2.2) The kernel of the representation of \(P\) is closed and of finite index, hence open, hence by \textit{Corps Locaux} IV Prop 14. & Prop. 1, \(I^{(\lambda)}\) acts trivially for \(\lambda \gg 0\).

The corollary (2.1.2.3) is Maschke’s theorem. In the definition of \(\text{pente}\), \(R_+ := \mathbb{R}_{\geq 0}\), and \(\lambda\) may equal 0 iff \(W\) is trivial. If \(H\) denotes the kernel of the representation \(P \to \text{GL}(V)\), then \(P/H\) is a finite group and \((P/H)^v = P^v/H\) (\textit{Corps Locaux} IV Prop. 14). The slope \(\lambda\) coincides with the largest real number \(v\) such that \((P/H)^v = \{1\}\); in the vocabulary of \textit{Corps Locaux} IV §3 Rmk. 1, this is the largest break in the filtration on \(P/H\). The canonical slope decomposition (2.1.2.4) of \(V\) is a decomposition as \(I\)- or \(G\)-module since if \(W\) is a simple \(P\)-submodule of \(V\) of slope \(\lambda\) and \(g \in G\), then \(gW\) is still simple as \(P\)-module and still of slope \(\lambda\), as the groups \(I^{(v)}\) are normal subgroups of \(G\), for all \(v \geq 0\). Therefore \(V_\lambda\) is preserved by \(G\).

(2.1.2.8) To see that \(L_\psi(1/\pi)\) has slope 1, it will suffice to show that if \(\Gamma := \text{Gal}(\eta'/\eta)\), then \(\Gamma = \Gamma_1\) and \(\Gamma_2 = \{1\}\). It would follow that \(q(1) = 1\) for this extension so that \(\Gamma^{1+\varepsilon} = \Gamma_1 = \Gamma\) and \(\Gamma^{1+\varepsilon} = \{1\}\) for all \(\varepsilon > 0\). With the criterion of \textit{Corps Locaux} IV §2 Prop. 5, it would suffice to show that for all \(s \in \Gamma\),

\[
\begin{align*}
  s(\pi')/\pi' &\equiv 1 \mod (\pi') \\
  s(\pi')/\pi' &\not\equiv 1 \mod (\pi')^2.
\end{align*}
\]

As \(s(\pi')/\pi' = 1/(1 + \alpha \pi')\) for \(\alpha \in \mathbb{F}_p\),

\[
  s(\pi')/\pi' \equiv 1 - \alpha \pi' \mod (\pi')^2.
\]

2.1.4. Schur’s lemma gives that every simple tame \(I\)-module has rank 1. The action of \(\text{Gal}(\bar{k}/k)\) on \(\hat{Z}(1)(\bar{k})\) by inner automorphisms coincides with the action of Galois on roots of unity; c.f. \textit{Stacks} tag 0BUB5.
2.1.5. The equivalence of the three conditions can be seen as follows: (i) trivially implies (ii) and (iii) as $V^I$ (resp. $V_1$) is the largest subobject (resp. quotient) on which $I$ acts trivially. The decomposition $V = \oplus V_\lambda$ permits us to assume $V = V_\lambda$. If $\lambda > 0$, then by definition $V^I_\lambda = 0$ and moreover $P$ acts nontrivially. Restricting the action of $P$ to a simple $P$-submodule $W$ of $V_\lambda$, one finds $W_P = 0$, hence $(V_\lambda)_P = 0$ and a fortiori $(V_\lambda)_I = 0$. Moreover, $V_\lambda$ has no nontrivial subquotient on which $I$ acts trivially, as such a subquotient would be a direct sum of simple $P$-modules of slope $\lambda$, so (i), (ii), and (iii) are automatically verified and we consider $V = V_0$, a tame $G$-module. The existence of a geometrically constant subquotient of $V$ as $G$-module implies the same as $I$-module, and therefore we need only show that (ii)$\Leftrightarrow$(iii) and the combination implies that there exists no $I$-module subquotient of the tame $I$-module $V$. As $T := I/P \approx \hat{\mathbb{Z}}(1)(\bar{k})$ is procyclic with topological generator, say, $t$, and the representation $V$ of $T$ is continuous,

$$0 \to V^T \to V \xrightarrow{t-1}\ V \to V_T \to 0$$

is exact, and shows that $V^T = 0 \Leftrightarrow V_T = 0$. Given $T$-submodules

$$V = V_0 \supset V_1 \supset V_2 \supset 0,$$

$V^T_1 = 0 \Rightarrow (V_1)_T = 0$ for each $i$, so if $V_1/V_2$ is $T$-invariant, the quotient $V_1 \to V_1/V_2$ factors through $(V_1)_T = 0$, and $V_1/V_2 = 0$. So, $V$ has no $T$-invariant subquotient.

Duality $V \leftrightarrow V^\vee$ sends $\mathcal{G}([0,\infty[)$ into itself as $(V^\vee)^I = (V_1)^\vee$.

2.2.1. Of course, the various functions are extended additively with sign, so that e.g. for a perverse sheaf $K$ on $X$, $s_x(K) \leq 0$ for $x \in |X|$.

(2.2.1.1) Let $x \xleftarrow{i} X \xrightarrow{j} X - x$. There is a distinguished triangle $(i^! = Ri^!$ etc.)

$$j^!K \to i^*K \to i^*j_*j^*K \to$$

As $K$ is perverse, $\mathcal{H}^{-1}(i^!K) = 0$ and $\mathcal{H}^i(j^*K) = 0$ for $j \neq -1$. Representing $j^*K$ by a complex $I$ of injectives in degrees $\geq -1$

$$0 \to \cdots \to 0 \to I_{-1} \xrightarrow{d^{-1}} I_0 \to I_1 \to \cdots$$
we have $\mathcal{H}^{-1}(j^*K) = \ker d^{-1}$. As $j_*$ is left exact,

$\mathcal{H}^{-1}(j_*j^*K) = \mathcal{H}^{-1}(j_*I) = \ker j_*(d^{-1}) = j_*(\ker d^{-1}) = j_*\mathcal{H}^{-1}(j^*K) = j_*j^*\mathcal{H}^{-1}(K)$,

so that

$$0 \to i^*\mathcal{H}^{-1}(K) \to i^*j_*(j^*\mathcal{H}^{-1}(K))$$

is exact, proving $\mathcal{H}^{-1}(K) \subset j_*j^*\mathcal{H}^{-1}(K)$ and the inequality $r(K) \leq r_x(K)$, as $r(\mathcal{H}^0(K)) = 0$, and showing that $\mathcal{H}^{-1}(K)$ is lisse at $x$ iff $I_x$ acts trivially on $\mathcal{H}^{-1}(K)$ if $r_x(\mathcal{H}^{-1}(K)) = r(\mathcal{H}^{-1}(K))$; of course $\mathcal{H}^0(K) \subset \ker r_x(\mathcal{H}^0(K)) = 0$. As $r_x(K) = r_x(\mathcal{H}^0(K)) - r(\mathcal{H}^{-1}(K))$ and $r(K) = -r(\mathcal{H}^{-1}(K))$, $r_x(K) \geq r(k)$ with equality iff (i) holds. Of course, $s_x$ measures wild ramification and so (ii)$\Leftrightarrow$(iii) trivially. See Reprise.

2.2.2. The tame quotient $\pi_1(\xi, \overline{\xi}) \to \pi_1(\xi, \overline{\xi})^{\text{mod}}$ has not actually been defined; ‘tame quotient’ in (2.1.1) meant $I \to I/P$. This tame quotient corresponds to $G \to G/P$.

(2.2.2.1) It suffices to show $\overline{i}_*^{\text{mod}}$ is an isomorphism in light of the short exact sequence of $\pi_1$ (SGA 1 6.11) which expresses the fundamental group as extension of $\text{Gal}(\overline{k}/k)$ by the geometric fundamental group.

(2.2.2.2) This mysterious theorem is found in Katz, Local-to-global extensions of representations of fundamental groups, where he also proves a cohomological formula for the $\ell$-adic Swan representation. In light of Reprise below, some cursory analysis of the meaning of this theorem can be made. The discussion doesn’t change if one replaces $k$ by $\overline{k}$. First of all, $\text{Gal}(\overline{k}/k(u))$ surjects onto $\pi_1(\mathbb{G}_{m,k}, \overline{\xi})$, and we can describe the kernel in terms of the monodromy at all geometric closed points of $\mathbb{G}_{m,k}$.

$$G := \text{Gal}(\overline{k}/k(u)) \to \pi_1(\mathbb{G}_{m,k}, \overline{\xi}) \to \pi_1(\mathbb{G}_{m,k}, \overline{\xi})^{\text{mod, co}} \to \pi_1(\mathbb{G}_{m,k}, \overline{\xi})^{\text{mod}}$$

With the notation (2.2.1), the kernel of the first map is topologically generated by $\{I_x\}_{x \in \mathbb{G}_{m,k}}$. The kernel of the second map is topologically generated by $P_{\infty}$. The kernel of the third map is topologically generated by $P_0$. (Both $P_{\infty}, P_0 \subset G$.) The injection $\pi_1(\xi, \overline{\xi}) \hookrightarrow G$ corresponds to the Galois extension $k(\xi)/k(u)$.
That \( \pi_1(\xi, \overline{\xi}) \) injects into \( \pi_1(\mathbf{G}_{m,k}, \overline{\xi})^{\text{mod,}\infty} \) corresponds simply to the statement that membership in the latter group puts no condition on the monodromy at 0. An object in the latter Galois category corresponds to a finite cover of \( \mathbb{P}^1 - \{0\} \) which is étale away from \( \infty \) and tamely ramified at infinity. The injection \( \pi_1(\xi, \overline{\xi}) \hookrightarrow \pi_1(\mathbf{G}_{m,k}, \overline{\xi})^{\text{mod,}\infty} \) corresponds to the statement that the constraints on monodromy at all closed points of \( \mathbb{P}^1 \) other than 0 imposed by membership in \( \pi_1(\mathbf{G}_{m,k}, \overline{\xi})^{\text{mod,}\infty} \) put in effect ‘no constraints’ on the monodromy at 0.

The isomorphism \( \pi_1(\xi \otimes_k \overline{k}, \overline{\xi})^{\text{mod}} \cong \pi_1(\mathbf{G}_{m,\overline{k}}, \overline{\xi})^{\text{mod}} \) is the statement that the Galois category of étale covers of the generic point of \( (\mathbb{A}^1_k)^{(0)} \) tamely ramified at the closed point is equivalent to the category of étale covers of \( \mathbf{G}_{m,\overline{k}} \) which are tamely ramified at 0 and \( \infty \). The proof given notes that both are isomorphic to \( \hat{\mathbb{Z}}(1)(\overline{k}) \), but this fact for \( \mathbf{G}_{m,\overline{k}} \) is not proved in (1.1.3.7), which introduces the Kummer coverings of \( \mathbf{G}_{m,\overline{k}} \). What’s needed is Grothendieck’s comparison theorem for curves (SGA 1 XIII 2.12), which immediately shows that the tame fundamental group of \( \mathbb{P}^1_k - \{\infty\} \) is trivial (this is the automorphism group of a fiber functor on the Galois category consisting of finite covers of \( \mathbb{P}^1_k \) étale over \( \mathbb{A}^1_k \) and tame at \( \infty \)) and that \( \pi_1(\mathbf{G}_{m,\overline{k}}, \overline{\xi})^{\text{mod}} \) is freely generated by a generator of the tame inertia at 0, hence is indeed isomorphic to \( \hat{\mathbb{Z}}(1)(\overline{k}) \). In light of this, we know that the Kummer coverings of (1.1.3.7) do exhaust the set of finite maps to \( \mathbb{P}^1_k \) étale over \( \mathbf{G}_{m,\overline{k}} \) and tamely ramified at 0 and \( \infty \).

Intermezzo I: Katz-Gabber extensions. The paper is by Katz, Local-to-global extensions of representations of fundamental groups.

(1.2.3) En effet, when you pull \( E \) back to \( X \otimes_K \mathbb{K}^{\text{sep}} \), the only monodromy is geometric. The map \( \pi_1(X \times_K \mathbb{K}^{\text{sep}}, \overline{x}) \rightarrow \text{Aut}(E(\overline{x})) \) is continuous with open kernel \( U \) corresponding to the Galois \( G \)-torsor \( Z \), where of course \( G = \pi_1(X \times_K \mathbb{K}^{\text{sep}}, \overline{x})/U \). Pulling back to \( Z \) kills the monodromy on \( E \) (i.e. \( E \) ‘splits’). Now the point is that in the exact sequence
\[
\varepsilon \rightarrow \pi_1(\overline{X}, \overline{x}) \rightarrow \pi_1(X, \overline{x}) \rightarrow \text{Gal}(\mathbb{K}^{\text{sep}}/K) \rightarrow \varepsilon,
\]
the surjection is a quotient map, a fortiori open. Let \( V \) be defined by the exact sequence
\[
e \rightarrow V \rightarrow \pi_1(X, \overline{x}) \rightarrow \text{Aut}(E(\overline{x})) \rightarrow e
\]
and let \( V' \) be the image of \( V \) in \( \text{Gal}(K^{\text{sep}}/K) \); \( V' \) is open and we let the finite Galois extension \( K' \) correspond to any such containing \( V' \) and over which \( Z \) is defined.

(1.3) A unique \( p \)-Sylow is the same as a normal \( p \)-Sylow. A normal \( p \)-Sylow in a group \( G \) is characteristic as it is the unique subgroup with its order (any \( p \)-subgroup of \( G \) is contained in it).

(1.3.2) In proof of \( 1 \Rightarrow 4 \), the statement that ‘the unique open normal subgroup of \( \pi_1(G_{m, L}, \overline{x}) \) of index \( N \geq 1 \) prime to \( p \) is the one corresponding to the \( N \)th power covering \( [N] \) of \( G_{m, L} \) by itself’ is easily seen: denoting such a subgroup by \( U \), \( U \) corresponds to a finite étale connected torsor \( T \) with group \( \pi_1(G_{m, L}, \overline{x})/U \), and the homomorphism from this group into the group opposite \( \text{Aut} T = \text{Aut} T(\overline{x}) \) is an isomorphism [SGA 1, 5.11]. So \( T \) is a cover of \( G_{m, L} \) tamely ramified at 0 and \( \infty \) since \( p \nmid | \text{Aut} T(\overline{x}) | \), and as in the note to (2.2.2), \( \pi_1(G_{m, L}, \overline{x}) \mod \cong \hat{\mathbb{Z}}(1)(L) \), so the statement is obvious.

Given a special \( E \to G_{m, K} \), the preimage of the unique \( p \)-Sylow of the geometric monodromy of \( E \) in \( \pi_1(G_{m, K^{\text{sep}}, \overline{x}}) \) is an open subgroup with index \( N_1 \) prime to \( p \); by the above claim, \( [N_1]^*E \) has geometric monodromy a \( p \)-group and is still at worst tamely ramified at 0 so that for some \( N_2 \), \( [N_2]^*[N_1]^*E \) extends to an étale cover of \( A^1_K \); now let \( N = N_1N_2 \).

The proof of \( 4 \Rightarrow 7 \) is self-evident after you observe that

\[
\begin{array}{ccc}
G_{m, K} & \xrightarrow{\text{Trans}_a} & G_{m, K} \\
\downarrow{[N]} & & \downarrow{[N]} \\
G_{m, K} & \xleftarrow{\text{Trans}_b} & G_{m, K}
\end{array}
\]

commutes, since the left composition corresponds to \( T \mapsto aT \mapsto aT^N \) as \( [N] \) is \( K \)-linear, while the right composition corresponds to \( T \mapsto T^N \mapsto (bT)^N = aT^N \).

(1.4) When Katz writes ‘an action of \( G \) on \( E \) covering its action on \( G_{m, K} \)’, for example, ‘covering’ means, in the language of [SGA 7, XIII 1.1], ‘compatible with.’ Moreover the action of \( G \) on \( E \) should be continuous.
In the proof of the main theorem, once reduced to \((N, K') = (1, K)\) over the field \(K'\), there is ‘nothing to prove’ when \(p = 1\) since in this case the words ‘monodromy group a \(p\)-group’ means ‘trivial monodromy group,’ and so putting

\[
A := \text{category of finite étale coverings of } A_{K'}^1 \text{ with trivial monodromy, and }
\]

\[
B := \text{category of finite étale coverings of } \text{Spec } K'((T^{-1})) \text{ with trivial monodromy,}
\]

both \(A\) and \(B\) are isomorphic to the category of finite étale coverings of \(\text{Spec } K'\); in the case of \(A\), this follows from the exact sequence \(\square\) in light of the fact that \(\pi_1(P^1_{K_{\text{sep}}}) = 1\); in the case of \(B\), this follows from the same exact sequence and the observation that \(K_{\text{sep}}((T^{-1}))\) is the function field of the strict henselization \(P^1_{(\infty)}\) of \(P^1_K\) at \(\infty\); in both cases, the condition ‘trivial monodromy’ means that the revêtement étale extends over \(\infty\) (resp. the closed point of the strict henselization), and \(\pi_1(P^1_{K_{\text{sep}}}) = 1 = \pi_1(P^1_{(\infty)})\) as any connected revêtement étale of the spectrum of a strictly henselian ring is trivial.

(1.4.2) If \(R\) is a ring, the idempotents of \(R[[x]]\) are in bijection with those of \(R\), for, given an idempotent \(f \in R[[x]]\), write \(f(x) = r_0 + f_1(x)\) with \(f_1(x)\) a power series with constant term 0; \(r_0\) must be idempotent and if \(f_1(x) \neq 0\) it has a nonzero term of lowest degree, say \(r_nx^n\). Then

\[
r_0 + f_1(x) = f(x) = f(x)^2 = r + 2r_0f_1(x) + f_1(x)^2
\]

so that \(2r_0r_n = r_n\) and therefore also \(2r_0r_n = 2r_0^2r_n = r_0r_n\) so \(r_0r_n = 0\), contradicting \(r_n \neq 0\).

Using this fact, we find that the idempotents of \(R\) are in bijection with those of \(R[T]\); writing \(R((T^{-1})) = R[T][[T^{-1}]]\) finds that the idempotents of \(R((T^{-1}))\) are in bijection with those of \(R[T]\) and so also with those of \(R\).

(1.4.4) It’s helpful to recall Arcata II (2.1).

(1.4.5) To see \(\mathcal{P}\) is surjective on \(T^{-1}R[[T^{-1}]]\), pick a power series in \(T^{-1}\) with no constant term \(b = \sum_{i>0} b_iT^{-i}\). To hit \(b\) with an element \(c\) of \(T^{-1}R[[T^{-1}]]\), we proceed in the usual way: given \(b\) at step \(i > 0\) with \(b_j = 0\) for \(j < i\) and \(c_1, \ldots, c_j\) already fixed, put \(c_i := b_i\) and replace \(b\) by \(b - \mathcal{P}(c_iT^{-i})\). Then \(c := \sum_{i>0} c_iT^{-i}\) has \(\mathcal{P}(c) = b\).
(1.4.6) Recall Serre, *Cohomologie Galoisienne* 2.3 Prop. 8 for

\[ H^*(\lim_{\leftarrow} G_i, \lim_{\rightarrow} A_i) = \lim_{\rightarrow} H^*(G_i, A_i). \]

(1.4.8) As tensor product commutes with colimits, the assertion about \( K^{\text{sep}} \otimes_K K((T^{-1})) \) is clear from the isomorphism \( K' \otimes_K K((T^{-1})) \sim K'((T^{-1})) \) which in turn follows from the basic fact that a finitely generated module \( M \) over a noetherian local ring \( A \) with maximal ideal \( \mathfrak{m} \) has \( \hat{M} \cong M \otimes_A \hat{A} \) for the \( \mathfrak{m} \)-adic topology; in our case \( A = K[T^{-1}], M = K'[T^{-1}], \) and the isomorphism of fields above is the statement over the generic fiber; i.e.

\[
K' \otimes_K K[[T^{-1}]] = K'[T^{-1}] \otimes_{K[[T^{-1}]]} K[[T^{-1}]] \sim K'[[T^{-1}]] \quad \text{and}
\]
\[
K' \otimes_K K((T^{-1})) = K'[T^{-1}] \otimes_{K[[T^{-1}]]} K((T^{-1})) \sim K'((T^{-1})).
\]

We see that the field \( K'((T^{-1})) \) is the fraction field of the henselian d.v.r. \( K'[[T^{-1}]] \) and the colimit of henselian local rings along local ring homomorphisms is henselian local, it suffices to observe that the colimit is also along étale, *a fortiori* unramified ring maps that the colimit is a henselian a d.v.r.

That the fraction field of a henselian d.v.r. has the same Galois theory as that of its completion is a result of Berkovich — [c.f. proof of the lemma in note to (2.4.1)]. This fact can also be obtained from the structure we have found for the Galois group: the unramified and tame parts are identical, and the free pro-p quotients can be shown to be mapped isomorphically: it amounts to showing that the map on étale \( H^1(-, \mathbb{Z}/p) \) from the (spectra of the) maximal pro-p Galois extension of \( K^{\text{sep}}((T^{-1})) \) to that of \( K^{\text{sep}} \otimes_K K((T^{-1})) \) is an isomorphism.

(1.4.10) The point is that the map of \( \pi_1 \)s of (1.4.7) factors as

\[
\pi_1(\text{Spec}(K^{\text{sep}}((T^{-1})), \overline{x}_1)) \rightarrow \pi_1(G_{m,K^{\text{sep}}}, \overline{y}_1) (tame \ at \ 0) \rightarrow \pi_1(G_{m,K^{\text{sep}}}, \overline{y}_1) (special)
\]

by functoriality of the inverse image. This composite is an isomorphism and its inverse, preceded by the (tame-at-0)-to-(special) quotient, provides the retraction. Why is the retraction unique? Identifying \( \pi_1(\text{Spec}(K^{\text{sep}}((T^{-1})), \overline{x}_1)) \) with its image under the canonical injection, retractions are in bijection with complements to \( \pi_1(\text{Spec}(K^{\text{sep}}((T^{-1})), \overline{x}_1)): \)
closed normal subgroups $N$ of $\pi_1(\mathbb{G}_{m,K}^{\text{sep}}, \bar{\eta}_1)(\text{tame at } 0)$ such that

$$N \cap \pi_1(\text{Spec}(K^{\text{sep}}((T^{-1})), \bar{x}_1)) = \{1\} \quad \text{and}$$

$$N \cdot \pi_1(\text{Spec}(K^{\text{sep}}((T^{-1})), \bar{x}_1)) = \pi_1(\mathbb{G}_{m,K}^{\text{sep}}, \bar{\eta}_1)(\text{tame at } 0).$$

Recall that $\pi_1(\mathbb{G}_{m,K}^{\text{sep}}, \bar{\eta}_1)(\text{special})$ is the maximal pro-‘group with unique $p$-Sylow subgroup’ quotient of $\pi_1(\mathbb{G}_{m,K}^{\text{sep}}, \bar{\eta}_1)(\text{tame at } 0)$. In the language of *Profinite Groups* by Ribes & Zalesskii, the class $\mathcal{C}$ of finite groups with unique $p$-Sylow subgroup is a *variety* of finite groups, hence *a fortiori* a *formation* of finite groups. Putting

$$R_\mathcal{C} := \bigcap \{N : N \text{ open normal subgroup of } \pi_1(\mathbb{G}_{m,K}^{\text{sep}}, \bar{\eta}_1)(\text{tame at } 0), \pi_1(\mathbb{G}_{m,K}^{\text{sep}}, \bar{\eta}_1)(\text{tame at } 0)/N \in \mathcal{C}\},$$

(3.4) of *op. cit.* says that $R_\mathcal{C}$ is a characteristic subgroup, the sequence

$$1 \to R_\mathcal{C} \to \pi_1(\mathbb{G}_{m,K}^{\text{sep}}, \bar{\eta}_1)(\text{tame at } 0) \to \pi_1(\mathbb{G}_{m,K}^{\text{sep}}, \bar{\eta}_1)(\text{special}) \to 1$$

is exact, and given any closed normal subgroup $K$ of $\pi_1(\mathbb{G}_{m,K}^{\text{sep}}, \bar{\eta}_1)(\text{tame at } 0)$ such that $\pi_1(\mathbb{G}_{m,K}^{\text{sep}}, \bar{\eta}_1)(\text{tame at } 0)/K$ is pro-$\mathcal{C}$, then $K$ is a subgroup of $R_\mathcal{C}$. This last point implies that the only $N$ as above is $R_\mathcal{C}$, hence the retraction is unique.

Since specialness is geometric, the following diagram of profinite groups and continuous homomorphisms is commutative with exact rows and columns (compare 1.3.3).

\[
\begin{array}{ccccccc}
1 & \rightarrow & R_\mathcal{C} & \rightarrow & \pi_1(\mathbb{G}_{m,K}^{\text{sep}}, \bar{\eta}_1)(\text{tame at } 0) & \rightarrow & \pi_1(\mathbb{G}_{m,K}^{\text{sep}}, \bar{\eta}_1)(\text{special}) & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \rightarrow & \pi_1(\mathbb{G}_{m,K}^{\text{sep}}, \bar{\eta}_1)(\text{tame at } 0) & \rightarrow & \pi_1(\mathbb{G}_{m,K}^{\text{sep}}, \bar{\eta}_1)(\text{tame at } 0) & \rightarrow & \text{Gal}(K^{\text{sep}}/K) & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \rightarrow & \pi_1(\mathbb{G}_{m,K}^{\text{sep}}, \bar{\eta}_1)(\text{special}) & \rightarrow & \pi_1(\mathbb{G}_{m,K}^{\text{sep}}, \bar{\eta}_1)(\text{special}) & \rightarrow & \text{Gal}(K^{\text{sep}}/K) & \rightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & & & & & & 1 & & 1
\end{array}
\]
In more words, clearly the kernel of the tame-to-special quotient for $K$ contains that for $K^{\text{sep}}$ (the functor of restriction to geometric monodromy corresponds to the functor 'reciprocal image along $\text{Spec } K^{\text{sep}} \to \text{Spec } K$'); to see it is no larger, note that the special coverings of $G_{m,K_i}$ are special coverings of $G_{m,K}$ for all Galois extensions $K_i$ of $K$; therefore an element of the kernel must fix all the special coverings of $G_{m,K_i}$ for each $i$, and $K^{\text{sep}} = \bigcup_i K_i$.

This diagram implies the uniqueness of the retraction for $K$ restricting to the given one for $K^{\text{sep}}$. In more words, as before, retractions of

$$\pi_1(G_{m,K}, \overline{y}_1) (\text{tame at 0}) \to \pi_1(G_{m,K}, \overline{y}_1) (\text{special})$$

are in bijection with closed normal subgroups $N$ satisfying the same conditions as before with $K^{\text{sep}}$ replaced by $K$ and $\overline{x}_1$ replaced by $\overline{x}$. The condition that the retraction must restrict to the given one for $K^{\text{sep}}$ implies that $N$ must contain $R_{\mathcal{C}}$. But as seen in the diagram above, $N = R_{\mathcal{C}}$. Therefore, as before, there is only one possible choice for $N$.

(1.6.4) Finitely presented since constructible – Th. finitude.

(1.6.6) See Katz, *Gauss Sums, Kloosterman Sums, and Monodromy Groups* (2.1.1) & (2.3.3).

(1.6.8) There are 3 claims to verify: (1) that the isomorphism class of a projective $\mathbb{Z}_l[G]$-module can be deduced by character; (2) that $\chi(A_{K}^1, -) = \chi_c(A_{K}^1, -)$, and (3) that $\chi_c(G_{m,K}, \mathfrak{f}^{\text{can}}) = -\text{swan}_\infty(\mathfrak{f})$.


$$R_K(G) \times R_K(G) \to \mathbb{Z}$$
and (14.6) shows that it is compatible with extension of field \( K \subset K' \), which induces an injection \( R_K(G) \to R_{K'}(G) \). Over a large enough field (e.g. the algebraic closure of \( K \)), the above bilinear form is nondegenerate and induces an isomorphism of \( R_K(G) \) onto its dual. The identity

\[
\text{Hom}_G(E, F) \cong \hat{E} \otimes_G F
\]

and the fact that the Swan representation is self-dual \([SGA5]\) Exp. X (3.8) & (4.4) show that indeed it suffices to show that

\[
\dim_{\mathbb{Q}_l}(H^1(A^1, j!: \mathcal{F}^{\text{can}})) = \text{swan}_\infty(\mathcal{F}),
\]

where the latter is the Swan conductor \( b(M) \) of Serre (19.3), in light of the isomorphism

\[
H^1(A^1, j!(\text{Reg}_{G; \mathbb{Z}_l})_r^{\text{can}}) \otimes_{\mathbb{Z}[G]} M \cong H^1(A^1, j! j^{\text{can}}).
\]

(2) The equality \( \chi(X, \mathcal{F}) = \chi_c(X, \mathcal{F}) \) for a proper smooth curve \( X \) over an algebraically closed field \( k \) was proven by Grothendieck \([SGA5]\) Exp. X (7.12), and later by Laumon for any \( X \) separated and finite type over \( k \) algebraically closed in his 1981 article \textit{Comparaison de caractéristiques d’Euler-Poincaré en cohomologie l-adique}.

(3) The formula \( \chi_c(G_m, \mathcal{F}^{\text{can}}) = - \text{swan}_\infty(\mathcal{F}) \) can be deduced from the Grothendieck-Ogg-Šavarevič formula as appears in Laumon (2.2.1.2). Laumon’s Swan conductor coincides with Serre’s Swan conductor \( b(M) \) after passing from upper to lower numbering

\[
s(M) = \sum_{\Lambda \in \Lambda(M)} \lambda, r(V_\lambda) = \int_0^\infty \text{codim } M^{G^i} d\lambda
\]

\[
= \int_0^\infty \frac{\text{codim } M^{G^r}}{[G_0 : G_r]} \, dr = \sum_{i=1}^\infty \frac{\text{codim } M^{G_i}}{[G_0 : G_i]} = b(M).
\]

Incidentally, if \( a(M) \) denotes the Artin conductor, putting

\[
t(M) := \int_{-1}^0 \text{codim } M^{G^i} d\lambda = \int_{-1}^0 \frac{\text{codim } M^{G^r}}{[G_0 : G_r]} \, dr = \text{codim } M^{G_0} = \text{codim } M^{I_\infty}
\]

allows us to write \( a(M) = t(M) + s(M) \), expressing the Artin conductor as the sum of the tame conductor \( t(M) \) and the Swan (or wild) conductor \( s(M) \).
2.3.1. (2.3.1.1) The negative sign before $\chi_{\pi}$ is due to the shift $[1]$ in the formula for Fourier transform. The contribution in the formula of Grothendieck-Ogg-Šafarevič from $a_\infty(K)$ is $r(K) - s(F_{\eta_\infty} \otimes \mathcal{L}(x.s')_{\eta_\infty})$ and of course $r_\infty(K') = 0$ as our $K$ on $\mathbb{P}^1$ is obtained as extension by zero of $K$ on $A$. The Swan conductor $s_\infty$ doesn’t care about the stalk of $K$ at $\infty$ but only about the restriction of $K$ to the generic point $\eta_\infty$ of the henselian trait $(\mathbb{P}^1_k)(\infty)$. As $\mathcal{L}$ is locally constant on $A$, $a_s(K \otimes \mathcal{L}(x.s')) = a_s(K)$ for $s \in S$.

To understand $\mathcal{L}(x.a'')_{\eta_\infty}$ when $a'' \neq 0$, let’s again write the Artin-Schreier covering

$$0 \to F_p \to G_{a,k} \xrightarrow{t^p-t} G_{a,k} \to 0.$$  

The induced map $k[t_1] \to k[t_2]$ on coordinate rings is $t_1 \mapsto t_2^p - t_2$, as can be readily seen by remembering what $F^*$ and $+$ do on the coordinate ring of $G_{a,k}$. The Artin-Schreier sheaf $\mathcal{A}$ is the sheaf of local sections of this covering. It is an $F_p$-torsor on $G_{a,k} = \text{Spec } k[t_1]$. As the above map on coordinates makes $t_2$ integral over $k[t_1]$ with equation of integral dependence $t_2^p - t_2 = t_1$, $\mathcal{A}$ is represented over $G_{a,k} = \text{Spec } k[t_1]$ by the revêtement étale $\text{Spec } k[t_1,t_2]/(t_2^p - t_2 - t_1)$. Pushing $\mathcal{A}$ by the character $\psi^{-1}$ gives $\mathcal{L}_\psi$ but it is conceptually easier to keep working with $\mathcal{A}$. The pullback of $\mathcal{A}$ to $A$ via $A \times a' \to A \times A' \to G_a$ corresponds to the revêtement étale of $A = \text{Spec } k[x]$ given by $\text{Spec } k[x,t]/(t^p - t - xa')$, this is the one-parameter family of Artin-Schreier coverings

$$t^p - t = xa', \quad a' \in \bar{k}.$$  

Let $F = \overline{F}_p$. Given $x \in A(F), x' \in A'(F)$, we can consider $x, x'$ as elements of $F_q$ for some $q = p^n$, $\mathcal{L}(xx') = \psi^{-1}(NF_{q}/F_p(x x'))$. When $k$ is merely a perfect field of characteristic $p$, $\mathcal{L}(x.a')$ corresponds to the push of the above $F_p$-torsor $\mathcal{A}$ by $\psi^{-1} : F_p \to \overline{Q}_p^\times$. The sheaf $\mathcal{A}$ corresponds to the representation of $\text{Gal}(\eta'/\eta)$ discussed in the note to (2.1.2.8); likewise $\mathcal{L}(x.a')_{\eta_\infty}$ corresponds to $L_\psi(1/xa')$; here $xa'$ corresponds to $\pi$ and indeed $1/xa'$ is a uniformizer for the strictly henselian trait $(\mathbb{P}^1_k)(\infty)$. So $\mathcal{L}(x.a')_{\eta_\infty}$ has slope 1.

If there is an $a'_1 \in \bar{k}$ such that

$$((F_{\eta_\infty})_1 \otimes \mathcal{L}(x.a'_1)_{\eta_\infty})^{(1)} \neq 0,$$  

...
then for all $a'_2 \neq a'_1 \in \overline{k}$,

$$((F_{\overline{\eta}_\infty})_1 \otimes \mathcal{L}(x.a'_2)_{\overline{\eta}_\infty})^{(1)}_\infty = ((F_{\overline{\eta}_\infty})_1 \otimes \mathcal{L}(x.a'_1)_{\overline{\eta}_\infty} \otimes \mathcal{L}(x.(a'_2 - a'_1))_{\overline{\eta}_\infty})^{(1)}_\infty = 0,$$

as $\mathcal{L}(x.(a'_2 - a'_1))_{\overline{\eta}_\infty}$ has slope 1.

Formula (i): in light of the above discussion, (2.1.2.7) gives that for almost all $a' \in \overline{k}$,

$$s(F_{\overline{\eta}_\infty} \otimes \mathcal{L}(x.a')_{\overline{\eta}_\infty}) = r((F_{\overline{\eta}_\infty})_{[0,1]} + r((F_{\overline{\eta}_\infty})_1) + s((F_{\overline{\eta}_\infty})_{1,\infty})).$$

By (2.2.1.1), $r(K') = r_s(K')$ for all but the finitely many $s' \in S'$. Combining these two facts with the Grothendieck-Ogg-Šafarevič formula for $r_s(K')$ gives

$$r(K') = r(F_{\overline{\eta}_\infty}) + \sum_{s \in S} \deg(s).a_s(K) - (r((F_{\overline{\eta}_\infty})_{[0,1]} + r((F_{\overline{\eta}_\infty})_1) + s((F_{\overline{\eta}_\infty})_{1,\infty})))$$

$$= \sum_{s \in S} \deg(s).a_s(K) + r((F_{\overline{\eta}_\infty})_{1,\infty}) - s((F_{\overline{\eta}_\infty})_{1,\infty}).$$

The difference between formulæ (i) and (ii) is in $s((F_{\overline{\eta}_\infty})_1 \otimes \mathcal{L}(x.s')_{\overline{\eta}_\infty})$; in the generic formula (i) we could discard the finitely many geometric points of $A'$ where this differs from $r((F_{\overline{\eta}_\infty})_1)$. Therefore,

$$r_s(K') - r(K') = -s((F_{\overline{\eta}_\infty})_1 \otimes \mathcal{L}(x.s')_{\overline{\eta}_\infty}) + r((F_{\overline{\eta}_\infty})_1).$$

The difference between formulæ (i) and (iii) is that in (i) we applied (2.1.2.7) while in (iii) we cannot, so that

$$r(K')_{0'} - r(K') = -s(F_{\overline{\eta}_\infty}) + (r((F_{\overline{\eta}_\infty})_{[0,1]} + r((F_{\overline{\eta}_\infty})_1) + s((F_{\overline{\eta}_\infty})_{1,\infty})))$$

$$= -s((F_{\overline{\eta}_\infty})_{[0,1]} + r((F_{\overline{\eta}_\infty})_{[0,1]}).$$

Intermezzo II: SGA 7 Exposé XIII. §1 In the definition of a compatible action of $G$ on a sheaf of sets $\mathcal{F}$ on $\overline{Y}$, to an étale $a : U \to \overline{Y}$ and $g \in G$ we must associate an isomorphism

$$\overline{Y} \times_{a(g),U} U \cong U \xrightarrow{u(g)} U \xrightarrow{a} \overline{Y} \xrightarrow{\mathcal{F}(U \to a)} \mathcal{F}(U \to \overline{Y})$$
so as to induce morphisms of sheaves \( \sigma(g) : u(g)^* \mathcal{F} \to \mathcal{F} \) in a compatible way so that \( \sigma(gh) = \sigma(g)\sigma(h) \). In effect, there is an obvious choice: \( \mathcal{F}(u(g)) \), and this choice explains why, given \( \mathcal{G} \) on \( Y \), the action of \( \text{Gal}(\overline{k}, k) \) on \( \mathcal{G} \) by transport of structure is compatible with the action of the same group on \( \overline{Y} \) (action ‘by transport of structure’ means the above action). \( A \text{ priori} \) the compatible action of \( G \) on \( \mathcal{F} \) need not even factor through \( u \).

(1.2.7) Let \( f : S' \to S \) be a surjective morphism of henselian traits. Then \( \eta' \mapsto \eta, s' \mapsto s \), and we can choose \( \overline{s}', \overline{s}, \overline{\eta}', \overline{\eta} \) so that \( \overline{s}' \to s \) factors through \( \overline{s} \) and likewise \( \overline{\eta}' \to \overline{\eta} \to \eta \) so that the diagram below commutes.

\[
\begin{array}{ccc}
Y \times_s \overline{s}' & \xrightarrow{f} & Y \times_s \overline{s} \\
\downarrow \overline{s}' & & \downarrow \overline{s} \\
Y \times_s s' & \xrightarrow{f} & Y
\end{array}
\]

This implies that for \( \mathcal{F} \) a sheaf on \( Y \), \( f^* \mathcal{F}_{\overline{s}'} := f^* \overline{s}'^* \mathcal{F} = \overline{s}'^* f^* \mathcal{F} \), so that \( \text{Gal}(\overline{s}'/s') \) acts on \( f^* \mathcal{F}_{\overline{s}'} \) via the homomorphism

\[
\text{Gal}(\overline{s}'/s') \to \text{Gal}(\overline{s}/s)
\]

induced by the restriction of \( f \) to \( s' \);

likewise, \( \text{Gal}(\overline{\eta}'/\eta') \) acts on \( f^* \mathcal{F}_{\overline{\eta}} \) via the homomorphism

\[
\text{Gal}(\overline{\eta}'/\eta') \to \text{Gal}(\overline{\eta}/\eta)
\]

induced by the restriction of \( f \) to \( \eta' \).

(1.3) Typo: \( \overline{X} := X \times_S \overline{S} \). Recall that \( \overline{S} \) is the normalization of \( S \) in \( \overline{\eta} \), which is the spectrum of a strictly henselian valuation ring with value group \( \mathbb{Q} \) so that the stalk at the closed point of \( \overline{S} \) coincides with global sections; its separably closed residue field may be an inseparable extension of \( k(s) \), but we take \( \overline{s} \) to denote the spectrum of this separably closed field, which can be considered as the closed point of \( \overline{S} \) or, by light abuse of notation (0.2.4), as defining a geometric point of \( S \).

It is perhaps worth comparing \( i^* j_* j^* \) with \( i^* j_* j^* \) in the definition of \( \Psi_{\eta} \). Let \( \mathcal{F} \) be a sheaf on \( Y \to S \); then \( \mathcal{G}_{\overline{\eta}} \) carries action of \( \text{Gal}(\overline{\eta}/\eta) \). In effect, \( i^* j_* j^* \mathcal{F} \) is \( \mathcal{G}_{\overline{\eta}} \), while \( i^* j_* j^* \mathcal{F} = \mathcal{G}_{\overline{\eta}} \) endowed with continuous action of \( \text{Gal}(\overline{\eta}/\eta) \) compatible with the action of the latter on \( Y_{\overline{s}} \). In galoisian terms of (1.2.2), this is the observation that \( \text{Gal}(\overline{\eta}/\overline{\eta}) = \{e\} \).
In (1.3.6.2), the $\psi$ is omitted on the left hand side because it induces an equivalence (see note to (2.1.8) below). The $X_s \times_s \eta$-component of both sides is readily computed from the definitions, keeping in mind that $\psi_{\eta}$ is a sheaf on $X_{\overline{\eta}}$; namely,

$$\Psi_{\eta}(f_*(\mathcal{F})) = i^* j_* (f_* (F)_{\eta}) = i^* j_* \Gamma(X_{\overline{\eta}}, \mathcal{F}),$$

which, as $\text{Gal}(\overline{\eta}/\eta)$-module, is simply $\Gamma(X_{\overline{\eta}}, \mathcal{F})$ (sitting on $\overline{s}$). With $f : X_{\overline{s}} \to \overline{s}$,

$$f_* \Psi_{\eta}(\mathcal{F}) = \Gamma(X_{\overline{s}}, \Psi_{\eta}(\mathcal{F})).$$

(1.3.8) The commutativity of $f_!$ with change of base does not require that $f$ be quasi-finite, only separated and locally of finite type [SGAA, Exp. XVII 6.1.4]. The morphism $f_! j_* \to j_* f_!$ follows from the definition of $f_!$ [SGAA, Exp. XVII 6.1.2] and the observation that the sections of $j_* f_!$ over $X'$ are the sections of $j_* f_!$ over $X_{\overline{\eta}}$ with support proper over $X'_{\overline{\eta}}$ while the sections of $f_! j_*$ are the same, only they must now have support proper over the larger space $X'_{\overline{s}}$. Therefore the map is induced simply by the inclusion $\eta \hookrightarrow S$. When $f$ is finite it is proper and all these maps are isomorphisms.

(1.3.9) Commute with direct image $\Rightarrow$ [SGAA, Exp. XVIII 3.1.12.3].

$$i^* f^! \to f^! i^* \Rightarrow \text{SGAA Exp. XVIII 3.1.14.2}.$$  

$f$ étale $\Rightarrow$ [SGAA, Exp. XVIII 3.1.8].

(1.3.10) The definition of this arrow is clarified with a diagram in [Th. finitude, 3.7], where the derived version of this map is shown to be an isomorphism, provided you assume $\mathcal{F}$ is of torsion prime to the residual characteristic of $S$.

(1.4) The business of securing the injectivity of $q'$ is the usual trick (SGA V Exp. XV p. 479). Namely, the cone of $sp^* K_s$ (where here $sp = sp \circ j$ in the sense of (1.2.2)) is the cone of the identity map on this complex; i.e. the complex $C(sp^* K_s) : = (sp^* K_s)^j \oplus (sp^* K_s)^{j-1}$ with differential $d^i(x, y) = (d^i x, x - d^{i-1} y)$. The identity map on this complex is homotopic to 0 via the homotopy $h(x, y) = (y, 0)$, so it is acyclic and $K'_\eta = K_\eta \oplus C(sp^* K_s)$ is homotopic to $K_\eta$. Replacing $q$ by $q' := (q, \text{id}, 0)$ (as $sp^* K_s$ coincides with $K_\tau$ with action of $\text{Gal}(\overline{\eta}/\eta)$ factoring through $\text{Gal}(\overline{s}/s)$ so that $q'$ remains equivariant), $q'$ is now a termwise split injection with termwise splitting given by projection to the second factor. In (1.4.2.2) $sp$ again coincides with $sp \circ j$ (1.2.2). En
6. LAUMON, TRANSFORMATION DE FOURIER

La transformation de Fourier, $\Phi(K)$, keep track of the sections of $K_\eta$ which do not come from residual extension of sections of $K_s$; bref, the discrepancy between $K_\eta$ and $K_s$.

(2.1.2) ‘l’image réciproque $F_\eta$ est acyclique’ $\Leftrightarrow \{SGAA\ \text{VII} \ 5.7\]}

(2.1.3) $\eta_{nr}$ is a strictly henselian trait coinciding with the normalization of $S$ in $\eta_{nr}$.

Claim: $X_{(x)} = X_{(x)} \times_{\eta_{nr}} S_{nr}$ is a strictly henselian trait coinciding with the normalization of $S$ in $\eta_{nr}$.

Claim: $X_{(x)} \times_{\eta_{nr}} S_{nr}$ is a strictly henselian trait coinciding with the normalization of $S$ in $\eta_{nr}$.

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Claim: $X_{(x)} \times_{\eta_{nr}} S_{nr}$ is a strictly henselian trait coinciding with the normalization of $S$ in $\eta_{nr}$.
In the case of a complex $K$ in $D^+(X, \Lambda)$ with $f$ smooth and the $\mathcal{H}^q(K)$ lisse, we can reduce to the previous paragraph with the help of the spectral sequence

$$E^{p,q}_{II} = R^p\Psi_\eta(\mathcal{H}^q(K_s)) \Rightarrow \mathcal{H}^{p+q}(K_s)$$

which finds that the map $sp^* i^* K \rightarrow R\Psi_\eta(K_s)$ is a quasi-isomorphism.

See also [Th. finitude, 2.12].

(2.1.6) $Rf_*$ is described as a functor $D^+(Y \times_s S, \Lambda) \rightarrow D^+(Y' \times_s S, \Lambda)$.

c) $f$ quasi-fini $\Rightarrow f!$ exact $\sim$ [SGAA, XVII 6.1.4].

d) $f$ quasi-fini $\Rightarrow Rf^!$ is the right derived functor of $f!$ $\sim$ [SGAA, XVIII 3.1.8 (i)].

(2.1.7) Climbing to the highest heights of pedantry, to derive (1.3.6.1), start by representing our $K$ in $D^+(X, \Lambda)$ by a complex of injectives $I_1$; as $f_*$ preserves injectives, $R\Psi Rf_* K = \Psi f_* I_1$. By (1.3.6.1) we find a morphism of complexes $\Psi f_* I_1 \rightarrow f_* \Psi I_1$. Taking a quasi-isomorphism $\Psi I_1 \rightarrow I_2$ into a complex of injectives $I_2$ composes to give

$$R\Psi Rf_* K = \Psi f_* I_1 \rightarrow f_* \Psi I_1 \rightarrow f_* I_2 = Rf_* R\Psi K.$$

The same method gives (2.1.7.2) and the first arrow in the below for (2.1.7.3).

$$Rf^! R\Psi = Rf_* j_* R\Psi \rightarrow Rf^* R\Psi j_* R\Psi \rightarrow Rf^* R\Psi j_* = R\Psi Rf^!.$$

For $f$ quasi-finite, $f^!$ is right adjoint to the exact functor $f!$ and so preserves injectives.

Deligne shows [Th. finitude, 3.7] that the formation of nearby cycles $R\Psi$ commutes with change of trait; i.e. the base-change morphism (2.1.7.5) is an isomorphism (for sheaves of torsion prime to the residual characteristic of $S$). (Confusingly, he calls $R^i\Psi_\eta$ vanishing cycles. This is surely because, according to Illusie, Grothendieck and vanishing cycles, when Grothendieck introduced the functors $R\Psi$, $R\Phi$, he called both these ‘functors of vanishing cycles.’)

(2.1.8) Typo: $K$ belongs in $D^+(X, \Lambda)$. When $X' = S$, $\Psi$ induces an equivalence of sheaves on $S$ with sheaves on the topos $s \times_s S$ (1.2.2 b)) and is therefore omitted. The $X_s \times_s \eta$-component of the left-hand side of (2.1.8.1) is

$$R\Psi_\eta Rf_* K = \tilde{i}^* \tilde{j}_* \tilde{r}^* Rf_* K = \tilde{i}^* \tilde{j}_* (Rf_* K)_\eta = \tilde{i}^* \tilde{j}_* R\Gamma(X_\eta, K).$$
As in the note to (1.3) above, as Gal(\(\mathfrak{n}/\mathfrak{n}\))-module, \(i^* j_* R\Gamma(X_{\mathfrak{n}}, K) = R\Gamma(X_{\mathfrak{n}}, K)\), but the left-hand side can be considered as sitting on \(\mathfrak{s}\) while the right-hand side sits on \(\mathfrak{n}\). This is pedantry. In any event, the \(X_s \times_s \mathfrak{n}\)-component of the right-hand side of (2.1.8.1) coincides with the right-hand side of (2.1.8.3) in light of (2.1.6.2), which in itself is pedantic and simply recognizes that the object \(R\Psi_{\mathfrak{n}}(K)\) sits on \(X_s\), and we take the stalk at the point \((\mathfrak{x}, \mathfrak{n})\) of \(X_s \times_s \mathfrak{n}\) as in (2.1.3). In other words, with \(f : X_s \to \mathfrak{s}\),

\[
R f_* \; R\Psi_{\mathfrak{n}}(K) = R f_* \; i^* j_* (K_{\mathfrak{n}}) = R\Gamma(X_{\mathfrak{s}}, i^* j_* (K_{\mathfrak{n}}) = R\Gamma(X_{\mathfrak{s}}, R\Psi_{\mathfrak{n}}(K)).
\]

The long exact sequence (2.1.8.9) exists when \(f\) is proper because in that case (2.1.8.3) is an isomorphism.

2.3.2. The translation of the note to (2.2.1.1) into \(G_{s'}\)-modules is that

\[
\mathcal{H}^{-1}(K_{s'}) \subset F_{\mathfrak{n}', \mathfrak{n}'},
\]

confirming the exactness on the left of the exact sequence of \(G_{s'}\)-modules.

(2.3.2.1) (i) The point is that the Artin-Schreier \(F_p\)-torsor \(A\) (c.f. note to 2.3.1) is trivialized by the base change \(G_{a,k} \overset{\psi^{-1}}{\longrightarrow} G_{a,k}\) by the Lang isogeny for \(G_{a,k}\), which is a revêtêtement étale. Pulling back to \(A \times_k A' \to G_{a,k}\) trivializes \(\mathcal{L}(x,x')\) over this revêtêtement étale and reduces to the stated theorem (see also note to (2.1.5) in Intermezzo II).

(In galoisian terms, if \(\mathfrak{x}\) is a geometric point of \(G_{a,k}\), \(\mathcal{L}_\psi\) is defined by

\[
\pi_1(G_{a,k}, \mathfrak{x}) \to F_p \overset{\psi^{-1}}{\longrightarrow} Q_{\ell}^\times
\]

The Artin-Schreier covering \(G_{a,k} \overset{\psi^{-1}}{\longrightarrow} G_{a,k}\) corresponds to the open subgroup of \(\pi_1(G_{a,k}, \mathfrak{x})\) which coincides with the kernel of this representation.)
(ii) Use (1.4.1.1) and [SGA 7, Exp. XIII 2.1.7.1] for $R\Psi$; proper base change for $sp^* i^*$ and (TR3) gives the desired isomorphism

$$
\begin{array}{ccccccc}
sp^* i^* \text{Rpr}_s & \longrightarrow & R\Psi_{\eta_s} \text{Rpr}_s & \longrightarrow & R\Phi \text{Rpr}_s & \longrightarrow \\
\downarrow & & \downarrow & & \downarrow & \\
\text{Rpr}_s sp^* i^* & \longrightarrow & \text{Rpr}_s R\Psi_{\eta_s} & \longrightarrow & \text{Rpr}_s R\Phi & \\
\end{array}
$$

Let $M := \text{pr}'(\alpha; K) \otimes \overline{\mathcal{L}}(x.x')[1]$. As $\text{Rpr}_s sp^* = sp^* \text{Rpr}_s$, this diagram computes to

$$
\begin{array}{ccccccc}
sp^* K'_s & \longrightarrow & R\Psi_{\eta_s'}(K') & \longrightarrow & R\Phi(K') & \longrightarrow \\
\downarrow & & \downarrow & & \downarrow & \\
sp^* R\Gamma(D_{s'}, i^*(M)) & \longrightarrow & R\Gamma(D_{s'}, R\Psi_{\eta_s'}(M)) & \longrightarrow & R\Phi(M)_{(\overline{\omega}, \overline{\omega}')} & \\
\end{array}
$$

where here $R\Gamma(D_{s'}, i^*(M)) = \text{Rpr}_s(i^* M)_{s'}$ for $\text{pr}' : D \times_k s' \to s'$. En effet, (i) gives that $R\Phi(M)$ is supported on $\infty \times_k \eta_{s'}$ (the topos; literally as a sheaf it sits on $\infty \times_k s' \hookrightarrow D_{s'}$); if $k$ denotes the inclusion of this point into $D \times_k \eta_{s'}$, $R\Phi(M) = k, L$ so that $R\text{pr}_s' R\Phi(M) = R\Gamma(D_{s'}, k, L) = L(\overline{\omega}, \overline{\omega}')$. Compare with [SGA 7, Exp. XIII 2.1.8] and note to (2.1.8) in Intermezzo II

(2.3.2.3) = (2.3.2.1) (iii) + the exact sequence at the top of the page + (2.3.1.1).

(2.3.3.1) Typos: in the first displayed equation of (ii) the stalk is at $(\overline{s}, \overline{\omega})$ not $(\overline{s}, \overline{\omega}')$.

In the first displayed equation of the proof obviously it is $D \times_k \overline{\omega}'$ not $D \times_k \overline{\omega}'$.

In the proof of (iii), the point is that as the support of $R\Psi_{\eta_{s'}} = R\Phi_{\eta_{s'}}$, is contained in $(S \cup \infty) \times_k \eta_{s'}$. Let $M_s$ denote the restriction of $R\Psi_{\eta_{s'}}(\text{pr}'(\alpha_i K) \otimes \overline{\mathcal{L}}(x.x')[1])$ to $s \times_k \overline{\omega}'$ (it is the fiber at $\eta_{s'}$ of a sheaf on the topos $s \times_k \eta_{s'}$, hence literally a sheaf on $s \times_k \overline{\omega}'$), and put $s : s \times_k D'_{(\infty')} \hookrightarrow D \times_k D'_{(\infty')} \xrightarrow{\text{pr}'} D'_{(\infty')}$ ($s \in S \cup \infty$). Then

$$R^{-1} \Gamma(D \times_k \overline{\omega}', R\Psi_{\eta_{s'}}(\text{pr}'(\alpha_i K) \otimes \overline{\mathcal{L}}(x.x')[1])) = \bigoplus_{s \times \overline{s}} s'_{\overline{s}}(M_s)
$$

with the notation of [SGA 7, 2.1.6]. Fix some $s \in S \cup \infty$. $s_{\overline{s}} : s \times_k \overline{s}' \to \overline{s}'$ is a revêtement étale that realizes $G_{s \times_k \overline{s}'} \hookrightarrow G_{\overline{s}'}$, as an open subgroup, and $s \times_k \overline{s}'$ is topologically a disjoint union of copies of $\overline{s}'$ indexed by the (algebraic) geometric
points centered on $s$ (there are deg$(s)$ many). As $G_{\infty'}$-module,

$$s_{\infty'}(M_s) = G_{\infty'} \otimes_{G_{\infty' \times k'}} M_s =: \text{Ind}_{G_{\infty' \times k'}}^{G_{\infty'}}(M_s).$$

The proof of (i) is given by (1.3.1.2), the proof of which is relegated to the paper of Katz-Laumon *Transformation de Fourier et Majoration de Sommes Exponentiels* and is discussed next.

**Intermezzo III: Katz-Laumon (2.4).** All notation and references in this section are to *Transformation de Fourier et Majoration de Sommes Exponentiels* unless otherwise noted. Let us first discuss how their proof of (2.4.1) gives the theorem (2.4.4) about universal strong local acyclicity. With the notation of the proof of (2.4.1), and letting $M$ throughout denote constant sheaf with value the $A$-module $M$, the translation trick allows us to replace $P^1 \times A^1$ by $P^1 \times G_m$. The claim is trivial away from $\infty \times G_m$. It suffices to show that $\tilde{\text{pr}}_1$ is universally locally acyclic rel. $M \otimes L^j A_X$. By (2.4.2), this will be true iff the same is true rel. $M \otimes \tilde{f}^* A_X$. By the projection formula for $\tilde{f}$ & Leray, this is the same as $\tilde{\text{pr}}_1 \circ \tilde{f}$ being universally locally acyclic rel. $M$. As proven, there is a neighborhood of $\infty \times G_m$ over which $\tilde{\text{pr}}_1 \circ \tilde{f}$ factors as a surjective radicial morphism followed by a smooth morphism; hence by universal local acyclicity of a smooth morphism, $\tilde{\text{pr}}_1 \circ \tilde{f}$ is universally locally acyclic rel. any locally constant sheaf, in particular rel. $M$.

In the proof itself, the formula for $[a]^* \mu^* \mathcal{L}_y$ depends on [Sommes trig., 1.7.1]. The extension of the Artin-Schreier covering $\tilde{X}$ is defined as the finite covering of $P^1 \times G_m$ defined by $X_0 T_1^q - X_0 T_1 T_0^{q-1} = X_1 T_0^q y$. It suffices to verify the various properties of $\tilde{X}$ when $S = F_q$. Note that $\tilde{X}$ is integral as $t^q - t - xy$ doesn’t have a root over $k(x, y)$, and $F_q$ acts transitively on the roots by $t \mapsto t + \gamma, \gamma \in F_q$. Pick any point Spec $F_q(y)$ of $G_m \otimes F_q$, corresponding to a choice of $y \in \overline{F}_q^X$ or $y$ as a transcendental generator of $F_q(y)/F_q$. On one hand, the point $\infty \in P^1_{F_q(y)}$ has one point of $\tilde{X} \times_{G_m} \text{Spec} F_q(y)$ lying over it, since when $X_0 = 0$, $T_0$ is nilpotent. On the other, $T_0 = 0$ implies $X_0 = 0$ as then $T_1$ is a unit. Therefore $T_0$ generates the maximal ideal of the local ring of the point of $\tilde{X} \times_{G_m} \text{Spec} F_q(y)$ lying over $\infty \times F_q F_q(y)$, which shows that every point of $\tilde{f}^{-1}(\infty \times_{F_q} F_q(y))$ is totally ramified over $\infty \times_{F_q} F_q(y)$, and also shows that regardless
of whether or not we extend scalars to $F := \overline{F}_q$, the following is true: The points of $\tilde{f}^{-1}(\infty \times G_m)$ are in bijection with the points of $\infty \times G_m$, hence also with the points of $G_m$; given a point $a \in G_m$, the maximal ideal $m$ of the corresponding point of $\tilde{X}$ is generated by $1 + \text{codim } a$ elements: if $a$ is closed, $m$ is generated by $T_0$ and the minimal polynomial of $y$, and if $a$ is the generic point, $m$ is generated by $T_0$ only. This implies that $\tilde{f}^{-1}(\infty \times_{F_q} G_m, F_q)$ is geometrically regular over $F_q$, hence, in light of the lemma in the note to [BBD] 2.2.1.0, smooth over $F_q$.

Checking that $\tilde{X}$ is étale over $A^1 \times G_m$ amounts to showing that for any choice of $x \in F, y \in F^\times$, the closed subscheme $Q \hookrightarrow P^1 F_q(x, y)$ cut out by $T_1^q - T_1 T_0^{q-1} = T_0^q xy$ is étale over $F_q(x, y)$. The subscheme $Q$ has empty intersection with $T_0 = 0$, and away from $T_0 = 0$, $Q$ is defined by the separable polynomial $t^q - t = xy$, and as above, $F_q$ acts transitively on the roots, so that $Q$ either is the disjoint union of $q$ copies of $\text{Spec } k(x, y)$ or is the spectrum of a separable field extension of $k(x, y)$. In any event: étale, and $\tilde{f}^{-1}(A^1 \times G_m)$ is smooth over $F_q$.

The proof of (2.4.2) hits a snag when it is only assumed, as stated, that there exists a nontrivial additive character $\psi : F_q \to A^\times$. The representation $f_* A_X$ coincides with the regular $A[F_q]$-module $A[F_q]$. If we write multiplicatively, this is the ring $A[x]/(x^q - 1)$ acting on itself as a free module of rank 1. The point is that of course the separable polynomial $x^q - 1$ need not split into linear factors over $A$ if it has one root distinct from 1 in $A$. What is needed is a primitive root of this polynomial; i.e. a root which is not a root of any polynomial $x^{p^m} - 1$, $m < n$ (suppose $q = p^n$). (Actually, $x^q - 1$ splits as a product of $n$ polynomials each of degree $p$, as $(x^{p^m} - 1)|(x^{p^{m+1}} - 1)$.) The existence of such a root in $A^\times$ implies that $A[F_q]$ decomposes completely into $q$ $A$-modules of rank 1 indexed by the $q$ distinct characters $F_q \to A^\times$; conversely, if such a root fails to exist, there will be irreducible factors of $x^q - 1$ of degree $d > 1$ corresponding to irreducible factors of $A[F_q]$ free of rank $d$ as $A$-module.

With this proviso, the decomposition of $f_* A_X$ is achieved, and all that is left to note for the decomposition of $\tilde{f}_* A_{\tilde{X}}$ is that the stalk over a point of the base of the latter are free $A$-modules of rank which coincides with the number of points in the geometric fiber [SGA4] VIII 5.5, and we have shown that every geometric point of $\infty \times G_m$ has precisely one point of $\tilde{X}$ lying over it, so that the stalk of $\tilde{f}_* A_{\tilde{X}}$ is free of rank 1 at every
geometric point of \( \infty \times G_m \). As the direct image of a constant sheaf on the generic point of a normal integral scheme is constant (c.f. remark at the end of [Reprise]),

\[
\tilde{f}_* A_X = \tilde{f}_* j'_* A_X = j_* f_* A_X = A_{P^1 \times G_m} \oplus \left( \bigoplus_{\psi' \neq 1} j_\psi^* \mathcal{L}_{\psi'} \right),
\]

and by rank considerations we must have \( j_\psi^* \mathcal{L}_{\psi'} = j_\psi^* \mathcal{L}_\psi \) for all nontrivial \( \psi \).

To see this last fact directly, observe that given a geometric point \( \bar{x} \) centered on a point \( x \in \bar{X} \) with image \( y \in \infty \times G_m \), and letting \( \bar{y} \) denote the image of \( \bar{x} \), let \( \bar{X}_{(\bar{y})} \), resp. \( \bar{Y}_{(\bar{y})} \) denote the strict henselizations of \( \bar{X} \), resp. \( \bar{Y} \), with \( \bar{Y} = P^1 \times G_m \) at \( \bar{x} \) and \( \bar{y} \), respectively. They are regular local rings. The map on generic points \( \eta_\tau \to \eta_{\bar{\tau}} \) is still given by \( t^q - \tau = x_0 y_0 \), as this still gives a finite extension of degree \( q \) of \( \bar{Y}_{(\bar{y})} \) so that if \( \bar{\xi} \) denotes the geometric point of \( G_m \), the uniformizer of \( \bar{Y}_{(\bar{y})} \times G_m \bar{\xi} \) is the \( q \)-th power of that of \( \bar{X}_{(\bar{y})} \times G_m \bar{\xi} \). It follows that provided \( \psi' \neq 1 \), \( \text{Gal}(\bar{\eta}_y / \eta_{\bar{\tau}}) \) acts nontrivially on the reciprocal image of \( \mathcal{L}_{\psi'} \) to \( \bar{Y}_{(\bar{y})} \), as \( \text{Gal}(\bar{\eta}_y / \eta_{\bar{\tau}}) \) does. As the stalk of \( j_\psi \mathcal{L}_{\psi'} \) at \( \bar{y} \) coincides with the invariants of the \( \text{Gal}(\bar{\eta}_y / \eta_{\bar{\tau}})^{-1} \)-module \( j_* \mathcal{L}_{\psi'} \), \( j_* (\mathcal{L}_{\psi'})_{\bar{y}} = 0 \).

In the last paragraph of the proof, we can do all this assuming \( S = F_q \). Fix coordinates \((\tau, y)\) on \( \tilde{f}^{-1}(U) \) via

\[
F_q[\tau, y, y^{-1}] \to F_q[\tau, \bar{\xi}, y, y^{-1}] / (\bar{\xi}(1 - \tau^{q-1}) - \tau^q y).
\]

The fiber of this map over the closed subscheme of \( A^1 \times G_m \) defined by \( \tau^{q-1} \) is empty. Therefore \( (\tau, y) \) factors through the complement \( U' \times G_m \) of this closed subscheme, which corresponds to inverting \( 1 - \tau^{q-1} \), after which the above map on functions becomes an isomorphism. As \( \bar{\xi} = \tau^q y / (1 - \tau^{q-1}) \) in \( (\tau, y) \) coordinates, \( \bar{p}_1 \circ \tilde{f} \) takes the stated form. The following diagram commutes.

\[
\begin{align*}
A^1 & \xrightarrow{\tau \mapsto y} U' \times G_m & \xrightarrow{\kappa} U' \times G_m \\
\tilde{f}^{-1}(U) & \xrightarrow{(\tau, y) \mapsto (\tau, y, y^{-1}, (1 - \tau^{q-1})^{-1})} F_q[\tau, y, y^{-1}, (1 - \tau^{q-1})^{-1}] & \xrightarrow{(\tau, y') \mapsto (\tau^q, y / (1 - \tau^{q-1}))} F_q[\tau, y', y'^{-1}, (1 - \tau^{q-1})^{-1}]
\end{align*}
\]
The map $\nu : U' \times G_m \to A^1$ is smooth since the coordinate ring of the former can be written as $S := F_q[\xi, y', a, b]/(ay' - 1, b(1 - \xi a) - 1)$, and

$$\det \begin{pmatrix} y' & -b\xi \\ 0 & 1 - \xi a \end{pmatrix} = y'(1 - \xi a)$$

is invertible in $S$ with inverse $ab$, so that $F_q[\xi] \to S$ is standard smooth and hence smooth (Stacks tags 00T6 & 00T7).

To verify that $\kappa$ is radicial and surjective, it suffices to show $\kappa$ is bijective and induces purely inseparable residual extensions. This can be done by verifying that for each $x \in U' \times G_m$, the scheme-theoretic fiber $\kappa^{-1}(x)$ is isomorphic to the spectrum of an artinian local ring with purely inseparable residual extension.

Picking a point $x$ amounts to choosing a field $F_q(\alpha, \beta)$ such that $\alpha^{q-1} \neq 1$; as this is trivially satisfied when $\alpha$ is transcendental and Frobenius is an isomorphism $F_q(\alpha)$ when $\alpha$ is algebraic, $\alpha^{q-1} \neq 1$ is equivalent to the condition that $\alpha \notin F_q$. The fiber $\kappa^{-1}(x)$ in this case is isomorphic to the spectrum of the finite $F_q(\alpha, \beta)$-algebra

$$A := F_q(\alpha, \beta)[\tau, y, y^{-1}, (1 - \tau^{q-1})^{-1}]/(\tau^q - \alpha, y - \beta(1 - \tau^{q-1}))$$

$$\cong F_q(\alpha, \beta)[\tau, (1 - \tau^{q-1})^{-1}]/(\tau^q - \alpha);$$

$\alpha$ (resp. $\beta$) is algebraic over $F_q$ iff $x$ does not map to the generic point of $U'$ (resp. $G_m$). If $\alpha$ is transcendental then

$$A \cong F_q(\alpha, \beta)[\tau]/(\tau^q - \alpha)$$

is a field which is purely inseparable over $F_q(\alpha, \beta)$. If $\alpha$ is algebraic then Frobenius induces an isomorphism of $F_q(\alpha)$ and there exists a $\sqrt[q]{\alpha} \in F_q(\alpha)$ so that $\tau^q - \alpha = (\tau - \sqrt[q]{\alpha})^q$; $A$ is therefore isomorphic to a localization of a local Artinian ring with trivial residual extension, and we need check only that $\kappa^{-1}(x)$ is nonempty; i.e. that $A \neq 0$. This is true iff $1 - \tau^{q-1}$ is not nilpotent, in which case it’s already a unit in $A$. As $\alpha \notin F_q$ by assumption and $\alpha \neq \sqrt[q]{\alpha}$ so that $(\sqrt[q]{\alpha})^{q-1} \neq 1$ and $1 - \tau^{q-1}$ indeed doesn’t go to zero under $\tau \mapsto \sqrt[q]{\alpha}$.

2.4.1. Back to Laumon. The point is that $\pi/\pi', \pi'/\pi$, and $1/\pi\pi'$ determine three distinct rational maps $T \times_k T' \to G_{a,k}$ (regular over $T \times \eta', \eta \times T', \eta \times \eta'$, respectively),
and \( \mathcal{L}_\psi(\pi/\pi') \), \( \mathcal{L}_\psi(\pi'/\pi) \), and \( \mathcal{L}_\psi(1/\pi\pi') \) are the reciprocal images of \( \mathcal{L}_\psi \) defined over these open loci of \( T \times T \) via these various maps.

By way of preface, it would seem from the discussion below that \( T \) implicitly is assumed to coincide with \( \text{Spec} \ k\{x\} \). Certainly it would seem that Néron desingularization should not be necessary for the result (2.4.2.1) cited as ‘immediate,’ or a somewhat esoteric result about the Galois group of generic points of henselian d.v.r.s to connect (2.4.2.2) to the discussion in (2.2.2). Perhaps there is an implicit assumption I’ve missed, or far simple ways to obtain these results with no assumption on \( T \) other than \( k[\pi] \subset R \subset k[[x]] \). As remarked in the note to (2.1.1) one must assume \( k \subset R \); in contrast with an equicharacteristic complete d.v.r., an equicharacteristic henselian d.v.r. need not contain a coefficient field.

(2.4.2.1) As (i), (ii), and (iii) are interchangeable, we dissect (i) as an example. The map \( \pi' : \eta' \to \eta_{\text{loc}} \) is specified by \( \pi' \mapsto 1/x' \). As \( 1/x' \) is a uniformizer for the local ring of \( D' \) at \( \infty' \) (and also its henselization \( D'_{(\infty')} \)), from the data of \( \pi' \mapsto 1/x' \) we get a morphism \( T' \to \text{Spec} \ O_{D',\infty'} \) and by the universal property of henselization (Raynaud, *Anneaux Locaux Henséliens*, VIII Déf. 1) a unique factorization of this morphism to a commutative triangle

\[
\begin{array}{ccc}
\text{Spec} O_{D',\infty'} & \leftarrow & D'_{(\infty')} \\
\downarrow & & \downarrow \pi' \\
T' & \rightarrow & \eta_{\text{loc}}
\end{array}
\]

which recovers \( \pi' \) on the generic fiber. Note that \( R\Psi_{\eta_{\text{loc}}} = R\Phi_{\eta_{\text{loc}}} \) as in (2.3.3.1). It is asserted that the base-change morphism

\[
(\pi \times \pi')^* R\Psi_{\eta_{\text{loc}}} \to R\Psi_{\eta'}(\pi \times \pi')^*
\]

is an isomorphism. As \( \pi \times \pi' = (1 \times \pi') \circ (\pi \times 1) \), we first deal with \( (1 \times \pi')^* \), which is the morphism induced by change of trait \( \pi' : T' \to D'_{(\infty')} \) and is an isomorphism by [Th. finitude 3.7]. On the other hand, letting \( V_0 := k[\pi]_{(\pi)} \) and \( T_0 := \text{Spec} \ V_0 \), the morphism \( \pi : T \to A \) factors as \( T \overset{\pi}{\to} T_0 \to A \) where the second morphism is ind-étale and therefore commutes with base change by passage to the limit. It doesn’t
seem ‘immediate’ to me that
\[(\pi_1 \times 1)^* R\Psi_{\eta'} \to R\Psi_{\eta'}(\pi_1 \times 1)^*\]
is an isomorphism. The cleanest way I know to show this is to invoke Néron desingularisation in the form \^[SGA 7, I 0.5.1\] In the paragraphs following the statement of this lemma (0.5.1), it’s explained how the conditions of the lemma are satisfied in the case of \(\pi_1 : T \to T_0\). The residual extension is trivial in our case; to see that the extension \(\eta/\eta_0\) is separable, it suffices to show that \(k(\eta)/k(\pi)\) is separable. For this one need only show that \(k(\pi^{1/p}) \otimes_{k(\pi)} k(\eta) \simeq k(\eta)[t]/(t^p - \pi)\) is separable. Assuming \(p > 1\), this is true iff \(\pi\) does not have a \(p^{th}\) root in \(k(\eta)\), which is true as \(\pi\) is a uniformizer. Néron says that \(\pi_1\) is a limit of smooth morphisms and therefore commutes with \(R\Psi\).

On a related note, I know of a funny way to show the more general

**Lemma.** — Let \(f : T \to T_0\) be a surjective morphism of henselian traits and suppose

1. \((\alpha)\) the special fiber \(T \times_{T_0} s_0\) is reduced,
2. \((\beta)\) \(k(s_0) \subset k(s)\) is purely inseparable, and
3. \((\gamma)\) the maximal purely inseparable extension of the completion \(\widehat{k(\eta_0)}\) is dense in \(k(\eta_1)\).

Then for every scheme \(X_0 \to T_0\), \(f\) induces an equivalence of topoi \(\tilde{X}_{\text{ét}} \to (\tilde{X}_0)_{\text{ét}}\), where \(X := X_0 \times_{T_0} T\).

This applies in the situation where \(T_0 = \text{Spec } R_0\) is equicharacteristic and \(R_0\) contains a coefficient field mapping isomorphically onto its residue field, which is assumed perfect, as assumed in (2.4.1), because of the inclusion \(k[\pi] \subset R_0 \subset k[[\pi]]\).

**Proof.** — By (2.4.1–2.4.3) in Berkovich, *Étale cohomology for non-Archimedean analytic spaces*, Publ. Math. IHES 78, reciprocal image \(L \mapsto L \otimes_{k(\eta_0)} k(\eta)\) induces an equivalence of categories of finite separable extensions of \(k(\eta_0)\) and \(k(\eta)\), and hence induces an isomorphism \(\text{Gal}(\overline{\eta}/\eta) \simeq \text{Gal}(\overline{\eta}/\eta_0)\). The same is true for the closed fiber (replacing \(\eta\) by \(s\)) if we restrict to finite Galois extensions by e.g. \^[Stacks tag 030M\]. Recall \^[SGAA, IV 9.5.4\], which says in our case that the category of sheaves on \(X \to T\)
is equivalent to the category of triples
\[(\mathcal{F}_s, \mathcal{F}_\eta, \varphi : \mathcal{F}_s \rightarrow i^* j_* \mathcal{F}_\eta) \quad \text{where} \quad \eta \xrightarrow{j} T \xleftarrow{i} s,\]
\((\mathcal{F}_s \text{ a sheaf on } X \times_T s, \mathcal{F}_\eta \text{ a sheaf on } X \times_T \eta) \) via the functor \(\mathcal{F} \mapsto (\mathcal{F}_s, \mathcal{F}_\eta, \varphi)\) where
\[\varphi : \mathcal{F}_s = i^* \mathcal{F} \rightarrow i^* j_* j^* \mathcal{F} = \text{sp}_s \mathcal{F}_\eta,\]
with quasi-inverse the functor which associates to \((\mathcal{F}_s, \mathcal{F}_\eta, \varphi)\) the object \(W\) defined by the cartesian square
\[
\begin{array}{ccc}
W & \xrightarrow{r} & j_* \mathcal{F}_\eta \\
\downarrow & & \downarrow \\
i_* \mathcal{F}_s & \xrightarrow{i_* \varphi} & i_* i^* j_* \mathcal{F}_\eta.
\end{array}
\]
Decorating everything with a subscript 0, the same is true for \(X_0 \rightarrow T_0\). Last, recall [SGA 7] XIII 1.3.3 (ii), which says that \(Y \rightarrow \text{Spec } k\) with separable closure \(\overline{k}\) and a sheaf \(\mathcal{F}\) on \(Y\), the functor \(\mathcal{F} \mapsto \overline{\mathcal{F}}\) induces an equivalence between the category of sheaves of sets on \(Y\) with that of sheaves of sets on \(\overline{Y}\) equipped with a continuous action of \(\text{Gal}(\overline{k}/k)\) compatible with the action of \(\text{Gal}(\overline{k}/k)\) on \(\overline{Y}\). \(\square\)

(2.4.2.2) By the Berkovich result in the proof of the previous lemma, we may assume \(T = \text{Spec } k\{x\}\); now invoke not (2.2.2.1) but rather the discussion following (2.2.2.2). A choice of retraction for \(i_*\) (in the notation of (2.2.2)) corresponds to an extension of \(V\) to a lisse \(\overline{Q}_l\)-sheaf \(\mathcal{F}\) on \(\mathbf{G}_{m,k}\). Let \(j : \mathbf{G}_{m,k} \hookrightarrow \mathbf{A}\); for (i) put \(K := j! \mathcal{F}\) and for (ii) and (iii) put \(\alpha j! \mathcal{F}[1]\). In all three cases, \(K\) is a perverse sheaf [BBD, 4.1.3] and \(\pi^* K = V_! [1]\).

2.5. In order to complete the proof (4.4) of Deligne’s main theorem in Weil II, it is only necessary to have (2.5.3.1) (i) (Kummer does not make an appearance). The proof of (2.5.3.1) (i) is an elegant synthesis of (2.3.2)–(2.4.2), and the only necessary auxiliary computation is that of \(\mathcal{F}(\overline{Q}_{\ell,A-[0]})\). There is an exact sequence of perverse sheaves on \(A\) [BBD, 4.1.10]
\[
0 \rightarrow \overline{Q}_{\ell,[0]} \rightarrow \overline{Q}_{\ell,A-[0]}[1] \rightarrow \overline{Q}_{\ell,A}[1] \rightarrow 0.
\]
Applying $\mathcal{F}$ gives the exact sequence of perverse sheaves on $A'$

$$0 \to \mathcal{Q}_{\ell,A'}[1] \to \mathcal{F}(\mathcal{Q}_{\ell,A-\{0\}})[1] \to \mathcal{Q}_{\ell,\{0\}}(-1) \to 0$$

which finds $\mathcal{F}(\mathcal{Q}_{\ell,A-\{0\}})|_{\eta_\infty'} = \mathcal{Q}_{\ell,\eta_\infty'}$ and completes the proof of (i).

4.2.2. (4.2.2.2) Write $\mathcal{F} := F$ to harmonize with Deligne’s notation. The condition that ‘the action of $I_s$ be unipotent of echelon 2.’ corresponds, in the vocabulary of \textbf{[Weil II] 1.7.2}, to the property that the filtration of local monodromy on $\mathcal{F}_\eta$ has at most 3 nonzero graded pieces; i.e. if $I$ does not act trivially on $\mathcal{F}_\eta$, then the nilpotent operator $N$ arising from the logarithm of the unipotent part of the local monodromy has $N \neq 0$ but $N^2 = 0$. In this case, as described in \textbf{[Weil II] 1.6.1}, the filtration has $M_1 = \mathcal{F}_\eta$, $M_0 = F^I_\eta$, and $M_{-1} = \text{im } N$, and $\text{Gr}_i^M = F^I_\eta/F^i_\eta$ which \textbf{[Weil II] 1.8.4} says is $\iota$-pure of weight $w + 1$.

(4.2.2.3) Recall the decomposition (1.6.14.3) of Weil II and the fact that there is a typo in it (c.f. note to 1.6.14). Recall that, as in the proof of \textbf{[Weil II] 1.8.4}, if $s \in S$, the fiber $(j_*\mathcal{F})_\tau$ coincides with $\ker N = \mathcal{F}^I_\eta$ where $N$ is the logarithm of the local monodromy (we can assume that all of $I$ acts unipotently). Recall that $N$ is compatible with the filtration of local monodromy, that $\text{Gr}_i^M(\ker N) \to P_i$ \textbf{[Weil II] 1.6.6}, and that $P_i = 0$ for $i > 0$ \textbf{[Weil II] 1.6.4}. The eigenvalues of $F_s$ on $j_*\mathcal{F}_\tau$ are in bijection with the eigenvalues of $F$ on the $P_i(\mathcal{F}_\eta)$, where $F$ is the conjugation class of liftings of Frobenius in the Weil group (or one such \textbf{[Weil II] 1.7.4}). Now Laumon’s proof goes through after you correct all the typos: $P_{-\iota}(\mathcal{F}_\eta)$ is $\iota$-pure of weight $w + 1$ \textbf{[Weil II] 1.8.4},

$$P_{-\iota}(\mathcal{F}_\eta) \simeq P_{-\iota}(\mathcal{F}_\eta)^\vee(i),$$

and $\alpha$ is an eigenvalue of Frobenius on some $P_{-\iota}(\mathcal{F}_\eta)$ (for this you need to know that the filtration of local monodromy is stable under the action of $W(\eta, \eta)$, which is true by \textbf{[Weil II] 1.7.5, 1.8.5]).

4.3.1. (4.3.1.1) It is implicitly claimed that $H^0(U, \mathcal{F}) = 0$, which is a consequence of the fact that $\mathcal{F}$ is assumed irreducible and not geometrically constant. If $\bar{u}$ is a
geometric point of $\overline{U} = U \times_{F_q} F$, there is an exact sequence
\[ e \to \pi_1(\overline{U}, \overline{u}) \to \pi_1(U, u) \to \text{Gal}(F/F_q) \to e \]
(SGA 1 6.1), and $\mathcal{F}$ is not geometrically constant if $\pi_1(\overline{U}, \overline{u})$ does not act trivially on $\mathcal{F}_{\overline{u}}$; i.e. the reciprocal image of $\mathcal{F}$ on $\overline{U}$ is not constant. As $\mathcal{F}_{\overline{u}}$ is stable under $\pi_1(U, u)$, it must be 0, as $\mathcal{F}$ is assumed irreducible, so $\Gamma(U, \mathcal{F}) = 0$.

4.3.2. Following Deligne, we decorate with a subscript 0 those objects over $F_q$ and remove this subscript to indicate the extension of scalars to $F$.

(4.3.2.1) Reduction to $X_0 = D_0$. We notate $j_0 : U_0 \hookrightarrow X_0$ and $\mathcal{F}_0$ lisse on $U_0$. The choice of a nonconstant meromorphic function on $X_0$ gives rise to a finite morphism $f_0 : X_0 \to D_0$. The only condition on the function is that it induce a morphism with nonempty étale locus. As $X_0$ is smooth, this is easy: just pick a point $x \in X_0$ and a generator of the local ring $\mathcal{O}_{X,x}$ (SGA 1 I 9.11). Leray gives

\[ R\Gamma(D, f_0^* j_* \mathcal{F}_0) \simeq R\Gamma(X, j_* \mathcal{F}_0). \]

It suffices to find some open $j'_0 : U'_0 \hookrightarrow D_0$ a lisse $\overline{Q}_\ell$-sheaf $\mathcal{F}_0'$ on $U'_0$, t-pure of weight $w$, and an identification $f_0^* j_0^* \mathcal{F}_0 \simeq j'_0^* \mathcal{F}_0'$ on $D_0$.

Lemma. — Let $\mathcal{F}$ be a lisse sheaf on a normal connected curve $S$ and $j : U \hookrightarrow S$ an open immersion. The unit of adjunction $\mathcal{F} \to j_* j^* \mathcal{F}$ is an isomorphism.

Admitting the lemma, pick some nonempty $U'_0$ such that $X_0 \times_{D_0} U'_0 \to U'_0$ is étale, and let $u_0 : V_0 := U_0 \cap (X_0 \times_{D_0} U'_0) \hookrightarrow U_0$. As the diagram
\[
\begin{array}{ccc}
V_0 & \xrightarrow{u_0} & U_0 & \xrightarrow{j_0} & X_0 \\
\downarrow{f_0|V_0} & & \downarrow{f_0} & & \\
U'_0 & \xrightarrow{j'_0} & D_0
\end{array}
\]
commutes,
\[ j'_0^*(f_0|V_0)_*(u_0^* \mathcal{F}_0) = f_0^* j_0^* u_0^* \mathcal{F}_0 \simeq f_0^* j_0^* \mathcal{F}_0, \]
and the sheaf $(f_0|V_0)_*(u_0^* \mathcal{F}_0)$ is lisse. More precisely, if $\overline{u}$ denotes a geometric generic point of $V_0$ (and its image in $U'_0$), and $u_0^* \mathcal{F}_0$ is defined by the monodromy representation...
\[ \pi_1(V_0, \eta) \rightarrow \text{Aut} \mathcal{F}_{\eta} \] the étale \( f_0|V_0 \) induces a morphism
\[ \pi_1(V_0, \eta) \rightarrow \pi_1(U'_0, \eta) \]
which is an injection of the former group onto the open subgroup of the latter corresponding to the revêtement étale \( f_0|V_0 \). The sheaf \((f_0|V_0)_*(u^*_0F_0)\) is defined by the induced representation
\[ \mathcal{F}_{\eta} \otimes_{\pi_1(V_0, \eta)} \pi_1(U'_0, \eta). \]
Therefore we take \( \mathcal{F}'_0 := (f_0|V_0)_*(u^*_0F_0) \).

**Proof of lemma.** — The lemma can be recovered as a corollary of the reprise below, and, perhaps more cheekily, from \([BBD]\ 4.3.2\) in light of the note to 1.4.2 above. A simple direct proof goes as follows. Checking the statement fiberwise at a geometric point \( s \) centered on a point \( s \in S - U \) reduces us to the setting of a henselian trait \((S, \eta, s, \eta, s)\) \([\text{Weil II} 0.6]\) and \( \mathcal{F} \) a lisse sheaf on \( S \). We have the usual exact sequence
\[ e \rightarrow I \rightarrow \text{Gal}(\eta/\eta) \rightarrow \text{Gal}(\eta/s) \rightarrow e. \]
where by SGA 1 V 8.2 & Arcata IV 2.2 the map \( \text{Gal}(\eta/\eta) \rightarrow \text{Gal}(\eta/s) \) factors as
\[ \text{Gal}(\eta/\eta) \rightarrow \pi_1(S, \eta) \twoheadrightarrow \pi_1(S, \eta) \twoheadrightarrow \pi_1(s, \eta) \cong \text{Gal}(\eta, s). \]
The stalk \( (\eta_!, \eta^* \mathcal{F})_x \) can be identified with \( \mathcal{F}^1_{\eta} \), where here \( \mathcal{F}^1_{\eta} \) is the \( \text{Gal}(\eta, \eta) \)-representation defined by \( \text{Gal}(\eta, \eta) \rightarrow \pi_1(S, \eta) \). The factorization above shows that \( I \) dies in this quotient and therefore acts trivially. \( \square \)

Returning to the proof of (4.3.2.1), Poincaré duality on \( D \) as stated rests on

**Theorem (Deligne, SGA 4\( \frac{1}{2} \) Dualité 1.3).** — Let \( j : U \hookrightarrow S \) be a dense open of a regular scheme \( S \) purely of dimension 1 and \( \mathcal{F} \) a locally constant constructible sheaf of \( \mathbb{Z}/n \)-modules on \( U \). One has \( D_{j_*\mathcal{F}} = j_*D\mathcal{F} \), i.e. \( \text{Hom}(j_*\mathcal{F}, \mathbb{Z}/n) = j_*\text{Hom}(\mathcal{F}, \mathbb{Z}/n_U) \) and \( \text{Ext}_i^j(j_*\mathcal{F}, \mathbb{Z}/n) = 0 \) for \( i > 0 \).

Let \( \overline{A} := A \otimes_{\mathbb{F}_q} \overline{k} \) and \( \overline{D} \) likewise. To calculate \( H^i_c(\overline{A}, j_*\mathcal{F}) \), the perfect pairing
\[ H^i_c(\overline{A}, j_*\mathcal{F}) \otimes H^{2-i}(\overline{A}, j_*\mathcal{F}) \rightarrow \mathbb{Q}_\ell(-1) \]
and Artin’s theorem give $H^0_c(\overline{A}, j_*\mathcal{F}) \approx 0$ and $H^2_c(\overline{A}, j_*\mathcal{F}) \approx M(-1)$, as $H^0(\overline{A}, j_*\mathcal{F}) = \bar{M}$. The exact sequence

$$0 = H^0_c(\overline{A}, j_*\mathcal{F}) \to H^0(\overline{D}, j_*\mathcal{F}) \to \mathcal{F}_\mathbb{F} \overset{\partial}{\to} H^1_c(\overline{A}, j_*\mathcal{F}) \to H^1(\overline{D}, j_*\mathcal{F}) \to 0$$

reduces the calculation of $H^1_c(\overline{A}, j_*\mathcal{F})$ to that of $H^1(\overline{D}, j_*\mathcal{F})$, as $\partial = 0$. Relative purity (Arcata V 3.4) gives that $j_*\mathcal{F}$ is constant on $\overline{A}$, and the standard computation $H^1(\overline{X}, \mathbb{Z}/\ell^n(1)) \approx \text{Pic}(\overline{X}/\overline{k})_{\ell^n}$ for a smooth connected curve $\overline{X}$ over $\overline{k}$ shows that $H^1(\overline{D}, \mathbb{Z}/\ell^n) = 0$ and hence the same is true for $H^1(\overline{D}, j_*\mathcal{F})$ (Arcata III 3.1, IV 6.2).

The action of local inertia on a normal curve $S$ determines the net locus (and therefore the étale locus by SGA 1 I 9.11) of the normalization of $S$ in a finite separable extension of its function field in the following way. Let $\overline{\eta}$ be a geometric point centered on the generic point $\eta$ of $S$, $L$ a finite Galois extension of the function field of $S$, and $S'$ the normalization of $S$ in $L$. Let $s$ be a geometric point centered on a closed point of $S$, $S(s)$ and $S(\overline{s})$ the spectra of the henselization and strict henselization, respectively, of the local ring of $S$ at $s$. Both are henselian traits. Let $S'(s) := S' \times_S S(s)$; $S'(s) \to S(s)$ is generically étale. As $S(s)$ is a henselian local ring, $S'(s)$ splits as a disjoint union of henselian local rings $S'(s)'$, indexed by $\xi \in \text{Spec } L \otimes_{k(\eta)} \overline{k}(\eta)$ (in bijection with the scheme-theoretic fiber of $S' \to S$ over $s$). Let $s'_i$ denote the corresponding closed point of $S'(s)'$ (and by abuse of notation the corresponding point in the scheme-theoretic fiber of $S' \to S$ over $s$). $S'(s)'$ is pro-étale over $S'$ and henselian, with the same residue field as $\mathcal{O}_{S', s'_i}$, therefore coincides with the henselization of $\mathcal{O}_{S', s'_i}$, hence is also normal. $S'(s)'$ finite and generically étale over $S(s)$, a fortiori integral. Hence $S'(s)'$ coincides with the normalization of $S(s)$ in $k(\xi)$. Considering $s'_i$ as a point in the scheme-theoretic fiber over $s$, $S' \to S$ is étale at $s'_i$ iff $S(s)' \to S(s)$ by faithfully flat descent (Stacks tag 02VN). Fixing some $i$ and letting $s' := s'_i, \xi := \xi_i$ we reduce to studying the normalization of the henselian trait $S(s)$ in a finite separable extension $k(\xi)$ of its function field. Let $\overline{\eta}_s$ be a geometric point centered on the generic point $\eta_s$ of $S(s)$. The local inertia at $s$ is defined by the exact sequence

$$e \to I_s \to \text{Gal}(\overline{\eta}/\eta_s) \to \text{Gal}(\overline{s}/s) \to e.$$

Compare the following lemma to [SGA 1] Exp. V §2.
Lemma. — \( S_{(s')} \rightarrow S_{(s)} \) is étale iff \( I_s \) acts trivially on \( S_{(s')} \).

Proof. — ‘⇒’ Arcata IV 2.2. ‘⇐’ \( k(\xi) \) is a finite separable extension of \( k(\eta_s) \); let \( K \) denote the Galois closure of \( k(\xi) \) in \( k(\eta) \), \( S_{(s')} = \text{Spec } A, S_{(s')} = \text{Spec } A' \), and \( B \) the normalization of \( A \) in \( K \). \( A \subset (A')^{I_s} = A' \subset B^{I_s} \) and \( A \subset B^{I_s} \) is étale (Stacks tag 09EH), so \( S_{(s')} \rightarrow S_{(s)} \) is étale.

A question that should be easy is: describe the category of lisse sheaves on a normal connected curve in terms of the Galois group of the function field. More precisely, given a lisse sheaf on an open subscheme \( S \) of a smooth complete curve \( \overline{S} \), describe the locus of \( S \) over which the sheaf can be extended to a lisse sheaf. In the case of finite coefficients, the answer goes like this. Let \( \eta \) be a geometric point centered on the generic point \( \eta \) of the curve \( S \). A l.c.c. sheaf \( \mathcal{F} \) on \( S \) is the same as a revêtement étale \( X \to S \); i.e. an open subgroup of \( \pi_1(S, \eta) \). \( S \) in turns corresponds to a closed subgroup \( Q \) of \( \text{Gal}(\eta/\eta) \) via the exact sequence

\[
e \to Q \to \text{Gal}(\eta/\eta) \to \pi_1(S, \eta) \to e.
\]

To each point \( s \in S \) we attach an inertia subgroup \( I_s \) which is a subgroup of \( \text{Gal}(\eta_s/\eta) \), where \( \eta_s \) is the generic point of \( S_{(s)} \). As \( S_{(s)} \) is a projective limit of revêtements étales of \( S \), \( I_s \) embeds into \( \text{Gal}(\eta/\eta) \) via

\[
I_s \subset \text{Gal}(\eta_s/\eta) \subset \text{Gal}(\eta/\eta).
\]

We must have that each \( I_s \) be contained in \( Q \), since as we saw above, the map \( \text{Gal}(\eta_s/\eta) \to \text{Gal}(\overline{S}/s) \) factors through the projection \( \text{Gal}(\eta_s/\eta) \to \pi_1(S, \eta) \). On the other hand, there may be other points in \( \overline{S} \) – \( S \) with nontrivial monodromy; i.e. the corresponding inertia acts nontrivially. This means that the direct image along \( S \to \overline{S} \) of the sheaf on \( S \) represented by \( X \) is not locally constant on a neighborhood of such a point. Geometrically, we can take the normalization \( X' \) of \( \overline{S} \) in the function field of \( X \); \( X' \times_S \overline{S} \cong X \) (SGA 1 I 10.2). \( X' \) is étale at a point if it is net there (SGA 1 I 9.11), so \( \mathcal{F} \) extends to a lisse sheaf over \( U \to \overline{S} \) iff \( X' \times_S U \to U \) is net. By the lemma above, this is true iff the inertia at each point \( u \in U \) acts trivially on the \( \text{Gal}(\eta/\eta_u) \)-representation corresponding to the sheaf on \( \overline{S}_{(u)} \) represented by \( X' \times_S \overline{S}_{(u)} \). Properly said, \( I_u \) acts on the fiber of this sheaf at \( \eta \). If \( I_u \) acts trivially, then \( X' \) is net in the fiber over \( u \) and \( X \) extends (via \( X' \)) to a revêtement
étale of $S \cup \{u\}$. (By the above lemma, is isomorphic to the direct image of $F$ under $S \hookrightarrow S \cup \{u\}$.) Thinking about $I_u$ as a subgroup of $\text{Gal}(\overline{\eta}, \eta)$, the condition that $I_u$ act trivially on $X' \times_S \overline{S}_{(u)}$ is equivalent to that $I_u$ act trivially on the $\text{Gal}(\overline{\eta}/\eta)$-representation $\mathcal{F}_{\overline{\eta}}$. Therefore, the locus of $\overline{S}$ over which $\mathcal{F}$ can be extended to a lisse sheaf coincides with the union of the $s \in \overline{S}$ such that $I_s$ acts trivially on $\mathcal{F}_{\overline{\eta}}$. This justifies the

Proposition. — Let $S$ be a normal connected curve with generic point $\eta$ and $\mathcal{C}$ the category with objects pairs $(\mathcal{F}, U)$ with $U$ a nonempty open of $S$ and $\mathcal{F}$ a l.c.c. sheaf on $U$, modulo the equivalence relation $(\mathcal{F}, U) \sim (\mathcal{G}, V)$ if $\mathcal{F}|U \cap V \cong \mathcal{G}|U \cap V$, and morphisms

$$\text{Hom}_\mathcal{C}((\mathcal{F}, U) \to (\mathcal{G}, V)) = \lim_{W \subset U \cap V} \text{Hom}(\mathcal{F}|W, \mathcal{G}|W),$$

limit taken over nonempty opens $W$ contained in $U \cap V$, $\mathcal{C}$ is equivalent to the category of finite separable extensions of the function field $k(\eta)$ of $S$ and $k(\eta)$-algebra morphisms. By Grothendieck’s Galois theory, this category is in turn equivalent to the category of finite sets with continuous $\text{Gal}(\overline{\eta}/\eta)$ action. Given such an extension $L$ of $k(\eta)$, the normalization $X$ of $S$ in $L$ is étale over a nonempty open $U \subset S$ and represents a l.c.c. sheaf on $U$. The maximal $U \subset S$ such that $X \times_S U \to U$ is étale coincides with the open subscheme of $S$ with closed points

$$|U| = \{s \in S : I_s \text{ acts trivially on } L\}$$

where here $I_s$ acts via $I_s \subset \text{Gal}(\overline{\eta}/\eta_s) \subset \text{Gal}(\overline{\eta}/\eta)$, $\eta_s$ the generic point of $S(\mathbf{s})$. If $\mathcal{F}$ is the sheaf of local sections for $X$ over this $U$, the pair $(\mathcal{F}, U)$ is distinguished in its class by the property that $U$ is maximal for the filtered partial order given by inclusion. Given $j : V \hookrightarrow U$ and $(\mathcal{G}, V) \sim (\mathcal{F}, U)$, $\mathcal{F} \to j_* \mathcal{G}$.

Corollary. — Given a lisse $(\mathbf{Z}_\ell, \mathbf{Q}_\ell, \mathbf{R}, E_\lambda, \mathbf{Q}_\ell)$-sheaf $\mathcal{F}$ on a nonempty open subscheme $U$ of a normal connected curve $S$ with geometric generic point $\eta$, the maximal locus $U' \subset S$ over which $\mathcal{F}$ extends to a lisse sheaf $\mathcal{F}'$ is defined by its set of closed points

$$|U'| = \{s \in S : I_s \text{ acts trivially on } \mathcal{F}_{\overline{\eta}}\}.$$ 

If $j : U \hookrightarrow U'$ denotes the open immersion, $\mathcal{F}' \to j_* \mathcal{F}$. 
Proof. — Disregarding the module structure, the sheaf $\mathcal{F}$ is represented by a projective system of revêtements étales, to each of which the proposition compatibly applies. □

Corollary. — Let $\mathcal{F}$ be a lisse sheaf on a normal connected curve $S$ and $j : U \hookrightarrow S$ an open immersion. The unit of adjunction $\mathcal{F} \to j_*j^*\mathcal{F}$ is an isomorphism.

Corollary. — Let $A \in \{\mathbb{Z}_\ell, \mathbb{Q}_\ell, \mathbb{R}, E_\lambda, \overline{\mathbb{Q}_\ell}\}$. The category $\mathcal{C}$ as above with l.c.c. sheaves replaced by lisse $A$-sheaves on a normal connected curve $S$ is equivalent to the full subcategory of $A$-modules of finite type with continuous action of $\text{Gal}(\overline{\eta}/\eta)$ generated by those $A$-modules $V$ with the property that $I_s$ acts trivially on $V$ for all but finitely many $s \in |S|$. 

Proof. — By SGA 5 Exp. VI (c.f. note to Sommes trig. 1.2), such a sheaf is equivalent the data of its monodromy representation, which is equivalent to a projective system of representations. Each representation in the projective system corresponds to a revêtement étale of some nonempty open $U \subset S$ together with the appropriate module structure on its fiber. By the proposition, each revêtement étale is unramified over an open set which can be calculated from the action of the local inertia. In order that there exist a nonempty open $U \subset S$ over which all the revêtements étales in the projective system are unramified (so that the corresponding projective system defines a lisse sheaf on $U$) it is necessary and sufficient that the local inertia at all but finitely many points act trivially on all the revêtements étales in the projective system. This is true for some $I_s$ iff $I_s$ acts trivially on the fiber of the lisse sheaf considered as $A$-module. □

Corollary. — The kernel of the map

$$\text{Gal}(\overline{\eta}/\eta) \to \pi_1(U, \overline{\eta})$$

is topologically generated by the subgroups $\{I_s : s \in |U|\}$.

Proof. — On one hand, the category of l.c.c. sheaves on $U$ is a Galois category with group $\pi_1(U, \overline{\eta})$. On the other, it is equivalent to the category of finite $\text{Gal}(\overline{\eta}/\eta)$-sets on which $I_s$ acts trivially for all $s \in U$. □

Remark. The same argument can be used to extend a lisse sheaf on a normal scheme over the generic point of a divisor. If the scheme is moreover smooth, the sheaf can be
extended over the points of codimension > 1 by the purity theorem of Zariski-Nagata (SGAA XVI 3.3).

**Remark.** It is a simple exercise to see that if \( g : \eta \rightarrow X \) is the inclusion of the generic point to a normal integral scheme \( X \), and \( M_\eta \) is a constant sheaf on \( \eta \) with constant value \( M \), then \( g_*M \) is likewise constant. Let \( M_X \) denote the constant sheaf on \( X \) with constant value \( M \). We need to show

\[
M_X \rightarrow g_*g^*M_X = g_*M_\eta.
\]

is an isomorphism. Fix a geometric point \( x \) of \( X \) and let \( X_{(x)} \) denote the (spectrum of the) strict henselization of \( X \) at \( x \); \( X_{(x)} \) is a normal domain as \( X \) is normal. The map on stalks is

\[
M \rightarrow \Gamma(\eta \times_X X_{(x)}, M_\eta).
\]

\( \eta_x := \eta \times_X X_{(x)} \) is the spectrum of the field of fractions of the strict henselization of \( X \) at \( x \), which is separable algebraic over \( k(\eta) \). Letting \( I_x := \text{Gal}(\eta/\eta_x) \),

\[
\Gamma(\eta \times_X X_{(x)}, M_\eta) = M^{I_x},
\]

where here we have identified \( M \) with the stalk of \( M_\eta \) at a geometric point centered on \( \eta \). So the condition that the map (†) be an isomorphism at the geometric point \( x \) is the same as \( I_x \) acting trivially on \( M \), considered as \( \text{Gal}(\eta/\eta_x) \)-module. In particular, this condition is satisfied by the constant sheaf, which corresponds to \( M \) with trivial action of Galois.

(We implicitly use here the description of étale morphism with normal integral target in EGA IV 18.10.7.)

**4.4.** The short exact sequence in the second paragraph is the exact sequence of the first paragraph of (2.3.2) in light of the fact that \( j'_*F' = j'_*F' \) is concentrated in degree \(-1\). The computation

\[
\mathcal{H}^{-1}(\mathcal{F}(j_*F[1])_{\eta'}) = H^1_c(A \otimes_{F_q} \overline{k}, j_*F)
\]
follows from proper base change along the closed immersion \(0' \hookrightarrow A'\) applied to either (1.4.1.1), yielding
\[ \mathcal{H}^{-1}(\mathcal{F}(j_*F[1])_{0'}) = H^1(\alpha, j_*F \otimes \mathcal{L}(x.0)) = H^1_c(A \times_{F_q} \bar{0}, j_*F) \]
or simply from (1.3.1.1) directly, yielding
\[ \mathcal{H}^{-1}(\mathcal{F}(j_*F[1])_{0'}) = \mathcal{H}^{-1}(R pr'_!(pr^* j_*F[1] \otimes \mathcal{L}(x.x'))_{0'}[1]) = H^1_c(A \times_{F_q} \bar{0}, j_*F \otimes \mathcal{L}(x.0)) = H^1_c(A \times_{F_q} \bar{k}, j_*F). \]

In any event, this \(G_{0'}\)-module coincides with the stalk of \(j'_*F'\) at \(0'\), which is what is expressed by the equation
\[ (F'_{\eta_{0'}})^{0'} = H^1_c(A \times_{F_q} \bar{k}, j_*F). \]

For the rightmost term of the short exact sequence, \(I_{0'}\) acts trivially on \(F_{\mathbb{F}}\) as \(F\) is by assumption unramified at infinity. Let \(\pi : \text{Spec } k\{\pi\} := T \to D\) and \(\pi' : \text{Spec } k\{\pi'\} := T' \to A_{(0')}\) be defined by \(\pi \mapsto 1/x\) and \(\pi' \mapsto x'\). We have
\[ R^{-1}\Phi_{\eta_{0'}}(j'_*F') = R^{-1}\Phi_{T_{0'}}(\pi, j_*F \otimes \mathcal{L}(x.x')[1])_{\mathbb{F}}(\mathbb{F}) = (\pi \times \pi')^* R^{-1}\Phi_{T_{0'}}(\pi, j_*F \otimes \mathcal{L}(x.x')[1]) = R^1\Phi_{T_{0'}} pr^*(F_{\mathbb{F}}) \otimes \mathcal{L}(\pi'/\pi)_{\mathbb{F}} \mathbb{F}^1(\mathbb{F}) (\mathbb{F}) = F_{\mathbb{F}} \otimes \mathcal{F}(\mathbb{F}) = F_{\mathbb{F}}(-1). \]

The second-to-last equality is true because \(F\) is assumed unramified at infinity, so that the action of \(G_{\infty} := \text{Gal}(\bar{\eta}_{0'}/\eta_{0'})\) on \(F_{\mathbb{F}}\) factors through \(G_{\infty} \twoheadrightarrow G_{\infty}/I_{\infty} = \text{Gal}(F/F_q)\), which is procyclic generated by Frobenius, and by Schur’s lemma \(F_{\mathbb{F}}\) splits as a direct sum of 1-dimensional torsion sheaves (in the sense of [Weil II, 1.2.7]) so that we may assume \(F_{\mathbb{F}} \cong \overline{Q}_\ell(b)\) for \(b \in Q_\ell\) an \(\ell\)-adic unit, which is the reciprocal image of a sheaf on \(\text{Spec } F_p\). As twisting by such a sheaf induces an exact autoequivalence of the category (or derived category) of \(\overline{Q}_\ell\)-sheaves, we find that \(\overline{Q}_\ell(b) \otimes \mathcal{F}(\mathbb{F}) = \mathcal{F}(\mathbb{F})(\overline{Q}_\ell(b) \otimes V)\) for any \(G_{\infty}\)-module \(V\).
A few words on the application of the Deligne lemmata: it is used here that $F$ is not ramified at infinity, so that if $\alpha$ denotes simultaneously $A \hookrightarrow D$ and $U \hookrightarrow U \cup \{\infty\}$ (depending on the context), $\alpha_* F$ is a lisse sheaf on $U \cup \{\infty\}$. This is not necessary to conclude by (4.2.2.1) that for each eigenvalue of Frobenius $\beta$ on $F_{\overline{\omega}}$,

$$w_q(\beta) \leq w,$$

but it is necessary to connect the corresponding statement for $F^\vee$ to that for $F$ in the following way. Namely, we find that for each eigenvalue of Frobenius $\gamma$ on $(F^\vee)_{\overline{\omega}}$,

$$w_q(\gamma) \leq -w,$$

and as $\alpha_* F$ and $\alpha_* F^\vee$ are both lisse, $(\alpha_* F)^\vee = \alpha_*(F^\vee)$ and

$$(F^\vee)_{\overline{\omega}} := (\alpha_* F^\vee)_{\overline{\omega}} = (\alpha_* \text{Hom}(F, \overline{Q}_\ell))_{\overline{\omega}} = \text{Hom}(\alpha_* F, \overline{Q}_\ell)_{\overline{\omega}} = \text{Hom}(F_{\overline{\omega}}, \overline{Q}_\ell) = (F_{\overline{\omega}})^\vee,$$

which allows us to conclude from $w_{N(\infty)}(\gamma) \leq -w$ that

$$w_q(\beta) \geq w.$$

This shows in our application that $F'$ is $t$-pure of $t$-weight $w$ and (4.2.2.1) gives that for each eigenvalue $\nu$ of $F_0'$ on $(j'_* F')_{\overline{\omega}} = H^1_c(A \times \overline{Q}_\ell, j'_* F)$,

$$w_q(\nu) \leq w + 1.$$

In the effort to show that $F'$ is $t$-pure, since $F$ is unramified at infinity, the $P_{\infty}$-module $(F \oplus \tilde{F}(-w))_{\overline{\infty}}$ is trivial; i.e. purely of slope 0, so that $\mathcal{F}_q(j_*(F \oplus \tilde{F}(-w))[1])$ is of the form $j'_i G'_i[1]$ and $\mathcal{F}_{q^{-1}}(j_*(F \oplus \tilde{F}(-w))[1])$ is of the form $j'_i G'_2[1]$ and their direct sum is of the form $j'_i G'[1]$ for the lisse $Q_\ell$-sheaf $G' := G'_1 \oplus G'_2$ on $A' - \{0\}$.

For the coup de grâce, it would be better to refer to (1.1.1), especially (1.1.1.3), and (1.2.1.2) than to (3.1). The point is that we can follow Rapport sur la formule des traces (1.6) and (3.1) (with $t^f$ replaced by $t$ so that degree is measured over $F_q$, not $F_p$, and shifting by $[1]$ to cancel the shift of $[1]$ on $G$ and in the definition of $\mathcal{F}$) to find that

$$\det(1 - t F^*_{x'}, G')$$

$$= \det(1 - t F^*_{x'}, H^0_c(A \times \overline{Q}_\ell, j'_* (F \oplus \tilde{F}(-w)) \otimes \mathcal{L}_q(x.x')[1]))$$

$$\times \det(1 - t F^*_{x'}, H^0_c(A \times \overline{Q}_\ell, j_* (F \oplus \tilde{F}(-w)) \otimes \mathcal{L}_{q^{-1}}(x.x')[1]))$$
Now the point is that as $G'$ is concentrated in degree zero, letting $\mathcal{L} = \mathcal{L}_\psi$ or $\mathcal{L}_\psi^{-1}$,

$$\mathrm{R}\Gamma(A \times_{F_q} \bar{x}', j_*(F \oplus \bar{\mathcal{F}}(-w)) \otimes \mathcal{L}(x.x')) = H^1_c(A \times_{F_q} \bar{x}', j_*(F \oplus \bar{\mathcal{F}}(-w)) \otimes \mathcal{L}(x.x'))[-1].$$

Grothendieck’s trace formula gives therefore that $\iota \mathrm{det}(1 - t F^*_x, G')$ can be computed as stated in terms of the polynomials $Q_{x,x'}$ in light of \textit{[Sommes trig., 1.7.6 & 1.7.7]} (compare (1.1.1.2) and (1.1.3.3)), which allow us to write for each $x \in |A \times_k x'|$

$$F'_x|(j_*(F \oplus \bar{\mathcal{F}}(-w)) \otimes \mathcal{L}_\psi(x.x'))_\pi = \psi(\mathrm{Tr}_{k(x)/F}(x.x')) F'_x|(j_*(F \oplus \bar{\mathcal{F}}(-w))_\pi).$$

If $P(t) \in \mathbb{R}[t]$ and $\alpha = e^{i\theta}$, then $P(\alpha t)P(\alpha^{-1}t) \in \mathbb{R}[t]$ as well, since $P(\alpha t)P(\overline{\alpha}t)$ is fixed by complex conjugation. Q.E.D.
Bibliography

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7. Théorèmes de finitude en cohomologie $\ell$-adique

1.3. Under the hypothesis of 1.1, $Rf_*$ is of finite cohomological dimension. In light of the description of the fiber of $f_*$ (Arcata 3.3), we may assume we are studying the cohomology of a scheme $X$ of finite type over $S$ the spectrum of a separably closed field (which has cohomological dimension 0) or a strictly henselian local DVR. SGAA X 4.2 proves this, but we have to satisfy two conditions. The condition (i) can be deduced from SGAA X 2.1, 2.3, which give bounds on the cohomological dimension of residue fields of points of $X$ depending on which points of $S$ they sit over; the condition (ii) follows from SGAA X 3.2, which gives a bound on the cohomological dimension of the function field of strict localizations of points of $X$.

1.6. Given a morphism of schemes $f : X \to Y$ and sheaves $\mathcal{F}, \mathcal{G}$ on $Y$, there is always a morphism

$$f^* \text{Hom}(\mathcal{F}, \mathcal{G}) \to \text{Hom}(f^*\mathcal{F}, f^*\mathcal{G}).$$

When $\mathcal{F}$ is locally constant constructible, this map is a bijection.

If $\mathcal{I}$ is an injective abelian sheaf on $X$ and $j : U \to X$ belongs to the topology on $X$, then $\mathcal{I}(U)$ is an injective object, since if $\underline{A}$ is the constant sheaf on $U$ with value $A$,

$$\text{Hom}(\underline{A}, \mathcal{I}(U)) = \text{Hom}(j_!\underline{A}, \mathcal{I}),$$

and $j_!$ is exact. The restriction $\mathcal{I}_U$ is also an injective sheaf, since for $\mathcal{F}$ on $U$

$$\text{Hom}(\mathcal{F}, \mathcal{I}_U) = \text{Hom}(j_!\mathcal{F}, \mathcal{I}).$$

Likewise, for any (geometric) point $x \in X$, $\mathcal{I}_x$ is an injective object, from the general fact that a inductive limit of injective modules over a noetherian ring is injective: this follows from Baer’s criterion in an obvious way and holds not only for injective abelian sheaves but also for injective sheaves of $\Lambda$-modules for $\Lambda$ a noetherian ring.

All this to say that $R\text{Hom}(\mathcal{F}, \mathcal{G})|_U = R\text{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$, and when $\mathcal{F}$ is l.c.c., $R\text{Hom}(\mathcal{F}, \mathcal{G})|_x = R\text{Hom}(\mathcal{F}|_x, \mathcal{G}|_x)$, and $R \text{Hom}(\mathcal{F}, \mathcal{G})$ is constructible (resp. l.c.c.) when $\mathcal{G}$ is. This also shows that for $\mathcal{F}$ constant, one can compute $R\text{Hom}$ by a projective resolution of the constant value of $\mathcal{F}$. 
1.11. $R\text{pr}_2_*\text{pr}_1^*K = b^*R\Gamma(X,K)$ is base change for $Ra_*$ along $b$. In the string of equalities, the first one is just Leray, and the second and last are again base change morphisms, discussed at length in SGAA XVII §4. Given

\[
\begin{array}{c}
X' \xrightarrow{g'} X \\
\downarrow f' \quad \downarrow f \\
S' \xrightarrow{g} S,
\end{array}
\]

at the level of complexes, the map $L \otimes g^*f_*K \to f'_*(f''^*L \otimes g''^*K)$ factors as

\[
L \otimes g^*f_*K \to L \otimes f'_*g''^*K \to f'_*f''^*L \otimes f'_*g''^*K \to f'_*(f''^*L \otimes g''^*K),
\]

where the last arrow comes from the fact that $F(U) \otimes G(U)$ are among the sections of $F \otimes G$ over $U$.

2.1. It is claimed that $R\text{Hom}(F, \mathbb{Z}/m) \leftarrow \text{Hom}(F, \mathbb{Z}/m)$ when $F$ is a l.c.c. sheaf (c.f. SGAA XVIII 3.2.6). There is evidently an arrow, and by the note to 1.6 (and since $\mathbb{Z}/m$ is an injective $\mathbb{Z}/m$-module), it is an isomorphism. It holds more generally with $\mathbb{Z}/m$ replaced by any l.c.c. sheaf locally isomorphic to $\mathbb{Z}/m$, such as one obtained by twisting.
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8. The derived category of perverse sheaves

1.1. The condition that \( k \) be algebraically closed when \( \text{D}(X) = \text{D}^b_c(X, \mathbf{Q}_\ell) \) is at first to secure \([BBD], 2.2.14\), and later to secure the existence of good neighborhoods.

2.1. The category \( \text{D}(\eta) \) coincides with the Verdier quotient of \( \text{D}(X) \) by the thick subcategory generated by the complexes acyclic at \( \eta \).

2.1.1. In \( \dim X = 0 \), the statement is clear as the abelian category of \( \mathbf{Q}_\ell \)-sheaves on such an \( X \) is semisimple.

To find the neighborhood \( U \), we proceed as follows: first shrink \( U \) so that \( M_U, N_U \) are lisse, and \( U \) is integral and contained in the smooth locus (\( X \) is reduced by assumption). By Artin’s good neighborhood theorem (\( k \) is algebraically closed), we find a \( Z \) smooth over \( k \) and a \( k \)-morphism \( f : U \to Z \) with geometrically irreducible fibers of dimension 1 (\( f \) is smooth of relative dimension 1 in the sense of Hartshorne, recalling EGA IV 17.8.2 or SGA 1 II 2.1). In particular, \( Z \) is regular and, as \( f(U) = Z \), irreducible hence integral. Replace \( Z \) by an affine open \( \text{Spec} A \) contained in it and \( U \) by an affine open \( \text{Spec} B \) contained in the preimage; the only thing that changes is that some fibers of \( f \) may now be empty; if a fiber is nonempty, it is still geometrically irreducible of dimension 1. As \( f \) is still smooth, hence in particular open, there is some \( a \in A \) with the property that \( \text{D}(a) \subset f(U) \subset Z \) and that the restriction of \( R^q f_* \text{Hom}(M_u, N_U) \) to \( \text{D}(a) \) is lisse for all \( q \). As \( f \) is dominant, the induced ring homomorphism \( A \to B \) is injective, the map of spectra \( \mathcal{U} : \text{Spec} B_a \to \text{Spec} A_a \) is surjective, and the \( R^q \mathcal{U}_* \text{Hom}(M_u, N_U) \) are lisse. Set \( U = \text{Spec} B_a, Z = \text{Spec} A_a; \mathcal{U} : U \to Z \) now has all the desired properties; it is smooth, affine, with 1-dimensional fibers (which are moreover geometrically irreducible), \( Z \) is smooth over \( k \) hence regular, and the \( L^q \) are lisse sheaves on \( Z \).

Now to see that \( L^q = 0 \) unless \( q = 0 \) or 1. The argument below works for finite coefficients; we let \( R \) be the ring of integers of a finite extension \( E_\ell \) of \( \mathbf{Q}_\ell \) with maximal ideal \( m \) and note that \( R/m^n \) is injective as a module over itself by Baer’s criterion. Let \( \Lambda := R/m^n \) and we consider constructible sheaves of \( \Lambda \)-modules. Let \( \mathcal{J} \) be any l.c.c.
8. THE DERIVED CATEGORY OF PERVERSE SHEAVES

sheaf of $\Lambda$-modules on $U$; we wish to show that $R^q\mathbb{I}\mathcal{J} = 0$ unless $q = 0, 1$. En effet,

$$R^q\mathbb{I}\mathcal{J}[2] \simeq R^q R\text{Hom}(\mathcal{J}^\vee(1), \Lambda(1)[2]) \sim R\text{Hom}(R^q(\mathcal{J}^\vee(1)), \Lambda)$$

$$= \text{Hom}(R^q(\mathcal{J}^\vee(1)), \Lambda)$$

and it reduces to showing that $R^q(\mathcal{J}^\vee(1))$ is concentrated in degrees 1 and 2, which will follow from proper base change from the corresponding fiberwise statement. In other words, we must show that given a l.c.c. sheaf $F$ on a smooth affine curve $Y$ over an algebraically closed field $k$, $H^i_c(Y, F)$ unless $i = 1, 2$. This follows immediately from Poincaré duality on $Y$

$$H^{2-i}(Y, F^\vee(1)) \sim H^i_c(Y, F)^\vee$$

and Artin’s theorem.

The abutment of the Leray spectral sequence

$$H^p(Z, L^q) \Rightarrow H^{p+q}(U, \text{Hom}(M_U, N_U)) \simeq H^{p+q}R\text{Hom}(\Omega^1, \text{Hom}(M_U, N_U))$$

is naturally isomorphic to

$$\text{Ext}^{p+q}_{D(U)}(M_U, N_U) \simeq H^{p+q}R\text{Hom}(\Omega^1, \text{RHom}(M_U, N_U))$$

because as $M_U$ is a lisse $\Omega^1$-sheaf, $\text{Hom}(M_U, -)$ is an exact functor. To see that $\text{Ext}^i(M_U, N_U) = 0$ for $i > 0$, note that since $M_U$ is lisse, for any geometric point $x$ of $U$, $\text{Hom}(M_U, N_U)_x = \text{Hom}((M_U)_x, (N_U)_x)$ and likewise for $R\text{Hom}$ (c.f. note to Th. fin. 1.6), so that $R\text{Hom}(M_U, N_U)$ is connective iff $\text{Hom}((M_U)_x, (N_U)_x)$ is, and we can compute the latter via a projective resolution of $(M_U)_x$. But $M_U$ is a lisse $\Omega^1$-sheaf, so that we can represent it on the level of a finite coefficient ring $\Lambda$ by a torsion-free sheaf, in which case $(M_U)_x$ is a projective $\Lambda$-module.

2.1.2. On the definition of the arrows

$$H^p(Z, R^0\mathbb{I}\mathbb{H}(M_U, N_U)) \rightarrow H^p(Y', R^0\mathbb{I}_{Y*}(\mathbb{H}(M_{U'}, Q_{U'}))) :$$
let \( j : Y' \hookrightarrow Z \) denote the immersion and its base extensions; then

\[
\begin{align*}
R\Gamma(Z, R^0j_\ast \hom(M_U, N_U)) & \sim R\Gamma(Z, j_\ast j^\ast R^0\hom(M_U, N_U)) \\
& \sim R\Gamma(Z, i_\ast i^\ast R^0\hom(M_U, N_U)) \\
& \sim R\Gamma(Z, j_\ast R^0\hom(M_{U_Y}, N_{U_Y})) \\
& \sim R\Gamma(Z, j_\ast Q_{U_Y}) \\
& \sim R\Gamma(Y', R^0\hom(M_{U_Y}, Q_{U_Y})) \\
& = R\Gamma(Y', R^0\hom(M_{U_Y}, Q_{U_Y}))
\end{align*}
\]

where the first arrow is (inconsequentially) an isomorphism since \( L^0 \) is lisse. (Of course \( j^\ast R^0\hom = H^0j^\ast R^0\hom \sim H^0R\hom j^\ast = R^0\hom j^\ast. \))

**Proof of (2.1.2) \Rightarrow (2.1.1).** In the proof, (2.1.2 b) is invoked with \( Z \) and replaced by \( Y \) and sheaves \( M_{U_Y} \) and \( P_{U_Y} \), where \( P_{U_Y} \) is lisse on \( U_Y \) but the condition that the \( R^q\hom(Y, M_{U_Y} \otimes P_{U_Y}) \) be lisse on \( Y \) is not secured by (2.1.2 a). The condition that the \( L^q \) be lisse is used in the proof of (2.1.2 b). Therefore (2.1.2 a) should say

(2.1.2 a) There exists an open \( Y \subset Z \), a lisse \( P_{U_Y} \) in \( M(U_Y) \) and an injection \( N_{U_Y} \hookrightarrow P_{U_Y} \) such that the sheaves \( R^q\hom(Y, M_{U_Y} \otimes P_{U_Y}) \) are lisse and the corresponding arrow \( R^1\hom(M_{U_Y}, N_{U_Y}) \rightarrow R^1\hom(M_{U_Y}, P_{U_Y}) \) is null.

Of course, the original (2.1.2 a) implies this statement.

As for the proof of (2.1.2) \Rightarrow (2.1.1), one must observe that restriction from \( U \) to \( U_Y \) and the arrow \( N_{U_Y} \hookrightarrow P_{U_Y} \) together induce a morphism of spectral sequences which is the zero map on \( H^{i-1}(Z, R^1\hom(M_U, N_U)) \rightarrow H^{i-1}(Y, R^1\hom(M_{U_Y}, P_{U_Y})) \) (this arrow is defined as the one in (2.1.2 b)), implying the first statement as the image of the \( E_2 \) page has

\[
\im(H^i(U, R^0\hom(M_U, N_U)) \rightarrow H^i(Y, R^0\hom(M_{U_Y}, P_{U_Y})))
\]

as the only nonzero term of the \( i \)th diagonal, so that its image in the abutment, which coincides with its image on the \( E_3 \) page, which coincides with its image in \( H^i(Y, R^0\hom(M_{U_Y}, P_{U_Y}))/d_2(H^{i-2}(Y, R^0\hom(M_{U_Y}, P_{U_Y}))) \) coincides with the image of \( \Ext^i_{D(U)}(M_U, N_U) \) in \( \Ext^i_{D(U_Y)}(M_{U_Y}, P_{U_Y}) \). This is in turn killed by (2.1.2 b) for \( i > 0 \).
Proof of 2.1.2 a.

\[ L^1 \otimes L^1 = \text{Hom}(L^1, L^1) = \text{Hom}(L^1, R^1 \mathcal{H}om(M, N)) \]

\[ \simeq H^1 \text{Hom}(L^1, R^1 \mathcal{H}om(M, N)) \simeq R^1 \mathcal{H}om(\mathcal{O}^* L^1, \text{Hom}(M, N)) \]

\[ \simeq R^1 \mathcal{H}om(\mathcal{O}^* L^1, M, N) \]

where the first isomorphism on the second line holds because \( L^1 \) is a lisse \( \mathcal{O}_\ell \)-sheaf so that \( \text{Hom}(L^1, -) \) is an exact functor, as discussed above.

The injection \( N \hookrightarrow P_{U_Y} \) induces the following map on \( L^1 \otimes L^1 \):

\[ L^1 \otimes L^1 \overset{\sim}{\to} R^1 \mathcal{H}om(\mathcal{O}^* L^1, M, N) \to R^1 \mathcal{H}om(\mathcal{O}^* L^1, M, P_{U_Y}); \quad \text{i.e.} \]

\[ \text{Hom}(L^1, L^1) = \text{Hom}(L^1, R^1 \mathcal{H}om(M, N)) \to \text{Hom}(L^1, R^1 \mathcal{H}om(M, P_{U_Y})). \]

If \( \alpha \) is the image of the global extension \( \tilde{\alpha} \), then the injection \( N \hookrightarrow P_{U_Y} \) defined by the extension of class \( \alpha \) will annihilate \( \tilde{\alpha} \) (c.f. note to BBD 3.1.17). As \( \alpha \) is the image of \( \tilde{\alpha} \) under a natural map (i.e. \( N \hookrightarrow P_{U_Y} \) induces a morphism of Leray spectral sequences), this injection likewise kills \( \alpha \). But \( L^1 \otimes L^1 = \text{Hom}(L^1, L^1) \), and under this isomorphism \( \alpha \mapsto \text{id} \), so that the global section \( \alpha \in H^0(Z, L^1 \otimes L^1) \) goes to zero precisely when the section \( \text{id} \in \text{Hom}(L^1, L^1) \) does; i.e. precisely when \( L^1 \) is annihilated.

\[ \text{Hom}(L^1, L^1) \ni \text{id} \mapsto L^1 = \text{Hom}(L^1, R^1 \mathcal{H}om(M, N)) \to \text{Hom}(L^1, R^1 \mathcal{H}om(M, P_{U_Y})) \]

The induction step is used to kill the obstruction in the following way: \( \dim Z = -1 + \dim X \) so the induction hypothesis gives \( Y \subset Z \) and \( \varphi : L^1_\gamma \hookrightarrow K_\gamma \) inducing zero on \( \text{Ext}^2_{\mathcal{D}(Z)}(L^{0*}, L^{1*}) \to \text{Ext}^2_{\mathcal{D}(Y)}(L^{0*}_\gamma, L^{1*}_\gamma). \) But

\[
\text{Ext}^2_{\mathcal{D}(Z)}(L^{0*}, L^{1*}) = H^2 R \text{Hom}(L^{0*}, L^{1*}) = H^2 R \text{Hom}(\mathcal{Q}_\ell, R \text{Hom}(L^{0*}, L^{1*}))
\]

\[
= H^2 R \text{Hom}(\mathcal{Q}_\ell, \text{Hom}(L^{0*}, L^{1*})) = H^2(Z, L^0 \otimes L^{1*}),
\]

and likewise on \( Y \).

Typos: it is clear from the above that

\[ K_\gamma \otimes L^1_\gamma \simeq R^1 \mathcal{H}om(\mathcal{O}^* K^*_\gamma \otimes M_{U_Y}, N_{U_Y}) \quad \text{and} \]

\[ \varphi(\alpha) \in \text{Ext}^1(\mathcal{O}^* K^*_\gamma \otimes M_{U_Y}, N_{U_Y}) \]
with $\mathbb{P}^* K_Y$ instead of $\mathbb{P} K_Y$ and $M_{U_Y}, N_{U_Y}$ in lieu of $M_U, N_U$. Finally, $P_{U_Y}$ is defined by an extension of $\mathbb{P}^* K_Y \otimes M_{U_Y}$, not $\mathbb{P} K_Y \otimes M_{U_Y}$, by $N_{U_Y}$.

The point is that, localizing about any point of the base $Y$, the sheaves $L^1$ and $K_Y$ become constant so that along any fiber, $L^1 \rightarrow K_Y$ admits a complement splitting $K_Y$. Therefore the monomorphism $N_{U_Y} \hookrightarrow P_{U_Y}$ killing $\varphi(\alpha)$ also kills $\alpha$ since it does so along every fiber, and it does so along every fiber because along any fiber there is a retraction to $\alpha \mapsto \varphi(\alpha)$.

Explanation of proof: our objective is to find an injection $N_{U_Y} \hookrightarrow P_{U_Y}$ such that the corresponding arrow $R^1 \mathbb{P} H^* (\mathbb{P} \text{Hom}(M_{U_Y}, N_{U_Y}) \rightarrow R^1 \mathbb{P} H^* (M_{U_Y}, P_{U_Y})$ is null. The first part of the proof is enough to construct such a $P_{U_Y}$ if we look over the strict henselization of $Y$ at a closed point $y \in Y$, because of course any lisse sheaf on $Y_y$ is constant so that if we set $Z = Y_y$, $\text{Ext}^1(\mathbb{P}^* L^1 \otimes M, N) = H^0(Y_y, L^1 \otimes L^1)$ since $R\Gamma(Y_y, -) = H^0(Y_y, -)$ is an exact functor so that
\[
\text{Ext}^1(\mathbb{P}^* L^1 \otimes M, N) = H^1 R\Gamma R\mathbb{P} H(\mathbb{P}^* L^1 \otimes M, N) \\
= R\Gamma H^1 \mathbb{P} H(\mathbb{P}^* L^1 \otimes M, N) = H^0(Y_y, R^1 \mathbb{P} H(\mathbb{P}^* L^1 \otimes M, N)) \\
= H^0(Y_y, L^1 \otimes L^1).
\]

Now the point is that even though it is simple to find a $P_{U_Y}$ looking over each closed fiber, the existence of a global $P_{U_Y}$ is obstructed by the possibly nonzero class $\partial(\alpha)$. Therefore the trick is to take an injection $\varphi : L^1_{Y_Y} \hookrightarrow K_Y$ into a lisse $K_Y$ on $Y \subset Z$ which kills $\partial(\alpha)$. Now, $\varphi(\alpha) \in H^0(Y, K_Y \otimes L_Y^1)$ is annihilated by $L_Y^1 \rightarrow R^1 \mathbb{P} H(M_{U_Y}, P_{U_Y})$ for the same reasons as before, but now the existence of $P_{U_Y}$ is guaranteed by design. The injection $\varphi$ induces an injection of global sections $H^0(Y, L^1_{Y_Y} \otimes L_Y^1) \hookrightarrow H^0(Y, K_Y \otimes L_Y^1)$.

$$
\begin{array}{ccc}
H^0(Y, L^1_{Y_Y} \otimes L_Y^1) & \xrightarrow{q \otimes \text{id}} & H^0(Y, K_Y \otimes L_Y^1) \\
\downarrow & & \downarrow \\
H^0(Y, L^1_{Y_Y} \otimes R^1 \mathbb{P} \text{Hom}(M_{U_Y}, P_{U_Y})) & \hookrightarrow & H^0(Y, K_Y \otimes R^1 \mathbb{P} \text{Hom}(M_{U_Y}, P_{U_Y}))
\end{array}
$$
As the rightmost vertical arrow kills $\varphi(\alpha)$, the leftmost vertical arrow must kill $\alpha$. As $\alpha$ corresponds to $\text{id}_{L^1} \in \text{Hom}(L^1_Y, L^1_Y)$, the fact that $\alpha$ goes to zero under \[ \text{Hom}(L^1_Y, L^1_Y) \rightarrow \text{Hom}(L^1_Y, R^1\mathcal{Y}^*\text{Hom}(M_{U_Y}, P_{U_Y})) \]
implies that the map $L^1_Y \rightarrow R^1\mathcal{Y}^*\text{Hom}(M_{U_Y}, P_{U_Y})$ is null.

**Proof of 2.1.2. b.** The $Q_{U_Y}$ of the statement of (2.1.2 b) is denoted $O_{U_Y}$ in the proof. About the cocartesian square: indeed pushouts exist in any topos, and if you like fancy words, any topos is an adhesive category, meaning that it has pullbacks and pushouts of monomorphisms, and pushout squares of monomorphisms are also pullback squares and are stable under pullback. In particular, $O_{U_Y}$ defined by a pushout, has $N_{U_Y} \leftarrow O_{U_Y}$ injective since $\mathcal{Y}^*L^0_Y \otimes M_{U_Y} \leftarrow \mathcal{Y}^*Q_{U_Y} \otimes M_{U_Y}$ is. The reason any Grothendieck topos is adhesive is as simple as you think it is, namely that the category of sets is adhesive and adhesivity is a condition on colimits and finite limits, hence preserved by functor categories and left-exact localizations (c.f. [nLab]).

The canonical arrow is given by composition

\[ \mathcal{Y}^*L^0_Y \otimes M_{U_Y} = \mathcal{Y}^*\mathcal{Y}^*Q^0_Y(M_{U_Y} \otimes N_{U_Y}) \otimes M_{U_Y} \rightarrow (M_{U_Y} \otimes N_{U_Y}) \otimes M_{U_Y} \rightarrow N_{U_Y}. \]

The pushout

\[
\begin{array}{ccc}
\mathcal{Y}^*L^0_Y \otimes M_{U_Y} & \leftarrow & \mathcal{Y}^*Q^0_Y \otimes M_{U_Y} \\
\downarrow & & \downarrow \\
N_{U_Y} & \leftarrow & O_{U_Y}
\end{array}
\]

defines by adjunction the commutative diagram

\[
\begin{array}{ccc}
\mathcal{Y}^*L^0_Y & \rightarrow & \mathcal{Y}^*Q^0_Y \\
\downarrow & & \downarrow \\
\text{Hom}(M_{U_Y}, N_{U_Y}) & \rightarrow & \text{Hom}(M_{U_Y}, O_{U_Y})
\end{array}
\]
and again by adjunction the commutative diagram

\[
\begin{array}{ccc}
L_Y^0 & \rightarrow & Q_Y^0 \\
\downarrow & & \downarrow \\
\mathbb{Y}_*\text{Hom}(M_{U_Y}, N_{U_Y}) & \rightarrow & \mathbb{Y}_*\text{Hom}(M_{U_Y}, O_{U_Y})
\end{array}
\]

Now to check that the leftmost vertical arrow in the diagram above is the identity it suffices to show that the identity is taken by adjunction (from the bottom diagram to the top) to the given arrow in the pushout diagram that defines \(O_{U_Y}\). The image of \(\text{id} : L_Y^0 \rightarrow \mathbb{Y}_*\text{Hom}(M_{U_Y}, N_{U_Y})\) under adjunction is given by the counit of the adjunction at the object \(\text{Hom}(M_{U_Y}, N_{U_Y})\)

\[
\varepsilon(\text{Hom}(M_{U_Y}, N_{U_Y})) : \mathbb{Y}_*\mathbb{Y}_*\text{Hom}(M_{U_Y}, N_{U_Y}) \rightarrow \text{Hom}(M_{U_Y}, N_{U_Y}).
\]

Now in general a morphism \(f : A \rightarrow \text{Hom}(B, C)\) is taken under tensor-hom adjunction to the morphism \(A \otimes B \rightarrow C\) which is the composition of \(f \otimes \text{id}\) followed by evaluation. This is exactly the definition of the canonical arrow above: \(\varepsilon(\text{Hom}(M_{U_Y}, N_{U_Y})) \otimes \text{id}\) followed by evaluation. For each \(i > 0\), applying the functor \(H^i(Y', -)\) to the last commutative diagram and precomposing with the arrow \(H^i(Z, L_Y^0) \rightarrow H^i(Y', L_Y^0)\) coming from the unit \(\eta(L^0) : L^0 \rightarrow \mathbb{Y}_*\mathbb{Y}_*L_Y^0 = \mathbb{Y}_*L_Y^0\) effaces \(H^i(Z, L_Y^0)\).

**Perversity.** All of the arguments above hold for any perversity.

**2.2.** By general considerations [BBD 1.2], the morphism \(N \rightarrow N'\) is a monomorphism if it completes to an exact triangle \((N, N', C)\) with \(C\) in \(M(X)\), or equivalently to an exact triangle \((K, N, N')\) with \(K[1]\) in \(M(X)\), so it suffices to show that \(p^*H^0K = 0\). As \(j_i^*\) is t-exact, it suffices to show that \(p^*H^0j_i^*K = 0\) for all \(i\). By hypothesis \(N_{U_i} \hookrightarrow N'_{U_i}\) so that \((N_{U_i}, j_i^*N', j_i^*K[1])\) has \(j_i^*K[1]\) in \(M(U_i)\).

To see that the injection \(N \hookrightarrow N'\) kills all \(\text{Ext}^j_{D(X)}(M, N)\), \(j > 0\), write

\[
\text{Ext}^j_{D(X)}(M, N) = \oplus_i \text{Ext}^j_{D(U_i)}(M, j_i^*N_{U_i}) = \oplus_i \text{Ext}^j_{D(U_i)}(j_i^*M, N_{U_i});
\]

this map sends \(q \in \text{Ext}^j_{D(X)}(M, N)\) to \(\oplus_i q_i\), where \(q_i \in \text{Ext}^j_{D(U_i)}(j_i^*M, N_{U_i})\) is given by

\[
q_i : M_{U_i}[-j] \rightarrow N_{U_i} \rightarrow N'_{U_i}.
\]
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This composition is zero by assumption.

**Perversity.** \( j_* \) no longer t-exact and the same argument doesn’t work if you replace \( j_* \) with \( p j_* \); \( j^* \) is still t-exact so that \( j^* p j_* = j^* p H^0 j_* = p H^0 j_* j_* = p H^0 = \text{id} \) on \( M(U_i) \) and \((j^*, p j_*)\) form an adjoint \( \text{pair} \) \( M(U_i) \leftrightarrow M(X) \) but not \( D(U_i) \leftrightarrow D(X) \).

**2.2.1.** We have
\[
\text{Ext}^i_{D(X)}(M, N) := \text{Hom}_{D(X)}(M, N[i]) = \text{Hom}_{D(Y)}(M, i_* i^* N[i]) = \text{Hom}_{D(Y)}(i^* M, i^* N[i]).
\]
Note that, although the embedding \( M(Y) \hookrightarrow M(X) \) is via the t-exact functor \( i_* \), in order to show \( I \) is an isomorphism we cannot naively make use of an adjunction between \( i^* \) and \( i_* \) as \( i^* \) is no longer t-exact. (It is true however that \((p^*, p i_*) \) form an adjoint pair [BBD] 1.4.16.)

The part about \( \text{Ext}^i_{D(X)}(M, N) \) coinciding with the set of connected components of \( E^i_{M(X)}(M, N) \) is (3.2.2) in Verdier, *Des catégories dérivées des catégories abéliennes*, Astérisque 239. The sequence of morphisms in \( E^i_{M(X)}(M, N) \) connecting \( C^* \) with \( \Phi_f(C^*) \) is as written
\[
C^* \rightarrow C^* \oplus \Xi_f(C_U) \rightarrow (C^* \oplus \Xi_f(C_U))/j_!(C_U) \leftarrow \Phi_f(C^*).
\]
(†)

Recall that the unipotent vanishing cycles functor \( \Phi_f : M(X) \rightarrow M(X) \) is defined as the \( H^0 \) of the complex of objects in \( M(X) \)
\[
j_! j^* K \rightarrow \Xi_f j^* K \oplus K \rightarrow j_* j^* K
\]
where \( K \) is a perverse sheaf on \( X \) (see §6 of Morel’s notes and the accompanying note).

The point is that the ends of \( C^* \) are supported on \( Y \), and for a perverse sheaf \( K \) in \( M(X) \) supported on \( Y \), the maps in (††) degenerate to
\[
K \rightarrow K \oplus 0 \rightarrow K \oplus 0 \leftarrow K = \Phi_f(K)
\]
with every map the identity.
2.2.2. Let’s show that given an adjunction of exact functors \( F \dashv G \) between abelian categories \( \mathcal{A} \xrightarrow{\mu} \mathcal{B} \), we have

\[
\text{Hom}_{D(\mathcal{A})}(FB, A) = \text{Hom}_{D(\mathcal{B})}(B, GA).
\]

The extension of the adjunction \( F \dashv G \) to complexes of objects of \( \mathcal{A} \) and \( \mathcal{B} \) is simple, and it is easy to verify that the adjunction sends homotopic morphisms to homotopic morphisms, hence descends to an adjunction \( K(\mathcal{A}) \cong K(\mathcal{B}) \). As the functors are both exact, they preserve quasi-isomorphisms, so to conclude on derived categories we write

\[
\xymatrix{ C \ar@{~}[d] \ar[r]^{f} & \ar@{~}[d] FC \ar[r]^{Ff} & \ar@{~}[d] FGA \ar[r]^{\varepsilon A} & A \ar@{~}[d] \ar[r]_{\eta B} & \ar@{~}[d] GFB \ar[r]_{Gf} & \ar@{~}[d] GFGA \ar[r]_{\varepsilon A} & GA \ar[r]_{f} & \ar@{~}[d] f \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r] & \ar@{~}[d] \ar[r}
generic point \( \eta \) in the rest of this paragraph. I see no reason why

\[
\ker(\text{Ext}^i_{M(X)}(M, N) \to \text{Ext}^i_{D(X)}(M, N)) = (\ker(\text{Ext}^i_{M(X)}(M, j_* N_U) \to \text{Ext}^i_{D(X)}(M, j_* N_U));
\]

take some stupid example like

\[
\begin{array}{cccccc}
0 & \to & 0 & \to & C_3 & \to & C_1 \oplus C_2 \oplus C_3 & \to & C_1 \oplus C_2 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & 0 & \to & C_2 \oplus C_3 & \to & C_1 \oplus C_2 \oplus C_3 & \to & C_1 & \to & 0.
\end{array}
\]

We instead make a slightly more subtle analysis. For each \( U \ni \eta \), we have the following commutative diagram with exact rows & columns.

\[
\begin{array}{cccccccc}
\text{Ext}^{i-1}_{M(U)}(M_U, N_U) & \to & \text{Ext}^{i-1}_{M(X)}(M, L^U) & \to & \text{Ext}^i_{M(X)}(M, N) & \to & \text{Ext}^i_{M(U)}(M_U, N_U) & \to & \text{Ext}^i_{M(X)}(M, L^U) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \beta^{-1} & & \downarrow & & \downarrow \beta' \\
\text{Ext}^{i-1}_{D(U)}(M_U, N_U) & \to & \text{Ext}^{i-1}_{D(X)}(M, L^U) & \to & \text{Ext}^i_{D(X)}(M, N) & \to & \text{Ext}^i_{D(U)}(M_U, N_U) & \to & \text{Ext}^i_{M(X)}(M, L^U)
\end{array}
\]

These diagrams form an inductive system with respect to étale neighborhoods of \( \eta \), and in the limit \( \beta^{-1} \) and \( \beta \) are isomorphisms. By the five lemma, \( \alpha \) is an isomorphism. (In more words, say we have an element \( x \in \ker \alpha \). Shrinking \( U \), we can make its image in \( \text{Ext}^i_{M(U)}(M_U, N_U) \) zero. Chasing the diagram, we produce an element \( y \in \ker \beta^{-1} \); shrinking \( U \) further we produce an element \( z \in \ker \beta^{-1} \) with \( \beta^{-1}(z) = y \) whose image in \( \text{Ext}^i_{M(X)}(M, N) \) equals \( x \). Hence \( x \) is zero. Given \( w \in \text{coker} \alpha \), \( \text{coker} \alpha \Leftarrow \text{coker} \beta \) injects by the strong four lemma. Shrinking \( U \), we can make the image of \( w \) in \( \text{coker} \beta \) zero so that \( w = 0 \), etc.)

**Reduction to the case \( X \) irreducible.** Let \( X_i \) denote the irreducible components of \( X \) with generic points \( \eta_i \). Since (2.2.3) covers the case when \( \dim \text{supp } N < \dim X \), (2.2.4) should more properly say that we consider an irreducible \( N \) supported at the generic point \( \eta \) of an irreducible component \( X_1 \) of \( X \) with \( \dim X_1 = \dim X \). This doesn’t preclude the possibility that \( N \) is supported at the generic points \( \eta_i \) of two (or more) such irreducible components \( X_i \) with \( \dim X_i = \dim X \) and possibly nonempty pairwise intersection. If so, we should pick open affine \( U \subset X \) with \( U = \bigsqcup_i U_i, \eta_i \in U_i \subset X_i \) so that the \( U_i \) are irreducible and have \( U_i \cap U_j = \emptyset \). We have to compute \( \text{Ext}^i(M_U, N_U) \) for
this $U$. Ordinary sheaves on $U$ decompose as a direct sum of sheaves each supported on one $U_i$, and morphisms between two such sheaves $\mathcal{F}, \mathcal{G}$ respect this decomposition so that $\text{Hom}(\mathcal{F}, \mathcal{G}) = \oplus_i \text{Hom}(\mathcal{F}_{U_i}, \mathcal{G}_{U_i})$. Objects of $D(U)$ are complexes of sheaves on $U$ and therefore decompose similarly into a direct sum; as $D = D^b$, in particular bounded below, $D(U)$ is equivalent to the homotopy category of bounded below complexes of injective sheaves, which likewise decompose, inducing a decomposition of the category $D(U) \cong \oplus_i D(U_i)$ and therefore a decomposition $D(U) \supset M(U) \cong \oplus_i M(U_i)$. Therefore

$$\text{Ext}^i_{D(U)}(M_U, N_U) = \oplus_i \text{Ext}^i_{D(U_i)}(M_{U_i}, N_{U_i}),$$

and we will have the same decomposition for $\text{Ext}^i_{M(U)}$ in light of $M(U) \cong \oplus_i M(U_i)$, provided we can show that $\text{Ext}^i_{M(U_i \sqcup U_j)}(M_{U_i}, N_{U_j}) = 0$ for $i \neq j$. Since $M(U_i \sqcup U_j) \cong M(U_i) \oplus M(U_j)$, this follows from Yoneda’s description of $\text{Ext}^i$: any acyclic complex

$$M_{U_i} \to L^{−p+1} \to \cdots \to L^0 \to N_{U_j}$$

will decompose as the direct sum of acyclic complexes

$$M_{U_i} \to L^{−p+1}_{U_i} \to \cdots \to L^0_{U_i} \to 0 \quad \text{and} \quad 0 \to L^{−p+1}_{U_j} \to \cdots \to L^0_{U_j} \to N_{U_j},$$

demonstrating a decomposition

$$\text{Ext}^i_{M(U_i \sqcup U_j)}(M_{U_i}, N_{U_j}) = \text{Ext}^i_{M(U_i)}(M_{U_i}, 0) \oplus \text{Ext}^i_{M(U_j)}(0, N_{U_j}) = 0.$$

**Epilogue: Madhav’s Constructible Sheaves.** Theorem 1 uses a transcendental input. The content of §3 does not, although the proof of Proposition 3.10 must be modified in order to work in the setting of constructible étale sheaves on a $k$-variety with $k$ of possibly nonzero characteristic.

**Proof of Theorem 1.** Arranging the projection $\pi$: follow the directions of the proof of the basic lemma (second form). The linear change of coordinates required to make $f$ monic in the last variable $x_n$ amounts to the substitutions $x_i \mapsto x_i + x_n$ for $i < n$ (which ensures that the coefficient of the highest power of $x_n$ appearing in $f$ is a nonzero scalar $\alpha \in k^\times$), followed by $x_n \mapsto \alpha^{-1/m} x_n$. If we work over a field which is not algebraically closed, $\gamma^m - 1/\alpha$ may not have any roots. However, we may replace $k$ by its perfect
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Closure (a universally integral, radicial, surjective extension), and then assume we need
an $m^{th}$ root of a nonzero scalar with $p \nmid m$; the minimal polynomial over $k$ for such
an element defines a finite separable extension $k' \supset k$; denote by $f$ the corresponding
map on spectra. If we can prove Theorem 1 for $A^n_k$, then we can deduce the result that
any constructible sheaf $\mathcal{F}$ admits a monomorphism $\mathcal{F} \hookrightarrow \mathcal{G}$ which induces the null map
$H^i(A^n_k, \mathcal{F}) \to H^i(A^n_k, \mathcal{G})$ by finding a monomorphism $\alpha : f^* \mathcal{F} \hookrightarrow \mathcal{G}$ with $H^i(A^n_k, \mathcal{G}) = 0$
for $i > 0$ and then taking the monomorphism $\mathcal{F} \xrightarrow{\text{res}} f_* f^* \mathcal{F} \xrightarrow{f_* \alpha} f_* \mathcal{G'} =: \mathcal{G}$.

In any event, the projection $\pi$ obtained by making these linear changes of coordinates
and then forgetting $x_n$ enjoys all the properties of the usual coordinate projection: $\pi$
smooth with fibers which are copies of $A^1_k$, hence $\pi$ is acyclic. If $\mathcal{F}$ is a constructible
sheaf on $A^n_k$, we know that after a linear change of coordinates inducing an automorphism
$f : A^n_k \to A^n_k$, $f^* \mathcal{F}$ satisfies the conditions of (2.2) & (1.3A) for $\pi : A^n \to A^{n-1}$ the
usual coordinate projection, so we get a monomorphism $f^* \mathcal{F} \hookrightarrow \mathcal{G}$ with $H^i(A^n_k, \mathcal{G}) = 0$
for $i > 0$. Now apply $f_*$. In this way we reduce to $\pi$ the usual coordinate projection
when applying (2.2) & (1.3A).

In the proof itself, both morphisms of short exact sequences are constructed by
forming $H$ and $G$ by pushout, and then forming cokernels. To see that the third vertical
maps constructed from the universal property of cokernels are isomorphisms, consider
by way of example the second diagram. Notate by $\alpha, \beta$ the nontrivial monomorphisms
of the upper and lower rows, respectively. Then coker $\alpha = \text{coker } \beta$ since to give a map
from $G$ that kills $H$ is the same as giving a map from $\pi^* J$ that kills $\pi^* \pi_* H$.

2.1. As $\mathcal{A} = \Delta^* \text{pr}^* \mathcal{A}$, the map $\text{pr}^* \mathcal{A} \to \Delta_* \mathcal{A}$ is of course the unit of adjunction.

2.2. As remarked in the note to the proof of Theorem 1 above, we may assume that
$\pi$ is the canonical projection. The relevant maps are defined by the cartesian square

$$
\begin{array}{ccc}
A^2_X & \xrightarrow{p_2} & A^1_X \\
\downarrow{p_1} & \Pi & \downarrow{\pi} \\
A^1_X & \xrightarrow{\pi} & X.
\end{array}
$$

Proof of (2) To see that $p_2|_Y$ is finite, note that it suffices to show that $p_2$ is finite when
restricted to $\Delta(A^1_X)$ and $p_1^{-1} V$ separately, as if $X \times A^1 = \text{Spec } A$ and $Y = \text{Spec } B$ with
ideals $I, J \subset B$ defining the closed loci of which $Y$ is the union (i.e. $I \cap J = 0$), the square
\[
\begin{array}{ccc}
B = B/I \cap J & \longrightarrow & B/I \\
\downarrow & & \downarrow \\
B/J & \longrightarrow & B/I \cup J
\end{array}
\]
is bicartesian, so that if $B/I$ and $B/J$ are finite as modules over $A$, so too is $B$. To check that $p_2|_{p_1^{-1}V}$ is finite, pull back the square above along $V \hookrightarrow A^{1}_X$ to produce another cartesian square

Now we need to show the vanishing of $p_2^*B$, which by (1.3A) is equivalent to the vanishing of global sections of $p_1^*A$ along every geometric fiber of $p_2$. Fix a geometric point $x \to A^{1}_X$ defining also the geometric point $x \to X$, and let $p_2^{-1}x \to A^2_X$ be the geometric fiber of $p_2$ above $x$; $p_2^{-1}x$ is isomorphic to the geometric fiber $F_x : \pi^{-1}x \to A^1_X$ of $\pi$, and the restriction of $p_1^*A$ to $p_2^{-1}x$ is therefore the same as the restriction of $A$ to $F_x$ (which is just a copy of $A^1_k$). Since $\pi|_V$ is surjective and $A|_V = 0$, $F^{-1}_xV = V_x$ is a nonempty closed locus. Denoting by $j_x : U_x \hookrightarrow F_x$ the complement of $V_x$ in the fiber, we have that $j_x^*(A|_{F_x})$ is locally constant and $A|_{F_x} = j_x^*j_x^*(A|_{F_x})$; such a sheaf on $A^1_k$ has no nonzero global section.

**Proof of (1)** In the notation of (1.3B), $M = X, M' = A^1_X, L = A^1_X, L' = A^2_X, g = \pi = f$, so that both $L \overset{f}{\to} M$ and $M' \overset{g}{\to} M$ coincide with $A^1_X \overset{\pi}{\to} X$. The business about $\pi_*A = 0$ reduces, after proper base change, to the same situation just discussed, namely to the fact that if $j : U \subset A^1_k$ is a nonempty open and $\mathcal{E}$ a lisse sheaf on $U$, $j_!\mathcal{E}$ has no nonzero global section. Again, it is crucial that $V$ intersect every fiber of $\pi$ nontrivially.

2.3. (As $R^p_{p_2^*}\Delta_* = id, R^q_{p_2^*}\Delta_*A = 0$ for $q > 0$.)
3.8. Here are the diagrams necessary to obtain the conclusion.

\[
\begin{align*}
\begin{array}{ccc}
FGB & \xleftarrow{Fu} & FA \\
\downarrow{v} & & \downarrow{t} \\
B & \xleftarrow{v} & B'
\end{array}
\end{align*}
\]

\[\eta : GB \leftrightarrow A\]
\[w : A \xrightarrow{\eta(A)} GFA \xrightarrow{Gt} GB'.\]

We want to show that the composition
\[
\begin{align*}
\begin{array}{ccc}
GB & \xrightarrow{u} & A \\
& & \xrightarrow{\eta(A)} GFA \\
& & \xrightarrow{Gt} GB'
\end{array}
\end{align*}
\]

coincides with \(Gv\). This follows from the below commutative diagram; that the dashed arrow coincides with the identity is a basic fact about adjunctions; c.f. Mac Lane IV.1 Th. 1.

\[
\begin{align*}
\begin{array}{ccc}
GB & \xleftarrow{u} & A \\
& & \xrightarrow{\eta(A)} GFA \\
& & \xrightarrow{Gt} GB'
\end{array}
\end{align*}
\]

From this general fact we deduce the admissibility of \(j_!Q\) on \(X\) from that of \(Q\) on \(U\) in the following way.

\((j_!, j^*):\) if \(\text{Ext}^q_{\text{Sh}(U)}(F, -)\) is effaceable on \(C(U)\), then \(\text{Ext}^q_{\text{Sh}(U)}(F, j^*(-))\) is effaceable on \(C(X)\). Let \(G\) be a constructible sheaf on \(X\) and \(F\) be a complex of injectives on \(\text{Sh}(X)\) resolving \(G\); \(j^*F\) is a complex of injectives on \(\text{Sh}(U)\) so that

\[
\text{R Hom}^c_{\text{Sh}(U)}(F, j^*(G)) = \text{Hom}^c_{\text{Sh}(U)}(F, j^*F) = \text{Hom}^c_{\text{Sh}(X)}(j_*F, F) = \text{R Hom}^c_{\text{Sh}(X)}(j_*F, G).
\]

Taking \(H^q\) finds

\[
\text{Ext}^q_{\text{Sh}(U)}(F, j^*G) = \text{Ext}^q_{\text{Sh}(X)}(j_*F, G),
\]
so that if the former is effaceable on $\mathcal{C}(X)$, the latter is.

3.10. Here are the necessary modifications: we may replace $k$ by its perfect closure, as the extension is radicial and doesn’t affect the étale topos. We may also take the reduced scheme structure on $X$ for the same reason so that $X$ becomes geometrically reduced ($\emptyset 35U$). Then the map $U' \cap Z_1 \to \mathbb{A}^d$ can be chosen to be generically finite étale. There is a nonempty Zariski open of $\mathbb{A}^d$ over which the map is finite étale (Hartshorne II.3 Ex. 3.7).

**Nori’s theorem for finite coefficients in characteristic $p$.** We wish to show that any constructible sheaf $\mathcal{F}$ is admissible. By dévissage one reduces to $\mathcal{F} = j_! \mathcal{F}'$ for $j$ the immersion of a locally closed stratum, and thereby (Nori 3.8 (a)) to the case $\mathcal{F}$ locally constant on $X$. By [SGAA VI 5.8], $\text{Ext}^q(\mathcal{F},-) \text{ commutes with filtered inductive limits}$ for all $q$ (in the language there, $\mathcal{F}$ is ‘de $q$-présentation finie’ for all $q$). Embedding $\mathcal{G}$ as a subsheaf of an injective sheaf $\mathcal{I}$ and passing to the limit along constructible subsheaves of $\mathcal{I}$ containing $\mathcal{G}$, one finds a constructible $\mathcal{G} \hookrightarrow \mathcal{G}'$ effacing $\text{Ext}^q(\mathcal{F}, \mathcal{G})$ for $q > 0$. 

Bibliography

[B1] On the derived category of perverse sheaves by Beilinson
[B2] How to glue perverse sheaves by Beilinson
[BBD] Faisceaux Pervers par Beilinson, Bernstein, Deligne & Gabber
[SGAA] SGA 4
9. How to glue perverse sheaves

Sasha’s exposé. Sasha gave (12/5/19) a lecture outlining how to glue perverse sheaves based on his article, which he says was written in a language somewhat better suited to \( \mathcal{D} \)-modules, so he gave an exposé tailored to perverse sheaves. This section is an attempt to write down the contents of his exposé.

\[
\begin{array}{ccc}
Y & \xrightarrow{i} & X & \xrightarrow{j} & U \\
\downarrow{f} & & & & \downarrow{f}
\end{array}
\]

A_1

The theorem is that the functor

\[
\mathcal{F} \mapsto (\mathcal{F}_Y, \mathcal{F}_U, \text{gluing datum})
\]

\[
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\Psi_{\text{un}} \mathcal{F}} & \mathcal{F}_U \\
\mathcal{F}_Y & \xrightarrow{\Phi_{\text{can}} \Phi_{\text{var}} \Psi_{\text{un}}(-1)} & \mathcal{F}_U
\end{array}
\]

is an equivalence of categories between perverse sheaves on \( X \) and perverse sheaves on \( U \) and \( Y \) with a gluing datum which consists of morphisms arising (up to a twist) from composing arrows coming from

\[
\Psi_{\text{un}} \xrightarrow{\Phi_{\text{can}}} \Phi_{\text{var}} \xrightarrow{\Psi_{\text{un}}(-1)}
\]

where var denotes variation. Choosing a generator for \( \mathbb{Z}_\ell(1) \), the composition of these arrows gives \( 1 - t \) (c.f. [SGA 7, Exp. XIII 1.4]). Now for the definition of the unipotent nearby and vanishing cycles.

\[
\begin{array}{ccc}
\overline{U} & \xrightarrow{\pi} & \hat{U} \\
Y & \xrightarrow{i} & X & \xrightarrow{j} & U \\
\downarrow{\pi} & & & & \downarrow{\pi}
\end{array}
\]

Here, \( \overline{U} \) is coming from the normalization of \( U \) in the separable closure of its generic point, while \( \hat{U} \) is the normalization of \( U \) in the extension of its generic point corresponding to the pro-\( \ell \) part, isomorphic to \( \mathbb{Z}_\ell(1) \). (In fact, this is all done on \( A^1 \setminus \{0\} \) and pulled back to \( U \) via \( f \). The Galois group at the generic point of \( A^1 \) of course has the structure...
of an extension of $\mathbb{Z}_\ell(1)$ by a pro-$p$ group, and the maximal $\text{pro-}\ell$ quotient corresponds
to an closed subgroup and hence to a Galois extension of the generic point. The
normalization of $\mathbb{A}^1 - \{0\}$ in this extension gives $\tilde{\mathbb{A}}^1$: it is the $\mathbb{Z}_\ell(1)$-torsor corresponding
to the Kummer sheaves, and its pullback to $U$ is also a $\mathbb{Z}_\ell(1)$-torsor, i.e. ‘logarithmic
sheaf’ since in the complex situation a choice of generator corresponds to a branch of
the logarithm – see the section on Poincaré duality in Weil I.) The usual nearby cycles
(we work with finite coefficients $F = \mathbb{Z}/\ell$; this changes nothing) is defined by
\[
\Psi(F_U) = i^* j_* \pi_* \pi^* F_U = F_U \otimes \pi_* \pi^* F_\ell,
\]
and we define the unramified nearby cycles
\[
\Psi^{\text{un}}(F_U) = i^* j_* (F_U \otimes \tilde{\pi}_* \tilde{\pi}^* F_\ell).
\]
Next define the Iwasawa algebra
\[
R := F_\ell[[\mathbb{Z}_\ell(1)]] = \limleftarrow F_\ell[\mathbb{Z}_\ell/\ell^n(1)].
\]
Choosing a generator $t$ of $\mathbb{Z}_\ell(1)$ allows us to write an isomorphism
\[
R \simeq F_\ell[[t - 1]];
\]
for the details see the note to 1.1. below. Inside of $R$ is the maximal ideal $m$ corresponding after a choice of generator to $(1 - t)$. It is an invertible module (zeros $\leftrightarrow$ poles). The
Iwasawa twist of a sheaf is the functor
\[
\mathbb{Z}_\ell(1)^\prime : \mathcal{F} \mapsto \mathcal{F} \otimes m =: \mathcal{F}(1)^\prime
\]
After a choice of generator, Iwasawa twist becomes Tate twist. Now recognize $\Psi^{\text{un}}(F_U)$
as a cone on the morphism $(j! \to j_*) (\mathcal{F}_U \otimes \tilde{\pi}_* \tilde{\pi}^* F_\ell)$. As $\Psi^{\text{un}}(F_U)[-1]$ is a perverse
sheaf, it is actually the kernel of the surjection $j! \to j_*$. Note that $H^0(U, \tilde{\pi}_* \tilde{\pi}^* F_\ell)$ can be
described as continuous functions from $U$ to $F_\ell$. Among them are obviously the constant
functions. Therefore inside $\tilde{\pi}_* \tilde{\pi}^* F_\ell$ sits a copy of $F_\ell$ and of course $\mathcal{F}_U \otimes F_\ell \simeq \mathcal{F}_U$. Define
the maximal extension $\Xi$ as the kernel of the composition
\[
j_! (\mathcal{F}_U \otimes \tilde{\pi}_* \tilde{\pi}^* F_\ell) \to j_* (\mathcal{F}_U \otimes \tilde{\pi}_* \tilde{\pi}^* F_\ell) \to j_* ((\mathcal{F}_U \otimes \tilde{\pi}_* \tilde{\pi}^* F_\ell)/\mathcal{F}_U),
\]
where the quotient by $\mathcal{F}_U$ refers to the constant functions described above. The second arrow remains an epimorphism as $j_*$ is exact (middle perversity). Note that the restriction of $\Xi$ to $U$ is $\mathcal{F}_U$. We have exact sequences

$$0 \to \Psi \to \Xi \to j_* \to 0$$
$$0 \to j_! \mathcal{F}_U \to \Xi \to \Psi(-1)' \to 0.$$

Now we can state a more refined version of the original theorem. We have complexes of perverse sheaves

$$j_! \mathcal{F}_U \to \Xi \mathcal{F}_U \oplus \mathcal{F} \to j_* \mathcal{F}$$
$$\Psi \to \Xi \oplus \Phi \to \Psi(-1)'$$

(probably the $\Psi$ and $\Phi$ should be unramified), both concentrated in degree zero. The $H^0$ of the first is $\Phi^{un}$ and of the second is $\mathcal{F}$.

**Remark.** About the lisse sheaves $I^{a,b}$: Sasha said that they are not really used in the glueing construction and that they have something to do with a unipotent nearby cycles construction that is nice with respect to Verdier duality.

**‘Glueing’ the glueing construction.** 1/10/20 Discussed with Sasha what happens if you try to extend this construction to the global situation of the complement of a Cartier divisor $D$; if you could do this, you could build up the category of perverse sheaves combinatorially. If $D$ is described by the vanishing of $f_1$ on $U_1$ and $f_2$ on $U_2$, the construction would glue on $U_1 \cap U_2$ if to the invertible function $f_1/f_2$ one could associate a canonical isomorphism of the construction for $f_1$ and the one for $f_2$ on $U_1 \cap U_2$; however, Sasha goes on to describe that isomorphisms between the two constructions are in bijection with choices of all $\ell$-power roots of $f_1/f_2$, and evidently $\mathbb{Z}_\ell(1)$ acts on these choices. So $\mathbb{Z}_\ell(1)$ acts on the set of isomorphisms between the constructions on $U_1 \cap U_2$. Closely related is Verdier’s notion of monodromic sheaf, which, if I’m remembering correctly, has the property that the sheaf and its translation by an element of $G_{m,k}$ are (abstractly) isomorphic, but does not provide a canonical isomorphism.
It seems to me that the choice of these $\ell$-power roots is in a way of trying to ‘glue’ the unipotent nearby cycles $\Psi_{\text{un}}^f$, which are by nature local. Because they are unipotent, they only depend on the tame nearby cycles $\Psi_{\text{t}}^f$, and gluing these should amount to something like extending the isomorphism of $U_1 \cap U_2$ onto itself to one between its universal tame covers. Therefore the crux of the issue seems to be the ‘globalization of nearby cycles.’

Now we go into a more detailed analysis of the paper itself section-by-section, trying to adapt it to the étale language of Sasha’s exposé.

1.1. Of course, $A_1^1_C$ is replaced by $A_1^1_k$ with char $k \neq \ell$ and the isomorphism $\mathbb{Z}((1) \simeq \pi_1((A^1 - \{0\})(C),1)$ is replaced by the isomorphism $\hat{\mathbb{Z}}((\overline{k}) \simeq \pi_1(G_{m,\overline{k}},1)^{\text{mod}}$, where the latter group is defined in [Laumon, 2.2.2] and corresponds to the étale coverings tamely ramified at $0$ and $\infty$.

Likewise, we will only consider the completed Iwasawa algebra ($A^\circ$ in the article and R in the exposé) at the prime $\ell$ $A^\circ = \mathbb{F}_\ell(\mathbb{Z}_\ell(1)) : = \varprojlim \mathbb{F}_\ell[\mathbb{Z}/\ell^n(1)]$

where as always $\mathbb{Z}/\ell^n(1)$ denotes the group of $\ell^n$ roots of unity (of the field $\overline{k}$) equipped with action of $\text{Gal}(\overline{k}/k)$, and $\mathbb{F}_\ell[\mathbb{Z}/\ell^n(1)]$ denotes the group algebra, where here $\mathbb{F}_\ell$ can be taken to be $\mathbb{Z}_\ell$ or $\mathbb{Z}/\ell^m$ for any $m$ and we will still have the isomorphism

$\mathbb{F}_\ell[[\overline{t} - 1]] \simeq A^\circ$

where $\overline{t}$ is a topological generator of $\mathbb{Z}_\ell(1)$. The proof that this is an isomorphism is the same of course as for the Iwasawa algebra of the $\ell$-adic integers, and the best source for this is Serre, Classes des Corps Cyclotomiques §6 (it is simply a matter of noting that the polynomial $(1 + T)^{\ell^n} - 1$ lies in $(\ell, T)^n \subset \mathbb{F}_\ell[[T]]$).

Let $\gamma := \overline{t} - 1$. In order to see $\text{Gr } A^\circ = \oplus_{i \geq 0} \mathbb{F}_\ell(i)$, we ask what is the action of Galois on $A^i/A^{i+1}$, which is free of rank 1 as $\mathbb{F}_\ell$-module. Galois acts via the cyclotomic character $\chi : \text{Gal}(\overline{k}/k) \to \mathbb{Z}_\ell^\times$, where after fixing an isomorphism of $\mathbb{F}_\ell$-modules $\mathbb{F}_\ell(1) \simeq \mathbb{F}_\ell$ (which amounts to a choice of generator $t \mapsto 1$), the action $t \mapsto gt$ of an
element \( g \in \text{Gal}(\overline{k}/k) \) corresponds to the map \( 1 \mapsto \chi(g) \). In particular, \( g\tilde{t} = \tilde{t}^{\chi(g)} \) and

\[
g_\gamma^n \mod \gamma^{n+1} = g((\gamma + 1) - 1)^n \mod \gamma^{n+1} \]

\[
= ((\gamma + 1)^{\chi(g)} - 1)^n \mod \gamma^{n+1} = \chi(g)^n \gamma^n \mod \gamma^{n+1}.
\]

This shows that \( A'/A^{i+1} \simeq \mathbf{F}_\ell(i) \). As \( A^o = \mathbf{F}_\ell([\gamma]), A = A_{(i-1)} = \mathbf{F}_\ell((\gamma)) \) and we find

\[
g_\gamma^{-n} \mod \gamma^{1-n} = g((\gamma+1)^{-1} - 1)^n \mod \gamma^{1-n} = ((\gamma+1)^{\chi(g)} - 1)^{-n} \mod \gamma^{1-n}
\]

\[
= \left( (\chi(g)\gamma)(1 + \left( \frac{\chi(g)}{2} \right)\gamma + \ldots) \right)^{-n} \mod \gamma^{1-n} = \chi(g)^{-n} \gamma^{-n} \mod \gamma^{1-n}.
\]

This shows that \( \text{Gr } A = \bigoplus_{i \in \mathbb{Z}} \mathbf{F}_\ell(i) \). These statements have been made in terms of a particular generator \( t \) but they are independent of the choice of generator; i.e. these isomorphisms are canonical.

Now we come to the pairing \( \langle \ , \ \rangle : A \times A \to \mathbf{F}_\ell(-1) \). Skew-symmetry: put

\[
a := f g^- = \frac{a_n}{\gamma^n} + \ldots + \frac{a_1}{\gamma} + \ldots.
\]

As \( g f^- = (f g^-)^- \), it amounts to showing that

\[
\text{res}_{\gamma=0} \left( a \frac{d\gamma}{\tilde{t}} \right) = -\text{res}_{\gamma=0} \left( a \frac{d\gamma}{\tilde{t}} \right).
\]

Note that

\[
\gamma^- = -\frac{\gamma}{\gamma + 1} = -\frac{\gamma}{\tilde{t}} \quad \text{and} \quad \frac{1}{\tilde{t}} = 1 - \gamma + \gamma^2 - \gamma^3 + \ldots
\]

so that

\[
\text{res}_{\gamma=0} \left( a \frac{d\gamma}{\tilde{t}} \right) = \text{res}_{\gamma=0} \left( \left( \frac{a_n}{\gamma^n} + \ldots + \frac{a_1}{\gamma} + \ldots \right) \left( 1 - \gamma + \gamma^2 - \gamma^3 + \ldots \right) d\gamma \right)
\]

\[
= a_1 + \ldots + (-1)^{n-1} a_n,
\]

\[
\text{res}_{\gamma=0} \left( a \frac{d\gamma}{\tilde{t}} \right) = \text{res}_{\gamma=0} \left( \left( \frac{-1)^n a_n \gamma^n}{\gamma^n} + \ldots + \frac{(-a_1 \tilde{t})}{\gamma} + \ldots \right) \frac{d\gamma}{\tilde{t}} \right)
\]

\[
= (-1)^n a_n \gamma^{n-1} + \ldots + \frac{(-a_1)}{\gamma} + \ldots
\]

\[
= -a_1 + \ldots + (-1)^n a_n.
\]
$Z_{\ell}(1)$-invariance is simply the statement that $\langle \tilde{f} i, \tilde{f}^i g \rangle = \langle f, g \rangle$, $i \in Z_{\ell}$, and is obvious. Likewise $(A^\iota)^+ = A^{-i}$ is obvious. The induced pairing on $Gr A$ should read

$$\langle S_i, S_{-i} \rangle = (-1)^{i+1} S_i \cdot S_{-i-1}, \quad S_j \in F_{\ell}(j)$$

because as a pairing $Gr A \times Gr A \to F_{\ell}(-1)$ it can be described simply as the piece of

$$(\tilde{f}, g) \mapsto (\tilde{f}, Gr(-)(\tilde{g})) \mapsto \tilde{f} Gr(-)(\tilde{g}) \mapsto Gr(\frac{1}{\ell}) (\tilde{f} Gr(-)(\tilde{g})) = \tilde{f} Gr(-)(\tilde{g})$$

in $Gr_{-1} A$. Some explanation: the involution map $a \mapsto a^-$ respects the filtration, as $\gamma^- = -\gamma/\tilde{t} = -\gamma(1 - \gamma + \cdots)$. Therefore it descends to a morphism $Gr(-) : Gr A \to Gr A$ determined by $\gamma \mapsto -\gamma$. Similarly, multiplication by $1/\tilde{t}$ respects the filtration on $A$ and descends to the identity map $Gr A \to Gr A$.

To obtain the isomorphism

$$A^a/A^b \simeq \text{Hom}(A^{-b}/A^{-a}, F_{\ell}(-1)),$$

simply induct on $b - a$, the case $b - a = 1$ just obtained, using the commutative diagram

$$\begin{array}{cccccc}
0 & \to & A^{b-1}/A^b & \to & A^a/A^b & \to & A^a/A^{b-1} & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \text{Hom}(A^{-b}/A^{-b+1}, F_{\ell}(-1)) & \to & \text{Hom}(A^{-b}/A^{-a}, F_{\ell}(-1)) & \to & \text{Hom}(A^{-b+1}/A^{-a}, F_{\ell}(-1)) & \to & 0
\end{array}$$

whose rows are exact as $A^a/A^b$ is free as $F_{\ell}$-module.

Finally, we should verify that the pairing is $\text{Gal}(\bar{k}, k)$-equivariant; i.e. that for $\sigma \in \text{Gal}(\bar{k}, k)$ we have $\langle \sigma f, \sigma g \rangle = \sigma \langle f, g \rangle = \chi(\sigma)^{-1} \langle f, g \rangle$. The point is that as $\chi(g) \in Z_{\ell}^\times$, $t$ generates $Z_{\ell}(1)$ if $\chi(g)t$ does, and

$$d \log(\overline{\sigma^{-1} t}) = d \log(\overline{t^{1/\chi(\sigma)}}) = \chi(\sigma)^{-1} d \log \tilde{t}, \quad \text{so that indeed}$$

$$\langle (\sigma f, \sigma g), t \rangle = \langle (f, g), \sigma^{-1} t \rangle = \langle (f, g), t/\chi(\sigma) \rangle = \langle \chi(\sigma)^{-1} (f, g), t \rangle.$$

We proceed to define the sheaf $I$ (and sheaves $I^{a,b}$) of $A$-modules on $G_{m,k}$ by endowing the Iwasawa algebra $A$ with an action of $\pi_1(G_{m,k}, 1)$, which acts via the quotient by the kernel of

$$\Gamma : \pi_1(G_{m,k}, 1) \to \pi_1(G_{m,k}, 1)^\text{mod} \simeq \hat{Z}(1)(\bar{k}) \to Z_{\ell}(1),$$
so that \( \pi_1(\mathbb{G}_{m,k}, 1) \) acts on \( I \) via \( G := \pi_1(\mathbb{G}_{m,k}, 1)/\ker \Gamma \); as \( \pi_1(\mathbb{G}_{m,k}, 1) \) can be described as the canonically split extension

\[
1 \to \pi_1(\mathbb{G}_{m,k}, 1) \to \pi_1(\mathbb{G}_{m,k}, 1) \to \text{Gal}(\overline{k}/k) \to 1,
\]


G can be described as the canonically split extension [SGA 1 Exp. IX 6.1]

\[
0 \to \mathbb{Z}_\ell(1) \to G \to \text{Gal}(\overline{k}/k) \to 1.
\]

The action of \( l \in \mathbb{Z}_\ell(1) \) on \( A \cong F_\ell[\mathbb{Z}_\ell(1)] \) is by multiplication by \( \bar{l} \in A \); the action of \( \text{Gal}(\overline{k}, k) \) is obvious.

‘Coincides on \( I_1 \) with the above \( \langle \ , \ \rangle \(': I_1 = A \) the stalk of \( I \) at 1.

A3. Generalities on \( \text{lim} \). In the below we use the words ‘exact pair,’ ‘short exact sequence,’ and ‘conflation’ interchangeably. Let \( (\mathcal{A}, \mathcal{E}) \) be an exact category with exact structure specified by class \( \mathcal{E} \) of exact pairs. \( \mathbb{Z} \times \mathbb{Z} \) has the product partial order \((a, b) \leq (c, d) \Leftrightarrow a \leq c \) and \( b \leq d \). Let’s verify the axioms of exact category (following Keller’s notes Derived categories and their uses) for \( \mathcal{A}_{\Pi} \). We notate an object of \( \mathcal{A}_{\Pi} \) by \( X_{ij} \). First note that the class \( \mathcal{E}_{\Pi} \) of exact pairs of \( \mathcal{A}_{\Pi} \) are the pairs \( X_{ij} \to Y_{ij} \to Z_{ij} \) such that the pair in \( \mathcal{A} \) corresponding to \((i, j)\) is in \( \mathcal{E} \) for all \((i, j) \in \Pi \). We verify that the composition of inflations is an inflation; this is true if given \( X_{ij}, Y_{ij} \) in \( \mathcal{A}_{\Pi} \) and inflation \( X_{ij} \to Y_{ij} \), we can find \( Z_{ij} \) in \( \mathcal{A}_{\Pi} \) making an exact pair \( X_{ij} \to Y_{ij} \to Z_{ij} \). We can find a \( Z_{ij} \) ‘pointwise’ for each \((i, j) \in \Pi \) from the exact structure \( \mathcal{E} \), and we need to join these ‘points’ to a natural transformation. This is done in a unique way from the universal property of cokernel and achieves \( \text{Ex1}^{\text{op}} \). Dually we can complete a diagram

\[
\begin{align*}
C_{ij}' & \quad \downarrow c_{ij} \\
B_{ij} & \quad \rightarrow C_{ij}
\end{align*}
\]
9. HOW TO GLUE PERVERSE SHEAVES

where \( p_{ij} \) comes from a deflation \( p : B_{ij} \to C_{ij} \) in \( \mathcal{E}^{\Pi} \) to a square

\[
\begin{array}{ccc}
A'_{ij} & \xrightarrow{i'_{ij}} & B'_{ij} \xrightarrow{p'_{ij}} C'_{ij} \\
\downarrow{b_{ij}} & & \downarrow{c_{ij}} \\
A_{ij} & \xrightarrow{i_{ij}} & B_{ij} \xrightarrow{p_{ij}} C_{ij}
\end{array}
\]
cartesian in \( \mathcal{A} \) where \( (i'_{ij}, p'_{ij}) \) is exact in \( \mathcal{E} \) and \( (i_{ij}, p_{ij}) \) is exact in \( \mathcal{E} \) coming from an exact pair \( (i, p) \) in \( \mathcal{E}^{\Pi} \). There is a unique way to make the \( B'_{ij} \) into an object \( B'_{ij} \) of \( A^{\Pi} \) compatible with the status of \( B'_{ij} \) as a limit of the first diagram for all \( (i, j) \in \Pi \); once this is done there is a unique way to make the \( A'_{ij} \) into an object \( A'_{ij} \) of \( A^{\Pi} \) from the universal property that \( i'_{ij} \) is a kernel of \( p'_{ij} \) for all \( (i, j) \in \Pi \). Finally we verify that \( B'_{ij} \) is a limit of the first diagram in \( A^{\Pi} \) since its morphisms for \( (i, j) \leq (i', j') \) were obtained from the universal property of limit of \( B'_{ij'} \). This secures Ex2.

Now we pass to the definition and properties of admissible objects. Typo: admissible objects are obviously objects of \( A^{\Pi} \), not \( A \). The full subcategory \( A^{\Pi}_a \subset A^{\Pi} \) would inherit an exact structure from the \( \mathcal{E}^{\Pi} \) on \( A^{\Pi} \) if it were closed under extensions in the sense that the existence of a short exact sequence \( X_{ij} \hookrightarrow Y_{ij} \twoheadrightarrow Z_{ij} \) in \( \mathcal{E}^{\Pi} \) with \( X_{ij} \) and \( Z_{ij} \) admissible would imply \( Y_{ij} \) admissible \[\text{Bühler}, 10.20\]. This would be secured if the statement ‘in any short exact sequence in \( A^{\Pi} \) if any two objects are admissible then the third is.’ The \( 3 \times 3 \) lemma \[\text{Bühler}, 3.6\] gives precisely this, modulo the additional condition that to be able to conclude that \( Y_{ij} \) is admissible from the admissibility of \( X_{ij} \) and \( Z_{ij} \), one must secure that for every \( i \leq j \leq k \) the sequence \( Y_{ij} \to Y_{ik} \to Y_{jk} \) is a complex. Writing the composition as \( Y_{ij} \to Y_{jj} \to Y_{kk} \), we see that the composition is zero if \( Y_{jj} \) is a zero object of \( A \). We will show that \( X_{jj} \) and \( Z_{jj} \) are zero objects of \( A \). Suppose \( X_{jj} \) is an admissible object of \( A^{\Pi} \). Let \( i = j = k \); we find that \( X_{jj} \xrightarrow{id} X_{jj} \xrightarrow{id} X_{jj} \) is short exact; in particular is a complex. Therefore \( X_{jj} \) is a zero object for all \( j \in \mathbb{Z} \). We therefore have an exact pair \( X_{jj} \to Y_{jj} \to Z_{jj} \) with \( X_{jj} \) and \( Z_{jj} \) zero objects of \( A \) so that both arrows in the exact pair are zero. In any additive category, of course the identity is a kernel for any zero morphism. As kernels are universal, \( X_{jj} \cong Y_{jj} \) and we have shown \( Y_{jj} \) is a zero object, and therefore that any extension of admissible objects is admissible. (Note that a consequence of the
admissibility criterion is that all maps \( X_{ij} \to X_{ik}, k \geq j \), are monomorphisms, and all maps \( X_{ik} \to X_{jk}, j \geq i \), are epimorphisms.)

\( \mathcal{A} \) embeds in \( \mathcal{A}^{\Pi}_a \) via the functor which to \( X \in \text{ob} \mathcal{A} \) associates the \( \Pi \)-object \( X_{ij} \) with

\[
X_{ij} = \begin{cases} 
X & i \leq -1, j \geq 0 \\
0 & \text{otherwise}
\end{cases}
\]

and all arrows between \( X \to X \) the identity, all other arrows 0. (There is a similar embedding for every \( i < j \).) ‘If \( \varphi \leq \psi \) i.e. \( \varphi(i) \leq \psi(i) \) for all \( i \),’ not ‘for any \( i \).’ A \( \Pi \)-object of \( \mathcal{A} \) is just a functor \( F : \Pi \to \mathcal{A} \), and \( \tilde{\varphi}(F) \) is the functor \( \Pi \to \Pi \to \Pi \).

The way \( \text{lim} \leftrightarrow \) is defined doesn’t work, but the simpler, obvious way to define it works flawlessly. The fewest conditions on the functions \( q \) necessary to make the stated arrows into a multiplicative system are probably that (i) \( \varphi : \mathbb{Z} \to \mathbb{Z} \) be surjective; and (ii) the function \( |\varphi(i) - i| \) have an upper bound. But who cares! The only functions we need to consider in our localization are the shift functions \( q_N(i) = i + N, N \in \mathbb{Z} \).

Let’s verify the axioms of multiplicative system for the system of morphisms \( \tilde{q}_N(X) \to \tilde{q}_M(X), X \in \text{ob} \mathcal{A}^{\Pi}_a, N \leq M \) which arise from the obvious morphism of functors \( \tilde{q}_N \to \tilde{q}_M \). (To be clear, this morphism of functors at an object \( X \) is defined by the arrows \( X_{i+N,j+N} \to X_{i+M,j+M} \) which are part of the data of \( X \) as \( \Pi \)-object.) If \( N \) is a natural number and \( X \) in \( \mathcal{A}^{\Pi}_a \), write \( X_N := \tilde{q}_N(X) \). Composition is trivial: \( X_A \to X_B = Y_C \to Y_D \) implies \( Y = X_{B-C} \) so the composition coincides with \( X_A \to X_{B+D-C} \). The following diagrams verify the remaining conditions of a multiplicative system when \( A \leq B \) are integers.

\[
\begin{array}{ccc}
\begin{array}{c}
X_A \\
\downarrow
\end{array} & \longrightarrow & \begin{array}{c}
Y \\
\downarrow
\end{array} & \begin{array}{c}
X_{A-B} \\
\downarrow
\end{array} & \longrightarrow & \begin{array}{c}
Y_A \\
\downarrow
\end{array} & X & \begin{array}{c}
f \\
g
\end{array} & \longrightarrow & Y & \begin{array}{c}
X_{B-A} \\
\downarrow
\end{array} & \longrightarrow & \begin{array}{c}
Y_B \\
\downarrow
\end{array} & \begin{array}{c}
f \\
g
\end{array} & \longrightarrow & Y_{B-A} \\
\end{array}
\]

We localize \( \mathcal{A} \) with respect to this multiplicative system to obtain the category \( \text{lim} \leftrightarrow \mathcal{A} \). The functor \( \text{lim} \leftrightarrow : \mathcal{A}^{\Pi}_a \to \text{lim} \leftrightarrow \mathcal{A} \) is surjective on (isomorphism classes) of objects. A
morphism \( \lim X \to \lim Y \) is represented by a diagram

\[
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow & & \downarrow \\
Y_A & \leftarrow & Y \\
\end{array}
\]

where \( A \geq 0 \) and \( Y \to Y_A \) is the morphism in the multiplicative system. Two such morphisms \( f : X \to Y_A \) and \( g : X \to Y_B \) agree if, supposing \( A \leq B \), the diagram

\[
\begin{array}{ccc}
X & \longrightarrow & Y_A & \longrightarrow & Y_C \\
\downarrow & & \downarrow & & \downarrow \\
X & \longrightarrow & Y_B & \longrightarrow & Y_C \\
\end{array}
\]

commutes; this is true iff \( g \) coincides with the composition \( X \xrightarrow{f} Y_A \to Y_B \).

The exact structure on \( \mathcal{A}_a^\Pi \) induces one on \( \lim \mathcal{A} \) as the localization functor \( \lim : \mathcal{A}_a^\Pi \to \lim \mathcal{A} \) preserves finite limits & colimits, and if \( \alpha : I \to \mathcal{A}_a^\Pi \) is an inductive or projective system in \( \mathcal{A}_a^\Pi \) indexed by a finite category \( I \) such that \( \lim \alpha \) (resp. \( \lim \alpha \)) exists in \( \mathcal{A}_a^\Pi \), then \( \lim(\lim \circ \alpha) \) (resp. \( \lim(\lim \circ \alpha) \)) exists in \( \lim \mathcal{A} \) and is isomorphic to \( \lim(\lim \alpha) \) (resp. \( \lim(\lim \alpha) \)); in particular, \( \lim \) commutes with kernels, cokernels, finite products and finite coproducts (c.f. Kashiwara & Schapira, Categories and Sheaves (7.1.22)). In other words, \( \lim : \mathcal{A}_a^\Pi \to \lim \mathcal{A} \) is exact and hence composes with the faithful exact embedding \( \mathcal{A} \hookrightarrow \mathcal{A}_a^\Pi \) to give a faithful exact embedding \( \lim : \mathcal{A} \hookrightarrow \lim \mathcal{A} \).

We will use the \( \lim \) construction in a particular way in what follows. Let \( \text{Fil}(\mathcal{A}) \) denote the filtered category of \( \mathcal{A} \) with objects sequences

\[
\ldots \to B^p \xrightarrow{j^p} B^{p-1} \to \ldots \quad p \in \mathbb{Z}
\]

of admissible monomorphisms in \( \mathcal{A} \) such that \( B^p = 0 \) for \( p \ll 0 \) and \( \text{coker } j^p = 0 \) for all \( p \gg 0 \). Morphisms in \( \text{Fil}(\mathcal{A}) \) are those compatible with the filtration and correspond bijectively to sequences \( f^p \in \text{Hom}_{\mathcal{A}}(B^p_1, B^p_2) \) such that \( f^{p-1}j^p_1 = j^p_2f^p \) for all \( p \in \mathbb{Z} \). The componentwise short exact sequences form an exact structure on \( \text{Fil}(\mathcal{A}) \). Given a
sequence
\[ \cdots \rightarrow A^p \rightarrow A^{p-1} \rightarrow \cdots \quad p \in \mathbb{Z} \]
of admissible monomorphisms in \( A \) (with no finiteness condition), let \( A^{a,b} := \text{coker}(A^b \rightarrow A^a) \) whenever \( b \geq a \), considered as object of \( \text{Fil}(A) \). Define the admissible \( \Pi \)-object \( X \) in \( \text{Fil}(A)_{a}^{\Pi} \) by \( X_{ij} := A^{-j,-i} \). When \( (i_1,j_1) \leq (i_2,j_2) \) with both in \( \Pi \) we have for transition morphism \( X_{i_1j_1} \rightarrow X_{i_2j_2} \) the canonical
\[
\text{coker}(A^{-i_1} \rightarrow A^{-j_1}) \rightarrow \text{coker}(A^{-i_2} \rightarrow A^{-j_2}),
\]
and \( X \) is admissible in view of exact sequences \( (i \leq j \leq k) \)
\[
\begin{array}{ccc}
X_{ij} & \hookrightarrow & X_{ik} \rightarrow X_{jk} \\
\downarrow & & \downarrow \\
A^{-j,-i} & \hookrightarrow & A^{-k,-i} \rightarrow A^{-k,-j}.
\end{array}
\]
Let \( \text{AM}(A) \) denote the subcategory of \( A \) with objects the objects of \( A \) and morphisms the admissible monomorphisms in \( A \). Let \( \text{Fil}_Z(A) \) denote the category with objects functors \( Z \rightarrow \text{AM}(A) \) and morphisms natural transformations of functors considered as functors \( Z \rightarrow A \) (\( \text{Fil}_Z(A) \) is just \( \text{Fil}(A) \) with no finiteness hypothesis). The above construction gives a functor
\[ T : \text{Fil}_Z(A) \rightarrow \text{Fil}(A)_{a}^{\Pi}. \]

1.2. Recall that \( A^\circ = F_\ell[\mathbb{Z}_\ell(1)] \) with fraction field \( A \); after choosing a generator \( t \) for \( \mathbb{Z}_\ell(1) \), \( A^\circ \) coincides with \( F_\ell((\bar{t} - 1)^i) \) while \( A \) coincides with \( F_\ell((\bar{t} - 1)) \). As before, \( F_\ell \) may coincide with \( \mathbb{Z}/\ell^n \) or \( \mathbb{Z}_\ell \), and we have \( A^i := (\bar{t} - 1)^i A^\circ \). It’s time to ‘see’ the \( \text{lim} \leftrightarrow \) construction in the construction of an étale avatar \( A_\text{ét} \) of \( A \). The point is that \( A \) as an étale sheaf on \( \text{Spec} \, k \) is far from constructible. Just as we obtain \( \ell \)-adic sheaves as projective systems of compatible constructible \( \mathbb{Z}/\ell^n \)-sheaves, we would like to obtain Laurent series as a suitable system of constructible (or \( \ell \)-adic) sheaves. This is what the \( \text{lim} \leftrightarrow \) construction does.

Let \( \mathcal{P}_k \) denote the category of \( F_\ell \)-sheaves on \( \text{Spec} \, k \). We would like to define an object \( \text{lim} A \) of \( \lim \mathcal{P}_k \) that behaves like the algebra \( A \) with action of Galois. Consider \( A \)
with its \( \mathbb{Z} \)-filtration as an object of \( \text{Fil}_\mathbb{Z}(\mathcal{P}_k) \) and let \( A_{\text{ét}} := T(A) \), where \( T \) is the functor defined at the end of \[ \text{the note to 1.1} \]. To see why \( A_{\text{ét}} \) is a good avatar of the field \( A \) of Laurent series, observe that

\[
\text{Hom}_{\lim \leftarrow} \mathcal{P}_k(\mathcal{F}_\ell, A_{\text{ét}}) = A
\]

with Galois acting on the left by transport of structure. Here the constant sheaf \( \mathcal{F}_\ell \) in \( \mathcal{P}_k \) is considered embedded via the embedding discussed in \[ \text{the note to A3} \] which is the particular case \( i = 0 \) of the system of embeddings \( \mathcal{P} \hookrightarrow \lim \mathcal{P} \) pictured in Figure 1 (where \((i, j)\) refers to the \( i \)th row and \( j \)th column). For, it is easily seen that \( \text{Hom}_{\lim \leftarrow} \mathcal{P}_k(\mathcal{F}_\ell, A_{\text{ét}}) \)

is in bijection with the data of a choice of \( i \in \mathbb{Z} \) and a morphism in \( \text{Fil}(\mathcal{P}_k)_a \) from \( \mathcal{F}_\ell \), embedded as in the left picture in Figure 1 to \( A_{\text{ét}} \), modulo the relation that two such morphisms are identified if they agree on their nontrivial overlap, as indicated in the right picture in Figure 1. If one associates to each nonzero Laurent series \( p \in A \) its initial form \( \text{in}(p) \in A^{a+a+1} \), there is a unique morphism in \( \text{Hom}_{\lim \leftarrow} \mathcal{P}_k(\mathcal{F}_\ell, A_{\text{ét}}) \) with \( \mathcal{F}_\ell \) embedded as in Figure 1 with \( i = a \) corresponding to \( p \). One recovers \( A^a \) via the morphisms that can be represented by a morphism in \( \text{Fil}(\mathcal{P}_k)_a \) with \( \mathcal{F}_\ell \) embedded in \( i \leq -a \). Galois acts continuously on \( \text{Hom}_{\lim \leftarrow} \mathcal{P}_k(\mathcal{F}_\ell, A_{\text{ét}}) = A \) for the topology induced by the filtration.
I do not see a way to make $A_{\text{ét}}$ into a ring, or a ring object. The issue is of course that if we were dealing with an ‘étalization’ of $A^\circ$, the multiplication would be easy to define as multiplication respects the filtration. Once we pass to fractions, however, the multiplication of two fractions no longer respects the topology. As I understand it, $A_{\text{ét}}$ should be understood in analogy with the adic formalism. The adic formalism, however, proceeds from the notion of $A$-linear category, where $A$ is a commutative ring so that every object of the category has endomorphism ring an $A$-algebra. To define the category of $\mathbb{Q}_\ell$-sheaves, one formally inverts the endomorphism ‘multiplication by $\ell$’ in the $\mathbb{Z}_\ell$-category. In our case, the algebra $A$ (or, more precisely, $A^\circ$) is not just a commutative ring, but a sheaf, a commutative ring with action of Galois. I have no doubt that $A^\circ$ can be ‘étalized’ in a very correct way, but it cannot be done directly from the formalism of SGA 5 for this reason. The matter is essentially this: the adic formalism in SGA 5 is enough to treat projective systems of étale sheaves which are modules for a constant sheaf of rings $\mathbb{Z}_\ell$, which does not really enter the picture directly as a sheaf of rings, but only via the commutative $A$ in the notion of $A$-linear category. Of course, the notion of ringed topos is foundational, and so the way it is done in the adic formalism is not really satisfactory. I wonder if the pro-étale topos allows one to satisfactorily define an ‘$\ell$-adic sheaf of rings’ which needn’t be constant. Our ‘étalization’ of $A^\circ$ should be such an ‘$\ell$-adic sheaf of rings.’

In the rest of the article, Sasha is trying to work ‘$\ell$-adically’ for an ‘nonconstant adic sheaf of rings’ $A$. I think the lim formalism isn’t quite up to the job, because I do not see how to make $A_{\text{ét}}$ into a ring object, and moreover I don’t see how to have it act on $I_{\text{ét}}$. However, I am not sure we actually need this. In the key lemma, forget the notion of $\pi^{-1}$ and just notice that if ker and coker are annihilated by a power of $\pi$ independent of $a,b$, say $N$, then both are killed by the map $id \to \tilde{\phi}_N$, which is an isomorphism in $\lim$.

1.4. Given $\pi \in A^{1*}$, let $\tilde{\pi} := \exp \pi$ and $\gamma := \tilde{\pi} - 1$ so that $A = F_\ell[[\gamma]]$ and $\pi = \log \tilde{\pi}$. (As remarked, if $t$ is a generator of the sheaf $\mathbb{Z}_\ell(1)$, $\pi = \log t$ is a logical choice.) The sheaf $I^{a,b}$ is defined by the action of the group $G$ defined in the note to (1.1) and in view of the canonically split extension $[1]$, it will suffice to show that if $l \in \mathbb{Z}_\ell(1), m \in \text{Gal}($k/k), $lm \sigma_{\pi}(x) = \sigma_{\pi}(lmx)$. Supposing $m = 1$, the statement is clear as $l$ acts trivially on $\mathbb{Z}_\ell(-n)$ (which is the reciprocal image of a sheaf on Spec $k$) and of course $\tilde{l}$
commutes with $\pi^n$ as both are elements of the Iwasawa algebra $A$. In general, $m$ acts by multiplication by $\chi(m)^{-n}$ on $\pi^{-n}$ and sends $\tilde{t} \mapsto \tilde{t}^{\chi(m)}$ so $m\pi = \log(\tilde{t}^{\chi(m)}) = \chi(m)\pi$. 
Bibliography

[B]    *How to glue perverse sheaves* par Beilinson

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10. Morel’s notes

0. Introduction. We use the notation \( G_{m,k} \leftarrow A^1_k \leftarrow \{0\} \). The nearby cycles functor \( \Psi_f \) is relative to the morphism \( f : X \to A^1_k \); let \( Y := f^{-1}(0) \) and \( U := X - Y \). We use the setup of (2.2.2) in Laumon’s Fourier transform article: write \( A^1_k = \text{Spec}(k[u]) \) with generic point \( \eta = \text{Spec}(k((u))) \) and Zariski trait \( (A^1_k)_0 := \text{Spec}(k[u]_{(u)}) \) at 0; let \( k[u] \) denote the henselization of the local ring \( k[u]_{(u)} \), let \( (A^1_k)_0 := \text{Spec}(k[u]) \) with generic point \( \xi = \text{Spec}(k((u)[u^{-1}])) \), so that the inclusion \( k[u,u^{-1}] \hookrightarrow k[u][u^{-1}] \) induces a morphism of schemes \( \iota : \xi \to G_{m,k} \). We also use \( i, j \) to denote the inclusion of the closed and generic points to \( (A^1_k)_0 \).

Fix a geometric point \( \rho : \tilde{\xi} \to \xi \), which via \( \xi \to \eta \) we also consider to be centered on \( \eta \). Given an object \( K \) of \( D^b_c(X_\xi) \), \( \text{Gal}(\tilde{\xi},\xi) \) acts on the nearby cycles \( R\Psi_\xi(K)[-1] \) of \( [?\text{, Exp. XIII}] \) relative to the morphism \( f \) (abusively letting \( \iota \) also denote the inclusion of \( \xi \) into \( A^1_k \)). (See the note to BBD Appendix A for a detailed comparison of two ways of writing \( R\Psi_\xi \!).

Let \( S \) denote \( (A^1_k)_0 \) with closed point 0. Let \( \tilde{\xi} \) denote the maximal tamely ramified extension of \( \xi \) and let \( \tilde{A}^1_k, \tilde{G}_{m,k}, \tilde{S} \) denote the normalisations of \( A^1_k, G_{m,k}, \) and \( S \), respectively, in \( \tilde{\xi} \). Likewise for tildes. Note that \( \tilde{0} \) is the spectrum of the separable closure of \( k \) and \( \tilde{S} \) is the spectrum of a valuation ring with value group \( \mathbb{Z}_{(p)} \) (the localization of \( \mathbb{Z} \) at the prime \( (p) \)) while \( \tilde{S} \) is the spectrum of a valuation ring with value group \( \mathbb{Q} \) and with residue field a purely inseparable extension of \( k(\tilde{s}) \). The commutative diagram on the right is obtained by localising the one on the left near 0.
Let $\pi$ denote the composition $\tilde{A}_k^1 \to \tilde{A}_k^1 \overset{\pi}{\to} A_k^1$ and its various base extensions to morphisms landing in $G_{m,k}$ or $S$. To get $\pi_1(G_{m,k},1)$ to act on nearby cycles, make the

**Definition.** For $K$ in $D^b_c(U)$, set $\Psi_f := i^*j_*\pi^*K[-1]$.

(We notate also by $i, j$ etc. the corresponding base extensions via the morphism $f$.)

As $i$ factors as $\{0\} \to (A^1_k)(0) \to A^1_k$, $\tilde{i}$ factors as $\tilde{\{0\}} \to \tilde{S} \to \tilde{A}_k^1$ so that $\Psi_f[1]$ coincides with the usual $\Psi_{\xi}$ for the henselian trait $S$. As $i$ also factors as $\{0\} \to (A^1_k) \to A^1_k$, $\Psi_f$ also coincides with $\tilde{i}^*\tilde{j}_*, K_{\xi}[-1]$ which carries an action of $\text{Gal}(\tilde{\xi}/\eta)$. (‘Coincide,’ means in this case ‘coincide’ in $D^b_c(Y_k)$, but these various ways of writing $\Psi_f$ endow it with actions by different groups; in this case, one is a subgroup of the other.) When $L$ is a lisse sheaf on $G_{m,k}$, this action of factors through the quotient $\text{Gal}(\tilde{\xi}/\eta) \to \pi_1(G_{m,k},1)$. For an arbitrary $Q_\ell$-sheaf, the lisse locus may be a proper subscheme of $G_{m,k}$ in which case the action of $\text{Gal}(\tilde{\xi}/\eta)$ may not factor through the stated quotient, but rather will factor through one with a smaller kernel. Regardless, Remark 0.2 about a local system $L$ is now obvious.

On the functorial exact triangle $\Psi_f \xrightarrow{T-1} \Psi_f \to i^*j_* \to$: this is discussed with more detail in the proof of $t$-exactness of (shifted) nearby and vanishing cycles in [BBD, Appendix A]. The point is, with $X \to S$ of finite type over a strictly henselian trait $(S, \eta, s)$ which is essentially of finite type over a field $k$ of characteristic $\neq \ell$, starting with $K \in D^b_c(X_\xi)$ and letting $G := \text{Gal}(\tilde{\xi}/\xi)$, $\tilde{K} \to \Gamma(G, \rho_*\rho^*K)$ and therefore

$$i^*j_*K = i^*j_*\Gamma(G, \rho_*\rho^*K).$$

Writing $G$ as an extension of $\mathbb{Z}_\ell(1)$ by a pro-group $Q$ of order prime to $\ell$, since invariants by $Q$ is an exact functor, $\Gamma(G, -)$ is represented on $D^b_c(X_\xi \times BG)$ by

$$K \mapsto \text{Cone}(K^Q \xrightarrow{T-1} K^Q)[-1]$$

(an explication of why can be found in [D, 10.7]; in short, it is because a $\mathbb{Z}_\ell(1)$-module admits a 2-step acyclic resolution by coinduced modules), so, keeping in mind

$\Psi_f := R\Psi_{\xi}[-1]$, in order to see that the triangle

$$\Psi_f(K^Q) \xrightarrow{T-1} \Psi_f(K^Q) \to i^*j_*(K^Q) \to$$
is distinguished, it suffices to see that $i^*j_*(K^Q) = (i^*j_*K)^Q$, which is clear, and then simply use functoriality.

When $S$ is no longer strictly henselian, the situation is the same, except of course $\text{Gal}(\overline{k}/k)$ could be nontrivial, so $R\Gamma(\pi_1(G_{m,\overline{k}}, 1), \Psi_f(K)) = i^*j_*(K)$, ‘packaged’ as a sheaf on $Y_{\overline{k}}$ with action of $\text{Gal}(\overline{k}/k)$ (recall [SGA 7, XIII 1.1.3]: the functor $\mathcal{F} \mapsto \{\mathcal{F} \text{ endowed with action of } \text{Gal}(\overline{k}/k)\}$ is an equivalence of categories). This is why in the notes Morel writes ‘the last term should be base changed from $Y$ to $Y_{\overline{k}}$’, but this is a distinction without a difference.

Morel takes $T$ to be a topological generator of the prime-to-$p$ quotient of $\pi_1(G_{m,\overline{k}}, 1)$, while in the above $T$ is a topological generator of the maximal pro-$\ell$ quotient $\mathbb{Z}_\ell(1)$ of $\text{Gal}(\xi/\xi)$. This is insignificant: the invariants under $Q$ functor is exact on a pro-$\ell$ module provided $Q$ has order prime to $\ell$ so we could take $T$ to be a topological generator of the prime-to-$p$ quotient and take invariants under the maximal pro-$p$ subgroup (which is a normal Sylow) or take $T$ to be a topological generator of the maximal pro-$\ell$ quotient and take invariants under everything else (which is the approach taken above). Let’s first analyse the maximal prime-to-$p$ quotient of both groups; we will see that they coincide, which will imply that the maximal pro-$\ell$ quotients also coincide.

The maximal prime-to-$p$ quotient of $\pi_1(G_{m,\overline{k}}, 1)$ coincides with $\pi_1(G_{m,\overline{k}}, 1)^{\text{mod}}$, and, after base changing to $\overline{k}$, the maximal prime-to-$p$ quotient of $\text{Gal}(\xi, \xi \otimes_k \overline{k})$ coincides with $\text{Gal}(\xi, \xi \otimes_k \overline{k})^{\text{mod}}$, the tame quotient of $\text{Gal}(\xi, \xi \otimes_k \overline{k})$. Both these groups are isomorphic to $\hat{\mathbb{Z}}(1)(\overline{k})$; c.f. [Laumon, 2.2.2.1] and [the note to 2.2.2.2]. To see that the prime-to-$p$ quotient of $\pi_1(G_{m,\overline{k}}, 1)$ coincides with $\pi_1(G_{m,\overline{k}}, 1)^{\text{mod}}$, simply recall that $\text{Gal}(\xi, \xi \otimes_k \overline{k}) \rightarrow \pi_1(G_{m,\overline{k}}, 1)$ and the former group admits a dévissage

$$1 \rightarrow P_0 \rightarrow \text{Gal}(\xi, \xi \otimes_k \overline{k}) \rightarrow \hat{\mathbb{Z}}(1)(\overline{k}) \rightarrow 1$$

where the wild inertia $P_0$ is a $p$-group so that the maximal prime-to-$p$ quotient of $\text{Gal}(\xi, \xi \otimes_k \overline{k})$ is $\hat{\mathbb{Z}}(1)(\overline{k})$; moreover the same is true of the maximal prime-to-$p$ quotient of the subgroup $\text{Gal}(\xi, \xi \otimes_k \overline{k}) \subset \text{Gal}(\xi, \xi \otimes_k \overline{k})$, and this subgroup surjects onto
\[ \text{Gal}(\xi, \xi \otimes \overline{k}) \mod \approx \hat{Z}(1)(\overline{k}). \]

We have a commutative diagram \cite{Laumon} 2.2.2

\[
\begin{array}{ccc}
\text{Gal}(\xi, \eta \otimes \overline{k}) & \longrightarrow & \pi_1(\mathbb{G}_m, 1) \\
\uparrow & & \uparrow R \\
\text{Gal}(\xi, \xi \otimes \overline{k}) & \longrightarrow & \text{Gal}(\xi, \xi \otimes \overline{k}) \mod \approx \hat{Z}(1)(\overline{k})
\end{array}
\]

and from the description of \( \text{Gal}(\xi, \eta \otimes \overline{k}) \) it follows that \( \pi_1(\mathbb{G}_m, 1) \mod \approx \hat{Z}(1)(\overline{k}) \) is the maximal prime-to-\( p \) quotient of \( \pi_1(\mathbb{G}_m, 1) \). Moreover, a topological generator of either the maximal prime-to-\( p \) or pro-\( \ell \) quotients of \( \text{Gal}(\xi, \xi \otimes \overline{k}) \) is carried onto the same in \( \pi_1(\mathbb{G}_m, 1) \).

Regardless of whether one takes \( T \) to be a topological generator of the maximal pro-\( \ell \) quotient or the maximal prime-to-\( p \) quotient (we will choose the latter to conform to Morel’s notes), there is still the issue that after extending scalars \( k \hookrightarrow \overline{k} \) the triangle

\[
\Psi_f(K^Q) \xrightarrow{T-1} \Psi_f(K^Q) \to i^* j_* (K^Q) \to
\]

is distinguished, where we have taken \( Q \)-invariants, whereas Morel doesn’t mention taking \( Q \)-invariants when she writes this triangle. This is most likely because she is using tamely-ramified nearby cycles \( \tilde{\Psi} \) to begin with.

**Definition.** For \( K \in D^b_c(U) \), let \( \tilde{\Psi}_f(K) := i^* \tilde{i}_* \tilde{\pi}^* K[-1] \) denote tame nearby cycles.

(Compare \cite{SGA7}, Exp. I 2.7.) Note that \( \tilde{\Psi}_f(K) = R\Gamma(Q, \Psi_f(K)) = \Psi_f(K)^Q = \Psi_f(K^Q), \) where \( Q \) is the wild inertia of \( \text{Gal}(\xi, \xi) \). Therefore, when Morel writes \( \Psi_f \), she is probably implicitly writing \( \tilde{\Psi}_f \), and we will do the same in what follows.

**1. Unipotent nearby cycles.** We retain the notation from the previous section. We write \( F_\ell \) for Morel’s \( F \). Now \( F_\ell \) can be \( \mathbb{Q}_\ell, E_\lambda, \overline{Q}_\ell \), etc.

(1.1) We construct the unipotent nearby cycles. To find that the endomorphism ring is finite over \( F_\ell \), simply use the fact that \( R\text{Hom} \) preserves constructibility and write

\[
\text{End}_{D^b_c}(\Psi_f K) = H^0 R\text{Hom}(\Psi_f K, \Psi_f K).
\]

Let \( P := \Psi_f K \) and \( E = \text{End} P \). We have \( F_\ell[T] \to E \) with kernel \( a(T) \), a nonconstant polynomial as \( \dim_{F_\ell} E < \infty \). Write \( a = bc \) with \( (T-1) \nmid b \) and \( c = (T-1)^m \) for some
\( m \in \mathbb{N} \). As \((b, c) = 1\), there exist \(x, y \in \text{F}_\ell[T]\) s.t. \(1 = xb + yc\) \((1 = \text{id}_P)\). As \(a = 0\) in \(E\), we have \((xb)(yc) = xya = 0\),

\[
xb = xb(xb + yc) = (xb)^2 + xya = (xb)^2,
\]

and similarly \((yc)^2 = yc\), all in \(E\). Let \(e_1 = xb\) and \(e_2 = yc\). We have found a pair of orthogonal idempotents \(e_1\) and \(e_2\) with \(e_1 + e_2 = 1 := \text{id}_P\), and by \([\text{the note to BBD} 2.2.18]\), the triangulated category \(D^b_c(Y)\) is Karoubi complete: every idempotent splits. More precisely, given an object \(P\) of this category and an idempotent \(e : P \rightarrow P\), there is an object \(P_e\) and a retraction \(P_e \rightarrow P \rightarrow P_e\) with \(pi = \text{id}_{P_e}\) and \(ip = e\). In particular, this means that \(i\) is a monomorphism, \(p\) an epimorphism, and \(P_e\) is simultaneously a limit and colimit of the diagram \(P \rightrightarrows P\), where the parallel morphisms are \(\text{id}_P\) and \(ip = e\), via the morphisms \(i\) and \(p\), respectively. Moreover, \(P_e\) is an absolute (co)limit, i.e. preserved by every functor.

Therefore from the \(e_i\) we get \(P_i, i_i, p_i\) \((i = 1, 2)\), and to verify that \(P = P_1 \oplus P_2\) it only remains \((\text{0103})\) to check that \(p_2 \circ i_1 = 0 = p_1 \circ i_2\). As \(i_2\) is a monomorphism and \(p_1\) an epimorphism, in order to show \(p_2 \circ i_1 = 0\) it suffices to write

\[
i_2 \circ p_j \circ i_1 \circ p_1 = e_2 \circ e_1 = 0.
\]

Therefore \(P = P_1 \oplus P_2\). Given any polynomial \(q(T)\), we consider \(q(T)\) as endomorphism of \(P\). As \(i_2 \circ p_2 \circ q(T) \circ i_1 \circ p_1 = e_2q(T)e_1 = q(T)e_2e_1 = 0\), in fact \(p_2 \circ q(T) \circ i_1 = 0\) and similarly \(p_1 \circ q(T) \circ i_2 = 0\). This means that \(q(T)\) descends to endomorphisms of \(P_1\) and \(P_2\), and the decomposition \(P = P_1 \oplus P_2\) is \(\text{F}_\ell[T]\)-equivariant. Moreover, to calculate \(q(T)\) on \(P_1\), it suffices to know \(p_1 \circ q(T) \circ i_1\).

We see that as \((T - 1)^m = c\) and \(a = bc = 0\) as endomorphism of \(P\), \((T - 1)^m e_1 = cxb = xya = 0\). Therefore \(p_1 \circ (T - 1)^m \circ i_1 = 0\) \((p_1\) is an epimorphism) and \((T - 1)^m = 0\) on \(P_1\). On the other hand, \(p_2 \circ (T - 1)(y(T - 1)^{m-1}) \circ i_2 = p_2 \circ e_2 \circ i_2 = \text{id}_{P_2}\) shows that \((T - 1)\) is invertible as endomorphism of \(P_2\). This shows that \((T - 1)\) is nilpotent on \(P_1\) and an automorphism of \(P_2\).
We set \( \Psi_f K := P_1 \) and \( \Psi_f^{nu} K := P_2 \), call the former the unipotent nearby cycles, and the latter the non-unipotent nearby cycles. We have proved a \( F_\ell[T] \)-linear decomposition

\[
\Psi_f K = \Psi_f^{un} K \oplus \Psi_f^{nu} K.
\]

Nothing in the above discussion changes if we replace \( \Psi_f \) by \( \tilde{\Psi}_f \), so we have similarly

\[
\tilde{\Psi}_f(K) = \tilde{\Psi}_f^{un}(K) \oplus \tilde{\Psi}_f^{nu}(K)
\]

and a distinguished triangle

\[
\tilde{\Psi}_f^{un} T^{-1} \longrightarrow \tilde{\Psi}_f^{un} \to i^* j_* \to .
\]

(That \( \tilde{\Psi}_f^{un} \) preserves constructibility is deduced trivially from the fact that \( \Psi_f^{un} \) does, and as taking \( Q \)-invariants is an exact functor on \( \ell \)-adic sheaves, \( H^i \tilde{\Psi}_f^{un} = H^i(\Psi_f^{un})^Q = (H^i\Psi_f^{un})^Q \); now recall that every subsheaf of a constructible sheaf of modules over a noetherian ring on a noetherian scheme is constructible.)

2. Some local systems on \( G_{m,k} \). The local systems \( L_a \) appear to be stand-ins for Sasha’s \( \text{Gr } I^{-a,b} \), which, by the way, come with the pairing \( \langle , \rangle \). Let’s look more closely at the analogy. The action of \( \pi_1(G_{m,k},1) \) on \( I^{-a,b} \) is also via the projection onto the maximal pro-\( \ell \) quotient \( Z_\ell(1) \), and \( t \in Z_\ell(1) \) acts on \( I^{-a,b} \) by \( \tilde{t} = \exp(\log \tilde{t}) \). The point is that \( \log \tilde{t} \in A^{1*} \) in Sasha’s notation; i.e. \( A^o \simeq F_\ell[[\log \tilde{t}]] \), and therefore \( \log \tilde{t} \) acts on \( \text{Gr } I^{-a,0} \) by Morel’s N. Therefore indeed Morel’s \( L_a \simeq \text{Gr } I^{-a,0} \). Morel claims an isomorphism \( L_a^\vee = \text{Gr } I^{0,a} \), but if we want to upgrade the naïve isomorphism of sheaves on \( \text{Spec } k \) to one on \( G_{m,k} \) we run into the issue that the logarithm of geometric monodromy acts on \( L_a^\vee \) by \( -\log \tilde{t} \), not \( \log \tilde{t} \) and so \( t \) acts by \( \exp(-\log \tilde{t}) \). Therefore a \( \pi_1(G_{m,k},1) \)-equivariant isomorphism is not provided by the identity morphism but rather by the isomorphism

\[
I^{0,a} = F \oplus F(1) \oplus \cdots \oplus F(a)
\]

\[
\begin{array}{cccc}
\downarrow \phi & \downarrow \text{id} & \downarrow -\text{id} & \downarrow (1)^\omega \text{id} \\
L_a^\vee & F \oplus F(1) \oplus \cdots \oplus F(a). & & \\
\end{array}
\]

In Sasha’s notation, \( (\text{Gr } I^{-a,0})^\vee \simeq \text{Gr } I^{0,a} \), and the maps \( \alpha, \beta \) are just the \( \text{Gr} \) of maps \( I^{-a,0} \hookrightarrow I^{-b,0} \) and \( I^{-b,0} \twoheadrightarrow I^{-a,b} \). To obtain the statement about \( D(\alpha_{a,b}) \), we see before passing to \( \text{Gr} \) that \( D(\alpha_{a,b})(-1)[-2] : I^{0,b} \to I^{0,a} \) is just reduction modulo \( I^a \). Passing to
Gr, this is the same as

$$\beta_{a,b}(b) : \mathcal{L}_b(b) = (\text{Gr } I^{-b,0})(b) \rightarrow (\text{Gr } I^{-a,0})(a) = \mathcal{L}_a(a).$$

3. **Beilinson’s construction of $\Psi_{un}$.** Everything is clear except perhaps at first glance the expression $\ker(N^a, \Psi_{un}^a K)$, as $N$ is not an endomorphism of $\Psi_{un}^a$ but rather sends it to $\Psi_{un}^a(-1)$. Write $N$ as a linear map $\Psi_{un}^a(1) \rightarrow \Psi_{un}^a$; what this means is that $N$ assigns, linearly in $F_\ell$, an endomorphism of $\Psi_{un}$ to an element of $F_\ell(1)$. Therefore there is an unambiguous meaning to $\ker(N^a, \Psi_{un}^a K)$: if $t_1, \ldots, t_a$ are $a$ nonzero elements of $F_\ell(1), \ker(Nt_a \ldots Nt_1, \Psi_{un}^a K)$ is independent of the choice of $t_i$.

(3.3) Returning to the various ways of defining $\Psi_f$, we settled upon $\tilde{\Psi}_{un}$. This functor with $f = \text{id}$, as a direct factor of tame cycles, no longer has the the property of sending every shifted local system $\mathcal{L}[1]$ on $G_{m,k}$ to $L[0]$ (where $L$ the representation of $\pi_1(G_{m,k}, 1)$ corresponding to $\mathcal{L}$). But it still does for these $\mathcal{L}_a$, as $\pi_1(G_{m,k}, 1)$ acts on $\mathcal{L}_a$ through the tame quotient, and moreover $T$ acts unipotently by construction.

(3.4) is indeed obvious, but it appears to have a typo. Start with $\Psi_{un}^a(K)$, express the latter as $\ker(N^{a+1}, \Psi_{un}^{a+1}(K))$ for $a \gg 0$ and then via the isomorphism $\gamma$ as $\ker(N, \Psi_{un}^a(K \otimes \mathcal{L}_a))$; i.e. for $a \gg 0$ the map

$$\Psi_{un}^a(K) \rightarrow \ker(N, \Psi_{un}^a(K \otimes \mathcal{L}_a))$$

$$x \mapsto (x, -Nx, \ldots, (-N)^a x)$$

is an embedding. The target is identified with $H^{-1}i^* j_*(K \otimes \mathcal{L}_a))$, which now has $N$ acting only by $1 \otimes N = 1 \otimes (\beta_{a,a+1} \circ \alpha_{a,a+1})$; i.e. by

$$(x, -Nx, \ldots, (-N)^a x) \mapsto (-N x, \ldots, (-N)^a x, 0);$$

via the above embedding, this means that $1 \otimes N = 1 \otimes (\beta_{a,a+1} \circ \alpha_{a,a+1})$ acts by $-N$, not $N$, on $\Psi_{un}^a(K)$.

(3.5) Sasha’s construction of $\Psi_{un}^a$ has the merit of behaving simply with respect to duality and admitting a description in terms of basic functors, but, as remarked above, whether it lands in $Y$ or $Y_K$ is insignificant since it lands in $Y_K$ with action of Gal($\overline{k}/k$) coming from the split-exact arithmetic-geometric sequence of $\pi_1$, and therefore can be regarded as a sheaf on $Y_k$ to begin with.
4. Duality. As before, $F_\ell = Q_\ell, E_\ell, \overline{Q}_\ell$, etc.

Lemma. — Let $f : X \to S$ be a separated morphism of finite type between schemes of finite type over $k$ and $\mathcal{L}$ a lisse $F_\ell$-sheaf on $S$. Then $f^!(\mathcal{L}) \cong f^* \mathcal{L} \otimes f^! F_\ell$.

Proof. — The counit of adjunction $f_* f^! \to \text{id}$ defines an arrow in the $H^0$ of
\[
R \text{Hom}(\mathcal{L} \otimes f^! F_\ell, \mathcal{L}) = R \text{Hom}(f_!(f^* \mathcal{L} \otimes f^! F_\ell), \mathcal{L}) = R \text{Hom}(f^* \mathcal{L} \otimes f^! F_\ell, f^! (\mathcal{L}))
\]
which is evidently locally an isomorphism. □

The lemma allows one to write, for $K \in D_\ell^b(X, F_\ell)$,
\[
\text{D}(K \otimes f^* \mathcal{L}_a) = R \text{Hom}(K, \text{D}(f^* \mathcal{L}_a^\vee)) = R \text{Hom}(K, f^! \text{D}(\mathcal{L}_a^\vee))
\]
\[
= R \text{Hom}(K, f^* \mathcal{L}_a^\vee \otimes f^! F_\ell)) = \text{D}(K) \otimes f^* \mathcal{L}_a^\vee \simeq \text{D}(K) \otimes f^* \mathcal{L}_a(a).
\]

(4.1) As $D$ is involutive and $\text{D}(X(a)) = \text{D}(X)(-a)$, the diagram

\[
\begin{array}{cccccc}
p^H i^* j_* (K \otimes \mathcal{L}_a) & \xrightarrow{\alpha_{a,a+1}} & p^H i^* j_* (K \otimes \mathcal{L}_{a+b+1}) & \xrightarrow{\beta_{b,a+1}} & p^H i^* j_* (K \otimes \mathcal{L}_b) \\
\uparrow & & \uparrow & & \uparrow \\
coker_a(K) & \xrightarrow{\alpha_{a,a+1}} & coker_{a+b+1}(K) & \xrightarrow{\beta_{b,a+1}} & coker_b(K)(-a-1) \\
\uparrow & & \uparrow & & \uparrow \\
\text{D} \text{ker}_a(\text{D}K)(a) & \xrightarrow{\text{D}(\text{ker}_{a+b+1}(\text{D}K)(a))} & \text{D} \text{ker}_{a+b+1}(\text{D}K)(a)(-b-1) & \xrightarrow{\text{D}(\text{ker}_b(\text{D}K)(a))} & \text{D} \text{ker}_b(\text{D}K)(a)(-b-1) \\
\uparrow & & \uparrow & & \uparrow \\
\text{D} \text{Psi}_f(\text{D}K)(-a) & \xrightarrow{\text{D}(\text{Psi}_{f+b+1}(\text{D}K)(-a))} & \text{D} \text{Psi}_f(\text{D}K)(-a-b-1) & \xrightarrow{\text{D}(\text{Psi}_f(\text{D}K)(-a-b-1))} & \text{D} \text{Psi}_f(\text{D}K)(-a-b-1)
\end{array}
\]
commutes, where the first line is isomorphic to the second by [BBD, 4.1.12.4], and we have used that the map $\alpha_{a,b}$ induces the map $\mathcal{L}_b^\vee = \mathcal{L}_b(b) \xrightarrow{\beta_{a,b}(b)} \mathcal{L}_a(a) = \mathcal{L}_a^\vee$ and vice versa. We need that $N^{b+1} = 0$ on $\text{Psi}_{f+b+1}(\text{D}K)$, and I don’t see why this is secured if $N^{b+1} = 0$ on $\text{Psi}_f(\text{D}K)$, so let’s modify the hypothesis to say ‘$N^{a+1} = 0$ on $\text{Psi}_f(\text{D}K)$ and $N^{b+1} = 0$ on $\text{Psi}_{f+b+1}(\text{D}K)$’.

5. The maximal extension functor. The $\beta$s in the commutative diagram defining $\gamma_{a,a-1}$ are $\beta_{a-1,a}$, not $\beta_{a,a+1}$.

10. MOREL’S NOTES

(5.1) It is of course the map $\beta_{a-1,a} : \mathcal{L}_a \to \mathcal{L}_{a-1}(-1)$ that is surjective. The snake lemma gives an isomorphism

$$\text{coker} \gamma_{a,a-1} \cong i^* \mathcal{H}^0 i^* \psi(K \otimes \mathcal{L}_{a-1})(-1).$$

To see that the last vertical arrow in the last commutative diagram of the proof is an isomorphism, it is easier to identify it with

$$i_\ast \psi_{\text{un}} K(-1) = \ker(j_!(K \otimes f^* \mathcal{L}_a)(-1)) \to j_\ast(K \otimes f^* \mathcal{L}_a(-1)) = i_\ast \psi_{\text{un}} K(-1),$$

which is the identity by the results of §3, noting that the square below commutes

$$\begin{array}{ccc}
\mathcal{L}_a & \xrightarrow{\alpha_{a,a+1}} & \mathcal{L}_{a+1} \\
\downarrow {\beta_{a-1,a}} & & \downarrow {\beta_{a,a+1}} \\
\mathcal{L}_{a-1}(-1) & \xrightarrow{\alpha_{a-1,a}(-1)} & \mathcal{L}_a(-1).
\end{array}$$

The remaining claims are now clear; given $0 \to K_1 \to K_2 \to K_3 \to 0$ exact, one deduces exactness of $0 \to \Xi_f K_1 \to \Xi_f K_2 \to \Xi_f K_3$ by finding an integer $a$ such that

$$\ker(j_!(K_i \otimes f^* \mathcal{L}_a)) \xrightarrow{\gamma_{a,a-1}} j_\ast(K_i \otimes f^* \mathcal{L}_{a-1})(-1) = \Xi_f K_i$$

for $i = 1, 2, 3$ and then applying the snake lemma to the following commutative diagram with exact rows

$$\begin{array}{ccc}
0 & \xrightarrow{j_!(K_1 \otimes f^* \mathcal{L}_a)} & j_!(K_2 \otimes f^* \mathcal{L}_a) & \xrightarrow{j_!(K_3 \otimes f^* \mathcal{L}_a)} & 0 \\
\downarrow {\gamma_{a,a-1}} & & \downarrow {\gamma_{a,a-1}} & & \downarrow {\gamma_{a,a-1}} \\
0 & \xrightarrow{j_\ast(K_1 \otimes f^* \mathcal{L}_a)} & j_\ast(K_2 \otimes f^* \mathcal{L}_a) & \xrightarrow{j_\ast(K_3 \otimes f^* \mathcal{L}_a)} & 0.
\end{array}$$

(5.4) Typo: it should read

$$\text{coker}(j_!(K \otimes f^* \mathcal{L}_b)(1) \to j_\ast(K \otimes f^* \mathcal{L}_{b-1})) = \text{coker}(j_!(K \otimes f^* \mathcal{L}_{b-1}) \to j_\ast(K \otimes f^* \mathcal{L}_{b-1}))$$

The point is, writing

$$\gamma_{a,a+1} = j_!(K \otimes \mathcal{L}_a) \to j_\ast(K \otimes \mathcal{L}_a) \xrightarrow{\alpha_{a,a+1}} j_\ast(K \otimes \mathcal{L}_{a+1}).$$
shows that \( \ker \gamma_{a,a+1} = \ker(j_1(K \otimes \mathcal{L}_a) \to j_*(K \otimes \mathcal{L}_a)) = \Psi^\text{un}_f(K) \) and the snake lemma begins with

\[
0 \to \Psi^\text{un}_f \xrightarrow{\alpha_{a,a+b+1}} \Psi^\text{un}_f \to \ker(\gamma_{b,b-1})(-a - 1) \to \cdots
\]

where \( \alpha_{a,a+b+1} \) induces an isomorphism of \( \Psi^\text{un}_f \). The dual statement about cokernels, after writing

\[
\gamma_{b,b-1}(a-1) = j_1(K \otimes \mathcal{L}_b)(a-1) \xrightarrow{\beta_{b,b-1}(a-1)} j_1(K \otimes \mathcal{L}_{b-1})(a-2) \to j_*(K \otimes \mathcal{L}_{b-1})(a-2),
\]

finds that

\[
coker \gamma_{b,b-1}(a-1) \simeq coker(j_1(K \otimes \mathcal{L}_{b-1})(-a - 2) \to j_*(K \otimes \mathcal{L}_{b-1})(a - 2)) \\
\simeq \Psi^\text{un}_f K(-a - 2)(-b + 1)
\]

so that, denoting \( \text{coker}(j_1(K \otimes \mathcal{L}_{a+b+1}) \to j_*(K \otimes \mathcal{L}_{a+b+1})) \) by \( \text{coker}(j_1 \to j_*) \), the snake lemma ends in

\[
\begin{array}{ccc}
\text{coker}(\gamma_{a,a+1}) & \xrightarrow{\beta_{b,a+b+1}} & \text{coker}(j_1 \to j_*) \\
\downarrow & & \downarrow \\
\text{coker}(\gamma_{a,a+1}) & \xrightarrow{\beta_{b-1,a+b+1}} & \text{coker}(j_1 \to j_*)(-a - 1) \\
\downarrow & & \downarrow \\
\text{coker}(\gamma_{a,a+1}) & \xrightarrow{\Psi^\text{un}_f K(-a - b - 1)} & \xrightarrow{\Psi^\text{un}_f K(-a - 2)(-b + 1)} \\
\end{array}
\]

using (4.1). Of course

\[
\beta_{b-1,a+b+1} : \text{coker}(j_1 \to j_*) \xrightarrow{\sim} \text{coker}(j_1 \to j_*)(-a - 1)
\]

is an isomorphism from the dual statement about \( \alpha \).

(5.6) Just apply \( \beta_{a,a+1} \) to the exact sequence of (5.5) for \( a \gg 0 \).

(5.7) If we denote the map \( i_*\Psi^\text{un}_f \to \Xi_f \) by \( \psi \), then the composition coincides with

\[
i_*\Psi^\text{un}_f D \xrightarrow{\psi} \Xi_f D = D\Xi_f \xrightarrow{D\psi} Di_*\Psi^\text{un}_f = i_*\Psi^\text{un}_f (-1)D,
\]

so that

\[
D(D\psi \circ \psi) = D\psi \circ DD\psi = D\psi \circ \psi.
\]
The morphism $\Xi_f \to i_\ast \Psi^\text{un}_f(-1)$ is induced by $\beta_{a-1,a} : j_!(K \otimes \mathcal{L}_a) \to j_!(K \otimes \mathcal{L}_{a-1})(-1)$ for $a \gg 0$. The dual morphism fits into a commutative diagram ($a \gg 0$)

\[
\begin{array}{ccc}
\text{coker}(j_!(K \otimes \mathcal{L}_{a-1})(a) \to j_!(K \otimes \mathcal{L}_{a-1})(a)) & \xrightarrow{\alpha_{a-1,a}(a)} & \text{coker}(\gamma_{a-1,a}(a)) \\
\uparrow & & \uparrow \\
\text{ker}(j_!(K \otimes \mathcal{L}_{a-1}) \to j_!(K \otimes \mathcal{L}_{a-1})) & \xrightarrow{\alpha_{a-1,a}} & \text{ker}(\gamma_{a,a-1}) \\
\uparrow & & \uparrow \\
i_\ast \Psi^\text{un}_f(K) & \leftarrow & \Xi_f(K).
\end{array}
\]

To see that the upper square commutes, it helps to recall that both vertical arrows are coboundaries in the snake lemma applied to two similar diagrams, and a morphism between these two diagrams can be built on $\alpha_{a-1,a}$ (applied to the upper right corners of the diagrams) and its twist (applied to the lower left corners). Therefore $D \upsilon \circ \upsilon = N : i_\ast \Psi^\text{un}_f \to \Psi^\text{un}_f(-1)$ and $DN = N$.

We can also study the composition $\Xi_f \to i_\ast \Psi^\text{un}_f(-1) \to \Xi_f(-1)$. The first map is induced by $\beta_{a,a-1}$ and the second by $\alpha_{a-1,a}(-1)$, as already remarked. We have (5.2) that

\[
\begin{align*}
\text{ker}(\gamma_{a,a-1}) & \xrightarrow{\beta_{a-1,a}} \text{ker}(\gamma_{a-1,a-2}(-1)) \\
\end{align*}
\]

induces $N$ on $\Xi_f$ for $a \gg 0$. Since $\gamma_{a,a-1}$ factors as $\beta_{a-1,a}$ followed by $j_!(K \otimes \mathcal{L}_{a-1})(-1) \to j_!(K \otimes \mathcal{L}_{a-1})(-1)$, $\beta_{a-1,a}$ applied to $\text{ker}(\gamma_{a,a-1})$ must actually factor as

\[
\begin{align*}
\text{ker}(\gamma_{a,a-1}) & \xrightarrow{\beta_{a-1,a}} \ker(j_!(K \otimes \mathcal{L}_{a-1}) \to j_!(K \otimes \mathcal{L}_{a-1}))(1) \leftarrow \text{ker}(\gamma_{a-1,a-2}(-1)).
\end{align*}
\]

The middle is $i_\ast \Psi^\text{un}_f(K)(-1)$, and if we postcompose the above morphism by $\alpha_{a-1,a}(-1)$ we don’t change it, as $\alpha_{a-1,a}$ induces an isomorphism $\Xi_fK \xrightarrow{\sim} \Xi_fK$ for $a \gg 0$. We have shown that $N : \Xi_f \to \Xi_f$ factors as

\[
\Xi_f \to i_\ast \Psi^\text{un}_f(-1) \leftarrow \Xi_f(-1)
\]
and the former map induces via the inclusion $i_*\Psi^\text{un}_f \hookrightarrow \Xi_f$ the morphism $N$ on $i_*\Psi^\text{un}_f$; i.e.

$$i_*\Psi^\text{un}_f \hookrightarrow \Xi_f \twoheadrightarrow i_*\Psi^\text{un}_f(-1) \quad \text{also composes to } N.$$  

Last, it is clear that the square

$$\begin{array}{ccc}
  j^{-1} & \hookrightarrow & \ker(\gamma_{a-1,a-2}) \\
  \downarrow & & \downarrow \\
  \Xi_f & \cong & \xi \\
  \downarrow & & \downarrow \\
  j_* & \hookleftarrow & \coker(\gamma_{a-1,a}(a))
\end{array}$$

commutes, where the leftmost down arrow is the canonical morphism, which shows that this morphism is also sent to itself by $D$.

6. The unipotent vanishing cycles functor.

Remark. The first paragraph discusses the construction of vanishing cycles. However, (6.2) shows that the functor $\Phi^\text{un}_f$ constructed is a direct factor of the the tame vanishing cycles which appears in the distinguished triangle

$$\Psi^\text{un}_f \xrightarrow{\can} \Phi^\text{un}_f \xrightarrow{} i^* \xrightarrow{}$$

where as before $\Psi^\text{un}_f = \tilde{\Psi}^\text{un}_f$.

In light of [BBD] 4.1.10.1, $\Psi^\text{un}_f$ sends $M(X)$ to $M(Y)$ because it has support in $Y$. (5.6) $\rightsquigarrow \epsilon(-1) \circ N = 0$. (5.2) $\rightsquigarrow N \circ \delta = 0$. It is easy to see that $\can(-1) \circ \text{var}$ induces the map $\Xi_f \rightarrow \Psi^\text{un}_f(-1) \hookrightarrow \Xi_f$. We studied this composition in the note to the previous section and showed that it coincides with $N$. 
There is an error in the second part of the proof that doesn’t change the conclusion. The following diagram is commutative with exact rows and columns.

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & i_*pH^{-1}i^*K \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\ker d^0 & \longrightarrow & K & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \ker \nu & \longrightarrow & i_*\Phi f^{\text{un}}K & \longrightarrow & i_*pH^0i^*K & \longrightarrow & 0 \\
\end{array}
\]

The snake lemma gives the exact sequence

\[
0 \rightarrow \ker \kappa \rightarrow j_*j^*K \rightarrow j_*j^*K/\ker d^0 \rightarrow \coker \kappa \rightarrow 0
\]

so that \( \kappa \) is surjective and \( \ker \kappa = i_*pH^{-1}i^*K \). This doesn’t change the conclusion that \( i_*\text{coker can} \sim i_*pH^0i^*K \). Moreover these conclusions for \( \ker(i_*\text{can}) \), \( i_*\text{coker can} \) imply the results without the \( i_* \) because \( i_* \) is t-exact and fully faithful.

7. **The functor \( \Omega_f \).** \( \Omega_f \) can be defined as \( \ker(\varepsilon + \text{adj}) \) or as \( \ker(\varepsilon - \text{adj}) \), and the isomorphism of complexes \( C'K \sim C^\ast K \) (6.1) carries the latter isomorphically onto the former. In particular, this gives an isomorphism \( \Omega_f \simeq \ker d^0 \), and, in light of the middle row of the diagram \( \dag \) above, the second short exact sequence. The first short exact sequence is (6.1) on the nose.

The argument that gives the quasi-isomorphism \((\Psi f^{\text{un}}j^* \rightarrow \Phi f^{\text{un}}) \rightarrow i^*\) actually establishes the existence of a distinguished triangle

\[
\Psi f^{\text{un}}j^* \rightarrow \Phi f^{\text{un}} \rightarrow i^* \rightarrow
\]

which is the one we expect from the ‘usual’ definition of \( \Phi f^{\text{un}} \). But while the first two functors take perverse sheaves to perverse sheaves, \( i^* \) has perverse amplitude in \([-1, 0]\).
It is strange to phrase this as a ‘quasi-isomorphism,’ since \( \Psi \) un is not in \( \mathcal{D}^b(Y) \), while \( i^* K \) is usually not a perverse sheaf. I prefer just to say that we have the distinguished triangle above.

8. Gluing. \( \mathcal{D}(c) \) is a complex because the composition of the two differentials gives \( N = N = 0 \).

(8.1) Both directions of the proof are wrong. The first part doesn’t work because \( \Phi^{un}_f \mathcal{D}(c) \) does not have the stated form. Note that the form that is claimed has \( b \circ a = 0 \) but this should instead give \( N \). The second part doesn’t work because the square on the right in the morphism of exact sequences doesn’t commute: tracing it down-right gives the canonical \( \Omega_f \to i_* \Psi^{un}_f j^* K(-1) \) while tracing it right-down gives the null morphism.

Fortunately, Sasha developed a clever device in his appendix on diads that gets the job done. Conforming to the language of the next section, we consider a category \( \mathcal{M}(X)^\sharp_1 \) of diads of perverse sheaves on \( X \). An object of this category is a diagram

\[
\begin{array}{cccc}
\alpha_\neq=(\alpha^1_\neq, \alpha^2_\neq) & \rightarrow & A \oplus B & \rightarrow \alpha_\neq=(\alpha^1_\neq, \alpha^2_\neq) \\
\downarrow & & \downarrow & \\
C_- & \rightarrow & \mathcal{A} & \rightarrow \rightarrow \rightarrow C_+
\end{array}
\]

in which \( \alpha^1_- \) is a monomorphism and \( \alpha^1_+ \) an epimorphism. Morphisms are defined pointwise and such a morphism of diads is a (mono/epi/iso)morphism if it is pointwise; one has \( (\mathcal{M}(X)^\sharp_1)^\circ = (\mathcal{M}(Y)^\circ)^\sharp_1 \) and exact functors \( \mathcal{M}(X) \to \mathcal{M}(Y) \) induce exact functors \( \mathcal{M}(X)^\sharp_1 \to \mathcal{M}(Y)^\sharp_1 \). The category \( \mathcal{M}(X)^\sharp_1 \) is endowed with an involutive autoequivalence \( r \) called the reflection functor, which we compute explicitly \[ \text{in the next section} \].

The category of perverse sheaves embeds into \( \mathcal{M}(X)^\sharp_1 \) via the functor \( C : \mathcal{M}(X) \hookrightarrow \mathcal{M}(X)^\sharp_1 \) which sends a perverse sheaf \( K \) to the diad

\[
j_! j^* K \rightarrow \Xi j^* K \oplus K \rightarrow j_* j^* K.
\]

Likewise the category of gluing data embeds into \( \mathcal{M}(X)^\sharp_1 \) via the functor \( D \) which sends a gluing datum \( c = (K_U, K_V, u, v) \) to the diad

\[
i_* \Psi^{un}_f K_U \rightarrow \Xi f K_U \oplus i_* K_Y \rightarrow i_* \Psi^{un}_f K_U(-1).
\]

Identifying the category of perverse sheaves and the category of gluing data with their respective essential images in \( \mathcal{M}(X)^\sharp_1 \) via \( C \) (resp. \( D \)), \( r \) exchanges these two
subcategories of $M(X)^\#_1$, and in doing so coincides with the functors $F$ and $G$. To verify this, we use our explicit computation of $r$ in the next section to check that
\[ r(C(K)) = i_* \Psi^\text{un} j^* K \xrightarrow{\text{can} \oplus \text{can}} \Xi f^* K \oplus i_* \Psi^\text{un} K \xrightarrow{\text{can} - \text{var}} i_* \Psi^\text{un} j^* K(-1), \]
\[ r(D(c)) = j_! K \mathcal{U} \delta \oplus \text{adj} \xrightarrow{\varepsilon - \text{adj}} \Xi f^* \mathcal{U} \oplus H^0(D(c)) \xrightarrow{\varepsilon - \text{adj}} j_* \mathcal{U}. \]

As $r$ is an involution, it induces an equivalence between the category of perverse sheaves and the category of gluing data, embedded as subcategories of $M(X)^\#_1$, completing the proof.

We now provide details of the proofs of the statements of Sasha’s appendix on monads and diads. Reich provides details for the key statement about the equivalence of diads of type $A^\#_1$ and $A^\#_2$ but his proof is wrong (he claims certain maps obtained from a diad of type $A^\#_2$ should yield an exact sequence when they don’t even form a complex).

**Sasha’s Appendix A1 & A2.** Let $\mathcal{A}$ denote an exact category.

**Proof of Sasha’s Lemma A1.** — Let $\mathcal{A}^b =: \mathcal{A}^b_0$ denote the category of monads and $\mathcal{A}^b_i$ Sasha’s $\mathcal{A}^\gamma_i$. Let $T_i : \mathcal{A}^b_i \rightarrow \mathcal{A}^b_{i+1}$ be the functors described, where $i \in \mathbb{Z}/3\mathbb{Z}$.

Define the functor $T_0^{-1} : \mathcal{A}^b_1 \rightarrow \mathcal{A}^b_0$ by
\[ (P_{-1} \xleftarrow{\gamma^{-1}} P_{0} \xrightarrow{\gamma^0} P_{1}) \mapsto (P_{-1} \xrightarrow{\gamma^0 \gamma^{-1}} P_{1} \rightarrow P_{1}/P_{0}). \]

$T_0^{-1}$ is clearly an inverse to $T_0$.

Turning to $\mathcal{A}^b_2$, first note that the following square is a pushout and pullback square, and therefore $\varepsilon_-$ is an admissible epimorphism and $\varepsilon_+$ an admissible monomorphism.

\[
\begin{array}{ccc}
L_- & \xrightarrow{\varepsilon_-} & B \\
\downarrow{\delta_-} & & \downarrow{\varepsilon_+} \\
A & \xrightarrow{\delta_+} & L_+
\end{array}
\]

This is true because to give a morphism to $A$ and $B$ which agrees on $L_+$ via $\delta_+$ and $-\varepsilon_+$ is the same as giving a kernel for $A \oplus B \xrightarrow{\delta_+ + \varepsilon_+} L_+$, and dually. We therefore have a commutative diagram in which the right square is bicartesian, and claim that the dashed
arrow is an isomorphism.

\[
\begin{array}{c}
\ker \varepsilon_- \to L_- \xrightarrow{\varepsilon_-} B \\
\downarrow \quad \downarrow \delta_- \quad \downarrow -\varepsilon_+ \\
\ker \delta_+ \to A \xrightarrow{\delta_+} L_+
\end{array}
\]

To give a morphism to \(A\) which \(\delta_+\) sends to zero is (since \(\varepsilon_+\) is a monomorphism) the same as giving a morphism to \(A \oplus B\) which is in the kernel of \(\delta_+ + \varepsilon_+\) and with null projection to \(B\). This is the same as giving a morphism to \(L_-\) which \(\varepsilon_-\) sends to zero, which is the same as giving a morphism to \(\ker \varepsilon_-\). Therefore every monad in \(A_2^b\) has \(\varepsilon_-\) an admissible epimorphism and \(\varepsilon_+\) an admissible monomorphism, and

\[
(L_- \xrightarrow{(\delta_- \varepsilon_-)} A \oplus B \xrightarrow{(\delta_+ \varepsilon_+)} L_+) \simeq (L_- \xrightarrow{(\delta_- \varepsilon_-)} A \oplus L_- / \ker \varepsilon_- \xrightarrow{(\delta_+ \varepsilon_+)} A / \ker \varepsilon_-),
\]

where \(\varepsilon_+\) is induced by \(\delta_-\). The composition of functors \(T_0 \circ T_2\) sends

\[
(L_- \xrightarrow{(\delta_- \varepsilon_-)} A \oplus B \xrightarrow{(\delta_+ \varepsilon_+)} L_+) \mapsto (\ker \varepsilon_- \xhookrightarrow{L_-} A),
\]

and by the above isomorphism is clearly an equivalence with inverse

\[
T_1 : (\ker \varepsilon_- \xhookrightarrow{L_-} A) \mapsto (L_- \xhookrightarrow{A \oplus L_- / \ker \varepsilon_-} A / \ker \varepsilon_-).
\]

Therefore \(T_1\) is also an equivalence with inverse \(T_0 \circ T_2\) and \(T_2\) is an equivalence with inverse \(T_1 \circ T_0\):

\[
\mathcal{P} := P_- \xhookrightarrow{\alpha_+} P_+ \\
\downarrow T_0 \\
P_- \xhookrightarrow{\ker \alpha_+} P \\
\downarrow T_1 \\
\ker \alpha_+ \xhookrightarrow{P \oplus H(\mathcal{P})} P/P_- \\
\downarrow T_2 \quad \downarrow \mathcal{P}.
\]

\[\square\]

In §A2 the arrows \(A \oplus B^i \to D_+\) are admissible epimorphisms, not monomorphisms.
Proof of Sasha’s Lemma A2. — Let $\mathcal{A}^\# =: \mathcal{A}^\#_0$ denote the category of diads and $\mathcal{A}^\#_1$ the related categories. Let $T^\#_0 : \mathcal{A}^\#_0 \to \mathcal{A}^\#_1$ and $T^\#_1 : \mathcal{A}^\#_1 \to \mathcal{A}^\#_2$ be as described; $T^\#_0$ is obviously an equivalence. Note that $\mathcal{A}^\#_1$ is a full subcategory of $\mathcal{A}^\#_0$ and $\mathcal{A}^\#_2$ is a full subcategory of $\mathcal{A}^\#_1$. Therefore $T^\#_1$ is just the restriction of $T_1 \circ T_0$ to $\mathcal{A}^\#_1$. We have to check that $T^\#_1$ actually lands in $\mathcal{A}^\#_2 \subset \mathcal{A}^\#_0$. Once that is established, if we can show that $T_2$ (the inverse of $T_1 \circ T_0$), restricted to $\mathcal{A}^\#_2$, lands in $\mathcal{A}^\#_1$, Lemma A1 then implies that $T^\#_1$ is an equivalence with inverse $T_2|_{\mathcal{A}^\#_2}$. We have

$$Q := C_- \xleftarrow{\alpha_+ = (\alpha^1_-, \alpha^2_-)} A \oplus B \xrightarrow{\alpha_+ = (\alpha^1_+, \alpha^2_+)} C_+ \xrightarrow{T_1 \circ T_0 = T^\#_1}$$

$$\ker(\alpha_+) \hookrightarrow A \oplus B \oplus H(Q) \twoheadrightarrow (A \oplus B)/C_-$$

and to show that $T^\#_1$ lands in $\mathcal{A}^\#_2$ we must secure that $\ker(\alpha_+) \hookrightarrow A \oplus H(Q)$ and $A \oplus H(Q) \twoheadrightarrow (A \oplus B)/C_-$. The commutative diagram below has exact rows.

$$\begin{array}{ccc}
C_- & \xrightarrow{\ker(\alpha_+)} & H(Q) \\
\downarrow & & \downarrow \\
A & \xrightarrow{\downarrow} & A \oplus H(Q) \xrightarrow{\downarrow} H(Q) \\
\downarrow & & \downarrow \\
C_+ & \xleftarrow{(A \oplus B)/C_-} & \ker(\alpha_+)/C_- \\
\end{array}$$

Both claims follow from the five lemma for exact categories.

It remains only to show that $T_2|_{\mathcal{A}^\#_2}$ lands in $\mathcal{A}^\#_1 \subset \mathcal{A}^\#_0$; then it will be an inverse to $T^\#_1$. We have

$$D_- \xleftarrow{(\gamma_-, \delta^1_-, \delta^2_-)} A \oplus B^1 \oplus B^2 \xrightarrow{(\gamma_+, \delta^1_+, \delta^2_+)} D_+ \xrightarrow{T_2}$$

$$\ker(\delta^2_-) \hookrightarrow A \oplus B^1 \twoheadrightarrow \coker((\gamma_-, \delta^1_-, \delta^2_-) : D_1 \to A \oplus B^1)$$

and we need to verify that $\ker \delta^2_- \to A$ is admissible monomorphism and $A \to \coker((\gamma_-, \delta^1_-, \delta^2_-) : D_1 \to A \oplus B^1)$ is admissible epimorphism. We know from A1 that $\delta^1_-$ and $\delta^2_- \oplus \delta^2_+$ are admissible epimorphisms while $\delta^1_-$ and $\delta^2_+$ are admissible monomorphisms.
If \( \ker \delta^2 \to A \) admitted a cokernel, then it would follow from the obscure axiom applied to the commutative diagram with exact rows

\[
\begin{array}{ccc}
\ker \delta^2 & \to & D_-
\\
\downarrow & & \downarrow
\\
A & \to & A \oplus B^2
\end{array}
\begin{array}{ccc}
& & \to
\\
& & \\
& & \\
\to & B^2 & \\
& & \\
& & \\
& & \\
D_- & \to & B^2
\end{array}
\]

that \( \ker \delta^2 \to A \) would be an admissible monomorphism. En effet,

\[
\text{coker}(\ker \delta^2 \to A) = \text{coker}(D_- \xrightarrow{(\gamma_- \delta^2)} A \oplus B^2),
\]

as both can be described as the colimit of the following diagram.

\[
\begin{array}{ccc}
D_- & \to & B^2
\\
\downarrow & & \\
\gamma_- & & \\
A & \to &
\end{array}
\]

This identification additionally shows that the diagram

\[
\begin{array}{ccc}
\ker \delta^2 & \to & A
\\
\downarrow & & \downarrow
\\
D_- & \to & A \oplus B^1
\end{array}
\begin{array}{ccc}
& & \to
\\
& & \\
& & \\
& & \\
& & \\
& & \\
\to & coker((\gamma_-, \delta^1_-) : D_- \to A \oplus B^1)
\end{array}
\]

commutes so that \( A \to \text{coker}((\gamma_-, \delta^1_-) : D_- \to A \oplus B^1) \) is an admissible epimorphism. (Moreover the square to the left in this diagram is bicartesian [Bühler 2.12].) We’ve proved that the functors

\[
\begin{array}{ccc}
\mathcal{A}^\sharp_1 & \xleftarrow{T^\sharp_1} & \mathcal{A}^\sharp_2
\end{array}
\]

are mutually inverse equivalences. □


The reflection functor. We wish to compute \( T_2|_{\mathcal{A}^\sharp_1} \circ r \circ T_1^\sharp \), which is an involution of the category \( \mathcal{A}^\sharp_1 \) (since \( r \) is an involution of \( \mathcal{A}^\sharp_2 \) and \( T_2|_{\mathcal{A}^\sharp_2} = (T_1^\sharp)^{-1} \)) that we will also
denote by \(r\); the diagram below illustrates the composition.

\[
\begin{align*}
\mathcal{Q} := C_+ & \xrightarrow{\alpha = (\alpha_1, \alpha_2)} A \oplus B \xrightarrow{\alpha = (\alpha_1, \alpha_2)} C_+ \\
\text{ker}(\alpha_+) & \xrightarrow{(\gamma, \delta_1, \delta_2)} A \oplus B \oplus H(Q) \xrightarrow{(\gamma, \delta_1, \delta_2)} (A \oplus B)/C_+ \\
\text{ker}(\alpha_+) & \xrightarrow{(\gamma, \delta_2, \delta_1)} A \oplus H(Q) \oplus B \xrightarrow{(\gamma, \delta_2, \delta_1)} (A \oplus B)/C_+ \\
\text{ker}(\delta_1) & \hookrightarrow A \oplus H(Q) \twoheadrightarrow \text{coker}(\ker(\alpha_+) \hookrightarrow A \oplus H(Q))
\end{align*}
\]

It is easy to see that \(\ker(\delta_1 : \ker(\alpha_+) \twoheadrightarrow B) = \ker(\alpha_1)\) because to give a morphism to \(\ker(\alpha_+)\) which goes to zero under \(\ker(\alpha_+) \hookrightarrow A \oplus B \twoheadrightarrow B\) is the same as giving a morphism to \(A\) which goes to zero under \(\alpha_+^1\). As for \(\text{coker}(\ker(\alpha_+) \hookrightarrow A \oplus H(Q))\), to give a map from \(A \oplus H(Q)\) that kills \(\ker(\alpha_+)\) is to give a map \(a_1\) from \(A\) and \(a_2\) from \(H(Q)\) such that \(a_1 - a_2\) kills \(\ker(\alpha_+)\); as \(C_+ \hookrightarrow \ker(\alpha_+) \twoheadrightarrow H(Q)\) is short exact, to give a map from \(A\) which coincides on \(\ker(\alpha_+)\) with a map from \(H(Q)\) is to give a map from \(A\) which annihilates \(C_+\). Therefore \(\text{coker}(\ker(\alpha_+) \hookrightarrow A \oplus H(Q)) = \text{coker}(C_+ \hookrightarrow A)\), so

\[
\begin{align*}
\ker(\alpha_1) & \hookrightarrow A \oplus H(Q) \twoheadrightarrow \text{coker}(\alpha_1^1) = A/C_+.
\end{align*}
\]

The map \(\ker(\alpha_1^1) \hookrightarrow A\) is the natural inclusion and \(A \twoheadrightarrow A/C_+\) the natural projection; the map \(\ker(\alpha_1^1) \hookrightarrow H(Q)\) coincides with \(a\) in the below commutative diagram (with rows that are not exact)
Finally, the map $H(Q) \to A/C_-$ coincides with $-b$ in the diagram above, the negative sign appearing because we have $-\gamma_0 : P_0/P_{-1} \to P_1/P_{-1}$ in the definition of the functor $T_1$ so that $\delta^2_+: H(Q) \to (A \oplus B)/C_-$ coincides with $-\iota$ in the composition

$$H(Q) \xrightarrow{\delta^2_+} (A \oplus B)/C_- \to A/C_-.$$
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