124 Final REVIEW OUTLINE ©

Matt J.

Outline

1. **Limits**
   (a) Bag of Tricks
      i. **L'Hospital**
   (b) Continuity
   (c) Limit Definition of Derivative
2. **Explicit Derivatives**
   (a) Drawing graph of f' versus f
   (b) Chain Rule, Product Rule, Quotient Rule
   (c) Trig and ArcTrig
   (d) Periodic Derivatives
3. **Implicit Derivatives**
   (a) Logarithmic Differentiation
   (b) Curves Defined by Relations
   (c) Parametric Derivatives
4. **Tangent Line Approximation**
   (a) For Functions
   (b) For Implicitly Defined Curves
   (c) Over or Underestimate?
5. **Related Rates**
   (a) With Angles
      i. Basic Trig
      ii. Law of Sines
      iii. Law of Cosines
   (b) Pythagorean
   (c) Similar Triangles
   (d) Volumes or other Geometric Formulas
   (e) Curves
6. **Critical Points**

(a) Max and Min
(b) Points of inflection
(c) Increasing and decreasing on intervals
(d) Concavity
(e) Vertical and Horizontal Asymptotes
(f) Drawing Graphs from This Data

7. **Optimization**
    (a) Perimeter
    (b) Area
    (c) Volume
    (d) With Functions

Stuff For Your Note Sheet

1. Formula for Parametrizing a circle $y(t) = A\sin(\omega t - \theta) + B$: $\quad x(t) = A\cos(\omega t - \theta) + C$
2. Trig Identities
    (a) Double and Half Angle Stuff
    (b) Wierd Sum Rules
    (c) Derivatives of arctrig functions
    (d) Law of Sines
    (e) Law of Cosines
    (f) Reference Triangles
3. Definitions of Continuity and Limit
4. Limit Definition of Derivative
5. Parametric First and Second Derivative Definitions
6. DUFADIP-U (*Diagram, Unknowns/Knows, Formula, Algebra, Differentiate, Isolate, Plug In*)
7. PLARHDTSLAW-C (*Picture, Label, Assess (Constraint/Formula), Relationship, Domain, Hone Down, Take Derivative, Set Equal to Zero, Local Max or Min?, Ascertain Globalness, Write a Sentence, Check you answered the question*)
8. Remember you must have an indeterminant quotient to apply L'höpital.
9. Remember how to prove a point is a global max or min when optimizing
    (a) If your domain is closed, i.e $[a, b]$, then confirm your critical point is a local min or local max via second derivative test or perhaps by checking the first derivative at nearby points. Then test all critical points and endpoints and make sure you say something about the domain being closed.
    (b) If your domain is not closed, i.e $(0, \infty)$ then you will most likely have only one critical point. Prove it is a local max or min and then state that since there is only one critical point you know it is a global max or min (Make sure you understand why).

Problems

1. **Limits**

   (a) Bag of Tricks

   i. L'Hospital

   $$\lim_{t\to 0}\frac{t\tan(t)}{1-\cos t}$$

   $$\lim_{x\to\infty} x\sin(\frac{\pi}{x})$$

   (b) Continuity

   i. A function f is continuous at a point a if and only if (1) a belongs to the domain of f and (2) $\lim_{x\to a^-} f(x) = f(a) = \lim_{x\to a^+} f(x)$

   Find c so that the function $F(x) = \begin{cases} ce^{x^2}, & x \geq \sqrt{2} \\ x^2, & x < \sqrt{2} \end{cases}$ *is continuous. Explain.*

   (c) Limit Definition of Derivative

   Calculate the derivatives of the following functions using the limit definition:

   $$f(x) = e^x$$

   To do the above you need to assume $(e^x)' = e^x$*, kinda circular but still good practice*

   $$g(t) = \frac{t}{3t+5}$$

2. **Explicit Derivatives**

   (a) Drawing graph of f' versus f *To practice, just draw something call it* f *then draw* f'

   (b) Chain Rule, Product Rule, Quotient Rule

   (c) Trig and ArcTrig

   (d) Periodic Derivatives

   $$f(x) = e^{2x} - x^n + 30x^{n-1} + \frac{1}{x}$$

   Find $f^{(n)}(x)$

3. **Implicit Derivatives**

   (a) Logarithmic Differentiation

   $$y = \ln(x)^{\ln(x)}$$

   $$y = (\sqrt{x})^{e^{\cos(\ln(x))}}$$

   (b) Strange Curves

2. (12 points) Consider the curve given by the following equation:

$$x + x^2 - 2y - \sin(x + y) = 0$$

(a) Calculate $\frac{dy}{dx}$.

(b) Check that the point $(-3, 3)$ is on the curve. Compute the equation of the tangent line to the curve at this point.

(c) Parametric

Note that $x(t) = e^{-t} + t$ in the following problem

4. (12 points) Consider the curve given by the following parametric equations:

$$\begin{aligned} x(t) &= e^{-t} + t \\ y(t) &= t\,e^{-\frac{t}{2}} \end{aligned}$$

(a) Calculate $\frac{dy}{dx}$.

(b) Find the coordinates of the point at which the tangent line to the above curve is horizontal.

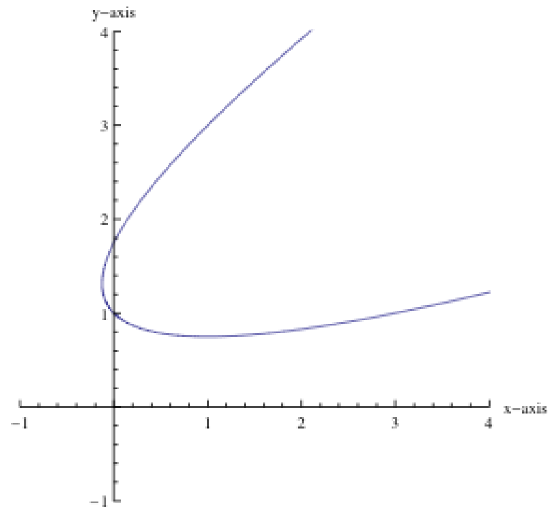
(c) Find the coordinates of the point at which the tangent line to the above curve is vertical.

5. (12 points)

Let C be the curve given by the parametric equations

$$\begin{cases} x(t) &= 2t^2 - t \\ y(t) &= t^2 + t + 1. \end{cases}$$

where $-\infty < t < \infty$.



.

(a) Find all points P on C for which the tangent line to C at P passes through the point $(1, 0)$.

(b) Now suppose that a particle is traveling along the curve C. Its position at time t is given by $(x(t), y(t))$. When is the particle heading directly towards the point $(1, 0)$?

4. **Tangent Line Approximation**

   (a) Single Variable

   8. (13 points) Find the linear approximation of the function $y = 2\ln(1+x) + e^x + 5x^2$ at the point $x = 0$ and use it to approximate the solution to the equation

$$1.01 = 2\ln(1+x) + e^x + 5x^2.$$

   (b) Implicit

   Estimate using tangent line approximation y_0 for the point $(-\pi, y_0)$ using the basepoint (-3,3) on the curve: $x + x^2 - 2y - \sin(x+y) = 0$.

   (c) Over or Underestimate?

   To check just take second derivative or use a graph. $f'' < 0$ then over, $f'' > 0$ then under. f'' is evaluated at basepoint

5. **Related Rates**

   (a) With Angles

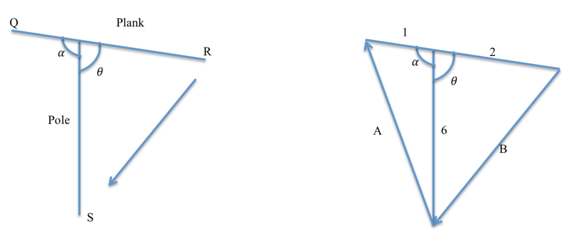
      i. Basic Trig

   4. (8 points) A UFO flies horizontally at a constant speed at an altitude of 15 km and passes directly over a tracking telescope on the ground. When the angle of elevation is $\pi/3$, this angle is decreasing at a rate of 0.1 rad/min. How fast is the UFO flying? (Include units in your final answer.)

      ii. Law of Sines

      iii. Law of Cosines

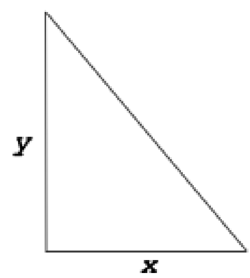
      A 3 meter plank is balanced on a 6 meter pole so that 1 meter lies on the left and 2 meters on the right side of the pole.



Suppose the point R is moving downward so that the distance between it and point S is shrinking at 3 m/s. When R is a distance of $4\sqrt{2}$ meters from S the law of cosines tells us that $\theta = \frac{\pi}{3}$ and that the distance from Q to S is $\sqrt{31}$. At this point in time find the rate the distance between Q and S is changing. That is, in the diagram on the right find $\frac{dA}{dt}$ for the initial conditions stated above. *The Law of Cosines is that $B^2 = C^2 + D^2 - 2CD\cos\theta$ with θ the angle opposite side B.*

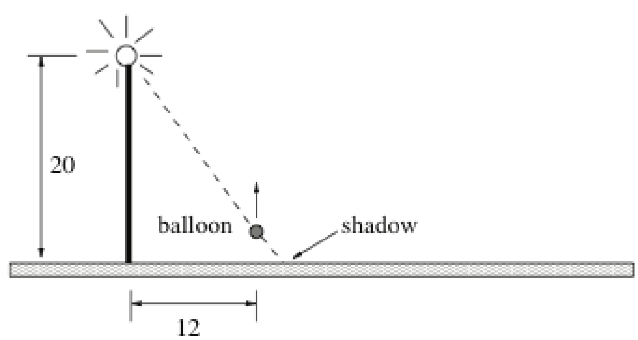
(b) Pythagorean

5. (12 points) The right triangle shown below is undergoing a transformation so that the side marked x is **increasing** at the rate of 3 cm/second and the side marked y is **decreasing** at the rate of 2 cm/second. How fast is the **perimeter** changing when $x = 50$ cm and $y = 120$ cm? Is the perimeter increasing or decreasing at that instant?



(c) Similar Triangles

3. A small helium balloon is rising at the rate of 8 ft/sec, a horizontaldistance of 12 feet from a 20 ft. lamppost. At what rate is the shadow of the balloon moving along the ground when the balloon is 5 feet above the ground?



(d) Volumes or other Geometric Formulas

(e) Curves

A marble is rolling along the curve

$$xe^{2\sin\theta} = 1$$

If the x coordinate of the marble is changing at a rate of 2 $\frac{cm}{sec}$ at the point $(x, \theta) = (1, \pi)$, then at what rate is the θ coordinate changing?

6. **Critical Points**

(a) Max and Min

(b) Points of inflection

(c) Increasing and decreasing on intervals

(d) Vertical and Horizontal Asymptotes

(e) Drawing Graphs from This Data

$$y = e^x(x-1)^2$$

$$y = \frac{x^3}{x^2-1}$$

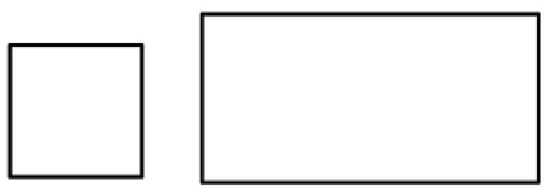
Do (a-d) with the two functions just listed

7. **Optimization**

(a) Perimeter Stuff

3. (12 points) A farmer has 136 meters of fencing. She wants to make two rectangular enclosures. One will be square. The other will have its long side twice as long as its short side. (Allow the possibility that all of the fencing could go to only one of the enclosures.)

For instance, the enclosures might look like this:



(a) What should the dimensions of the enclosures be to make the combined total area of the enclosures as **small** as possible?

(b) What should the dimensions of the enclosures be to make the combined total area of the enclosures as **large** as possible?

(b) Area

8. [**10 points total**] A rectangular poster is to have an area of 2700 cm^2 with 3-cm margins at the bottom and sides and a 5-cm margin at the top. What dimensions will give the largest printed area (the area inside the margins)? Please show your work, and justify that your solution truly is a maximum.

(c) Volume

5. [**14 points**] A mass of clay of volume $\frac{4}{3}\pi\ \text{in}^3$ is formed into two spheres. How should the clay be divided to make the total surface area of the two spheres is

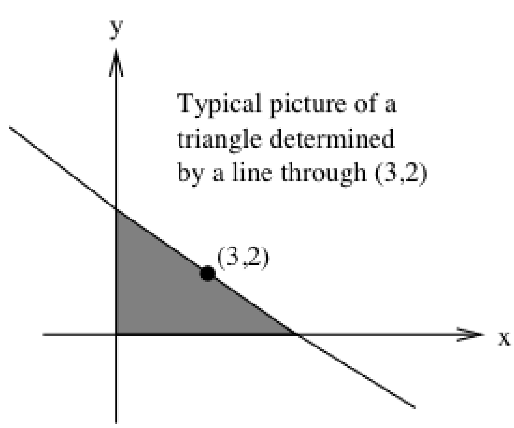
(a) a maximum?

(b) a minimum?

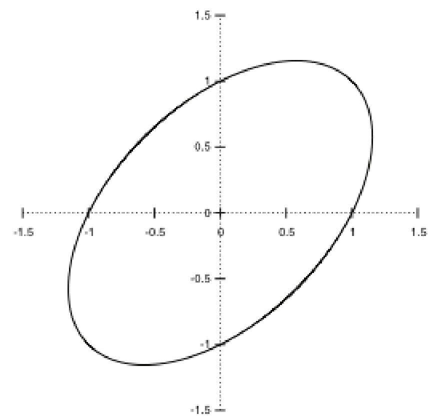
Note: the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$ and its total surface area is $4\pi r^2$.

(d) With Curves

7. (12 points) Find the equation of the line passing through the point (3,2) which cuts off the triangle of least area from the first quadrant.



6. (10 points) Find the point on the ellipse $x^2 + y^2 - xy = 1$ with the largest y coordinate.



(e) Random Stuff That is Hard to Anticipate

Gordo wants two positive real numbers such that their product is 100 and the sum of the first number with the square of the second number is as small as possible. What should Gordo choose?